.

STRONG ORDER 1 CONVERGENCE OF THE EULER-MARUYAMA METHOD FOR RANDOM ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. It is well known that the Euler-Maruyama method of approximating a random ordinary differential equation $\mathrm{d}X_t/\mathrm{d}t = f(t, X_t, Y_t)$ driven by a stochastic process $\{Y_t\}_t$ with θ-Hölder sample paths is estimated to be of strong order θ with respect to the time step, provided $f = f(t, x, y)$ is sufficiently regular. Here, we show that, in common situations, it is possible to exploit further conditions on the noise and prove that the strong convergence is actually of order 1, regardless of the Hölder regularity of the sample paths. This applies to additive or multiplicative Itô noises (such as Wiener, Ornstein-Uhlenbeck, and Geometric Brownian process); to point-process noises (such as Poisson point processes and Hawkes self-exciting processes, which are not even continuous and have jump-type discontinuities); and to transport-type processes. The order 1 convergence is based on two main ideas: First, we do not estimate directly the local error and, instead, add up the local steps and work directly with an accumulated global error. Secondly, we assume either a control of the total variation of the sample paths of the noise or that the noise is an Itô process. In the first case, the noise-sensitive part of the global error is bounded by the time step multiplied by a term involving the expectation of the total variation. In the second case, we exploit the Itô isometry to bound that part of the global error.

1. Introduction

Consider the following initial value problem for a **random ordinary differential equation (RODE)**:

$$\begin{cases} \dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t), \qquad 0 \leq t \leq T, \\ X_t\big|_{t=0} = X_0, \end{cases} \tag{1.1}$$

on a time interval $I = [0, T]$, with $T > 0$, and where the noise $\{Y_t\}_{t\in I}$ is a given stochastic process. The sample space is denoted by Ω.

The Euler-Maruyama method for solving this initial value problem consists in approximating the solution on a uniform time mesh $t_j = j\Delta t_N$, $j = 0, \ldots, N$, with fixed

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time step $\Delta t_N = T/N$, for a given $N \in \mathbb{N}$. In such a mesh, the Euler-Maruyama scheme takes the form

$$X_{t_j}^N = X_{t_{j-1}}^N + \Delta t_N f(t_{j-1}, X_{t_{j-1}}^N, Y_{t_{j-1}}), \qquad j = 1, \ldots, N, \tag{1.2}$$

with the initial condition

$$X_0^N = X_0. \tag{1.3}$$

Notice $t_j = j\Delta t_N = jT/N$ also depends on N, but we do not make this dependency explicit, for the sake of notational simplicity.

When the noise $\{Y_t\}_{t\in I}$ has θ-Hölder continuous sample paths, it can be show [3], under further suitable conditions, that the Euler-Maruyama scheme converges strongly with order θ with respect to the time step, i.e. there exists a constant $C \geq 0$ such that

$$\max_{j=0,\ldots,N} \mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq C\Delta t_N^\theta, \qquad \forall N \in \mathbb{N}, \tag{1.4}$$

where $\mathbb{E}[\cdot]$ indicates the expectation of a random variable on Ω.

Our aim is to show that, in many classical examples, it is possible to exploit further conditions that yield in fact a strong order 1 convergence, with the sample paths still being Hölder continuous or even discontinuous. This is the case, for instance, when the noise is a point process, a transport process, or an Itô process.

The first main idea of the proof is to not estimate the local error and, instead, work with an explicit formula for the global error, namely (see Lemma 3.1)

$$\begin{aligned} X_{t_j} - X_{t_j}^N = X_0 - X_0^N & \\ &+ \int_0^{t_j} \left(f(s, X_s, Y_s) - f(s, X_{\tau^N(s)}, Y_s)\right) \,\mathrm{d}s \\ &+ \int_0^{t_j} \left(f(s, X_{\tau^N(s)}, Y_s) - f(s, X_{\tau^N(s)}^N, Y_s)\right) \,\mathrm{d}s \\ &+ \int_0^{t_j} \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})\right) \,\mathrm{d}s, \end{aligned} \tag{1.5}$$

for $j = 1, \ldots, N$, where τ^N is a piecewise constant function with jumps at the mesh points t_j (see Equation (3.2)).

The first term vanishes due to the initial condition $X_0^N = X_0$. The second term only depends on the solution and can be easily estimated with natural regularity conditions on the term $f = f(t, x, y)$. The third term is handled solely with the typical required condition on $f = f(t, x, y)$ of being uniformly globallly Lipschitz continuity with respect to x. With those, we obtain the following basic bound for the global error (see Lemma 4.1)

$$
\begin{aligned}
|X_{t_j} - X_{t_j}^N| \leq \bigg(|X_0 - X_0^N| + L_X \int_0^{t_j} |X_s - X_{\tau^N(s)}| \;\mathrm{d}s \\
\left| \int_0^{t_j} \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)}) \right) \;\mathrm{d}s \right| \bigg) e^{L_X t_j}.
\end{aligned}
\tag{1.6}
$$

The only problematic, noise-sensitive term is the last one. The classical analysis is to use an assumed θ-Hölder regularity of the noise sample paths and estimate the local error as

$$
\mathbb{E}\left[\left| f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)}) \right| \right] \leq C\Delta t^\theta.
$$

Instead, we look at the whole global error formula (1.5) and assume, for the last term, that the steps of the process given by $F_t = f(t, X_{\tau^N(t)}^N, Y_t)$ can be controlled in a suitable sense. In order to give the main idea, let us assume for the moment that the sample paths of $\{F_t\}_{t\ inI}$ satisfy

$$
F_s - F_\tau = \int_\tau^s \;\mathrm{d}F_\xi,
$$

either in the sense of a Riemann-Stieltjes integral or of an Itô integral. The first sense fits the case of noises with bounded total variation, while the second one fits the case of an Itô noise. In any case, we bound the global error term using integration by parts,

$$
\begin{aligned}
\int_0^{t_j} \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)}) \right) \;\mathrm{d}s &= \int_0^{t_j} \int_{\tau^N(s)}^s \;\mathrm{d}F_\xi \;\mathrm{d}s \\
&= \int_0^{t_j} \int_\xi^{\tau^N(\xi)+\Delta t_N} \;\mathrm{d}s \;\mathrm{d}F_\xi \\
&= \int_0^{t_j} (\tau^N(\xi) + \Delta t_N - \xi) \;\mathrm{d}F_\xi.
\end{aligned}
$$

Then, we find that

$$
\begin{aligned}
\mathbb{E}\left[\left| \int_0^{t_j} \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)}) \right) \;\mathrm{d}s \right| \right] \\
\leq \Delta t_N \mathbb{E}\left[\int_0^{t_j} \;\mathrm{d}F_\xi \right],
\end{aligned}
$$

which yields the strong order 1 convergence provided the remaining expectation is finite.

In the case of an Itô integral, this is exactly what we assume, because the Itô integral is not order preserving; the bound on the remaining expectation is obtained via Itô isometry. In the case of bounded variation, however, we can relax the above condition and work not with $\{F_t\}_{t\in I}$ itself but with a bound on the step of the form

$$|f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})| \leq \bar{F}_s - \bar{F}_{\tau^N(s)}.$$

Only this bounding process $\{\bar{F}_t\}_{t\in I}$ is required to have sample paths of bounded variation, which is usually easier to check. These two cases are treated in Section 5 (for the bounded variation case; see Lemma 5.1 and Theorem 5.1) and Section 6 (for the Itô noise case; see Lemma 6.1 and Theorem 6.1).

The conditions in Theorem 5.1 and Theorem 6.1 are not readily verifiable, but Theorem 5.2 and Theorem 6.2 give more explicit conditions for each of the two cases. Essentially, $f = f(t,x,y)$ is required to have minimal regularity in the sense of differentiability and growth conditions and the noise $\{Y_t\}_{t\in I}$ is either required to have sample paths of bounded variation or to be an Itô noise.

We end the paper with a few explicit examples and their numerical implementation, illustrating the strong order 1 convergence.

2. Pathwise solution

For the notion and main results on pathwise solution for RODEs, we refer the reader to [4, Section 2.1]. We start with a fundamental set of conditions that imply the existence and uniqueness of pathwise solutions of the RODE (1.1) in the sense of Carathéodory:

Hypothesis 2.1. *We consider a function $f = f(t,x,y)$ defined on $I \times \mathbb{R} \times \mathbb{R}$ and a real-valued stochastic process $\{Y_t\}_{t\in I}$, where $I = [0,T]$, $T > 0$. We make the following standing hypotheses.*

(i) *f is globally Lipschitz continuous on x, uniformly in t and y, i.e. there exists a constant $L_X \geq 0$ such that*

$$|f(t,x_1,y) - f(t,x_2,y)| \leq L_X|x_1 - x_2|, \quad \forall t \in I,\ \forall x_1, x_2, y \in \mathbb{R}. \tag{2.1}$$

(ii) *We also assume that $(t,x) \mapsto f(t,x,Y_t)$ satisfies the Carathéodory conditions:*
   (a) *The mapping $x \mapsto f(t,x,Y_t(\omega))$ is continuous on $x \in \mathbb{R}$, for almost every $(t,\omega) \in I \times \Omega$;*
   (b) *The mapping $t \mapsto f(t,x,Y_t(\omega))$ is Lebesgue measurable in $t \in I$, for each $x \in \mathbb{R}$ and each sample path $t \mapsto Y_t(\omega)$;*
   (c) *The bound $|f(t,x,Y_t)| \leq M_t + L_X|x|$ holds for all $t \in I$ and all $x \in \mathbb{R}$, where $\{M_t\}_{t\in I}$ is a real stochastic process with Lebesgue integrable sample paths $t \mapsto M_t(\omega)$ on $t \in I$.*

Under these assumptions, for each sample value in Ω, the integral equation

$$X_t = X_0 + \int_0^t f(s, X_s, Y_s)\,\mathrm{d}s \tag{2.2}$$

has a unique solution, in the Lebesgue sense, for the realization $X_0 = X_0(\omega)$ of the initial condition and the sample path $t \mapsto Y_t(\omega)$ of the noise process (see [2, Theorem 1.1]). Moreover, the mapping $(t, \omega) \mapsto X_t(\omega)$ is measurable (see [4, Section 2.1.2]) and, hence, give rise to a well-defined stochastic process $\{X_t\}_{t\in I}$.

Each sample path solution $t \mapsto X_t(\omega)$ is bounded by

$$|X_t| \leq \left(|X_0| + \int_0^t M_s \,\mathrm{d}s\right) e^{L_X t}, \quad \forall t \in I. \tag{2.3}$$

For the strong convergence of the Euler-Maruyama approximation, we also need to control the expectation of the solution above, among other things. With that in mind, we have the following useful result.

Lemma 2.1. *Under Hypothesis 2.1, suppose further that*

$$\mathbb{E}[|X_0|] < \infty \tag{2.4}$$

and

$$\int_0^T \mathbb{E}[|M_s|] \,\mathrm{d}s < \infty \tag{2.5}$$

Then,

$$\mathbb{E}[|X_t|] \leq \left(\mathbb{E}[|X_0|] + \int_0^t \mathbb{E}[|M_s|] \,\mathrm{d}s\right) e^{L_X t}, \quad t \in I. \tag{2.6}$$

Proof. Thanks to (2.3), the result is straightfoward □

Remark 2.1. When $f = f(t, x, y)$ is continuous on all three variables, as well as uniformly globally Lipscthiz continuous in x, and the sample paths of $\{Y_t\}_{t\geq 0}$ are continuous, then the integrand in (2.2) is continuous in t and the integral becomes a Riemann integral. In this case, the integral form (2.2) of the pathwise solutions of (1.1) holds in the Riemann sense.

Remark 2.2. In special *dissipative* cases, depending on the structure of the equation, we might not need the second condition (2.5) and only require $\mathbb{E}[|X_0|] < \infty$. More generally, when some bounded, positively invariant region exists and is of interest, we may truncate the nonlinear term to achieve the desired global conditions for the equation with the truncated term, but which coincide with the original equation in the region of interest. But we leave these cases to be handled in the applications.

3. Integral formula for the global pathwise error

In this section, we derive the following integral formula for the global error:

Lemma 3.1. *Under Hypothesis 2.1, the Euler-Maruyama approximation (1.2) for any pathwise solution of the random ordinary differential equation (1.1) satisfies the* *global error formula*

$$
\begin{aligned}
X_{t_j} - X_{t_j}^N = X_0 - X_0^N & \\
& + \int_0^{t_j} \left(f(s, X_s, Y_s) - f(s, X_{\tau^N(s)}, Y_s)\right) \;\mathrm{d}s \\
& + \int_0^{t_j} \left(f(s, X_{\tau^N(s)}, Y_s) - f(s, X_{\tau^N(s)}^N, Y_s)\right) \;\mathrm{d}s \\
& + \int_0^{t_j} \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})\right) \;\mathrm{d}s,
\end{aligned} \tag{3.1}
$$

for $j = 1, \ldots, N$, *where* τ^N *is the piecewise constant jump function along the time mesh:*

$$
\tau^N(t) = \max_j\{j\Delta t_N;\; j\Delta t_N \leq t\} = \left[\frac{t}{\Delta t_N}\right]\Delta t_N = \left[\frac{tN}{T}\right]\frac{T}{N}. \tag{3.2}
$$

Proof. Under Hypothesis 2.1, the solutions of (1.1) are pathwise solutions in the Lebesgue sense of (2.2). With that in mind, we first obtain an expression for a single time step, from time t_{j-1} to $t_j = t_{j-1} + \Delta t_N$.

For notational simplicity, we momentarily write $t = t_{j-1}$ and $\tau = \Delta t_N$, so that $t_j = t + \tau$. The exact pathwise solution satisfies

$$
X_{t+\tau} = X_t + \int_t^{t+\tau} f(s, X_s, Y_s)\;\mathrm{d}s.
$$

The Euler-Maruyama step is given by

$$
X_{t+\tau}^N = X_t^N + \tau f(t, X_t^N, Y_t).
$$

Subtracting, we obtain

$$
X_{t+\tau} - X_{t+\tau}^N = X_t - X_t^N + \int_t^{t+\tau} \left(f(s, X_s, Y_s) - f(t, X_t^N, Y_t)\right)\;\mathrm{d}s.
$$

We arrange the integrand as

$$
\begin{aligned}
f(s, X_s, Y_s) - f(t, X_t^N, Y_t) = & f(s, X_s, Y_s) - f(s, X_t, Y_s) \\
& + f(s, X_t, Y_s) - f(s, X_t^N, Y_s) \\
& + f(s, X_t^N, Y_s) - f(t, X_t^N, Y_t).
\end{aligned}
$$

This yields

$$\begin{aligned} X_{t+\tau} - X_{t+\tau}^N =& X_t - X_t^N \\ &= \int_t^{t+\tau} \left(f(s, X_s, Y_s) - f(s, X_t, Y_s)\right) \,\mathrm{d}s \\ &\quad + \int_t^{t+\tau} \left(f(s, X_t, Y_s) - f(s, X_t^N, Y_s)\right) \,\mathrm{d}s \\ &\quad + \int_t^{t+\tau} \left(f(s, X_t^N, Y_s) - f(t, X_t^N, Y_t)\right) \,\mathrm{d}s. \end{aligned}$$

Going back to the notation $t = t_{j-1}$ and $t + \tau = t_j$, the above identity reads

$$\begin{aligned} X_{t_j} - X_{t_j}^N =& X_{t_{j-1}} - X_{t_{j-1}}^N \\ &= \int_{t_{j-1}}^{t_j} \left(f(s, X_s, Y_s) - f(s, X_{t_{j-1}}, Y_s)\right) \,\mathrm{d}s \\ &\quad + \int_{t_{j-1}}^{t_j} \left(f(s, X_{t_{j-1}}, Y_s) - f(s, X_{t_{j-1}}^N, Y_s)\right) \,\mathrm{d}s \\ &\quad + \int_{t_{j-1}}^{t_j} \left(f(s, X_{t_{j-1}}^N, Y_s) - f(t_{j-1}, X_{t_{j-1}}^N, Y_{t_{j-1}})\right) \,\mathrm{d}s. \end{aligned} \tag{3.3}$$

Now we iterate the time steps (3.3) to find that

$$\begin{aligned} X_{t_j} - X_{t_j}^N = X_0 - X_0^N & \\ &+ \sum_{i=1}^{j} \left(\int_{t_{i-1}}^{t_i} \left(f(s, X_s, Y_s) - f(s, X_{t_i}, Y_s)\right) \,\mathrm{d}s \right. \\ &+ \int_{t_{i-1}}^{t_i} \left(f(s, X_{t_{i-1}}, Y_s) - f(s, X_{t_{i-1}}^N, Y_s)\right) \,\mathrm{d}s \\ &\left. + \int_{t_{i-1}}^{t_i} \left(f(s, X_{t_{i-1}}^N, Y_s) - f(t_{i-1}, X_{t_{i-1}}^N, Y_{t_{i-1}})\right) \,\mathrm{d}s \right). \end{aligned}$$

Using the jump function τ^N defined by (3.2), we may rewrite the above expression as in (3.1). □

Remark 3.1. Strictly speaking, we only need condition (ii) from Hypothesis 2.1 in order to deduce (4.1), but since we need (i) for the strong convergence anyways, it is simpler to state the result as in Lemma 4.1.

4. Basic estimate for the global pathwise error

Here we derive an estimate, under minimal hypotheses, that is the basis for the estimates in specific cases.

Lemma 4.1. *Under Hypothesis 2.1, the global error (3.1) is estimated as*

$$
\begin{aligned}
|X_{t_j} - X_{t_j}^N| \leq \Bigg(& |X_0 - X_0^N| + L_X \int_0^{t_j} |X_s - X_{\tau^N(s)}| \,\mathrm{d}s \\
& \left| \int_0^{t_j} \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)}) \right) \,\mathrm{d}s \right| \Bigg) e^{L_X t_j}.
\end{aligned}
\tag{4.1}
$$

for $j = 1, \ldots, N$, where τ^N is given by (3.2).

Proof. We estimate the first two integrals in (3.1). For the first one, we use (2.1), so that

$$
|f(s, X_s, Y_s) - f(s, X_t, Y_s)| \leq L_X |X_s - X_t|,
$$

for $t, s \in I$, and, in particular, for $t = \tau^N(s)$. Hence,

$$
\left| \int_0^{t_j} \left(f(s, X_s, Y_s) - f(s, X_{\tau^N(s)}, Y_s) \right) \,\mathrm{d}s \right| \leq L_X \int_0^{t_j} |X_s - X_{\tau^N(s)}| \,\mathrm{d}s.
$$

For the second term, we use again (2.1), so that

$$
|f(s, X_t, Y_s) - f(s, X_t^N, Y_s)| \leq L_X |X_t - X_t^N|,
$$

for any $t, s \in I$, and, in particular, for $t = \tau^N(s)$. Hence,

$$
\begin{aligned}
\left| \int_0^{t_j} \left(f(s, X_{\tau^N(s)}, Y_s) - f(s, X_{\tau^N(s)}^N, Y_s) \right) \,\mathrm{d}s \right| & \leq L_X \int_0^{t_j} |X_{\tau^N(s)} - X_{\tau^N(s)}^N| \,\mathrm{d}s \\
& \leq L_X \sum_{i=0}^{j-1} |X_{t_i} - X_{t_i}^N| \Delta t_N.
\end{aligned}
$$

With these two estimates, we bound (3.1) as

$$
\begin{aligned}
|X_{t_j} - X_{t_j}^N| \leq |X_0 - X_0^N| & \\
& + L_X \int_0^{t_j} |X_s - X_{\tau^N(s)}| \,\mathrm{d}s \\
& + L_X \sum_{i=0}^{j-1} |X_{t_i} - X_{t_i}^N| \Delta t_N \\
& + \left| \int_0^{t_j} \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)}) \right) \,\mathrm{d}s \right|.
\end{aligned}
$$

Using the discrete version of the Gronwall Lemma, we prove (4.1). □

The first term in the right hand side of (4.1) usually vanishes since in general we take $X_0^N = X_0$, but it suffices to assume that X_0^N approximates X_0 to order Δt_N, which is useful for lower order approximations or for the discretization of (random) partial differential equations.

The third term in (4.1) is the more delicate one that will be handled differently in the next sections.

As for the second term, which only concerns the solution itself, not the approximation, we use the following simple but useful general result.

Lemma 4.2. *Under Hypothesis 2.1, it follows that*

$$\int_0^{t_j} \left|X_s - X_{\tau^N(s)}\right| \,\mathrm{d}s \leq \Delta t_N \int_0^{t_j} (M_s + L_X|X_s|)\,\mathrm{d}s. \tag{4.2}$$

Proof. By assumption, we have $|f(t, X_t, Y_t)| \leq M_t + L_X|X_t|$, for all $t \in I$ and all sample paths. Thus,

$$\left|X_s - X_{\tau^N(s)}\right| = \left|\int_{\tau^N(s)}^s f(\xi, X_\xi, Y_\xi)\,\mathrm{d}\xi\right| \leq \int_{\tau^N(s)}^s (M_\xi + L_X|X_\xi|)\,\mathrm{d}\xi.$$

Integrating over $[0, t_j]$ and using integration by parts,

$$\begin{aligned}
\int_0^{t_j} \left|X_s - X_{\tau^N(s)}\right| \,\mathrm{d}s &\leq \int_0^{t_j}\int_{\tau^N(s)}^s (M_\xi + L_X|X_\xi|)\,\mathrm{d}\xi\,\mathrm{d}s \\
&= \int_0^{t_j}\int_{\xi}^{\tau^N(\xi)+\Delta t_N} (M_\xi + L_X|X_\xi|)\,\mathrm{d}s\,\mathrm{d}\xi \\
&= \int_0^{t_j} (\tau^N(\xi) + \Delta t_N - \xi)(M_\xi + L_X|X_\xi|)\,\mathrm{d}\xi.
\end{aligned}$$

Using that $\tau^N(\xi) \leq \xi$ and that the remaining terms are non-negative, we have $\tau^N(\xi) + \Delta t_N - \xi \leq \Delta t_N$ and we obtain exactly (4.2). □

Combining the two previous results we obtain

Proposition 4.1. *Under Hypothesis 2.1, suppose further that* (2.4) *and* (2.5) *hold and that, for some constant* $C_0 \geq 0$,

$$\mathbb{E}[|X_0 - X_0^N|] \leq C_0\Delta t_N, \qquad N \in \mathbb{N}. \tag{4.3}$$

Then, for every $j = 0, \ldots, N$,

$$\begin{aligned}
&\mathbb{E}\left[|X_{t_j} - X_{t_j}^N|\right] \\
&\quad\leq \left(C_0\Delta t_N + \Delta t_N L_X\left(\mathbb{E}[|X_0|] + \int_0^{t_j}\mathbb{E}[M_\xi]\,\mathrm{d}\xi\right)e^{L_X t_j}\right. \\
&\qquad\left.\mathbb{E}\left[\left|\int_0^{t_j}\left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})\right)\,\mathrm{d}s\right|\right]\right)e^{L_X t_j}.
\end{aligned} \tag{4.4}$$

Proof. Under Hypothesis 2.1, Lemma 4.2 applies and estimate (4.2) holds. Using (2.4) and (2.5), that estimate yields

$$\int_0^{t_j}\mathbb{E}[|X_s - X_{\tau^N(s)}]|\,\mathrm{d}s \leq \Delta t_N \int_0^{t_j}(\mathbb{E}[M_s] + L_X\mathbb{E}[|X_s|])\,\mathrm{d}s.$$

Using now (2.3), we obtain

$$
\begin{aligned}
&\int_0^{t_j} \mathbb{E}[|X_s - X_{\tau^N(s)}|]\,\mathrm{d}s \\
&\qquad \leq \Delta t_N \int_0^{t_j} \left(\mathbb{E}[M_s] + L_X\left(\mathbb{E}[|X_0|] + \int_0^s \mathbb{E}[M_\xi]\,\mathrm{d}\xi\right)e^{L_X s}\right)\,\mathrm{d}s \\
&\qquad \leq \Delta t_N \left(\int_0^{t_j} \mathbb{E}[M_s]\,\mathrm{d}s + L_X\int_0^{t_j}\left(\mathbb{E}[|X_0|] + \int_0^{t_j} \mathbb{E}[M_\xi]\,\mathrm{d}\xi\right)e^{L_X s}\,\mathrm{d}s\right) \\
&\qquad = \Delta t_N \left(\int_0^{t_j} \mathbb{E}[M_s]\,\mathrm{d}s + \left(\mathbb{E}[|X_0|] + \int_0^{t_j} \mathbb{E}[M_\xi]\,\mathrm{d}\xi\right)\left(e^{L_X t_j} - 1\right)\right).
\end{aligned}
$$

Thus,

$$
\int_0^{t_j} \mathbb{E}[|X_s - X_{\tau^N(s)}|]\,\mathrm{d}s \leq \Delta t_N \left(\mathbb{E}[|X_0|] + \int_0^{t_j} \mathbb{E}[M_\xi]\,\mathrm{d}\xi\right)e^{L_X t_j}. \tag{4.5}
$$

Now we turn our attention to Lemma 4.1. Taking the expectation of the global error formula (4.1) gives

$$
\begin{aligned}
\mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq \Bigg(&\mathbb{E}\left[\left|X_0 - X_0^N\right|\right] + L_X\int_0^{t_j} \mathbb{E}\left[\left|X_s - X_{\tau^N(s)}\right|\right]\,\mathrm{d}s \\
&\mathbb{E}\left[\left|\int_0^{t_j}\left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})\right)\,\mathrm{d}s\right|\right]\Bigg)e^{L_X t_j}.
\end{aligned}
$$

Using now estimate (4.5) and condition (4.3), we find (4.4), which completes the proof. □

5. The case of monotonic sample path bounds

Here, the noise $\{Y_t\}_{t\in I}$ is *not* assumed to be an Itô noise and f is not assumed to be differentiable, but, instead, that the steps can be controlled by monotonic nondecreasing processes with finite expected growth. This fits well with the typical case of point processes, such as renewal-reward processes, Hawkes process, and such.

More precisely, we have the following result:

Lemma 5.1. *Besides Hypothesis 2.1, suppose that, for all* $0 \leq s \leq T$,

$$
|f(s, X_{\tau^N(s)}, Y_s) - f(\tau^N(s), X_{\tau^N(s)}, Y_{\tau^N(s)})| \leq \bar{F}_s - \bar{F}_{\tau^N(s)}, \tag{5.1}
$$

where $\{\bar{F}_t\}$ *is a real stochastic process satisfying*

$$
\mathbb{E}[|\bar{F}_t|] \text{ is monotonic nondecreasing and bounded in } t \in I. \tag{5.2}
$$

Then,

$$
\mathbb{E}\left[\left|\int_0^t \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})\right)\,\mathrm{d}s\right|\right] \leq (\mathbb{E}[\bar{F}_t] - \mathbb{E}[\bar{F}_0])\Delta t_N, \tag{5.3}
$$

for all $0 \leq t \leq T$ *and every* $N \in \mathbb{R}$.

Proof. Let $N \in \mathbb{R}$. From the assumption (5.1) we have

$$\mathbb{E}\left[|f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})|\right] \leq \mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_{\tau^N(s)}],$$

for every $0 \leq s \leq T$. Thus, upon integration,

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t \left(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})\right)\,\mathrm{d}s\right|\right] \\
&\qquad \leq \int_0^t \mathbb{E}\left[|f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)})|\right]\,\mathrm{d}s \\
&\qquad \leq \int_0^t \left(\mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_{\tau^N(s)}]\right)\,\mathrm{d}s.
\end{aligned}$$

Now we need to bound the right hand side. When $0 \leq t \leq t_1 = \Delta t_N$, we have $\tau^N(s) = 0$ for all $0 \leq s < t_1$, so that,

$$\int_0^t (\mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_{\tau^N(s)}])\,\mathrm{d}s = \int_0^t (\mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_0])\,\mathrm{d}s.$$

Using the monotonicity and the condition that $t \leq \Delta t_N$,

$$\begin{aligned}
\int_0^t (\mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_{\tau^N(s)}])\,\mathrm{d}s &\leq \int_0^t (\mathbb{E}[\bar{F}_t] - \mathbb{E}[\bar{F}_0])\,\mathrm{d}s \\
&= (\mathbb{E}[\bar{F}_t] - \mathbb{E}[\bar{F}_0])t \leq (\mathbb{E}[\bar{F}_t] - \mathbb{E}[\bar{F}_0])\Delta t_N.
\end{aligned}$$

When $\Delta t_N \leq t \leq T$, we split the integration of the second term at time $s = t_1 = \Delta t_N$ and write

$$\int_0^t (\mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_{\tau^N(s)}])\,\mathrm{d}s = \int_0^t \mathbb{E}[\bar{F}_s]\,\mathrm{d}s - \int_0^{t_1} \mathbb{E}[\bar{F}_{\tau^N(s)}]\,\mathrm{d}s - \int_{t_1}^t \mathbb{E}[\bar{F}_{\tau^N(s)}]\,\mathrm{d}s$$

Using the monotonicity together with the fact that $s - \Delta t_N \leq \tau^N(s) \leq s$ for all $\Delta t_N \leq s \leq T$,

$$\begin{aligned}
\int_0^t (\mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_{\tau^N(s)}])\,\mathrm{d}s &\leq \int_0^t \mathbb{E}[\bar{F}_s]\,\mathrm{d}s - \int_0^{\Delta t_N} \mathbb{E}[\bar{F}_0]\,\mathrm{d}s - \int_{\Delta t_N}^t \mathbb{E}[\bar{F}_{s-\Delta t_N}]\,\mathrm{d}s \\
&= \int_0^t \mathbb{E}[\bar{F}_s]\,\mathrm{d}s - \int_0^{\Delta t_N} \mathbb{E}[\bar{F}_0]\,\mathrm{d}s - \int_0^{T-\Delta t_N} \mathbb{E}[\bar{F}_s]\,\mathrm{d}s \\
&= \int_{t-\Delta t_N}^t \mathbb{E}[\bar{F}_s]\,\mathrm{d}s - \mathbb{E}[\bar{F}_0]\Delta t_N.
\end{aligned}$$

Using again the monotonicity yields

$$\int_0^t (\mathbb{E}[\bar{F}_s] - \mathbb{E}[\bar{F}_{\tau^N(s)}])\,\mathrm{d}s \leq \int_{t-\Delta t_N}^t \mathbb{E}[\bar{F}_t]\,\mathrm{d}s - \mathbb{E}[\bar{F}_0]\Delta t_N = (\mathbb{E}[\bar{F}_t] - \mathbb{E}[\bar{F}_0])\Delta t_N.$$

Putting the estimates together proves (5.3). $\square$

Theorem 5.1. *Under Hypothesis 2.1, suppose also that (2.4), (2.5), (4.3), (5.1), and (5.2) hold. Then, the Euler-Maruyama scheme (1.2)-(1.3) is of strong order 1, i.e.*

$$\max_{j=0,\ldots,N} \mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq C\Delta t_N, \qquad \forall N \in \mathbb{N}, \tag{5.4}$$

for a constant C given by

$$C = \left(C_0 + L_X\left(\mathbb{E}[|X_0|] + \int_0^T \mathbb{E}[M_\xi]\,\mathrm{d}\xi\right)e^{L_X T} + (\mathbb{E}[\bar{F}_T] - \mathbb{E}[\bar{F}_0])\right)e^{L_X T} \tag{5.5}$$

Proof. Under Hypothesis 2.1, the Lemma 4.1 applies and the global error estimate (4.1) holds.

Thanks to (2.4), (2.5), and (4.3), the Proposition 4.1 applies and the global error is bounded according to (4.4).

With assumptions (5.1) and (5.2), Lemma 5.1 applies and the last term in (4.4) is bounded according to (5.3). Using (5.3) in (4.4) yields

$$\begin{aligned}\mathbb{E}\left[|X_{t_j} - X_{t_j}^N|\right] \leq \Big(C_0\Delta t_N + \Delta t_N L_X\left(\mathbb{E}[|X_0|] + \int_0^{t_j} \mathbb{E}[M_\xi]\,\mathrm{d}\xi\right)e^{L_X t_j} \\ + (\mathbb{E}[\bar{F}_{t_j}] - \mathbb{E}[\bar{F}_0])\Delta t_N\Big)e^{L_X t_j}.\end{aligned}$$

Since this holds for every $j = 0, \ldots, N$, we obtain the desired (5.4). ☐

The conditions of Theorem 5.1, especially (5.1)-(5.2), are not readily verified, but the following result gives more explicit conditions.

Theorem 5.2. *Suppose that $f = f(t, x, y)$ is uniformly globally Lipschitz continuous in x and is continuously differentiable in (t, y), with partial derivatives $\partial_t f$ and $\partial_y f$ with at most linear growth in x and y, i.e.*

$$|\partial_t f(t, x, y)| \leq C_1 + C_2|x| + C_3|y|, \quad |\partial_y f(t, x, y)| \leq C_4 + C_5|x| + C_6|y|, \tag{5.6}$$

in $(t, x, y) \in I \times \mathbb{R} \times \mathbb{R}$, for suitable constants $C_1, C_2, C_3, C_4 \geq 0$. Assume, further, that the sample paths of $\{Y_t\}_{t\in I}$ are of bounded variation $V(\{Y_t\}_{t\in I}; I)$, on I, with finite quadratic mean,

$$\mathbb{E}[V(\{Y_t\}_{t\in I}; I)^2] < \infty, \tag{5.7}$$

and with

$$\mathbb{E}[|X_0|^2] < \infty. \tag{5.8}$$

Then, the Euler-Maruyama scheme is of strong order 1, i.e.

$$\max_{j=0,\ldots,N} \mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq C\Delta t_N, \qquad \forall N \in \mathbb{N}, \tag{5.9}$$

for a suitable constant $C \geq 0$.

Proof. Notice that

$$
\begin{aligned}
|f(t,x,y)| \leq |f(t,x,y) - f(t,0,y)| + |f(t,0,y) - f(0,0,y)| + |f(0,0,y) - f(0,0,0)| \\
\leq L_X|x| + C_1 + C_3|y| + C_4 + C_6|y|.
\end{aligned}
$$

Thus,

$$
|f(t,x,Y_t)| \leq M_t + L_X|x|,
$$

where

$$
M_t = C_1 + C_4 + (C_3 + C_6)|Y_t|.
$$

Since the sample paths of $\{Y_t\}_{t\in I}$ are of bounded variation, the process $\{M_t\}_{t\in I}$ has integrable sample paths. This means that we are under the Hypothesis 2.1. Moreover, thanks to (5.7), we see that

$$
\mathbb{E}[|Y_t|] \leq \mathbb{E}[|Y_t|^2] \leq \mathbb{E}[V(\{Y_t\}_{t\in I}; I)^2] < \infty.
$$

Then, thanks to the Lyapunov inequality $\mathbb{E}[|Y_t|] \leq \mathbb{E}[|Y_t|^2]^{1/2}$, we see that $\{M_t\}_{t\in I}$ satisfies (2.5). By assumption, (2.4) also holds, so that, from (2.3), we have

$$
K_X = \sup_{t\in I} \mathbb{E}[|X_t|^2] < \infty.
$$

Now, in order to apply Theorem 5.1, it remains to verify (5.1)-(5.2). We have

$$
\begin{aligned}
|f(s,X_\tau,Y_s) - f(\tau,X_\tau,Y_\tau)| = \left| \int_\tau^s \partial_t f(\xi,X_\tau,Y_\xi)\,\mathrm{d}\xi + \int_\tau^s \partial_y f(\xi,X_\tau,Y_\xi)\,\mathrm{d}Y_\xi \right| \\
\leq C_1(s-\tau) + C_2(s-\tau)|X_\tau| + (C_3 + C_4|X_\tau|)V(\{Y_t\}_{t\in I}; \tau, s).
\end{aligned}
$$

Thus, (5.1) holds with

$$
\bar{F}_t = (C_1 + C_2|X_{\tau^N(t)}|)t + (C_3 + C_4|X_{\tau^N(t)}|)V(\{Y_t\}_{t\in I}; 0, t).
$$

It is clear that, not only the expectation, but all the sample paths of $\{F_t\}_{t\in I}$ are monotonic non-decreasing in $t \in I$, with $\bar{F}_0 = 0$. Moreover, thanks to (5.7), and using the Cauchy-Schwarz inequality in the last term, we have

$$
\mathbb{E}[\bar{F}_T] \leq (C_1 + C_2K_1)T + (C_3 + C_4K_1)\mathbb{E}[V(\{Y_t\}_{t\in I}; 0, T)^2] < \infty.
$$

Thus, Theorem 5.1 applies and we deduce the strong order 1 convergence of the Euler-Maruyama method. □

Remark 5.1. The conditions (5.7) and (5.8) on the finite mean square of the total variation of the noise and of the initial condition can be relaxed provided we have a better control on the growth of the $\partial_y f(t,x,y)$ with respect to x. More precisely, if

$$
|\partial_y f(t,x,y)| \leq C_4 + C_5|x|^{p-1} + C_6|y|,
$$

and

$$
\mathbb{E}[V(\{Y_t\}_{t\in I}; T, 0)^p] < \infty,
$$

along with

$$
\mathbb{E}[|X_0|^p] < \infty,
$$

with $1 \leq p < \infty$, then the process $\{\bar{F}_t\}_{t\in I}$ becomes

$$\bar{F}_t = (C_1 + C_2|X_{\tau^N(t)}|)t + (C_3 + C_4|X_{\tau^N(t)}|^{p-1})V(\{Y_t\}_{t\in I}; 0, t).$$

Applying the Hölder inequality yields

$$\bar{F}_t \leq (C_1 + C_2|X_{\tau^N(t)}|)t + C_3 V(\{Y_t\}_{t\in I}; 0, t) + C_4\frac{p-1}{p}|X_{\tau^N(t)}|^p + \frac{C_4}{p}V(\{Y_t\}_{t\in I}; 0, t)^p.$$

With that, the required conditions on $\{\bar{F}_t\}_{t\in I}$ are met and we are allowed to apply Theorem 5.1 and deduce the strong order 1 convergence of the Euler-Maruyama method.

Remark 5.2. One particular example that fits the conditions of Theorem 5.2 is when $f = f(t, x, y)$ is semi-separable, i.e.

$$f(t, x, y) = a(t, y)h(x) + b(t, y), \tag{5.10}$$

where $a = a(t, y)$ and $b = b(t, y)$ are continuously differentiable on $I \times \mathbb{R}$ with uniformly bounded first derivatives, $a = a(t, y)$ itself is uniformly bounded, and $h = h(x)$ is globally Lipschitz continous on $\mathbb{R}$.

Since $a = a(t, x)$ is uniformly bounded and $h = h(x)$ is globally Lipschitz continuous, it follows that $f = f(t, x, y)$ is uniformly globally Lipschitz continuous in x. Moreover, it is continuously differentiable in (t, y), with partial derivatives $\partial_t f$ and $\partial_y f$ given by

$$\partial_t f = \partial_t a(t, y)h(x) + \partial_t b(t, y), \qquad \partial_y f = \partial_y a(t, y)h(x) + \partial_t b(t, y)$$

Since the partial derivatives of $a = a(t, y)$ and $b = b(t, y)$ are uniformly bounded and h is Lipstchiz, it follows that the partial derivatives $\partial_t f$ and $\partial_y f$ have at most linear growth. Thus, (5.6) is satisfies.

6. The case of an Itô noise

Here, as explained in the Introduction, we assume the process given by $F_t = f(s, X_{\tau^N(s),Y_s})$ is an Itô process, which, in applications, follows from assuming that $f = f(t, x, y)$ is sufficiently regular and that the noise $\{Y_t\}_{t\in I}$ is itself an Itô process.

With that in mind, we first have the following result.

Lemma 6.1. *Besides Hypothesis 2.1, suppose that $F_t^N = f(t, X_{\tau^N(t)}, Y_t)$ is an Itô noise, satisfying*

$$\mathrm{d}F_t^N = A_t\,\mathrm{d}t + B_t\,\mathrm{d}W_t, \tag{6.1}$$

for a Wiener process $\{W_t\}_{t\in 0}$ and stochastic processes $\{A_t\}_{t\in I}$, $\{B_t\}_{t\in I}$ adapted to $\{W_t\}_{t\geq 0}$ and such that

$$\int_0^T \mathbb{E}[|A_t|]\,\mathrm{d}t < \infty, \quad \int_0^T \mathbb{E}[|B_t|^2]\,\mathrm{d}t < \infty. \tag{6.2}$$

Then,

$$\mathbb{E}\left[\left|\int_0^t \Big(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X_{\tau^N(s)}, Y_{\tau^N(s)})\Big)\;\mathrm{d}s\right|\right]$$
$$\leq \Delta t_N \left(\int_0^t \mathbb{E}[|A_\xi|]\;\mathrm{d}\xi + \left(\int_0^t \mathbb{E}[|B_\xi|^2]\;\mathrm{d}\xi\right)^{1/2}\right), \quad (6.3)$$

for all $0 \leq t \leq T$ *and every* $N \in \mathbb{R}$.

Proof. We write

$$f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)}) = \int_{\tau^N(s)}^s A_\xi\;\mathrm{d}\xi + \int_{\tau^N(s)}^s B_\xi\;\mathrm{d}W_\xi.$$

Upon integration,

$$\int_0^t \Big(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X_{\tau^N(s)}, Y_{\tau^N(s)})\Big)\;\mathrm{d}s$$
$$= \int_0^t \left(\int_{\tau^N(s)}^s A_\xi\;\mathrm{d}\xi + \int_{\tau^N(s)}^s B_\xi\;\mathrm{d}W_\xi\right)\;\mathrm{d}s$$

Using integration by parts,

$$\int_0^t \Big(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X_{\tau^N(s)}, Y_{\tau^N(s)})\Big)\;\mathrm{d}s$$
$$= \int_0^t \int_\xi^{\tau^N(\xi)+\Delta t_N} A_\xi\;\mathrm{d}s\;\mathrm{d}\xi + \int_0^t \int_\xi^{\tau^N(\xi)+\Delta t_N} B_\xi\;\mathrm{d}s\;\mathrm{d}W_\xi$$
$$= \int_0^t (\tau^N(\xi) + \Delta t_N - \xi) A_\xi\;\mathrm{d}\xi + \int_0^t (\tau^N(\xi) + \Delta t_N - \xi) B_\xi\;\mathrm{d}W_\xi.$$

Taking the absolute mean and using the Itô isometry on the second term yields

$$\mathbb{E}\left[\left|\int_0^t \Big(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X_{\tau^N(s)}, Y_{\tau^N(s)})\Big)\;\mathrm{d}s\right|\right]$$
$$\leq \int_0^t |\tau^N(\xi) + \Delta t_N - \xi|\mathbb{E}[|A_\xi|]\;\mathrm{d}\xi + \left(\int_0^t (\tau^N(\xi) + \Delta t_N - \xi)^2 \mathbb{E}[|B_\xi|^2]\;\mathrm{d}\xi\right)^{1/2}.$$

Since $|\tau^N(\xi) + \Delta t_N - \xi| \leq \Delta t_N$, we find

$$\mathbb{E}\left[\left|\int_0^t \Big(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X_{\tau^N(s)}, Y_{\tau^N(s)})\Big)\;\mathrm{d}s\right|\right]$$
$$\leq \Delta t_N \left(\int_0^t \mathbb{E}[|A_\xi|]\;\mathrm{d}\xi + \left(\int_0^t \mathbb{E}[|B_\xi|^2]\;\mathrm{d}\xi\right)^{1/2}\right),$$

which completes the proof. $\square$

Combining the estimate in Lemma 6.1 with the previous estimate for the global error we obtain the following main result.

Theorem 6.1. *Under Hypothesis 2.1, suppose also that (2.4), (2.5), (4.3), (6.1), and (6.2) hold. Then, the Euler-Maruyama scheme (1.2)-(1.3) is of strong order 1, i.e.*

$$\max_{j=0,\ldots,N} \mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq C\Delta t_N, \qquad \forall N \in \mathbb{N}, \tag{6.4}$$

for a constant C given by

$$C = \left(C_0 + L_X\left(\mathbb{E}[|X_0|] + \int_0^T \mathbb{E}[M_\xi]\,\mathrm{d}\xi\right)e^{L_X T}\right.$$
$$\left. + \left(\int_0^T \mathbb{E}[|A_\xi|]\,\mathrm{d}\xi + \left(\int_0^T \mathbb{E}[|B_\xi|^2]\,\mathrm{d}\xi\right)^{1/2}\right)\right)e^{L_X T} \tag{6.5}$$

Proof. Under Hypothesis 2.1, the Lemma 4.1 applies and the global error estimate (4.1) holds.

Thanks to (2.4), (2.5), and (4.3), the Proposition 4.1 applies and the global error is bounded according to (4.4).

With assumptions (6.1) and (6.2), Lemma 6.1 applies and the last term in (4.4) is bounded according to (6.3). Using (6.3) in (4.4) yields

$$\mathbb{E}\left[|X_{t_j} - X_{t_j}^N|\right] \leq \left(C_0\Delta t_N + \Delta t_N L_X\left(\mathbb{E}[|X_0|] + \int_0^{t_j} \mathbb{E}[M_\xi]\,\mathrm{d}\xi\right)e^{L_X t_j}\right.$$
$$\left. + \Delta t_N\left(\int_0^{t_j} \mathbb{E}[|A_\xi|]\,\mathrm{d}\xi + \left(\int_0^{t_j} \mathbb{E}[|B_\xi|^2]\,\mathrm{d}\xi\right)^{1/2}\right)\right)e^{L_X t_j}.$$

Since this holds for every $j = 0, \ldots, N$, we obtain the desired (6.4). $\square$

In practice, conditions (6.1)-(6.2) follows from assuming sufficent regularity on $f = f(t, x, y)$ and an Itô noise, as given by the following result.

Theorem 6.2. *Let $f = f(t, x, y)$ be twice continuously differentiable with uniformly bounded derivatives. Suppose that the noise $\{Y_t\}_{t\in I}$ is an Itô noise,*

$$\mathrm{d}Y_t = a(t, Y_t)\,\mathrm{d}t + b(t, Y_t)\,\mathrm{d}W_t, \tag{6.6}$$

with $a = a(t, y)$ and $b = b(t, y)$ continuous and satisfying

$$|a(t, y)| \leq A_M + A_Y|y|, \qquad |b(t, y)| \leq B_M + B_Y|y|. \tag{6.7}$$

Assume the bounds (2.4), (4.3), and

$$\mathbb{E}[|Y_0|] < \infty \tag{6.8}$$

Then, the Euler-Maruyama scheme is of strong order 1, i.e.

$$\max_{j=0,\ldots,N} \mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq C\Delta t_N, \qquad \forall N \in \mathbb{N}, \tag{6.9}$$

for a suitable constant $C \geq 0$.

Proof. Let us start by showing that Hypothesis 2.1 is valid. Since $f = f(t, x, y)$ is (twice) continuously differentiable with, in particular, bounded derivative in x, then it is uniformly globally Lipscthiz in x. Since $a = a(t, y)$ and $b = b(t, y)$ are continuous, the the noise has continuous sample paths. Thus, the remaining condition in Hypothesis 2.1 to be verified is (iic).

From (6.6) and (6.7), we have

$$Y_t = \int_0^t a(s, Y_s)\,\mathrm{d}s + \int_0^s b(t, Y_s)\,\mathrm{d}W_s.$$

Using the Itô formula, we have

$$\mathrm{d}Y_t^2 = \left(2a(t, Y_t)Y_t + b(t, Y_t)^2\right)\,\mathrm{d}t + 2b(t, Y_t)Y_t\,\mathrm{d}W_t.$$

Thus

$$Y_t^2 = Y_0^2 + \int_0^t \left(2a(s, Y_s)Y_s + b(t, Y_s)^2\right)\,\mathrm{d}s + \int_0^t 2b(s, Y_s)Y_s\,\mathrm{d}W_s.$$

Taking the expectation,

$$\mathbb{E}[|Y_t|^2] = \mathbb{E}[|Y_0|^2] + \int_0^t \left(2a(s, Y_s)Y_s + b(t, Y_s)^2\right)\,\mathrm{d}s.$$

Using (6.7), this yields

$$\begin{aligned}\mathbb{E}[|Y_t|^2] &\leq \mathbb{E}[|Y_0|^2] + \int_0^t \left(2\mathbb{E}[(A_M + A_Y|Y_s|)|Y_s|] + \mathbb{E}[(B_M + B_Y|Y_s|)^2]\right)\,\mathrm{d}s \\ &\leq \mathbb{E}[|Y_0|^2] + \int_0^t \left(4(A_M^2 + (1 + A_Y)\mathbb{E}[|Y_s|^2]) + 2(B_M^2 + B_Y^2\mathbb{E}[|Y_s|^2])\right)\,\mathrm{d}s\end{aligned}$$

By the Gronwall Lemma,

$$\mathbb{E}[|Y_t|^2] \leq \left(\mathbb{E}[|Y_0|^2] + (4A_M^2 + 2B_M^2)t\right) e^{(4(1+A_Y)+2B_Y^2)t}.$$

Thus,

$$\sup_{t\in I} \mathbb{E}[|Y_t|^2] \leq \left(\mathbb{E}[|Y_0|^2] + (4A_M^2 + 2B_M^2)T\right) e^{(4(1+A_Y)+2B_Y^2)T}. \tag{6.10}$$

Since $f = f(t, x, y)$ is Lipschitz in x and twice continuously differentiable in (t, y) with uniformly bounded first order derivatives, we have the bound

$$|f(t, x, y)| \leq |f(0, 0, 0)| + L_X|x| + L_T|t| + L_Y|y|.$$

Thus,

$$|f(t, x, Y_t)| \leq M_t + L_X|x|$$

with

$$M_t = |f(0, 0, 0)| + L_T|t| + L_Y|y|.$$

Thanks to (6.10), we see that

$$\int_0^T M_t \,\mathrm{d}t < \infty.$$

Therefore, we are under the condition of (2.1).

Now, in view of Theorem 6.1, it remains to prove that $F_t^N = f(t, X_{\tau^N(t)}, Y_t)$ is an Itô noise (6.1), with the bounds (6.2). The fact that it is an Itô noise follows from the Itô formula and the fact the smoothness of $f = f(t, x, y)$. Indeed, since $(t, y) \mapsto f(t, x, y)$ is twice continuously differentiable, for each fixed x, the Itô formula is applicable and yields

$$\mathrm{d}f(t, x, Y_t) = \left(\partial_t f(t, x, Y_t) + a(t, Y_t)\partial_y f(t, x, Y_t) + \frac{b(t, Y_t)^2}{2}\partial_{yy} f(t, x, Y_t)\right) \mathrm{d}t \\ + b(t, Y_t)\partial_y f(t, x, Y_t) \,\mathrm{d}W_t, \quad (6.11)$$

for every fixed $x \in \mathbb{R}$. This means (6.1) holds with

$$A_t = \partial_t f(t, x, Y_t) + a(t, Y_t)\partial_y f(t, x, Y_t) + \frac{b(t, Y_t)^2}{2}\partial_{yy} f(t, x, Y_t)$$

and

$$B_t = b(t, Y_t)\partial_y f(t, x, Y_t).$$

It remains to show that $\{A_t\}_{t\in I}$ is mean integrable and that $\{B_t\}_{t\in I}$ is square mean integrable. Since $f = f(t, x, y)$ has uniformly bounded derivatives, we have

$$|A_t| \leq L_T + L_Y(A_M + A_Y|Y_t|) + 2L_{YY}(B_M^2 + B_Y^2|Y_t|^2),$$

and

$$|B_t| \leq L_Y(B_M + B_Y|Y_t|),$$

for a suitable constants $L_{YY} \geq 0$. Now, thanks to (6.10), we see that (6.2) is satisfied.

Therefore, all the conditions of Theorem 6.1 are met and we deduce the strong order 1 convergence of the Euler-Maruyama method. □

Remark 6.1. In the case that the diffusion term $b = b(t, y)$, in (6.6), is actually independent of y, then the noise is an additive noise and the Euler-Maruyama scheme is of strong order 1, otherwise, it has always been regarded to be of order 1/2 (CITATION...). Here, however, we deduce, under the conditions of Theorem 6.2, that even if $b = b(t, y)$ depends on y, the strong convergence of the Euler-Maruyama scheme is actually of order 1.

7. Applications

In this section, we describe a few explicit examples that fall into one of the cases considered above and, hence, the Euler-Maruyama method exhibits a strong order one convergence.

7.1. **Earthquake and other impulse driven models.** See Neckel and Rupp pg 582 as a starting point of a model driven by a transport process as the source of ground motion excitation (both Kanai-Tajima and Bogdanoff, check out also the Clough-Penzien).

7.2. **RODE with Itô noise.** Here, we consider a class of RODEs treated in [1, Chapter 3], namely the (1.1) with a noise $\{Y_t\}_{t\in I}$ satisfying

$$\mathrm{d}Y_t = a(Y_t)\;\mathrm{d}t + b(Y_t)\;\mathrm{d}W_t \tag{7.1}$$

with the assumptions that

the functions $a = a(y)$, $b = b(y)$, and $f = f(t,x,y)$ are twice continuously differentiable with all partial derivatives uniformly bounded. (7.2)

7.3. **Population dynamics.** Our first example is a population dynamics model modified from [4, Section 15.2],

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = Z_t X_t(r - X_t) \tag{7.3}$$

where $r > 0$ is constant and $\{Z_t\}_{t\geq}$ is a stochastic process playing the role of a random growth parameter and given by

$$Z_t = \lambda(1 + \varepsilon\sin(Y_t)),$$

where $0 < \varepsilon < 1$ and $\{Y_t\}_{t\geq 0}$ is an Itô noise process satisfying (6.6). We assume Y_t has an analytic formula so we do not need to approximate the coupled stochastic differential equation system for (X_t, Y_t). This setting includes a Wiener process, an Orstein-Uhlenbeck process and a geometric Brownian motion process.

We suppose the initial condition is non-negative and bounded almost surely:

$$0 \leq X_0 \leq R,$$

for some $R > r$.

The noise process $\{Z_t\}_{t\geq 0}$ itself satisfies

$$0 < \lambda - \varepsilon \leq Z_t \leq \lambda + \varepsilon < 2\lambda, \qquad \forall t \geq 0.$$

Define

$$f(t,x,z) = zx(r - x)$$

and notice that $f(t,x,z)x \geq 0$, for $x \geq 0$ and $z \geq 0$, and $f(t,x,z)x \leq 0$, for $x \geq r$ and $z \geq 0$. Hence the interval $[0,R]$ in x is positively invariant and the pathwise solutions of (7.3) are almost surely bounded as well, with

$$0 \leq X_t \leq R,$$

for all $t \geq 0$.

The function $f = f(t,x,z)$ is continuously differentiable infinitely many times and with

$$\left|\frac{\partial f}{\partial x}(t,x,y)\right| = |y(r - 2x)| \leq 2\lambda(2R - r),$$

for $|x| \leq R$ and $0 \leq z \leq 2\lambda$. In turn, the function $z = z(y) = \lambda(1 + \varepsilon \sin(y))$ is also continuously differentiable infinitely many times and is uniformly bounded along with all its derivatives.

The right hand side of (7.3) is not globally Lipschitz, but, for the sake of analysis, since X_t and Y_t are bounded, the right hand side can be modified to a twice continuously differentiable, uniformly globally Lipschitz function $\tilde{f}(t, x, y)$ that coincides with $f(t, x, y)$ for $(t, x, y) \in \mathbb{R} \times [0, R] \times [0, 2\lambda]$ and satisfies (2.1) with

$$L_X = 2\lambda(2R - r).$$

Thus, the RODE (7.3) with $0 \leq X_0 \leq R$ almost surely, for some $R > r$, is equivalent to the RODE

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = \tilde{f}(t, X_t, Y_t). \tag{7.4}$$

With $\tilde{f} = \tilde{f}(t, x, y)$, the Hypothesis 2.1 hold. Moreover, it follows from (2.3) (notice $M_t = 0$ here) that

$$|X_t| \leq |X_0| e^{2\lambda(2R - r)t} \leq R e^{2\lambda(2R-r)T}, \qquad 0 \leq t \leq T.$$

almost surely.

OLD Condition on $\{F_t\}$ SHOULD BE FIXED is satisfied with

$$F_t?!?!?!$$

CAREFULL, WE CHANGED THE NOTATION, MUST FIX FROM HERE ON AND CHECK EVERYTHING.

The Itô formula applied to $Y_t = g(O_t)$, where $g(\eta) = \lambda(1+\varepsilon \sin(\eta))$ implies $\{Y_t\}_{t\geq 0}$ is an Itô process with

$$\mathrm{d}Y_t = \left((\theta_1 - \theta_2 O_t) g'(O_t) + \frac{\theta_3^2}{2} g''(O_t) \right) \mathrm{d}t + \theta_3 g'(O_t) \,\mathrm{d}W_t.$$

We have

$$g'(\eta) = \lambda\varepsilon \cos(\eta), \quad g''(\eta) = -\lambda\varepsilon \sin(\eta)$$

hence both are uniformly bounded. Therefore, all the conditions of Theorem 6.2 hold and the Euler-Maruyama method is of strong order 1.

7.4. **Drug delivery.**

7.5. **Earthquake model.**

7.6. **Point-process noise.**

Appendix A. Discrete Grownwall Lemma

We end this section by abstracting away the Gronwall type inequality we use (this is probably written somewhere, and I need to find the source):

Lemma A.1. *Let $(e_j)_j$ be a (finite or infinite) sequence of positive numbers satisfying*

$$e_j \leq a \sum_{i=0}^{j-1} e_i + b, \tag{A.1}$$

with $e_0 = 0$, where $a, b > 0$. Then,

$$e_j \leq b e^{aj}, \qquad \forall j. \tag{A.2}$$

Proof. The result is trivially true for $j = 0$. Suppose, by induction, that the result is true up to $j - 1$. Then,

$$e_j \leq a \sum_{i=0}^{j-1} b e^{ai} + b = b \left(a \sum_{i=0}^{j-1} e^{ai} + 1 \right).$$

Using that $1 + a \leq e^a$, we have $a \leq e^a - 1$, hence

$$e_j \leq b \left((e^a - 1) \sum_{i=0}^{j-1} e^{ia} + 1 \right).$$

Using that $\sum_{i=0}^{j-1} \theta^i = (\theta^j - 1)(\theta - 1)$, with $\theta = e^a$, we see that

$$(e^a - 1) \sum_{i=0}^{j-1} e^{ia} \leq e^{ja} - 1,$$

so that

$$e_j \leq b e^{ja},$$

which completes the induction. $\square$

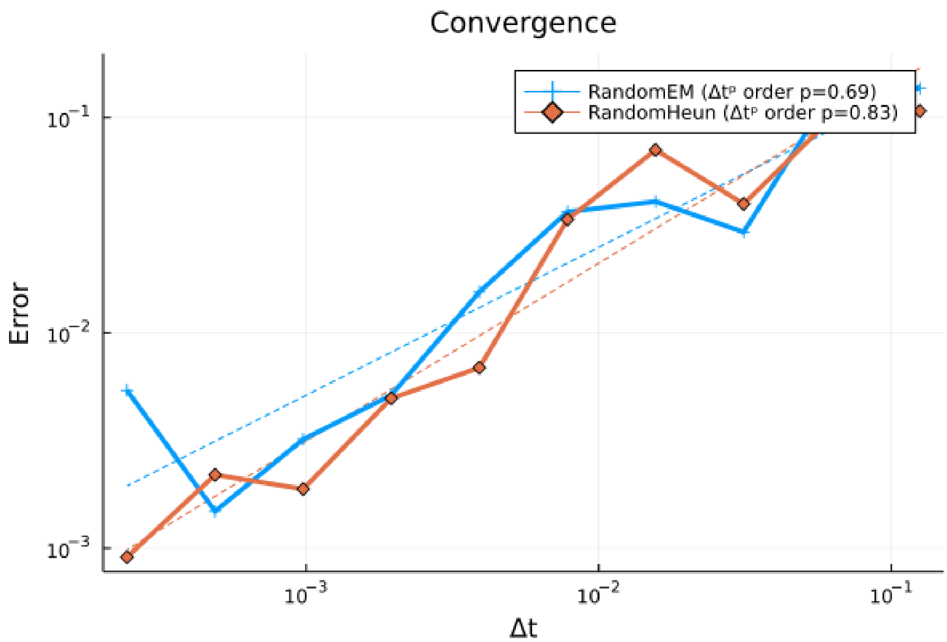
Appendix B. Numerical examples

B.1. **Lower-order converge.** For a lower order convergence, below order 1, we take the noise $\{Y_t\}_t$ to be the transport process defined by

$$Y_t = \sin(t/Z)^{1/3},$$

where Z is a beta random variable $Z \sim B(\alpha, \beta)$. Notice Z takes values strictly within $(0, 1)$ and, hence, $\sin(t/Z)$ can have arbitrarily high frequencies and, hence, go through the critic value $y = 0$ extremely often.

(Need to remove the Heun method and do more tests).



Acknowledgments

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