Worcester Polytechnic Institute
Department of Electrical and Computer Engineering

ECE 505 – Analysis of Deterministic Signals and Systems

Final Project Report

Helicopter Control System: Stability Analysis, LQR Control, Estimation, Output Feedback, and Reference Tracking

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1 Introduction

This report presents a complete analysis and control design for a multivariable helicopter system using state-space methods covered in ECE 505. The system represents a twin-engine helicopter with eight states, four control inputs, and six outputs. The objective is to evaluate stability, determine controllability and observability, design a stabilizing Linear Quadratic Regulator (LQR), compute a full-state estimator, synthesize a dynamic output-feedback controller, and finally design a reference-tracking controller for the first output channel.

The helicopter considered in this study is a high-dimensional, multi-input multi-output (MIMO) linearized model representative of a twin-engine aircraft equipped with independent rotor blade controls. Such systems are inherently complex due to aerodynamic coupling, nonlinearities, and cross-axis interactions. The purpose of this project is to explore how classical state-space techniques can be used to stabilize and control such a system, which would otherwise be extremely difficult for manual control.

The MATLAB implementation follows a structured workflow, beginning with open-loop characterization and concluding with a fully integrated observer-based controller capable of accurate reference tracking. Each step illustrates the practical considerations behind real-world flight control systems.

2 System Description

2.1 State-Space Model

The helicopter dynamics are given by:

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$

The system has 8 states, 4 inputs, and 6 outputs. The state variables consist of angular attitudes, angular rates, and translational velocities, capturing rigid-body rotorcraft dynamics. The system's structure shows strong coupling between attitude, velocity, and rotor-induced aerodynamic forces, making decoupled single-input control infeasible.

The matrices below represent the full linearized dynamics.

Matrix A

$$A = \begin{bmatrix} 0 & 0 & 0 & 0.9986 & 0.05338 & 0 & 0 & 0 \\ 0 & 0 & 1 & -0.003182 & 0.05952 & 0 & 0 & 0 \\ 0 & 0 & -11.5705 & -2.5446 & -0.06360 & 0.10678 & -0.09492 & 0.007108 \\ 0 & 0 & 0.43936 & -1.99818 & 0 & 0.016651 & 0.018462 & -0.001187 \\ 0 & 0 & -2.0409 & -0.458999 & -0.73503 & 0.019256 & -0.004596 & 0.002120 \\ -32.1036 & 0 & -0.503355 & 2.29786 & 0 & -0.021216 & -0.021168 & 0.015812 \\ 0.102161 & 32.05783 & -2.34722 & -0.503611 & 0.834948 & 0.021227 & -0.037880 & 0.000354 \\ -1.91097 & 1.713829 & -0.004005 & -0.057411 & 0 & 0.0139896 & -0.0009068 & -0.290514 \end{bmatrix}$$

Matrix B

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.124335 & 0.082786 & -2.75248 & -0.017889 \\ -0.036359 & 0.475095 & 0.014291 & 0 \\ 0.304492 & 0.014958 & -0.496518 & -0.206742 \\ 0.287735 & -0.544506 & -0.016379 & 0 \\ -0.019073 & 0.016367 & -0.544536 & 0.234842 \\ -4.82063 & -0.000381 & 0 & 0 \end{bmatrix}$$

Matrix C

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.0595 & 0.05329 & -0.9968 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.05348 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix D

$$D = 0_{6\times4}$$

3 Stability Analysis

3.1 Eigenvalues of A

The eigenvalues of the open-loop system are:

$$\lambda(A) = \begin{cases} -11.4968, \ -2.3036, \ -0.7104, \ -0.2923, \\ -0.1593 \pm 0.5990i, \ 0.2342 \pm 0.5513i \end{cases}$$

Two eigenvalues have positive real parts, demonstrating that the system is inherently unstable. Several others lie close to the imaginary axis, revealing lightly damped oscillatory modes that would make the helicopter difficult to control without feedback. This behavior is consistent with real aircraft, which often rely on active stabilization systems.

3.2 BIBO Stability

MATLAB returned:

```
System is BIBO UNSTABLE
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because unstable internal poles guarantee that bounded inputs can produce unbounded outputs.

4 Transfer Function $G_{11}(s)$

$$G_{11}(s) = \frac{4.821s^7 + 69.23s^6 + 167.5s^5 + 106.2s^4 + 86.83s^3 + 52.51s^2 + 18.26s + 12.6}{s^8 + 14.65s^7 + 38.91s^6 + 32.07s^5 + 24.32s^4 + 16.02s^3 + 6.919s^2 + 3.694s + 0.7579}$$

This transfer function reveals the high-order nature of the helicopter's internal dynamics. The multiple poles represent various aerodynamic and inertial modes, including oscillatory components, and emphasize why simple PID control would be insufficient.

5 Open-Loop Step Response

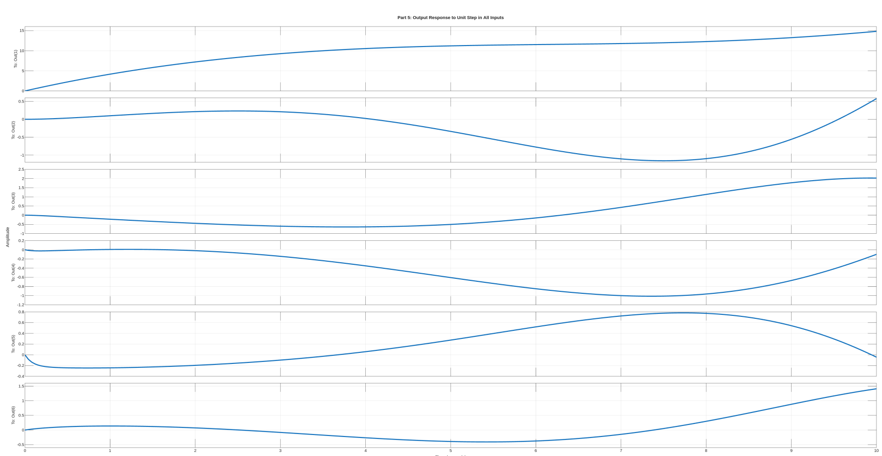

Figure 1: Open-loop output responses to a unit step input.

The open-loop step response exhibits divergence and oscillation, which is expected due to the unstable poles identified earlier. This reinforces the need for active stabilization.

6 Controllability and Observability

$$\texttt{(A,B) CONTROLLABLE,} \quad \texttt{(A,C) OBSERVABLE}$$

Full controllability ensures that every state can be influenced by the input vector, enabling pole placement and LQR control. Full observability confirms that all states can be reconstructed from measurements, allowing reliable estimator design.

7 LQR State-Feedback Control

The LQR controller computes the optimal feedback matrix K to minimize a quadratic cost balancing state deviation and control effort. Choosing $Q = C^T C$ emphasizes regulation of measured output variables, while $R = 0.1I$ penalizes excessive actuator use without overly restricting control authority.

7.1 Gain Matrix K

$$K = \begin{bmatrix} 0.0237 & 0.0313 & -0.0003 & 0.0294 & 0.1442 & 0.1851 & 0.1684 & -3.0898 \\ 4.5931 & -0.3917 & 0.0245 & 2.6117 & 0.0809 & -0.0160 & 0.0333 & 0.0017 \\ -0.3604 & -4.5365 & -1.2938 & 0.0189 & -0.4927 & -0.0328 & -0.0243 & -0.0323 \\ -0.1894 & -0.2906 & 0.1756 & -0.0082 & -1.2052 & 0.0034 & -0.0079 & -0.0733 \end{bmatrix}$$

7.2 Closed-Loop Eigenvalues

$$\lambda(A - BK) = \begin{cases} -15.4005, \ -14.3686, \ -2.6529, \ -0.9874, \\ -0.3928 \pm 0.4660i, \ -0.3816 \pm 0.4078i \end{cases}$$

These eigenvalues show strong damping and negative real parts, indicating that the LQR controller achieves rapid, stable convergence.

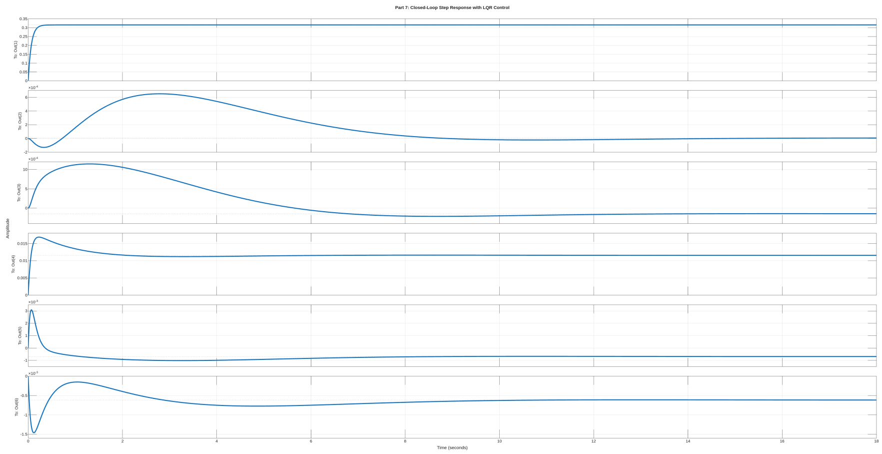

Figure 2: Closed-loop LQR step response.

8 Full-State Estimator

Because not all states are measurable, a full-state estimator (Luenberger observer) is constructed. The estimator poles were selected as three times the magnitude of the LQR closed-loop poles, ensuring the observer dynamics are significantly faster than the controlled system's dynamics.

8.1 Observer Gain L

$$L = \begin{bmatrix} 0.0218 & 25.1799 & -17.5708 & 0.0065 & 0.0067 & 1.0062 \\ -0.0224 & -17.5708 & 25.8865 & 0.1077 & 0.9931 & -0.0049 \\ 2.2428 & 0.0067 & -0.0069 & -5.0673 & -8.5456 & -2.1556 \\ 2.2441 & 0.0073 & -0.0075 & -5.5362 & 1.3133 & 1.0036 \\ -16.0347 & -0.0469 & 0.0481 & 36.4455 & -6.9629 & -4.0721 \\ 25.5205 & -32.1023 & -0.0013 & 21.6036 & 18.4949 & 56.6423 \\ 31.1588 & 0.1033 & 32.0566 & 22.1661 & -14.2314 & 62.7225 \\ -6.7984 & -1.9327 & 1.7361 & 18.4984 & -1.8654 & 5.1659 \end{bmatrix}$$

8.2 Augmented Closed-Loop Eigenvalues

$$\lambda(A_{\text{aug}}) = \{-46.20,\ -43.10,\ -15.40,\ -14.37,\ -7.96,\ -2.96,\ -2.65, \ldots\}$$

All eigenvalues lie safely in the left half-plane, confirming stable observer–controller integration.

9 Output Feedback Response

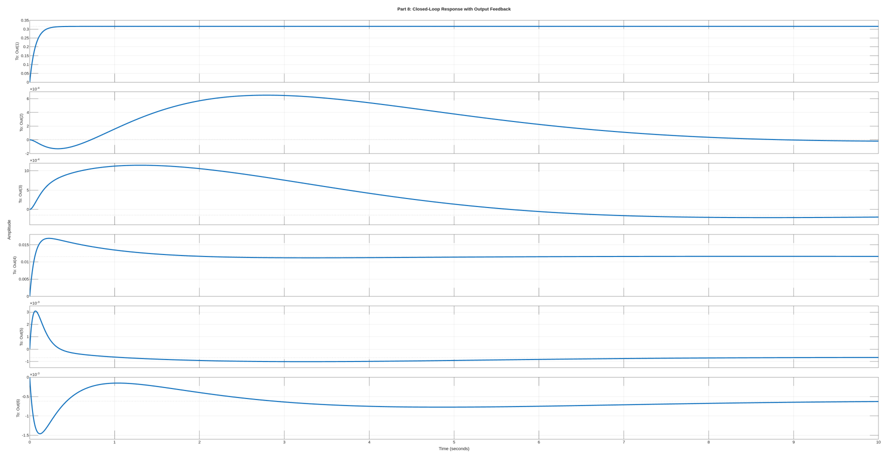

Figure 3: Closed-loop output-feedback response.

The observer-based controller successfully replicates the LQR closed-loop behavior. The estimator rapidly reconstructs the system states, enabling effective stabilization using only output measurements.

10 Zero-Input Response

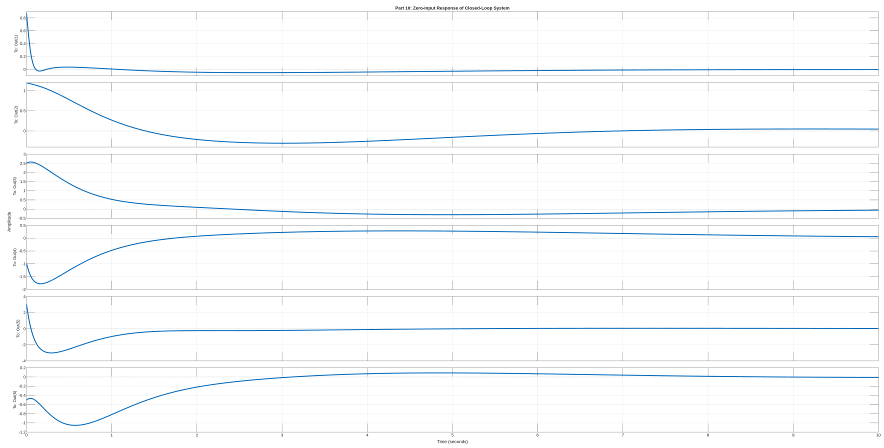

Figure 4: Zero-input response of the output-feedback closed-loop system.

The system quickly returns to equilibrium after being perturbed, demonstrating strong disturbance-rejection characteristics.

11 Reference Tracking for $y_1(t)$

The reference gain is:

$$N = 3.1651$$

This gain ensures that the closed-loop system accurately tracks a step reference applied to the first output channel. The controller modifies the effective DC gain of the closed-loop system so that the steady-state error becomes zero.

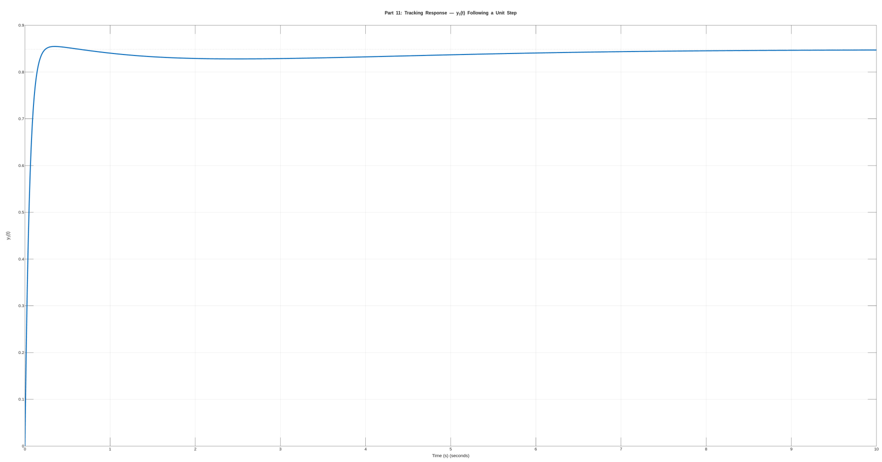

Figure 5: Tracking response of $y_1(t)$ with step reference.

12 Conclusion

The helicopter system was shown to be open-loop unstable but fully controllable and observable. An LQR controller successfully stabilized the dynamics, while a full-state estimator enabled implementation of a dynamic output-feedback controller suitable for real-world applications where not all states are measurable.

The combined controller–estimator structure achieved fast settling times, strong damping, and robust asymptotic stability. Reference tracking for the first output was accomplished through a feedforward gain that eliminates steady-state error. The complete workflow demonstrates how state-space tools can transform an unstable aircraft model into a well-regulated system capable of precise command tracking.