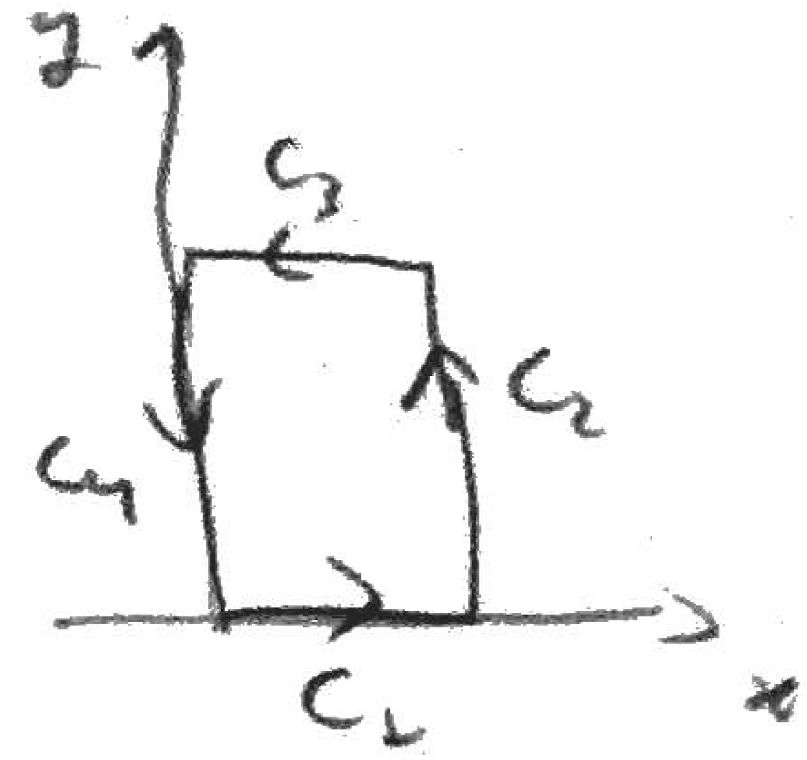
Segunda Prova de Cálculo III

1°) Duas maneiras de resolver

Diretamente:



$$C = C_1 + C_2 + C_3 + C_4$$

Em C_1 : $y = 0$ e $dy = 0$ $\qquad \sigma_1(t) = (t, 0)$, $0 \leq t \leq 1$

C_2 : $x = 1$ e $dx = 0$ $\qquad \sigma_2(t) = (1, t)$, $0 \leq t \leq 3$

C_3 : $y = 3$ e $dy = 0$ $\qquad \sigma_3(t) = (1-t, 3)$, $0 \leq t \leq 1$

C_4 : $x = 0$ e $dx = 0$ $\qquad \sigma_4(t), (0, 3-t)$, $0 \leq t \leq 3$

Logo

$$\int_C ye^x dx + 2e^x dy = \int_{C_1} 0 + \int_{C_2} 2e\,dy + \int_{C_3} 3e^x dx + \int_{C_4} 2dy$$

$$= 0 + \int_0^3 2e\,dy - \int_0^1 3e^x dx - \int_0^3 2dy$$

$$= 0 + 6e - 3e^x\Big]_{x=0}^{x=1} - 6$$

$$= 0 + 6e - 3(e-1) - 6 = 3(e-1)$$

Portanto

$$\boxed{\int_C ye^x dx + 2e^x dy = 3(e-1)}$$

1)

Via teorema de Green

$P = ye^x \quad , \quad Q = 2e^x$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2e^x - e^x = e^x$$

$$\int_C ye^x dx + 2e^x dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D e^x \, dA$$

$$= \int_0^1 \int_0^3 e^x \, dy \, dx$$

$$= 3\int_0^1 e^x \, dx$$

$$= 3 \, e^x \Big]_0^1 = 3(e-1)$$

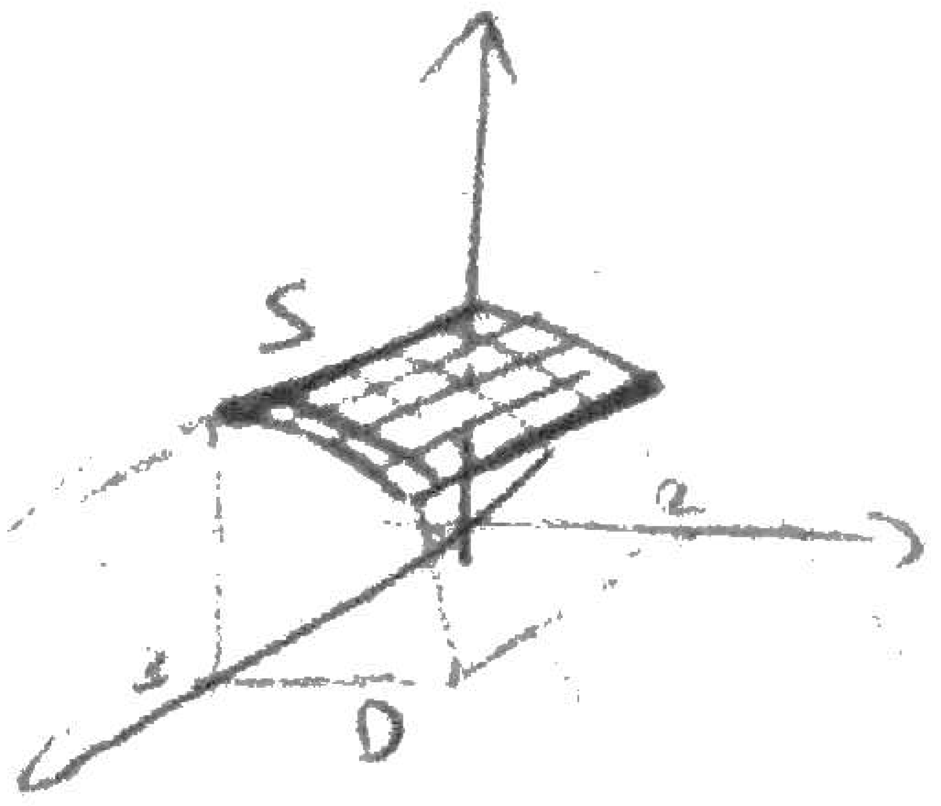
onde $D = \{(x,y) : 0 \leq x \leq 1 , \ 0 \leq y \leq 3\}$

de modo que $\partial D = C$

Portanto,

$$\boxed{\int_C ye^x dx + 2e^x dy = 3(e-1)}$$

2ª) $F = (e^{xz}\cos(xy),\ x,\ -xy)$



Superfície:

$S = \{(x,y,z);\ 0 \le x \le 3,\ 0 \le y \le 2,\ z = 9 - y^2\}$

Parametrização:

$D = \{(x,y);\ 0 \le x \le 3,\ 0 \le y \le 2\}$

$$r(x,y) = (x, y, 9 - y^2)$$

$$r_x = (1, 0, 0)$$
$$r_y = (0, 1, -2y)$$

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & -2y \end{vmatrix} = (0, 2y, 1)$$

Fluxo:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(e^{x(9-y^2)}\cos(xy), x, -xy\right) \cdot (0, 2y, 1)\, dA$$

$$= \iint_D (2xy - xy)\, dA$$

$$= \iint_D xy\, dA$$

$$= \int_0^3 \int_0^2 xy\, dy\, dx$$

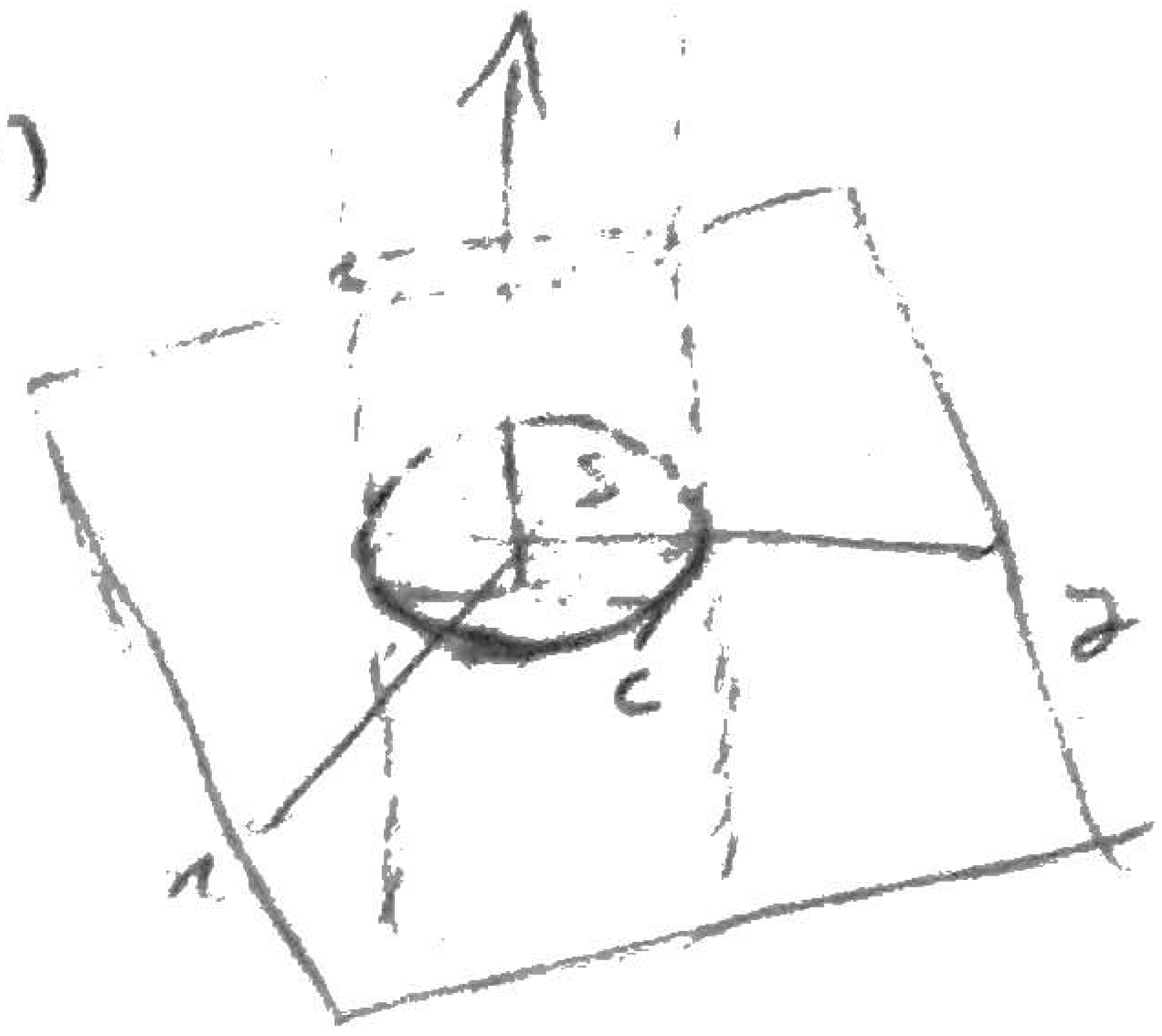
$$= \int_0^3 x \frac{y^2}{2}\Big]_{y=0}^{y=2} dx = \int_0^3 x\left(\frac{4}{2} - \frac{0}{2}\right) dx$$

$$= \int_0^3 2x\, dx = x^2\Big]_{x=0}^{x=3} = 3^2 - 0 = 9$$

Logo

$$\boxed{\iint_S \bar{F}\, d\bar{s} = 9}$$

3) (a)



$C = \{(x,y,z) : x^2+y^2=1,\ z=-x-y\}$

$C = \partial S$ onde $S = \{(x,y,z) : x^2+y^2 \leq 1,\ z = -x-y\}$

Pelo Teorema de Stokes

$$\int_C \vec{F}\, d\vec{r} = \iint_S \overrightarrow{\text{rot}\, F} \cdot d\vec{S}$$

$$\text{rot}\, F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 2xz & 2xy & yz \end{vmatrix} = (z-0, 2x-0, 2y-0)$$
$$= (z, 2x, 2y)$$

Parametrização de S:

$D = \{(x,y) : x^2+y^2 \leq 1\}$ $\qquad r_x = (1, 0, -1)$

$r(x,y) = (x, y, -x-y)$ $\qquad r_y = (0, 1, -1)$

$$\Rightarrow r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = (1, 1, 1)$$

Logo

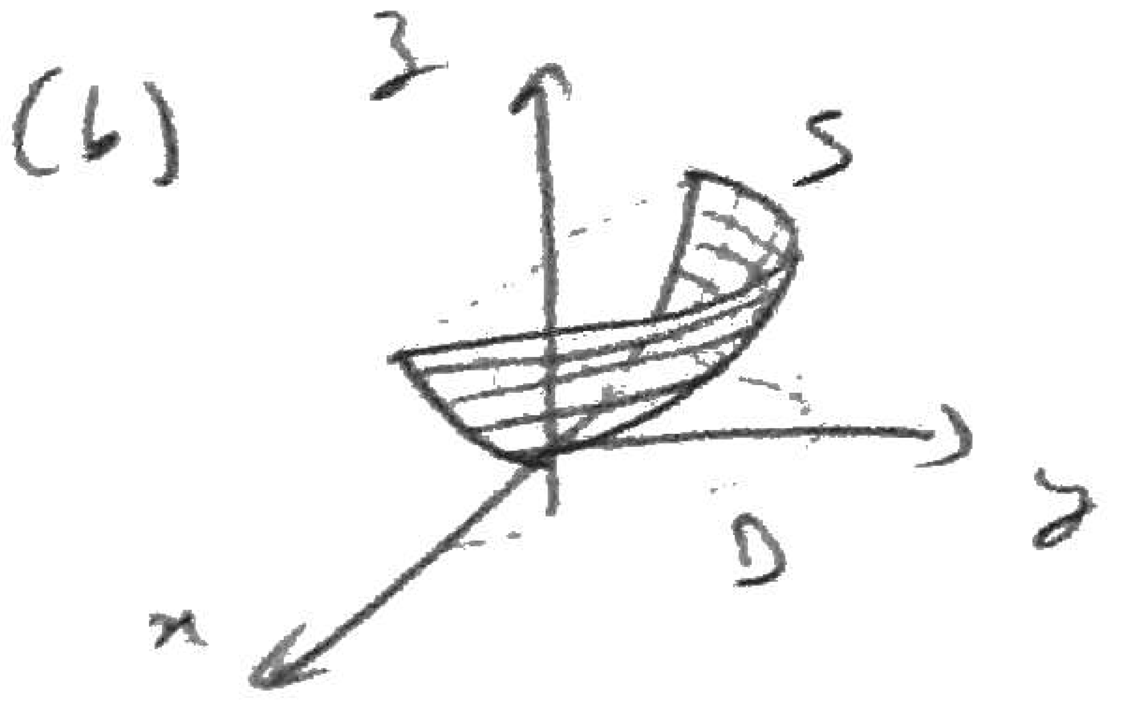
$$\int_C \vec{F}\, d\vec{r} = \iint_D (-x-y, 2x, 2y) \cdot (1,1,1)\, dA = \iint_D -x-y+2x+2y\, dA$$
$$= \iint_D x+y\, dA$$

Em coordenadas cartesianas: $\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x+y)\, dy\, dx}$

OU

Em coordenadas polares $\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi}\int_0^1 (r\cos\theta + r\sin\theta)\, r\, dr\, d\theta}$

3) (b)



$$\begin{cases} z = x^2+y^2 \\ z = 16 \end{cases} \Leftrightarrow \begin{cases} x^2+y^2 = 16 \end{cases}$$

$S = \{(x,y,z) : x^2+y^2 \leq 16 \;, y \geq 0, \; z = x^2+y^2\}$

Parametrização:

$D = \{(x,y) : x^2+y^2 \leq 16 \,, y \geq 0\}$

$r(x,y) = (x, y, x^2+y^2)$

$r_x = (1, 0, 2x) \qquad r_y = (0, 1, 2y)$

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} = (-2x, -2y, 1)$$

$\|r_x \times r_y\| = \sqrt{1 + 4x^2 + 4y^2}$

$$\text{Área de } S = \iint_S \|r_x \times r_y\| \, dA$$

Em Coordenadas Cartesianas:

$$\boxed{\text{Área} = \int_{-4}^{4} \int_{0}^{\sqrt{16-x^2}} \sqrt{1+4x^2+4y^2} \; dy\, dx}$$

OU Em Coordenadas Polares:

$$\text{Área} = \int_0^1 \int_0^{\pi} \sqrt{1+4r^2} \; r \, d\theta \, dr$$

4)

(a) $S_2 = \{(x,y,z);\ x^2+y^2 \leq 1,\ z=0\}$

Parametrização:

$r(x,y) = (x, y, 0)$, $D = \{(x,y);\ x^2+y^2 \leq 1\}$

$r_x = (1, 0, 0)$ $\quad r_y = (0, 0, 1)$

$r_x \times r_y = (0, 0, 1)$

$F(r(x,y)) = (\text{sen}(yz), x^2 e^z, z)\big|_{z=0} = (0, x^2, 0)$

$\text{Fluxo} = \iint_{S_2} \vec{F}\, d\vec{s} = \iint_D (0, x^2, 0)\cdot(0,0,1)\, dA$

$= \iint_D 0\, dA = 0$

Logo

$$\boxed{\text{Fluxo por } S_2 = \iint_{S_2} \vec{F}\, d\vec{s} = 0}$$

4) (b) $S_1 = \{(x,y,z): x^2+y^2+z^2=1, z \geq 0\}$

$S_2 = \{(x,y,z), x^2+y^2 \leq 1, z=0\}$

$S_1 + S_2 = \partial V$

onde

$V = \{(x,y,z): x^2+y^2 \leq 1, 0 \leq z \leq 1-x^2+y^2\}$

onde S_1 está orientada com a normal "para cima" e S_2 com a normal "para baixo"

Pelo Teorema da divergência

$$\iiint_V \operatorname{div}\vec{F}\, dV = \iint_{\partial V} \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F}\, d\vec{S} + \iint_{S_2} \vec{F} \cdot dS$$

Como $\iint_{S_2} \vec{F}\, dS = 0$ pelo item (a), temos

$$\iint_{S_1} \vec{F}\, d\vec{S} = \iiint_V \operatorname{div} F\, dV$$

Agora, $\operatorname{div} F = \frac{\partial}{\partial x}(\operatorname{sen}(yz)) + \frac{\partial}{\partial y}(x^2 e^z) + \frac{\partial}{\partial z} z = 0+0+1 = 1$

Logo

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iiint_V dV = \text{Volume de } V$$

$= \text{Volume da meia-esfera de raio } 1 = \frac{2\pi}{3}$

Portanto

$$\boxed{\text{Fluxo por } S_1 = \iint_{S_1} \vec{F} \cdot dS = \frac{2\pi}{3}}$$