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CONDITIONS FOR THE STRONG ORDER 1 CONVERGENCE OF THE EULER-MARUYAMA APPROXIMATION FOR RANDOM ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. It is well known that the Euler-Maruyama method of approximating a random ordinary differential equation $\mathrm{d}X_t/\mathrm{d}t = f(t, X_t, Y_t)$ driven by a stochastic process $\{Y_t\}_t$ with θ-Hölder sample paths is estimated to be of strong order θ with respect to the time step, provided $f = f(t, x, w)$ is sufficiently regular. Here, we show that, in common situations, it is possible to exploit "hidden" conditions on the noise and prove that the strong convergence is actually of order 1, regardless of much regularity on the sample paths. This applies to Itô process noises (such as Wiener, Ornstein-Uhlenbeck, and Geometric Brownian process), which are Hölder continuous, and to point processes (such as Poisson point processes and Hawkes self-exciting processes), which are not even continuous and have jump-type discontinuities, as well as to transport processes. The order 1 convergence follows from not estimating directly the local error, but, instead, adding up the local steps and estimating the compound error. In the case of an Itô noise, the compound error is then estimated via Itô formula and the Itô isometry. In the case of a point process or a transport process, a monotonic bound is exploited. We HOPEFULLY complement the result by giving examples where some of the conditions are not met and the order of convergence seems indeed to be less than 1.

1. Introduction

Consider the following initial value problem for a **random ordinary differential equation (RODE)**:

$$
\begin{cases} \dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t), \qquad 0 \leq t \leq T, \\ X_t|_{t=0} = X_0, \end{cases} \tag{1.1}
$$

on a time interval $I = [0, T]$, with $T > 0$, and where the noise $\{Y_t\}_{t\in I}$ is a given stochastic process. The sample space is denoted by Ω.

The Euler-Maruyama method for solving this initial value problem consists in approximating the solution on a uniform time mesh $t_j = j\Delta t_N$, $j = 0, \ldots, N$, with fixed time step $\Delta t_N = T/N$, for a given $N \in \mathbb{N}$. In such a mesh, the Euler-Maruyama

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scheme takes the form

$$X_{t_j}^N = X_{t_{j-1}}^N + \Delta t_N f(t_{j-1}, X_{t_{j-1}}^N, Y_{t_{j-1}}), \qquad j = 1, \ldots, N, \tag{1.2}$$

with the initial condition

$$X_0^N = X_0. \tag{1.3}$$

Notice $t_j = j\Delta t_N = jT/N$ also depends on N, but we do not make this dependency explicit, for the sake of notational simplicity.

When the noise $\{Y_t\}_{t\in I}$ has θ-Hölder continuous sample paths, it can be show [2], under further suitable conditions, that the Euler-Maruyama scheme converges strongly with order θ with the time step, i.e. there exists a constant $C \geq 0$ such that

$$\max_{j=0,\ldots,N} \mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq C\Delta t_N^\theta, \qquad \forall N \in \mathbb{N}, \tag{1.4}$$

where $\mathbb{E}[\cdot]$ indicates the expectation of a random variable on Ω.

Our aim is to show that, in many classical examples, it is possible to exploit further "hidden" conditions that yield in fact a strong order 1 convergence, even when the sample paths are still Hölder continuous or have jump discontinuities. This is the case, for instance, when the noise is an Itô noise, or when the equation is semi-separable and the noise is a point process or a transport process.

More precisely, for the semi-separable case, we assume f is of the form

$$f(t, x, y) = a(t, y)h(x) + b(t, y).$$

In this case, we assume the processes $\{a(t, Y_t)\}_{t\in I}$ and $\{b(t, Y_t)\}_{t\in I}$ have their steps bounded by a monotonic process, which typically happens for point processes, i.e.

$$|a(t+\tau, Y_{t+\tau}) - a(t, Y_t)| \leq A_t, \quad |b(t+\tau, Y_{t+\tau}) - b(t, Y_t)| \leq B_t,$$

where $\{A_t\}_{t\in I}$ and $\{B_t\}_{t\in I}$ have monotonically non-decreasing sample paths. Under further suitable conditions (see Corollary 5.1), we show that the Euler-Maruyama method is of strong order 1, i.e. (1.4) holds with $\theta = 1$.

Even if the structure of the equation is not exactly semi-separable but it is somehow possible to bound the steps $|f(t+\tau, X_t, Y_{t+\tau}) - a(t, X_t, Y_t)|$ by a suitable process with monotonic non-decreasing sample paths, it is possible to prove the strong order 1 convergence (see Theorem 5.1).

For the Itô noise case, we consider a general equation of the form (1.1), with a noise defined as an **Itô process** $\{Y_t\}_{t\geq 0}$, satisfying

$$\mathrm{d}Y_t = A_t \,\mathrm{d}t + B_t \,\mathrm{d}W_t, \tag{1.5}$$

We are not solving for Y_t, nor approximating it numerically, otherwise we would actually need to consider a system of stochastic differential equations. Instead, we assume it is a known process that can be computed analytically, such as a Wiener process, an Ornstein-Uhlenbeck process, or a geometric Brownian motion. With those in mind, A_t and B_t may be originally given in terms of $\{W_t\}_{t\geq 0}$ and $\{Y_t\}_{t\in I}$, but the general assumption is only given in terms of $\{A_t\}_{t\in I}$ and $\{B_t\}_{t\in I}$.

In the case that $f = f(t, x, y)$ is twice continuously differentiable, the Itô formula applies and we show, under suitable conditions on $\{A_t\}_{t\in I}$, $\{B_t\}_{t\in I}$ and the derivatives of f, that the Euler-Maruyama method is of strong order 1, i.e. (1.4) holds with $\theta = 1$.

In order to make the main idea clear cut, here are the options we have for estimating the error:

(i) If the local error e_j, at the jth time step, is bounded as

$$\mathbb{E}[|e_j|] \lesssim \Delta t_N^{3/2},$$

as usual for a 1/2-Hölder noise, then adding them up leads to

$$\sum \mathbb{E}[|e_j|] \lesssim N\Delta t_N^{3/2} = T\Delta t_N^{1/2}.$$

(ii) If we use the Itô isometry locally, we still get the local error as

$$\mathbb{E}[|e_j|] \leq \mathbb{E}[|e_j|^2]^{1/2} \lesssim \left(\Delta t_N^{2(3/2)}\right)^{1/2} = \Delta t_N^{3/2},$$

and adding that up still leads to an error of order Δt_N^{θ}.

(iii) If, instead, we first add the terms up, then $\sum e_j$ becomes an integral over $[0, T]$ with respect to the Wiener noise, so that we can use the Itô isometry on the added up term and obtain

$$\begin{aligned}\mathbb{E}\left[\left|\sum e_j\right|\right] \lesssim \left(\mathbb{E}\left[\left|\sum e_j\right|^2\right]\right)^{1/2} &= \left(\sum \mathbb{E}[|e_j|^2]\right)^{1/2} \\ &= \left(\sum \Delta t_N^3\right)^{1/2} = \left(\Delta t_N^2\right)^{1/2} = \Delta t_N.\end{aligned}$$

and we finally get the error to be of order 1.

2. Pathwise solution

For the notion and main results on pathwise solution for RODEs, we refer the reader to [3, Section 2.1]. We consider two sets of hypotheses, namely Hypothesis 2.1 and Hypothesis 2.2, each suitable to one of the two main cases we consider, namely the case in which the steps are bounded by processes with monotonic expected values and the case with Itô type noises.

We start with the following hypotheses, which imply the existence and uniqueness of pathwise solutions of the RODE (1.1) in the sense of Carathéodory.

Hypothesis 2.1. *We consider a function $f = f(t, x, y)$ defined on $I \times \mathbb{R} \times \mathbb{R}$ and a real-valued stochastic process $\{Y_t\}_{t\in I}$, where $I = [0, T]$, $T > 0$. We make the following standing hypotheses.*

(i) *f is globally Lipschitz continuous on x, uniformly in t and y, i.e. there exists a constant $L_X \geq 0$ such that*

$$|f(t, x_1, y) - f(t, x_2, y)| \leq L_X|x_1 - x_2|, \quad \forall t \in [0, T],\ \forall x_1, x_2, y \in \mathbb{R}. \tag{2.1}$$

(ii) *We also assume that $(t,x) \mapsto f(t,x,Y_t)$ satisfies the Carathéodory conditions:*
  (a) *The mapping $x \mapsto f(t,x,y)$ is continuous on $x \in \mathbb{R}$, for almost every $(t,y) \in I \times \mathbb{R}$;*
  (b) *The mapping $t \mapsto f(t,x,Y_t)$ is Lebesgue measurable in $t \in [0,T]$, for each $x \in \mathbb{R}$ and each sample path $t \mapsto Y_t(\omega)$;*
  (c) *The bound $|f(t,x,Y_t)| \leq M_t + L_X|x|$ holds for all $t \in I$ and all $x \in \mathbb{R}$, where $\{M_t\}_{t\in I}$ is a real stochastic process with Lebesgue integrable sample paths $t \mapsto M_t(\omega)$ on $t \in [0,T]$.*

Under these assumptions, for each sample value $\omega \in \Omega$, the integral equation

$$X_t = X_0 + \int_0^t f(s, X_s, Y_s)\,\mathrm{d}s \tag{2.2}$$

has a unique solution, in the Lebesgue sense, for the realization $X_0 = X_0(\omega)$ of the initial condition and the sample path $t \mapsto Y_t(\omega)$ of the noise process (see [1, Theorem 1.1]). Moreover, the mapping $(t,\omega) \mapsto X_t(\omega)$ is measurable (see [3, Section 2.1.2]) and, hence, give rise to a well-defined stochastic process $\{X_t\}_{t\in I}$.

Each sample path solution $t \mapsto X_t(\omega)$ is bounded by

$$|X_t| \leq \left(|X_0| + \int_0^t M_s\,\mathrm{d}s\right) e^{L_X t}, \quad \forall t \in I. \tag{2.3}$$

For the strong convergence of the Euler-Maruyama approximation, we also need to control the expectation of the solution above, among other things. With that in mind, we have the following useful result.

Lemma 2.1. *Under the Hypothesis 2.1, suppose further that*

$$\mathbb{E}[|X_0|] < \infty \tag{2.4}$$

and

$$\int_0^T \mathbb{E}[|M_s|]\,\mathrm{d}s < \infty \tag{2.5}$$

Then,

$$\mathbb{E}[|X_t|] \leq \left(\mathbb{E}[|X_0|] + \int_0^t \mathbb{E}[|M_s|]\,\mathrm{d}s\right) e^{L_X t}, \quad t \in I. \tag{2.6}$$

Proof. Thanks to (2.3), the result is straightfoward □

In special *dissipative* cases, depending on the structure of the equation, we might not need the second condition and only require $\mathbb{E}[|X_0|] < \infty$, but we do not exploit this case here.

When $f = f(t,x,y)$ is continuous on all three variables, as well as uniformly globally Lipscthiz continuous in x, and the sample paths of $\{Y_t\}_{t\geq 0}$ are continuous, then the integrand in (2.2) is continuous in t and the integral becomes a Riemann integral. In this case, the integral form (2.2) of the pathwise solutions of (1.1) holds in the Riemann sense.

This setting is assumed in the analysis of the case of Itô noise. With that in mind, we define, more precisely, a second set of hypotheses, on top of the first one above.

Hypothesis 2.2. *We consider a function $f = f(t, x, y)$ defined on $I \times \mathbb{R} \times \mathbb{R}$ and a real-valued stochastic process $\{Y_t\}_{t\in I}$, where $I = [0, T]$, $T > 0$. Besides the Hypothesis 2.1, we also assume*

(i) *$f = f(t, x, y)$ is continuous on $I \times \mathbb{R} \times \mathbb{R}$.*
(ii) *For each $x \in \mathbb{R}$, the map $(t, y) \mapsto f(t, x, y)$ is twice continuously differentiable on $I \times \mathbb{R}$.*
(iii) *The sample paths $t \mapsto Y_t(\omega)$ of the noise process are continuous on I.*
(iv) *The noise process $\{Y_t\}_{t\in I}$ is an Itô noise in the sense of satisfying*
$$\mathrm{d}Y_t = A_t\,\mathrm{d}t + B_t\,\mathrm{d}W_t, \tag{2.7}$$
where $\{W_t\}_{t\geq 0}$ is a standard Wiener process and $\{A_t\}_{t\in I}$ and $\{B_t\}_{t\in I}$ are stochastic processes adapted to $\{W_t\}_{t\geq 0}$.

3. Integral formula for the global pathwise error

In this section, we derive the following integral formula for the global error:

Lemma 3.1. *Under the Hypothesis 2.1, the Euler-Maruyama approximation (1.2) for any pathwise solution of the random ordinary differential equation (1.1) satisfies the global error formula*
$$\begin{aligned} X_{t_j} - X_{t_j}^N = X_0 - X_0^N& \\ &+ \int_0^{t_j} \left(f(s, X_s, Y_s) - f(s, X_{\tau^N(s)}, Y_s)\right)\,\mathrm{d}s \\ &+ \int_0^{t_j} \left(f(s, X_{\tau^N(s)}, Y_s) - f(s, X_{\tau^N(s)}^N, Y_s)\right)\,\mathrm{d}s \\ &+ \int_0^{t_j} \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})\right)\,\mathrm{d}s, \end{aligned} \tag{3.1}$$
for $j = 1, \ldots, N$, where τ^N is the piecewise constant jump function along the time mesh:
$$\tau^N(t) = \max_j\{j\Delta t_N;\ j\Delta t_N \leq t\} = \left[\frac{t}{\Delta t_N}\right]\Delta t_N = \left[\frac{tN}{T}\right]\frac{T}{N}. \tag{3.2}$$

Proof. Under the Hypothesis 2.1, the solutions of (1.1) are pathwise solutions in the Lebesgue sense of (2.2). With that in mind, we first obtain an expression for a single time step, from time t_{j-1} to $t_j = t_{j-1} + \Delta t_N$.

For notational simplicity, we momentarily write $t = t_{j-1}$ and $\tau = \Delta t_N$, so that $t_j = t + \tau$. The exact pathwise solution satisfies
$$X_{t+\tau} = X_t + \int_t^{t+\tau} f(s, X_s, Y_s)\,\mathrm{d}s.$$

The Euler-Maruyama step is given by

$$X_{t+\tau}^N = X_t^N + \tau f(t, X_t^N, Y_t).$$

Subtracting, we obtain

$$X_{t+\tau} - X_{t+\tau}^N = X_t - X_t^N + \int_t^{t+\tau} \left(f(s, X_s, Y_s) - f(t, X_t^N, Y_t)\right) \,\mathrm{d}s.$$

We arrange the integrand as

$$\begin{aligned} f(s, X_s, Y_s) - f(t, X_t^N, Y_t) =& f(s, X_s, Y_s) - f(s, X_t, Y_s) \\ &+ f(s, X_t, Y_s) - f(s, X_t^N, Y_s) \\ &+ f(s, X_t^N, Y_s) - f(t, X_t^N, Y_t). \end{aligned}$$

This yields

$$\begin{aligned} X_{t+\tau} - X_{t+\tau}^N =& X_t - X_t^N \\ &= \int_t^{t+\tau} (f(s, X_s, Y_s) - f(s, X_t, Y_s)) \,\mathrm{d}s \\ &\quad + \int_t^{t+\tau} \left(f(s, X_t, Y_s) - f(s, X_t^N, Y_s)\right) \,\mathrm{d}s \\ &\quad + \int_t^{t+\tau} \left(f(s, X_t^N, Y_s) - f(t, X_t^N, Y_t)\right) \,\mathrm{d}s. \end{aligned}$$

Going back to the notation $t = t_{j-1}$ and $t + \tau = t_j$, the above identity reads

$$\begin{aligned} X_{t_j} - X_{t_j}^N =& X_{t_{j-1}} - X_{t_{j-1}}^N \\ &= \int_{t_{j-1}}^{t_j} \left(f(s, X_s, Y_s) - f(s, X_{t_{j-1}}, Y_s)\right) \,\mathrm{d}s \\ &\quad + \int_{t_{j-1}}^{t_j} \left(f(s, X_{t_{j-1}}, Y_s) - f(s, X_{t_{j-1}}^N, Y_s)\right) \,\mathrm{d}s \\ &\quad + \int_{t_{j-1}}^{t_j} \left(f(s, X_{t_{j-1}}^N, Y_s) - f(t_{j-1}, X_{t_{j-1}}^N, Y_{t_{j-1}})\right) \,\mathrm{d}s. \end{aligned} \tag{3.3}$$

Now we iterate the time steps (3.3) to find that

$$
\begin{aligned}
X_{t_j} - X_{t_j}^N = X_0 - X_0^N & \\
& + \sum_{i=1}^{j} \left(\int_{t_{i-1}}^{t_i} \left(f(s, X_s, Y_s) - f(s, X_{t_i}, Y_s)\right) \,\mathrm{d}s \right. \\
& + \int_{t_{i-1}}^{t_i} \Big(f(s, X_{t_{i-1}}, Y_s) - f(s, X_{t_{i-1}}^N, Y_s) \Big) \,\mathrm{d}s \\
& \left. + \int_{t_{i-1}}^{t_i} \Big(f(s, X_{t_{i-1}}^N, Y_s) - f(t_{i-1}, X_{t_{i-1}}^N, Y_{t_{i-1}}) \Big) \,\mathrm{d}s \right).
\end{aligned}
$$

Using the jump function τ^N, we may rewrite the above expression as in (3.1). □

Remark 3.1. Strictly speaking, we only need condition (ii) from Hypothesis 2.1 in order to deduce (4.1), but since we need (i) for the strong convergence anyways, it is simpler to state the result as in Lemma 4.1.

4. Basic estimate for the global pathwise error

Here we derive an estimate, under minimal hypotheses, that will be the basis for the estimates in specific cases.

Lemma 4.1. *Under the Hypothesis 2.1, the global error (3.1) is estimated as*

$$
\begin{aligned}
|X_{t_j} - X_{t_j}^N| \leq \Bigg(& |X_0 - X_0^N| + L_X \int_0^{t_j} |X_s - X_{\tau^N(s)}| \,\mathrm{d}s \\
& \left| \int_0^{t_j} \Big(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)}) \Big) \,\mathrm{d}s \right| \Bigg) e^{L_X t_j}.
\end{aligned}
\tag{4.1}
$$

for $j = 1, \ldots, N$, where τ^N is given by (3.2).

Proof. We estimate the first two integrals in (3.1). For the first one, we use (2.1), so that

$$
|f(s, X_s, Y_s) - f(s, X_t, Y_s)| \leq L_X |X_s - X_t|,
$$

for $t, s \in [0, T]$, and, in particular, for $t = \tau^N(s)$. Hence,

$$
\left| \int_0^{t_j} \left(f(s, X_s, Y_s) - f(s, X_{\tau^N(s)}, Y_s) \right) \,\mathrm{d}s \right| \leq L_X \int_0^{t_j} |X_s - X_{\tau^N(s)}| \,\mathrm{d}s.
$$

For the second term, we use again (2.1), so that

$$
|f(s, X_t, Y_s) - f(s, X_t^N, Y_s)| \leq L_X |X_t - X_t^N|,
$$

again for any $t, s \in [0, T]$, and, in particular, for $t = \tau^N(s)$. Hence,

$$\begin{aligned}\left|\int_0^{t_j} \left(f(s, X_{\tau^N(s)}, Y_s) - f(s, X_{\tau^N(s)}^N, Y_s)\right) \,\mathrm{d}s\right| &\leq L_X \int_0^{t_j} |X_{\tau^N(s)} - X_{\tau^N(s)}^N| \,\mathrm{d}s \\ &\leq L_X \sum_{i=0}^{j-1} |X_{t_i} - X_{t_i}^N| \Delta t_N.\end{aligned}$$

With these two estimates, we bound (3.1) as

$$\begin{aligned}|X_{t_j} - X_{t_j}^N| \leq |X_0 - X_0^N| & \\ &+ L_X \int_0^{t_j} |X_s - X_{\tau^N(s)}| \,\mathrm{d}s \\ &+ L_X \sum_{i=0}^{j-1} |X_{t_i} - X_{t_i}^N| \Delta t_N \\ &+ \left|\int_0^{t_j} \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})\right) \,\mathrm{d}s\right|.\end{aligned}$$

Using the discrete version of the Gronwall Lemma, we prove (4.1). □

The first term in the right hand side of (4.1) usually vanishes since in general we take $X_0^N = X_0$, but it suffices to assume that X_0^N approximates X_0 to order Δt_N, which is useful for lower order approximations or for the discretization of (random) partial differential equations.

The third term in (4.1) is the more delicate one that will be handled differently in the next sections.

As for the second term, which only concerns the solution itself, not the approximation, we use the following simple but useful general result.

Lemma 4.2. *Under the Hypothesis 2.1, it follows that*

$$\int_0^{t_j} \left|X_s - X_{\tau^N(s)}\right| \,\mathrm{d}s \leq \Delta t_N \int_0^{t_j} (M_s + L_X |X_s|) \,\mathrm{d}s. \tag{4.2}$$

Proof. By assumption, we have $|f(t, X_t, Y_t)| \leq M_t + L_X |X_t|$, for all $t \in I$ and all sample paths. Thus,

$$\left|X_s - X_{\tau^N(s)}\right| = \left|\int_{\tau^N(s)}^s f(\xi, X_\xi, Y_\xi) \,\mathrm{d}\xi\right| \leq \int_{\tau^N(s)}^s (M_\xi + L_X |X_\xi|) \,\mathrm{d}\xi.$$

Integrating over $[0, t_j]$ and using integration by parts

$$\begin{aligned}\int_0^{t_j} \left|X_s - X_{\tau^N(s)}\right| \mathrm{d}s &\le \int_0^{t_j} \int_{\tau^N(s)}^s (M_\xi + L_X|X_\xi|) \,\mathrm{d}\xi \,\mathrm{d}s \\ &= \int_0^{t_j} \int_\xi^{\tau^N(\xi)+\Delta t_N} (M_\xi + L_X|X_\xi|) \,\mathrm{d}s \,\mathrm{d}\xi \\ &= \int_0^{t_j} (\tau^N(\xi) + \Delta t_N - \xi)(M_\xi + L_X|X_\xi|) \,\mathrm{d}\xi.\end{aligned}$$

Using that $\tau^N(\xi) \le \xi$ and that the remaining terms are non-negative, we have $\tau^N(\xi) + \Delta t_N - \xi \le \Delta t_N$ and we obtain exactly (4.2)

Taking the expectation of both sides of (4.2) and using (2.3) and (**??**) we find (4.7). □

Combining the two previous results we obtain

Proposition 4.1. *Under the Hypothesis 2.1, suppose further that*

(i) *the initial condition is strongly bounded, i.e.*

$$\mathbb{E}[|X_0|] < \infty; \tag{4.3}$$

(ii) *the non-negative process $\{M_t\}_{t\in I}$ is strongly integrable, i.e.*

$$\int_0^T \mathbb{E}[M_s] \,\mathrm{d}s < \infty; \tag{4.4}$$

(iii) *for some constant $C_0 \ge 0$,*

$$\mathbb{E}[|X_0 - X_0^N|] \le C_0 \Delta t_N, \qquad N \in \mathbb{N}. \tag{4.5}$$

Then

$$\begin{aligned}&\mathbb{E}\left[|X_{t_j} - X_{t_j}^N|\right] \\ &\quad\le \Bigg(C_0\Delta t_N + \Delta t_N L_X \left(\mathbb{E}[|X_0|] + \int_0^{t_j} \mathbb{E}[M_\xi] \,\mathrm{d}\xi\right) e^{L_X t_j} \\ &\qquad \mathbb{E}\left[\left|\int_0^{t_j} \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})\right) \,\mathrm{d}s\right|\right]\Bigg) e^{L_X t_j}.\end{aligned} \tag{4.6}$$

Proof. Under Hypothesis 2.1, Lemma 4.2 applies and estimate (4.2) holds. Using (4.3) and (4.4), that estimate yields

$$\int_0^{t_j} \mathbb{E}[|X_s - X_{\tau^N(s)}|] \,\mathrm{d}s \le \Delta t_N \int_0^{t_j} (\mathbb{E}[M_s] + L_X \mathbb{E}[|X_s|]) \,\mathrm{d}s.$$

Using now (2.3), we obtain

$$
\begin{aligned}
\int_0^{t_j} & \mathbb{E}[|X_s - X_{\tau^N(s)}|] \;\mathrm{d}s \\
& \leq \Delta t_N \int_0^{t_j} \left(\mathbb{E}[M_s] + L_X \left(\mathbb{E}[|X_0|] + \int_0^s \mathbb{E}[M_\xi] \;\mathrm{d}\xi \right) e^{L_X s} \right) \;\mathrm{d}s \\
& \leq \Delta t_N \left(\int_0^{t_j} \mathbb{E}[M_s] \;\mathrm{d}s + L_X \int_0^{t_j} \left(\mathbb{E}[|X_0|] + \int_0^{t_j} \mathbb{E}[M_\xi] \;\mathrm{d}\xi \right) e^{L_X s} \;\mathrm{d}s \right) \\
& = \Delta t_N \left(\int_0^{t_j} \mathbb{E}[M_s] \;\mathrm{d}s + \left(\mathbb{E}[|X_0|] + \int_0^{t_j} \mathbb{E}[M_\xi] \;\mathrm{d}\xi \right) \left(e^{L_X t_j} - 1 \right) \right).
\end{aligned}
$$

Thus,

$$
\int_0^{t_j} \mathbb{E}[|X_s - X_{\tau^N(s)}|] \;\mathrm{d}s \leq \Delta t_N \left(\mathbb{E}[|X_0|] + \int_0^{t_j} \mathbb{E}[M_\xi] \;\mathrm{d}\xi \right) e^{L_X t_j}. \tag{4.7}
$$

Now we turn our attention to Lemma 4.1. Taking the expectation of the global error formula (4.1) gives

$$
\begin{aligned}
\mathbb{E}\left[|X_{t_j} - X_{t_j}^N| \right] \leq \Bigg(& \mathbb{E}\left[|X_0 - X_0^N| \right] + L_X \int_0^{t_j} \mathbb{E}\left[|X_s - X_{\tau^N(s)}| \right] \;\mathrm{d}s \\
& \mathbb{E}\left[\left| \int_0^{t_j} \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)}) \right) \;\mathrm{d}s \right| \right] \Bigg) e^{L_X t_j}.
\end{aligned}
$$

Using now estimate (4.7) and condition (4.5), we find (4.6), which completes the proof. □

5. The case of monotonic sample path bounds

Here, the noise $\{Y_t\}_{t\in I}$ is *not* assumed to be an Itô noise and f is not assumed to be differentiable, but, instead, that the steps can be controlled by monotonic nondecreasing processes with finite expected growth. This fits well with the typical case of point processes, such as renewal-reward processes, Hawkes process, and such.

More precisely, we have the following result:

Lemma 5.1. *Besides the Hypothesis 2.1, suppose that, for all $0 \leq \tau \leq s \leq T$,*

$$
\mathbb{E}\left[|f(s, X_\tau, Y_s) - f(\tau, X_\tau, Y_\tau)| \right] \leq e(s) - e(\tau), \tag{5.1}
$$

where $e = e(s)$ is a monotonically non-decreasing function in $s \in I$. Then,

$$
\mathbb{E}\left[\left| \int_0^t \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)}) \right) \;\mathrm{d}s \right| \right] \leq (e(t) - e(0))\Delta t_N, \tag{5.2}
$$

for all $0 \leq t \leq T$ and every $N \in \mathbb{R}$.

Proof. Let $N \in \mathbb{R}$. From the assumption (5.1) we have

$$\mathbb{E}\left[|f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})|\right] \leq e(s) - e(\tau^N(s)),$$

for every $0 \leq s \leq T$. Thus, upon integration,

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t \left(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})\right) \, \mathrm{d}s\right|\right] \\
&\qquad \leq \int_0^t \mathbb{E}\left[|f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})\right] \, \mathrm{d}s \\
&\qquad \leq \int_0^t \left(e(s) - e(\tau^N(s))\right) \, \mathrm{d}s.
\end{aligned}$$

Now we need to bound the right hand side. When $0 \leq t \leq t_1 = \Delta t_N$, we have $\tau^N(s) = 0$ for all $0 \leq s < t_1$, so that,

$$\int_0^t (e(s) - e(\tau^N(s))) \, \mathrm{d}s = \int_0^t (e(s) - e(0)) \, \mathrm{d}s.$$

Using the monotonicity,

$$\int_0^t (e(s) - e(\tau^N(s))) \, \mathrm{d}s \leq \int_0^t (e(t) - e(0)) \, \mathrm{d}s = (e(t) - e(0))t \leq (e(t) - e(0))\Delta t_N.$$

When $\Delta t_N \leq t \leq T$, we split the integration of the second term at time $s = t_1 = \Delta t_N$ and write

$$\int_0^t (e(s) - e(\tau^N(s))) \, \mathrm{d}s = \int_0^t e(s) \, \mathrm{d}s - \int_0^{t_1} e(\tau^N(s)) \, \mathrm{d}s - \int_{t_1}^t e(\tau^N(s)) \, \mathrm{d}s$$

Using the monotonicity together with the fact that $s - \Delta t_N \leq \tau^N(s) \leq s$ for all $\Delta t_N \leq s \leq T$,

$$\begin{aligned}
\int_0^t (e(s) - e(\tau^N(s))) \, \mathrm{d}s &\leq \int_0^t e(s) \, \mathrm{d}s - \int_0^{\Delta t_N} e(0) \, \mathrm{d}s - \int_{\Delta t_N}^t e(s - \Delta t_N) \, \mathrm{d}s \\
&= \int_0^t e(s) \, \mathrm{d}s - \int_0^{\Delta t_N} e(0) \, \mathrm{d}s - \int_0^{T - \Delta t_N} e(s) \, \mathrm{d}s \\
&= \int_{t - \Delta t_N}^t e(s) \, \mathrm{d}s - e(0)\Delta t_N.
\end{aligned}$$

Using again the monotonicity yields

$$\int_0^t (e(s) - e(\tau^N(s))) \, \mathrm{d}s \leq \int_{t - \Delta t_N}^t e(t) \, \mathrm{d}s - e(0)\Delta t_N = (e(t) - e(0))\Delta t_N.$$

Putting the estimates together proves (5.2). $\square$

Lemma 5.2. *Besides the Hypothesis 2.1, suppose that, for all $0 \leq \tau \leq s \leq T$,*

$$|f(s, X_\tau, Y_s) - f(\tau, X_\tau, Y_\tau)| \leq G_s - G_\tau, \tag{5.3}$$

where $\{G_t\}_{t\in I}$ is a real random process with monotonically non-decreasing sample paths. Then,

$$\left|\int_0^t \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})\right)\;\mathrm{d}s\right| \leq (G_t - G_0)\Delta t_N, \tag{5.4}$$

for all $0 \leq t \leq T$ and every $N \in \mathbb{R}$.

Proof. Let $N \in \mathbb{R}$. From the assumption (5.3) we have

$$|f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})| \leq G_s - G_{\tau^N(s)},$$

for every $0 \leq s \leq T$. Thus, upon integration,

$$\left|\int_0^t \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})\right)\;\mathrm{d}s\right| \leq \int_0^t (G_s - G_{\tau^N(s)})\;\mathrm{d}s.$$

Now we need to bound the right hand side. When $0 \leq t \leq t_1 = \Delta t_N$, we have $\tau^N(s) = 0$ for all $0 \leq s < t_1$, so that,

$$\int_0^t (G_s - G_{\tau^N(s)})\;\mathrm{d}s = \int_0^t (G_s - G_0)\;\mathrm{d}s.$$

Using the monotonicity,

$$\int_0^t (G_s - G_{\tau^N(s)})\;\mathrm{d}s \leq \int_0^t (G_t - G_0)\;\mathrm{d}s = (G_t - G_0)t \leq (G_t - G_0)\Delta t_N.$$

When $\Delta t_N \leq t \leq T$, we split the integration of the second term at time $s = t_1 = \Delta t_N$ and write

$$\int_0^t (G_s - G_{\tau^N(s)})\;\mathrm{d}s = \int_0^t G_s\;\mathrm{d}s - \int_0^{t_1} G_{\tau^N(s)}\;\mathrm{d}s - \int_{t_1}^t G_{\tau^N(s)}\;\mathrm{d}s$$

Using the monotonicity together with the fact that $s - \Delta t_N \leq \tau^N(s) \leq s$ for all $\Delta t_N \leq s \leq T$,

$$\begin{aligned}
\int_0^t (G_s - G_{\tau^N(s)})\;\mathrm{d}s &\leq \int_0^t G_s\;\mathrm{d}s - \int_0^{\Delta t_N} G_0\;\mathrm{d}s - \int_{\Delta t_N}^t G_{s-\Delta t_N}\;\mathrm{d}s \\
&= \int_0^t G_s\;\mathrm{d}s - \int_0^{\Delta t_N} G_0\;\mathrm{d}s - \int_0^{T-\Delta t_N} G_s\;\mathrm{d}s \\
&= \int_{t-\Delta t_N}^t G_s\;\mathrm{d}s - G_0\Delta t_N.
\end{aligned}$$

Using again the monotonicity yields

$$\int_0^t (G_s - G_{\tau^N(s)})\;\mathrm{d}s \leq \int_{t-\Delta t_N}^t G_t\;\mathrm{d}s - G_0\Delta t_N = (G_t - G_0)\Delta t_N.$$

Putting the estimates together proves (5.4). □

Theorem 5.1. *Under the Hypothesis 2.1, suppose further that, for all $0 \leq \tau \leq s \leq T$, we have*

$$|f(s, X_s, Y_s)| \leq F_t, \tag{5.5}$$

and

$$|f(s, X_\tau, Y_s) - f(\tau, X_\tau, Y_\tau)| \leq G_s - G_\tau, \tag{5.6}$$

where $\{F_t\}_{t\in I}$ and $\{G_t\}_{t\in I}$ are real random process with $\{G_t\}_{t\in I}$ having monotonically non-decreasing sample paths. Assume, finally, that

$$\sup_{0\leq\tau<t\leq T} \frac{1}{t-\tau}\int_\tau^t \mathbb{E}[F_s]\,\mathrm{d}s < \infty, \qquad \mathbb{E}[(G_T - G_0)] < \infty.$$

Then, the Euler-Maruyama scheme (1.2)-(1.3) is of strong order 1, i.e.

$$\max_{j=0,\ldots,N} \mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq C\Delta t_N, \qquad \forall N \in \mathbb{N}, \tag{5.7}$$

for a constant $C \geq 0$ given by

$$C = \left(\frac{T}{2}\sup_{0\leq\tau<t\leq T}\frac{1}{t-\tau}\int_\tau^t \mathbb{E}[F_s]\,\mathrm{d}s + \mathbb{E}[(G_T - G_0)]\right)e^{L_X T}. \tag{5.8}$$

Proof. Under the supplied hypotheses, Lemma 4.1 applies and the global error estimate (4.1) holds. Since $X_0^N = X_0$, the first term on the right hand side vanishes and we have two terms left to estimate. The second term is handled via Lemma 4.2. For the third term, we apply Lemma 5.2 and use the estimate (5.4). Putting the estimates together, we bound the global error by

$$|X_{t_j} - X_{t_j}^N| \leq \left(\int_0^{t_j}\int_{\tau^N(s)}^s F_\sigma\,\mathrm{d}\sigma\,\mathrm{d}s + (G_t - G_0)\Delta t_N\right)e^{L_X t_j}.$$

Taking the expectation and using again Lemma 4.2, we find that

$$\begin{aligned}\mathbb{E}\left[|X_{t_j} - X_{t_j}^N|\right] &\leq \left(\frac{t_j}{2}K\Delta t_N + \mathbb{E}[(G_{t_j} - G_0)]\Delta t_N\right)e^{L_X t_j} \\ &\leq \left(\frac{T}{2}K + \mathbb{E}[(G_T - G_0)]\right)e^{L_X T}\Delta t_N,\end{aligned}$$

where K is given in (**??**). This proves (5.7) with the constant C given by (5.8). □

One typical case in which a bound such as that in Theorem 5.1 is possible is that of a linear equation, or, when f is *semi-separable*. More precisely, we have the following result.

Corollary 5.1. *Suppose that $f = f(t, x, y)$ is of the form*

$$f(t, x, y) = a(t, y)h(x) + b(t, y), \tag{5.9}$$

where $a = a(t, y)$, $h = h(x)$, and $b = b(t, y)$ are continous on $[0, T] \times \mathbb{R}$ and h is globally Lipschitz continous in $x \in \mathbb{R}$, uniformly in $t \in I$. Assume, further, that

$$
|a(s, Y_s) - a(\tau, Y_\tau)| \leq A_s - A_\tau, \quad |b(s, Y_s) - b(\tau, Y_\tau)| \leq B_s - B_\tau, \quad |h(X_t)| \leq H_t,
$$

where $\{A_t\}_{t\in I}$, $\{B_t\}_{t\in I}$ and $\{H_t\}_{t\in I}$ are nonnegative stochastic processes with monotonic non-decreasing sample paths. Suppose that

$$
\mathbb{E}[(A_T - A_0)] < \infty, \quad \mathbb{E}[(B_T - B_0)] < \infty, \quad \mathbb{E}[H_t] < \infty.
$$

Then, the Euler-Maruyama scheme is of strong order 1, i.e.

$$
\max_{j=0,\ldots,N} \mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq C\Delta t_N, \qquad \forall N \in \mathbb{N}, \tag{5.10}
$$

for a suitable constant $C \geq 0$.

Proof. We have

$$
\begin{aligned}
|f(s, X_\tau, Y_s) - f(\tau, X_\tau, Y_\tau)| &\leq |a(s, Y_s) - a(\tau, Y_\tau)||h(X_\tau)| + |b(s, Y_s) - b(\tau, Y_\tau)| \\
&\leq (A_s - A_\tau)H_\tau + B_s - B_\tau \\
&\leq A_s H_s - A_\tau H_\tau + B_s - B_\tau \\
&= G_s - G_\tau,
\end{aligned}
$$

for $G_t = A_t H_t$. Notice $\{G_t\}_{t\geq 0}$ also has nondecreasing monotonic sample paths, and with

$$
\mathbb{E}[G_T - G_0] < \infty.
$$

Thus, Theorem 5.1 applies and the strong order 1 convergence holds. □

Remark 5.1. In many applications, it is possible to bound

$$
f(t, x, y) \leq C(1 + |x|^a + |y|^b),
$$

for suitable $a, b \geq 1$, in which case

$$
f(t, X_\tau, Y_t) \leq C(1 + |X_\tau|^a + G_t^b)
$$

where $G_t = \sup_{0\leq s\leq t} |Y_t|$ is monotonically nondecreasing, and we just need the bounds

$$
\mathbb{E}[(|X_t|)^a] < \infty, \qquad \mathbb{E}[(\sup_{0\leq t\leq T} |Y_t|)^b] < \infty.
$$

6. The case of an Itô noise

Here, we assume the noise $\{Y_t\}_{t\in I}$ is an **Itô process**, i.e. satisfying

$$
\mathrm{d}Y_t = A_t \,\mathrm{d}t + B_t \,\mathrm{d}W_t, \tag{6.1}
$$

where $\{W_t\}_{t\geq 0}$ is a Wiener process and $\{A_t\}_{t\in I}$ and $\{B_t\}_{t\in I}$ are stochastic processes adapted to $\{W_t\}_{t\geq 0}$. As mentioned in the Introduction, we are not solving for Y_t, otherwise we would actually have a system of stochastic differential equations. Instead, we assume it is a known process, with analytic solution to be used in the Euler-Maruyama approximation of (1.1). In theory, A_t and B_t are allowed to be originally given in terms of $\{W_t\}_{t\geq 0}$ and $\{Y_t\}_{t\geq 0}$. For example, Y_t may be a Wiener process, an Ornstein-Uhlenbeck process or a geometric Brownian process. At this point, we only assume that $\{A_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$ satisfy

$$\mathbb{E}[|A_t|] \leq M_A, \quad \mathbb{E}[|B_t|] \leq M_B, \qquad \forall t \in [0,T]. \tag{6.2}$$

We also assume that $(t,y) \mapsto f(t,x,y)$ is twice continuously differentiable, for each fixed x, so the Itô formula is applicable and yields

$$\begin{aligned} \mathrm{d}f(t,x,Y_t) = \left(\partial_t f(t,x,Y_t) + A_t \partial_y f(t,x,Y_t) + \frac{B_t^2}{2}\partial_{yy} f(t,x,Y_t)\right)\,\mathrm{d}t \\ + B_t \partial_y f(t,x,Y_t)\,\mathrm{d}W_t, \end{aligned} \tag{6.3}$$

for every fixed $x \in \mathbb{R}$.

We need to estimate the global error (4.1). The first term in the right hand side vanishes with the assumption that $X_0^N = X_0$. The second term is bounded according to Lemma 4.2. Our main concern here is in estimating the last term. For that, we have the following results.

Lemma 6.1. *Under the Hypothesis 2.2, the bound*

$$\int_0^{t_j} \left(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})\right)\,\mathrm{d}s = I^1 + I^2, \tag{6.4}$$

holds for each $j = 0, 1, \ldots, N$, where

$$\begin{aligned} I^1 = \int_0^{t_j} & (\tau^N(s) + \Delta t_N - s) \\ & \left(\partial_s f(s, X^N_{\tau^N(s)}, Y_s) + A_s \partial_y f(s, X^N_{\tau^N(s)}, Y_s) + \frac{B_s^2}{2}\partial_{yy} f(s, X^N_{\tau^N(s)}, Y_s)\right)\,\mathrm{d}s \end{aligned}$$

and

$$I^2 = \int_0^{t_j} (\tau^N(s) + \Delta t_N - s) B_s \partial_y f(s, X^N_{\tau^N(s)}, Y_s)\,\mathrm{d}W_s.$$

Proof. We use the Itô formula on $Z_s = f(s, X_t^N, Y_s)$ and write, for any $0 \leq t < t+\tau \leq T$,

$$\begin{aligned} \int_t^{t+\tau} \left(f(s, X_t^N, Y_s) - f(t, X_t^N, Y_t)\right)\,\mathrm{d}s &= \int_t^{t+\tau}\int_t^s \mathrm{d}Z_\xi\,\mathrm{d}s \\ &= \int_t^{t+\tau}\int_t^s \left(\partial_\xi f(\xi, X_t^N, Y_\xi) + A_\xi \partial_y f(\xi, X_t^N, Y_\xi) + \frac{B_\xi^2}{2}\partial_{yy} f(\xi, X_t^N, Y_\xi)\right)\,\mathrm{d}\xi\,\mathrm{d}s \\ &\quad + \int_t^{t+\tau}\int_t^s B_\xi \partial_y f(\xi, X_t^N, Y_\xi)\,\mathrm{d}W_\xi\,\mathrm{d}s. \end{aligned}$$

Using Fubini's Theorem, the first double integral is rewritten as

$$\int_t^{t+\tau}\int_\xi^{t+\tau}\left(\partial_\xi f(\xi, X_t^N, Y_\xi) + A_\xi \partial_y f(\xi, X_t^N, Y_\xi) + \frac{B_\xi^2}{2}\partial_{yy} f(\xi, X_t^N, Y_\xi)\right)\,\mathrm{d}s\,\mathrm{d}\xi$$
$$= \int_t^{t+\tau}(t+\tau-\xi)\left(\partial_\xi f(\xi, X_t^N, Y_\xi) + A_\xi \partial_y f(\xi, X_t^N, Y_\xi) + \frac{B_\xi^2}{2}\partial_{yy} f(\xi, X_t^N, Y_\xi)\right)\,\mathrm{d}\xi.$$

Using now Fubini's Theorem on the last integral, we find

$$\begin{aligned}\int_t^{t+\tau}\int_t^s B_\xi \partial_y f(\xi, X_t^N, Y_\xi)\,\mathrm{d}W_\xi\,\mathrm{d}s &= \int_t^{t+\tau}\int_\xi^{t+\tau} B_\xi \partial_y f(\xi, X_t^N, Y_\xi)\,\mathrm{d}s\,\mathrm{d}W_\xi \\ &= \int_t^{t+\tau}(t+\tau-\xi)B_\xi \partial_y f(\xi, X_t^N, Y_\xi)\,\mathrm{d}W_\xi.\end{aligned}$$

Thus, for $\tau = \Delta t_N$, $t = t_{j-1} = (j-1)\Delta t_N$, and $t+\tau = t_{j-1} + \Delta t_N = t_j$, we have

$$\int_{t_j}^{t_{j+1}}\left(f(s, X_{t_j}^N, Y_s) - f(t, X_{t_j}^N, Y_t)\right)\,\mathrm{d}s = I_j^1 + I_j^2,$$

where

$$I_j^1 = \int_{t_j}^{t_{j+1}}(t_{j+1}-\xi)\left(\partial_\xi f(\xi, X_{t_j}^N, Y_\xi) + A_\xi \partial_y f(\xi, X_{t_j}^N, Y_\xi) + \frac{B_\xi^2}{2}\partial_{yy} f(\xi, X_{t_j}^N, Y_\xi)\right)\,\mathrm{d}\xi,$$

and

$$I_j^2 = \int_{t_j}^{t_{j+1}}(t_{j+1}-\xi)B_\xi \partial_y f(\xi, X_{t_j}^N, Y_\xi)\,\mathrm{d}W_\xi.$$

Summing them up in j and writing s for ξ leads to (6.4). $\square$

Lemma 6.2. *Under the Hypothesis 2.2, suppose, moreover, that*

$$\begin{aligned}K_1 = \sup_{N\in\mathbb{N}}\int_0^T \mathbb{E}\Big[\Big|&\partial_s f(s, X_{\tau^N(s)}^N, Y_s) \\ &+ A_s\partial_y f(s, X_{\tau^N(s)}^N, Y_s) + \frac{B_s^2}{2}\,\partial_{yy} f(s, X_{\tau^N(s)}^N, Y_s)\Big|\Big]\,\mathrm{d}s < \infty\end{aligned} \tag{6.5}$$

and

$$K_2 = \sup_{N\in\mathbb{N}}\left(\int_0^T \mathbb{E}\left[\left|B_s\partial_y f(s, X_{\tau^N(s)}^N, Y_s)\right|^2\right]\,\mathrm{d}s\right)^{1/2} < \infty. \tag{6.6}$$

Then,

$$\int_0^{t_j}\mathbb{E}\left[\left|f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})\right|\right]\,\mathrm{d}s \leq C\Delta_N \tag{6.7}$$

for each $j = 0, 1, \ldots, N$, and a suitable constant $C > 0$.

Proof. Taking the expectation of the absolute value of (6.4), we see we need to estimate $\mathbb{E}[|I^1|]$ and $\mathbb{E}[|I^2|]$, defined in Lemma 6.1. The estimate of the first term is straightfoward and leads to

$$
\begin{aligned}
\mathbb{E}[|I^1|] \leq &\int_0^{t_j} |\tau^N(s) + \Delta t_N - s| \\
&\qquad \left|\partial_s f(s, X^N_{\tau^N(s)}, Y_s) + A_s \partial_y f(s, X^N_{\tau^N(s)}, Y_s) + \frac{B_s^2}{2}\partial_{yy} f(s, X^N_{\tau^N(s)}, Y_s)\right| \,\mathrm{d}s \\
&\leq \Delta t_N \int_0^T \left|\partial_s f(s, X^N_{\tau^N(s)}, Y_s) + A_s \partial_y f(s, X^N_{\tau^N(s)}, Y_s) + \frac{B_s^2}{2}\partial_{yy} f(s, X^N_{\tau^N(s)}, Y_s)\right| \,\mathrm{d}s \\
&\leq K_1 \Delta t_N,
\end{aligned}
$$

where K_1 is given by (6.5). For the second term, we use the Lyapunov inequality and the Itô isometry to find that

$$
\begin{aligned}
\mathbb{E}[|I^2|] &\leq \sqrt{\mathbb{E}[|I^2|^2]} \\
&= \left(\int_{t_j}^{t_{j+1}} \mathbb{E}\left[\left((\tau^N(s) + \Delta t_N - \xi) B_\xi \partial_y f(\xi, X^N_{t_j}, Y_\xi)\right)^2\right] \,\mathrm{d}\xi\right)^{1/2} \\
&\leq \Delta t_N \left(\int_{t_j}^{t_{j+1}} \mathbb{E}\left[\left(B_\xi \partial_y f(\xi, X^N_{t_j}, Y_\xi)\right)^2\right] \,\mathrm{d}\xi\right)^{1/2} \\
&\leq K_2 \Delta t_N,
\end{aligned}
$$

for K_2 given by (6.6). Putting the two estimates together, we find (6.7). $\square$

Gathering the conditions, we obtain the following result.

Theorem 6.1. *Under the Hypothesis 2.2, suppose, moreover, that* (**??**), (6.5) *and* (6.6) *hold. Then, the Euler-Maruyama scheme* (1.2)-(1.3) *is of strong order 1, i.e.*

$$
\max_{j=0,\ldots,N} \mathbb{E}\left[\left|X_{t_j} - X^N_{t_j}\right|\right] \leq C \Delta t_N, \qquad \forall N \in \mathbb{N}, \tag{6.8}
$$

for a suitable constant $C \geq 0$

The required conditions in Theorem 6.1, especially conditions (6.5) and (6.6) of Lemma 6.2, are not readily checked. We present, below, some particular cases with more explicit bounds that imply those.

Corollary 6.1. *Yeah, corollaries are good.*

Remark 6.1. When the noise $\{Y_t\}_{t\in I}$ is a Wiener process, then $A_t = 0$ and $B_t = 1$ and the bounds (6.5) and (6.6) become

$$
K_1 = \int_0^T \mathbb{E}\left[\left|\partial_s f(s, X^N_{\tau^N(s)}, Y_s) + \frac{1}{2}\partial_{yy} f(s, X^N_{\tau^N(s)}, Y_s)\right|\right] \,\mathrm{d}s < \infty \tag{6.9}
$$

and

$$K_2 = \left(\int_0^{t_j} \mathbb{E}\left[\left| \partial_y f(s, X^N_{\tau^N(s)}, Y_s) \right|^2 \right] \,\mathrm{d}s \right)^{1/2} < \infty. \tag{6.10}$$

Remark 6.2. When the noise $\{Y_t\}_{t\in I}$ is a Ornstein-Uhlenbeck process, then $A_t = -\nu Y_t$ and $B_t = \sigma$, for $\nu \in \mathbb{R}$, $\sigma > 0$, and the bounds (6.5) and (6.6) become

$$K_1 = \int_0^T \mathbb{E}\left[\left| \partial_s f(s, X^N_{\tau^N(s)}, Y_s) - \nu \partial_y f(s, X^N_{\tau^N(s)}, Y_s) + \frac{\sigma^2}{2} \partial_{yy} f(s, X^N_{\tau^N(s)}, Y_s) \right| \right] \,\mathrm{d}s < \infty \tag{6.11}$$

and

$$K_2 = \left(\int_0^{t_j} \mathbb{E}\left[\left| \sigma \partial_y f(s, X^N_{\tau^N(s)}, Y_s) \right|^2 \right] \,\mathrm{d}s \right)^{1/2} < \infty. \tag{6.12}$$

Remark 6.3. When the noise $\{Y_t\}_{t\in I}$ is a geometric Brownian motion process, then $A_t = \mu Y_t$ and $B_t = \sigma Y_t$, for $\mu \in \mathbb{R}$, $\sigma > 0$, and the bounds (6.5) and (6.6) become

$$K_1 = \int_0^T \mathbb{E}\left[\left| \partial_s f(s, X^N_{\tau^N(s)}, Y_s) + \mu Y_s \partial_y f(s, X^N_{\tau^N(s)}, Y_s) + \frac{\sigma^2 Y_s^2}{2} \partial_{yy} f(s, X^N_{\tau^N(s)}, Y_s) \right| \right] \,\mathrm{d}s < \infty \tag{6.13}$$

and

$$K_2 = \left(\int_0^{t_j} \mathbb{E}\left[\left| \sigma Y_s \partial_y f(s, X^N_{\tau^N(s)}, Y_s) \right|^2 \right] \,\mathrm{d}s \right)^{1/2} < \infty. \tag{6.14}$$

7. Applications

In this section, we describe a few explicit examples that fall into one of the cases considered above and, hence, the Euler-Maruyama method exhibits a strong order one convergence.

7.1. **Population dynamics.** Our first example is a population dynamics model modified from [3, Section 15.2],

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = Y_t X_t (r - X_t) \tag{7.1}$$

where $r > 0$ is constant and $\{Y_t\}_{t\geq}$ is a stochastic process playing the role of a random growth parameter and given by

$$Y_t = \lambda(1 + \varepsilon \sin(O_t)),$$

where $0 < \varepsilon < 1$ and $\{O_t\}_{t\geq 0}$ is an Ornstein-Uhlenbeck process given by

$$\mathrm{d}O_t = (\theta_1 - \theta_2 O_t)\,\mathrm{d}t + \theta_3 \,\mathrm{d}W_t,$$

with $\theta_1, \theta_2, \theta_3 > 0$ constant and $\{W_t\}_{t\geq 0}$ being a standard Wiener process. We do not need to approximate the coupled stochastic differential equation system for (X_t, O_t) since the Ornstein-Uhlenbeck process has an analytic solution, given by

$$O_t = \frac{\theta_1}{\theta_2} + e^{-\theta_2(t-s)}\left(O_s - \frac{\theta_1}{\theta_2}\right) + \theta_3 \int_s^t e^{-\theta_2(t-\sigma)}\,\mathrm{d}W_\sigma.$$

We suppose the initial condition is non-negative and bounded almost surely:

$$0 \leq X_0 \leq R,$$

for some $R > r$.

The noise process $\{\Lambda_t\}_{t\geq 0}$ itself satisfies

$$0 < \lambda - \varepsilon \leq Y_t \leq \lambda + \varepsilon < 2\lambda, \qquad \forall t \geq 0.$$

Define

$$f(t, x, y) = yx(r - x)$$

and notice that $f(t, x, y)x \geq 0$, for $x \geq 0$ and $y \geq 0$, and $f(t, x, y)x \leq 0$, for $x \geq r$ and $y \geq 0$. Hence the interval $[0, R]$ in x is positively invariant and the pathwise solutions of (7.1) are almost surely bounded as well, with

$$0 \leq X_t \leq R,$$

for all $t \geq 0$.

The function $f = f(t, x, y)$ is continuously differentiable infinitely many times and with

$$\left|\frac{\partial f}{\partial x}(t, x, y)\right| = |y(r - 2x)| \leq 2\lambda(2R - r),$$

for $|x| \leq R$ and $0 \leq y \leq 2\lambda$.

The right hand side of (7.1) is not globally Lipschitz, but, for the sake of analysis, since X_t and Y_t are bounded, the right hand side can be modified to a twice continuously differentiable, uniformly globally Lipschitz function $\tilde{f}(t, x, y)$ that coincides with $f(t, x, y)$ for $(t, x, y) \in \mathbb{R} \times [0, R] \times [0, 2\lambda]$ and satisfies (2.1) with

$$L_X = 2\lambda(2R - r).$$

Thus, the RODE (7.1) with $0 \leq X_0 \leq R$ almost surely, for some $R > r$, is equivalent to the RODE

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = \tilde{f}(t, X_t, Y_t). \tag{7.2}$$

With $\tilde{f} = \tilde{f}(t, x, y)$, the Hypothesis 2.2 hold. Moreover, it follows from (2.3) (notice $M_t = 0$ here) that

$$|X_t| \leq |X_0|e^{2\lambda(2R-r)t} \leq Re^{2\lambda(2R-r)T}, \qquad 0 \leq t \leq T.$$

almost surely.

OLD Condition on $\{F_t\}$ SHOULD BE FIXED is satisfied with

$$F_t ?!?!?!$$

The Itô formula applied to $Y_t = g(O_t)$, where $g(\eta) = \lambda(1+\varepsilon\sin(\eta))$ implies $\{Y_t\}_{t\geq 0}$ is an Itô process with

$$\mathrm{d}Y_t = \left((\theta_1 - \theta_2 O_t)g'(O_t) + \frac{\theta_3^2}{2}g''(O_t)\right)\,\mathrm{d}t + \theta_3 g'(O_t)\,\mathrm{d}W_t.$$

We have

$$g'(\eta) = \lambda\varepsilon\cos(\eta), \quad g''(\eta) = -\lambda\varepsilon\sin(\eta)$$

hence both are uniformly bounded. Therefore, condition (6.5) is satisfied with

$$K_1 \leq \lambda\varepsilon\int_0^T \left((\theta_1 - \theta_2\mathbb{E}[|O_t|] + \frac{\theta_3^2}{2})\right)\,\mathrm{d}s < \infty,$$

while (6.6) is satisfied with

$$K_2 \leq \lambda\varepsilon\theta_3 T^{1/2} < \infty.$$

Hence, all the conditions of Theorem 6.1 hold and the Euler-Maruyama method is of strong order 1.

7.2. **Drug delivery.**

7.3. **Earthquake model.**

7.4. **Point-process noise.**

Appendix A. Discrete Grownwall Lemma

We end this section by abstracting away the Gronwall type inequality we use (this is probably written somewhere, and I need to find the source):

Lemma A.1. *Let $(e_j)_j$ be a (finite or infinite) sequence of positive numbers satisfying*

$$e_j \leq a\sum_{i=0}^{j-1} e_i + b, \tag{A.1}$$

with $e_0 = 0$, where $a, b > 0$. Then,

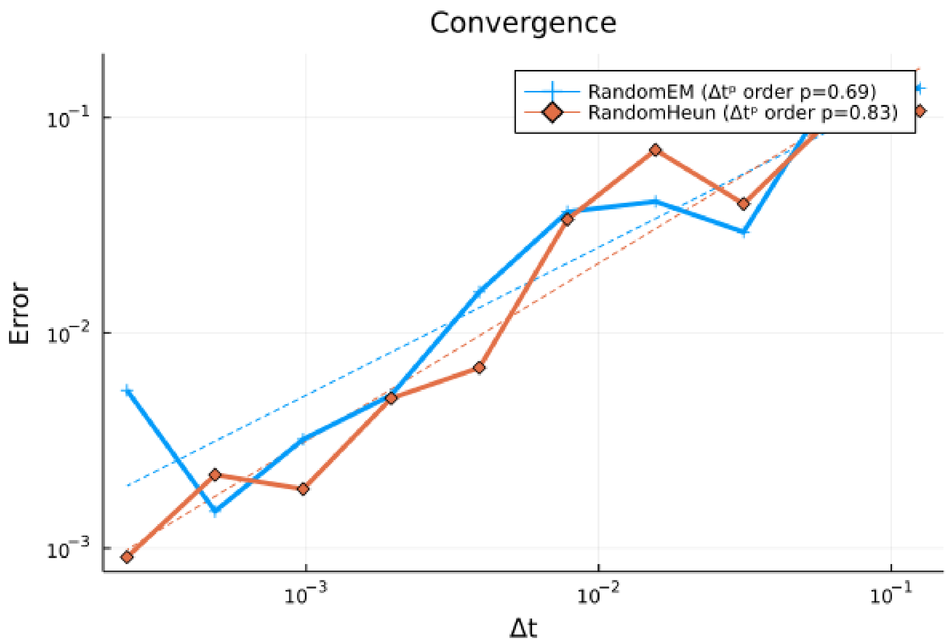
$$e_j \leq be^{aj}, \qquad \forall j. \tag{A.2}$$

Proof. The result is trivially true for $j = 0$. Suppose, by induction, that the result is true up to $j-1$. Then,

$$e_j \leq a\sum_{i=0}^{j-1} be^{ai} + b = b\left(a\sum_{i=0}^{j-1} e^{ai} + 1\right).$$

Using that $1 + a \leq e^a$, we have $a \leq e^a - 1$, hence

$$e_j \leq b\left((e^a - 1)\sum_{i=0}^{j-1} e^{ia} + 1\right).$$



Using that $\sum_{i=0}^{j-1} \theta^i = (\theta^j - 1)(\theta - 1)$, with $\theta = e^a$, we see that

$$(e^a - 1) \sum_{i=0}^{j-1} e^{ia} \leq e^{ja} - 1,$$

so that

$$e_j \leq b e^{ja},$$

which completes the induction. □

Appendix B. Numerical examples

B.1. **Lower-order converge.** For a lower order convergence, below order 1, we take the noise $\{Y_t\}_t$ to be the transport process defined by

$$Y_t = \sin(t/Z)^{1/3},$$

where Z is a beta random variable $Z \sim B(\alpha, \beta)$. Notice Z takes values strictly within $(0,1)$ and, hence, $\sin(t/Z)$ can have arbitrarily high frequencies and, hence, go through the critic value $y = 0$ extremely often.

(Need to remove the Heun method and do more tests).

Acknowledgments

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