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CONDITIONS FOR THE STRONG ORDER 1 CONVERGENCE OF THE EULER-MARUYAMA APPROXIMATION FOR RANDOM ODES

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ABSTRACT. It is well known that the Euler-Maruyama method of approximating a random ordinary differential equation $\mathrm{d}X_t/\mathrm{d}t = f(t, X_t, Y_t)$ driven by a stochastic process $\{Y_t\}_t$ with θ-Hölder sample paths is estimated to be of strong order θ with respect to the time step, provided $f = f(t, x, w)$ is sufficiently regular. Here, we show that, in common situations, it is possible to exploit "hidden" conditions and prove that the strong convergence is actually of order 1, even if the sample paths are still θ-Hölder continuous. This includes the case of an Itô process noise with finite mean drift and diffusion (which includes a Wiener, or an Orstein-Uhlenbeck, or a Geometric Brownian process) and the case when f has a separable homogeneous part .... The order 1 convergence follows from not estimating directly the local error, but, instead, adding up the local steps and estimating the compound error. In the case of an Itô noise, the compound error is then estimated via Itô formula and the Itô isometry. We complement the result by giving examples where some of the conditions are not met and the order of convergence seems indeed to be less than 1.

1. Introduction

Consider the following initial value problem for a **random ordinary differential equation (RODE)**:

$$\begin{cases} \dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t), \qquad 0 \leq t \leq T, \\ X_t|_{t=0} = X_0, \end{cases} \tag{1.1}$$

where the noise $\{Y_t\}_{t\in I}$ is a real stochastic process with continuous sample paths on the time interval $I = [0, T]$; the evolution law function $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous; and the initial condition X_0 is a real random variable. The sample space is denoted by Ω. We also treat systems of random ordinary equations, as discussed later in the article, but we start with the scalar case, in order to present the main ideas.

The Euler-Maruyama method for solving this initial value problem on the time interval $I = [0, T]$ consists in approximating the solution on a uniform time mesh

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$t_j = j\Delta t$, $j = 0, \ldots, N$, with fixed time step $\Delta t = T/N$, for a given $N \in \mathbb{N}$. In such a mesh, the Euler-Maruyama scheme takes the form

$$X_{t_j}^N = X_{t_{j-1}}^N + \Delta t f(t_{j-1}, X_{t_{j-1}}^N, Y_{t_{j-1}}), \qquad j = 1, \ldots, N, \tag{1.2}$$

with the initial condition

$$X_0^N = X_0. \tag{1.3}$$

Notice both $\Delta t = \Delta t_N = T/N$ and $t_j = t_j^N = j\Delta t_N = jT/N$ depend on N, but we sometimes do not make this dependency explicit, for the sake of notational simplicity.

When the noise $\{Y_t\}_{t\in I}$ has θ-Hölder continuous sample paths, it can be show, under suitable conditions on $f = f(t,x,y)$, that the Euler-Maruyama scheme converges strongly with order θ with the time step, i.e. there exists $C > 0$ such that

$$\mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq C\Delta t_N^\theta, \qquad \forall N \in \mathbb{N}, \tag{1.4}$$

where $\mathbb{E}[\cdot]$ indicates the expectation of a random variable on Ω (see []).

Our aim is to show that, in many classical examples, it is possible to exploit further "hidden" conditions that yield in fact a strong order 1 convergence, even when the sample paths are still Hölder continuous. This is the case, for instance, when the noise is an Itô noise. Another case is when the equation is linear and the Hölder-continuous noise is only present in the non-homogeneous term.

For the linear case, we assume, more precisely, that the equation is of the form

$$\begin{cases} \dfrac{\mathrm{d}X_t}{\mathrm{d}t} = g(t)X_t + h(t, Y_t), \qquad 0 \leq t \leq T, \\ X_t|_{t=0} = X_0. \end{cases} \tag{1.5}$$

For the Itô noise case, we consider a general equation of the form (1.1),

$$\begin{cases} \dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t), \qquad 0 \leq t \leq T, \\ X_t|_{t=0} = X_0, \end{cases} \tag{1.6}$$

with a noise defined as an **Itô process** $\{Y_t\}_{t\geq 0}$, satisfying

$$\mathrm{d}Y_t = A_t \,\mathrm{d}t + B_t \,\mathrm{d}W_t, \tag{1.7}$$

where $\{W_t\}_{t\geq 0}$ is a Wiener process and $\{A_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$ are stochastic processes adapted to the $\{W_t\}_{t\geq 0}$. We are not solving for Y_t, otherwise we would actually have a system of stochastic differential equations. Instead, we assume it is a known process, and we allow A_t and B_t to actually be given in terms of $\{W_t\}_{t\geq 0}$ and $\{Y_t\}_{t\geq 0}$. For example, Y_t may be an Orstein-Uhlenbeck process or a geometric Brownian process.

In the case that $f = f(t, x, y)$ is twice continuously differentiable, the Itô formula is applicable and yields

$$\mathrm{d}f(t,x,Y_t) = \left(\partial_t f(t,x,Y_t) + A_t \partial_y f(t,x,Y_t) + \frac{B_t^2}{2}\partial_{yy} f(t,x,Y_t)\right)\,\mathrm{d}t + B_t \partial_y f(t,x,Y_t)\,\mathrm{d}W_t, \quad (1.8)$$

for every fixed $x \in \mathbb{R}$.

We show that, if the expectations of $\{A_t\}_t$ and $\{B_t\}_t$ are uniformly bounded in time on $[0,T]$ and $\partial_t f$, $\partial_x f$, $\partial_y f$, and $\partial_{yy} f$ are uniformly bounded on $[0,T] \times \mathbb{R} \times \mathbb{R}$, then the Euler-Maruyama method is of strong order 1, i.e. there exists $C > 0$ such that

$$\mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq C\Delta t, \qquad \forall N \in \mathbb{N}, \Delta t = \frac{T}{N}, \quad (1.9)$$

where $\mathbb{E}[\cdot]$ indicates the expectation of a random variable on Ω (see Theorem 6.1).

We summarize here the main tricks we use to accomplish such error estimate:

(i) We assume the noise is an Itô process, so we can use the Itô isometry at some point;
(ii) We use the Itô formula to separate the most problematic/rough part of the noise;
(iii) We do not estimate this problematic term locally at each time step;
(iv) Instead, we add up the difference equation for the time steps and write the error in terms of a time integral of this rough part of the noise;
(v) We then use the Itô isometry to estimate this integral term by Δt;

In order to make the main idea clear cut, here are the options we have for estimating the rough part of the noise:

(i) If the local error e_j of the rough part of the noise, at the jth time step, is bounded as
$$\mathbb{E}[|e_j|] \lesssim \Delta t^{3/2},$$
as usual for a 1/2-Hölder noise, then adding them up leads to
$$\sum \mathbb{E}[|e_j|] \lesssim N\Delta t^{3/2} = T\Delta t^{1/2}.$$
(ii) If we use the Itô isometry locally, we still get the local error as
$$\mathbb{E}[|e_j|] \leq \mathbb{E}[|e_j|^2]^{1/2} \lesssim \left(\Delta t^{2(3/2)}\right)^{1/2} = \Delta t^{3/2},$$
and adding that up still leads to an error of order Δt^{θ}.
(iii) If, instead, we first add the terms up, then $\sum e_j$ becomes an integral over $[0,T]$ with respect to the Wiener noise, so that we can use the Itô isometry

on the added up term and obtain

$$\mathbb{E}\left[\left|\sum e_j\right|\right] \lesssim \left(\mathbb{E}\left[\left|\sum e_j\right|^2\right]\right)^{1/2} = \left(\sum \mathbb{E}[|e_j|^2]\right)^{1/2}$$
$$= \left(\sum \Delta t^3\right)^{1/2} = \left(\Delta t^2\right)^{1/2} = \Delta t.$$

and we finally get the error to be of order 1.

2. Integral formula for the global pathwise error

In this section, we derive the following integral formula for the global error:

Lemma 2.1. *Suppose $f = f(t, x, y)$ is continuous on $[0, T] \times \mathbb{R} \times \mathbb{R}$. Then, the Euler-Maruyama approximation (1.2) for any pathwise solution of the random ordinary differential equation (1.1) satisfies the global error formula*

$$\begin{aligned} X_{t_j} - X_{t_j}^N = X_0 - X_0^N & \\ & + \int_0^{t_j} \left(f(s, X_s, Y_s) - f(s, X_{\tau^N(s)}, Y_s)\right) \;\mathrm{d}s \\ & + \int_0^{t_j} \left(f(s, X_{\tau^N(s)}, Y_s) - f(s, X_{\tau^N(s)}^N, Y_s)\right) \;\mathrm{d}s \\ & + \int_0^{t_j} \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})\right) \;\mathrm{d}s, \end{aligned} \tag{2.1}$$

for $j = 1, \ldots, N$, where τ^N is the piecewise constant jump function along the time mesh:

$$\tau^N(t) = \max_j \{j\Delta t_N;\ j\Delta t_N \leq t\} = \left[\frac{t}{\Delta t_N}\right] \Delta t_N = \left[\frac{tN}{T}\right] \frac{T}{N}. \tag{2.2}$$

Proof. First, we obtain an expression for a single time step, from time t_{j-1} to $t_j = t_{j-1} + \Delta t$. For notational simplicity, we momentarily write $t = t_{j-1}$ and $\tau = \Delta t$, so that $t_j = t + \tau$. The exact pathwise solution satisfies

$$X_{t+\tau} = X_t + \int_t^{t+\tau} f(s, X_s, Y_s) \;\mathrm{d}s.$$

The Euler-Maruyama step is given by

$$X_{t+\tau}^N = X_t^N + \tau f(t, X_t^N, Y_t).$$

Subtracting, we obtain

$$X_{t+\tau} - X_{t+\tau}^N = X_t - X_t^N + \int_t^{t+\tau} \left(f(s, X_s, Y_s) - f(t, X_t^N, Y_t)\right) \;\mathrm{d}s.$$

We arrange the integrand as

$$\begin{aligned}f(s,X_s,Y_s)-f(t,X_t^N,Y_t)=&f(s,X_s,Y_s)-f(s,X_t,Y_s)\\&+f(s,X_t,Y_s)-f(s,X_t^N,Y_s)\\&+f(s,X_t^N,Y_s)-f(t,X_t^N,Y_t).\end{aligned}$$

This yields

$$\begin{aligned}X_{t+\tau}-X_{t+\tau}^N=&X_t-X_t^N\\&=\int_t^{t+\tau}\left(f(s,X_s,Y_s)-f(s,X_t,Y_s)\right)\,\mathrm{d}s\\&\quad+\int_t^{t+\tau}\Big(f(s,X_t,Y_s)-f(s,X_t^N,Y_s)\Big)\,\mathrm{d}s\\&\quad+\int_t^{t+\tau}\Big(f(s,X_t^N,Y_s)-f(t,X_t^N,Y_t)\Big)\,\mathrm{d}s.\end{aligned}$$

Going back to the notation $t=t_{j-1}$ and $t+\tau=t_j$, the above identity reads

$$\begin{aligned}X_{t_j}-X_{t_j}^N=&X_{t_{j-1}}-X_{t_{j-1}}^N\\&=\int_{t_{j-1}}^{t_j}\Big(f(s,X_s,Y_s)-f(s,X_{t_{j-1}},Y_s)\Big)\,\mathrm{d}s\\&\quad+\int_{t_{j-1}}^{t_j}\Big(f(s,X_{t_{j-1}},Y_s)-f(s,X_{t_{j-1}}^N,Y_s)\Big)\,\mathrm{d}s\\&\quad+\int_{t_{j-1}}^{t_j}\Big(f(s,X_{t_{j-1}}^N,Y_s)-f(t_{j-1},X_{t_{j-1}}^N,Y_{t_{j-1}})\Big)\,\mathrm{d}s.\end{aligned}\tag{2.3}$$

Now we iterate the time steps (2.3) to find that

$$\begin{aligned}X_{t_j}-X_{t_j}^N=&X_0-X_0^N\\&+\sum_{i=1}^{j}\left(\int_{t_{i-1}}^{t_i}\left(f(s,X_s,Y_s)-f(s,X_{t_i},Y_s)\right)\,\mathrm{d}s\right.\\&+\int_{t_{i-1}}^{t_i}\Big(f(s,X_{t_{i-1}},Y_s)-f(s,X_{t_{i-1}}^N,Y_s)\Big)\,\mathrm{d}s\\&\left.+\int_{t_{i-1}}^{t_i}\Big(f(s,X_{t_{i-1}}^N,Y_s)-f(t_{i-1},X_{t_{i-1}}^N,Y_{t_{i-1}})\Big)\,\mathrm{d}s\right).\end{aligned}$$

Using the jump function τ^N, we may rewrite the above expression as in (2.1). □

3. Basic estimate

Here we derive an estimate, under minimal hypotheses, that will be the basis for the estimates in specific cases.

Lemma 3.1. *Suppose $f = f(t,x,y)$ is continuous on $I \times \mathbb{R} \times \mathbb{R}$ and is uniformly globally Lipschitz continuous on x, i.e. there exists a constant $L_x \geq 0$ such that*

$$|f(t,x_1,y) - f(t,x_2,y)| \leq L_x|x_1 - x_2|, \qquad \forall t \in [0,T],\ \forall x_1,x_2,y \in \mathbb{R}. \tag{3.1}$$

Then, the global error (2.1) is estimated as

$$\begin{aligned} |X_{t_j} - X_{t_j}^N| \leq \Bigg(&|X_0 - X_0^N| + L_x \int_0^{t_j} |X_s - X_{\tau^N(s)}| \,\mathrm{d}s \\ &\left|\int_0^{t_j} \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)}) \right) \,\mathrm{d}s\right| \Bigg) e^{L_x t_j}. \end{aligned} \tag{3.2}$$

for $j = 1, \ldots, N$, where τ^N is given by (2.2).

Proof. We estimate the first two integrals in (2.1). For the first one, we use (3.1), so that

$$|f(s,X_s,Y_s) - f(s,X_t,Y_s)| \leq L_x|X_s - X_t|,$$

for $t, s \in [0,T]$, and, in particular, for $t = \tau^N(s)$. Hence,

$$\left|\int_0^{t_j} \left(f(s,X_s,Y_s) - f(s,X_{\tau^N(s)},Y_s) \right) \,\mathrm{d}s\right| \leq L_x \int_0^{t_j} |X_s - X_{\tau^N(s)}| \,\mathrm{d}s.$$

For the second term, we use again (6.2), so that

$$|f(s,X_t,Y_s) - f(s,X_t^N,Y_s)| \leq L_x|X_t - X_t^N|,$$

again for any $t, s \in [0,T]$, and, in particular, for $t = \tau^N(s)$. Hence,

$$\begin{aligned} \left|\int_0^{t_j} \left(f(s,X_{\tau^N(s)},Y_s) - f(s,X_{\tau^N(s)}^N,Y_s) \right) \,\mathrm{d}s\right| &\leq L_x \int_0^{t_j} |X_{\tau^N(s)} - X_{\tau^N(s)}^N| \,\mathrm{d}s \\ &\leq L_x \sum_{i=0}^{j-1} |X_{t_i} - X_{t_i}^N| \Delta t. \end{aligned}$$

With these two estimates, we bound (2.1) as

$$\begin{aligned} |X_{t_j} - X_{t_j}^N| \leq |X_0 - X_0^N| & \\ &+ L_x \int_0^{t_j} |X_s - X_{\tau^N(s)}| \,\mathrm{d}s \\ &+ L_x \sum_{i=0}^{j-1} |X_{t_i} - X_{t_i}^N| \Delta t \\ &+ \left|\int_0^{t_j} \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)}) \right) \,\mathrm{d}s\right|. \end{aligned}$$

Using the discrete version of the Gronwall Lemma, we prove (3.2). $\square$

4. The case of an Itô noise

Here, we assume noise $\{Y_t\}_{t\in I}$ is defined as an Itô process, i.e. satisfying (1.7), and we assume $\{A_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$ satisfy

$$\mathbb{E}[|A_t|] \leq M_A, \quad \mathbb{E}[|B_t|] \leq M_B, \qquad \forall t \in [0,T]. \tag{4.1}$$

As mentioned in the Introduction, $\{A_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$ may depend on Y_t itself, such as in an Orstein-Uhlenbeck process or a geometric Brownian motion process.

We also assume that $f = f(t,x,y)$ is twice continuously differentiable, so the Itô formula (1.8) is applicable.

5. The case of a separable equation

Here, the noise $\{Y_t\}_{t\in I}$ is *not* assumed to be an Itô noise, and, instead, we assume the term $f(t,x,y)$ can be written as

$$f(t,x,y) = g_1(t,y)h(t,x) + g_2(t,y)$$

In particular, this applies to linear equations.

We assume h is uniformly globally Lipschitz in both x and t.

6. Strong order of convergence

We assume $f = f(t,x,y)$ is twice continuously differentiable with

$$L_t = \sup_{t,x,y} |\partial_t f(t,x,y)| < \infty \tag{6.1}$$

$$L_x = \sup_{t,x,y} |\partial_x f(t,x,y)| < \infty \tag{6.2}$$

$$L_y = \sup_{t,x,y} |\partial_y f(t,x,y)| < \infty \tag{6.3}$$

$$L_{yy} = \sup_{t,x,y} |\partial_y^2 f(t,x,y)| < \infty, \tag{6.4}$$

where the suprema are taken for $(t,x,y) \in [0,T]\times\mathbb{R}\times\mathbb{R}$. The first three condition (6.1), (6.2), and (6.3) imply that f has an at most linear growth:

$$\sup_{t,x,y} |f(t,x,y)| \leq M_0 + L(|t| + |x| + |y|), \tag{6.5}$$

for suitable nonnegative constants M_0, L.

We also assume the drift and diffusion of the Itô process $\{Y_t\}_t$ are uniformly bounded,

$$M_A = \sup_{\omega} \sup_{t,x,y} |A_t(\omega)| < \infty, \tag{6.6}$$

$$M_B = \sup_{\omega} \sup_{t,x,y} |B_t(\omega)| < \infty, \tag{6.7}$$

where the suprema are taken for $t \in [0,T]$ and for samples in all sample space $\omega \in \Omega$.

6.1. **A single step.** Here we obtain an expression for a single time step which will be suitable for a proper estimate later on. For the sake of notational simplicity, we consider a single time step from a time t to a time $t+\tau$. Later on we take $t = t_{j-1}$ and $\tau = \Delta t$, with $t_j = t_{j-1} + \Delta t$.

The exact solution satisfies, for any $t, \tau \geq 0$,

$$X_{t+\tau} = X_t + \int_t^{t+\tau} f(s, X_s, Y_s)\;\mathrm{d}s.$$

The Euler-Maruyama step is given by

$$X_{t+\tau}^N = X_t^N + \tau f(t, X_t^N, Y_t).$$

Subtracting, we obtain

$$X_{t+\tau} - X_{t+\tau}^N = X_t - X_t^N + \int_t^{t+\tau} \big(f(s, X_s, Y_s) - f(t, X_t^N, Y_t) \big)\;\mathrm{d}s.$$

We arrange the integrand as

$$\begin{aligned} f(s, X_s, Y_s) - f(t, X_t^N, Y_t) = & f(s, X_s, Y_s) - f(s, X_t, Y_s) \\ & + f(s, X_t, Y_s) - f(s, X_t^N, Y_s) \\ & + f(s, X_t^N, Y_s) - f(t, X_t^N, Y_t). \end{aligned}$$

This yields

$$\begin{aligned} X_{t+\tau} - X_{t+\tau}^N = & X_t - X_t^N \\ & = \int_t^{t+\tau} \left(f(s, X_s, Y_s) - f(s, X_t, Y_s) \right)\;\mathrm{d}s \\ & \quad + \int_t^{t+\tau} \big(f(s, X_t, Y_s) - f(s, X_t^N, Y_s) \big)\;\mathrm{d}s \\ & \quad + \int_t^{t+\tau} \big(f(s, X_t^N, Y_s) - f(t, X_t^N, Y_t) \big)\;\mathrm{d}s. \end{aligned}$$

For the integral of the last pair of terms, we use the Itô formula on $Z_s = f(s, X_t^N, Y_s)$ and write

$$\begin{aligned} \int_t^{t+\tau} \big(f(s, X_t^N, Y_s) - f(t, X_t^N, Y_t) \big)\;\mathrm{d}s = \int_t^{t+\tau} \int_t^s \mathrm{d}Z_\xi\;\mathrm{d}s \\ = \int_t^{t+\tau} \int_t^s \left(\partial_\xi f(\xi, X_t^N, Y_\xi) + A_\xi \partial_y f(\xi, X_t^N, Y_\xi) + \frac{B_\xi^2}{2} \partial_{yy} f(\xi, X_t^N, Y_\xi) \right)\;\mathrm{d}s\;\mathrm{d}t \\ + \int_t^{t+\tau} \int_t^s B_\xi \partial_y f(\xi, X_t^N, Y_\xi)\;\mathrm{d}W_\xi\;\mathrm{d}s. \end{aligned}$$

Using Fubini's Theorem, the last integral is rewritten as

$$\begin{aligned}\int_t^{t+\tau}\int_t^s B_\xi \partial_y f(\xi, X_t^N, Y_\xi)\,\mathrm{d}W_\xi\,\mathrm{d}s &= \int_t^{t+\tau}\int_\xi^{t+\tau} B_\xi \partial_y f(\xi, X_t^N, Y_\xi)\,\mathrm{d}s\,\mathrm{d}W_\xi \\ &= \int_t^{t+\tau} (t+\tau-\xi) B_\xi \partial_y f(\xi, X_t^N, Y_\xi)\,\mathrm{d}W_\xi. \end{aligned} \tag{6.8}$$

We rearrange these terms and write, for $\tau = \Delta t$ and $t = t_{j-1} = (j-1)\Delta t$,

$$X_{t_j} - X_{t_j}^N = X_{t_{j-1}} - X_{t_{j-1}}^N + I_{j-1}^1 + I_{j-1}^2 + I_{j-1}^3, \tag{6.9}$$

where

$$I_j^1 = \int_{t_j}^{t_{j+1}} \Big(f(s, X_{t_j}, Y_s) - f(s, X_{t_j}^N, Y_s) \Big)\,\mathrm{d}s,$$

$$\begin{aligned} I_j^2 = &\int_{t_j}^{t_{j+1}} \big(f(s, X_s, Y_s) - f(s, X_{t_j}, Y_s) \big)\,\mathrm{d}s \\ &+ \int_{t_j}^{t_{j+1}}\int_{t_j}^s \left(\partial_\xi f(\xi, X_{t_j}^N, Y_\xi) + A_\xi \partial_y f(\xi, X_{t_j}^N, Y_\xi) + \frac{B_\xi^2}{2}\partial_{yy} f(\xi, X_{t_j}^N, Y_\xi) \right)\,\mathrm{d}t, \end{aligned}$$

and

$$I_j^3 = \int_{t_j}^{t_{j+1}} (t_{j+1} - \xi) B_\xi \partial_y f(\xi, X_{t_j}^N, Y_\xi)\,\mathrm{d}W_\xi.$$

6.2. **Local estimates.** The term I_j^1 is estimated using that $f = f(t, x, y)$ is globally Lipschitz in x, so that

$$\big| f(s, X_t, Y_s) - f(s, X_t^N, Y_s) \big| \le L_x |X_t - X_t^N|.$$

Hence,

$$\begin{aligned}\left| \int_{t_j}^{t_{j+1}} \Big(f(s, X_{t_j}, Y_s) - f(s, X_{t_j}^N, Y_s) \Big)\,\mathrm{d}s \right| &\le \int_{t_j}^{t_{j+1}} \Big| f(s, X_{t_j}, Y_s) - f(s, X_{t_j}^N, Y_s) \Big|\,\mathrm{d}s \\ &\le L_x |X_{t_j} - X_{t_j}^N| \Delta t. \end{aligned}$$

This means

$$\big| I_j^1 \big| \le L_x |X_{t_j} - X_{t_j}^N| \Delta t. \tag{6.10}$$

For I_j^2, the first term is estimated as

$$|f(s, X_s, Y_s) - f(s, X_t, Y_s)| \le L_x |X_s - X_t| \le L_x \int_t^s |f(\sigma, X_\sigma, Y_\sigma)|\,\mathrm{d}\sigma \le L_x M_f (s - t).$$

This yields, upon integration,

$$\left|\int_{t_j}^{t_{j+1}} \big(f(s, X_s, Y_s) - f(s, X_{t_j}, Y_s)\big) \;\mathrm{d}s\right| \leq \int_{t_j}^{t_{j+1}} \big|f(s, X_s, Y_s) - f(s, X_{t_j}, Y_s)\big| \;\mathrm{d}s$$
$$\leq \frac{L_x M_f}{2}\Delta t^2.$$

The double integral is estimated as

$$\left|\int_{t_j}^{t_{j+1}}\int_{\xi}^{t_{j+1}} \left(\partial_\xi f(\xi, X_{t_j}^N, Y_\xi) + A_\xi \partial_y f(\xi, X_{t_j}^N, Y_\xi) + \frac{B_\xi^2}{2}\partial_{yy} f(\xi, X_{t_j}^N, Y_\xi)\right) \;\mathrm{d}t\right|$$
$$\leq \int_{t_j}^{t_{j+1}}\int_{\xi}^{t_{j+1}} \left(L_t + M_A L_y + \frac{M_B^2}{2}L_{yy}\right) \;\mathrm{d}t$$
$$= \frac{1}{2}\tau^2\left(L_t + M_A L_y + \frac{M_B^2}{2}L_{yy}\right). \quad (6.11)$$

Hence,

$$\left|I_j^2\right| \leq M\Delta t^2, \quad (6.12)$$

where

$$M = \frac{1}{2}\left(L_x M_f + L_t + M_A L_y + \frac{M_B^2}{2}L_{yy}\right).$$

Remark 6.1. Notice that, at this point, we did not estimate the last integral, otherwise we are not able to obtain the strong order 1 estimate, only 1/2. Indeed, if we use Fubini and the Itô isometry in the last integral, we find

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_t^{t+\tau}\int_t^s B_\xi \partial_y f(\xi, X_t^N, Y_\xi)\;\mathrm{d}W_\xi\;\mathrm{d}s\right)^2\right] = \mathbb{E}\left[\left(\int_t^{t+\tau}\int_\xi^{t+\tau} B_\xi \partial_y f(\xi, X_t^N, Y_\xi)\;\mathrm{d}s\;\mathrm{d}W_\xi\right)^2\right] \\
&= \int_t^{t+\tau}\mathbb{E}\left[\left(\int_\xi^{t+\tau} B_\xi \partial_y f(\xi, X_t^N, Y_\xi)\;\mathrm{d}s\right)^2\right]\;\mathrm{d}\xi \leq \int_t^{t+\tau}\left(\int_\xi^{t+\tau} M_B^2 L_y\;\mathrm{d}s\right)^2\;\mathrm{d}\xi \\
&\leq \int_t^{t+\tau} M_B^2 L_y (t+\tau-\xi)^2\;\mathrm{d}\xi = -\frac{1}{3}M_B^2 L_y^2 (t+\tau-\xi)^3\Big]_t^{t+\tau} = \frac{1}{3}M_B^2 L_y^2 \tau^3,
\end{aligned}$$

so that

$$\sqrt{\mathbb{E}\left[\left(\int_t^{t+\tau}\int_t^s B_\xi \partial_y f(\xi, X_t^N, Y_\xi)\;\mathrm{d}W_\xi\;\mathrm{d}s\right)^2\right]} \leq \frac{\sqrt{3}}{3}M_B L_y \tau^{3/2}. \quad (6.13)$$

After adding up n times, we end up with a $\tau^{1/2}$ estimate, which is not sufficient.

6.3. **Integral estimate.** The third term I_j^3 is not estimated for each j separately. Instead, we estimate its summation over j. Notice

$$\sum_{i=0}^{j-1} I_i^3 = \sum_{i=0}^{j-1}\int_{t_i}^{t_{i+1}} (t_{i+1} - \xi) B_\xi \partial_y f(\xi, X_{t_i}^N, Y_\xi)\;\mathrm{d}W_\xi$$
$$= \int_0^{t_j} \left([\xi/\Delta t + 1]\Delta t - \xi\right) B_\xi \partial_y f(\xi, X_{[\xi/\Delta t]\Delta t}^N, Y_\xi)\;\mathrm{d}W_\xi,$$

where $[r]$ denotes the largest integer below a real number r.

For this term, we estimate its strong norm, i.e. first moment. This is estimated using the Lyapunov inequality, the Itô formula and the Itô isometry, as follows

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^{t_j} ([\xi/\Delta t + 1]\Delta t - \xi)\, B_\xi \partial_y f(\xi, X^N_{[\xi/\Delta t]\Delta t}, Y_\xi)\, \mathrm{d}W_\xi\right|\right] \\
&\quad \leq \mathbb{E}\left[\left(\int_0^{t_j} ([\xi/\Delta t + 1]\Delta t - \xi)\, B_\xi \partial_y f(\xi, X^N_{[\xi/\Delta t]\Delta t}, Y_\xi)\, \mathrm{d}W_\xi\right)^2\right]^{1/2} \\
&\quad = \left(\int_0^{t_j} \mathbb{E}\left[\left(([\xi/\Delta t + 1]\Delta t - \xi)\, B_\xi \partial_y f(\xi, X^N_{[\xi/\Delta t]\Delta t}, Y_\xi)\right)^2\right]\, \mathrm{d}\xi\right)^{1/2} \\
&\quad \leq \left(\int_0^{t_j} \left(([\xi/\Delta t + 1]\Delta t - \xi)^2\, \mathbb{E}\left[\left(B_\xi \partial_y f(\xi, X^N_{[\xi/\Delta t]\Delta t}, Y_\xi)\right)^2\right]\right)\, \mathrm{d}\xi\right)^{1/2} \\
&\quad \leq \left(\int_0^{t_j} \Delta t^2 M_B^2 L_y^2\, \mathrm{d}\xi\right)^{1/2}.
\end{aligned}$$

Thus,

$$\mathbb{E}\left[\left|\sum_{i=0}^{j-1} I_j^3\right|\right] \leq M_B L_y t_j^{1/2} \Delta t. \tag{6.14}$$

6.4. **Iterating the steps.** Iterating (6.9) and assuming that $X_0^N = X_0$, we find

$$X_{t_j} - X_{t_j}^N = \sum_{i=0}^{j-1} I_j^1 + \sum_{i=0}^{j-1} I_j^2 + \sum_{i=0}^{j-1} I_j^3. \tag{6.15}$$

We estimate the first moment as

$$\mathbb{E}\left[|X_{t_j} - X_{t_j}^N|\right] \leq \sum_{i=0}^{j-1} \mathbb{E}\left[|I_j^1|\right] + \sum_{i=0}^{j-1} \mathbb{E}\left[|I_j^2|\right] + \mathbb{E}\left[\left|\sum_{i=0}^{j-1} I_j^3\right|\right]. \tag{6.16}$$

Using (6.10), (6.12), and (6.14), we obtain

$$\begin{aligned}
\mathbb{E}\left[|X_{t_j} - X_{t_j}^N|\right] &\leq L_x \sum_{i=0}^{j-1} \mathbb{E}\left[|X_{t_j} - X_{t_j}^N|\right] \Delta t + \sum_{i=0}^{j-1} C\Delta t^2 + M_B L_y t_j \Delta t \\
&\leq L_x \sum_{i=0}^{j-1} \mathbb{E}\left[|X_{t_i} - X_{t_i}^N|\right] \Delta t + C_T \Delta t, \qquad (6.17)
\end{aligned}$$

where

$$C_T = M + M_B L_y T^{1/2}.$$

Now, we show by induction that

$$\mathbb{E}\left[|X_{t_j} - X_{t_j}^N|\right] \leq C_T e^{L_x t_j} \Delta t.$$

This is trivially true for $j = 0$. Now suppose it is true up to $j-1$. It follows from (6.17) that

$$\mathbb{E}\left[|X_{t_j} - X_{t_j}^N|\right] \leq L_x \sum_{i=0}^{j-1} C_T \Delta t e^{L_x t_i} \Delta t + C_T \Delta t = C_T \Delta t \left(1 + L_x \Delta t \sum_{i=0}^{j-1} e^{L_x t_i}\right).$$

Using that $1 + r \leq e^r$, with $r = L_x \Delta t$ and $t_i + \Delta t = t_{i+1}$, we see that

$$L_x \Delta t \leq e^{L_x \Delta t} - 1,$$

which telescopes the sum and yields

$$\mathbb{E}\left[|X_{t_j} - X_{t_j}^N|\right] \leq C_T \Delta t \left(1 + (e^{L_x \Delta t} - 1) \sum_{i=0}^{j-1} e^{L_x t_i}\right) = C_T \Delta t \left(1 + (e^{L_x j \Delta t} - 1)\right).$$

Hence,

$$\mathbb{E}\left[|X_{t_j} - X_{t_j}^N|\right] \leq C_T e^{L_x t_j} \Delta t,$$

which completes the induction. Hence, we have proved the following result.

Theorem 6.1. *Consider the initial value problem (1.1), on a time interval* $[0, T]$, *with* $T > 0$, *and assume the noise is given by (1.7), with (6.6) and (6.7). Suppose* $f = f(t, x, y)$ *is twice continuously differentiable, with (6.5)-(6.4). Let* $\{X_t\}_{t \geq 0}$ *be the solution of (1.1). Let* $N \in \mathbb{N}$ *and let* $\{X_{t_j}^N\}_{j=0,\ldots,N}$ *be the solution of the Euler-Maruyama method (1.2)-(1.3). Then,*

$$\mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq C_T e^{L_x t_j} \Delta t, \qquad j = 0, \ldots, N, \ \forall N \in \mathbb{N}, \Delta t = \frac{T}{N}, \tag{6.18}$$

where

$$C_T = \frac{1}{2}\left(L_x M_f + L_t + M_A L_y + \frac{M_B^2}{2} L_{yy}\right) + M_B L_y T^{1/2}. \tag{6.19}$$

We end this section by abstracting away the Gronwall type inequality we use (this is probably written somewhere, and I need to find the source):

Lemma 6.1. *Let* $(e_j)_j$ *be a (finite or infinite) sequence of positive numbers satisfying*

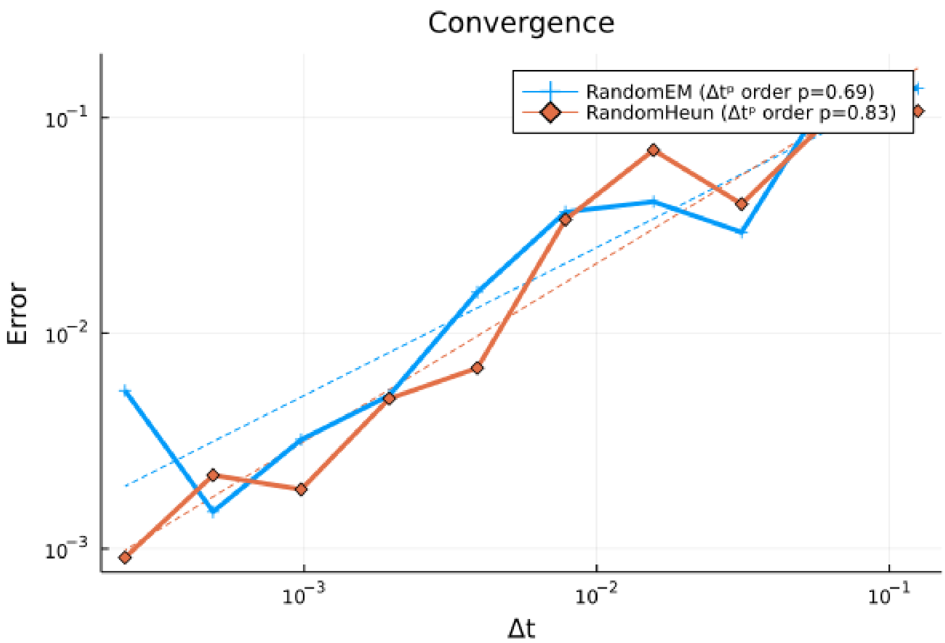
$$e_j \leq a \sum_{i=0}^{j-1} e_i + b, \tag{6.20}$$

with $e_0 = 0$, *where* $a, b > 0$. *Then,*

$$e_j \leq b e^{aj}, \qquad \forall j. \tag{6.21}$$

Proof. The result is trivially true for $j = 0$. Suppose, by induction, that the result is true up to $j-1$. Then,

$$e_j \leq a \sum_{i=0}^{j-1} b e^{ai} + b = b\left(a \sum_{i=0}^{j-1} e^{ai} + 1\right).$$



Using that $1 + a \leq e^a$, we have $a \leq e^a - 1$, hence

$$e_j \leq b \left((e^a - 1) \sum_{i=0}^{j-1} e^{ia} + 1 \right).$$

Using that $\sum_{i=0}^{j-1} \theta^i = (\theta^j - 1)(\theta - 1)$, with $\theta = e^a$, we see that

$$(e^a - 1) \sum_{i=0}^{j-1} e^{ia} \leq e^{ja} - 1,$$

so that

$$e_j \leq b e^{ja},$$

which completes the induction. □

7. Special cases

7.1. **Non-homogeneous term of bounded variation.** Consider a RODE of the form

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = g(t, Y_t, X_t) + h(t, Y_t),$$

where g is globally Lipschitz and $t \mapsto h(t, Y_t)$ is of bounded variation.

8. Numerical examples

8.1. **Lower-order converge.** For a lower order convergence, below order 1, we take the noise $\{Y_t\}_t$ to be the transport process defined by

$$Y_t = \sin(t/Z)^{1/3},$$

where Z is a beta random variable $Z \sim B(\alpha, \beta)$. Notice Z takes values strictly within $(0,1)$ and, hence, $\sin(t/Z)$ can have arbitrarily high frequencies and, hence, go through the critic value $y = 0$ extremely often.

(Need to remove the Heun method and do more tests).

9. Estimate on the solution

We assume $f = f(t,x,y)$ is continuous in all variables and is Lipschitz continuous in each variable, i.e. there exist constants $L_t, L_x, L_y \geq 0$ such that

$$|f(t_1,x,y) - f(t_2,x,y)| \leq L_t|t_1 - t_2|, \tag{9.1}$$

$$|f(t,x_1,y) - f(t,x_2,y)| \leq L_x|x_1 - x_2|, \tag{9.2}$$

$$|f(t,x,y_1) - f(t,x,y_2)| \leq L_y|y_1 - y_2|, \tag{9.3}$$

for all $t, t_1, t_2 \in I$, $x, x_1, x_2 \in \mathbb{R}$ and $y, y_1, y_2 \in \mathbb{R}$. By the continuity of $f = f(t,x,y)$, we also have

$$M_0 = \sup_{t\in I} |f(t,0,0)| < \infty.$$

These conditions imply that f has an at most linear growth in x and y:

$$|f(t,x,y)| \leq M_0 + L_x|x| + L_y|y|, \tag{9.4}$$

for every $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}$.

We assume the initial condition has a bounded first moment:

$$\mathbb{E}[|X_0|] \leq C_0 < \infty. \tag{9.5}$$

As for the noise, we assume, for now, that

$$\mathbb{E}[|Y_t|] \leq M_Y, \qquad \forall t \in [0,T]. \tag{9.6}$$

With the assumed regularity on $f = f(t,x,y)$, the solutions of (1.1) are pathwise solutions, so that

$$X_t = X_0 + \int_0^t f(s, X_s, Y_s)\;\mathrm{d}s.$$

Using (9.4), we estimate each solution with

$$|X_t| \leq |X_0| + \int_0^t (M_0 + L_x|X_s| + L_y|Y_s|)\;\mathrm{d}s.$$

Using Gronwall's lemma, we find

$$|X_t| \leq \left(|X_0| + M_0 t + L_y \int_0^t |Y_s|\;\mathrm{d}s\right) e^{L_x t}, \quad t \in [0,T]. \tag{9.7}$$

In particular, taking the expectation,

$$\mathbb{E}[|X_t|] \leq \left(\mathbb{E}[|X_0|] + M_0 t + L_y \int_0^t \mathbb{E}[|Y_s|]\ \mathrm{d}s\right) e^{L_x t}, \quad t \in [0, T].$$

Using hypotheses (9.5) and (9.6), we find that

$$\mathbb{E}[|X_t|] \leq (C_0 + (M_0 + L_y M_Y) t)\, e^{L_x t}, \quad t \in [0, T].$$

hence,

$$\mathbb{E}[|X_t|] \leq M_X, \qquad t \in [0, T], \tag{9.8}$$

with

$$M_X = (C_0 + (M_0 + L_y M_Y) T) e^{L_x T}. \tag{9.9}$$

Similarly, we write, for $t \geq t_0 > 0$,

$$X_t - X_{t_0} = \int_{t_0}^t f(s, X_s, Y_s)\ \mathrm{d}s.$$

Using (9.4), we estimate

$$\begin{aligned} |X_t - X_{t_0}| &\leq \int_{t_0}^t (M_0 + L_x |X_s| + L_y |Y_s|)\ \mathrm{d}s \\ &\leq L_x \int_{t_0}^t |X_s|\ \mathrm{d}s + L_y \int_{t_0}^t |Y_s|\ \mathrm{d}s + M_0 (t - t_0). \end{aligned}$$

Using (9.7), we obtain

$$|X_t - X_{t_0}| \leq L_x \int_{t_0}^t \left(|X_0| + M_0 s + L_y \int_0^s |Y_\sigma|\ \mathrm{d}\sigma\right) e^{L_x s}\ \mathrm{d}s + L_y \int_{t_0}^t |Y_s|\ \mathrm{d}s + M_0 (t - t_0) \tag{9.10}$$

Appendix

The heart of the matter is the following. Think of τ as the time-step Δt, but we use τ for simplicity. Let $y = y(t)$ be a θ-Hölder continuous function, with Hölder constant C. Then, we can do the usual "local"-type estimate

$$\begin{aligned} \left|\int_0^T (y(t + \tau) - y(t))\ \mathrm{d}t\right| &\leq \int_0^T |y(t + \tau) - y(t)|\ \mathrm{d}t \\ &\leq C \int_0^T \tau^\theta\ \mathrm{d}t \\ &= C \tau^\theta T, \end{aligned}$$

which yields an order θ approximation, with respect to the "time step" τ.

However, we can also integrate first, so that

$$
\begin{aligned}
\left|\int_0^T (y(t+\tau) - y(t))\ \mathrm{d}t\right| &= \left|\int_0^T y(t+\tau)\ \mathrm{d}t - \int_0^T y(t)\ \mathrm{d}t\right| \\
&= \left|\int_\tau^{T+\tau} y(t)\ \mathrm{d}t - \int_0^T y(t)\ \mathrm{d}t\right| \\
&= \left|\int_T^{T+\tau} y(t)\ \mathrm{d}t - \int_0^\tau y(t)\ \mathrm{d}t\right| \\
&= \left| y(T)\tau + \int_T^{T+\tau} (y(t) - y(T))\ \mathrm{d}t \right. \\
&\qquad \left. - y(0)\tau - \int_0^\tau (y(t) - y(0))\ \mathrm{d}t \right| \\
&\leq |y(T) - y(0)|\tau + C\int_T^{T+\tau} |t-T|^\theta\ \mathrm{d}t + |y(0)|\tau + \int_0^\tau t^\theta\ \mathrm{d}t \\
&\leq |y(T) - y(0)|\tau + \frac{C}{1+\theta}\tau^{1+\theta} + \frac{C}{1+\theta}\tau^{1+\theta} \\
&\leq CT^\theta\tau + \frac{2C}{1+\theta}\tau^{1+\theta}.
\end{aligned}
$$

Hence,

$$
\left|\int_0^T (y(t+\tau) - y(t))\ \mathrm{d}t\right| \leq C\tau\left(T^\theta + \frac{2}{1+\theta}\tau^\theta\right),
$$

which reveals the order 1 convergence.

Well, but, actually, we don't have $y(t+\tau) - y(t)$ in the integrand. We have $y(t) - y(\tau^N(t))$, where $\tau^N(t)$ picks the largest $j\tau$ smaller than or equal to t, i.e. $\tau^N(t) = \max\{j\tau;\ j\tau \leq t, j\}$. Then we need to estimate

$$
\left|\int_0^T (y(t) - y(\tau^m(t)))\ \mathrm{d}t\right|.
$$

In this case,

$$
\begin{aligned}
\left|\int_0^T (y(t) - y(\tau^m(t)))\ \mathrm{d}t\right| &\leq \left(\int_0^T |y(t) - y(\tau^m(t))|^{1/\theta}\ \mathrm{d}t\right)^\theta \left(\int_0^T 1^{1/(1-\theta)}\ \mathrm{d}t\right)^{1-\theta} \\
&\leq \left(\int_0^T \left|C\tau^\theta\right|^{1/\theta}\ \mathrm{d}t\right)^\theta T^{1-\theta} \\
&\leq CT^\theta\tau^\theta T^{1-\theta}.
\end{aligned}
$$

Thus,

$$
\left|\int_0^T (y(t) - y(\tau^m(t)))\ \mathrm{d}t\right| \leq CT\tau^\theta. \tag{9.11}
$$

Ops, we should find another way. And even if we find a way, keep in mind that what we really have is a term $t \mapsto y(t, x(t))$ and what we need to estimate is

$$\int_0^T \big(y(t, x(\tau^N(t)) - y(\tau^N(t), x(\tau^N(t)))\big)\;\mathrm{d}t.$$

Suppose f has a separable homogeneous part, i.e.

$$f(t,x) = g(t)h(x) + g_0(t),$$

and that g and g_0 are the differences between two nondecreasing functions:

$$g(t) = g_u(t) - g_d(t), \qquad g_0(t) = g_{0u}(t) - g_{0d}(t).$$

For simplicity, we will consider $g_0 = 0$, but a nonzero g_0 can be handled in a similar way, as discussed below.

So, the idea is to write the global error as

$$\begin{aligned}
&\int_0^T \big(f(t, x(\tau^N(t)) - f(\tau^N(t), x(\tau^N(t)))\big)\;\mathrm{d}t \\
&\quad = \int_0^T \big(g_u(t)h(x(\tau^N(t))) - g_u(\tau^N(t))h(x(\tau^N(t)))\big)\;\mathrm{d}t \\
&\qquad - \int_0^T \big(g_d(t)h(x(\tau^N(t))) - g_d(\tau^N(t))h(x(\tau^N(t)))\big)\;\mathrm{d}t \\
&\quad = \int_0^T (g_u(t) - g_u(\tau^N(t)))h(x(\tau^N(t)))\;\mathrm{d}t \\
&\qquad - \int_0^T (g_d(t) - g_d(\tau^N(t)))h(x(\tau^N(t)))\;\mathrm{d}t
\end{aligned}$$

and estimate the integral in each g_i, $i = u, d$, by

$$\left|\int_0^T (g_i(t) - g_i(\tau^N(t)))h(x(\tau^N(t)))\;\mathrm{d}t\right| \leq \sup_{t\in[0,T]}\{|h(x(\tau^N(t))|\}\int_0^T (g_i(t) - g_i(\tau^N(t)))\;\mathrm{d}t.$$

Now, the remaining integral we split at time $t_1 = \Delta t$ and use the monotonicity:

$$\begin{aligned}
\int_0^T (g_i(t) - g_i(\tau^N(t)))\;\mathrm{d}t &= \int_0^T g_i(t)\;\mathrm{d}t - \int_0^{t_1} g_i(\tau^N(t))\;\mathrm{d}t - \int_{t_1}^T g_i(\tau^N(t))\;\mathrm{d}t \\
&\leq \int_0^T g_i(t)\;\mathrm{d}t - \int_0^{\Delta t} g_i(0)\;\mathrm{d}t - \int_{\Delta t}^T g_i(t - \Delta t)\;\mathrm{d}t \\
&= \int_0^T g_i(t)\;\mathrm{d}t - \int_0^{\Delta t} g_i(0)\;\mathrm{d}t - \int_0^{T-\Delta t} g_i(t)\;\mathrm{d}t \\
&= \int_{T-\Delta t}^T g_i(t)\;\mathrm{d}t - g_i(0)\Delta t \\
&\leq (g_i(T) - g_i(0))\Delta t.
\end{aligned}$$

Therefore,

$$\left|\int_0^T \big(f(t, x(\tau^N(t)) - f(\tau^N(t), x(\tau^N(t)))\big)\ \mathrm{d}t\right| \\ \leq (g_u(T) - g_u(0) + g_d(T) - g_d(0)) \sup_{t\in[0,T]} \{|h(x(\tau^N(t))|\}\Delta t. \quad (9.12)$$

But we are not quite there. This is sufficient for an ODE, but not for a RODE, when the time dependency includes the noise, $f = f(t, x, y)$, i.e. we actually need to handle $t \mapsto f(t, X_{\tau^N(t)}, Y_t) = g(t, Y_t)h(X_{\tau^N(t)})$. In this case, the decomposition $g = g_u - g_d$ depends on $\{Y_t\}_{t\in I}$ in a nontrivial way.

We may, nevertheless, assume that, for each sample path, $t \mapsto g^Y(t) = g(t, Y_t(\omega))$ itself decomposes into a difference between two nondecreasing functions and then we assume they have bounded first moments:

$$\mathbb{E}\left[(g_i^Y(T) - g_i^Y(0))\right] < \infty.$$

But in order for this to be meaningful, we need to show that some reasonable/interesting examples satisfy these conditions, e.g. we should try $g(t, Y_t) = Y_t$ with some transport noise $Y_t = (\sin(\Omega t))^{1/3}$ or a process with Takagi-Landsberg sample paths with varing Hurst coefficients, or whatever.

Acknowledgments

References

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