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THE EULER-MARUYAMA SCHEME FOR RANDOM ORDINARY DIFFERENTIAL EQUATIONS IS OF STRONG ORDER 1

RICARDO M. S. ROSA

ABSTRACT. It is well known that the Euler-Maruyama method of approximating a random ordinary differential equation $\mathrm{d}X_t/\mathrm{d}t = f(t, X_t, Y_t)$ driven by a stochastic process $\{Y_t\}_t$ with θ-Hölder sample paths is estimated to be of strong order θ with respect to the time step, provided e.g. $f = f(t, x, w)$ is globally Lipschitz. Here, we show that, if $\{Y_t\}_t$ is an Itô process with bounded drift and diffusion and f is twice continuously differentiable and is bounded along with its derivatives, then the Euler-Maruyama method is actually of strong order 1. The estimate follows from not estimating part of the local error and, instead, adding up the local steps and estimating the compound error via the Itô formula and the Itô isometry. We complement the result by giving examples where some of the conditions are not met and the order of convergence seems indeed to be less than 1.

1. INTRODUCTION

We consider the following initial value problem for a **random ordinary differential equation (RODE)**:

$$\begin{cases} \dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t), & 0 \leq t \leq T, \\ X_t\big|_{t=0} = X_0, \end{cases} \tag{1.1}$$

where $\{Y_t\}_{t\in I}$ is a real stochastic process with continuous sample paths on the time interval $I = [0, T]$; $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous; and X_0 is a real random variable. The sample space is denoted by Ω. A similar result holds for systems of random ordinary equations, as discussed later in the article, but we start with the scalar case to keep the ideas simple and clear.

The Euler-Maruyama method for solving this initial value problem on the time interval $I = [0, T]$ consists in approximating the solution on a uniform time mesh $t_j = j\Delta t$, $j = 0, \ldots, N$, with fixed time step $\Delta t = T/N$, for a given $N \in \mathbb{N}$. Then, the Euler-Maruyama scheme for the approximation takes the form

$$X_{t_j}^N = X_{t_{j-1}}^N + \Delta t f(t_{j-1}, X_{t_{j-1}}^N, Y_{t_{j-1}}), \qquad j = 1, \ldots, N, \tag{1.2}$$

Date: September 28, 2022.

2000 *Mathematics Subject Classification.* 76D05, 76D06, 35Q30, 37L99.

Key words and phrases. random ordinary differential equations, Euler-Maruyama method, strong convergence, Itô process.

with the initial condition

$$X_0^N = X_0. \tag{1.3}$$

One of our interests is to drive the equation with an **Itô process** $\{Y_t\}_{t\geq 0}$, satisfying

$$\mathrm{d}Y_t = A_t\,\mathrm{d}t + B_t\,\mathrm{d}W_t, \tag{1.4}$$

where $\{W_t\}_{t\geq 0}$ is a Wiener process and $\{A_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$ are stochastic processes adapted to the $\{W_t\}_{t\geq 0}$. We are not solving for Y_t, otherwise we would actually have a system of stochastic differential equations. Instead, we assume it is a known process, and we allow A_t and B_t to actually be given in terms of $\{W_t\}_{t\geq 0}$ and $\{Y_t\}_{t\geq 0}$. For example, Y_t may be a geometric Brownian process or some function of a Wiener or geometric Brownian process.

In the case that $f = f(t,x,y)$ is twice continuously differentiable, the Itô formula is applicable and yields

$$\begin{aligned} \mathrm{d}f(t,x,Y_t) = \left(\partial_t f(t,x,Y_t) + A_t\partial_y f(t,x,Y_t) + \frac{B_t^2}{2}\partial_{yy} f(t,x,Y_t)\right)\,\mathrm{d}t \\ + B_t\partial_y f(t,x,Y_t)\,\mathrm{d}W_t, \end{aligned} \tag{1.5}$$

for every fixed $x\in\mathbb{R}$.

We show that, if the expectations of $\{A_t\}_t$ and $\{B_t\}_t$ are uniformly bounded in time on $[0,T]$ and $\partial_t f$, $\partial_x f$, $\partial_y f$, and $\partial_{yy} f$ are uniformly bounded on $[0,T]\times\mathbb{R}\times\mathbb{R}$, then the Euler-Maruyama method is of strong order 1, i.e. there exists $C>0$ such that

$$\mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq C\Delta t, \qquad \forall N\in\mathbb{N}, \Delta t = \frac{T}{N}, \tag{1.6}$$

where $\mathbb{E}[\cdot]$ indicates the expectation of a random variable on Ω (see Theorem 2.1).

We summarize here the main tricks we use to accomplish such error estimate:

(i) We assume the noise is an Itô process, so we can use the Itô isometry at some point;
(ii) We use the Itô formula to separate the most problematic/rough part of the noise;
(iii) We do not estimate this problematic term locally at each time step;
(iv) Instead, we add up the difference equation for the time steps and write the error in terms of a time integral of this rough part of the noise;
(v) We then use the Itô isometry to estimate this integral term by Δt;

In order to make the main idea clear cut, here are the options we have for estimating the rough part of the noise:

(i) If the local error e_j of the rough part of the noise, at the jth time step, is bounded as
$$\mathbb{E}[|e_j|] \lesssim \Delta t^{3/2},$$
as usual for a 1/2-Hölder noise, then adding them up leads to
$$\sum \mathbb{E}[|e_j|] \lesssim N\Delta t^{3/2} = T\Delta t^{1/2}.$$

(ii) If we use the Itô isometry locally, we still get the local error as

$$\mathbb{E}[|e_j|] \leq \mathbb{E}[|e_j|^2]^{1/2} \lesssim \left(\Delta t^{2(3/2)}\right)^{1/2} = \Delta t^{3/2},$$

and adding that up still leads to an error of order Δt^θ.

(iii) If, instead, we first add the terms up, then $\sum e_j$ becomes an integral over $[0,T]$ with respect to the Wiener noise, so that we can use the Itô isometry on the added up term and obtain

$$\begin{aligned}\mathbb{E}\left[\left|\sum e_j\right|\right] \lesssim \left(\mathbb{E}\left[\left|\sum e_j\right|^2\right]\right)^{1/2} &= \left(\sum \mathbb{E}[|e_j|^2]\right)^{1/2} \\ &= \left(\sum \Delta t^3\right)^{1/2} = \left(\Delta t^2\right)^{1/2} = \Delta t.\end{aligned}$$

and we finally get the error to be of order 1.

2. Strong order of convergence

We assume $f = f(t,x,y)$ is twice continuously differentiable with

$$L_t = \sup_{t,x,y} |\partial_t f(t,x,y)| < \infty \tag{2.1}$$

$$L_x = \sup_{t,x,y} |\partial_x f(t,x,y)| < \infty \tag{2.2}$$

$$L_y = \sup_{t,x,y} |\partial_y f(t,x,y)| < \infty \tag{2.3}$$

$$L_{yy} = \sup_{t,x,y} |\partial_y^2 f(t,x,y)| < \infty, \tag{2.4}$$

where the suprema are taken for $(t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}$. The first three condition (2.1), (2.2), and (2.3) imply that f has an at most linear growth:

$$\sup_{t,x,y} |f(t,x,y)| \leq M_0 + L(|t| + |x| + |y|), \tag{2.5}$$

for suitable nonnegative constants M_0, L.

We also assume the drift and diffusion of the Itô process $\{Y_t\}_t$ are uniformly bounded,

$$M_A = \sup_\omega \sup_{t,x,y} |A_t(\omega)| < \infty, \tag{2.6}$$

$$M_B = \sup_\omega \sup_{t,x,y} |B_t(\omega)| < \infty, \tag{2.7}$$

where the suprema are taken for $t \in [0,T]$ and for samples in all sample space $\omega \in \Omega$.

2.1. **A single step.** Here we obtain an expression for a single time step which will be suitable for a proper estimate later on. For the sake of notational simplicity, we consider a single time step from a time t to a time $t+\tau$. Later on we take $t = t_{j-1}$ and $\tau = \Delta t$, with $t_j = t_{j-1} + \Delta t$.

The exact solution satisfies, for any $t, \tau \geq 0$,

$$X_{t+\tau} = X_t + \int_t^{t+\tau} f(s, X_s, Y_s)\,\mathrm{d}s.$$

The Euler-Maruyama step is given by

$$X_{t+\tau}^n = X_t^n + \tau f(t, X_t^n, Y_t).$$

Subtracting, we obtain

$$X_{t+\tau} - X_{t+\tau}^n = X_t - X_t^n + \int_t^{t+\tau} \left(f(s, X_s, Y_s) - f(t, X_t^n, Y_t)\right)\,\mathrm{d}s.$$

We arrange the integrand as

$$\begin{aligned} f(s, X_s, Y_s) - f(t, X_t^n, Y_t) = & f(s, X_s, Y_s) - f(s, X_t, Y_s) \\ & + f(s, X_t, Y_s) - f(s, X_t^n, Y_s) \\ & + f(s, X_t^n, Y_s) - f(t, X_t^n, Y_t). \end{aligned}$$

This yields

$$\begin{aligned} X_{t+\tau} - X_{t+\tau}^n = & X_t - X_t^n \\ & = \int_t^{t+\tau} \left(f(s, X_s, Y_s) - f(s, X_t, Y_s)\right)\,\mathrm{d}s \\ & \quad + \int_t^{t+\tau} \left(f(s, X_t, Y_s) - f(s, X_t^n, Y_s)\right)\,\mathrm{d}s \\ & \quad + \int_t^{t+\tau} \left(f(s, X_t^n, Y_s) - f(t, X_t^n, Y_t)\right)\,\mathrm{d}s. \end{aligned}$$

For the integral of the last pair of terms, we use the Itô formula on $Z_s = f(s, X_t^n, Y_s)$ and write

$$\begin{aligned} \int_t^{t+\tau} & \left(f(s, X_t^n, Y_s) - f(t, X_t^n, Y_t)\right)\,\mathrm{d}s = \int_t^{t+\tau} \int_t^s \mathrm{d}Z_\xi\,\mathrm{d}s \\ & = \int_t^{t+\tau} \int_t^s \left(\partial_\xi f(\xi, X_t^n, Y_\xi) + A_\xi \partial_y f(\xi, X_t^n, Y_\xi) + \frac{B_\xi^2}{2} \partial_{yy} f(\xi, X_t^n, Y_\xi)\right)\,\mathrm{d}s\,\mathrm{d}t \\ & \quad + \int_t^{t+\tau} \int_t^s B_\xi \partial_y f(\xi, X_t^n, Y_\xi)\,\mathrm{d}W_\xi\,\mathrm{d}s. \end{aligned}$$

Using Fubini's Theorem, the last integral is rewritten as

$$\begin{aligned}\int_t^{t+\tau}\int_t^s B_\xi \partial_y f(\xi, X_t^n, Y_\xi)\,\mathrm{d}W_\xi\,\mathrm{d}s &= \int_t^{t+\tau}\int_\xi^{t+\tau} B_\xi \partial_y f(\xi, X_t^n, Y_\xi)\,\mathrm{d}s\,\mathrm{d}W_\xi \\ &= \int_t^{t+\tau}(t+\tau-\xi) B_\xi \partial_y f(\xi, X_t^n, Y_\xi)\,\mathrm{d}W_\xi. \end{aligned} \tag{2.8}$$

We rearrange these terms and write, for $\tau = \Delta t$ and $t = t_{j-1} = (j-1)\Delta t$,

$$X_{t_j} - X_{t_j}^n = X_{t_{j-1}} - X_{t_{j-1}}^n + I_{j-1}^1 + I_{j-1}^2 + I_{j-1}^3, \tag{2.9}$$

where

$$I_j^1 = \int_{t_j}^{t_{j+1}} \Big(f(s, X_{t_j}, Y_s) - f(s, X_{t_j}^n, Y_s) \Big)\,\mathrm{d}s,$$

$$\begin{aligned} I_j^2 = &\int_{t_j}^{t_{j+1}} \big(f(s, X_s, Y_s) - f(s, X_{t_j}, Y_s) \big)\,\mathrm{d}s \\ &+ \int_{t_j}^{t_{j+1}}\int_{t_j}^s \left(\partial_\xi f(\xi, X_{t_j}^n, Y_\xi) + A_\xi \partial_y f(\xi, X_{t_j}^n, Y_\xi) + \frac{B_\xi^2}{2}\partial_{yy} f(\xi, X_{t_j}^n, Y_\xi) \right)\,\mathrm{d}t, \end{aligned}$$

and

$$I_j^3 = \int_{t_j}^{t_{j+1}} (t_{j+1} - \xi) B_\xi \partial_y f(\xi, X_{t_j}^n, Y_\xi)\,\mathrm{d}W_\xi.$$

2.2. **Local estimates.** The term I_j^1 is estimated using that $f = f(t, x, y)$ is globally Lipschitz in x, so that

$$|f(s, X_t, Y_s) - f(s, X_t^n, Y_s)| \leq L_x |X_t - X_t^n|.$$

Hence,

$$\begin{aligned}\left| \int_{t_j}^{t_{j+1}} \Big(f(s, X_{t_j}, Y_s) - f(s, X_{t_j}^n, Y_s) \Big)\,\mathrm{d}s \right| &\leq \int_{t_j}^{t_{j+1}} \Big| f(s, X_{t_j}, Y_s) - f(s, X_{t_j}^n, Y_s) \Big|\,\mathrm{d}s \\ &\leq L_x |X_{t_j} - X_{t_j}^n| \Delta t. \end{aligned}$$

This means

$$\big| I_j^1 \big| \leq L_x |X_{t_j} - X_{t_j}^n| \Delta t. \tag{2.10}$$

For I_j^2, the first term is estimated as

$$|f(s, X_s, Y_s) - f(s, X_t, Y_s)| \leq L_x |X_s - X_t| \leq L_x \int_t^s |f(\sigma, X_\sigma, Y_\sigma)|\,\mathrm{d}\sigma \leq L_x M_f (s-t).$$

This yields, upon integration,

$$\left|\int_{t_j}^{t_{j+1}} \left(f(s,X_s,Y_s) - f(s,X_{t_j},Y_s)\right)\,\mathrm{d}s\right| \leq \int_{t_j}^{t_{j+1}} \left|f(s,X_s,Y_s) - f(s,X_{t_j},Y_s)\right|\,\mathrm{d}s$$
$$\leq \frac{L_x M_f}{2}\Delta t^2.$$

The double integral is estimated as

$$\left|\int_{t_j}^{t_{j+1}}\int_{\xi}^{t_{j+1}} \left(\partial_\xi f(\xi, X_{t_j}^n, Y_\xi) + A_\xi \partial_y f(\xi, X_{t_j}^n, Y_\xi) + \frac{B_\xi^2}{2}\partial_{yy} f(\xi, X_{t_j}^n, Y_\xi)\right)\,\mathrm{d}t\right|$$
$$\leq \int_{t_j}^{t_{j+1}}\int_{\xi}^{t_{j+1}} \left(L_t + M_A L_y + \frac{M_B^2}{2}L_{yy}\right)\,\mathrm{d}t$$
$$= \frac{1}{2}\tau^2\left(L_t + M_A L_y + \frac{M_B^2}{2}L_{yy}\right). \quad (2.11)$$

Hence,

$$\left|I_j^2\right| \leq M\Delta t^2, \quad (2.12)$$

where

$$M = \frac{1}{2}\left(L_x M_f + L_t + M_A L_y + \frac{M_B^2}{2}L_{yy}\right).$$

Remark 2.1. Notice that, at this point, we did not estimate the last integral, otherwise we are not able to obtain the strong order 1 estimate, only 1/2. Indeed, if we use Fubini and the Itô isometry in the last integral, we find

$$\mathbb{E}\left[\left(\int_t^{t+\tau}\int_t^s B_\xi \partial_y f(\xi, X_t^n, Y_\xi)\,\mathrm{d}W_\xi\,\mathrm{d}s\right)^2\right] = \mathbb{E}\left[\left(\int_t^{t+\tau}\int_\xi^{t+\tau} B_\xi \partial_y f(\xi, X_t^n, Y_\xi)\,\mathrm{d}s\,\mathrm{d}W_\xi\right)^2\right]$$
$$= \int_t^{t+\tau}\mathbb{E}\left[\left(\int_\xi^{t+\tau} B_\xi \partial_y f(\xi, X_t^n, Y_\xi)\,\mathrm{d}s\right)^2\right]\mathrm{d}\xi \leq \int_t^{t+\tau}\left(\int_\xi^{t+\tau} M_B^2 L_y\,\mathrm{d}s\right)^2\mathrm{d}\xi$$
$$\leq \int_t^{t+\tau} M_B^2 L_y (t+\tau-\xi)^2\,\mathrm{d}\xi = -\frac{1}{3}M_B^2 L_y^2 (t+\tau-\xi)^3\Big]_t^{t+\tau} = \frac{1}{3}M_B^2 L_y^2 \tau^3,$$

so that

$$\sqrt{\mathbb{E}\left[\left(\int_t^{t+\tau}\int_t^s B_\xi \partial_y f(\xi, X_t^n, Y_\xi)\,\mathrm{d}W_\xi\,\mathrm{d}s\right)^2\right]} \leq \frac{\sqrt{3}}{3}M_B L_y \tau^{3/2}. \quad (2.13)$$

After adding up n times, we end up with a $\tau^{1/2}$ estimate, which is not sufficient.

2.3. **Integral estimate.** The third term I_j^3 is not estimated for each j separately. Instead, we estimate its summation over j. Notice

$$\sum_{i=0}^{j-1} I_i^3 = \sum_{i=0}^{j-1}\int_{t_i}^{t_{i+1}} (t_{i+1}-\xi) B_\xi \partial_y f(\xi, X_{t_i}^n, Y_\xi)\,\mathrm{d}W_\xi$$
$$= \int_0^{t_j} \left([\xi/\Delta t + 1]\Delta t - \xi\right) B_\xi \partial_y f(\xi, X_{[\xi/\Delta t]\Delta t}^n, Y_\xi)\,\mathrm{d}W_\xi,$$

where $[r]$ denotes the largest integer below a real number r.

For this term, we estimate its strong norm, i.e. first moment. This is estimated using the Lyapunov inequality, the Itô formula and the Itô isometry, as follows

$$
\begin{aligned}
&\mathbb{E}\left[\left|\int_0^{t_j} ([\xi/\Delta t+1]\Delta t-\xi)\, B_\xi \partial_y f(\xi, X^n_{[\xi/\Delta t]\Delta t}, Y_\xi)\,\mathrm{d}W_\xi\right|\right] \\
&\quad \leq \mathbb{E}\left[\left(\int_0^{t_j} ([\xi/\Delta t+1]\Delta t-\xi)\, B_\xi \partial_y f(\xi, X^n_{[\xi/\Delta t]\Delta t}, Y_\xi)\,\mathrm{d}W_\xi\right)^2\right]^{1/2} \\
&\quad = \left(\int_0^{t_j} \mathbb{E}\left[\left(([\xi/\Delta t+1]\Delta t-\xi)\, B_\xi \partial_y f(\xi, X^n_{[\xi/\Delta t]\Delta t}, Y_\xi)\right)^2\right]\,\mathrm{d}\xi\right)^{1/2} \\
&\quad \leq \left(\int_0^{t_j} \left(([\xi/\Delta t+1]\Delta t-\xi)^2\, \mathbb{E}\left[\left(B_\xi \partial_y f(\xi, X^n_{[\xi/\Delta t]\Delta t}, Y_\xi)\right)^2\right]\right)\,\mathrm{d}\xi\right)^{1/2} \\
&\quad \leq \left(\int_0^{t_j} \Delta t^2 M_B^2 L_y^2\,\mathrm{d}\xi\right)^{1/2}.
\end{aligned}
$$

Thus,

$$
\mathbb{E}\left[\left|\sum_{i=0}^{j-1} I_j^3\right|\right] \leq M_B L_y t_j^{1/2}\Delta t. \tag{2.14}
$$

2.4. **Iterating the steps.** Iterating (2.9) and assuming that $X_0^n = X_0$, we find

$$
X_{t_j} - X_{t_j}^n = \sum_{i=0}^{j-1} I_j^1 + \sum_{i=0}^{j-1} I_j^2 + \sum_{i=0}^{j-1} I_j^3. \tag{2.15}
$$

We estimate the first moment as

$$
\mathbb{E}\left[|X_{t_j} - X_{t_j}^n|\right] \leq \sum_{i=0}^{j-1}\mathbb{E}\left[|I_j^1|\right] + \sum_{i=0}^{j-1}\mathbb{E}\left[|I_j^2|\right] + \mathbb{E}\left[\left|\sum_{i=0}^{j-1} I_j^3\right|\right]. \tag{2.16}
$$

Using (2.10), (2.12), and (2.14), we obtain

$$
\begin{aligned}
\mathbb{E}\left[|X_{t_j} - X_{t_j}^n|\right] &\leq L_x \sum_{i=0}^{j-1}\mathbb{E}\left[|X_{t_j} - X_{t_j}^n|\right]\Delta t + \sum_{i=0}^{j-1} C\Delta t^2 + M_B L_y t_j \Delta t \\
&\leq L_x \sum_{i=0}^{j-1}\mathbb{E}\left[|X_{t_i} - X_{t_i}^n|\right]\Delta t + C_T\Delta t,
\end{aligned} \tag{2.17}
$$

where

$$
C_T = M + M_B L_y T^{1/2}.
$$

Now, we show by induction that

$$
\mathbb{E}\left[|X_{t_j} - X_{t_j}^n|\right] \leq C_T e^{L_x t_j}\Delta t.
$$

This is trivially true for $j = 0$. Now suppose it is true up to $j-1$. It follows from (2.17) that

$$\mathbb{E}\left[|X_{t_j} - X_{t_j}^n|\right] \leq L_x \sum_{i=0}^{j-1} C_T \Delta t e^{L_x t_i} \Delta t + C_T \Delta t = C_T \Delta t \left(1 + L_x \Delta t \sum_{i=0}^{j-1} e^{L_x t_i}\right).$$

Using that $1 + r \leq e^r$, with $r = L_x \Delta t$ and $t_i + \Delta t = t_{i+1}$, we see that

$$L_x \Delta t \leq e^{L_x \Delta t} - 1,$$

which telescopes the sum and yields

$$\mathbb{E}\left[|X_{t_j} - X_{t_j}^n|\right] \leq C_T \Delta t \left(1 + (e^{L_x \Delta t} - 1) \sum_{i=0}^{j-1} e^{L_x t_i}\right) = C_T \Delta t \left(1 + (e^{L_x j \Delta t} - 1)\right).$$

Hence,

$$\mathbb{E}\left[|X_{t_j} - X_{t_j}^n|\right] \leq C_T e^{L_x t_j} \Delta t,$$

which completes the induction. Hence, we have proved the following result.

Theorem 2.1. *Consider the initial value problem (1.1), on a time interval* $[0, T]$, *with* $T > 0$, *and assume the noise is given by (1.4), with (2.6) and (2.7). Suppose* $f = f(t, x, y)$ *is twice continuously differentiable, with (2.5)-(2.4). Let* $\{X_t\}_{t \geq 0}$ *be the solution of (1.1). Let* $N \in \mathbb{N}$ *and let* $\{X_{t_j}^N\}_{j=0,\ldots,N}$ *be the solution of the Euler-Maruyama method (1.2)-(1.3). Then,*

$$\mathbb{E}\left[\left|X_{t_j} - X_{t_j}^N\right|\right] \leq C_T e^{L_x t_j} \Delta t, \qquad j = 0, \ldots, N, \ \forall N \in \mathbb{N}, \Delta t = \frac{T}{N}, \tag{2.18}$$

where

$$C_T = \frac{1}{2}\left(L_x M_f + L_t + M_A L_y + \frac{M_B^2}{2} L_{yy}\right) + M_B L_y T^{1/2}. \tag{2.19}$$

We end this section by abstracting away the Gronwall type inequality we use (this is probably written somewhere, and I need to find the source):

Lemma 2.1. *Let* $(e_j)_j$ *be a (finite or infinite) sequence of positive numbers satisfying*

$$e_j \leq a \sum_{i=0}^{j-1} e_i + b, \tag{2.20}$$

with $e_0 = 0$, *where* $a, b > 0$. *Then,*

$$e_j \leq b e^{aj}, \qquad \forall j. \tag{2.21}$$

Proof. The result is trivially true for $j = 0$. Suppose, by induction, that the result is true up to $j - 1$. Then,

$$e_j \leq a \sum_{i=0}^{j-1} b e^{ai} + b = b\left(a \sum_{i=0}^{j-1} e^{ai} + 1\right).$$

Using that $1 + a \leq e^a$, we have $a \leq e^a - 1$, hence

$$e_j \leq b \left((e^a - 1) \sum_{i=0}^{j-1} e^{ia} + 1 \right).$$

Using that $\sum_{i=0}^{j-1} \theta^i = (\theta^j - 1)(\theta - 1)$, with $\theta = e^a$, we see that

$$(e^a - 1) \sum_{i=0}^{j-1} e^{ia} \leq e^{ja} - 1,$$

so that

$$e_j \leq b e^{ja},$$

which completes the induction. □

3. Special cases

3.1. Non-homogeneous term of bounded variation. Consider a RODE of the form

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = g(t, Y_t, X_t) + h(t, Y_t),$$

where g is globally Lipschitz and $t \mapsto h(t, Y_t)$ is of bounded variation.

4. Numerical examples

4.1. Lower-order converge. For a lower order convergence, below order 1, we take the noise $\{Y_t\}_t$ to be the transport process defined by

$$Y_t = \sin(t/Z)^{1/3},$$

where Z is a beta random variable $Z \sim B(\alpha, \beta)$. Notice Z takes values strictly within $(0, 1)$ and, hence, $\sin(t/Z)$ can have arbitrarily high frequencies and, hence, go through the critic value $y = 0$ extremely often.

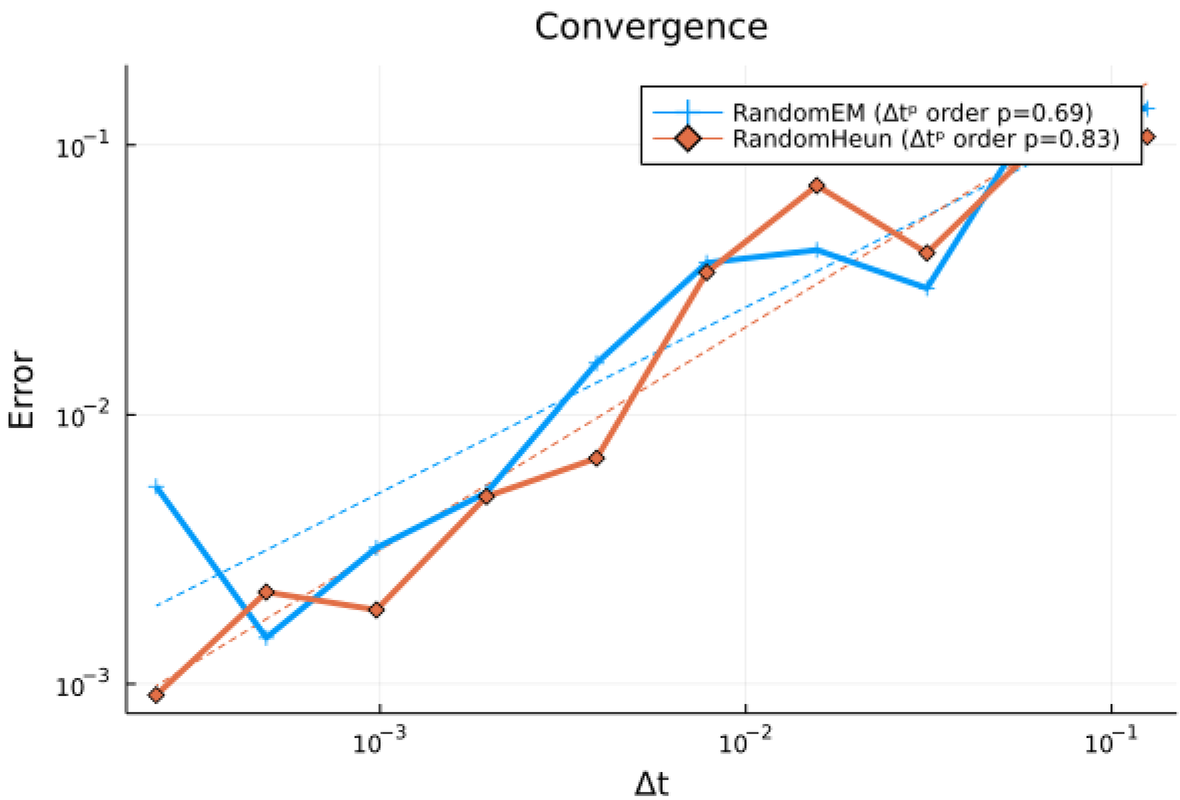
(Need to remove the Heun method and do more tests).

Appendix

The heart of the matter is the following. Think of τ as the time-step Δt, but we use τ for simplicity. Let $y = y(t)$ be a θ-Hölder continuous function, with Hölder constant C. Then, we can do the usual "local"-type estimate

$$\begin{aligned}
\left| \int_0^T (y(t+\tau) - y(t)) \, \mathrm{d}t \right| &\leq \int_0^T |y(t+\tau) - y(t)| \, \mathrm{d}t \\
&\leq C \int_0^T \tau^\theta \, \mathrm{d}t \\
&= C \tau^\theta T,
\end{aligned}$$

which yields an order θ approximation, with respect to the "time step" τ.



However, we can also integrate first, so that

$$\begin{aligned}
\left|\int_0^T (y(t+\tau) - y(t))\;\mathrm{d}t\right| &= \left|\int_0^T y(t+\tau)\;\mathrm{d}t - \int_0^T y(t)\;\mathrm{d}t\right| \\
&= \left|\int_\tau^{T+\tau} y(t)\;\mathrm{d}t - \int_0^T y(t)\;\mathrm{d}t\right| \\
&= \left|\int_T^{T+\tau} y(t)\;\mathrm{d}t - \int_0^\tau y(t)\;\mathrm{d}t\right| \\
&= \Bigg|y(T)\tau + \int_T^{T+\tau} (y(t) - y(T))\;\mathrm{d}t \\
&\qquad - y(0)\tau - \int_0^\tau (y(t) - y(0))\;\mathrm{d}t\Bigg| \\
&\leq |y(T) - y(0)|\tau + C\int_T^{T+\tau} |t - T|^\theta\;\mathrm{d}t + |y(0)|\tau + \int_0^\tau t^\theta\;\mathrm{d}t \\
&\leq |y(T) - y(0)|\tau + \frac{C}{1+\theta}\tau^{1+\theta} + \frac{C}{1+\theta}\tau^{1+\theta} \\
&\leq CT^\theta\tau + \frac{2C}{1+\theta}\tau^{1+\theta}.
\end{aligned}$$

Hence,

$$\left|\int_0^T (y(t+\tau) - y(t))\ \mathrm{d}t\right| \leq C\tau\left(T^\theta + \frac{2}{1+\theta}\tau^\theta\right),$$

which reveals the order 1 convergence.

Well, but, actually, we don't have $y(t+\tau) - y(t)$ in the integrand. We have $y(t) - y(\tau^n(t))$, where $\tau^n(t)$ picks the largest $j\tau$ smaller than or equal to t, i.e. $\tau^n(t) = \max\{j\tau;\ j\tau \leq t, j\}$. Then we need to estimate

$$\left|\int_0^T (y(t) - y(\tau^m(t)))\ \mathrm{d}t\right|.$$

In this case,

$$\begin{aligned}
\left|\int_0^T (y(t) - y(\tau^m(t)))\ \mathrm{d}t\right| &\leq \left(\int_0^T |y(t) - y(\tau^m(t))|^{1/\theta}\ \mathrm{d}t\right)^\theta \left(\int_0^T 1^{1/(1-\theta)}\ \mathrm{d}t\right)^{1-\theta} \\
&\leq \left(\int_0^T \left|C\tau^\theta\right|^{1/\theta}\ \mathrm{d}t\right)^\theta T^{1-\theta} \\
&\leq CT^\theta\tau^\theta T^{1-\theta}.
\end{aligned}$$

Thus,

$$\left|\int_0^T (y(t) - y(\tau^m(t)))\ \mathrm{d}t\right| \leq CT\tau^\theta. \tag{4.1}$$

Ops, we should find another way. And even if we find a way, keep in mind that what we really have is a term $t \mapsto y(t, x(t))$ and what we need to estimate is

$$\int_0^T (y(t, x(\tau^n(t)) - y(\tau^n(t), x(\tau^n(t))))\ \mathrm{d}t.$$

The Itô isometry is a very special thing, that let us pass the square to the inside of the integral. So we should look for other ways. For instance, if it is separable, $y(t,x) = g(t)F(x)$, with $g(t)$ of a scalar with bounded variation (the part $F(x)$ may be a vector and the equation be a system), and $g(t) = g_1(t) - g_2(t)$ with each nondecreasing term g_1, g_2 Hölder continuous with exponent θ, than we can write

$$y(t, x(\tau^n(t)) - y(\tau^n(t), x(\tau^n(t))) = (g_1(t) - g_1(\tau^n(t)))x(\tau^n(t)) - (g_2(t) - g_2(\tau^n(t)))x(\tau^n(t)),$$

squeeze each term,

$$g_1(t) - g_1(\tau^n(t)) \leq g_1(t) - g_1(t-\tau), \quad g_2(t) - g_2(\tau^n(t)) \leq g_2(t) - g_2(t-\tau),$$

and hopefully mimick the first estimate where we only had $y(t)$, but even that is not clear because of the presence of $x(\tau^n(t))$ that prevents us to write the little integrals as a single integral over the whole interval. But we need to try.

Acknowledgments

References

(Ricardo M. S. Rosa) Instituto de Matemática, Universidade Federal do Rio de Janeiro, Brazil

Email address, R. Rosa: rrosa@im.ufrj.br