.

# STRONG ORDER 1 CONVERGENCE OF THE EULER-MARUYAMA METHOD FOR RANDOM ORDINARY DIFFERENTIAL EQUATIONS

RICARDO M. S. ROSA AND PETER E. KLOEDEN

ABSTRACT. It is well known that the Euler-Maruyama method of approximating a random ordinary differential equation $\mathrm{d}X_t/\mathrm{d}t = f(t, X_t, Y_t)$ driven by a stochastic process $\{Y_t\}_t$ with $\theta$-Hölder sample paths is estimated to be of strong order $\theta$ with respect to the time step, provided $f = f(t, x, y)$ is sufficiently regular. Here, we show that, in common situations, it is possible to exploit further conditions on the noise and prove that the strong convergence is actually of order 1, regardless of the Hölder regularity of the sample paths. This applies to Itô process noises (such as Wiener, Ornstein-Uhlenbeck, and Geometric Brownian process), which have Hölder continuous sample paths; to point-process noises (such as Poisson point processes and Hawkes self-exciting processes), which are not even continuous and have jump-type discontinuities, and to transport-type processes. The order 1 convergence is based on two main ideas: First of all, we do not estimate directly the local error and, instead, add up the local steps and work directly with an accumulated global error. Secondly, we assume either a control of the total variation of the sample paths of the noise or that the noise is an Itô process, in which case we exploit the Iso isometry on the global error term.

## 1. Introduction

Consider the following initial value problem for a **random ordinary differential equation (RODE)**:

$$
\begin{cases} \dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t), \qquad 0 \leq t \leq T, \\ X_t\big|_{t=0} = X_0, \end{cases} \tag{1.1}
$$

on a time interval $I = [0, T]$, with $T > 0$, and where the noise $\{Y_t\}_{t\in I}$ is a given stochastic process. The sample space is denoted by $\Omega$.

The Euler-Maruyama method for solving this initial value problem consists in approximating the solution on a uniform time mesh $t_j = j\Delta t_N$, $j = 0, \ldots, N$, with fixed time step $\Delta t_N = T/N$, for a given $N \in \mathbb{N}$. In such a mesh, the Euler-Maruyama

---

*Date*: October 31, 2022.

2000 *Mathematics Subject Classification.* 76D05, 76D06, 35Q30, 37L99.

*Key words and phrases.* random ordinary differential equations, Euler-Maruyama method, strong convergence, Itô process, point process.

scheme takes the form

$$X_{t_j}^N = X_{t_{j-1}}^N + \Delta t_N f(t_{j-1}, X_{t_{j-1}}^N, Y_{t_{j-1}}), \qquad j = 1, \ldots, N, \tag{1.2}$$

with the initial condition

$$X_0^N = X_0. \tag{1.3}$$

Notice $t_j = j\Delta t_N = jT/N$ also depends on $N$, but we do not make this dependency explicit, for the sake of notational simplicity.

When the noise $\{Y_t\}_{t\in I}$ has $\theta$-Hölder continuous sample paths, it can be show [3], under further suitable conditions, that the Euler-Maruyama scheme converges strongly with order $\theta$ with respect to the time step, i.e. there exists a constant $C \geq 0$ such that

$$\max_{j=0,\ldots,N} \mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq C\Delta t_N^\theta, \qquad \forall N \in \mathbb{N}, \tag{1.4}$$

where $\mathbb{E}[\cdot]$ indicates the expectation of a random variable on $\Omega$.

Our aim is to show that, in many classical examples, it is possible to exploit further conditions that yield in fact a strong order 1 convergence, with the sample paths still being Hölder continuous or even discontinuous. This is the case, for instance, when the noise is an Itô noise, or a point process or a transport process.

The first main of the proof is to not estimate the local error and, instead, work with an explicit formula for the global error, namely (see Lemma 3.1)

$$\begin{aligned} X_{t_j} - X_{t_j}^N = X_0 - X_0^N & \\ & + \int_0^{t_j} \left(f(s, X_s, Y_s) - f(s, X_{\tau^N(s)}, Y_s)\right) \,\mathrm{d}s \\ & + \int_0^{t_j} \left(f(s, X_{\tau^N(s)}, Y_s) - f(s, X_{\tau^N(s)}^N, Y_s)\right) \,\mathrm{d}s \\ & + \int_0^{t_j} \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})\right) \,\mathrm{d}s, \end{aligned} \tag{1.5}$$

for $j = 1, \ldots, N$, where $\tau^N$ is a piecewise constant function with jumps at the mesh points $t_j$. The first term vanishes due to the initial condition $X_0^N = X_0$. The second term only depends on the solution and can be easily estimated with natural regularity conditions on the term $f = f(t, x, y)$. The third term is handled solely with the typical required condition on $f = f(t, x, y)$ of being uniformly globallly Lipschitz continuity with respect to $x$. With those, we obtain the following basic bound for the global error (see Lemma 4.1)

$$\begin{aligned} |X_{t_j} - X_{t_j}^N| \leq \Bigg( & |X_0 - X_0^N| + L_X \int_0^{t_j} |X_s - X_{\tau^N(s)}| \,\mathrm{d}s \\ & \left|\int_0^{t_j} \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})\right) \,\mathrm{d}s\right| \Bigg) e^{L_X t_j}. \end{aligned} \tag{1.6}$$

The only problematic term is the last one. The classical analysis is to use an assumed $\theta$-Hôlder regularity of noise and estimate the local error as

$$|f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})| \leq C\Delta t^\theta.$$

Instead, we keep the global error formula and assume the steps of $F_t = f(t, X^N_{\tau^N(t)}, Y_t)$ are somehow integrable. In order to give the main idea, let us assume for the moment that the sample paths of $\{F_t\}_{t\ inI}$ satisfy

$$F_s - F_\tau = \int_\tau^s \mathrm{d}F_\xi,$$

either in the sense of a Riemann-Stieltjes integral or of an Itô integral. The first sense fits the case of noises with bounded total variation. The second one fits the case of an Itô noise. In any case, we bound the global error term using integration by parts,

$$\begin{aligned}\int_0^{t_j} \left(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})\right) \mathrm{d}s &= \int_0^{t_j} \int_{\tau^N(s)}^s \mathrm{d}F_\xi \,\mathrm{d}s \\ &= \int_0^{t_j} \int_\xi^{\tau^N(\xi)+\Delta t_N} \mathrm{d}s \,\mathrm{d}F_\xi.\end{aligned}$$

Then, we find that

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^{t_j} \left(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})\right) \mathrm{d}s\right|\right] \\ \leq \Delta t_N \mathbb{E}\left[\int_0^{t_j} \mathrm{d}F_\xi\right] = \Delta t_N \mathbb{E}\left[F_{t_j} - F_0\right],\end{aligned}$$

which yields the strong order 1 convergence.

That is the main idea. In the case of an Itô integral, that is exactly what we assume, because the Itô integral is not order preserving. In the case of bounded variation, however, we can relax the above condition and work not with $\{F_t\}_{t\in I}$ directly but with a bound on the step of the form

$$|f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})| \leq \bar{F}_s - \bar{F}_{\tau^N(s)},$$

where only this bounding process $\{\bar{F}_t\}_{t\in I}$.

and then either of two assumptions on the stochastic process $\{F_t\}_{t\geq 0}$ given by $F_t = f(t, X^N_{\tau^N(t)}, Y_t)$:

(i) The sample paths of $\{F_t\}_{t\in I}$ are of bounded variation $V(\{F_t\}_I; I)$, with finite mean $\mathbb{E}[V(\{F_t\}_I; I)] < \infty$ (see Lemma 6.1 and Theorem 6.1 and notice we actually require less than that, but that is the main idea); or

(ii) The process $\{F_t\}_{t\in I}$ is an Itô process, $\mathrm{d}F_t = A_t\,\mathrm{d}t + B_t\,\mathrm{d}W_t$, with $t \mapsto \mathbb{E}[A_t]$ integrable on $I$ and $t \mapsto \mathbb{E}[B_t]$ square integrable on $I$ (see Lemma 6.1 and Theorem 6.1).

In the first case, we control the steps with

$$f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)}) = \int_s^t \,\mathrm{d}F_\xi$$

In the second case, we control the steps with

$$f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)}) = \int_s^t A_\xi \,\mathrm{d}\xi + \int_s^t B_\xi \,\mathrm{d}W_\xi.$$

Indeed, many common Hölder functions are actually of bounded variation, such as $t \mapsto t^{1/3}$ and $t \mapsto \sin(\omega t)^{1/5}$. This is a common situation when the noise is a transport process $Y_t = g(t, Y_0)$, for a random vector $Y_0 = Y_0(\omega)$. Even functions with jump discontinuities, such as those in point-processes, are of bounded variation. Being of bounded variation, it is possible to control their steps and obtain the strong order 1 convergence.

In the case of an Itô noise, the sample paths are typically Hölder continuous with exponent $1/2$ and are not of bounded variation. In this case, however, the Itô isometry reveals futher regularity that can be used to yield a strong order 1 convergence.

More precisely, for the semi-separable case, we assume $f$ is of the form

$$f(t, x, y) = a(t, y)h(x) + b(t, y).$$

In this case, we assume the processes $\{a(t, Y_t)\}_{t\in I}$ and $\{b(t, Y_t)\}_{t\in I}$ have their steps bounded by a monotonic process, which typically happens for point processes, i.e.

$$|a(t+\tau, Y_{t+\tau}) - a(t, Y_t)| \leq A_t, \quad |b(t+\tau, Y_{t+\tau}) - b(t, Y_t)| \leq B_t,$$

where $\{A_t\}_{t\in I}$ and $\{B_t\}_{t\in I}$ have monotonically non-decreasing sample paths. Under further suitable conditions (see Theorem 5.2), we show that the Euler-Maruyama method is of strong order 1, i.e. (1.4) holds with $\theta = 1$.

Even if the structure of the equation is not exactly semi-separable but it is somehow possible to bound the steps $|f(t+\tau, X_t, Y_{t+\tau}) - a(t, X_t, Y_t)|$ by a suitable process with monotonic non-decreasing sample paths, it is possible to prove the strong order 1 convergence (see Theorem 5.1).

For the Itô noise case, we consider a general equation of the form (1.1), with a noise defined as an **Itô process** $\{Y_t\}_{t\geq 0}$, satisfying

$$\mathrm{d}Y_t = A_t \,\mathrm{d}t + B_t \,\mathrm{d}W_t, \tag{1.7}$$

We are not solving for $Y_t$, nor approximating it numerically, otherwise we would actually need to consider a system of stochastic differential equations. Instead, we assume it is a known process that can be computed analytically, such as a Wiener process, an Ornstein-Uhlenbeck process, or a geometric Brownian motion. With those in mind, $A_t$ and $B_t$ may be originally given in terms of $\{W_t\}_{t\geq 0}$ and $\{Y_t\}_{t\in I}$, but the general assumption is only given in terms of $\{A_t\}_{t\in I}$ and $\{B_t\}_{t\in I}$.

In the case that $f = f(t, x, y)$ is twice continuously differentiable, the Itô formula applies and we show, under suitable conditions on $\{A_t\}_{t \in I}$, $\{B_t\}_{t \in I}$ and the derivatives of $f$, that the Euler-Maruyama method is of strong order 1, i.e. (1.4) holds with $\theta = 1$.

In order to make the main idea clear cut, here are the options we have for estimating the error:

(i) If the local error $e_j$, at the $j$th time step, is bounded as
$$\mathbb{E}[|e_j|] \lesssim \Delta t_N^{3/2},$$
as usual for a 1/2-Hölder noise, then adding them up leads to
$$\sum \mathbb{E}[|e_j|] \lesssim N \Delta t_N^{3/2} = T \Delta t_N^{1/2}.$$

(ii) If we use the Itô isometry locally, we still get the local error as
$$\mathbb{E}[|e_j|] \leq \mathbb{E}[|e_j|^2]^{1/2} \lesssim \left(\Delta t_N^{2(3/2)}\right)^{1/2} = \Delta t_N^{3/2},$$
and adding that up still leads to an error of order $\Delta t_N^{\theta}$.

(iii) If, instead, we first add the terms up, then $\sum e_j$ becomes an integral over $I$ with respect to the Wiener noise, so that we can use the Itô isometry on the added up term and obtain
$$\mathbb{E}\left[\left|\sum e_j\right|\right] \lesssim \left(\mathbb{E}\left[\left|\sum e_j\right|^2\right]\right)^{1/2} = \left(\sum \mathbb{E}[|e_j|^2]\right)^{1/2}$$
$$= \left(\sum \Delta t_N^3\right)^{1/2} = \left(\Delta t_N^2\right)^{1/2} = \Delta t_N.$$
and we finally get the error to be of order 1.

## 2. Pathwise solution

For the notion and main results on pathwise solution for RODEs, we refer the reader to [4, Section 2.1]. We consider two sets of hypotheses, namely Hypothesis 2.1 and Hypothesis 2.2, each suitable to one of the two main cases we consider, namely the case in which the steps are bounded by processes with monotonic expected values and the case with Itô type noises.

We start with the following hypotheses, which imply the existence and uniqueness of pathwise solutions of the RODE (1.1) in the sense of Carathéodory.

**Hypothesis 2.1.** *We consider a function $f = f(t, x, y)$ defined on $I \times \mathbb{R} \times \mathbb{R}$ and a real-valued stochastic process $\{Y_t\}_{t \in I}$, where $I = [0, T]$, $T > 0$. We make the following standing hypotheses.*

(i) *$f$ is globally Lipschitz continuous on $x$, uniformly in $t$ and $y$, i.e. there exists a constant $L_X \geq 0$ such that*
$$|f(t, x_1, y) - f(t, x_2, y)| \leq L_X |x_1 - x_2|, \quad \forall t \in I, \ \forall x_1, x_2, y \in \mathbb{R}. \tag{2.1}$$

(ii) *We also assume that $(t, x) \mapsto f(t, x, Y_t)$ satisfies the Carathéodory conditions:*
   (a) *The mapping $x \mapsto f(t, x, y)$ is continuous on $x \in \mathbb{R}$, for almost every $(t, y) \in I \times \mathbb{R}$;*
   (b) *The mapping $t \mapsto f(t, x, Y_t)$ is Lebesgue measurable in $t \in I$, for each $x \in \mathbb{R}$ and each sample path $t \mapsto Y_t(\omega)$;*
   (c) *The bound $|f(t, x, Y_t)| \leq M_t + L_X|x|$ holds for all $t \in I$ and all $x \in \mathbb{R}$, where $\{M_t\}_{t\in I}$ is a real stochastic process with Lebesgue integrable sample paths $t \mapsto M_t(\omega)$ on $t \in I$.*

Under these assumptions, for each sample value $\omega \in \Omega$, the integral equation

$$X_t = X_0 + \int_0^t f(s, X_s, Y_s)\,\mathrm{d}s \tag{2.2}$$

has a unique solution, in the Lebesgue sense, for the realization $X_0 = X_0(\omega)$ of the initial condition and the sample path $t \mapsto Y_t(\omega)$ of the noise process (see [2, Theorem 1.1]). Moreover, the mapping $(t, \omega) \mapsto X_t(\omega)$ is measurable (see [4, Section 2.1.2]) and, hence, give rise to a well-defined stochastic process $\{X_t\}_{t\in I}$.

Each sample path solution $t \mapsto X_t(\omega)$ is bounded by

$$|X_t| \leq \left(|X_0| + \int_0^t M_s\,\mathrm{d}s\right) e^{L_X t}, \quad \forall t \in I. \tag{2.3}$$

For the strong convergence of the Euler-Maruyama approximation, we also need to control the expectation of the solution above, among other things. With that in mind, we have the following useful result.

**Lemma 2.1.** *Under Hypothesis 2.1, suppose further that*

$$\mathbb{E}[|X_0|] < \infty \tag{2.4}$$

*and*

$$\int_0^T \mathbb{E}[|M_s|]\,\mathrm{d}s < \infty \tag{2.5}$$

*Then,*

$$\mathbb{E}[|X_t|] \leq \left(\mathbb{E}[|X_0|] + \int_0^t \mathbb{E}[|M_s|]\,\mathrm{d}s\right) e^{L_X t}, \quad t \in I. \tag{2.6}$$

*Proof.* Thanks to (2.3), the result is straightfoward □

In special *dissipative* cases, depending on the structure of the equation, we might not need the second condition and only require $\mathbb{E}[|X_0|] < \infty$, but we do not exploit this case here.

When $f = f(t, x, y)$ is continuous on all three variables, as well as uniformly globally Lipscthiz continuous in $x$, and the sample paths of $\{Y_t\}_{t\geq 0}$ are continuous, then the integrand in (2.2) is continuous in $t$ and the integral becomes a Riemann integral. In this case, the integral form (2.2) of the pathwise solutions of (1.1) holds in the Riemann sense.

This setting is assumed in the analysis of the case of Itô noise. With that in mind, we define, more precisely, a second set of hypotheses, on top of the first one above.

**Hypothesis 2.2.** *We consider a function $f = f(t, x, y)$ defined on $I \times \mathbb{R} \times \mathbb{R}$ and a real-valued stochastic process $\{Y_t\}_{t\in I}$, where $I = [0, T]$, $T > 0$. Besides Hypothesis 2.1, we also assume*

(i) *$f = f(t, x, y)$ is continuous on $I \times \mathbb{R} \times \mathbb{R}$.*
(ii) *For each $x \in \mathbb{R}$, the map $(t, y) \mapsto f(t, x, y)$ is twice continuously differentiable on $I \times \mathbb{R}$.*
(iii) *The sample paths $t \mapsto Y_t(\omega)$ of the noise process are continuous on $I$.*
(iv) *The noise process $\{Y_t\}_{t\in I}$ is an Itô noise in the sense of satisfying*

$$\mathrm{d}Y_t = A_t \,\mathrm{d}t + B_t \,\mathrm{d}W_t, \tag{2.7}$$

*where $\{W_t\}_{t\geq 0}$ is a standard Wiener process and $\{A_t\}_{t\in I}$ and $\{B_t\}_{t\in I}$ are stochastic processes adapted to $\{W_t\}_{t\geq 0}$.*

## 3. Integral formula for the global pathwise error

In this section, we derive the following integral formula for the global error:

**Lemma 3.1.** *Under Hypothesis 2.1, the Euler-Maruyama approximation (1.2) for any pathwise solution of the random ordinary differential equation (1.1) satisfies the global error formula*

$$\begin{aligned} X_{t_j} - X_{t_j}^N = X_0 - X_0^N & \\ & + \int_0^{t_j} \big(f(s, X_s, Y_s) - f(s, X_{\tau^N(s)}, Y_s)\big) \,\mathrm{d}s \\ & + \int_0^{t_j} \Big(f(s, X_{\tau^N(s)}, Y_s) - f(s, X_{\tau^N(s)}^N, Y_s)\Big) \,\mathrm{d}s \\ & + \int_0^{t_j} \Big(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})\Big) \,\mathrm{d}s, \end{aligned} \tag{3.1}$$

*for $j = 1, \ldots, N$, where $\tau^N$ is the piecewise constant jump function along the time mesh:*

$$\tau^N(t) = \max_j\{j\Delta t_N;\ j\Delta t_N \leq t\} = \left[\frac{t}{\Delta t_N}\right] \Delta t_N = \left[\frac{tN}{T}\right] \frac{T}{N}. \tag{3.2}$$

*Proof.* Under Hypothesis 2.1, the solutions of (1.1) are pathwise solutions in the Lebesgue sense of (2.2). With that in mind, we first obtain an expression for a single time step, from time $t_{j-1}$ to $t_j = t_{j-1} + \Delta t_N$.

For notational simplicity, we momentarily write $t = t_{j-1}$ and $\tau = \Delta t_N$, so that $t_j = t + \tau$. The exact pathwise solution satisfies

$$X_{t+\tau} = X_t + \int_t^{t+\tau} f(s, X_s, Y_s) \,\mathrm{d}s.$$

The Euler-Maruyama step is given by

$$X_{t+\tau}^N = X_t^N + \tau f(t, X_t^N, Y_t).$$

Subtracting, we obtain

$$X_{t+\tau} - X_{t+\tau}^N = X_t - X_t^N + \int_t^{t+\tau} \left( f(s, X_s, Y_s) - f(t, X_t^N, Y_t) \right) \,\mathrm{d}s.$$

We arrange the integrand as

$$\begin{aligned} f(s, X_s, Y_s) - f(t, X_t^N, Y_t) =& f(s, X_s, Y_s) - f(s, X_t, Y_s) \\ & + f(s, X_t, Y_s) - f(s, X_t^N, Y_s) \\ & + f(s, X_t^N, Y_s) - f(t, X_t^N, Y_t). \end{aligned}$$

This yields

$$\begin{aligned} X_{t+\tau} - X_{t+\tau}^N =& X_t - X_t^N \\ & = \int_t^{t+\tau} \left( f(s, X_s, Y_s) - f(s, X_t, Y_s) \right) \,\mathrm{d}s \\ & \quad + \int_t^{t+\tau} \left( f(s, X_t, Y_s) - f(s, X_t^N, Y_s) \right) \,\mathrm{d}s \\ & \quad + \int_t^{t+\tau} \left( f(s, X_t^N, Y_s) - f(t, X_t^N, Y_t) \right) \,\mathrm{d}s. \end{aligned}$$

Going back to the notation $t = t_{j-1}$ and $t + \tau = t_j$, the above identity reads

$$\begin{aligned} X_{t_j} - X_{t_j}^N =& X_{t_{j-1}} - X_{t_{j-1}}^N \\ & = \int_{t_{j-1}}^{t_j} \left( f(s, X_s, Y_s) - f(s, X_{t_{j-1}}, Y_s) \right) \,\mathrm{d}s \\ & \quad + \int_{t_{j-1}}^{t_j} \left( f(s, X_{t_{j-1}}, Y_s) - f(s, X_{t_{j-1}}^N, Y_s) \right) \,\mathrm{d}s \\ & \quad + \int_{t_{j-1}}^{t_j} \left( f(s, X_{t_{j-1}}^N, Y_s) - f(t_{j-1}, X_{t_{j-1}}^N, Y_{t_{j-1}}) \right) \,\mathrm{d}s. \end{aligned} \tag{3.3}$$

Now we iterate the time steps (3.3) to find that

$$
\begin{aligned}
X_{t_j} - X_{t_j}^N = X_0 - X_0^N & \\
& + \sum_{i=1}^{j} \left( \int_{t_{i-1}}^{t_i} \left( f(s, X_s, Y_s) - f(s, X_{t_i}, Y_s) \right) \,\mathrm{d}s \right. \\
& + \int_{t_{i-1}}^{t_i} \Big( f(s, X_{t_{i-1}}, Y_s) - f(s, X_{t_{i-1}}^N, Y_s) \Big) \,\mathrm{d}s \\
& \left. + \int_{t_{i-1}}^{t_i} \Big( f(s, X_{t_{i-1}}^N, Y_s) - f(t_{i-1}, X_{t_{i-1}}^N, Y_{t_{i-1}}) \Big) \,\mathrm{d}s \right).
\end{aligned}
$$

Using the jump function $\tau^N$, we may rewrite the above expression as in (3.1). □

**Remark 3.1.** Strictly speaking, we only need condition (ii) from Hypothesis 2.1 in order to deduce (4.1), but since we need (i) for the strong convergence anyways, it is simpler to state the result as in Lemma 4.1.

## 4. Basic estimate for the global pathwise error

Here we derive an estimate, under minimal hypotheses, that will be the basis for the estimates in specific cases.

**Lemma 4.1.** *Under Hypothesis 2.1, the global error (3.1) is estimated as*

$$
\begin{aligned}
|X_{t_j} - X_{t_j}^N| \leq \Bigg( & |X_0 - X_0^N| + L_X \int_0^{t_j} |X_s - X_{\tau^N(s)}| \,\mathrm{d}s \\
& \left| \int_0^{t_j} \Big( f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)}) \Big) \,\mathrm{d}s \right| \Bigg) e^{L_X t_j}.
\end{aligned}
\tag{4.1}
$$

*for $j = 1, \ldots, N$, where $\tau^N$ is given by (3.2).*

*Proof.* We estimate the first two integrals in (3.1). For the first one, we use (2.1), so that

$$
|f(s, X_s, Y_s) - f(s, X_t, Y_s)| \leq L_X |X_s - X_t|,
$$

for $t, s \in I$, and, in particular, for $t = \tau^N(s)$. Hence,

$$
\left| \int_0^{t_j} \left( f(s, X_s, Y_s) - f(s, X_{\tau^N(s)}, Y_s) \right) \,\mathrm{d}s \right| \leq L_X \int_0^{t_j} |X_s - X_{\tau^N(s)}| \,\mathrm{d}s.
$$

For the second term, we use again (2.1), so that

$$
|f(s, X_t, Y_s) - f(s, X_t^N, Y_s)| \leq L_X |X_t - X_t^N|,
$$

again for any $t, s \in I$, and, in particular, for $t = \tau^N(s)$. Hence,

$$
\begin{aligned}
\left| \int_0^{t_j} \left( f(s, X_{\tau^N(s)}, Y_s) - f(s, X^N_{\tau^N(s)}, Y_s) \right) \,\mathrm{d}s \right| &\leq L_X \int_0^{t_j} |X_{\tau^N(s)} - X^N_{\tau^N(s)}| \,\mathrm{d}s \\
&\leq L_X \sum_{i=0}^{j-1} |X_{t_i} - X^N_{t_i}| \Delta t_N.
\end{aligned}
$$

With these two estimates, we bound (3.1) as

$$
\begin{aligned}
|X_{t_j} - X^N_{t_j}| \leq |X_0 - X^N_0| & \\
&+ L_X \int_0^{t_j} |X_s - X_{\tau^N(s)}| \,\mathrm{d}s \\
&+ L_X \sum_{i=0}^{j-1} |X_{t_i} - X^N_{t_i}| \Delta t_N \\
&+ \left| \int_0^{t_j} \left( f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)}) \right) \,\mathrm{d}s \right|.
\end{aligned}
$$

Using the discrete version of the Gronwall Lemma, we prove (4.1). □

The first term in the right hand side of (4.1) usually vanishes since in general we take $X^N_0 = X_0$, but it suffices to assume that $X^N_0$ approximates $X_0$ to order $\Delta t_N$, which is useful for lower order approximations or for the discretization of (random) partial differential equations.

The third term in (4.1) is the more delicate one that will be handled differently in the next sections.

As for the second term, which only concerns the solution itself, not the approximation, we use the following simple but useful general result.

**Lemma 4.2.** *Under Hypothesis 2.1, it follows that*

$$
\int_0^{t_j} \left| X_s - X_{\tau^N(s)} \right| \,\mathrm{d}s \leq \Delta t_N \int_0^{t_j} (M_s + L_X |X_s|) \,\mathrm{d}s. \tag{4.2}
$$

*Proof.* By assumption, we have $|f(t, X_t, Y_t)| \leq M_t + L_X |X_t|$, for all $t \in I$ and all sample paths. Thus,

$$
\left| X_s - X_{\tau^N(s)} \right| = \left| \int_{\tau^N(s)}^s f(\xi, X_\xi, Y_\xi) \,\mathrm{d}\xi \right| \leq \int_{\tau^N(s)}^s (M_\xi + L_X |X_\xi|) \,\mathrm{d}\xi.
$$

Integrating over $[0, t_j]$ and using integration by parts,

$$\begin{aligned}
\int_0^{t_j} \left|X_s - X_{\tau^N(s)}\right| \,\mathrm{d}s &\leq \int_0^{t_j} \int_{\tau^N(s)}^{s} (M_\xi + L_X|X_\xi|) \,\mathrm{d}\xi \,\mathrm{d}s \\
&= \int_0^{t_j} \int_{\xi}^{\tau^N(\xi)+\Delta t_N} (M_\xi + L_X|X_\xi|) \,\mathrm{d}s \,\mathrm{d}\xi \\
&= \int_0^{t_j} (\tau^N(\xi) + \Delta t_N - \xi)(M_\xi + L_X|X_\xi|) \,\mathrm{d}\xi.
\end{aligned}$$

Using that $\tau^N(\xi) \leq \xi$ and that the remaining terms are non-negative, we have $\tau^N(\xi) + \Delta t_N - \xi \leq \Delta t_N$ and we obtain exactly (4.2). ☐

Combining the two previous results we obtain

**Proposition 4.1.** *Under Hypothesis 2.1, suppose further that* (2.4) *and* (2.5) *hold and that, for some constant* $C_0 \geq 0$,

$$\mathbb{E}[|X_0 - X_0^N|] \leq C_0 \Delta t_N, \qquad N \in \mathbb{N}. \tag{4.3}$$

*Then, for every* $j = 0, \ldots, N$,

$$\begin{aligned}
&\mathbb{E}\left[|X_{t_j} - X_{t_j}^N|\right] \\
&\quad \leq \left(C_0 \Delta t_N + \Delta t_N L_X \left(\mathbb{E}[|X_0|] + \int_0^{t_j} \mathbb{E}[M_\xi] \,\mathrm{d}\xi\right) e^{L_X t_j}\right. \\
&\qquad\quad \left.\mathbb{E}\left[\left|\int_0^{t_j} \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})\right) \,\mathrm{d}s\right|\right]\right) e^{L_X t_j}.
\end{aligned} \tag{4.4}$$

*Proof.* Under Hypothesis 2.1, Lemma 4.2 applies and estimate (4.2) holds. Using (2.4) and (2.5), that estimate yields

$$\int_0^{t_j} \mathbb{E}[|X_s - X_{\tau^N(s)}]| \,\mathrm{d}s \leq \Delta t_N \int_0^{t_j} (\mathbb{E}[M_s] + L_X \mathbb{E}[|X_s|]) \,\mathrm{d}s.$$

Using now (2.3), we obtain

$$\begin{aligned}
&\int_0^{t_j} \mathbb{E}[|X_s - X_{\tau^N(s)}]| \,\mathrm{d}s \\
&\quad \leq \Delta t_N \int_0^{t_j} \left(\mathbb{E}[M_s] + L_X \left(\mathbb{E}[|X_0|] + \int_0^s \mathbb{E}[M_\xi] \,\mathrm{d}\xi\right) e^{L_X s}\right) \,\mathrm{d}s \\
&\quad \leq \Delta t_N \left(\int_0^{t_j} \mathbb{E}[M_s] \,\mathrm{d}s + L_X \int_0^{t_j} \left(\mathbb{E}[|X_0|] + \int_0^{t_j} \mathbb{E}[M_\xi] \,\mathrm{d}\xi\right) e^{L_X s} \,\mathrm{d}s\right) \\
&\quad = \Delta t_N \left(\int_0^{t_j} \mathbb{E}[M_s] \,\mathrm{d}s + \left(\mathbb{E}[|X_0|] + \int_0^{t_j} \mathbb{E}[M_\xi] \,\mathrm{d}\xi\right) \left(e^{L_X t_j} - 1\right)\right).
\end{aligned}$$

Thus,

$$\int_0^{t_j} \mathbb{E}[|X_s - X_{\tau^N(s)}|] \,\mathrm{d}s \leq \Delta t_N \left(\mathbb{E}[|X_0|] + \int_0^{t_j} \mathbb{E}[M_\xi] \,\mathrm{d}\xi\right) e^{L_X t_j}. \tag{4.5}$$

Now we turn our attention to Lemma 4.1. Taking the expectation of the global error formula (4.1) gives

$$\mathbb{E}\left[|X_{t_j} - X_{t_j}^N|\right] \leq \left(\mathbb{E}\left[|X_0 - X_0^N|\right] + L_X \int_0^{t_j} \mathbb{E}\left[|X_s - X_{\tau^N(s)}|\right] \,\mathrm{d}s \right.$$
$$\left. \mathbb{E}\left[\left|\int_0^{t_j} \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})\right) \,\mathrm{d}s\right|\right]\right) e^{L_X t_j}.$$

Using now estimate (4.5) and condition (4.3), we find (4.4), which completes the proof. □

## 5. The case of monotonic sample path bounds

Here, the noise $\{Y_t\}_{t\in I}$ is *not* assumed to be an Itô noise and $f$ is not assumed to be differentiable, but, instead, that the steps can be controlled by monotonic nondecreasing processes with finite expected growth. This fits well with the typical case of point processes, such as renewal-reward processes, Hawkes process, and such.

More precisely, we have the following result:

**Lemma 5.1.** *Besides Hypothesis 2.1, suppose that, for all $0 \leq s \leq T$,*

$$|f(s, X_{\tau^N(s)}, Y_s) - f(\tau^N(s), X_{\tau^N(s)}, Y_{\tau^N(s)})| \leq \bar{F}_s - \bar{F}_{\tau^N(s)}, \tag{5.1}$$

*where $\{\bar{F}_t\}$ is a real stochastic process satisfying*

$$\mathbb{E}[|\bar{F}_t|] \text{ is monotonic nondecreasing and bounded in } t \in I. \tag{5.2}$$

*Then,*

$$\mathbb{E}\left[\left|\int_0^t \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})\right) \,\mathrm{d}s\right|\right] \leq (\mathbb{E}[\bar{F}_t] - \mathbb{E}[\bar{F}_0])\Delta t_N, \tag{5.3}$$

*for all $0 \leq t \leq T$ and every $N \in \mathbb{R}$.*

*Proof.* Let $N \in \mathbb{R}$. From the assumption (5.1) we have

$$\mathbb{E}\left[|f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})|\right] \leq \mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_{\tau^N(s)}],$$

for every $0 \leq s \leq T$. Thus, upon integration,

$$
\begin{aligned}
\mathbb{E}\left[\left|\int_0^t \left(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})\right) \,\mathrm{d}s\right|\right] \\
&\leq \int_0^t \mathbb{E}\left[|f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})|\right] \,\mathrm{d}s \\
&\leq \int_0^t \left(\mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_{\tau^N(s)}]\right) \,\mathrm{d}s.
\end{aligned}
$$

Now we need to bound the right hand side. When $0 \leq t \leq t_1 = \Delta t_N$, we have $\tau^N(s) = 0$ for all $0 \leq s < t_1$, so that,

$$
\int_0^t (\mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_{\tau^N(s)}]) \,\mathrm{d}s = \int_0^t (\mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_0]) \,\mathrm{d}s.
$$

Using the monotonicity and the condition that $t \leq \Delta t_N$,

$$
\begin{aligned}
\int_0^t (\mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_{\tau^N(s)}]) \,\mathrm{d}s &\leq \int_0^t (\mathbb{E}[\bar{F}_t] - \mathbb{E}[\bar{F}_0]) \,\mathrm{d}s \\
&= (\mathbb{E}[\bar{F}_t] - \mathbb{E}[\bar{F}_0])t \leq (\mathbb{E}[\bar{F}_t] - \mathbb{E}[\bar{F}_0])\Delta t_N.
\end{aligned}
$$

When $\Delta t_N \leq t \leq T$, we split the integration of the second term at time $s = t_1 = \Delta t_N$ and write

$$
\int_0^t (\mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_{\tau^N(s)}]) \,\mathrm{d}s = \int_0^t \mathbb{E}[\bar{F}_s] \,\mathrm{d}s - \int_0^{t_1} \mathbb{E}[\bar{F}_{\tau^N(s)}] \,\mathrm{d}s - \int_{t_1}^t \mathbb{E}[\bar{F}_{\tau^N(s)}] \,\mathrm{d}s
$$

Using the monotonicity together with the fact that $s - \Delta t_N \leq \tau^N(s) \leq s$ for all $\Delta t_N \leq s \leq T$,

$$
\begin{aligned}
\int_0^t (\mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_{\tau^N(s)}]) \,\mathrm{d}s &\leq \int_0^t \mathbb{E}[\bar{F}_s] \,\mathrm{d}s - \int_0^{\Delta t_N} \mathbb{E}[\bar{F}_0] \,\mathrm{d}s - \int_{\Delta t_N}^t \mathbb{E}[\bar{F}_{s-\Delta t_N}] \,\mathrm{d}s \\
&= \int_0^t \mathbb{E}[\bar{F}_s] \,\mathrm{d}s - \int_0^{\Delta t_N} \mathbb{E}[\bar{F}_0] \,\mathrm{d}s - \int_0^{T-\Delta t_N} \mathbb{E}[\bar{F}_s] \,\mathrm{d}s \\
&= \int_{t-\Delta t_N}^t \mathbb{E}[\bar{F}_s] \,\mathrm{d}s - \mathbb{E}[\bar{F}_0]\Delta t_N.
\end{aligned}
$$

Using again the monotonicity yields

$$
\int_0^t (\mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_{\tau^N(s)}]) \,\mathrm{d}s \leq \int_{t-\Delta t_N}^t \mathbb{E}[\bar{F}_t] \,\mathrm{d}s - \mathbb{E}[\bar{F}_0]\Delta t_N = (\mathbb{E}[\bar{F}_t] - \mathbb{E}[\bar{F}_0])\Delta t_N.
$$

Putting the estimates together proves (5.3). $\square$

**Theorem 5.1.** *Under Hypothesis 2.1, suppose also that* (2.4), (2.5), (4.3), (5.1), *and* (5.2) *hold. Then, the Euler-Maruyama scheme* (1.2)-(1.3) *is of strong order 1, i.e.*

$$
\max_{j=0,\ldots,N} \mathbb{E}\left[\left|X_{t_j} - X^N_{t_j}\right|\right] \leq C\Delta t_N, \qquad \forall N \in \mathbb{N}, \tag{5.4}
$$

*for a constant $C$ given by*

$$C = \left( C_0 + L_X \left( \mathbb{E}[|X_0|] + \int_0^T \mathbb{E}[M_\xi] \,\mathrm{d}\xi \right) e^{L_X T} + (\mathbb{E}[\bar{F}_T] - \mathbb{E}[\bar{F}_0]) \right) e^{L_X T} \tag{5.5}$$

*Proof.* Under Hypothesis 2.1, the Lemma 4.1 applies and the global error estimate (4.1) holds.

Thanks to (2.4), (2.5), and (4.3), the Proposition 4.1 applies and the global error is bounded according to (4.4).

With assumptions (5.1) and (5.2), Lemma 5.1 applies and the last term in (4.4) is bounded according to (5.3). Using (5.3) in (4.4) yields

$$\begin{aligned}\mathbb{E}\left[|X_{t_j} - X_{t_j}^N|\right] \leq \Bigg( C_0 \Delta t_N + \Delta t_N L_X \left( \mathbb{E}[|X_0|] + \int_0^{t_j} \mathbb{E}[M_\xi] \,\mathrm{d}\xi \right) e^{L_X t_j} \\ + (\mathbb{E}[\bar{F}_{t_j}] - \mathbb{E}[\bar{F}_0]) \Delta t_N \Bigg) e^{L_X t_j}.\end{aligned}$$

Since this holds for every $j = 0, \ldots, N$, we obtain the desired (5.4). □

The conditions of Theorem 5.1, especially (5.1)-(5.2), are not readily verified, but the following result gives more explicit conditions.

**Theorem 5.2.** *Suppose that $f = f(t,x,y)$ is uniformly globally Lipschitz continuous in $x$ and is continuously differentiable in $(t,y)$, with partial derivatives $\partial_t f$ and $\partial_y f$ with at most linear growth in $x$ and $y$, i.e.*

$$|\partial_t f(t,x,y)| \leq C_1 + C_2|x| + C_3|y|, \quad |\partial_y f(t,x,y)| \leq C_4 + C_5|x| + C_6|y|, \tag{5.6}$$

*in $(t,x,y) \in I \times \mathbb{R} \times \mathbb{R}$, for suitable constants $C_1, C_2, C_3, C_4 \geq 0$. Assume, further, that the sample paths of $\{Y_t\}_{t\in I}$ are of bounded variation $V(\{Y_t\}_{t\in I}; I)$, on $I$, with finite quadratic mean,*

$$\mathbb{E}[V(\{Y_t\}_{t\in I}; I)^2] < \infty, \tag{5.7}$$

*and with*

$$\mathbb{E}[|X_0|^2] < \infty. \tag{5.8}$$

*Then, the Euler-Maruyama scheme is of strong order 1, i.e.*

$$\max_{j=0,\ldots,N} \mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq C \Delta t_N, \qquad \forall N \in \mathbb{N}, \tag{5.9}$$

*for a suitable constant $C \geq 0$.*

*Proof.* Notice that

$$\begin{aligned}|f(t,x,y)| &\leq |f(t,x,y) - f(t,0,y)| + |f(t,0,y) - f(0,0,y)| + |f(0,0,y) - f(0,0,0)| \\ &\leq L_X|x| + C_1 + C_3|y| + C_4 + C_6|y|.\end{aligned}$$

Thus,

$$|f(t,x,Y_t)| \leq M_t + L_X|x|,$$

where
$$M_t = C_1 + C_4 + (C_3 + C_6)|Y_t|.$$
Since the sample paths of $\{Y_t\}_{t\in I}$ are of bounded variation, the process $\{M_t\}_{t\in I}$ has integrable sample paths. This means that we are under the Hypothesis 2.1. Moreover, thanks to (5.7), we see that
$$\mathbb{E}[|Y_t|] \le \mathbb{E}[|Y_t|^2] \le \mathbb{E}[V(\{Y_t\}_{t\in I}; I)^2] < \infty.$$
Then, thanks to the Lyapunov inequality $\mathbb{E}[|Y_t|] \le \mathbb{E}[|Y_t|^2]^{1/2}$, we see that $\{M_t\}_{t\in I}$ satisfies (2.5). By assumption, (2.4) also holds, so that, from (2.3), we have
$$K_X = \sup_{t\in I} \mathbb{E}[|X_t|^2] < \infty.$$

Now, in order to apply Theorem 5.1, it remains to verify (5.1)-(5.2). We have
$$\begin{aligned}|f(s, X_\tau, Y_s) - f(\tau, X_\tau, Y_\tau)| &= \left| \int_\tau^s \partial_t f(\xi, X_\tau, Y_\xi)\,\mathrm{d}\xi + \int_\tau^s \partial_y f(\xi, X_\tau, Y_\xi)\,\mathrm{d}Y_\xi \right| \\ &\le C_1(s-\tau) + C_2(s-\tau)|X_\tau| + (C_3 + C_4|X_\tau|)V(\{Y_t\}_{t\in I}; \tau, s).\end{aligned}$$
Thus, (5.1) holds with
$$\bar{F}_t = (C_1 + C_2|X_{\tau^N(t)}|)t + (C_3 + C_4|X_{\tau^N(t)}|)V(\{Y_t\}_{t\in I}; 0, t).$$
It is clear that, not only the expectation, but all the sample paths of $\{F_t\}_{t\in I}$ are monotonic non-decreasing in $t \in I$, with $\bar{F}_0 = 0$. Moreover, thanks to (5.7), and using the Cauchy-Schwarz inequality in the last term, we have
$$\mathbb{E}[\bar{F}_T] \le (C_1 + C_2 K_1)T + (C_3 + C_4 K_1)\mathbb{E}[V(\{Y_t\}_{t\in I}; 0, T)^2] < \infty.$$
Thus, Theorem 5.1 and we deduce the strong order 1 convergence of the Euler-Maruyama method. □

**Remark 5.1.** The conditions (5.7) and (5.8) on the finite mean square of the total variation of the noise and of the initial condition can be relaxed provided we have a better control on the growth of the $\partial_y f(t, x, y)$ with respect to $x$. More precisely, if
$$|\partial_y f(t, x, y)| \le C_4 + C_5|x|^{p-1} + C_6|y|,$$
and
$$\mathbb{E}[V(\{Y_t\}_{t\in I}; T, 0)^p] < \infty,$$
along with
$$\mathbb{E}[|X_0|^p] < \infty,$$
with $1 \le p < \infty$, then the process $\{\bar{F}_t\}_{t\in I}$ becomes
$$\bar{F}_t = (C_1 + C_2|X_{\tau^N(t)}|)t + (C_3 + C_4|X_{\tau^N(t)}|^{p-1})V(\{Y_t\}_{t\in I}; 0, t).$$
Applying the Hölder inequality yields
$$\bar{F}_t \le (C_1 + C_2|X_{\tau^N(t)}|)t + C_3 V(\{Y_t\}_{t\in I}; 0, t) + C_4\frac{p-1}{p}|X_{\tau^N(t)}|^p + \frac{C_4}{p}V(\{Y_t\}_{t\in I}; 0, t)^p.$$

With that, the required conditions on $\{\bar{F}_t\}_{t\in I}$ are met and we are allowed to apply Theorem 5.1 and deduce the strong order 1 convergence of the Euler-Maruyama method.

**Remark 5.2.** One particular example that fits the conditions of Theorem 5.2 is when $f = f(t,x,y)$ is semi-separable, i.e.

$$f(t,x,y) = a(t,y)h(x) + b(t,y), \tag{5.10}$$

where $a = a(t,y)$ and $b = b(t,y)$ are continuously differentiable on $I \times \mathbb{R}$ with uniformly bounded first derivatives, $a = a(t,y)$ itself is uniformly bounded, and $h = h(x)$ is globally Lipschitz continous on $\mathbb{R}$.

Since $a = a(t,x)$ is uniformly bounded and $h = h(x)$ is globally Lipschitz continuous, it follows that $f = f(t,x,y)$ is uniformly globally Lipschitz continuous in $x$. Moreover, it is continuously differentiable in $(t,y)$, with partial derivatives $\partial_t f$ and $\partial_y f$ given by

$$\partial_t f = \partial_t a(t,y)h(x) + \partial_t b(t,y), \qquad \partial_y f = \partial_y a(t,y)h(x) + \partial_t b(t,y)$$

Since the partial derivatives of $a = a(t,y)$ and $b = b(t,y)$ are uniformly bounded and $h$ is Lipstchiz, it follows that the partial derivatives $\partial_t f$ and $\partial_y f$ have at most linear growth. Thus, (5.6) is satisfies.

## 6. The case of an Itô noise

In Lemma 5.1, we assumed a control of the steps of $s \mapsto f(s, X_{\tau^N(s),Y_s})$ by the steps of a process $\{\bar{F}_s^N\}_s$ with non-decreasing expectation. A particular case, presented in Theorem 5.2, is when the steps are controlled by the steps of a process with bounded variation. In this case, we find that

$$|f(s, X_{\tau^N(s),Y_s}) - f(\tau, X_{\tau^N(\tau),Y_\tau})| \leq \int_\tau^s \mathrm{d}\bar{F}_\xi^N.$$

An alternative condition considered in this section is to assume that $s \mapsto f(s, X_{\tau^N(s),Y_s})$ itself is an Itô process and, as such,

$$f(s, X_{\tau^N(s),Y_s}) - f(\tau, X_{\tau^N(\tau),Y_\tau}) = \int_\tau^s A_\xi \,\mathrm{d}\xi + \int_\tau^s B_\xi \,\mathrm{d}W_\xi,$$

for suitable $\{A_t\}_{t\in I}$ and $\{B_t\}_{t\in I}$.

In practice, we may assume $f = f(t,x,y)$ is twice continuously differentiable in $(t,y)$ and the noise $\{Y_t\}_{t\in I}$ itself is an Itô noise, so that, by the Itô formula, the steps above are themselves Itô noises.

With that in mind, we first have the following result.

**Lemma 6.1.** *Besides Hypothesis 2.1, suppose that $F_t^N = f(t, X_{\tau^N(t)}, Y_t)$ is an Itô noise, satisfying*

$$\mathrm{d}F_t^N = A_t \,\mathrm{d}t + B_t \,\mathrm{d}W_t, \tag{6.1}$$

*for a Wiener process* $\{W_t\}_{t\in 0}$ *and stochastic processes* $\{A_t\}_{t\in I}$, $\{B_t\}_{t\in I}$ *adapted to* $\{W_t\}_{t\geq 0}$ *and such that*

$$\int_0^T \mathbb{E}[|A_t|]\;\mathrm{d}t < \infty, \quad \int_0^T \mathbb{E}[|B_t|^2]\;\mathrm{d}t < \infty. \tag{6.2}$$

*Then,*

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^t \left(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X_{\tau^N(s)}, Y_{\tau^N(s)})\right)\;\mathrm{d}s\right|\right] \\ \leq \Delta t_N \left(\int_0^t \mathbb{E}[|A_\xi|]\;\mathrm{d}\xi + \left(\int_0^t \mathbb{E}[|B_\xi|^2]\;\mathrm{d}\xi\right)^{1/2}\right),\end{aligned} \tag{6.3}$$

*for all* $0 \leq t \leq T$ *and every* $N \in \mathbb{R}$.

*Proof.* We write

$$f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)}) = \int_{\tau^N(s)}^s A_\xi\;\mathrm{d}\xi + \int_{\tau^N(s)}^s B_\xi\;\mathrm{d}W_\xi.$$

Upon integration,

$$\begin{aligned}\int_0^t \left(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X_{\tau^N(s)}, Y_{\tau^N(s)})\right)\;\mathrm{d}s \\ = \int_0^t \left(\int_{\tau^N(s)}^s A_\xi\;\mathrm{d}\xi + \int_{\tau^N(s)}^s B_\xi\;\mathrm{d}W_\xi\right)\;\mathrm{d}s\end{aligned}$$

Using integration by parts,

$$\begin{aligned}&\int_0^t \left(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X_{\tau^N(s)}, Y_{\tau^N(s)})\right)\;\mathrm{d}s \\ &\quad = \int_0^t \int_\xi^{\tau^N(\xi)+\Delta t_N} A_\xi\;\mathrm{d}s\;\mathrm{d}\xi + \int_0^t \int_\xi^{\tau^N(\xi)+\Delta t_N} B_\xi\;\mathrm{d}s\;\mathrm{d}W_\xi \\ &\quad = \int_0^t (\tau^N(\xi) + \Delta t_N - \xi) A_\xi\;\mathrm{d}\xi + \int_0^t (\tau^N(\xi) + \Delta t_N - \xi) B_\xi\;\mathrm{d}W_\xi.\end{aligned}$$

Taking the absolute mean and using the Itô isometry on the second term yields

$$\begin{aligned}&\mathbb{E}\left[\left|\int_0^t \left(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X_{\tau^N(s)}, Y_{\tau^N(s)})\right)\;\mathrm{d}s\right|\right] \\ &\quad \leq \int_0^t |\tau^N(\xi) + \Delta t_N - \xi| \mathbb{E}[|A_\xi|]\;\mathrm{d}\xi + \left(\int_0^t (\tau^N(\xi) + \Delta t_N - \xi)^2 \mathbb{E}[|B_\xi|^2]\;\mathrm{d}\xi\right)^{1/2}.\end{aligned}$$

Since $|\tau^N(\xi) + \Delta t_N - \xi| \leq \Delta t_N$, we find

$$\mathbb{E}\left[\left|\int_0^t \left(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X_{\tau^N(s)}, Y_{\tau^N(s)})\right)\,\mathrm{d}s\right|\right] \\ \leq \Delta t_N \left(\int_0^t \mathbb{E}[|A_\xi|]\,\mathrm{d}\xi + \left(\int_0^t \mathbb{E}[|B_\xi|^2]\,\mathrm{d}\xi\right)^{1/2}\right),$$

which completes the proof. □

**Theorem 6.1.** *Under Hypothesis 2.1, suppose also that (2.4), (2.5), (4.3), (6.1), and (6.2) hold. Then, the Euler-Maruyama scheme (1.2)-(1.3) is of strong order 1, i.e.*

$$\max_{j=0,\ldots,N} \mathbb{E}\left[\left|X_{t_j} - X^N_{t_j}\right|\right] \leq C\Delta t_N, \qquad \forall N \in \mathbb{N}, \tag{6.4}$$

*for a constant $C$ given by*

$$C = \left(C_0 + L_X\left(\mathbb{E}[|X_0|] + \int_0^T \mathbb{E}[M_\xi]\,\mathrm{d}\xi\right)e^{L_X T} \\ + \left(\int_0^T \mathbb{E}[|A_\xi|]\,\mathrm{d}\xi + \left(\int_0^T \mathbb{E}[|B_\xi|^2]\,\mathrm{d}\xi\right)^{1/2}\right)\right)e^{L_X T} \tag{6.5}$$

*Proof.* Under Hypothesis 2.1, the Lemma 4.1 applies and the global error estimate (4.1) holds.

Thanks to (2.4), (2.5), and (4.3), the Proposition 4.1 applies and the global error is bounded according to (4.4).

With assumptions (6.1) and (6.2), Lemma 6.1 applies and the last term in (4.4) is bounded according to (6.3). Using (6.3) in (4.4) yields

$$\mathbb{E}\left[|X_{t_j} - X^N_{t_j}|\right] \leq \left(C_0\Delta t_N + \Delta t_N L_X\left(\mathbb{E}[|X_0|] + \int_0^{t_j} \mathbb{E}[M_\xi]\,\mathrm{d}\xi\right)e^{L_X t_j} \\ + \Delta t_N\left(\int_0^{t_j} \mathbb{E}[|A_\xi|]\,\mathrm{d}\xi + \left(\int_0^{t_j} \mathbb{E}[|B_\xi|^2]\,\mathrm{d}\xi\right)^{1/2}\right)\right)e^{L_X t_j}.$$

Since this holds for every $j = 0, \ldots, N$, we obtain the desired (6.4). □

**Corollary 6.1.** *Let $f = f(t, x, y)$ be twice continuously differentiable with uniformly bounded derivatives. Suppose that the noise $\{Y_t\}_{t\in I}$ is an Itô noise,*

$$\mathrm{d}Y_t = a(t, Y_t)\,\mathrm{d}t + b(t, Y_t)\,\mathrm{d}W_t, \tag{6.6}$$

*with $a = a(t, y)$ and $b = b(t, y)$ continuous and satisfying*

$$|a(t, y)| \leq A_M + A_Y|y|, \qquad |b(t, y)| \leq B_M + B_Y|y|. \tag{6.7}$$

*Assume the bounds (2.4), (4.3), and*

$$\mathbb{E}[|Y_0|] < \infty \tag{6.8}$$

*Then, the Euler-Maruyama scheme is of strong order 1, i.e.*

$$\max_{j=0,\ldots,N} \mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq C\Delta t_N, \qquad \forall N \in \mathbb{N}, \tag{6.9}$$

*for a suitable constant $C \geq 0$.*

*Proof.* Let us start by showing that Hypothesis 2.1 is valid. Since $f = f(t,x,y)$ is (twice) continuously differentiable with, in particular, bounded derivative in $x$, then it is uniformly globally Lipscthiz in $x$. Since $a = a(t,y)$ and $b = b(t,y)$ are continuous, the the noise has continuous sample paths. Thus, the remaining condition in Hypothesis 2.1 to be verified is (iic).

From (6.6) and (6.7), we have

$$Y_t = \int_0^t a(s, Y_s)\,\mathrm{d}s + \int_0^s b(t, Y_s)\,\mathrm{d}W_s.$$

Using the Itô formula, we have

$$\mathrm{d}Y_t^2 = \left(2a(t,Y_t)Y_t + b(t,Y_t)^2\right)\,\mathrm{d}t + 2b(t,Y_t)Y_t\,\mathrm{d}W_t.$$

Thus

$$Y_t^2 = Y_0^2 + \int_0^t \left(2a(s,Y_s)Y_s + b(t,Y_s)^2\right)\,\mathrm{d}s + \int_0^t 2b(s,Y_s)Y_s\,\mathrm{d}W_s.$$

Taking the expectation,

$$\mathbb{E}[|Y_t|^2] = \mathbb{E}[|Y_0|^2] + \int_0^t \left(2a(s,Y_s)Y_s + b(t,Y_s)^2\right)\,\mathrm{d}s.$$

Using (6.7), this yields

$$\begin{aligned}
\mathbb{E}[|Y_t|^2] &\leq \mathbb{E}[|Y_0|^2] + \int_0^t \left(2\mathbb{E}[(A_M + A_Y|Y_s|)|Y_s|] + \mathbb{E}[(B_M + B_Y|Y_s|)^2)\right)\,\mathrm{d}s \\
&\leq \mathbb{E}[|Y_0|^2] + \int_0^t \left(4(A_M^2 + (1 + A_Y)\mathbb{E}[|Y_s|^2]) + 2(B_M^2 + B_Y^2\mathbb{E}[|Y_s|^2])\right)\,\mathrm{d}s
\end{aligned}$$

By the Gronwall Lemma,

$$\mathbb{E}[|Y_t|^2] \leq \left(\mathbb{E}[|Y_0|^2] + (4A_M^2 + 2B_M^2)t\right)e^{(4(1+A_Y)+2B_Y^2)t}.$$

Thus,

$$\sup_{t\in I}\mathbb{E}[|Y_t|^2] \leq \left(\mathbb{E}[|Y_0|^2] + (4A_M^2 + 2B_M^2)T\right)e^{(4(1+A_Y)+2B_Y^2)T}. \tag{6.10}$$

Since $f = f(t,x,y)$ is Lipschitz in $x$ and twice continuously differentiable in $(t,y)$ with uniformly bounded first order derivatives, we have the bound

$$|f(t,x,y)| \leq |f(0,0,0)| + L_X|x| + L_T|t| + L_Y|y|.$$

Thus,

$$|f(t,x,Y_t)| \leq M_t + L_X|x|$$

with
$$M_t = |f(0,0,0)| + L_T|t| + L_Y|y|.$$
Thanks to (6.10), we see that
$$\int_0^T M_t \,\mathrm{d}t < \infty.$$
Therefore, we are under the condition of (2.1).

Now, in view of Theorem 6.1, it remains to prove that $F_t^N = f(t, X_{\tau^N(t)}, Y_t)$ is an Itô noise (6.1), with the bounds (6.2). The fact that it is an Itô noise follows from the Itô formula and the fact the smoothness of $f = f(t,x,y)$. Indeed, since $(t,y) \mapsto f(t,x,y)$ is twice continuously differentiable, for each fixed $x$, the Itô formula is applicable and yields

$$\mathrm{d}f(t,x,Y_t) = \left(\partial_t f(t,x,Y_t) + a(t,Y_t)\partial_y f(t,x,Y_t) + \frac{b(t,Y_t)^2}{2}\partial_{yy} f(t,x,Y_t)\right)\,\mathrm{d}t \\ + b(t,Y_t)\partial_y f(t,x,Y_t)\,\mathrm{d}W_t, \quad (6.11)$$

for every fixed $x \in \mathbb{R}$. This means (6.1) holds with
$$A_t = \partial_t f(t,x,Y_t) + a(t,Y_t)\partial_y f(t,x,Y_t) + \frac{b(t,Y_t)^2}{2}\partial_{yy} f(t,x,Y_t)$$
and
$$B_t = b(t,Y_t)\partial_y f(t,x,Y_t).$$
It remains to show that $\{A_t\}_{t\in I}$ is mean integrable and that $\{B_t\}_{t\in I}$ is square mean integrable. Since $f = f(t,x,y)$ has uniformly bounded derivatives, we have
$$|A_t| \leq L_T + L_Y(A_M + A_Y|Y_t|) + 2L_{YY}(B_M^2 + B_Y^2|Y_t|^2),$$
and
$$|B_t| \leq L_Y(B_M + B_Y|Y_t|),$$
for a suitable constants $L_{YY} \geq 0$. Now, thanks to (6.10), we see that (6.2) is satisfied.

Therefore, all the conditions of Theorem 6.1 are met and we deduce the strong order 1 convergence of the Euler-Maruyama method. □

## 7. Applications

In this section, we describe a few explicit examples that fall into one of the cases considered above and, hence, the Euler-Maruyama method exhibits a strong order one convergence.

### 7.1. **Earthquake and other impulse driven models.** 
See Neckel and Rupp pg 582 as a starting point of a model driven by a transport process as the source of ground motion excitation (both Kanai-Tajima and Bogdanoff, check out also the Clough-Penzien).

7.2. **RODE with Itô noise.** Here, we consider a class of RODEs treated in [1, Chapter 3], namely the (1.1) with a noise $\{Y_t\}_{t\in I}$ satisfying

$$\mathrm{d}Y_t = a(Y_t)\,\mathrm{d}t + b(Y_t)\,\mathrm{d}W_t \tag{7.1}$$

with the assumptions that

$$\text{the functions } a = a(y),\ b = b(y), \text{ and } f = f(t,x,y) \text{ are twice continuously differentiable with all partial derivatives uniformly bounded.} \tag{7.2}$$

In the case that the diffusion term $b = b(y)$ is actually independent of $y$, then the noise is an additive noise and the Euler-Maruyama scheme is of strong order 1, otherwise, it is thought to be of order 1/2 (CITATION...). Here, however, we deduce that, even if $b = b(y)$ depends on $y$, the strong convergence of the Euler-Maruyama scheme is of order 1, under the assumptions (7.2).

7.3. **Population dynamics.**

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = \mu X_t(r - X_t)cos(Y_t), \tag{7.3}$$

where $\mu, r > 0$ are constant and $\{Y_t\}_{t\geq 0}$ is an Itô noise process satisfying (6.6), which includes a Wiener process, an Orstein-Uhlenbeck process or a geometric Brownian motion process.

Define

$$f(t,x,y) = x(r-x)cos(y).$$

We have

$$f(t,x,y)x = x^2(r-x)cos(y)$$

and notice that $f(t,x,y)x \geq 0$, for $x \geq 0$ and $y \geq 0$, and $f(t,x,y)x \leq 0$, for $x \geq r$ and $y \geq 0$. Hence the interval $[0,R]$ in $x$ is positively invariant and the pathwise solutions of (7.4) are almost surely bounded as well, with

$$0 \leq X_t \leq R,$$

for all $t \geq 0$.

7.4. **Population dynamics.** Our first example is a population dynamics model modified from [4, Section 15.2],

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = Z_t X_t(r - X_t) \tag{7.4}$$

where $r > 0$ is constant and $\{Z_t\}_{t\geq}$ is a stochastic process playing the role of a random growth parameter and given by

$$Z_t = \lambda(1 + \varepsilon\sin(Y_t)),$$

where $0 < \varepsilon < 1$ and $\{Y_t\}_{t\geq 0}$ is an Itô noise process satisfying (6.6). We assume $Y_t$ has an analytic formula so we do not need to approximate the coupled stochastic differential equation system for $(X_t, Y_t)$. This setting includes a Wiener process, an Orstein-Uhlenbeck process and a geometric Brownian motion process.

We suppose the initial condition is non-negative and bounded almost surely:

$$0 \leq X_0 \leq R,$$

for some $R > r$.

The noise process $\{Z_t\}_{t\geq 0}$ itself satisfies

$$0 < \lambda - \varepsilon \leq Z_t \leq \lambda + \varepsilon < 2\lambda, \qquad \forall t \geq 0.$$

Define

$$f(t,x,z) = zx(r-x)$$

and notice that $f(t,x,z)x \geq 0$, for $x \geq 0$ and $z \geq 0$, and $f(t,x,z)x \leq 0$, for $x \geq r$ and $z \geq 0$. Hence the interval $[0,R]$ in $x$ is positively invariant and the pathwise solutions of (7.4) are almost surely bounded as well, with

$$0 \leq X_t \leq R,$$

for all $t \geq 0$.

The function $f = f(t,x,z)$ is continuously differentiable infinitely many times and with

$$\left|\frac{\partial f}{\partial x}(t,x,y)\right| = |y(r-2x)| \leq 2\lambda(2R-r),$$

for $|x| \leq R$ and $0 \leq z \leq 2\lambda$. In turn, the function $z = z(y) = \lambda(1 + \varepsilon\sin(y))$ is also continuously differentiable infinitely many times and is uniformly bounded along with all its derivatives.

The right hand side of (7.4) is not globally Lipschitz, but, for the sake of analysis, since $X_t$ and $Y_t$ are bounded, the right hand side can be modified to a twice continuously differentiable, uniformly globally Lipschitz function $\tilde{f}(t,x,y)$ that coincides with $f(t,x,y)$ for $(t,x,y) \in \mathbb{R} \times [0,R] \times [0,2\lambda]$ and satisfies (2.1) with

$$L_X = 2\lambda(2R-r).$$

Thus, the RODE (7.4) with $0 \leq X_0 \leq R$ almost surely, for some $R > r$, is equivalent to the RODE

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = \tilde{f}(t, X_t, Y_t). \tag{7.5}$$

With $\tilde{f} = \tilde{f}(t,x,y)$, the Hypothesis 2.2 hold. Moreover, it follows from (2.3) (notice $M_t = 0$ here) that

$$|X_t| \leq |X_0| e^{2\lambda(2R-r)t} \leq Re^{2\lambda(2R-r)T}, \qquad 0 \leq t \leq T.$$

almost surely.

OLD Condition on $\{F_t\}$ SHOULD BE FIXED is satisfied with

$$F_t ?!?!?!$$

CAREFULL, WE CHANGED THE NOTATION, MUST FIX FROM HERE ON AND CHECK EVERYTHING.

The Itô formula applied to $Y_t = g(O_t)$, where $g(\eta) = \lambda(1+\varepsilon\sin(\eta))$ implies $\{Y_t\}_{t\geq 0}$ is an Itô process with

$$\mathrm{d}Y_t = \left((\theta_1 - \theta_2 O_t)g'(O_t) + \frac{\theta_3^2}{2}g''(O_t)\right)\,\mathrm{d}t + \theta_3 g'(O_t)\,\mathrm{d}W_t.$$

We have

$$g'(\eta) = \lambda\varepsilon\cos(\eta), \quad g''(\eta) = -\lambda\varepsilon\sin(\eta)$$

hence both are uniformly bounded. Therefore, all the conditions of Corollary 6.1 hold and the Euler-Maruyama method is of strong order 1.

### 7.5. **Drug delivery.**

### 7.6. **Earthquake model.**

### 7.7. **Point-process noise.**

## Appendix A. Discrete Grownwall Lemma

We end this section by abstracting away the Gronwall type inequality we use (this is probably written somewhere, and I need to find the source):

**Lemma A.1.** *Let $(e_j)_j$ be a (finite or infinite) sequence of positive numbers satisfying*

$$e_j \leq a\sum_{i=0}^{j-1} e_i + b, \tag{A.1}$$

*with $e_0 = 0$, where $a, b > 0$. Then,*

$$e_j \leq be^{aj}, \qquad \forall j. \tag{A.2}$$

*Proof.* The result is trivially true for $j = 0$. Suppose, by induction, that the result is true up to $j-1$. Then,

$$e_j \leq a\sum_{i=0}^{j-1} be^{ai} + b = b\left(a\sum_{i=0}^{j-1} e^{ai} + 1\right).$$

Using that $1 + a \leq e^a$, we have $a \leq e^a - 1$, hence

$$e_j \leq b\left((e^a - 1)\sum_{i=0}^{j-1} e^{ia} + 1\right).$$

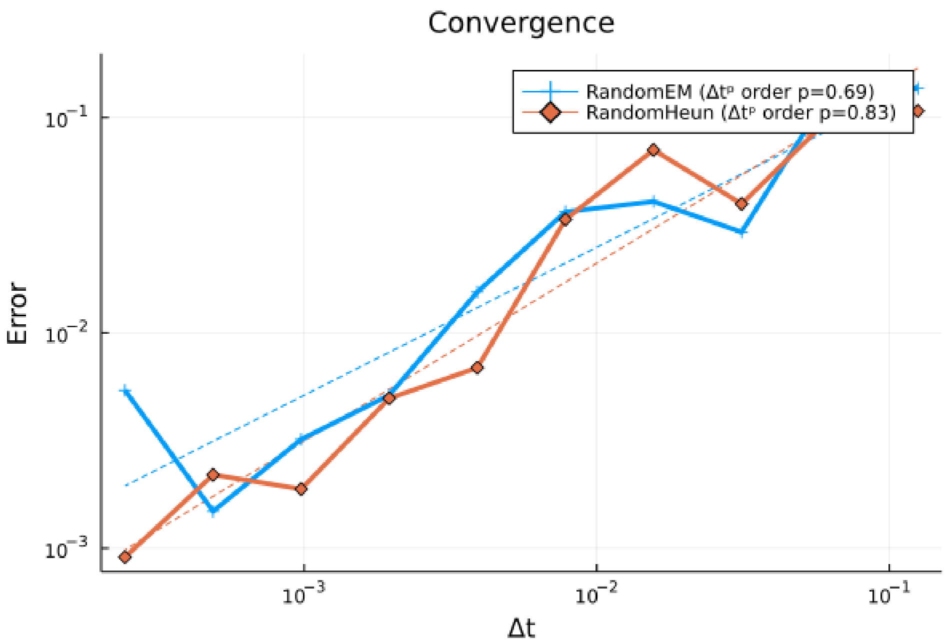
Using that $\sum_{i=0}^{j-1}\theta^i = (\theta^j - 1)(\theta - 1)$, with $\theta = e^a$, we see that

$$(e^a - 1)\sum_{i=0}^{j-1} e^{ia} \leq e^{ja} - 1,$$

so that

$$e_j \leq be^{ja},$$

which completes the induction. □




## Appendix B. Numerical examples

B.1. **Lower-order converge.** For a lower order convergence, below order 1, we take the noise $\{Y_t\}_t$ to be the transport process defined by

$$Y_t = \sin(t/Z)^{1/3},$$

where $Z$ is a beta random variable $Z \sim B(\alpha, \beta)$. Notice $Z$ takes values strictly within $(0, 1)$ and, hence, $\sin(t/Z)$ can have arbitrarily high frequencies and, hence, go through the critic value $y = 0$ extremely often.

(Need to remove the Heun method and do more tests).

## Acknowledgments

## References


[1] Y. Asai, *Numerical Methods for Random Ordinary Differential Equations and their Applications in Biology and Medicine.* Dissertation, Institut fÃ¼r Mathematik, Goethe UniversitÃ¤t Frankfurt am Main, 2016.

[2] E. A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations,* New York: McGraw-Hill, 1987.

[3] L. Grüne, P.E. Kloeden, Higher order numerical schemes for affinely controlled nonlinear systems, *Numer. Math.* 89 (2001), 669–690.

[4] X. Han & P. E. Kloeden (2017), *Random Ordinary Differential Equations and Their Numerical Solution,* Probability Theory and Stochastic Modelling, vol. 85, Springer Singapore. DOI: 10.1007/978-981-10-6265-0.

(Ricardo M. S. Rosa) Instituto de Matemática, Universidade Federal do Rio de Janeiro, Brazil

(Peter E. Kloeden) Mathematics Department, University of Tubingen, Germany

*Email address*, R. M. S. Rosa: `rrosa@im.ufrj.br`

*Email address*, P. E. Kloeden: `kloeden@math.uni-frankfurt.de`