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IMPROVED ERROR ESTIMATE FOR THE ORDER OF STRONG CONVERGENCE OF THE EULER METHOD FOR RANDOM ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. It is well known that the Euler method for approximating the solutions of a random ordinary differential equation $\mathrm{d}X_t/\mathrm{d}t = f(t, X_t, Y_t)$ driven by a stochastic process $\{Y_t\}_t$ with θ-Hölder sample paths is estimated to be of strong order θ with respect to the time step, provided $f = f(t, x, y)$ is sufficiently regular and with suitable bounds. Here, we show that it is possible to exploit further conditions on the noise and prove that, in many typical cases, the strong convergence is actually of order 1, regardless of the Hölder regularity of the sample paths. This applies to additive or multiplicative Itô noises (such as Wiener, Ornstein-Uhlenbeck, and Geometric Brownian motion process); to point-process noises (such as Poisson point processes and Hawkes self-exciting processes, which are not even continuous and have jump-type discontinuities); and to transport-type processes with sample paths of bounded variation. The order 1 convergence is based on a novel approach, resting on three main ideas: First, we do not estimate directly the local error and, instead, add up the local steps and work directly with an accumulated global error, leading to formula with an iterated integral, the outer one spanning the whole interval and the inner one spanning a time step. Secondly, we use Fubini theorem to switch the order of the iterated integral, moving the critical regularity to the large scale time, easing the regularity requirement on the small scale of the time step. Finally, we assume either a control of the total variation of the sample paths of the noise (as in many point processes and transport process) or that the noise is an Itô process (such as Wiener, Ornstein-Uhlenbeck, and Geometric Brownian motion) in order to bound the large scale term via Itô isometry. We complement the work with examples with fractional Brownian motion noise with Hurst parameter $0 < H < 1/2$ for which the order of convergence is $H + 1/2$, hence lower than the attained order 1 in the previous examples, but still higher than the order H of convergence expected from previous works.

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1. Introduction

Consider the following initial value problem for a **random ordinary differential equation (RODE)**:

$$
\begin{cases} \dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t), & 0 \leq t \leq T, \\ X_t\big|_{t=0} = X_0, \end{cases} \tag{1.1}
$$

on a time interval $I = [0, T]$, with $T > 0$, and where the noise $\{Y_t\}_{t\in I}$ is a given stochastic process. The sample space is denoted by Ω.

The Euler method for solving this initial value problem consists in approximating the solution on a uniform time mesh $t_j = j\Delta t_N$, $j = 0, \ldots, N$, with fixed time step $\Delta t_N = T/N$, for a given $N \in \mathbb{N}$. In such a mesh, the Euler scheme takes the form

$$
X_{t_j}^N = X_{t_{j-1}}^N + \Delta t_N f(t_{j-1}, X_{t_{j-1}}^N, Y_{t_{j-1}}), \qquad j = 1, \ldots, N, \tag{1.2}
$$

with the initial condition

$$
X_0^N = X_0. \tag{1.3}
$$

Notice $t_j = j\Delta t_N = jT/N$ also depends on N, but we do not make this dependency explicit, for the sake of notational simplicity.

When the noise $\{Y_t\}_{t\in I}$ has θ-Hölder continuous sample paths, it can be show [12], under suitable regularity conditions on f, that the Euler scheme converges strongly with order θ with respect to the time step, i.e. there exists a constant $C \geq 0$ such that

$$
\max_{j=0,\ldots,N} \mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq C\Delta t_N^\theta, \qquad \forall N \in \mathbb{N}, \tag{1.4}
$$

where $\mathbb{E}[\cdot]$ indicates the expectation of a random variable on Ω.

Our aim is to show that, in many classical examples, it is possible to exploit further conditions that yield in fact a higher strong order convergence, with the sample paths still being Hölder continuous or even discontinuous. This is the case, for instance, when the noise is a point process, a transport process, or an Itô process, for which the convergence is of strong order 1. It is also the case for fractional Brownian motion noise with Hurst parameter H, for which the sample paths are H-Hölder continuous, but the strong convergence is of order 1 only when $1/2 \leq H < 1$, dropping to order $H + 1/2$, when $0 < H < 1/2$, which is still higher the Hölder exponent H of the sample paths.

The first main idea of the proof is to not estimate the local error and, instead, work with an explicit formula for the global error, namely (see Lemma 3.1)

$$
\begin{aligned}
X_{t_j} - X_{t_j}^N = X_0 - X_0^N \\
&+ \int_0^{t_j} \Big(f(s, X_s, Y_s) - f(s, X_{\tau^N(s)}^N, Y_s) \Big) \,\mathrm{d}s \\
&+ \int_0^{t_j} \Big(f(s, X_{\tau^N(s)}^N, Y_s) - f(s, X_{\tau^N(s)}^N, Y_s) \Big) \,\mathrm{d}s \\
&+ \int_0^{t_j} \Big(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)}) \Big) \,\mathrm{d}s,
\end{aligned} \tag{1.5}
$$

for $j = 1, \ldots, N$, where τ^N is a piecewise constant function with jumps at the mesh points t_j (see Equation (3.2)).

The first term vanishes due to the initial condition $X_0^N = X_0$. The second term only depends on the solution and can be easily estimated with natural regularity conditions on the term $f = f(t, x, y)$. The third term is handled solely with the typical required condition on $f = f(t, x, y)$ of being uniformly globally Lipschitz continuity with respect to x. With those, we obtain the following basic bound for the global error (see Lemma 4.2)

$$
\begin{aligned}
|X_{t_j} - X_{t_j}^N| \leq \Bigg(|X_0 - X_0^N| + L_X \int_0^{t_j} |X_s - X_{\tau^N(s)}| \,\mathrm{d}s \\
\left| \int_0^{t_j} \Big(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)}) \Big) \,\mathrm{d}s \right| \Bigg) e^{L_X t_j}.
\end{aligned} \tag{1.6}
$$

The only problematic, noise-sensitive term is the last one. The classical analysis is to use an assumed θ-Hölder regularity of the noise sample paths and estimate the local error as

$$
\mathbb{E}\left[\left| f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)}) \right| \right] \leq C \Delta t_N^\theta.
$$

Instead, we look at the whole noise error

$$
\mathbb{E}\left[\left| \int_0^{t_j} \Big(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)}) \Big) \,\mathrm{d}s \right| \right]
$$

and assume that the steps of the process given by $F_t = f(t, X_{\tau^N(t)}^N, Y_t)$ can be controlled in a suitable global way. In order to give the main idea, let us assume for the moment that the sample paths of $\{F_t\}_{t \in I}$ satisfy

$$
F_s - F_\tau = \int_\tau^s \mathrm{d}F_\xi,
$$

either in the sense of a Riemann-Stieltjes integral or of an Itô integral. The first sense fits the case of noises with bounded total variation, while the second one fits the case of an Itô noise. In any case, we bound the global error term using the Fubini Theorem,

$$\begin{aligned}\int_0^{t_j} \left(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})\right)\,\mathrm{d}s &= \int_0^{t_j}\int_{\tau^N(s)}^{s} \mathrm{d}F_\xi\,\mathrm{d}s \\ &= \int_0^{t_j}\int_{\xi}^{\tau^N(\xi)+\Delta t_N} \mathrm{d}s\,\mathrm{d}F_\xi \\ &= \int_0^{t_j} (\tau^N(\xi)+\Delta t_N - \xi)\,\mathrm{d}F_\xi.\end{aligned}$$

Then, we find that

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^{t_j} \left(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})\right)\,\mathrm{d}s\right|\right] \\ \leq \mathbb{E}\left[\left|\int_0^{t_j} (\tau^N(\xi)+\Delta t_N - \xi)\,\mathrm{d}F_\xi\right|\right].\end{aligned}$$

In the case of an Itô noise, we assume

$$\mathrm{d}F_t = A_t\,\mathrm{d}t + B_t\,\mathrm{d}W_t,$$

with adapted processes $\{A_t\}_t$, $\{B_t\}_t$, which may actually depend on $\{Y_t\}_t$, so that multiplicative noise is allowed. Then we bound the right hand side using the Lyapunov inequality and the Itô isometry:

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^{t_j} (\tau^N(\xi)+\Delta t_N - \xi)\,\mathrm{d}F_\xi\right|\right] &\leq \left(\int_0^{t_j} (\tau^N(\xi)+\Delta t_N - \xi)^2\mathbb{E}[A_\xi^2]\,\mathrm{d}\xi\right)^{1/2} \\ &\quad + \int_0^{t_j} (\tau^N(\xi)+\Delta t_N - \xi)\mathbb{E}[B_\xi]\,\mathrm{d}\xi \\ &\leq \Delta t_N\left(\left(\int_0^{t_j}\mathbb{E}[A_\xi^2]\,\mathrm{d}\xi\right)^{1/2} + \int_0^{t_j}\mathbb{E}[B_\xi]\,\mathrm{d}\xi\right).\end{aligned}$$

which yields the strong order 1 convergence, provided the integrals are finite.

In the case of noises with bounded variation, we may actually relax the above condition and assume the steps are bounded by a process $\{\bar{F}_t\}_{t\in I}$ with monotonic non-decreasing sample paths,

$$|f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})| \leq \bar{F}_s - \bar{F}_{\tau^N(s)}.$$

Using the monotonicity, this yields

$$\mathbb{E}\left[\left|\int_0^{t_j} (\tau^N(\xi)+\Delta t_N - \xi)\,\mathrm{d}\bar{F}_\xi\right|\right] \leq \Delta t_N\left(\mathbb{E}[\bar{F}_{t_j}] - \mathbb{E}[\bar{F}_0]\right),$$

yielding, again, strong order 1 convergence.

These two cases are treated in Section 5 (for the bounded variation case; see Lemma 5.1 and Theorem 5.1) and Section 6 (for the Itô noise case; see Lemma 6.1 and Theorem 6.1).

The conditions in Theorem 5.1 and Theorem 6.1 are not readily verifiable, but Theorem 5.2 and Theorem 6.2 give more explicit conditions for each of the two cases. Essentially, $f = f(t, x, y)$ is required to have minimal regularity in the sense of differentiability and growth conditions and the noise $\{Y_t\}_{t\in I}$ is either required to have sample paths of bounded variation or to be an Itô noise.

We complement this work with a few explicit examples and their numerical implementation, illustrating the strong order 1 convergence in the cases above. We also include an example with a fractional Brownian motion noise (fBm), for which the order of convergence drops to $H + 1/2$, when the Hurst parameter is in the range $0 < H < 1/2$. We do not have a general proof of this order of convergence in the case of fBm noise, but, in the example considered, we essentially have (see (8.13) and (8.15))

$$F_s - F_\tau \sim \int_\tau^s (s-\tau)^{H-1/2}\,\mathrm{d}W_\xi + \text{higher order term}.$$

In this case, disregarding the higher order term,

$$\begin{aligned}
\int_0^{t_j} &\Big(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)}) \Big)\,\mathrm{d}s \\
&\sim \int_0^{t_j}\int_{\tau^N(s)}^s (s-\tau^N(s))^{H-1/2}\,\mathrm{d}W_\xi\,\mathrm{d}s \\
&= \int_0^{t_j}\int_\xi^{\tau^N(\xi)+\Delta t_N} (s-\tau^N(s))^{H-1/2}\,\mathrm{d}s\,\mathrm{d}W_\xi \\
&\sim \int_0^{t_j} (\tau^N(\xi)+\Delta t_N - \tau^N(\xi))^{H+1/2}\,\mathrm{d}W_\xi \\
&= (\Delta t_N)^{H+1/2}\int_0^{t_j}\,\mathrm{d}W_\xi.
\end{aligned}$$

which, upon taking the expectation of the absolute value, yields a strong convergence of order $H + 1/2$.

2. Pathwise solutions

For the notion and main results on pathwise solution for RODEs, we refer the reader to [13, Section 2.1] (see also [18, Section 3.3]).

We start with a fundamental set of conditions that imply the existence and uniqueness of pathwise solutions of the RODE (1.1) in the sense of Carathéodory:

Hypothesis 2.1. *We consider a function $f = f(t, x, y)$ defined on $I \times \mathbb{R} \times \mathbb{R}$ and a real-valued stochastic process $\{Y_t\}_{t\in I}$, where $I = [0, T]$, $T > 0$. We make the following standing hypotheses.*

(i) *f is globally Lipschitz continuous on x, uniformly in t and y, i.e. there exists a constant $L_X \geq 0$ such that*
$$|f(t, x_1, y) - f(t, x_2, y)| \leq L_X|x_1 - x_2|, \quad \forall t \in I, \ \forall x_1, x_2, y \in \mathbb{R}. \tag{2.1}$$
(ii) *The mapping $(t, x) \mapsto f(t, x, Y_t)$ satisfies the Carathéodory conditions:*
  (a) *The mapping $x \mapsto f(t, x, Y_t(\omega))$ is continuous on $x \in \mathbb{R}$, for almost every $(t, \omega) \in I \times \Omega$;*
  (b) *The mapping $t \mapsto f(t, x, Y_t(\omega))$ is Lebesgue measurable in $t \in I$, for each $x \in \mathbb{R}$ and each sample path $t \mapsto Y_t(\omega)$;*
  (c) *The bound $|f(t, x, Y_t)| \leq M_t + L_X|x|$ holds for all $t \in I$ and all $x \in \mathbb{R}$, where $\{M_t\}_{t\in I}$ is a real stochastic process with Lebesgue integrable sample paths $t \mapsto M_t(\omega)$ on $t \in I$.*

Under these assumptions, for each sample value in Ω, the integral equation
$$X_t = X_0 + \int_0^t f(s, X_s, Y_s) \, \mathrm{d}s \tag{2.2}$$
has a unique solution, in the Lebesgue sense, for the realizations $X_0 = X_0(\omega)$, of the initial condition, and $t \mapsto Y_t(\omega)$, of the noise process (see [7, Theorem 1.1]). Moreover, the mapping $(t, \omega) \mapsto X_t(\omega)$ is measurable (see [13, Section 2.1.2]) and, hence, give rise to a well-defined stochastic process $\{X_t\}_{t\in I}$.

Each sample path solution $t \mapsto X_t(\omega)$ is bounded by
$$|X_t| \leq \left(|X_0| + \int_0^t M_s \, \mathrm{d}s\right) e^{L_X t}, \quad \forall t \in I. \tag{2.3}$$

For the strong convergence of the Euler approximation, we also need to control the expectation of the solution above, among other things. With that in mind, we have the following useful result.

Lemma 2.1. *Under Hypothesis 2.1, suppose further that*
$$\mathbb{E}[|X_0|] < \infty \tag{2.4}$$
and
$$\int_0^T \mathbb{E}[|M_s|] \, \mathrm{d}s < \infty \tag{2.5}$$
Then,
$$\mathbb{E}[|X_t|] \leq \left(\mathbb{E}[|X_0|] + \int_0^t \mathbb{E}[|M_s|] \, \mathrm{d}s\right) e^{L_X t}, \quad t \in I. \tag{2.6}$$

Proof. Thanks to (2.3), the result is straightfoward □

Remark 2.1. When $f = f(t, x, y)$ is continuous on all three variables, as well as uniformly globally Lipschitz continuous in x, and the sample paths of $\{Y_t\}_{t\geq 0}$ are continuous, then the integrand in (2.2) is continuous in t and the integral becomes a Riemann integral. In this case, the integral form (2.2) of the pathwise solutions of (1.1) holds in the Riemann sense.

Remark 2.2. In special *dissipative* cases, depending on the structure of the equation, we might not need the second condition (2.5) and only require $\mathbb{E}[|X_0|] < \infty$. More generally, when some bounded, positively invariant region exists and is of interest, we may truncate the nonlinear term to achieve the desired global conditions for the equation with the truncated term, but which coincide with the original equation in the region of interest. But we leave these cases to be handled in the applications.

3. Integral formula for the global pathwise error

In this section, we derive the following integral formula for the global error:

Lemma 3.1. *Under Hypothesis 2.1, the Euler approximation (1.2) for any pathwise solution of the random ordinary differential equation (1.1) satisfies the global error formula*

$$\begin{aligned} X_{t_j} - X_{t_j}^N = X_0 - X_0^N & \\ &+ \int_0^{t_j} \left(f(s, X_s, Y_s) - f(s, X_{\tau^N(s)}^N, Y_s) \right) \,\mathrm{d}s \\ &+ \int_0^{t_j} \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(s, X_{\tau^N(s)}^N, Y_s) \right) \,\mathrm{d}s \\ &+ \int_0^{t_j} \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)}) \right) \,\mathrm{d}s, \end{aligned} \tag{3.1}$$

for $j = 1, \ldots, N$, where τ^N is the piecewise constant jump function along the time mesh:

$$\tau^N(t) = \max_j \{ j\Delta t_N;\ j\Delta t_N \leq t \} = \left[\frac{t}{\Delta t_N} \right] \Delta t_N = \left[\frac{tN}{T} \right] \frac{T}{N}. \tag{3.2}$$

Proof. Under Hypothesis 2.1, the solutions of (1.1) are pathwise solutions in the Lebesgue sense of (2.2). With that in mind, we first obtain an expression for a single time step, from time t_{j-1} to $t_j = t_{j-1} + \Delta t_N$.

For notational simplicity, we momentarily write $t = t_{j-1}$ and $\tau = \Delta t_N$, so that $t_j = t + \tau$. The exact pathwise solution satisfies

$$X_{t+\tau} = X_t + \int_t^{t+\tau} f(s, X_s, Y_s) \,\mathrm{d}s.$$

The Euler step is given by

$$X_{t+\tau}^N = X_t^N + \tau f(t, X_t^N, Y_t).$$

Subtracting, we obtain

$$X_{t+\tau} - X_{t+\tau}^N = X_t - X_t^N + \int_t^{t+\tau} \left(f(s, X_s, Y_s) - f(t, X_t^N, Y_t) \right) \,\mathrm{d}s.$$

We arrange the integrand as

$$\begin{aligned} f(s, X_s, Y_s) - f(t, X_t^N, Y_t) =& f(s, X_s, Y_s) - f(s, X_t, Y_s) \\ & + f(s, X_t, Y_s) - f(s, X_t^N, Y_s) \\ & + f(s, X_t^N, Y_s) - f(t, X_t^N, Y_t). \end{aligned}$$

This yields

$$\begin{aligned} X_{t+\tau} - X_{t+\tau}^N =& X_t - X_t^N \\ & = \int_t^{t+\tau} \left(f(s, X_s, Y_s) - f(s, X_t, Y_s)\right) \,\mathrm{d}s \\ & \quad + \int_t^{t+\tau} \Big(f(s, X_t, Y_s) - f(s, X_t^N, Y_s)\Big) \,\mathrm{d}s \\ & \quad + \int_t^{t+\tau} \Big(f(s, X_t^N, Y_s) - f(t, X_t^N, Y_t)\Big) \,\mathrm{d}s. \end{aligned}$$

Going back to the notation $t = t_{j-1}$ and $t + \tau = t_j$, the above identity reads

$$\begin{aligned} X_{t_j} - X_{t_j}^N =& X_{t_{j-1}} - X_{t_{j-1}}^N \\ & = \int_{t_{j-1}}^{t_j} \Big(f(s, X_s, Y_s) - f(s, X_{t_{j-1}}, Y_s)\Big) \,\mathrm{d}s \\ & \quad + \int_{t_{j-1}}^{t_j} \Big(f(s, X_{t_{j-1}}, Y_s) - f(s, X_{t_{j-1}}^N, Y_s)\Big) \,\mathrm{d}s \\ & \quad + \int_{t_{j-1}}^{t_j} \Big(f(s, X_{t_{j-1}}^N, Y_s) - f(t_{j-1}, X_{t_{j-1}}^N, Y_{t_{j-1}})\Big) \,\mathrm{d}s. \end{aligned} \tag{3.3}$$

Now we iterate the time steps (3.3) to find that

$$\begin{aligned} X_{t_j} - X_{t_j}^N = & X_0 - X_0^N \\ & + \sum_{i=1}^{j} \left(\int_{t_{i-1}}^{t_i} \left(f(s, X_s, Y_s) - f(s, X_{t_i}, Y_s)\right) \,\mathrm{d}s \right. \\ & + \int_{t_{i-1}}^{t_i} \Big(f(s, X_{t_{i-1}}, Y_s) - f(s, X_{t_{i-1}}^N, Y_s)\Big) \,\mathrm{d}s \\ & \left. + \int_{t_{i-1}}^{t_i} \Big(f(s, X_{t_{i-1}}^N, Y_s) - f(t_{i-1}, X_{t_{i-1}}^N, Y_{t_{i-1}})\Big) \,\mathrm{d}s \right). \end{aligned}$$

Using the jump function τ^N defined by (3.2), we may rewrite the above expression as in (3.1). □

Remark 3.1. Strictly speaking, we only need condition (ii) from Hypothesis 2.1 in order to deduce (4.3), but since we need (i) for the strong convergence anyways, it is simpler to state the result as in Lemma 4.2.

4. Basic estimate for the global pathwise error

Here we derive an estimate, under minimal hypotheses, that is the basis for the estimates in specific cases. For that, we use a discrete version of the Grownwall lemma. Here we state a particular case of a result that can be found in [9] (see also [6]).

Lemma 4.1 (Discrete Gronwall Lemma)**.** *Let $(e_j)_j$ be a (finite or infinite) sequence of positive numbers satisfying*

$$e_j \leq a \sum_{i=0}^{j-1} e_i + b, \tag{4.1}$$

for every j, with $e_0 = 0$, and where $a, b > 0$. Then,

$$e_j \leq b e^{aj}, \qquad \forall j. \tag{4.2}$$

Proof. This follows from [9, Lemma V.2.4] by taking $n = j$, $a_n = e_j$, $b_n = 0$, $c_n = b$, and $\lambda = a$. For the sake of completeness, we give a direct proof for this particular case.

The result is trivially true for $j = 0$. Suppose, by induction, that the result is true up to $j - 1$. Then,

$$e_j \leq a \sum_{i=0}^{j-1} b e^{ai} + b = b \left(a \sum_{i=0}^{j-1} e^{ai} + 1 \right).$$

Using that $1 + a \leq e^a$, we have $a \leq e^a - 1$, hence

$$e_j \leq b \left((e^a - 1) \sum_{i=0}^{j-1} e^{ia} + 1 \right).$$

Using that $\sum_{i=0}^{j-1} \theta^i = (\theta^j - 1)(\theta - 1)$, with $\theta = e^a$, we see that

$$(e^a - 1) \sum_{i=0}^{j-1} e^{ia} \leq e^{ja} - 1,$$

so that

$$e_j \leq b e^{ja},$$

which completes the induction. □

We are now ready to start proving our basic estimate for the global pathwise error.

Lemma 4.2. *Under Hypothesis 2.1, the global error (3.1) is estimated as*

$$\begin{aligned} |X_{t_j} - X_{t_j}^N| \leq \Bigg(& |X_0 - X_0^N| + L_X \int_0^{t_j} |X_s - X_{\tau^N(s)}| \,\mathrm{d}s \\ & \left| \int_0^{t_j} \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)}) \right) \,\mathrm{d}s \right| \Bigg) e^{L_X t_j}. \end{aligned} \tag{4.3}$$

for $j = 1, \ldots, N$, *where* τ^N *is given by* (3.2).

Proof. We estimate the first two integrals in (3.1). For the first one, we use (2.1), so that

$$|f(s, X_s, Y_s) - f(s, X_t, Y_s)| \leq L_X |X_s - X_t|,$$

for $t, s \in I$, and, in particular, for $t = \tau^N(s)$. Hence,

$$\left| \int_0^{t_j} \left(f(s, X_s, Y_s) - f(s, X^N_{\tau^N(s)}, Y_s) \right) \, \mathrm{d}s \right| \leq L_X \int_0^{t_j} |X_s - X_{\tau^N(s)}| \, \mathrm{d}s.$$

For the second term, we use again (2.1), so that

$$|f(s, X_t, Y_s) - f(s, X^N_t, Y_s)| \leq L_X |X_t - X^N_t|,$$

for any $t, s \in I$, and, in particular, for $t = \tau^N(s)$. Hence,

$$\begin{aligned} \left| \int_0^{t_j} \left(f(s, X^N_{\tau^N(s)}, Y_s) - f(s, X^N_{\tau^N(s)}, Y_s) \right) \, \mathrm{d}s \right| &\leq L_X \int_0^{t_j} |X^N_{\tau^N(s)} - X^N_{\tau^N(s)}| \, \mathrm{d}s \\ &\leq L_X \sum_{i=0}^{j-1} |X_{t_i} - X^N_{t_i}| \Delta t_N. \end{aligned}$$

With these two estimates, we bound (3.1) as

$$\begin{aligned} |X_{t_j} - X^N_{t_j}| \leq |X_0 - X^N_0| & \\ &+ L_X \int_0^{t_j} |X_s - X_{\tau^N(s)}| \, \mathrm{d}s \\ &+ L_X \sum_{i=0}^{j-1} |X_{t_i} - X^N_{t_i}| \Delta t_N \\ &+ \left| \int_0^{t_j} \left(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)}) \right) \, \mathrm{d}s \right|. \end{aligned}$$

Using the discrete version of the Gronwall Lemma, we prove (4.3). □

The first term in the right hand side of (4.3) usually vanishes since in general we take $X^N_0 = X_0$, but it suffices to assume that X^N_0 approximates X_0 to order Δt_N, which is useful for lower order approximations or for the discretization of (random) partial differential equations.

The third term in (4.3) is the more delicate one that will be handled differently in the next sections.

As for the second term, which only concerns the solution itself, not the approximation, we use the following simple but useful general result.

Lemma 4.3. *Under Hypothesis 2.1, it follows that*

$$\int_0^{t_j} \left| X_s - X_{\tau^N(s)} \right| \, \mathrm{d}s \leq \Delta t_N \int_0^{t_j} (M_s + L_X |X_s|) \, \mathrm{d}s. \tag{4.4}$$

Proof. By assumption, we have $|f(t, X_t, Y_t)| \leq M_t + L_X|X_t|$, for all $t \in I$ and all sample paths. Thus,

$$\left|X_s - X_{\tau^N(s)}\right| = \left|\int_{\tau^N(s)}^s f(\xi, X_\xi, Y_\xi)\,\mathrm{d}\xi\right| \leq \int_{\tau^N(s)}^s (M_\xi + L_X|X_\xi|)\,\mathrm{d}\xi.$$

Integrating over $[0, t_j]$ and using Fubini's theorem to exchange the order of integration,

$$\begin{aligned}
\int_0^{t_j} \left|X_s - X_{\tau^N(s)}\right|\,\mathrm{d}s &\leq \int_0^{t_j}\int_{\tau^N(s)}^s (M_\xi + L_X|X_\xi|)\,\mathrm{d}\xi\,\mathrm{d}s \\
&= \int_0^{t_j}\int_\xi^{\tau^N(\xi)+\Delta t_N} (M_\xi + L_X|X_\xi|)\,\mathrm{d}s\,\mathrm{d}\xi \\
&= \int_0^{t_j} (\tau^N(\xi) + \Delta t_N - \xi)(M_\xi + L_X|X_\xi|)\,\mathrm{d}\xi.
\end{aligned}$$

Using that $\tau^N(\xi) \leq \xi$ and that the remaining terms are non-negative, we have $\tau^N(\xi) + \Delta t_N - \xi \leq \Delta t_N$ and we obtain exactly (4.4). □

Combining the two previous results we obtain

Proposition 4.1. *Under Hypothesis 2.1, suppose further that (2.4) and (2.5) hold and that, for some constant $C_0 \geq 0$,*

$$\mathbb{E}[|X_0 - X_0^N|] \leq C_0\Delta t_N, \qquad N \in \mathbb{N}. \tag{4.5}$$

Then, for every $j = 0, \ldots, N$,

$$\begin{aligned}
&\mathbb{E}\left[|X_{t_j} - X_{t_j}^N|\right] \\
&\quad\leq \left(C_0\Delta t_N + \Delta t_N L_X\left(\mathbb{E}[|X_0|] + \int_0^{t_j}\mathbb{E}[M_\xi]\,\mathrm{d}\xi\right)e^{L_X t_j}\right. \\
&\qquad\left.\mathbb{E}\left[\left|\int_0^{t_j}\left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})\right)\,\mathrm{d}s\right|\right]\right)e^{L_X t_j}.
\end{aligned} \tag{4.6}$$

Proof. Estimate (4.6) is obtained by taking the expectation of (4.3) in Lemma 4.2 and properly estimating the first two terms on the right hand side. The first term is handled with the assumption (4.5). We just need to take care of the second term.

Under Hypothesis 2.1, Lemma 4.3 applies and estimate (4.4) holds. Using (2.4) and (2.5), that estimate yields

$$\int_0^{t_j}\mathbb{E}[|X_s - X_{\tau^N(s)}|]\,\mathrm{d}s \leq \Delta t_N\int_0^{t_j}(\mathbb{E}[M_s] + L_X\mathbb{E}[|X_s|])\,\mathrm{d}s.$$

Using now (2.3), we obtain

$$\begin{aligned}
&\int_0^{t_j} \mathbb{E}[|X_s - X_{\tau^N(s)}|]\,\mathrm{d}s \\
&\qquad \leq \Delta t_N \int_0^{t_j} \left(\mathbb{E}[M_s] + L_X \left(\mathbb{E}[|X_0|] + \int_0^s \mathbb{E}[M_\xi]\,\mathrm{d}\xi \right) e^{L_X s} \right)\,\mathrm{d}s \\
&\qquad \leq \Delta t_N \left(\int_0^{t_j} \mathbb{E}[M_s]\,\mathrm{d}s + L_X \int_0^{t_j} \left(\mathbb{E}[|X_0|] + \int_0^{t_j} \mathbb{E}[M_\xi]\,\mathrm{d}\xi \right) e^{L_X s}\,\mathrm{d}s \right) \\
&\qquad = \Delta t_N \left(\int_0^{t_j} \mathbb{E}[M_s]\,\mathrm{d}s + \left(\mathbb{E}[|X_0|] + \int_0^{t_j} \mathbb{E}[M_\xi]\,\mathrm{d}\xi \right) \left(e^{L_X t_j} - 1 \right) \right).
\end{aligned}$$

Thus,

$$\int_0^{t_j} \mathbb{E}[|X_s - X_{\tau^N(s)}|]\,\mathrm{d}s \leq \Delta t_N \left(\mathbb{E}[|X_0|] + \int_0^{t_j} \mathbb{E}[M_\xi]\,\mathrm{d}\xi \right) e^{L_X t_j}. \tag{4.7}$$

Now we look at Lemma 4.2. Taking the expectation of the global error formula (4.3) gives

$$\begin{aligned}
\mathbb{E}\left[|X_{t_j} - X_{t_j}^N| \right] \leq \Bigg(&\mathbb{E}\left[|X_0 - X_0^N| \right] + L_X \int_0^{t_j} \mathbb{E}\left[|X_s - X_{\tau^N(s)}| \right]\,\mathrm{d}s \\
&\mathbb{E}\left[\left| \int_0^{t_j} \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)}) \right)\,\mathrm{d}s \right| \right] \Bigg) e^{L_X t_j}.
\end{aligned}$$

Using now estimate (4.7) and condition (4.5), we find (4.6), which completes the proof. $\square$

5. The case of noise with sample paths of bounded variation

Here, the noise $\{Y_t\}_{t\in I}$ is *not* assumed to be an Itô noise and f is not assumed to be differentiable, but, instead, that the steps can be controlled by monotonic nondecreasing processes with finite expected growth. This fits well with the typical case of point processes, such as renewal-reward processes, Hawkes process, and the like.

More precisely, we have the following result:

Lemma 5.1. *Besides Hypothesis 2.1, suppose that, for all* $0 \leq s \leq T$,

$$|f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})| \leq \bar{F}_s - \bar{F}_{\tau^N(s)}, \tag{5.1}$$

where $\{\bar{F}_t\}$ *is a real stochastic process with monotonic nondecreasing sample paths and with*

$$\mathbb{E}[\bar{F}_t] \text{ uniformly bounded on } t \in I. \tag{5.2}$$

Then,

$$\mathbb{E}\left[\left|\int_0^t \left(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})\right)\,\mathrm{d}s\right|\right] \\ \leq (\mathbb{E}[\bar{F}_t] - \mathbb{E}[\bar{F}_0])\Delta t_N, \quad (5.3)$$

for all $0 \leq t \leq T$ *and every* $N \in \mathbb{R}$.

Proof. Let $N \in \mathbb{R}$. From the assumption (5.1) we have

$$\mathbb{E}\left[|f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})|\right] \leq \mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_{\tau^N(s)}],$$

for every $0 \leq s \leq T$. Thus, upon integration,

$$\begin{aligned}\mathbb{E}&\left[\left|\int_0^t \left(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})\right)\,\mathrm{d}s\right|\right] \\ &\leq \int_0^t \mathbb{E}\left[|f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})\right]\,\mathrm{d}s \\ &\leq \int_0^t \left(\mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_{\tau^N(s)}]\right)\,\mathrm{d}s.\end{aligned}$$

Now we need to bound the right hand side. When $0 \leq t \leq t_1 = \Delta t_N$, we have $\tau^N(s) = 0$ for all $0 \leq s < t_1$, so that,

$$\int_0^t (\mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_{\tau^N(s)}])\,\mathrm{d}s = \int_0^t (\mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_0])\,\mathrm{d}s.$$

Using the monotonicity of $\{\bar{F}_t\}$ and the condition that $t \leq \Delta t_N$,

$$\begin{aligned}\int_0^t (\mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_{\tau^N(s)}])\,\mathrm{d}s &\leq \int_0^t (\mathbb{E}[\bar{F}_t] - \mathbb{E}[\bar{F}_0])\,\mathrm{d}s \\ &= (\mathbb{E}[\bar{F}_t] - \mathbb{E}[\bar{F}_0])t \leq (\mathbb{E}[\bar{F}_t] - \mathbb{E}[\bar{F}_0])\Delta t_N.\end{aligned}$$

When $\Delta t_N \leq t \leq T$, we split the integration of the second term at time $s = t_1 = \Delta t_N$ and write

$$\int_0^t (\mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_{\tau^N(s)}])\,\mathrm{d}s = \int_0^t \mathbb{E}[\bar{F}_s]\,\mathrm{d}s - \int_0^{t_1} \mathbb{E}[\bar{F}_{\tau^N(s)}]\,\mathrm{d}s - \int_{t_1}^t \mathbb{E}[\bar{F}_{\tau^N(s)}]\,\mathrm{d}s$$

Using the monotonicity together with the fact that $s - \Delta t_N \leq \tau^N(s) \leq s$ for all $\Delta t_N \leq s \leq T$,

$$\begin{aligned}\int_0^t (\mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_{\tau^N(s)}]) \;\mathrm{d}s &\leq \int_0^t \mathbb{E}[\bar{F}_s] \;\mathrm{d}s - \int_0^{\Delta t_N} \mathbb{E}[\bar{F}_0] \;\mathrm{d}s - \int_{\Delta t_N}^t \mathbb{E}[\bar{F}_{s-\Delta t_N}] \;\mathrm{d}s \\ &= \int_0^t \mathbb{E}[\bar{F}_s] \;\mathrm{d}s - \int_0^{\Delta t_N} \mathbb{E}[\bar{F}_0] \;\mathrm{d}s - \int_0^{T-\Delta t_N} \mathbb{E}[\bar{F}_s] \;\mathrm{d}s \\ &= \int_{t-\Delta t_N}^t \mathbb{E}[\bar{F}_s] \;\mathrm{d}s - \mathbb{E}[\bar{F}_0]\Delta t_N.\end{aligned}$$

Using again the monotonicity yields

$$\int_0^t (\mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_{\tau^N(s)}]) \;\mathrm{d}s \leq \int_{t-\Delta t_N}^t \mathbb{E}[\bar{F}_t] \;\mathrm{d}s - \mathbb{E}[\bar{F}_0]\Delta t_N = (\mathbb{E}[\bar{F}_t] - \mathbb{E}[\bar{F}_0])\Delta t_N.$$

Putting the estimates together and using the boundedness (5.2) prove (5.3). □

Theorem 5.1. *Under Hypothesis 2.1, suppose also that* (2.4), (2.5), (4.5), (5.1), *and* (5.2) *hold. Then, the Euler scheme* (1.2)-(1.3) *is of strong order 1, i.e.*

$$\max_{j=0,\ldots,N} \mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq C\Delta t_N, \qquad \forall N \in \mathbb{N}, \tag{5.4}$$

for a constant C given by

$$C = \left(C_0 + L_X\left(\mathbb{E}[|X_0|] + \int_0^T \mathbb{E}[M_\xi] \;\mathrm{d}\xi\right)e^{L_X T} + (\mathbb{E}[\bar{F}_T] - \mathbb{E}[\bar{F}_0])\right)e^{L_X T} \tag{5.5}$$

Proof. Under Hypothesis 2.1, the Lemma 4.2 applies and the global error estimate (4.3) holds.

Thanks to (2.4), (2.5), and (4.5), the Proposition 4.1 applies and the global error is bounded according to (4.6).

With assumptions (5.1) and (5.2), Lemma 5.1 applies and the last term in (4.6) is bounded according to (5.3). Using (5.3) in (4.6) yields

$$\begin{aligned}\mathbb{E}\left[|X_{t_j} - X_{t_j}^N|\right] \leq \bigg(C_0\Delta t_N + \Delta t_N L_X\left(\mathbb{E}[|X_0|] + \int_0^{t_j} \mathbb{E}[M_\xi] \;\mathrm{d}\xi\right)e^{L_X t_j} \\ + (\mathbb{E}[\bar{F}_{t_j}] - \mathbb{E}[\bar{F}_0])\Delta t_N\bigg)e^{L_X t_j}.\end{aligned}$$

Since this holds for every $j = 0, \ldots, N$, we obtain the desired (5.4). □

The conditions of Theorem 5.1, especially (5.1)-(5.2), are not readily verifiable, but the following result gives more explicit conditions.

Theorem 5.2. *Suppose that $f = f(t, x, y)$ is uniformly globally Lipschitz continuous in x and is continuously differentiable in (t, y), with partial derivatives $\partial_t f$ and $\partial_y f$ with at most linear growth in x and y, i.e.*

$$|\partial_t f(t, x, y)| \leq C_1 + C_2|x| + C_3|y|, \quad |\partial_y f(t, x, y)| \leq C_4 + C_5|x| + C_6|y|, \tag{5.6}$$

in $(t,x,y) \in I \times \mathbb{R} \times \mathbb{R}$, *for suitable constants* $C_1, C_2, C_3, C_4 \geq 0$. *Assume, further, that the sample paths of* $\{Y_t\}_{t\in I}$ *are of bounded variation* $V(\{Y_t\}_{t\in I}; I)$, *on* I, *with finite quadratic mean,*

$$\mathbb{E}[V(\{Y_t\}_{t\in I}; I)^2] < \infty, \tag{5.7}$$

and with

$$\mathbb{E}[|X_0|^2] < \infty. \tag{5.8}$$

Then, the Euler scheme is of strong order 1, i.e.

$$\max_{j=0,\ldots,N} \mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq C\Delta t_N, \qquad \forall N \in \mathbb{N}, \tag{5.9}$$

for a suitable constant $C \geq 0$.

Proof. Notice that

$$\begin{aligned} |f(t,x,y)| \leq |f(t,x,y) - f(t,0,y)| + |f(t,0,y) - f(0,0,y)| + |f(0,0,y) - f(0,0,0)| \\ \leq L_X|x| + C_1 + C_3|y| + C_4 + C_6|y|. \end{aligned}$$

Thus,

$$|f(t,x,Y_t)| \leq M_t + L_X|x|,$$

where

$$M_t = C_1 + C_4 + (C_3 + C_6)|Y_t|.$$

Since the sample paths of $\{Y_t\}_{t\in I}$ are of bounded variation, the process $\{M_t\}_{t\in I}$ has integrable sample paths. This means that we are under the Hypothesis 2.1. Moreover, thanks to (5.7), we see that

$$\mathbb{E}[|Y_t|] \leq \mathbb{E}[|Y_t|^2] \leq \mathbb{E}[V(\{Y_t\}_{t\in I}; I)^2] < \infty.$$

Then, thanks to the Lyapunov inequality $\mathbb{E}[|Y_t|] \leq \mathbb{E}[|Y_t|^2]^{1/2}$, we see that $\{M_t\}_{t\in I}$ satisfies (2.5). By assumption, (2.4) also holds, so that, from (2.3), we have

$$K_X = \sup_{t\in I} \mathbb{E}[|X_t|^2] < \infty.$$

Now, in order to apply Theorem 5.1, it remains to verify (5.1)-(5.2). We have

$$\begin{aligned} |f(s,X_\tau,Y_s) - f(\tau,X_\tau,Y_\tau)| = \left|\int_\tau^s \partial_t f(\xi,X_\tau,Y_\xi)\,\mathrm{d}\xi + \int_\tau^s \partial_y f(\xi,X_\tau,Y_\xi)\,\mathrm{d}Y_\xi\right| \\ \leq C_1(s-\tau) + C_2(s-\tau)|X_\tau| + (C_3 + C_4|X_\tau|)V(\{Y_t\}_{t\in I}; \tau, s). \end{aligned}$$

Thus, (5.1) holds with

$$\bar{F}_t = (C_1 + C_2|X^N_{\tau^N(t)}|)t + (C_3 + C_4|X^N_{\tau^N(t)}|)V(\{Y_t\}_{t\in I}; 0, t).$$

It is clear that all the sample paths of $\{F_t\}_{t\in I}$ are monotonic non-decreasing in $t \in I$, with $\bar{F}_0 = 0$. Moreover, thanks to (5.7), and using the Cauchy-Schwarz inequality in the last term, we have

$$\mathbb{E}[\bar{F}_T] \leq (C_1 + C_2K_1)T + (C_3 + C_4K_1)\mathbb{E}[V(\{Y_t\}_{t\in I}; 0, T)^2] < \infty.$$

Thus, Theorem 5.1 applies and we deduce the strong order 1 convergence of the Euler method. □

Remark 5.1. The conditions (5.7) and (8.21) on the finite mean square of the total variation of the noise and of the initial condition can be relaxed provided we have a better control on the growth of the $\partial_y f(t,x,y)$ with respect to x. More precisely, if

$$|\partial_y f(t,x,y)| \leq C_4 + C_5|x|^{p-1} + C_6|y|,$$

and

$$\mathbb{E}[V(\{Y_t\}_{t\in I}; T, 0)^p] < \infty,$$

along with

$$\mathbb{E}[|X_0|^p] < \infty,$$

with $1 \leq p < \infty$, then the process $\{\bar{F}_t\}_{t\in I}$ becomes

$$\bar{F}_t = (C_1 + C_2|X^N_{\tau^N(t)}|)t + (C_3 + C_4|X^N_{\tau^N(t)}|^{p-1})V(\{Y_t\}_{t\in I}; 0, t).$$

Applying the Hölder inequality yields

$$\bar{F}_t \leq (C_1 + C_2|X^N_{\tau^N(t)}|)t + C_3V(\{Y_t\}_{t\in I}; 0, t) + C_4\frac{p-1}{p}|X^N_{\tau^N(t)}|^p + \frac{C_4}{p}V(\{Y_t\}_{t\in I}; 0, t)^p.$$

With that, the required conditions on $\{\bar{F}_t\}_{t\in I}$ are met and we are allowed to apply Theorem 5.1 and deduce the strong order 1 convergence of the Euler method.

Remark 5.2. One particular example that easily yields (5.6) is when $f = f(t,x,y)$ is semi-separable, i.e.

$$f(t,x,y) = a(t,y)h(x) + b(t,y), \tag{5.10}$$

where $a = a(t,y)$ and $b = b(t,y)$ are continuously differentiable on $I \times \mathbb{R}$ with uniformly bounded first derivatives, $a = a(t,y)$ itself is uniformly bounded, and $h = h(x)$ is globally Lipschitz continous on $\mathbb{R}$.

Since $a = a(t,x)$ is uniformly bounded and $h = h(x)$ is globally Lipschitz continuous, it follows that $f = f(t,x,y)$ is uniformly globally Lipschitz continuous in x. Moreover, it is continuously differentiable in (t,y), with partial derivatives $\partial_t f$ and $\partial_y f$ given by

$$\partial_t f = \partial_t a(t,y)h(x) + \partial_t b(t,y), \qquad \partial_y f = \partial_y a(t,y)h(x) + \partial_t b(t,y)$$

Since the partial derivatives of $a = a(t,y)$ and $b = b(t,y)$ are uniformly bounded and h is Lipschitz, it follows that the partial derivatives $\partial_t f$ and $\partial_y f$ have at most linear growth. Thus, (5.6) is satisfies and Theorem 5.2 applies. But this special form (5.10) is by no means necessary, and the result applies to more general terms $f = f(t,x,y)$, as stated in the theorem.

6. The case of an Itô noise

Here, as explained in the Introduction, we assume the process given by $F_t = f(s, X_{\tau^N(s),Y_s})$ is an Itô process, which, in applications, follows from assuming that $f = f(t, x, y)$ is sufficiently regular and that the noise $\{Y_t\}_{t\in I}$ is itself an Itô process.

With that in mind, we first have the following result.

Lemma 6.1. *Besides Hypothesis 2.1, suppose that $F_t^N = f(t, X_{\tau^N(t)}^N, Y_t)$ is an Itô noise, satisfying*

$$\mathrm{d}F_t^N = A_t\,\mathrm{d}t + B_t\,\mathrm{d}W_t, \tag{6.1}$$

for a Wiener process $\{W_t\}_{t\geq 0}$ and stochastic processes $\{A_t\}_{t\in I}$, $\{B_t\}_{t\in I}$ adapted to $\{W_t\}_{t\geq 0}$ and such that

$$\int_0^T \mathbb{E}[|A_t|]\,\mathrm{d}t < \infty, \quad \int_0^T \mathbb{E}[|B_t|^2]\,\mathrm{d}t < \infty. \tag{6.2}$$

Then,

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^t \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})\right)\,\mathrm{d}s\right|\right]& \\ \leq \Delta t_N \left(\int_0^t \mathbb{E}[|A_\xi|]\,\mathrm{d}\xi + \left(\int_0^t \mathbb{E}[|B_\xi|^2]\,\mathrm{d}\xi\right)^{1/2}\right)&, \end{aligned}\tag{6.3}$$

for all $0 \leq t \leq T$ and every $N \in \mathbb{R}$.

Proof. We write

$$f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)}) = \int_{\tau^N(s)}^s A_\xi\,\mathrm{d}\xi + \int_{\tau^N(s)}^s B_\xi\,\mathrm{d}W_\xi.$$

Upon integration,

$$\begin{aligned}\int_0^t \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})\right)\,\mathrm{d}s& \\ = \int_0^t \left(\int_{\tau^N(s)}^s A_\xi\,\mathrm{d}\xi + \int_{\tau^N(s)}^s B_\xi\,\mathrm{d}W_\xi\right)\,\mathrm{d}s&.\end{aligned}$$

Exchanging the order of integration, according to Fubini's theorem, yields

$$\begin{aligned}\int_0^t &\left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})\right)\,\mathrm{d}s \\ &= \int_0^t \int_\xi^{\tau^N(\xi)+\Delta t_N} A_\xi\,\mathrm{d}s\,\mathrm{d}\xi + \int_0^t \int_\xi^{\tau^N(\xi)+\Delta t_N} B_\xi\,\mathrm{d}s\,\mathrm{d}W_\xi \\ &= \int_0^t (\tau^N(\xi) + \Delta t_N - \xi)A_\xi\,\mathrm{d}\xi + \int_0^t (\tau^N(\xi) + \Delta t_N - \xi)B_\xi\,\mathrm{d}W_\xi.\end{aligned}$$

Taking the absolute mean and using the Itô isometry [19] on the second term gives

$$\mathbb{E}\left[\left|\int_0^t \left(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})\right)\,\mathrm{d}s\right|\right]$$
$$\leq \int_0^t |\tau^N(\xi) + \Delta t_N - \xi| \mathbb{E}[|A_\xi|]\,\mathrm{d}\xi + \left(\int_0^t (\tau^N(\xi) + \Delta t_N - \xi)^2 \mathbb{E}[|B_\xi|^2]\,\mathrm{d}\xi\right)^{1/2}.$$

Since $|\tau^N(\xi) + \Delta t_N - \xi| \leq \Delta t_N$, we find

$$\mathbb{E}\left[\left|\int_0^t \left(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})\right)\,\mathrm{d}s\right|\right]$$
$$\leq \Delta t_N \left(\int_0^t \mathbb{E}[|A_\xi|]\,\mathrm{d}\xi + \left(\int_0^t \mathbb{E}[|B_\xi|^2]\,\mathrm{d}\xi\right)^{1/2}\right),$$

which completes the proof. □

Combining the estimate in Lemma 6.1 with the previous estimate for the global error we obtain the following main result.

Theorem 6.1. *Under Hypothesis 2.1, suppose also that* (2.4), (2.5), (4.5), (6.1), *and* (6.2) *hold. Then, the Euler scheme* (1.2)-(1.3) *is of strong order 1, i.e.*

$$\max_{j=0,\ldots,N} \mathbb{E}\left[\left|X_{t_j} - X^N_{t_j}\right|\right] \leq C\Delta t_N, \qquad \forall N \in \mathbb{N}, \tag{6.4}$$

for a constant C given by

$$C = \left(C_0 + L_X\left(\mathbb{E}[|X_0|] + \int_0^T \mathbb{E}[M_\xi]\,\mathrm{d}\xi\right) e^{L_X T} + \left(\int_0^T \mathbb{E}[|A_\xi|]\,\mathrm{d}\xi + \left(\int_0^T \mathbb{E}[|B_\xi|^2]\,\mathrm{d}\xi\right)^{1/2}\right)\right) e^{L_X T} \tag{6.5}$$

Proof. Under Hypothesis 2.1, the Lemma 4.2 applies and the global error estimate (4.3) holds.

Thanks to (2.4), (2.5), and (4.5), the Proposition 4.1 applies and the global error is bounded according to (4.6).

With assumptions (6.1) and (6.2), Lemma 6.1 applies and the last term in (4.6) is bounded according to (6.3). Using (6.3) in (4.6) yields

$$\mathbb{E}\left[|X_{t_j} - X^N_{t_j}|\right] \leq \left(C_0\Delta t_N + \Delta t_N L_X\left(\mathbb{E}[|X_0|] + \int_0^{t_j} \mathbb{E}[M_\xi]\,\mathrm{d}\xi\right) e^{L_X t_j} + \Delta t_N\left(\int_0^{t_j} \mathbb{E}[|A_\xi|]\,\mathrm{d}\xi + \left(\int_0^{t_j} \mathbb{E}[|B_\xi|^2]\,\mathrm{d}\xi\right)^{1/2}\right)\right) e^{L_X t_j}.$$

Since this holds for every $j = 0, \ldots, N$, we obtain the desired (6.4). □

In practice, conditions (6.1)-(6.2) follows from assuming sufficent regularity on $f = f(t, x, y)$ and an Itô noise, as given by the following result.

Theorem 6.2. *Let $f = f(t, x, y)$ be twice continuously differentiable with uniformly bounded derivatives. Suppose that the noise $\{Y_t\}_{t\in I}$ is an Itô noise,*

$$\mathrm{d}Y_t = a(t, Y_t)\ \mathrm{d}t + b(t, Y_t)\ \mathrm{d}W_t, \tag{6.6}$$

with $a = a(t, y)$ and $b = b(t, y)$ continuous and satisfying

$$|a(t, y)| \leq A_M + A_Y|y|, \qquad |b(t, y)| \leq B_M + B_Y|y|. \tag{6.7}$$

Assume the bounds (2.4), (4.5), and

$$\mathbb{E}[|Y_0|] < \infty \tag{6.8}$$

Then, the Euler scheme is of strong order 1, i.e.

$$\max_{j=0,\ldots,N} \mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq C\Delta t_N, \qquad \forall N \in \mathbb{N}, \tag{6.9}$$

for a suitable constant $C \geq 0$.

Proof. Let us start by showing that Hypothesis 2.1 is valid. Since $f = f(t, x, y)$ is (twice) continuously differentiable with, in particular, bounded derivative in x, then it is uniformly globally Lipschitz in x. Since $a = a(t, y)$ and $b = b(t, y)$ are continuous, the the noise has continuous sample paths. Thus, the remaining condition in Hypothesis 2.1 to be verified is (iic).

From (6.6) and (6.7), we have

$$Y_t = \int_0^t a(s, Y_s)\ \mathrm{d}s + \int_0^s b(t, Y_s)\ \mathrm{d}W_s.$$

Using the Itô formula, we have

$$\mathrm{d}Y_t^2 = \left(2a(t, Y_t)Y_t + b(t, Y_t)^2\right)\ \mathrm{d}t + 2b(t, Y_t)Y_t\ \mathrm{d}W_t.$$

Thus

$$Y_t^2 = Y_0^2 + \int_0^t \left(2a(s, Y_s)Y_s + b(t, Y_s)^2\right)\ \mathrm{d}s + \int_0^t 2b(s, Y_s)Y_s\ \mathrm{d}W_s.$$

Taking the expectation,

$$\mathbb{E}[|Y_t|^2] = \mathbb{E}[|Y_0|^2] + \int_0^t \left(2a(s, Y_s)Y_s + b(t, Y_s)^2\right)\ \mathrm{d}s.$$

Using (6.7), this yields

$$\begin{aligned}\mathbb{E}[|Y_t|^2] &\leq \mathbb{E}[|Y_0|^2] + \int_0^t \left(2\mathbb{E}[(A_M + A_Y|Y_s|)|Y_s|] + \mathbb{E}[(B_M + B_Y|Y_s|)^2\right)\ \mathrm{d}s \\ &\leq \mathbb{E}[|Y_0|^2] + \int_0^t \left(4(A_M^2 + (1 + A_Y)\mathbb{E}[|Y_s|^2]) + 2(B_M^2 + B_Y^2\mathbb{E}[|Y_s|^2])\right)\ \mathrm{d}s\end{aligned}$$

By the classical Gronwall Lemma [11],

$$\mathbb{E}[|Y_t|^2] \leq \left(\mathbb{E}[|Y_0|^2] + (4A_M^2 + 2B_M^2)t\right) e^{(4(1+A_Y)+2B_Y^2)t}.$$

Thus,

$$\sup_{t\in I} \mathbb{E}[|Y_t|^2] \leq \left(\mathbb{E}[|Y_0|^2] + (4A_M^2 + 2B_M^2)T\right) e^{(4(1+A_Y)+2B_Y^2)T}. \tag{6.10}$$

Since $f = f(t, x, y)$ is Lipschitz in x and twice continuously differentiable in (t, y) with uniformly bounded first order derivatives, we have the bound

$$|f(t, x, y)| \leq |f(0, 0, 0)| + L_X|x| + L_T|t| + L_Y|y|.$$

Thus,

$$|f(t, x, Y_t)| \leq M_t + L_X|x|$$

with

$$M_t = |f(0, 0, 0)| + L_T|t| + L_Y|y|.$$

Thanks to (6.10), we see that

$$\int_0^T M_t \,\mathrm{d}t < \infty.$$

Therefore, we are under the condition of (2.1).

Now, in view of Theorem 6.1, it remains to prove that $F_t^N = f(t, X_{\tau^N(t)}^N, Y_t)$ is an Itô noise (6.1), with the bounds (6.2). The fact that it is an Itô noise follows from the Itô formula and the smoothness of $f = f(t, x, y)$. Indeed, since $(t, y) \mapsto f(t, x, y)$ is twice continuously differentiable, for each fixed x, the Itô formula is applicable and yields

$$\begin{aligned} \mathrm{d}f(t, x, Y_t) = \left(\partial_t f(t, x, Y_t) + a(t, Y_t)\partial_y f(t, x, Y_t) + \frac{b(t, Y_t)^2}{2}\partial_{yy} f(t, x, Y_t)\right) \,\mathrm{d}t \\ + b(t, Y_t)\partial_y f(t, x, Y_t) \,\mathrm{d}W_t, \end{aligned} \tag{6.11}$$

for every fixed $x \in \mathbb{R}$. This means (6.1) holds with

$$A_t = \partial_t f(t, x, Y_t) + a(t, Y_t)\partial_y f(t, x, Y_t) + \frac{b(t, Y_t)^2}{2}\partial_{yy} f(t, x, Y_t)$$

and

$$B_t = b(t, Y_t)\partial_y f(t, x, Y_t).$$

It remains to show that $\{A_t\}_{t\in I}$ is mean integrable and that $\{B_t\}_{t\in I}$ is square mean integrable. Since $f = f(t, x, y)$ has uniformly bounded derivatives, we have

$$|A_t| \leq L_T + L_Y(A_M + A_Y|Y_t|) + 2L_{YY}(B_M^2 + B_Y^2|Y_t|^2),$$

and

$$|B_t| \leq L_Y(B_M + B_Y|Y_t|),$$

for a suitable constants $L_{YY} \geq 0$. Now, thanks to (6.10), we see that (6.2) is satisfied.

Therefore, all the conditions of Theorem 6.1 are met and we deduce the strong order 1 convergence of the Euler method. □

Remark 6.1. When the diffusion term $b = b(t, y) = b(t)$, in (6.6), is actually independent of y, then the noise is an additive noise and in this case the Euler scheme is well known to be of strong order 1 [14]. In the more general $b = b(t, y)$ case, however, the Euler scheme has always been regarded to be of order 1/2 [12] (see also [23] for mean square convergence). Here, though, we deduce, under the conditions of Theorem 6.2, that even if $b = b(t, y)$ depends on y, the strong convergence of the Euler scheme is actually of order 1.

7. The mixed case with Itô and bounded variation noises

Of course, it is possible to mix the two cases and have the following result combining Theorem 5.1 and Theorem 6.1.

Theorem 7.1. *Under Hypothesis 2.1, suppose also that* (2.4), (2.5), (4.5)*. Suppose, moreover, that* $F_t^N = f(t, X_{\tau^N(t)}^N, Y_t)$ *can be split into a sum* $F_t^N = G_t^N + H_t^N$ *where* $\{G_t^N\}_{t\in I}$ *satisfies* (6.1) *and* (6.2) *and where the steps of* $\{H_t^N\}_{t\in I}$ *are bounded by a real stochastic process* $\{\bar{H}_t\}$ *with monotonic non-decreasing sample paths, i.e.*

$$|H_s^N - H_{\tau^N(s)}^N| \leq \bar{H}_s^N - \bar{H}_{\tau^N(s)}^N \tag{7.1}$$

with

$$\mathbb{E}[\bar{H}_t] \textit{ uniformly bounded on } t \in I. \tag{7.2}$$

Then, the Euler scheme (1.2)-(1.3) *is of strong order 1, i.e. there exists a constant* $C \geq 0$ *such that*

$$\max_{j=0,\ldots,N} \mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq C\Delta t_N, \qquad \forall N \in \mathbb{N}. \tag{7.3}$$

We ommit the proof since it is just a combination of Lemma 5.1 and Lemma 6.1. As a consequence, we also have the following more explicit result, which is a combination of Theorem 5.2 and Theorem 6.2.

Theorem 7.2. *Suppose that* $f = f(t, x, y)$ *is twice continuously differentiable with uniformly bounded derivatives. Assume, further, that the sample paths of* $\{Y_t\}_{t\in I}$ *are made of two components, one of bounded variation with finite quadratic mean, as in* (5.7)*, and another an Itô noise satisfying* (6.6) *and* (6.8)*. Assume, moreover, that* (8.21) *holds. Then, the Euler scheme is of strong order 1, i.e.*

$$\max_{j=0,\ldots,N} \mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq C\Delta t_N, \qquad \forall N \in \mathbb{N}, \tag{7.4}$$

for a suitable constant $C \geq 0$.

8. Numerical examples

In this section, we illustrate the strong order 1 convergence with a few examples that fall into one of the cases considered above. We also illustrate the $H + 1/2$ order of convergence in the case of a fractional Brownian motion noise with Hurst parameter $0 < H < 1/2$.

For estimating the order of convergence, we use the Monte Carlo method, computing a number of numerical approximations $\{X_{t_j}^N(\omega_m)\}_{j=0,\ldots,N}$, of sample path solutions $\{X_t(\omega_m)\}_{t\in I}$, for samples $\omega_1,\ldots,\omega_M$, and taking the maximum in time of the average of their absolute differences at the mesh points:

$$\epsilon^N = \max_{j=0,\ldots,n} \frac{1}{M}\sum_{m=1}^M \left|X_{t_j}(\omega_m) - X_{t_j}^N(\omega_m)\right|. \tag{8.1}$$

Here are the main parameters for the error estimate:

(i) $M \in \mathbb{N}$ is the number of samples for the Monte Carlo estimate of the strong error, typically $M = 1,000$.
(ii) The time interval $[0,T]$ for the initial-value problem, typically with $T = 1.0$.
(iii) The initial condition X_0, which is typically $X_0 \sim \mathcal{N}(0,1)$.
(iv) A series of time steps $\Delta_i = T/(N_i - 1)$, with $N_i = 2^{n_i}$, most often with $n_i = 4,\ldots,10$, hence $N_i = 32, 64,\ldots,1024$.
(v) A number N_{tgt} of mesh points for a finer discretization to compute a target solution path.
(vi) The target solution path is either a pathwise solution with an exact distribution, when available, or a higher-order approximation, thanks to the choice of N_{tgt}. The target resolution is typically $N_{\mathrm{tgt}} = \max_i\{N_i^2\}$, e.g. $N = 2^{20} = 1,048,576$, unless an exact pathwise solution is available, in which case a coarser mesh of the order of can be used.

And here is the method:

(i) For each sample $m = 1,\ldots,M$, we first generate a discretization $\{Y_{t_j}\}_{j=0,N_{\mathrm{tgt}}}$ of a sample path of the noise on the finest grid $\{t_j^{N_{\mathrm{tgt}}}\}$, with N_{tgt} points, using either an exact distribution for the noise or an approximation in a much finer mesh.
(ii) Next, we use the values of the noise at the target time mesh to generate the target solution $\{X_{t_j}\}_{j=0,N_{\mathrm{tgt}}}$, still on the fine mesh. This is constructed either using the Euler approximation itself, keeping in mind that the mesh is sufficiently fine, or by an exact distributions of the solution, when available.
(iii) Then, for each time step $N_i = 2^{n_i}$ in the selected range, we compute the Euler approximation using the computed noise values at the corresponding coarser mesh.
(iv) We then compare each approximation $\{X_{t_j}^{N_i}\}_{j=0,\ldots,N_i}$ to the values of the target path on that coarse mesh and update the strong error

$$\epsilon_{t_j}^{N_i} = \frac{1}{M}\sum_{m=1}^M \left|X_{t_j}(\omega_m) - X_{t_j}^{N_i}(\omega_m)\right|$$

at each mesh point.

(v) At the end of all the simulations, we take the maximum in time, on each corresponding coarse mesh, to obtain the error for each mesh,

$$\epsilon^{N_i} = \max_{j=0,\ldots,N_i} \epsilon_{t_j}^{N_i}$$

(vi) Finally, we fit $(\Delta_i, \epsilon^{N_i})$ to the power law $C\Delta_i^p$, via linear least-square regression in log scale, for suitable C and p, with p giving the order of convergence.

As for the implementation itself, all code is written in the Julia language [3]. Julia is a high-performance programming language, suitable for scientific computing and computationally-demanding applications.

Julia has a performant and feature-rich `DifferentialEquations.jl` ecosystem of packages for solving differential equations [20], including random and stochastic differential equations, as well as delay equations, differential-algebraic equations, jump diffusions, partial differential equations, neural differential equations. It also has packages to seemlessly compose such equations in optimization problems, Bayesian parameter estimation, global sensitivity analysis, uncertainty quantification, and so on.

Although all the source code for the `DifferentialEquations.jl` ecosystem is available on the Github platform [10], it involves a quite large ecosystem of packages, with an intricate interplay between them. Hence, for the numerical results presented below, we chose not to use this ecosystem and, instead, implement our own code, with the minimum methods necessary for the convergence estimates. This is done mostly for the sake of transparency, in such a way that checking the accuracy of the implementation, for publication purposes, would be easier. All the source code for the numerical simulations presented below are in the Github repository [21].

8.1. **Homogeneous linear equation with Wiener noise.** We start by considering the Euler approximation of one of the simplest Random ODEs, that of a linear homogenous equation with a Wiener process as the coefficient:

$$\begin{cases} \dfrac{\mathrm{d}X_t}{\mathrm{d}t} = W_t X_t, \qquad 0 \leq t \leq T, \\ \left.X_t\right|_{t=0} = X_0. \end{cases} \tag{8.2}$$

Since the noise is simply a Wiener process, the corresponding RODE can be turned into an SDE with an additive noise. In this case, the Euler-Maruyama approximation for the noise part of the SDE is distributionally exact and the Euler-Maruyama method for the SDE is equivalent to the Euler method for the RODE. Moreover, it is known that the Euler-Maruyama method for an SDE with additive noise is of strong order 1. Nevertheless, we illustrate the strong convergence directly for the Euler method for this RODE equation, for the sake of completeness.

Equation (8.2) has the explicit solution

$$X_t = e^{\int_0^t W_s \;\mathrm{d}s} X_0. \tag{8.3}$$

When we compute an approximate solution via Euler's method, however, we only draw the realizations $\{W_{t_i}\}_{i=0}^n$ of a sample path, on the mesh points. We cannot compute the exact integral $\int_0^{t_j} W_s \;\mathrm{d}s$ just from these values, and, in fact, an exact solutions is not uniquely defined from them. We can, however, find its exact distribution and use that to draw feasible exact solutions and use them to estimate the error.

First we break down the sum into parts:

$$\int_0^{t_j} W_s \;\mathrm{d}s = \sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} W_s \;\mathrm{d}s. \tag{8.4}$$

On each mesh interval $[t_i, t_{i+1}]$, we consider the process

$$B_t^i = W_t - W_{t_i} - \frac{t - t_i}{t_{i+1} - t_i}(W_{t_{i+1}} - W_{t_i}) \tag{8.5}$$

which is a Brownian bridge on that mesh interval, vanishing at the extremes, and independent of W_{t_i} and $W_{t_{i+1}}$. Then,

$$\begin{aligned}\int_{t_i}^{t_{i+1}} W_s \;\mathrm{d}s &= \int_{t_i}^{t_{i+1}} B_s^i \;\mathrm{d}s + \int_{t_i}^{t_{i+1}} \left(W_{t_i} + \frac{s - t_i}{t_{i+1} - t_i}(W_{t_{i+1}} - W_{t_i})\right) \;\mathrm{d}s \\ &= \frac{1}{2}\left(W_{t_i} + W_{t_{i+1}}\right)(t_{i+1} - t_i) + Z_i,\end{aligned}$$

where

$$Z_i = \int_{t_i}^{t_{i+1}} B_s^i \;\mathrm{d}s. \tag{8.6}$$

Notice the first term is the trapezoidal rule while the second term is a Gaussian with zero mean. We need to compute the variance of Z_i to completely characterize it. By translation, it suffices to consider a Brownian bridge $\{B_t\}_{t\in[0,\tau]}$ on an interval $[0, \tau]$, with $\tau = \Delta t_N$. This is obtained from $B_t = W_t - (t/\tau)W_\tau$. We have, since $\mathbb{E}[W_t W_s] = \min\{t, s\}$, that

$$\mathbb{E}[B_t B_s] = \min\{t, s\} - \frac{ts}{\tau}.$$

FIGURE 1. Euler approximation of $\mathrm{d}X_t/\mathrm{d}t = W_tX_t$ with $X_0 = 1.0$, on $[0, T]$, and a few sample paths of exact solutions compatible with the given realizations of the noise on the mesh points.

Notice $\mathbb{E}[B_tB_s]$ is symmetric in ts, so we can write

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^\tau B_s\,\mathrm{d}s\right)^2\right] &= \mathbb{E}\left[\int_0^\tau\int_0^\tau B_sB_t\,\mathrm{d}s\,\mathrm{d}\right] \\
&= \int_0^\tau\int_0^\tau \mathbb{E}[B_sB_t]\,\mathrm{d}s\,\mathrm{d}t \\
&= 2\int_0^\tau\int_0^t \mathbb{E}[B_sB_t]\,\mathrm{d}s\,\mathrm{d}t \\
&= 2\int_0^\tau\int_0^t \left(s - \frac{ts}{\tau}\right)\,\mathrm{d}s\,\mathrm{d}t \\
&= \frac{2}{\tau}\int_0^\tau\int_0^t (\tau - t)s\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

Hence,

$$\mathbb{E}\left[\left(\int_0^\tau B_s\,\mathrm{d}s\right)^2\right] = \frac{2}{\tau}\int_0^\tau (\tau - t)\frac{t^2}{2}\,\mathrm{d}t = \frac{2}{\tau}\left(\tau\frac{t^3}{6} - \frac{t^4}{8}\right)\Big|_{t=0}^{\tau} = \frac{\tau^3}{12}.$$

Back to Z_i, this means that

$$Z_i \sim \mathcal{N}\left(0, \frac{(t_{i+1} - t_i)^3}{12}\right) = \frac{\sqrt{(t_{i+1} - t_i)^3}}{\sqrt{12}}\mathcal{N}(0, 1). \tag{8.7}$$

For a normal variable $N \sim \mathcal{N}(\mu, \sigma)$, the expectation of the random variable e^N is $\mathbb{E}[e^N] = e^{\mu + \sigma^2/2}$. Hence,

$$\mathbb{E}[e^{Z_i}] = e^{((t_{i+1} - t_i)^3)/24}. \tag{8.8}$$

This is the contribution of this random variable to the mean of the exact solution. But we actually draw directly Z_i and use $e^{\sum_i Z_i}$.

Hence, once an Euler approximation of (8.2) is computed, along with realizations $\{W_{t_i}\}_{i=0}^n$ of a sample path of the noise, we consider an exact solution given by

$$X_{t_j} = X_0 e^{\sum_{i=0}^{j-1}\left(\frac{1}{2}\left(W_{t_i} + W_{t_{i+1}}\right)(t_{i+1} - t_i) + Z_i\right)}, \tag{8.9}$$

for realizations Z_i drawn from a normal distributions given by (8.7). Figure 8.1 shows an approximate solution and a few sample paths of exact solutions associated with the given realizations of the noise on the mesh points.

Table 8.1 shows the estimated strong error obtained from a thousand sample paths for each chosen time step, with initial condition $X_0 \sim \mathcal{N}(0, 1)$, on the interval $[0, T]$. Figure 8.4 illustrates the order of convergence.

N	dt	error
16	0.0667	0.0194
32	0.0323	0.0108
64	0.0159	0.00546
128	0.00787	0.00278
256	0.00392	0.00139
512	0.00196	0.000687
1024	0.000978	0.000345
2048	0.000489	0.000172
4096	0.000244	8.62e-5
8192	0.000122	4.66e-5
16384	6.1e-5	2.45e-5

TABLE 1. Mesh points (N), time steps (dt), and strong error (error) of the Euler method for $\mathrm{d}X_t/\mathrm{d}t = W_t X_t$, with initial condition $X_0 \sim \mathcal{N}(0,1)$ and a standard Wiener process noise $\{W_t\}_t$, on the time interval (0.0, 1.0), based on $M = 1000$ sample paths for each fixed time step, with the target solution calculated with 65536 points.

8.2. **Non-homogeneous linear system of RODEs with different types of noises.** Now we consider a system of linear equations with a series of different types of noises. For most of these noises, the current knowledge expects a lower order of strong convergence than the strong order 1 we prove here. The aim of this section is to illustrate this improvement at once, for all such noises.

The system of equation takes the form

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{X}_t}{\mathrm{d}t} = -\left\|\mathbf{Y}_t\right\|^2 \mathbf{X}_t + \mathbf{Y}_t, \qquad 0 \leq t \leq T, \\ \mathbf{X}_t\big|_{t=0} = \mathbf{X}_0, \end{cases} \tag{8.10}$$

where $\{\mathbf{X}_t\}_t$ is a vector-valued process and $\{\mathbf{Y}_t\}_t$ is a given vector-valued noise process with the same dimension as $\mathbf{X}_t$. Each coordinate of $\{\mathbf{Y}_t\}_t$ is a scalar noise process independent of the noises in the other coordinates. The scalar noises used in the following simulations are the following, in the order of coordinates of $\mathbf{Y}_t$:

(i) A standard Wiener process;
(ii) An Ornstein-Uhlenbeck process with drift $\nu = 0.3$, diffusion $\sigma = 0.5$, and initial condition $y_0 = 0.2$;
(iii) A geometric Brownian motion process with drift $\mu = 0.3$, diffusion coefficient $\sigma = 0.5$, and initial condition $y_0 = 0.2$;
(iv) A compound Poisson process with rate $\lambda = 5.0$ and jump law following an exponential distribution with scale $\theta = 0.5$;

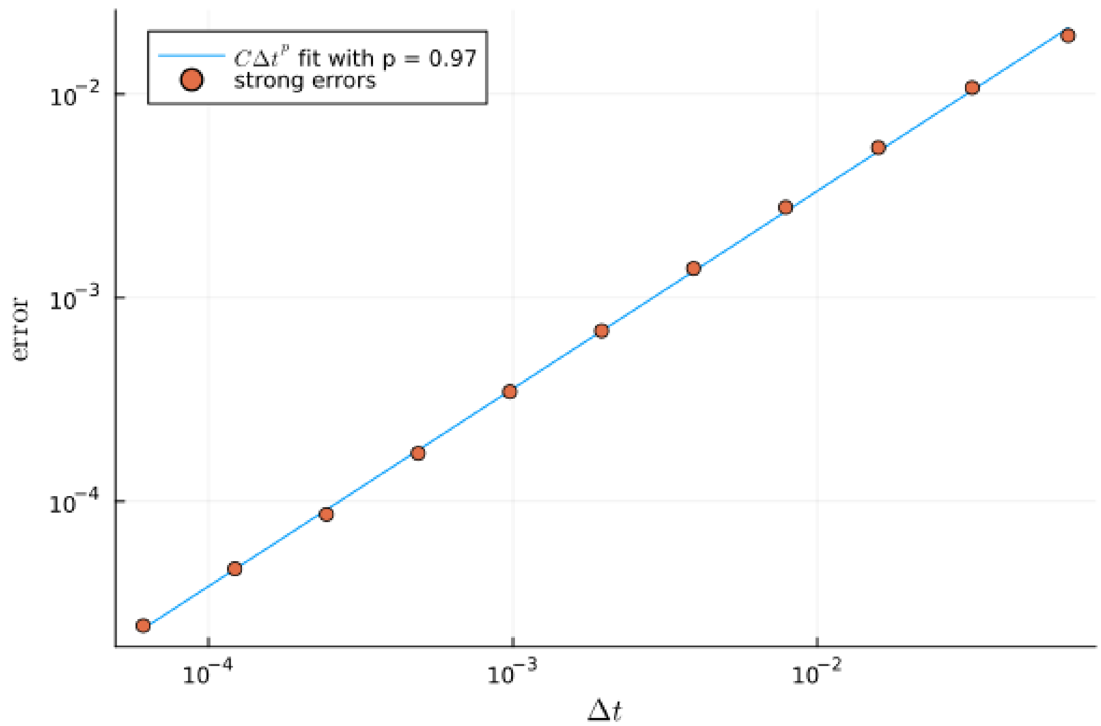

FIGURE 2. Order of convergence $p = 0.98$ of the strong error of the Euler method for $\mathrm{d}X_t/\mathrm{d}t = W_t X_t$, on $[0, T] = [0.0, 1.0]$, with $X_0 \sim \mathcal{N}(0, 1)$, and a standard Wiener process noise $\{W_t\}_t$, computed with $M = 1000$ sample paths, for each $\Delta t = 1/N$, $N = 16, 32, 64, 128, 256, 512, 1024$.

(v) A Poisson step process with rate $\lambda = 5.0$ and step law following a Uniform distribution within the unit interval;
(vi) An exponentially decaying Hawkes process with initial rate $\lambda_0 = 3.0$, base rate $a = 2.0$, exponential decay rate $\delta = 3.0$, and jump law following an exponential distribution with scale $\theta = 0.5$;
(vii) A transport process of the form $t \mapsto \sum_{i=1}^{6} \sin^{1/3}(\omega_i t)$, where the frequencies ω_i are independent random variables following a Gamma distribution with shape parameter $\alpha = 7.5$ and scale $\theta = 2.0$;
(viii) A fractional Brownian motion process with Hurst parameter $H = 0.6$ and initial condition $y_0 = 0.2$.

Table 8.2 shows the estimated strong error obtained from a 200 sample paths for each chosen time step, with initial condition $\mathbf{X}_0 \sim \mathcal{N}(\mathbf{0}, \mathrm{I})$, i.e. normally distributed on each coordinate, independently of the other coordinates, and on the time interval $[0.0, 1.0]$. Figure 8.4 illustrates the order of convergence.

N	dt	error
64	0.0159	0.22
128	0.00787	0.106
256	0.00392	0.0525
512	0.00196	0.0262

TABLE 2. Mesh points (N), time steps (dt), and strong error (error) of the Euler method for $\mathrm{d}\mathbf{X}_t/\mathrm{d}t = -\|\mathbf{Y}_t\|^2\mathbf{X}_t + \mathbf{Y}_t$, with initial condition $\mathbf{X}_0 \sim \mathcal{N}(\mathbf{0}, \mathrm{I})$ and vector-valued noise $\{Y_t\}_t$ with all the implemented noises, on the time interval (0.0, 1.0), based on $M = 200$ sample paths for each fixed time step, with the target solution calculated with 262144 points.

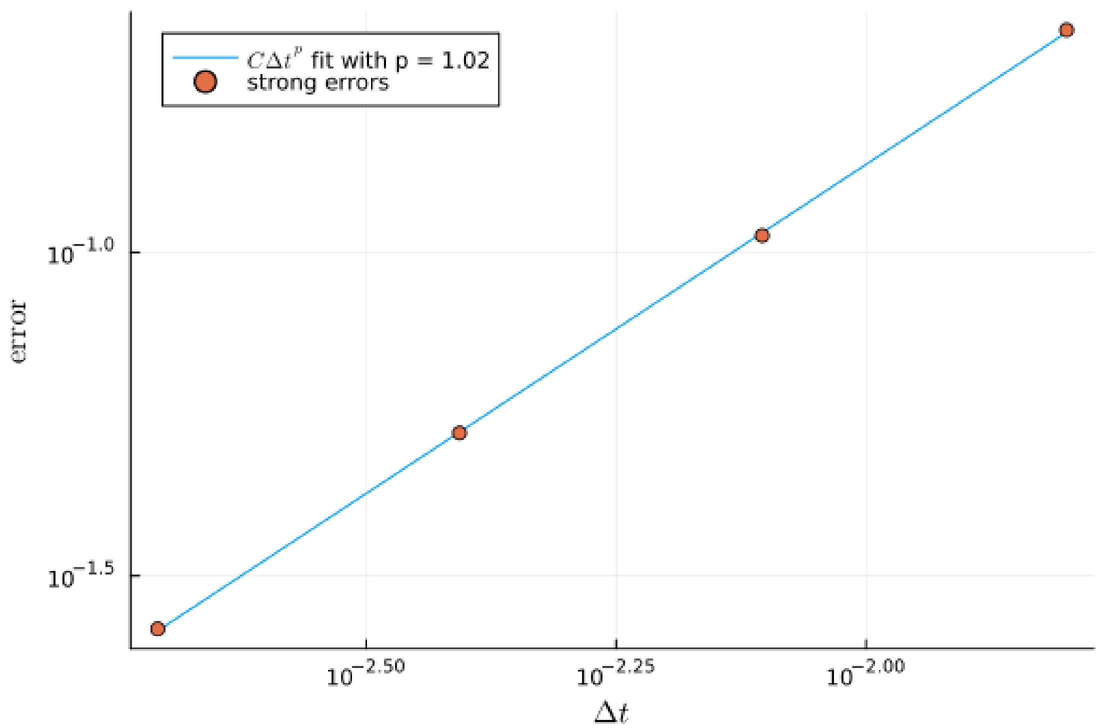

FIGURE 3. Order of convergence $p = 1.02$ of the strong error of the Euler method for $\mathrm{d}\mathbf{X}_t/\mathrm{d}t = -\left\|\mathbf{Y}_t\right\|^2\mathbf{X}_t + \mathbf{Y}_t$, on $[0, T] = [0.0, 1.0]$, with $\mathbf{X}_0 \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$, and a vector-valued noise with different types of noise processes, computed with $M = 200$ sample paths, for each $\Delta t = 1/N$, with $N = 64$, 128, 256, and 512.

8.3. **Fractional Brownian motion noise.** Here, we consider again a linear equation, of the form

$$\begin{cases} \dfrac{\mathrm{d}X_t}{\mathrm{d}t} = -X_t + B_t^H, & 0 \leq t \leq T, \\ X_t|_{t=0} = X_0, \end{cases} \tag{8.11}$$

except now the noise $\{B_t^H\}_t$ is assumed to be a fractional Brownian motion (fBm) with Hurst parameter $0 < H < 1$. We show that, for $0 < H < 1/2$, the order of convergence is $H + 1/2$. The same seems to hold for a nonlinear dependency on the fBm, but the proof is more involved, depending on a fractional Itô formula (see [4, Theorem 4.2.6], [2, Theorem 4.1]and [17, Theorem 2.7.4]), based on the Wick Itô Skorohod (WIS) integral (see [4, Chapter 4]). A corresponding WIS isometry is also needed (see e.g. [4, Theorem 4.5.6]), involving Malliavin calculus and fractional derivatives. For these reasons, we leave the nonlinear case to a subsequent work and focus on this simple linear example, which suffices to illustrate the peculiarity of the dependence on H of the order of convergence.

Although the above linear equation has the explicit solution

$$X_t = e^{-t}X_0 + \int_0^t e^{-(t-s)}B_s^H \,\mathrm{d}s, \tag{8.12}$$

computing a distributionally exact solution of this form is a delicate process. Thus we check the convergence numerically by solving the equation with the Euler method itself, but on a much finer mesh. Nevertheless, the proof is done rigorously below, with the framework developed in the first sections.

Indeed, we need to estimate the last term of (4.6), in Proposition 4.1, involving the steps of the term $f(t, x, y) = -x + y$, which in this case reduce to

$$f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)}) = B_s^H - B_{\tau^N(s)}^H, \tag{8.13}$$

for $0 \leq s \leq T$. There are several ways to represent an fBm (see e.g. [4, 17]). We will use the formula [16, eq. (2.1)], [4, eq. (1.1)]

$$B_t^H = \frac{1}{\Gamma(H+1/2)}\left(\int_{-\infty}^0 \left((t-s)^{H-1/2} - (-s)^{H-1/2}\right)\,\mathrm{d}W_s \right.$$
$$\left. + \int_0^t (t-s)^{H-1/2}\,\mathrm{d}W_s\right), \tag{8.14}$$

where $\Gamma(\cdot)$ is the well-known Gamma function. For the step, (8.14) means that

$$B_s^H - B_{\tau^N(s)}^H = \frac{1}{\Gamma(H+1/2)}\left(\int_{-\infty}^{\tau^N(s)} \left((s-\xi)^{H-1/2} - (\tau^N(s)-\xi)^{H-1/2}\right)\,\mathrm{d}W_\xi \right.$$
$$\left. + \int_{\tau^N(s)}^s (s-\xi)^{H-1/2}\,\mathrm{d}W_\xi\right). \tag{8.15}$$

Then, using Fubini's Theorem to exchange the order of integration,

$$
\begin{aligned}
&\int_0^{t_j} \Big(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)}) \Big)\ \mathrm{d}s \\
&\quad = \frac{1}{\Gamma(H+1/2)} \int_0^{t_j} \int_{-\infty}^{\tau^N(s)} \big((s-\xi)^{H-1/2} - (\tau^N(s)-\xi)^{H-1/2} \big)\ \mathrm{d}W_\xi\ \mathrm{d}s \\
&\qquad + \frac{1}{\Gamma(H+1/2)} \int_0^{t_j} \int_{\tau^N(s)}^{s} (s-\xi)^{H-1/2}\ \mathrm{d}W_\xi\ \mathrm{d}s \\
&\quad = \frac{1}{\Gamma(H+1/2)} \int_{-\infty}^{0} \int_0^{t_j} \big((s-\xi)^{H-1/2} - (\tau^N(s)-\xi)^{H-1/2} \big)\ \mathrm{d}s\ \mathrm{d}W_\xi \\
&\qquad + \frac{1}{\Gamma(H+1/2)} \int_0^{t_j} \int_{\tau^N(\xi)+\Delta t_N}^{t_j} \big((s-\xi)^{H-1/2} - (\tau^N(s)-\xi)^{H-1/2} \big)\ \mathrm{d}s\ \mathrm{d}W_\xi \\
&\qquad + \frac{1}{\Gamma(H+1/2)} \int_0^{t_j} \int_{\xi}^{\tau^N(\xi)+\Delta t_N} (s-\xi)^{H-1/2}\ \mathrm{d}s\ \mathrm{d}W_\xi
\end{aligned}
\tag{8.16}
$$

For the first term, notice $\sigma \mapsto 1/(\sigma-\xi)^{H-1/2}$ is continuously differentiable in $\sigma > \xi$, so that

$$
(s-\xi)^{H-1/2} - (\tau^N(s)-\xi)^{H-1/2} = -(H-1/2)\int_{\tau^N(s)}^{s} (\sigma-\xi)^{H-3/2}\ \mathrm{d}\sigma.
$$

Thus,

$$
\int_0^{t_j} \big((s-\xi)^{H-1/2} - (\tau^N(s)-\xi)^{H-1/2} \big)\ \mathrm{d}s = (H-1/2)\int_0^{t_j}\int_{\tau^N(s)}^{s} (\sigma-\xi)^{H-3/2}\ \mathrm{d}\sigma\ \mathrm{d}s.
$$

Exchanging the order of integration yields

$$
\begin{aligned}
\int_0^{t_j} \big((s-\xi)^{H-1/2} & - (\tau^N(s)-\xi)^{H-1/2} \big)\ \mathrm{d}s \\
& = (H-1/2)\int_0^{t_j}\int_{\sigma}^{\tau^N(\sigma)+\Delta t_N} (\sigma-\xi)^{H-3/2}\ \mathrm{d}s\ \mathrm{d}\sigma \\
& = (H-1/2)\int_0^{t_j} \big(\tau^N(\sigma) + \Delta t_N - \sigma \big) (\sigma-\xi)^{H-3/2}\ \mathrm{d}\sigma.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\left| \int_0^{t_j} \big((s-\xi)^{H-1/2} - (\tau^N(s)-\xi)^{H-1/2} \big)\ \mathrm{d}s \right| \\
&\qquad\qquad \leq (1/2 - H)\int_0^{t_j} \Delta t_N (\sigma-\xi)^{H-3/2}\ \mathrm{d}\sigma.
\end{aligned}
$$

Now, using the Lyapunov inequality and the Itô isometry, and using the same trick as above,

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_{-\infty}^0 \int_0^{t_j} \left((s-\xi)^{H-1/2} - (\tau^N(s)-\xi)^{H-1/2}\right)\,\mathrm{d}s\,\mathrm{d}W_\xi\right|\right] \\
&\qquad \leq \left(\int_{-\infty}^0 \left(\int_0^{t_j} \left((s-\xi)^{H-1/2} - (\tau^N(s)-\xi)^{H-1/2}\right)\,\mathrm{d}s\right)^2 \mathrm{d}\xi\right)^{1/2} \\
&\qquad \leq \Delta t_N \left(\int_{-\infty}^0 \left((1/2-H)\int_0^{t_j} (\sigma-\xi)^{H-3/2}\,\mathrm{d}\sigma\right)^2 \mathrm{d}\xi\right)^{1/2} \\
&\qquad \leq (1/2-H)\Delta t_N \left(\int_{-\infty}^0 \left(\int_0^{T} (\sigma-\xi)^{H-3/2}\,\mathrm{d}\sigma\right)^2 \mathrm{d}\xi\right)^{1/2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{\Gamma(H+1/2)}\Delta t_N \mathbb{E}\left[\left|\int_{-\infty}^0 \int_0^{t_j} \left((s-\xi)^{H-1/2} - (\tau^N(s)-\xi)^{H-1/2}\right)\,\mathrm{d}s\,\mathrm{d}W_\xi\right|\right] \\
\leq C_H^{(1)}\Delta t_N, \quad (8.17)
\end{aligned}$$

for a suitable constant $C_H^{(1)}$. We see this term is order 1.

The second term is similar.

$$\begin{aligned}
&\int_{\tau^N(\xi)+\Delta t_N}^{t_j} \left((s-\xi)^{H-1/2} - (\tau^N(s)-\xi)^{H-1/2}\right)\,\mathrm{d}s \\
&\qquad = (H-1/2)\int_{\tau^N(\xi)+\Delta t_N}^{t_j} \int_{\tau^N(s)}^{s} (\sigma-\xi)^{H-3/2}\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\qquad = (H-1/2)\int_{\tau^N(\xi)+\Delta t_N}^{t_j} \int_{\sigma}^{\tau^N(\sigma)+\Delta t_N} (\sigma-\xi)^{H-3/2}\,\mathrm{d}s\,\mathrm{d}\sigma \\
&\qquad = (H-1/2)\int_{\tau^N(\xi)+\Delta t_N}^{t_j} \left(\tau^N(\sigma)+\Delta t_N-\sigma\right)(\sigma-\xi)^{H-3/2}\,\mathrm{d}\sigma.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left|\int_{\tau^N(\xi)+\Delta t_N}^{t_j} \left((s-\xi)^{H-1/2} - (\tau^N(s)-\xi)^{H-1/2}\right)\,\mathrm{d}s\right| \\
&\qquad\qquad \leq (1/2-H)\Delta t_N \int_{\tau^N(\xi)+\Delta t_N}^{t_j} (\sigma-\xi)^{H-3/2}\,\mathrm{d}\sigma.
\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^{t_j}\int_{\tau^N(\xi)+\Delta t_N}^{t_j}\left((s-\xi)^{H-1/2}-(\tau^N(s)-\xi)^{H-1/2}\right)\,\mathrm{d}s\,\mathrm{d}W_\xi\right|\right] \\ \leq \left(\int_0^{t_j}\left(\int_{\tau^N(\xi)+\Delta t_N}^{t_j}\left((s-\xi)^{H-1/2}-(\tau^N(s)-\xi)^{H-1/2}\right)\,\mathrm{d}s\right)^2\,\mathrm{d}\xi\right)^{1/2} \\ \leq \Delta t_N(1/2-H)\left(\int_0^{t_j}\left(\int_{\tau^N(\xi)+\Delta t_N}^{T}(\sigma-\xi)^{H-3/2}\,\mathrm{d}\sigma\right)^2\,\mathrm{d}\xi\right)^{1/2}.\end{aligned}$$

Therefore,

$$\frac{1}{\Gamma(H+1/2)}\mathbb{E}\left[\left|\int_0^{t_j}\int_{\tau^N(\xi)+\Delta t_N}^{t_j}\left((s-\xi)^{H-1/2}-(\tau^N(s)-\xi)^{H-1/2}\right)\,\mathrm{d}s\,\mathrm{d}W_\xi\right|\right] \\ \leq C_H^{(2)}\Delta t_N, \quad (8.18)$$

for a possibly different constant $C_H^{(2)}$. This term is also of order 1.

For the last term, we have

$$0 \leq \int_\xi^{\tau^N(\xi)+\Delta t_N}(s-\xi)^{H-1/2}\,\mathrm{d}s = \frac{1}{H+1/2}(\tau^N(\xi)+\Delta t_N-\xi)^{H+1/2} \\ \leq \frac{1}{H+1/2}\Delta t_N^{H+1/2}.$$

so that, using the Lyapunov inequality and the Itô isometry

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^{t_j}\int_\xi^{\tau^N(\xi)+\Delta t_N}(s-\xi)^{H-1/2}\,\mathrm{d}s\,\mathrm{d}W_\xi\right|\right] \\ \leq \left(\int_0^{t_j}\left(\int_\xi^{\tau^N(\xi)+\Delta t_N}(s-\xi)^{H-1/2}\,\mathrm{d}s\right)^2\,\mathrm{d}\xi\right)^{1/2} \\ \leq \left(\int_0^{t_j}\Delta t_N^{2H+1}\,\mathrm{d}\xi\right)^{1/2} \leq t_j^{1/2}\Delta t_N^{H+1/2}.\end{aligned}$$

Therefore,

$$\frac{1}{\Gamma(H+1/2)}\mathbb{E}\left[\left|\int_0^{t_j}\int_\xi^{\tau^N(\xi)+\Delta t_N}(s-\xi)^{H-1/2}\,\mathrm{d}s\,\mathrm{d}W_\xi\right|\right] \leq C_H^{(3)}\Delta t_N^{H+1/2}, \quad (8.19)$$

for a third constant $C_H^{(3)}$.

H	p
0.1	0.630713
0.2	0.759896
0.3	0.855504
0.4	0.942058
0.5	1.0012
0.7	1.00544
0.9	0.99782

TABLE 3. Hurst parameter H and order p of strong convergence for a number of Hurst values.

Putting the three estimates (8.17), (8.18), (8.19) in (8.16) we find that

$$\mathbb{E}\left[\left|\int_0^{t_j}\left(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})\right)\,\mathrm{d}s\right|\right] \\ \leq C_H^{(4)}\Delta t_N + C_H^{(3)}\Delta t_N^{H+1/2}, \quad (8.20)$$

where $C_H^{(4)} = C_H^{(1)} + C_H^{(2)}$. Applying this estimate to Proposition 4.1 shows that the Euler method is of order $H + 1/2$, when $0 < H < 1/2$, and is of order 1, when $1/2 \leq H < 1$, having in mind that $H = 1/2$ corresponds to the classical Wiener process.

In summary, we have proved the following result.

Theorem 8.1. *Consider the equation (8.11) where $\{B_t^H\}_t$ is a fractional Brownian motion (fBm) with Hurst parameter $0 < H < 1$. Suppose the initial condition X_0 satisfies*

$$\mathbb{E}[|X_0|^2] < \infty. \quad (8.21)$$

Then, the Euler scheme for this initial value problem is of strong order $H + 1/2$, for $0 < H < 1/2$, and is of order 1, for $1/2 \leq H < 1$. More precisely,

$$\max_{j=0,\ldots,N} \mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq C_1\Delta t_N + C_2\Delta t_N^{H+1/2}, \qquad \forall N \in \mathbb{N}, \quad (8.22)$$

for suitable constants $C_1, C_2 \geq 0$.

As for the numerics, the Euler approximation is implemented for (8.11) with several values of H. We fix the time interval as $[0, T] = [0.0, 1.0]$, set the resolution for the target approximation to $N_{\mathrm{tgt}} = 2^19$, choose the time steps for the convergence test as $\Delta t = 1/N$, $N = 64$, 128, 256, and 512, and use $M = 200$ samples for the Monte-Carlo estimate of the strong error. Table 8.3 shows the obtained convergence estimates, for a series of Hurst parameters, which is also illustrated in Figure 8.3.

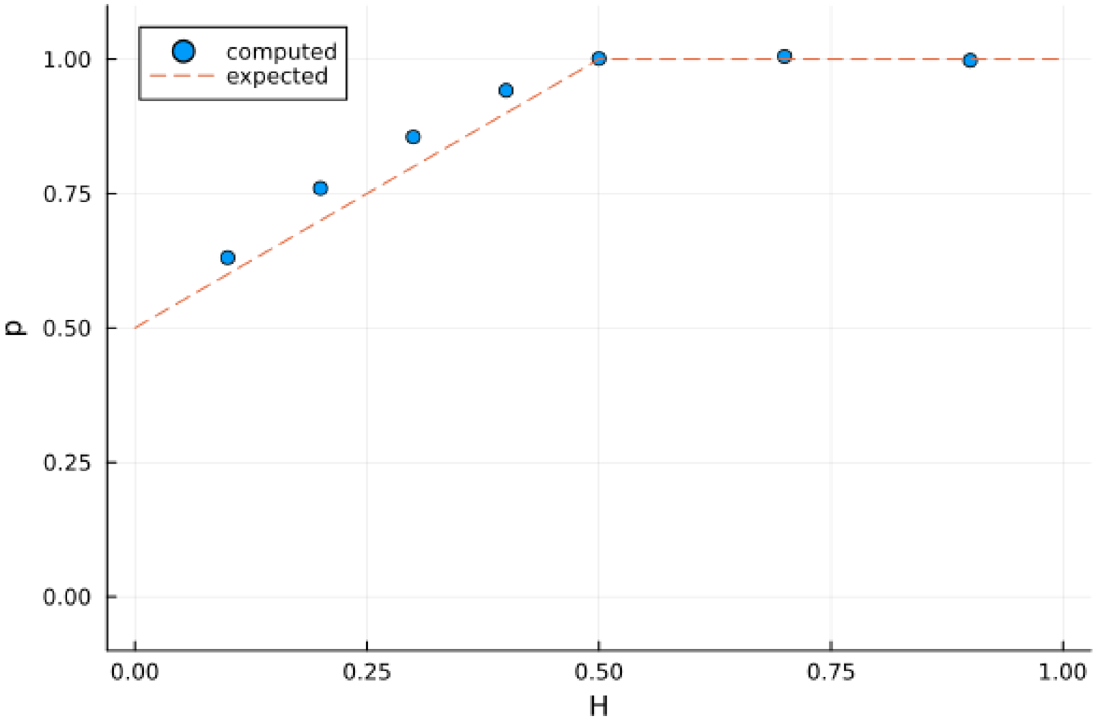

FIGURE 4. Order p of strong convergence for each value of the Hurst parameter H (scattered plot) along with the theoretical value $p = \min\{H + 1/2, 1\}$ (dashed line).

8.4. **Population dynamics with harvest.** Here, we consider a population dynamics modelled by a logistic equation with random coefficients, loosely inspired by [13, Section 15.2], with an extra term representing harvest:

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = G_t X_t (r - X_t) - H_t. \tag{8.23}$$

Here, $r > 0$ is constant, $\{G_t\}_{t\geq 0}$ is a stochastic process playing the role of a random growth parameter, and $\{H_t\}_{t\geq 0}$ is a nonnegative point process playing the role of the harvest term. More specifically, $\{G_t\}_{t\geq 0}$ is given by

$$G_t = \gamma(1 + \varepsilon \sin(Z_t)),$$

where $0 < \varepsilon < 1$ and $\{Z_t\}_{t\geq 0}$ is a geometric Brownian motion process, hence of the form (6.6)-(6.7). The harvest term $\{H_t\}_{t\geq 0}$ is a "Poisson step" process of the form

$$H_t = S_{N_t}$$

where $\{N_t\}_{t\geq 0}$ is a Poisson point-process with rate λ, $S_0 = 0$, and the S_i, for $i = 1, 2, \ldots$, are independent and identically distributed random variables.

We suppose the initial condition is non-negative and bounded almost surely:

$$0 \leq X_0 \leq R,$$

for some $R > r$.

The noise process $\{Z_t\}_{t\geq 0}$ itself satisfies

$$0 < \lambda - \varepsilon \leq Z_t \leq \lambda + \varepsilon < 2\lambda, \qquad \forall t \geq 0.$$

Define $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ by

$$f(t, x, y) = \Big\{ \gamma(1 + \varepsilon \sin(y_1))x(r - x) - y_2, \quad x > 0, 0, \quad x \leq 0.$$

The equation (8.23) becomes

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t),$$

where $\{Y_t\}_{t\geq 0}$ is the vector-valued process with $Y_t = (Z_t, H_t)$.

Notice that $f(t, x, y) = 0$, for $x < 0$ and arbitrary $y = (y_1, y_2)$, and $f(t, x, y) < 0$, for $x \geq r$, $y_2 \geq 0$, and arbitrary y_1. Since the noise $y_2 = H_t$ is always nonnegative, we see that the interval $0 \leq x \leq R$ is positively invariant and attracts the orbits with a nonnegative initial condition. Thus, the pathwise solutions of the initial-value problem under consideration are almost surely bounded as well.

The function $f = f(t, x, y)$ is continuously differentiable infinitely many times and with bounded derivatives within the positively invariant interval. Hence, within the region of interest, all the conditions of Theorem 7.2 hold and the Euler method is of strong order 1.

Below, we simulate numerically the solutions of the above problem, with $\gamma = 1.0$, $\varepsilon = 0.3$, $r = 1.0$, and $\alpha = \gamma/2 = 0.5$. The geometric Brownian motion process $\{Z_t\}_{t\geq 0}$ is taken with drift coefficient $\mu = 1.0$, diffusion coefficient $\sigma = 0.8$, and initial condition $y_0 = 1.0$. The Poisson process $\{N_t\}_{t\geq 0}$ is taken with rate $\lambda = 15.0$. And the step process $\{H_t\}_{t\geq 0}$ is taken with steps following a Beta distribution with shape parameters $\alpha = 5.0$ and $\beta = 7.0$. The initial condition X_0 is taken to be a Beta distribution with shape parameters $\alpha = 7.0$ and $\beta = 5.0$. We take $M = 1000$ samples for the Monte-Carlo estimate of the strong error of convergence. For the target solution, we solve the equation with a time mesh with $N_{\text{tgt}} = 2^18$ points, while for the approximations we take $N = 2^i$, for $i = 4, \ldots, 9$.

Table 8.4 shows the estimated strong error obtained for each mesh resolution, while Figure **??** illustrates the order of convergence, estimated to be of order

8.5. **Kanai-Tajimi Earthquake model.** The Kanai-Tajimi model for the oscillations of a single-storey building is usually written as the second-order equation (see [18, Chapter 18])

$$\ddot{x} + 2\zeta_g\omega_g\dot{x} - \omega_g^2 x = \dot{\xi},$$

N	dt	error
16	0.0667	0.0118
32	0.0323	0.00553
64	0.0159	0.00269
128	0.00787	0.00133
256	0.00392	0.000657
512	0.00196	0.000325

TABLE 4. Mesh points (N), time steps (dt), and strong error (error) of the Euler method for population dynamics, with initial condition $X_0 \sim \mathrm{Beta}(7.0, 5.0)$ and gBm and step process noises, on the time interval (0.0, 1.0), based on $m = 1000$ sample paths for each fixed time step, with the target solution calculated with 262144 points.

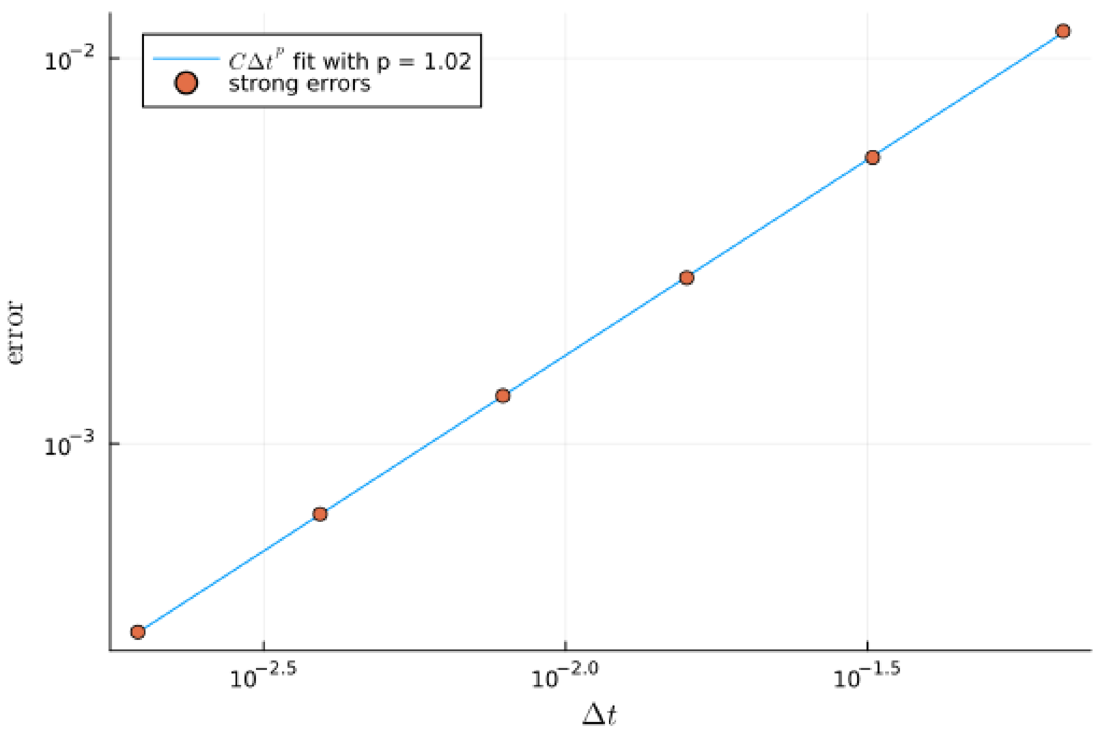

FIGURE 5. Order of convergence of the strong error of the Euler method for equation (8.23).

where ζ_g and ω_g are damping and frequency model parameters, taken to be $\zeta_g = 0.64$ and $\omega_g = 15.56$ `rad/s`, respectively, and where $\dot{\xi}$ is a white noise excitation. This can be written as a system of stochastic differential equations as

$$\begin{cases} \mathrm{d}x = -y\ \mathrm{d}t, \\ \mathrm{d}y = (-2\zeta_g\omega_g y + \omega_g^2 x)\ \mathrm{d}t + \ \mathrm{d}W_t, \end{cases} \tag{8.24}$$

where $\{W_t\}_t$ is a standard Wiener process.

The Kanai-Tajimi model can also be written as a system of random ordinary differential equations, taking the form

$$\begin{cases} \dot{z}_1 = -z_2 + O_t, \\ \dot{z}_2 = -2\zeta_g\omega_g(z_2 + O_t) + \omega_g^2 z_1 + O_t, \end{cases} \tag{8.25}$$

where $\{O_t\}_t$ is an Orstein-Uhlenbeck process satisfying

$$\mathrm{d}O_t = -O_t\ \mathrm{d}t + \mathrm{d}W_t, \tag{8.26}$$

which can be solved exactly.

8.6. **Earthquake and other impulse driven models.** Now we consider a simple structural model driven by an earthquake-type random disturbance in the form of a transport process, inspired by the model in [5] (see also [18, Chapter 18] with this and other models). There are a number of models for earthquake-type forcing, such as the Kanai-Tajima and the Clough-Penzien. We chose the Bogdanoff-Goldberg-Bernard model due to the different type of noise, so we can illustrate the improved convergence in the case of a transport process noise.

8.7. **A random Fisher-KPP nonlinear PDE driven by boundary noise.** [15]

Acknowledgments

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