Improved order of convergence of the Euler method for random ordinary differential equations driven by semi-martingale noises

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VI Workshop on Fluids and PDE
Celebrating the 60th birthdays of Helena J. Nussenzveig Lopes and Milton C. Lopes Filho
Unicamp - Campinas - October 23-27, 2023

October 26, 2023

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Helena and Milton

- Congratulations on your 60th birthday!
- Thanks for building and keeping this "Fluid"community together for so long!

Sorry, no fluids this time

- Current fluid-related work:
  - Energy cascade in 3D channel flows (with J. Klewicki and S. Zimmerman)
  - Optimal minimax bounds for ensemble averages of NSE and other evolutionary system (with R. Temam)
  - Convergence of statistical solutions (with A. Bronzi and C. Mondaini)
- Today:
  - Euler method for Random ODEs (with P. Kloeden)
    no fluids... hardly any PDE...

Work

- *"Improved order of convergence of the Euler method for random ordinary differential equations driven by semi-martingale noises"*
- Joint work with Peter Kloeden (University of Tübingen, Germany)
- arXiv:2306.15418
- github rmsrosa/rode_conv_em (reproducible Julia code)

What is the story about that?

- I was teaching theory and numerics for SDEs and RODEs
- Wanted to illustrate the half order of convergence of Euler for SDEs and RODEs
- Ok for SDEs
- But couldn't find example for RODEs

Order of convergence

- Order p if

$$\text{error} \sim \Delta t^p.$$

- Thus,

$$\Delta t \mapsto \frac{1}{2}\Delta t \quad \Longrightarrow \quad \text{error} \mapsto \left(\frac{1}{2}\right)^p \text{error}$$

- Decimals

$$\Delta t \mapsto \frac{1}{10}\Delta t \quad \Longrightarrow \quad \text{error} \mapsto \left(\frac{1}{10}\right)^p \text{error}$$

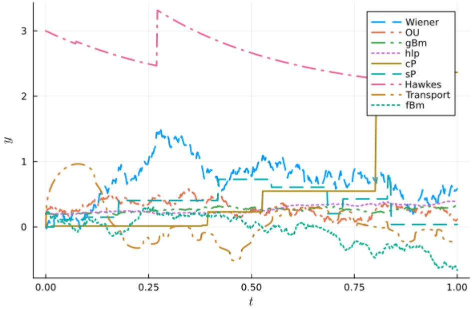
What is a RODE?

Random Ordinary Differential Equation in the picture:

- ODE:
$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x);$$
- SDE:
$$\mathrm{d}X_t = f(t, X_t)\,\mathrm{d}t + g(t, X_t)\,\mathrm{d}W_t;$$
- RODE:
$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t);$$
- where
  - W_t = Wiener process (Hölder continuous paths),
  - $\mathrm{d}W_t/\mathrm{d}t$ = white noise (very irregular, distribution),
  - Y_t = noise process (varied, but not so irregular as white noise).

Why RODEs?

- When noise is not just an Itô noise... and can be computed exactly
- For example
  - Ornstein-Uhlenbeck (colored noise) process
  - geometric Brownian motion
  - fractional Brownian motion
  - Transport process
  - Point process (Poisson, Hawkes, etc.)
  - time-changed Wiener process



Ornstein-Uhlenbeck approximates white noise

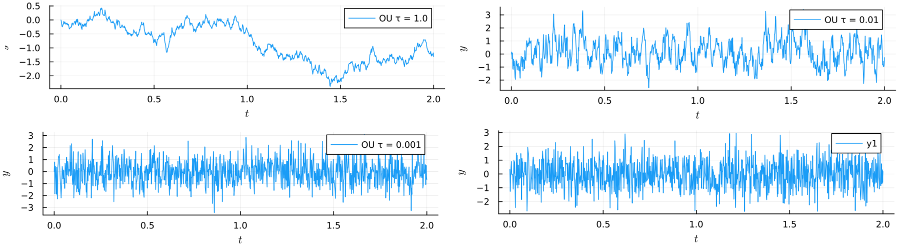
Ornstein-Uhlenbeck

$$\tau \mathrm{d}O_t = -dt + \zeta \, \mathrm{d}W_t$$

with statistics

$$\mathbb{E}[O_t] = O_0 e^{-\frac{t}{\tau}}, \quad \mathrm{Var}(O_t) = \frac{\varsigma^2}{2\tau}, \quad \mathrm{Cov}(O_t, O_s) = \frac{\varsigma^2}{2\tau} e^{-\frac{|t-s|}{\tau}}.$$

approximates white noise as time scale $\tau \to 0$, with $\zeta^2/2\tau = 1$.



E.g. population growth with random perturbation

- ODE: deterministic growth $\mu_t = \mu$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mu x$$

- SDE: growth perturbed by white noise $\mu_t = \mu + \sigma \frac{\mathrm{d}W_t}{\mathrm{d}t}$

$$\mathrm{d}X_t = \mu X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}W_t$$

- RODE: growth perturbed by colored OU noise $\mu_t = \mu + \sigma O_t$

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = (\mu + \sigma O_t) X_t$$

Actuarial risk model for an insurance company

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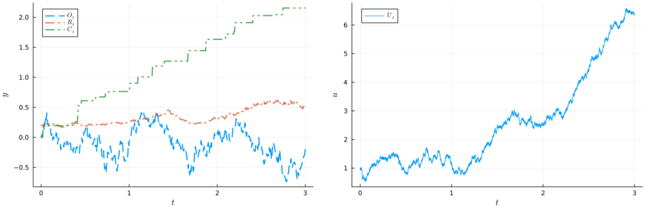
- Surplus $U_t = U_0 + \gamma t - \sum_{i=1}^{N_t} C_i$; claims $C_t = \sum_{i=1}^{N_t} C_i$, premium γ:
- Jump differential equation formulation: $\mathrm{d}U_t = \gamma\,\mathrm{d}t - \mathrm{d}C_t$
- Premium perturbed by white noise + random (gBm) interest rate R_t

$$
\mathrm{d}U_t = (\gamma + R_t U_t)\,\mathrm{d}t + \varepsilon\,\mathrm{d}W_t - \mathrm{d}C_t.
$$

- Write $X_t = U_t - C_t - O_t$ with OU process $\mathrm{d}O_t = -\nu O_t\,\mathrm{d}t + \varepsilon\,\mathrm{d}W_t$
- Get RODE model

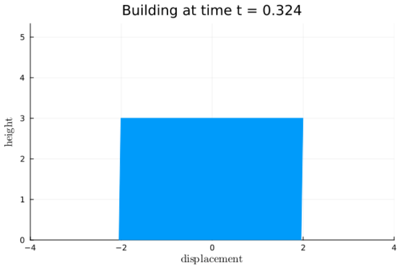
$$
\frac{\mathrm{d}X_t}{\mathrm{d}t} = R_t X_t + R_t(C_t + O_t) + \nu O_t + \gamma.
$$

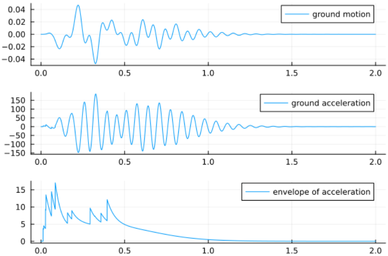


Earthquake ground-shaking model

- Mechanical structure (ceiling of a one-storey building)

$$\begin{cases} \ddot{X}_t + 2\zeta_0\omega_0\dot{X}_t + \omega_0^2 X_t = -\ddot{M}_t \\ X_0 = 0, \quad \dot{X}_0 = 0 \end{cases}$$





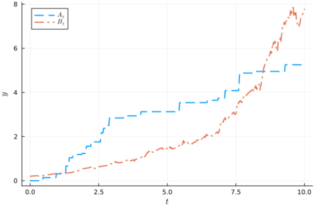
- Ground motion $M_t = \sum_{i=1}^k \gamma_i (t-\tau_i)_+^2 e^{-\delta_i (t-\tau_i)} \cos(\omega_i (t-\tau_i))$,

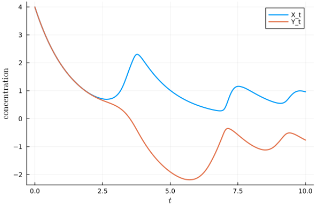
Toggle-switch gene-expression model

- X_t and Y_t are protein products of interacting genes

$$
\begin{cases}
\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = \left(A_t + \dfrac{X_t^4}{a^4 + X_t^4}\right)\left(\dfrac{b^4}{b^4 + Y_t^4}\right) - \mu X_t, \\
\dfrac{\mathrm{d}Y_t}{\mathrm{d}t} = \left(B_t + \dfrac{Y_t^4}{c^4 + Y_t^4}\right)\left(\dfrac{d^4}{d^4 + X_t^4}\right) - \nu Y_t,
\end{cases}
$$

- a, b, c, d threshold parameters; μ, λ decay rates
- External (noise) activation parameters $\{A_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$ (e.g. Compound Poisson and GBm)





Random Fisher-KPP PDE

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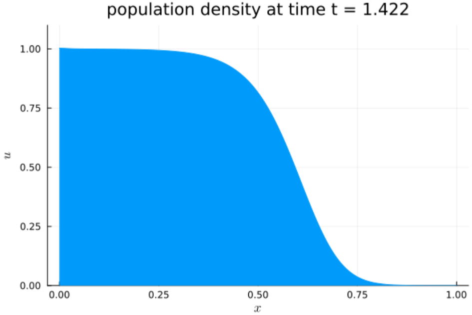
- $u = u(t,x)$ population density

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mu \dfrac{\partial^2 u}{\partial x^2} + \lambda u \left(1 - \dfrac{u}{u_m}\right), & (t,x) \in (0,\infty) \times (0,1), \\ \dfrac{\partial u}{\partial x}(t,0) = -Y_t, \quad \dfrac{\partial u}{\partial x}(t,1) = 0, \end{cases}$$

- random incoming migrations Y_t as a colored OU noise modulated by an exponentially decaying Hawkes process



Convergence of the Euler method

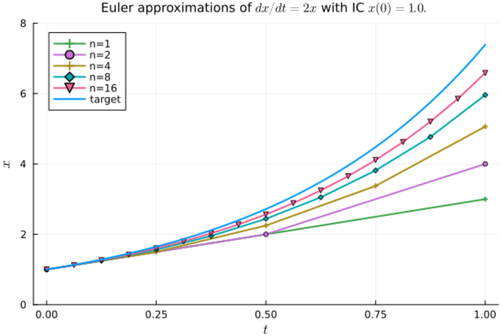
- ODE: typically order 1
- SDE: typically order 1/2
- RODE with θ-Hölder noise: order θ ?

Remarks:

- ODE is order 1 for smooth RHS $f(t, x)$
- ODE is *apparently* order θ for f θ-Hölder in time
- Wiener noise has *almost* 1/2-Hölder paths
- But noises have more structure than just pathwise Hölder regularity...

Euler method for scalar or systems of ODEs

- The *Euler method* is one of the simplest for ODEs $\dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t, x)$.



- $x_{j+1}^n = x_j^n + f(t_j, x_j^n)\Delta t$, where $\Delta t = T/n$.
- Converges with order 1: $\Delta t \mapsto \frac{1}{2}\Delta t \Rightarrow \mathrm{error} \mapsto \frac{1}{2}\mathrm{error}$

Euler-Maruyama for scalar or system of SDEs

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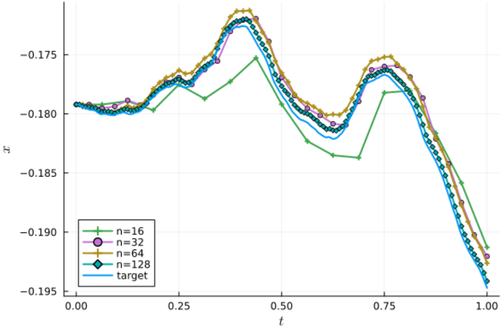
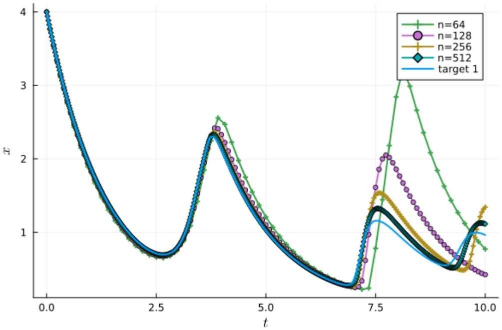
- *Euler-Maruyama* method for SDEs $\mathrm{d}X_t = f(t, X_t)\,\mathrm{d}t + g(t, X_t)\,\mathrm{d}W_t$,



- $X_{j+1}^n = X_j^n + f(t_j, X_j^n)\Delta t + g(t_j, X_j^n)\Delta W_j$.
- $\Delta t = T/n$, $\Delta W_j \sim (\Delta t)^{1/2}\mathcal{N}(0,1)$.
- Converges *strongly* with order 1/2: $\Delta t \mapsto \frac{1}{2}\Delta t \Rightarrow \text{error} \mapsto \left(\frac{1}{2}\right)^{1/2}\text{error}$

Euler method for RODEs

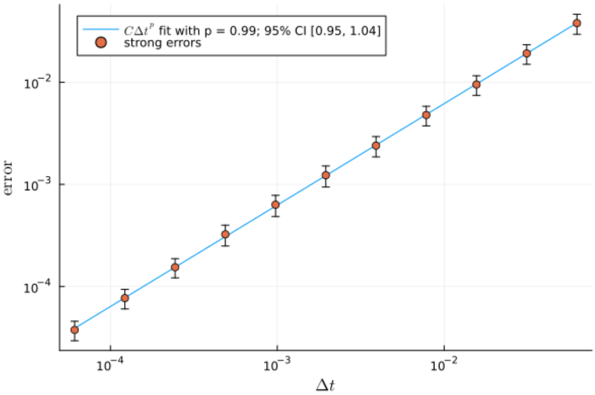
- What about for RODEs $\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t)$?



- $X_{j+1}^n = X_j^n + f(t_j, X_j^n, Y_{t_j})\Delta t$, where $\Delta t = T/n$.
- Expected to converge *strongly* with order θ for noises $\{Y_t\}_t$ with θ-Hölder continuous sample paths:

$$\Delta t \mapsto \frac{1}{2}\Delta t \Rightarrow \text{error} \mapsto \frac{1}{2^\theta}\text{error}$$

Looking for an example



- Example

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = W_t X_t$$

- Same with a few other equations and noises...
- Need an expert...

Looking for an expert...

Overview of Julia's SciML

Search docs

SciML: Open Source Software for Scientific Machine Learning with Julia

- Where to Start?

SciML: Open Source Software for Scientific Machine Learning with Julia

Edit on GitHub

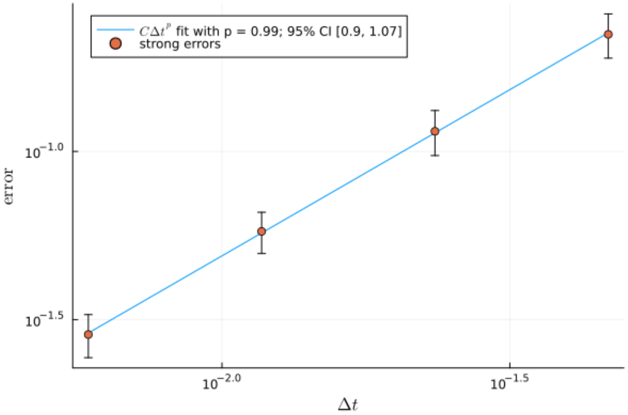
SciML: Differentiable Modeling and Simulation Combined with Machine Learning

The SciML organization is a collection of tools for solving equations and modeling systems developed in the Julia programming language with bindings to other languages such as R and Python. The organization provides well-maintained tools which compose together as a coherent ecosystem. It has a coherent development principle, unified APIs over large collections of equation solvers, pervasive differentiability and sensitivity analysis, and features many of the highest performance and parallel implementations one can find.

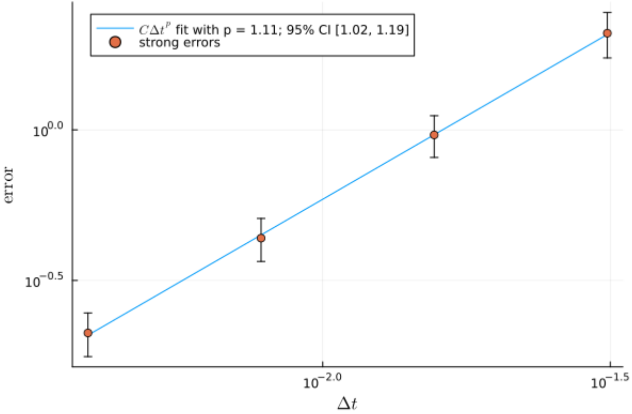
Scientific Machine Learning (SciML) = Scientific Computing + Machine Learning

Looking for an expert...

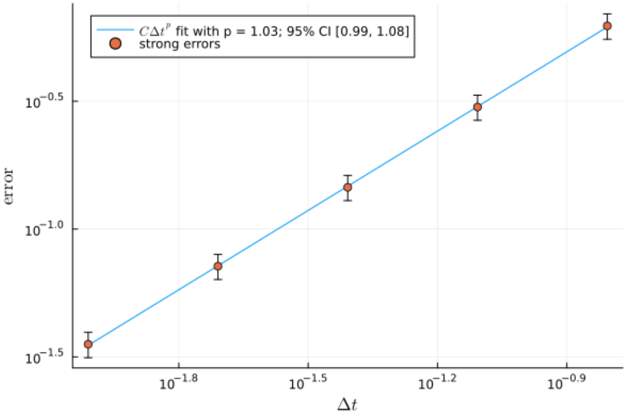
Order of convergence - risk model



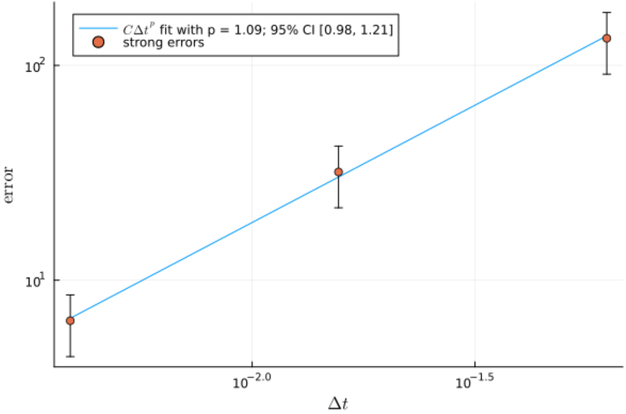
Order of convergence - Earthquake



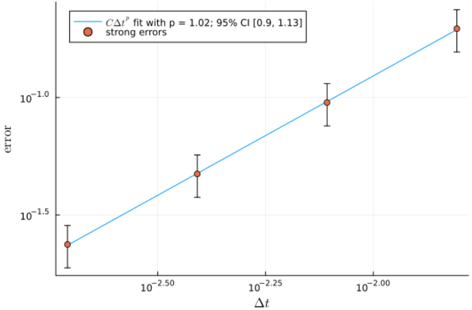
Order of convergence - toggle switch model

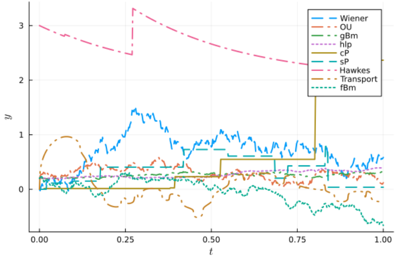


Order of convergence - random Fisher-KPP



Order of convergence - linear with “all” noises

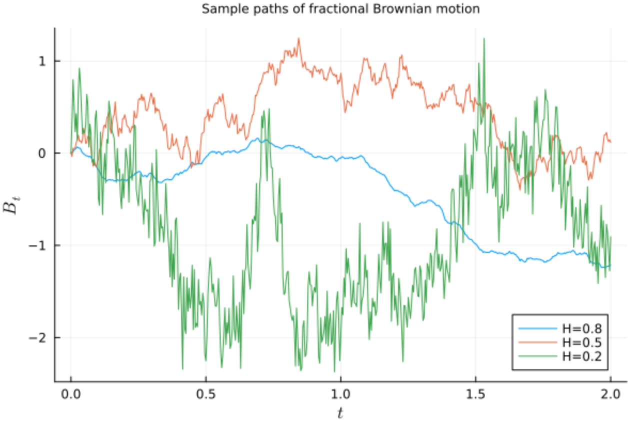




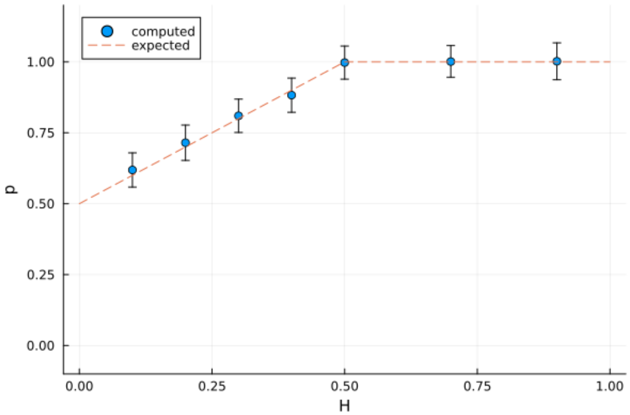
$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = \|Y_t\|^2 X_t + Y_t$$

Fractional Brownian motion noise

- Covariance $\mathbb{E}[B_t^{(H)} B_s^{(H)}] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$
  - $0 < H < 1/2$: steps are negatively correlated (rougher)
  - $H = 1/2$: steps are not correlated (Wiener process)
  - $1/2 < H < 1$: steps are positively correlated (less rough)
  - Pathwise Hölder regularity $|B_t^{(H)} - B_s^{(H)}| \leq c|t-s|^{H-\varepsilon}$



Order of convergence - linear equation with fBm



$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = -X_t + B_t^{(H)}$$

Order of convergence - ODE

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- Consider $x = x(t)$ solution on $[0, T]$ of ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x).$$

- x_j^N is an approximation at times $t_j = j\Delta t_N$, $j = 0, \ldots, N$.
- $\Delta t_N = T/N$ is the (fixed) time step
- Method is order p if there exists C such that

$$\max_{j=0,\ldots,N} \|x(t_j) - x_j^N\| \leq C\Delta t_N^p.$$

Order of convergence - RODE

- Consider solution $\{X_t\}_t$ of RODE

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t)$$

- Approximation X_j^N of X_{t_j} at $t_j = j\Delta t_N$
- Method is of *pathwise* order p if there exists $C = C(\omega) > 0$ such that

$$\max_j \|X_{t_j}(\omega) - X_j^N(\omega)\| \leq C(\omega)\Delta t_N^p$$

- Method is of *strong* order p if there exists $C = C(\omega) > 0$ such that

$$\max_j \mathbb{E}[\|X_{t_j} - X_j^N\|] \leq C\Delta t_N^p$$

Estimating the order of convergence

- Given a RODE

$$\begin{cases} \dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t), \\ X_t|_{t=0} = X_0 \end{cases}$$

- For each sample IC $X_0(\omega_m)$ and sample noise $\{Y_t(\omega_m)\}_t$
- Get sample solution $X_t(\omega_m)$ (distributionally exact or on a refined mesh)
- Get sample approximation $X_j^N(\omega_m)$
- Estimate via Monte-Carlo

$$\epsilon_N = \max_j \mathbb{E}[\|X_{t_j} - X_j^N\|] \approx \max_j \frac{1}{M} \sum_m \|X_{t_j}(\omega_m) - X_j^n(\omega_m)\|$$

- Estimate error via standard deviation of the samples
- Fit the errors to a power law $\epsilon_N \sim (1/N)^p$.

Euler for ODEs

- $\dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t, x)$ (equation)

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- $x(t + \Delta t) = x(t) + \displaystyle\int_t^{t+\Delta t} f(s, x(s))\,\mathrm{d}s$ (integral form)

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- $x(t_j + \Delta t) \approx x(t_j) + f(t_j, x(t_j))\Delta t$ (First order approximation)

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- $\Rightarrow \epsilon_j = \displaystyle\int_{t_j}^{t_j+\Delta t} (f(s, x(s)) - f(t_j, x(t_j)))\,\mathrm{d}s$ (local truncation error)

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Euler for ODEs

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Euler for ODEs

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- Summing up the local errors:

$$\epsilon = \sum_j \epsilon_j \lesssim \sum_j \Delta t^2 \lesssim O(\Delta t).$$

Euler for ODEs

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- Hence order 1

Euler for ODEs with Hölder regularity

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Euler for ODEs with Hölder regularity

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- $\dfrac{\mathrm{d}x}{\mathrm{d}t} = f(t, x)$ (equation)
- $x(t + \Delta t) = x(t) + \displaystyle\int_t^{t+\Delta t} f(s, x(s))\,\mathrm{d}s$ (integral form)
- $x(t_j + \Delta t) \approx x(t_j) + f(t_j, x(t_j))\Delta t$ (First order approximation)
- $\Rightarrow \epsilon_j = \displaystyle\int_{t_j}^{t_j+\Delta t} (f(s, x(s)) - f(t_j, x(t_j)))\,\mathrm{d}s$ (local truncation error)
- $|f(s, x(s)) - f(t_j, x(t_j))| \lesssim (\Delta t)^\theta$ (if f θ-Hölder in time)

Euler for ODEs with Hölder regularity

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Euler for ODEs with Hölder regularity

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$$
\epsilon = \sum_j \epsilon_j \lesssim \sum_j \Delta t^(1+\theta) \lesssim O(\Delta t^\theta).
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Euler for ODEs with Hölder regularity

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- Hence order θ

Euler-Maruyama for SDEs

- $\mathrm{d}X = f(t, X_t)\,\mathrm{d}t + g(t, X_t)\,\mathrm{d}W_t$

Euler-Maruyama for SDEs

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- (Actually do squared norm, global error, and Itô, but that is the idea)

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- $\epsilon_j \lesssim \Delta t^{3/2} + \Delta t^{1/2}\Delta W_j$.

Euler-Maruyama for SDEs

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- Summing up:

$$\epsilon = \sum_j \epsilon_j \lesssim \sum_j (\Delta t^{1+1/2} + \Delta t^{1/2}\Delta W) \lesssim O(\Delta t^{1/2}).$$

Euler-Maruyama for SDEs

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- Hence order 1/2

Euler for RODEs - Hölder noise

- $$\frac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t)$$

Euler for RODEs - Hölder noise

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- $\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t)$
- $X_{t+\Delta t} = X_t + \displaystyle\int_t^{t+\Delta t} f(s, X_s, Y_s)\,\mathrm{d}s$

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- $X_{t_j+\Delta t} = X_{t_j} + f(t_j, X_{t_j}, Y_{t_j})\Delta t$

Euler for RODEs - Hölder noise

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- $\Rightarrow \epsilon_j = \displaystyle\int_{t_j}^{t_j+\Delta t} (f(s, X_s, Y_s) - f(t_j, X_{t_j}, Y_{t_j}))\,\mathrm{d}s$

Euler for RODEs - Hölder noise

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- Hence order θ for θ-Hölder noises

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- Hence order θ for θ-Hölder noises
- See Wang, Cao, Han, & P. Kloeden (2021)

Euler for RODEs - Itô diffusion

- RODE augmented with SDE

$$\begin{cases} \mathrm{d}X_t = f(t, X_t, Y_t)\,\mathrm{d}t \\ \mathrm{d}Y_t = \mu(t, Y_t)\,\mathrm{d}t + \sigma(t, Y_t)\,\mathrm{d}W_t \end{cases}$$

- Euler-Maruyama reduces to Euler for RODE part
- Euler-Maruyama for additive noise $\sigma = \sigma(t)$ is order 1
- Milstein method is order 1
- Milstein method reduces to Euler for RODE part
- In any case, Euler is order 1
- See Wang, Cao, Han, & P. Kloeden (2021)

Euler for RODEs - idea

- $\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t).$

Euler for RODEs - idea

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- $\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t)$.
- $\Rightarrow \epsilon_j = \displaystyle\int_{t_j}^{t_j+\Delta t} \left(f(s, X_s, Y_s) - f(t_j, X_{t_j}, Y_{t_j})\right) \mathrm{d}s$

Euler for RODEs - idea

- $\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t).$
- $\Rightarrow \epsilon_j = \displaystyle\int_{t_j}^{t_j+\Delta t} (f(s, X_s, Y_s) - f(t_j, X_{t_j}, Y_{t_j}))\,\mathrm{d}s$
- $f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}) = \displaystyle\int_{t_j(s)}^{s} \mathrm{d}F_s$

Euler for RODEs - idea

- $\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t).$
- $\Rightarrow \epsilon_j = \displaystyle\int_{t_j}^{t_j+\Delta t} (f(s, X_s, Y_s) - f(t_j, X_{t_j}, Y_{t_j}))\,\mathrm{d}s$
- $f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}) = \displaystyle\int_{t_j(s)}^{s} \mathrm{d}F_s$
- Using Fubini

$$
\begin{aligned}
\epsilon &= \int_0^t (f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}))\,\mathrm{d}s \\
&= \int_0^t \int_{t_j(s)}^{s} \mathrm{d}F_\tau\,\mathrm{d}s = \int_0^t \int_\tau^{t_j(\tau)+\Delta t} ;\mathrm{d}s\,\mathrm{d}F_\tau \leq \int_0^t \Delta t\,\mathrm{d}F_\tau \\
&\leq C\Delta t.
\end{aligned}
$$

Euler for RODEs - idea

- $\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t).$
- $\Rightarrow \epsilon_j = \displaystyle\int_{t_j}^{t_j+\Delta t} (f(s, X_s, Y_s) - f(t_j, X_{t_j}, Y_{t_j}))\,\mathrm{d}s$
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&\leq C\Delta t.
\end{aligned}
$$

- Hence order 1

Euler for RODEs - Itô process noise

- $\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t)$, where $\mathrm{d}Y_t = A_t\,\mathrm{d}t + B_t\,\mathrm{d}W_t$

Euler for RODEs - Itô process noise

- $\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t)$, where $\mathrm{d}Y_t = A_t\,\mathrm{d}t + B_t\,\mathrm{d}W_t$
- $\Rightarrow \epsilon_j = \displaystyle\int_{t_j}^{t_j+\Delta t} (f(s, X_s, Y_s) - f(t_j, X_{t_j}, Y_{t_j}))\,\mathrm{d}s$ (local truncation error)

Euler for RODEs - Itô process noise

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- $\Rightarrow \epsilon = \displaystyle\int_0^t (f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}))\,\mathrm{d}s$ (global error)
- $f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}) = \displaystyle\int_{t_j(s)}^s \tilde{A}_\tau\,\mathrm{d}\tau + \int_{t_j}^s \tilde{B}_\tau\,\mathrm{d}W_\tau$ (Itô f)

Euler for RODEs - Itô process noise

- $\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t)$, where $\mathrm{d}Y_t = A_t\,\mathrm{d}t + B_t\,\mathrm{d}W_t$
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- $\Rightarrow \epsilon = \displaystyle\int_0^t (f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}))\,\mathrm{d}s$ (stochastic Fubini)

$$
\begin{aligned}
&= \int_0^t \int_{t_j(s)}^s \tilde{A}_\tau\,\mathrm{d}\tau + \int_{t_j}^\tau \tilde{B}_\tau\,\mathrm{d}W_\tau\,\mathrm{d}s \\
&= \int_0^t \int_\tau^{t_j(\tau)+\Delta t} \tilde{A}_\tau\,\mathrm{d}s\,\mathrm{d}\tau + \int_0^t \int_\tau^{t_j(\tau)+\Delta t} \tilde{B}_\tau\,\mathrm{d}s\,\mathrm{d}W_\tau \\
&\leq \left(\int_0^t \tilde{A}_\tau\,\mathrm{d}\tau + \int_0^t \tilde{B}_\tau \mathrm{d}W_\tau \right) \Delta t
\end{aligned}
$$

Euler for RODEs - Itô process noise

- $\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t)$, where $\mathrm{d}Y_t = A_t\,\mathrm{d}t + B_t\,\mathrm{d}W_t$
- $\Rightarrow \epsilon = \displaystyle\int_0^t (f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}))\,\mathrm{d}s$ (global error)
- $f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}) = \displaystyle\int_{t_j(s)}^s \tilde{A}_\tau\,\mathrm{d}\tau + \int_{t_j}^s \tilde{B}_\tau\,\mathrm{d}W_\tau$ (Itô f)
- $\Rightarrow \epsilon = \displaystyle\int_0^t (f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}))\,\mathrm{d}s$ (stochastic Fubini)

$$
\begin{aligned}
&= \int_0^t \int_{t_j(s)}^s \tilde{A}_\tau\,\mathrm{d}\tau + \int_{t_j}^\tau \tilde{B}_\tau\,\mathrm{d}W_\tau\,\mathrm{d}s \\
&= \int_0^t \int_\tau^{t_j(\tau)+\Delta t} \tilde{A}_\tau\,\mathrm{d}s\,\mathrm{d}\tau + \int_0^t \int_\tau^{t_j(\tau)+\Delta t} \tilde{B}_\tau\,\mathrm{d}s\,\mathrm{d}W_\tau \\
&\leq \left(\int_0^t \tilde{A}_\tau\,\mathrm{d}\tau + \int_0^t \tilde{B}_\tau \mathrm{d}W_\tau\right)\Delta t
\end{aligned}
$$

- Hence order 1

Euler for RODEs - Noise with bounded variation

- Finite variation process:

$$
\begin{aligned}
f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})
&= \int_{\tau^N(s)}^s D_\xi f(\xi, X^N_{\tau^N(s)}, Y_{\xi-})\,\mathrm{d}\xi \\
&\quad + \int_{\tau^N(s)+}^s D_y f(\xi, X^N_{\tau^N(s)}, Y_{\xi-})\,\mathrm{d}Y_\xi \\
&\quad + \sum_{\tau^N(s)<\xi\leq s} \Big(f(\xi, X^N_{\tau^N(s)}, Y_\xi) - f(\xi, X^N_{\tau^N(s)}, Y_{\xi-}) \\
&\qquad\qquad - D_y f(\xi, X^N_{\tau^N(s)}, Y_{\xi-})\Delta Y_\xi \Big).
\end{aligned}
$$

- Same idea, just use Fubini to get order 1

Euler for RODEs - Noise semi-martingale

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- $\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t)$
- $\{Y_t\}_t$ semi-martingale
- $Y_t = N_t + B_t$ (finite variation + local martingale)
- Essentially combine the ideas
- Use Fubini to move "irregularity"to large scales
- Not smooth but integrable somehow
- Get order 1
- The "chain rule formula" (next slide) is more involved but it works...

Global error via chain rule for semimartingales

$$
\begin{aligned}
&\int_0^{t_j} \left(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)}) \right) \,\mathrm{d}s \\
&\quad = \int_0^{t_j} \int_{\tau^N(s)}^s D_\xi f(\xi, X^N_{\tau^N(s)}, Y_{\xi-}) \,\mathrm{d}\xi \,\mathrm{d}s \\
&\qquad + \int_0^{t_j} \int_{\tau^N(s)^+}^s D_y f(\xi, X^N_{\tau^N(s)}, Y_{\xi-}) \,\mathrm{d}Y_\xi \,\mathrm{d}s \\
&\qquad + \int_0^{t_j} \sum_{\tau^N(s) < \xi \leq s} \Big(f(\xi, X^N_{\tau^N(s)}, Y_\xi) - f(\xi, X^N_{\tau^N(s)}, Y_{\xi-}) \\
&\qquad\qquad\qquad\qquad - D_y f(\xi, X^N_{\tau^N(s)}, Y_{\xi-}) \Delta Y_\xi \Big) \,\mathrm{d}s \\
&\qquad + \frac{1}{2} \int_0^{t_j} \int_{\tau^N(s)}^s D_{yy} f(\xi, X^N_{\tau^N(s)}, Y_{\xi-}) \,\mathrm{d}[Y, Y]^c_\xi \,\mathrm{d}s,
\end{aligned}
$$

where $[Y, Y]^c_\xi$ is the continuous part of the quadratic variation of the process.

Fractional Brownian motion noise

- Only done for the linear equation

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = -X_t + B_t^{(H)}$$

- Same idea of using Fubini
- But local error has a $(t-s)^{H-1/2}$ kernel
- This leads to an $\mathcal{O}(\Delta t + \Delta t^{H+1/2})$ error
- For $H = 1/2$, fBm is Wiener, recover order 1
- For $1/2 < H \leq 1$, it is smoother, also get order 1
- For $0 < H < 1/2$, get lower order $H + 1/2 < 1$, but still better than previously know order H.

Outlook

- General result for fBm (not just that linear equation)
- Volterra stochastic noise
- Higher order methods for non-Itô noises
- Improved order of convergence for other methods?

THANKS!