Optimal minimax bounds for the Navier-Stokes equations and other infinite dimensional dissipative systems

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Based on

"Optimal minimax bounds for time and ensemble averages of dissipative infinite-dimensional systems with applications to the incompressible Navier-Stokes equations"

Joint work with Roger Temam (Indiana University)

Submitted to Pure and Applied Functional Analysis

Dedicated to the memory of Ciprian Foias

Estimates

Estimates are desirable for many purposes:

- existence and uniqueness of solutions;
- blow-up of solutions
- stabilization results
- estimating dimensions of invariant sets;
- localizing local or global attractors;
- assessing real quantities
    - energy, drag coefficient, mechanical stress, chemical concentration, infected population, pharmaceutical dosage, etc.

Techniques

Usual techniques:

- Energy-type estimates
- Variational inequalities

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Numerically approaches:

- Monte-Carlo estimates for classical evolution methods
- Linear programming/sums of squares for mini-max estimate

Differential equation

Suppose we are interested on a differential equation on a phase space X,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = F(u),$$

and on estimating the **mean** of a quantity $\phi : X \to \mathbb{R}$ over a **compact** and **positively-invariant** subset $B \subset X$:

$$\sup_{u_0 \in B} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \phi(u(t)) \, \mathrm{d}t.$$

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Numerically, can use Monte-Carlo (solve multiple solutions for long time intervals, using a decent integration method, ...)

Or, work only on the phase space, without solving the equation...

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Now, if we find a **auxilliary function** V such that

$$\phi(u) + F(u) \cdot \nabla V(u) \leq C, \quad \forall u \in B,$$

then

$$\overline{\phi + F \cdot \nabla V} \leq C, \quad \text{hence} \quad \bar{\phi} \leq C.$$

Convex optimization problem

The problem

$$\text{Find } V \text{ s.th. } \phi(u) + F(u)\cdot\nabla V(u) \leq C, \ \forall u \in B, \text{ for } \textit{best possible } C,$$

can be written as a convex optimization problem

$$\sup \bar{\phi} \leq \inf_{(C,V)\in\mathbb{R}\times\mathcal{C}^1,\ S_{C,V}(u)\geq 0} C,$$

where $S_{C,V}(u) = C - \phi(u) - F(u)\cdot\nabla V(u)$.

(Optimization of linear map $(C,V)\mapsto C$ over a convex set, since $\mathbb{R}\times\mathcal{C}^1$ is convex and $S_{C,V}(u)$ is linear in C and V.)

Optimization with Sum of Squares (SoS)

If ϕ, F **polynomials**, we narrow minimization over **polynomials** V such that

$$S_{C,V}(u) = C - \phi(u) - F(u) \cdot \nabla V(u) = \texttt{SoS},$$

where $\texttt{SoS} = \sum_i p_i(u)^2$, for other polynomials p_i.

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Question: How large is the set of $\texttt{SoS}$?

Historical background on Sum of Squares

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- Hilbert's 17th problem (1900) is about decomposing positive polynomials into SoS of rational functionals.
- Hilbert had already noticed not all positive polynomials are SoSs.
- Reznik (2000) - survey of SoS and Hilbert's 17th problem
- Shor 1980s, 1990s, Choi, Lam, Reznik 1990s - SoS polynomial decomposition
- Parrilo (2000s) - several applications: Lyapunov functions, control, etc.
- Several SoS MATLAB toolbox solvers (2000s)
- Papachristodoulou, Peet (2006) - applications to PDEs
- Yu, Kashima, Imura (2008)- local stability of 2D fluid flows
- Goulart, Chernyshenko (2012) - global stability of fluid flows
- Fantuzzi, Goluskin, Doering, Goulart, Chernyshenko, Huang, Papachristodoulou (2010s) ...

Bounds for the van der Pol limit cycle

From "Bounds for Deterministic and Stochastic Dynamical Systems using Sum-of-Squares Optimization", by G. Fantuzzi, D. Goluskin, D. Huang, and S. I. Chernyshenko, in SIAM J. Applied Dynamical Systems, Vol. 15, No. 4, pp. 1962–1988.

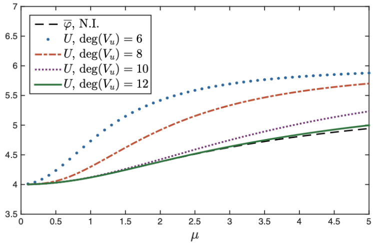
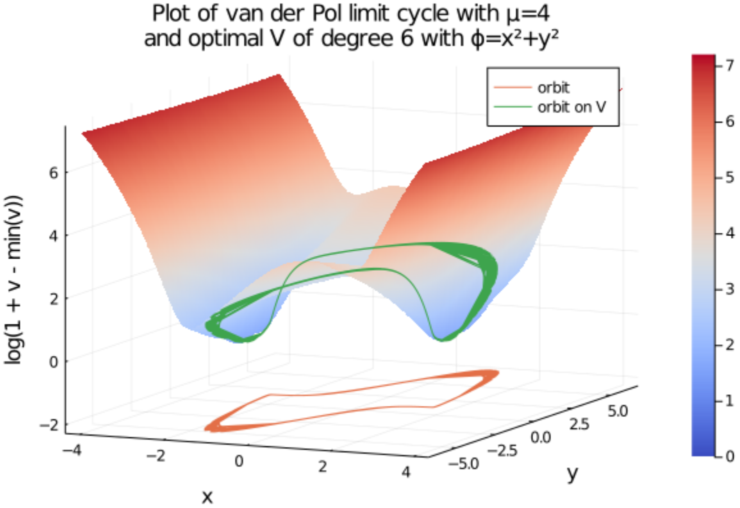
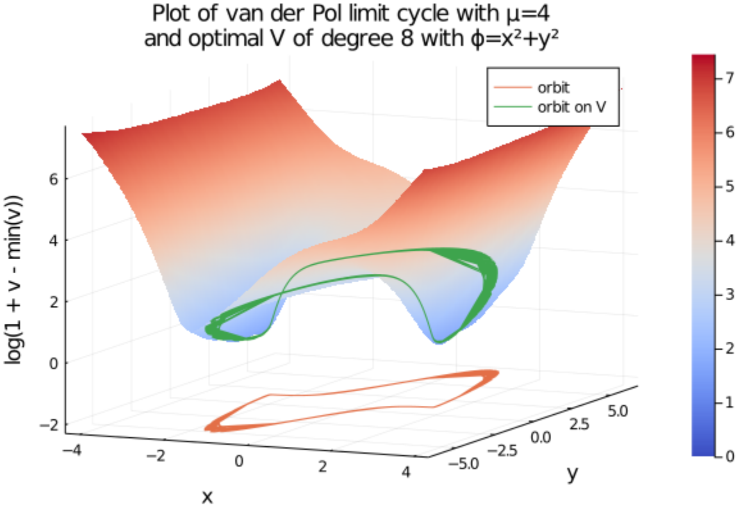


Figure 2. *Optimal upper bounds on* $\overline{\varphi} = \overline{x^2 + y^2}$ *for the van der Pol oscillator computed with the upper bound problem of* (2.9) *for different degrees of* V_u. *The time average* $\overline{\varphi}$ *obtained by numerical integration (N.I.) of the system is also shown.*

Auxiliary function with degree 6 for van der Pol



Auxiliary function with degree 8 for van der Pol



Auxiliary function with degree 10 for van der Pol

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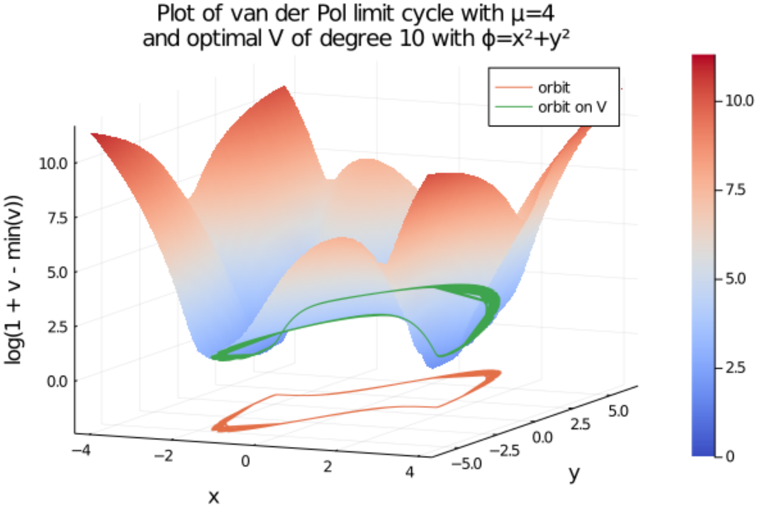

Minimax problem

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The problem

Find V and the *best possible* C s.th. $\phi(u) + F(u) \cdot \nabla V(u) \leq C, \ \forall u \in B,$

can also be written as the minimax problem

$$\sup_{u_0 \in B} \bar{\phi}(u_0) \leq \min_{V \in \mathcal{C}^1(B)} \max_{u \in B} \left\{ \phi(u) + F(u) \cdot \nabla V(u) \right\}.$$

Optimality of the minimax formula

Tobasco-Goluskin-Doering (2018):

It turns out, the minimax formula is optimal and is achieved!

$$\max_{u_0 \in B} \bar{\phi}(u_0) = \min_{V \in \mathcal{C}^1(B)} \max_{u \in B} \{\phi(u) + F(u) \cdot \nabla V(u)\} .$$

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Conditions:

- $X = \mathbb{R}^n$
- $F : \mathbb{R}^n \to \mathbb{R}^n$ continuously differentiable
- $B \subset \mathbb{R}^n$ compact
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Questions:

- What about infinite-dimensional systems? Like 2D NSE.
- What about 3DNSE?

Proof of optimality in the finite-dimensional case

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&= \max_{\mu\in\mathcal{P}_{\mathcal{E}\text{-}\mathrm{inv}}(B)} \int_B \phi\,\mathrm{d}\mu \ \text{(Lagrangian = Eulerian invariance } \langle F\cdot\nabla V\rangle = 0\text{)}
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&= \inf_{V\in\mathcal{C}^1(B)} \max_{u\in B} \{\phi(u) + F(u)\cdot\nabla V(u)\} \text{ (extreme at Dirac delta)}
\end{aligned}
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Concepts for PDEs

X Hausdorff space; W Banach space; $W \subset X \subset W'$ continuous.

Definition

A **cylindrical test functional** on W' is $\Psi : W' \to \mathbb{R}$ of the form

$$\Psi(u) = \psi(\langle u, w_1 \rangle_{W',W}, \ldots, \langle u, w_m \rangle_{W',W}), \qquad \forall u \in W',$$

where $w_1, \ldots, w_m \in W$, $m \in \mathbb{N}$, and $\psi \in \mathcal{C}_c^1(\mathbb{R}^m)$. Space $\mathcal{T}^{\mathrm{cyl}}(W')$.

Definition

A **weak stationary statistical solution** is a Borel probability measure μ on X such that, for any $\mathcal{T}^{\mathrm{cyl}}(W')$, the map $u \mapsto \langle F(u), \Psi'(u) \rangle_{W',W}$ is μ-integrable and

$$\int_X \langle F(u), \Psi'(u) \rangle_{W',W} \,\mathrm{d}\mu(v) = 0.$$

Space $\mathcal{P}_{\mathrm{wsss}}(E)$ of those carried by Borel subset E.

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Infinite-dimensional dissipative semigroup

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&= \max_{\mu \in \mathcal{P}_{\mathrm{inv}}(B)} \int_B \phi \,\mathrm{d}\mu \\
&= \max_{\mu \in \mathcal{P}_{\mathrm{inv}}(B \cap K)} \int_{B \cap K} \phi \,\mathrm{d}\mu \ (K \text{ compact attracting; } B \text{ normal})
\end{aligned}
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&= \sup_{\mu\in\mathcal{P}(B\cap K)} \inf_{V\in\mathcal{T}_{\mathrm{cyl}}} \int_{B\cap K} \phi + F\cdot\nabla V\,\mathrm{d}\mu \\
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&= \inf_{V\in\mathcal{T}_{\mathrm{cyl}}} \max_{u\in B\cap K} \{\phi(u) + F(u)\cdot\nabla V(u)\} \quad \text{(extreme at Dirac delta)}
\end{aligned}
$$

3D NSE

$$\sup_{u \in \mathcal{U}(B)} \bar{\phi}(u) =$$

3D NSE

$$\sup_{u \in \mathcal{U}(B)} \bar{\phi}(u) = \sup_{u \in \mathcal{U}(B)} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \phi(u(t)) \,\mathrm{d}t \text{ (definition)}$$

$$
\begin{aligned}
\sup_{u\in\mathcal{U}(B)} \bar{\phi}(u) &= \sup_{u\in\mathcal{U}(B)} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \phi(u(t))\,\mathrm{d}t \\
&\leq \max_{\mu\in\mathcal{P}_{\mathsf{wsss}}(B)} \int_{B\cap K} \phi\,\mathrm{d}\mu \text{ (Bogoliubov-Krylov; } K \text{ comp. attr.; } B \text{ normal)}
\end{aligned}
$$

3D NSE

$$\sup_{u\in\mathcal{U}(B)} \bar{\phi}(u) = \sup_{u\in\mathcal{U}(B)} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \phi(u(t))\,\mathrm{d}t$$

$$\leq \max_{\mu\in\mathcal{P}_{\mathrm{wsss}}(B)} \int_{B\cap K} \phi\,\mathrm{d}\mu$$

$$= \sup_{\mu\in\mathcal{P}(B\cap K)} \inf_{V\in\mathcal{T}_{\mathrm{cyl}}} \int_{B\cap K} \phi + F\cdot\nabla V\,\mathrm{d}\mu \quad \left(\inf_V = \begin{cases} 0, & \mu\in\mathcal{P}_{\mathrm{wsss}}(B\cap K) \\ -\infty, & \mu\notin\mathcal{P}_{\mathrm{wsss}}(B\cap K)\end{cases}\right)$$

3D NSE

$$
\begin{aligned}
\sup_{u\in\mathcal{U}(B)} \bar{\phi}(u) &= \sup_{u\in\mathcal{U}(B)} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \phi(u(t))\,\mathrm{d}t \\
&\leq \max_{\mu\in\mathcal{P}_{\mathrm{wsss}}(B)} \int_{B\cap K} \phi \,\mathrm{d}\mu \\
&= \sup_{\mu\in\mathcal{P}(B\cap K)} \inf_{V\in\mathcal{T}_{\mathrm{cyl}}} \int_{B\cap K} \phi + F\cdot\nabla V \,\mathrm{d}\mu \\
&= \inf_{V\in\mathcal{T}_{\mathrm{cyl}}} \sup_{\mu\in\mathcal{P}(B\cap K)} \int_{B\cap K} \phi + F\cdot\nabla V \,\mathrm{d}\mu \quad \text{(minimax principle)}
\end{aligned}
$$

3D NSE

$$
\begin{aligned}
\sup_{u\in\mathcal{U}(B)} \bar{\phi}(u) &= \sup_{u\in\mathcal{U}(B)} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \phi(u(t))\,\mathrm{d}t \\
&\leq \max_{\mu\in\mathcal{P}_{\mathrm{wsss}}(B)} \int_{B\cap K} \phi\,\mathrm{d}\mu \\
&= \sup_{\mu\in\mathcal{P}(B\cap K)} \inf_{V\in\mathcal{T}_{\mathrm{cyl}}} \int_{B\cap K} \phi + F\cdot\nabla V\,\mathrm{d}\mu \\
&= \inf_{V\in\mathcal{T}_{\mathrm{cyl}}} \sup_{\mu\in\mathcal{P}(B\cap K)} \int_{B\cap K} \phi + F\cdot\nabla V\,\mathrm{d}\mu \\
&= \inf_{V\in\mathcal{T}_{\mathrm{cyl}}} \max_{u\in B\cap K} \{\phi(u) + F(u)\cdot\nabla V(u)\} \text{ (extreme at Dirac delta)}
\end{aligned}
$$

Thm I

Theorem

Suppose K is a compact and metrizable subset of X and F is continuous on K. Assume the set $\mathcal{P}_{\mathsf{wsss}}(K)$ of weak stationary statistical solutions carried by K is not empty. Let $\phi \in \mathcal{C}(K)$. Then,

$$\max_{\mu \in \mathcal{P}_{\mathsf{wsss}}(K)} \int_K \phi(u)\, d\mu(u) = \inf_{\Psi \in \mathcal{T}_{\mathsf{cyl}}(W')} \max_{u \in K} \left\{ \phi(u) + \langle F(u), \Psi'(u) \rangle_{W',W} \right\}.$$

Thm II

Theorem

Let B be a positively invariant set for $\{S(t)\}_{t\geq 0}$ which is closed in X and normal, and suppose that there exists a compact and metrizable subset K of X which attracts the points of B. Suppose F is continuous on K. Let $\phi \in \mathcal{C}_b(B)$. Then,

$$\max_{\mu\in\mathcal{P}_{\mathbf{wsss}}(K)} \int_K \phi(u)\,d\mu(u) = \inf_{\Psi\in\mathcal{T}_{\mathbf{cyl}}(W')} \max_{u\in B\cap K} \left\{\phi(u) + \langle F(u), \Psi'(u)\rangle_{W',W}\right\}.$$

Suppose, further, that $\mathcal{P}_{\mathbf{wsss}}(K) = \mathcal{P}_{inv}(B\cap K)$. Then

$$\max_{u_0\in B} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \phi(S(t)u_0)\,dt = \inf_{\Psi\in\mathcal{T}_{\mathbf{cyl}}(W')} \max_{u\in B\cap K} \left\{\phi(u) + \langle F(u), \Psi'(u)\rangle_{W',W}\right\}.$$

Thm III

Theorem

For the 3D NSE, with $X = H_w$, $W = D(A)$, and $K \supset \mathcal{A}_w$ compact, $B \subset H$ bounded and positively invariant, $\phi \in \mathcal{C}_b(B)$,

$$
\begin{aligned}
\sup_{u \in \mathcal{U}(B)} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \phi(S(t)u_0)\, dt \\
&\leq \sup_{\mu \in \mathcal{P}_{fpsss}(K)} \int_K \phi(u)\, d\mu(u) \\
&\leq \inf_{\Psi \in \mathcal{T}_{cyl}(W')} \max_{u \in B \cap K} \left\{ \phi(u) + \langle F(u), \Psi'(u) \rangle_{W',W} \right\}.
\end{aligned}
$$