GMBE Derivation

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1 Notation

Let:

- $\mathbb{S}^{(N)}$ denote a set of cardinality N
- $\mathcal{P}\left(\mathbb{S}^{(N)}\right)$ be the powerset of the set $\mathbb{S}^{(N)}$.
- $\mathcal{U}^{(n)}\left(\mathbb{S}^{(N)}\right)$ is the subset of $\mathcal{P}\left(\mathbb{S}^{(N)}\right)$, whose elements are (unions of) n-length combinations of elements taken from $\mathbb{S}^{(N)}$
- $\mathcal{I}^{(n)}\left(\mathbb{S}^{(N)}\right)$ is similar to $\mathcal{U}^{(n)}\left(\mathbb{S}^{(N)}\right)$ except that instead of taking unions of n-length combinations one takes intersections.
- By analogy to powerset we define $\mathcal{I}\left(\mathbb{S}^{(N)}\right)$ to be the set: $\left\{\mathcal{I}^{(m)}\left(\mathbb{S}^{(N)}\right) \mid m \in [0, N]\right\}$.

2 Problem Descriptions

Ultimately we have a function f, which maps the powerset of a set $\mathbb{S}^{(N)}$ to a scalar $f\left(\mathcal{P}\left(\mathbb{S}^{(N)}\right)\right)$; however, the computational cost of computing $f\left(\mathcal{P}\left(\mathbb{S}^{(N)}\right)\right)$ scales exponentially with N. Instead we propose a series of systematic approximations to $f\left(\mathcal{P}\left(\mathbb{S}^{(N)}\right)\right)$.

In the first order approximation we form a family of sets $\mathbb{F}^{(m)}$ which contains m subsets of $\mathbb{S}^{(N)}$. The only stipulation on the members of $\mathbb{F}^{(m)}$ is that:

$$\mathbb{S}^{(N)} = \bigcup_{f_i^{(1)} \in \mathbb{F}^{(m)}} f_i^{(1)} \tag{1}$$

must be true. Of note we do not assume that the various $f_i^{(1)}$ are disjoint. Using $\mathbb{F}^{(m)}$, we then approximate $\mathcal{P}\left(\mathbb{S}^{(N)}\right)$ via:

$$\mathcal{P}\left(\mathbb{S}^{(N)}\right) \approx \bigcup_{f_i^{(1)} \in \mathbb{F}^{(m)}} \mathcal{P}\left(f_i^{(1)}\right). \tag{2}$$

Using the inclusion-exclusion principle (IEP), we can then approximate $f\left(\mathcal{P}\left(\mathbb{S}^{(N)}\right)\right)$ via:

$$\begin{aligned} f\left(\mathcal{P}\left(\mathbb{S}^{(N)}\right)\right) \approx & \sum_{i=1}^{m} f\left(\mathcal{P}\left(f_i^{(1)}\right)\right) - \sum_{i=1}^{m-1}\sum_{j=i+1}^{m} f\left(\mathcal{P}\left(f_i^{(1)} \cap f_j^{(1)}\right)\right) + \cdots + \\ & (-1)^{m-1} f\left(\mathcal{P}\left(f_1^{(1)} \cap f_2^{(1)} \cap \cdots \cap f_m^{(1)}\right)\right), \end{aligned} \tag{3}$$

where $f_i^{(1)} \cap f_j^{(1)}$ is the intersection of sets $f_i^{(1)}$ and $f_j^{(1)}$. For brevity, and because there is a one-to-one mapping between $\mathbb{S}^{(N)}$ and its powerset, we define a new function g such that:

$$g\left(\mathbb{S}^{(N)}\right) \equiv f\left(\mathcal{P}\left(\mathbb{S}^{(N)}\right)\right). \tag{4}$$

In terms of g, our first order approximation, $g\left(\mathbb{F}^{(m)}\right)$, becomes:

$$\begin{aligned} g\left(\mathbb{F}^{(m)}\right) =& \sum_{i=1}^{m} g\left(f_i^{(1)}\right) - \sum_{i=1}^{m-1}\sum_{j=i+1}^{m} g\left(f_i^{(1)} \cap f_j^{(1)}\right) + \cdots + \\ & (-1)^{m-1} g\left(f_1^{(1)} \cap f_2^{(1)} \cap \cdots \cap f_m^{(1)}\right). \end{aligned} \tag{5}$$

To further simplify the equation we write the intersections in terms of the various $\mathcal{I}^{(j)}\left(\mathbb{F}^{(m)}\right)$:

$$g\left(\mathbb{F}^{(m)}\right) = \sum_{j=1}^{m} \sum_{s_i^{(1,j)} \in \mathcal{I}^{(j)}\left(\mathbb{F}^{(m)}\right)} (-1)^{j-1} g\left(s_i^{(1,j)}\right). \tag{6}$$

As a second order approximation to $\mathcal{P}\left(\mathbb{S}^{(N)}\right)$ we take all pairwise unions of the initial m subsets:

$$\mathcal{P}\left(\mathbb{S}^{(N)}\right) \approx \bigcup_{f_i^{(2)} \in \mathcal{U}^{(2)}\left(\mathbb{F}^{(m)}\right)} \mathcal{P}\left(f_i^{(2)}\right). \tag{7}$$

Using the IEP the second order approximation to $g\left(\mathbb{S}^{(N)}\right)$, $g\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(m)}\right)\right)$, is:

$$g\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(m)}\right)\right) = \sum_{j=1}^{\left|\mathcal{U}^{(2)}\left(\mathbb{F}^{(m)}\right)\right|} \sum_{s_i^{(2,j)} \in \mathcal{I}^{(j)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(m)}\right)\right)} (-1)^{j-1} g\left(s_i^{(2,j)}\right). \tag{8}$$

The above readily generalizes to an ℓ-order approximation to $\mathcal{P}\left(\mathbb{S}^{(N)}\right)$:

$$\mathcal{P}\left(\mathbb{S}^{(N)}\right) \approx \bigcup_{f_i^{(\ell)} \in \mathcal{U}^{(\ell)}\left(\mathbb{F}^{(m)}\right)} \mathcal{P}\left(f_i^{(l)}\right). \tag{9}$$

and the ℓ-order approximation to $g\left(\mathbb{S}^{(N)}\right)$, $g\left(\mathcal{U}^{(\ell)}\left(\mathbb{F}^{(m)}\right)\right)$:

$$g\left(\mathcal{U}^{(\ell)}\left(\mathbb{F}^{(m)}\right)\right) = \sum_{j=1}^{\left|\mathcal{U}^{(\ell)}\left(\mathbb{F}^{(m)}\right)\right|} \sum_{s_i^{(\ell,j)} \in \mathcal{I}^{(j)}\left(\mathcal{U}^{(\ell)}\left(\mathbb{F}^{(m)}\right)\right)} (-1)^{j-1} g\left(s_i^{(\ell,j)}\right). \tag{10}$$

If we flatten the nested families of sets we end up with subsets of $\mathbb{S}^{(N)}$ which are written as intersections-of-unions of the elements of $\mathbb{F}^{(m)}$. Since intersection distributes over union, it should be possible to rewrite $g\left(\mathcal{U}^{(\ell)}\left(\mathbb{F}^{(m)}\right)\right)$ in terms of unions-of-intersections of the elements of $\mathbb{F}^{(m)}$. Inverting this relationship is the first problem we must tackle.

3 Problem One

Our overall strategy is deduce the general form of $g\left(\mathcal{U}^{(\ell)}\left(\mathbb{F}^{(m)}\right)\right)$, by first determining the general form for select ℓ values. Writing the order-one approximation in terms of the elements of $\mathbb{F}^{(m)}$ is trivial (it is just Eq. 6). The first non-trivial order is $\ell = 2$.

3.1 $\ell = 2$

To deduce the general form for $\ell = 2$ we systematically consider increasing values of m. For $m = 1$, there is no $\ell = 2$ term, and $m = 2$ has a trivial $\ell = 2$ term:

$$g\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(2)}\right)\right) = g\left(f_1 \cup f_2\right). \tag{11}$$

Hence the first non-trivial m is $m = 3$.

For $m = 3$ we have:

$$\begin{aligned} g\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(3)}\right)\right) =& g\left(f_{12}\right) + g\left(f_{13}\right) + g\left(f_{23}\right) - g\left(f_{12} \cap f_{13}\right) - g\left(f_{12} \cap f_{23}\right) - \\ & g\left(f_{13} \cap f_{23}\right) + g\left(f_{12} \cap f_{13} \cap f_{23}\right), \end{aligned} \tag{12}$$

where we have further simplified the notation by defining:

$$f_{ijk\cdots} \equiv f_i \cup f_j \cup f_k \cup \cdots \tag{13}$$

Noting that:

$$(a \cup b) \cap (a \cup c) = a \cup (b \cap c) \tag{14}$$

and that:

$$\begin{aligned} (a \cup b) \cap (a \cup c) \cap (b \cup c) &= (a \cup (b \cap c)) \cap (b \cup c) \\ &= (a \cap (b \cup c)) \cup ((b \cap c) \cap (b \cup c)) \\ &= (a \cap b) \cup (a \cap c) \cup (b \cap c), \end{aligned} \tag{15}$$

we can rewrite Eq. (12) as:

$$\begin{aligned} g\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(3)}\right)\right) =& g\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(2)}\right)\right) + g\left(f_{13}\right) + g\left(f_{23}\right) - g\left(f_3 \cup s_{12}\right) - g\left(f_2 \cup s_{13}\right) - \\ & g\left(f_1 \cup s_{23}\right) + g\left(s_{12} \cup s_{13} \cup s_{23}\right) \end{aligned} \tag{16}$$

where we have defined:

$$s_{ijk\cdots} \equiv f_i \cap f_j \cap f_k \cap \cdots \tag{17}$$

and identified the $m = 2$ approximation.

For $m = 4$ there are 63 terms. Stemming from the $\mathcal{I}^{(1)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(4)}\right)\right)$ family of sets are 6 trivial terms:

$$\begin{aligned} \sum_{f_i^{(2)} \in \mathcal{I}^{(1)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(4)}\right)\right)} g\left(f_i^{(2)}\right) =& g\left(f_{12}\right) + g\left(f_{13}\right) + g\left(f_{23}\right) + g\left(f_{14}\right) + g\left(f_{24}\right) + g\left(f_{34}\right) \\ =& \sum_{f_i^{(2)} \in \mathcal{I}^{(1)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(3)}\right)\right)} g\left(f_i^{(2)}\right) + \sum_{f_i^{(2)} \in \mathcal{I}^{(1)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(4)}\right)\right) \setminus \mathcal{I}^{(1)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(3)}\right)\right)} g\left(f_i^{(2)}\right), \end{aligned} \tag{18}$$

where in the second line we separated the six terms into the three that are from the $\mathcal{I}^{(1)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(3)}\right)\right)$ family of sets and the three which are unique to the $\mathcal{I}^{(1)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(4)}\right)\right)$ family of sets. This is straightforward to generalize to arbitrary m:

$$\sum_{f_i^{(2)} \in \mathcal{I}^{(1)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(m)}\right)\right)} g\left(f_i^{(2)}\right) = \sum_{f_i^{(2)} \in \mathcal{I}^{(1)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(m-1)}\right)\right)} g\left(f_i^{(2)}\right) + \sum_{f_i^{(2)} \in \mathcal{I}^{(1)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(m)}\right)\right) \setminus \mathcal{I}^{(1)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(m-1)}\right)\right)} g\left(f_i^{(2)}\right). \tag{19}$$

Stemming from $\mathcal{I}^{(2)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(4)}\right)\right)$ are 15 terms:

$$\begin{aligned}\sum_{s_i^{(2,2)}\in\mathcal{I}^{(2)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(4)}\right)\right)} g\left(s_i^{(2,2)}\right) =& g\left(f_{12}\cap f_{13}\right)+g\left(f_{12}\cap f_{23}\right)+g\left(f_{12}\cap f_{14}\right)+g\left(f_{12}\cap f_{24}\right)+\\ & g\left(f_{12}\cap f_{34}\right)+g\left(f_{13}\cap f_{23}\right)+g\left(f_{13}\cap f_{14}\right)+g\left(f_{13}\cap f_{24}\right)+\\ & g\left(f_{13}\cap f_{34}\right)+g\left(f_{23}\cap f_{14}\right)+g\left(f_{23}\cap f_{24}\right)+g\left(f_{23}\cap f_{34}\right)+\\ & g\left(f_{14}\cap f_{24}\right)+g\left(f_{14}\cap f_{34}\right)+g\left(f_{24}\cap f_{34}\right),\end{aligned} \tag{20}$$

which respectively simplify to:

$$\begin{aligned}\sum_{s_i^{(2,2)}\in\mathcal{I}^{(2)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(4)}\right)\right)} g\left(s_i^{(2,2)}\right) =& g\left(f_1\cup s_{23}\right)+g\left(f_2\cup s_{13}\right)+g\left(f_1\cup s_{24}\right)+g\left(f_2\cup s_{14}\right)+\\ & g\left(s_{13}\cup s_{23}\cup s_{14}\cup s_{24}\right)+g\left(f_3\cup s_{12}\right)+g\left(f_1\cup s_{34}\right)+\\ & g\left(s_{12}\cup s_{23}\cup s_{14}\cup s_{34}\right)+g\left(f_3\cup s_{14}\right)+g\left(s_{12}\cup s_{13}\cup s_{24}\cup s_{34}\right)+\\ & g\left(f_2\cup s_{34}\right)+g\left(f_3\cup s_{24}\right)+g\left(f_4\cup s_{12}\right)+g\left(f_4\cup s_{13}\right)+g\left(f_4\cup s_{23}\right),\end{aligned} \tag{21}$$

where we have used:

$$\begin{aligned}(a\cup b)\cap(c\cup d) &= (a\cap(c\cup d))\cup(b\cap(c\cup d))\\ &= (a\cap c)\cup(a\cap d)\cup(b\cap c)\cup(b\cap d)\end{aligned} \tag{22}$$

Identifying the terms which were present for $\mathcal{I}^{(2)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(3)}\right)\right)$:

$$\begin{aligned}\sum_{s_i^{(2,2)}\in\mathcal{I}^{(2)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(4)}\right)\right)} g\left(s_i^{(2,2)}\right) =& \sum_{s_i^{(2,2)}\in\mathcal{I}^{(2)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(3)}\right)\right)} g\left(s_i^{(2,2)}\right)+g\left(f_1\cup s_{24}\right)+g\left(f_2\cup s_{14}\right)+\\ & g\left(s_{13}\cup s_{23}\cup s_{14}\cup s_{24}\right)+g\left(f_1\cup s_{34}\right)+\\ & g\left(s_{12}\cup s_{23}\cup s_{14}\cup s_{34}\right)+g\left(f_3\cup s_{14}\right)+g\left(s_{12}\cup s_{13}\cup s_{24}\cup s_{34}\right)+\\ & g\left(f_2\cup s_{34}\right)+g\left(f_3\cup s_{24}\right)+g\left(f_4\cup s_{12}\right)+g\left(f_4\cup s_{13}\right)+g\left(f_4\cup s_{23}\right).\end{aligned} \tag{23}$$

Stemming from $\mathcal{I}^{(3)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(4)}\right)\right)$ are 20 terms:

$$\begin{aligned}\sum_{s_i^{(2,3)}\in\mathcal{I}^{(3)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(4)}\right)\right)} g\left(s_i^{(2,3)}\right) =& g\left(f_{12}\cap f_{13}\cap f_{23}\right)+g\left(f_{12}\cap f_{13}\cap f_{14}\right)+g\left(f_{12}\cap f_{13}\cap f_{24}\right)+\\ & g\left(f_{12}\cap f_{13}\cap f_{34}\right)+g\left(f_{12}\cap f_{23}\cap f_{14}\right)+g\left(f_{12}\cap f_{23}\cap f_{24}\right)+\\ & g\left(f_{12}\cap f_{23}\cap f_{34}\right)+g\left(f_{12}\cap f_{14}\cap f_{24}\right)+g\left(f_{12}\cap f_{14}\cap f_{34}\right)+\\ & g\left(f_{12}\cap f_{24}\cap f_{34}\right)+g\left(f_{13}\cap f_{23}\cap f_{14}\right)+g\left(f_{13}\cap f_{23}\cap f_{24}\right)+\\ & g\left(f_{13}\cap f_{23}\cap f_{34}\right)+g\left(f_{13}\cap f_{14}\cap f_{24}\right)+g\left(f_{13}\cap f_{14}\cap f_{34}\right)+\\ & g\left(f_{13}\cap f_{24}\cap f_{34}\right)+g\left(f_{23}\cap f_{14}\cap f_{24}\right)+g\left(f_{23}\cap f_{14}\cap f_{34}\right)+\\ & g\left(f_{23}\cap f_{24}\cap f_{34}\right)+g\left(f_{14}\cap f_{24}\cap f_{34}\right),\end{aligned} \tag{24}$$

which respectively simplify to:

$$\sum_{s_i^{(2,3)}\in\mathcal{I}^{(3)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(4)}\right)\right)} g\left(s_i^{(2,3)}\right) = g\left(s_{12}\cup s_{13}\cup s_{23}\right)+g\left(f_1\cup s_{234}\right)+g\left(s_{12}\cup s_{14}\cup s_{23}\right)+ \\ g\left(s_{13}\cup s_{14}\cup s_{23}\right)+g\left(s_{12}\cup s_{13}\cup s_{24}\right)+g\left(f_2\cup s_{134}\right)+ \\ g\left(s_{13}\cup s_{23}\cup s_{24}\right)+g\left(s_{12}\cup s_{14}\cup s_{24}\right)+g\left(s_{13}\cup s_{14}\cup s_{24}\right)+ \\ g\left(s_{14}\cup s_{23}\cup s_{24}\right)+g\left(s_{12}\cup s_{13}\cup s_{34}\right)+g\left(s_{12}\cup s_{23}\cup s_{34}\right)+ \\ g\left(f_3\cup s_{124}\right)+g\left(s_{12}\cup s_{14}\cup s_{34}\right)+g\left(s_{13}\cup s_{14}\cup s_{34}\right)+ \\ g\left(s_{14}\cup s_{23}\cup s_{34}\right)+g\left(s_{12}\cup s_{24}\cup s_{34}\right)+g\left(s_{13}\cup s_{24}\cup s_{34}\right)+ \\ g\left(s_{23}\cup s_{24}\cup s_{34}\right)+g\left(f_4\cup s_{123}\right). \tag{25}$$

Identifying the terms which stem from $\mathcal{I}^{(2)}\left(\mathcal{U}^{(3)}\left(\mathbb{F}^{(3)}\right)\right)$:

$$\sum_{s_i^{(2,3)}\in\mathcal{I}^{(2)}\left(\mathcal{U}^{(3)}\left(\mathbb{F}^{(4)}\right)\right)} g\left(s_i^{(2,3)}\right) = \sum_{s_i^{(2,3)}\in\mathcal{I}^{(2)}\left(\mathcal{U}^{(3)}\left(\mathbb{F}^{(3)}\right)\right)} g\left(s_i^{(2,3)}\right)+g\left(f_1\cup s_{234}\right)+g\left(s_{12}\cup s_{14}\cup s_{23}\right)+ \\ g\left(s_{13}\cup s_{14}\cup s_{23}\right)+g\left(s_{12}\cup s_{13}\cup s_{24}\right)+g\left(f_2\cup s_{134}\right)+ \\ g\left(s_{13}\cup s_{23}\cup s_{24}\right)+g\left(s_{12}\cup s_{14}\cup s_{24}\right)+g\left(s_{13}\cup s_{14}\cup s_{24}\right)+ \\ g\left(s_{14}\cup s_{23}\cup s_{24}\right)+g\left(s_{12}\cup s_{13}\cup s_{34}\right)+g\left(s_{12}\cup s_{23}\cup s_{34}\right)+ \\ g\left(f_3\cup s_{124}\right)+g\left(s_{12}\cup s_{14}\cup s_{34}\right)+g\left(s_{13}\cup s_{14}\cup s_{34}\right)+ \\ g\left(s_{14}\cup s_{23}\cup s_{34}\right)+g\left(s_{12}\cup s_{24}\cup s_{34}\right)+g\left(s_{13}\cup s_{24}\cup s_{34}\right)+ \\ g\left(s_{23}\cup s_{24}\cup s_{34}\right)+g\left(f_4\cup s_{123}\right). \tag{26}$$

At this point we have identified all terms which are also present in $g\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(3)}\right)\right)$ and remaining terms are unique to $g\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(4)}\right)\right)$.

Stemming from $\mathcal{I}^{(4)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(4)}\right)\right)$ are 15 terms:

$$\sum_{s_i^{(2,4)}\in\mathcal{I}^{(4)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(4)}\right)\right)} g\left(s_i^{(2,4)}\right) = g\left(f_{12}\cap f_{13}\cap f_{23}\cap f_{14}\right)+g\left(f_{12}\cap f_{13}\cap f_{23}\cap f_{24}\right)+ \\ g\left(f_{12}\cap f_{13}\cap f_{23}\cap f_{34}\right)+g\left(f_{12}\cap f_{13}\cap f_{14}\cap f_{24}\right)+ \\ g\left(f_{12}\cap f_{13}\cap f_{14}\cap f_{34}\right)+g\left(f_{12}\cap f_{13}\cap f_{24}\cap f_{34}\right)+ \\ g\left(f_{12}\cap f_{23}\cap f_{14}\cap f_{24}\right)+g\left(f_{12}\cap f_{23}\cap f_{14}\cap f_{34}\right)+ \\ g\left(f_{12}\cap f_{23}\cap f_{24}\cap f_{34}\right)+g\left(f_{12}\cap f_{14}\cap f_{24}\cap f_{34}\right)+ \\ g\left(f_{13}\cap f_{23}\cap f_{14}\cap f_{24}\right)+g\left(f_{13}\cap f_{23}\cap f_{14}\cap f_{34}\right)+ \\ g\left(f_{13}\cap f_{23}\cap f_{24}\cap f_{34}\right)+g\left(f_{13}\cap f_{14}\cap f_{24}\cap f_{34}\right)+ \\ g\left(f_{23}\cap f_{14}\cap f_{24}\cap f_{34}\right), \tag{27}$$

which respectively simplify to:

$$\sum_{s_i^{(2,4)} \in \mathcal{I}^{(4)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(4)}\right)\right)} g\left(s_i^{(2,4)}\right) = \begin{aligned}[t] & g\left(s_{12} \cup s_{13} \cup s_{234}\right) + g\left(s_{12} \cup s_{23} \cup s_{134}\right) + g\left(s_{13} \cup s_{23} \cup s_{124}\right) + \\ & g\left(s_{12} \cup s_{14} \cup s_{234}\right) + g\left(s_{13} \cup s_{14} \cup s_{234}\right) + g\left(s_{14} \cup s_{23}\right) + \\ & g\left(s_{12} \cup s_{24} \cup s_{134}\right) + g\left(s_{13} \cup s_{24}\right) + g\left(s_{23} \cup s_{24} \cup s_{134}\right) + \\ & g\left(s_{14} \cup s_{24} \cup s_{123}\right) + g\left(s_{12} \cup s_{34}\right) + g\left(s_{13} \cup s_{34} \cup s_{124}\right) + \\ & g\left(s_{23} \cup s_{34} \cup s_{124}\right) + g\left(s_{14} \cup s_{34} \cup s_{123}\right) + g\left(s_{24} \cup s_{34} \cup s_{123}\right), \end{aligned} \tag{28}$$

Stemming from $\mathcal{I}^{(5)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(4)}\right)\right)$ are 6 terms:

$$\sum_{s_i^{(2,5)} \in \mathcal{I}^{(5)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(4)}\right)\right)} g\left(s_i^{(2,5)}\right) = \begin{aligned}[t] & g\left(f_{12} \cap f_{13} \cap f_{23} \cap f_{14} \cap f_{24}\right) + g\left(f_{12} \cap f_{13} \cap f_{23} \cap f_{14} \cap f_{34}\right) + \\ & g\left(f_{12} \cap f_{13} \cap f_{23} \cap f_{24} \cap f_{34}\right) + g\left(f_{12} \cap f_{13} \cap f_{14} \cap f_{24} \cap f_{34}\right) + \\ & g\left(f_{12} \cap f_{23} \cap f_{14} \cap f_{24} \cap f_{34}\right) + g\left(f_{13} \cap f_{23} \cap f_{14} \cap f_{24} \cap f_{34}\right), \end{aligned} \tag{29}$$

which respectively simplify to:

$$\sum_{s_i^{(2,5)} \in \mathcal{I}^{(5)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(4)}\right)\right)} g\left(s_i^{(2,5)}\right) = \begin{aligned}[t] & g\left(s_{12} \cup s_{134} \cup s_{234}\right) + g\left(s_{13} \cup s_{124} \cup s_{234}\right) + g\left(s_{23} \cup s_{124} \cup s_{134}\right) + \\ & g\left(s_{14} \cup s_{123} \cup s_{234}\right) + g\left(s_{24} \cup s_{123} \cup s_{134}\right) + g\left(s_{34} \cup s_{123} \cup s_{124}\right), \end{aligned} \tag{30}$$

Finally there is one term stemming from $\mathcal{I}^{(6)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(4)}\right)\right)$:

$$\sum_{s_i^{(2,6)} \in \mathcal{I}^{(6)}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(4)}\right)\right)} g\left(s_i^{(2,6)}\right) = g\left(f_{12} \cap f_{13} \cap f_{23} \cap f_{14} \cap f_{24} \cap f_{34}\right) = g\left(s_{123} \cup s_{124} \cup s_{134} \cup s_{234}\right) \tag{31}$$

At this point we make a number of observations:

- There are $2^{\binom{m}{2}} - 1$ terms. So $m = 5$ and 6 respectively contain 1023, and 32,768 terms. Meaning explicitly writing out higher-orders is impractical.
- The expressions involve unions of elements from $\mathcal{U}^{(2)}\left(\mathbb{F}^{(m)}\right)$, $\mathcal{U}^{(1)}\left(\mathbb{F}^{(m)}\right)$, and $\mathcal{I}\left(\mathcal{U}^{(1)}\left(\mathbb{F}^{(m)}\right)\right)$ instead of intersections of elements from $\mathcal{I}\left(\mathcal{U}^{(2)}\left(\mathbb{F}^{(m)}\right)\right)$.
- Algorithmically this allows us to reuse information from the dramatically smaller $\mathcal{I}\left(\mathcal{U}^{(1)}\left(\mathbb{F}^{(m)}\right)\right)$ family of sets (namely which elements of $\mathcal{I}\left(\mathcal{U}^{(1)}\left(\mathbb{F}^{(m)}\right)\right)$ are empty) to decrease the number of terms we must consider.