EXPONENTIAL DECAY FOR SOFT POTENTIALS NEAR MAXWELLIAN

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Abstract. Consider both soft potentials with angular cutoff and Landau collision kernels in the Boltzmann theory inside a periodic box. We prove that any smooth perturbation near a given Maxwellian approaches to zero at the rate of $e^{-\lambda t^p}$ for some $\lambda > 0$ and $0 < p < 1$. Our method is based on an unified energy estimate with appropriate exponential velocity weight. Our results extend the classical Caflisch result [2] to the case of very soft potential and Coulomb interactions, and also improve the recent "almost exponential" decay results by [4, 12].

1. Introduction

In this article, we are concerned with soft potentials and Landau collision kernels in the Boltzmann theory for dynamics of dilute particles in a periodic box. Recall the Boltzmann equation as

$$\partial_t F + v \cdot \nabla_x F = Q(F, F), \;\; F(0, x, v) = F_0(x, v), \tag{1}$$

where $F(t, x, v)$ is the spatially periodic distribution function for the particles at time $t \geq 0$, position $x = (x_1, x_2, x_3) \in \mathbb{T}^3$ and velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. The l.h.s. of this equation models the transport of particles and the operator on the r.h.s. models the effect of collisions on the transport:

$$Q(F, G) \equiv \int_{\mathbb{R}^3 \times S^2} |u - v|^\gamma B(\theta)\{F(u')G(v') - F(u)G(v)\} du d\omega.$$

Here $F(u) = F(t, x, u)$ etc. The exponent is $\gamma = 1 - \frac{4}{s}$ with $1 < s < 4$; we assume

$$-3 < \gamma < 0,$$

(soft potentials) and $B(\theta)$ satisfies the Grad angular cutoff assumption:

$$0 < B(\theta) \leq C|\cos\theta|. \tag{2}$$

Moreover, the post-collisional velocities satisfy

$$v' = v + [(u - v) \cdot \omega]\omega, \qquad u' = u - [(u - v) \cdot \omega]\omega, \tag{3}$$

$$|u|^2 + |v|^2 = |u'|^2 + |v'|^2, \tag{4}$$

And θ is defined by $\cos\theta = [u - v] \cdot \omega / |u - v|$.

On the other hand, the Landau equation is formally obtained in a singular limit of the Boltzmann equation. It can also be written as (1) but the collision operator

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is given by

$$Q(F,G) = \nabla_v \cdot \left\{ \int_{\mathbb{R}^3} \phi(v-u)[F(u)\nabla_v G(v) - G(v)\nabla_u F(u)]du \right\}$$
$$= \sum_{i,j=1}^{3} \partial_i \int_{\mathbb{R}^3} \phi^{ij}(v-u)[F(u)\partial_j G(v) - G(v)\partial_j F(u)]du,$$

where $\partial_i = \partial_{v_i}$ etc. The non-negative matrix ϕ is given by

$$\phi^{ij}(v) = \left\{ \delta_{ij} - \frac{v_i v_j}{|v|^2} \right\} |v|^{2+\gamma}.$$

We assume soft potentials, which means $-3 \leq \gamma < -2$. The original Landau collision operator with Coulombic interactions corresponds to $\gamma = -3$.

Denote the steady state Maxwellian by

$$\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}.$$

We perturb around the Maxwellian as

$$F(t,x,v) = \mu(v) + \sqrt{\mu(v)} f(t,x,v).$$

Then the initial value problem (1) can be rewritten as

$$[\partial_t + v\cdot\nabla_x]f + Lf = \Gamma[f,f], \ f(0,x,v) = f_0(x,v), \tag{5}$$

where L is the linear part of the collision operator, Q, and Γ is the non-linear part.

For the Boltzmann equation, the standard linear operator [5] is

$$Lg = \nu(v)g - Kg, \tag{6}$$

where the collision frequency is

$$\nu(v) = \int B(\theta)|v-u|^\gamma \mu(u) du d\omega. \tag{7}$$

The operators K and Γ, in the Boltzmann case, are defined in (17) and (46).

For the Landau equation, the linear operator [9] is

$$Lg = -Ag - Kg, \tag{8}$$

with A, K and Γ defined in (44), (45) and (46). The Landau collision frequency is

$$\sigma^{ij}(v) = \phi^{ij} * \mu = \int_{\mathbb{R}^3} \phi^{ij}(v-u)\mu(u)du. \tag{9}$$

We remark that $\sigma^{ij}(v)$ is a positive self-adjoint matrix [3].

Notation: Let $\langle \cdot, \cdot \rangle$ denote the standard $L^2(\mathbb{R}^3)$ inner product. We also use $(\cdot,\cdot)$ to denote the standard $L^2(\mathbb{T}^3 \times \mathbb{R}^3)$ inner product with corresponding L^2 norm $\|\cdot\|$. Define a weight function in v by

$$w = w(\ell,\vartheta)(v) \equiv (1+|v|^2)^{\tau\ell/2} \exp\left(\frac{q}{4}(1+|v|^2)^{\frac{\vartheta}{2}}\right). \tag{10}$$

Here $\tau < 0$, $\ell \in \mathbb{R}$, $0 < q$ and $0 \leq \vartheta \leq 2$. Denote weighted L^2 norms as

$$|g|^2_{\ell,\vartheta} \equiv \int_{\mathbb{R}^3} w^2(\ell,\vartheta)|g|^2 dv, \ ||g||^2_{\ell,\vartheta} \equiv \int_{\mathbb{T}^3} |g|^2_{\ell,\vartheta} dx.$$

For the Boltzmann equation, define the weighted dissipation norm as

$$
|g|^2_{\boldsymbol{\nu},\ell,\vartheta} \equiv \int_{\mathbb{R}^3} w^2(\ell,\vartheta)\nu(v)|g(v)|^2 dv, \tag{11}
$$

$$
||g||^2_{\boldsymbol{\nu},\ell,\vartheta} \equiv \int_{\mathbb{T}^3} |g|^2_{\boldsymbol{\nu},\ell,\vartheta} dx.
$$

For the Landau equation, define the weighted dissipation norm as

$$
|g|^2_{\boldsymbol{\sigma},\ell,\vartheta} \equiv \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} w^2(\ell,\vartheta) \left\{ \sigma^{ij}\partial_i g \partial_j g + \sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}|g|^2 \right\} dv, \tag{12}
$$

$$
||g||^2_{\boldsymbol{\sigma},\ell,\vartheta} \equiv \int_{\mathbb{T}^3} |g|^2_{\boldsymbol{\sigma},\ell,\vartheta} dx.
$$

Since our proof of decay does not depend upon any properties which are specific to either dissipation norm, sometimes we unify the notation as $||g||_{\boldsymbol{D},\ell,\vartheta}$, which denotes either $||g||_{\boldsymbol{\nu},\ell,\vartheta}$ or $||g||_{\boldsymbol{\sigma},\ell,\vartheta}$. If $\vartheta = 0$ then we drop the index, e.g. $||g||_{\boldsymbol{D},\ell,0} = ||g||_{\boldsymbol{D},\ell}$ and the same for the other norms.

Next define a high order derivative

$$
\partial^\alpha_\beta \equiv \partial_t^{\alpha^0}\partial_{x_1}^{\alpha^1}\partial_{x_2}^{\alpha^2}\partial_{x_3}^{\alpha^3}\partial_{v_1}^{\beta^1}\partial_{v_2}^{\beta^2}\partial_{v_3}^{\beta^3}
$$

where $\alpha = [\alpha^0, \alpha^1, \alpha^2, \alpha^3]$ is the multi-index related to the space-time derivative and $\beta = [\beta^1, \beta^2, \beta^3]$ is the multi-index related to the velocity derivatives. If each component of β is not greater than that of β_1's, we denote by $\beta \leq \beta_1$; $\beta < \beta_1$ means $\beta \leq \beta_1$ and $|\beta| < |\beta_1|$. We also denote $\begin{pmatrix} \beta \\ \beta_1 \end{pmatrix}$ by $C^{\beta_1}_\beta$.

Fix $N \geq 8$ and $l \geq 0$. An "**Instant Energy functional**" satisfies

$$
\frac{1}{C}\mathcal{E}_{l,\vartheta}(g)(t) \leq \sum_{|\alpha|+|\beta|\leq N} ||\partial^\alpha_\beta g(t)||^2_{|\beta|-l,\vartheta} \leq C\mathcal{E}_{l,\vartheta}(g)(t).
$$

If g is independent of $t \geq 0$, then the temporal derivatives are defined through equation (5). Further, the "**Dissipation Rate**" is given by

$$
\mathcal{D}_{l,\vartheta}(g)(t) \equiv \sum_{|\alpha|+|\beta|\leq N} ||\partial^\alpha_\beta g(t)||^2_{\boldsymbol{D},|\beta|-l,\vartheta}.
$$

We will also write $\mathcal{E}_{l,0}(g)(t) = \mathcal{E}_l(g)(t)$ and $\mathcal{D}_{l,0}(g)(t) = \mathcal{D}_l(g)(t)$. We note from (10) that for $l > 0$ these norms contain a polynomial factor $(1+|v|^2)^{-\tau l/2}$. The weight factor $(1+|v|^2)^{\tau|\beta|/2}$ (dependant on the number of velocity derivatives) is designed to control the streaming term $v \cdot \nabla_x f$.

If initially $F_0(x,v) = \mu(v) + \sqrt{\mu(v)} f_0(x,v)$ has the same mass, momentum and total energy as the Maxwellian μ, then formally for any $t \geq 0$ we have

$$
\int_{\mathbb{T}^3\times\mathbb{R}^3} f(t)\mu^{1/2} = \int_{\mathbb{T}^3\times\mathbb{R}^3} v_i f(t)\mu^{1/2} = \int_{\mathbb{T}^3\times\mathbb{R}^3} |v|^2 f(t)\mu^{1/2} = 0. \tag{13}
$$

We are now ready to state the main result.

Theorem 1. *Let $N \geq 8$, $l \geq 0$, $0 \leq \vartheta \leq 2$ and $0 < q$. If $\vartheta = 2$ then further assume $q < 1$. Choose initial data $F_0(x,v) = \mu(v) + \sqrt{\mu(v)} f_0(x,v)$ such that $f_0(x,v)$ satisfies (13). In* (10)*, for the Boltzmann case assume $\tau \leq \gamma$ and for the Landau case assume $\tau \leq -1$.*

Then there exists an instant energy functional $\mathcal{E}_{l,\vartheta}(f)(t)$ such that if $\mathcal{E}_{l,\vartheta}(f_0)$ is sufficiently small, then the unique global solution to (1) in both the Boltzmann case and the Landau case satisfies

$$\frac{d}{dt}\mathcal{E}_{l,\vartheta}(f)(t) + \mathcal{D}_{l,\vartheta}(f)(t) \leq 0. \tag{14}$$

In particular,

$$\sup_{0\leq s\leq\infty} \mathcal{E}_{l,\vartheta}(f)(s) \leq \mathcal{E}_{l,\vartheta}(f_0). \tag{15}$$

Moreover, if $\vartheta > 0$, then there exists $\lambda > 0$ such that

$$\mathcal{E}_{l}(f)(t) \leq Ce^{-\lambda t^p}\mathcal{E}_{l,\vartheta}(f_0),$$

where in the Boltzmann case

$$p = p(\vartheta, \gamma) = \frac{\vartheta}{\vartheta - \gamma},$$

and in the Landau case

$$p = p(\vartheta, \gamma) = \frac{\vartheta}{\vartheta - (2+\gamma)}.$$

In the second authors papers [8, 9], (14) was established with $\tau = \gamma$ in the Boltzmann case, $\tau = 2 + \gamma$ in the Landau case and $\vartheta = l = 0$. We extended (14) to the case $l \geq 0$ in [12]. There we used (14) and (15) to establish the following theorem via direct interpolation for $\vartheta = 0$.

Theorem 2. *Assume everything from Theorem 1 and fix $k > 0$. In addition, if $\mathcal{E}_{l+k,\vartheta}(f_0)$ is sufficiently small then*

$$\mathcal{E}_{l,\vartheta}(f)(t) \leq C_{l,k}\left(1 + \frac{t}{k}\right)^{-k}\mathcal{E}_{l+k,\vartheta}(f_0).$$

For $\vartheta > 0$, the proof of Theorem 2 is exactly the same as in [12]. Also in [12], for the Landau case we assumed Coulomb interactions ($\gamma = -3$). Still the proof of Theorem 2 in the Landau case for $\tau \leq -1$ and $-3 \leq \gamma < -2$ is exactly the same. However, we remark that for $\tau < 2 + \gamma$ the interpolations used to prove Theorem 2 are not optimal. The polynomial decay rate can be improved as τ grows smaller than γ by using tighter interpolations. But this is somewhat superficial because as τ grows smaller we are implicitly using a larger weight in our norms.

The main difficulty in proving any kind of decay for soft potentials is caused by the lack of a spectral gap for the linear operator (6) and (8). In the Boltzmann case, the dominant part of the linear operator (6) is of the form

$$\frac{1}{C}(1+|v|^2)^{\gamma/2} \leq \nu(v) \leq C(1+|v|^2)^{\gamma/2}, \quad C > 0. \tag{16}$$

From another point of view, at high velocities the dissipation is much weaker than the instant energy. However, Theorem 1 and Theorem 2 show that given explicit control over $f(t, x, v)$ at high velocities, no matter how weak, we can obtain a precise decay rate. On the other hand, we believe it very unlikely that existence of solutions can be established in this setting with a weight bigger than (10) with $\vartheta = 2$. From this point of view, Theorem 1 and Theorem 2 together form a rather satisfactory theory of convergence rates to Maxwellian for soft potentials and Landau operators, in a close to equilibrium context.

The constants in our estimates are certainly not optimal or explicit in all cases. However, $p = \frac{\vartheta}{\vartheta-\gamma}$ comes from the following simplification of the Boltzmann equation [1]:

$$\partial_t f(t,|v|) + |v|^\gamma f(t,|v|) = 0, \quad -3 < \gamma < 0, \quad |v| > 0.$$

Consider initial data with rapid decay as required by our norms

$$f(0,|v|) = e^{-c|v|^\vartheta}, \ c > 0, \ 0 < \vartheta \le 2.$$

Then the solution to this system is exactly

$$f(t,|v|) = e^{-c|v|^\vartheta - t|v|^\gamma}$$

By splitting into $\{|v| \ge t^{p/\vartheta}\}$ and $\{|v| < t^{p/\vartheta}\}$ we have

$$c_0 e^{-c_1 t^p} \le \int_{|v|>0} |f(t,|v|)|^2 d|v| \le c_2 e^{-c_3 t^p},$$

with $c_i > 0$ $(i = 0,1,2,3)$.

The study of trend to Maxwellians is important in kinetic theory both from physical and mathematical standpoints. In a periodic box, it was Ukai [14] who obtained exponential convergence (with $p = 1$), and hence constructed the first global in time solutions in the spatially inhomogeneous Boltzmann theory. He treated the case of a cutoff hard potential. In 1980, Caflisch [1, 2] established exponential decay (with the same $p(2,\gamma)$) as well as global in time solutions for the Boltzmann equation with potentials which are not too soft $(-1 < \gamma < 0)$. About the same time, in the whole space setting, also for cutoff soft potentials with $\gamma > -1$, Ukai and Asano [15] obtained the rate $O(t^{-\alpha})$ with $0 < \alpha < 1$; their optimal case in $\mathbb{R}^3$ yields $\alpha = 3/4$. In these early investigations, a sufficiently fast time decay of the linearized Boltzmann equation around a Maxwellian played the crucial role in bootstrapping to the full nonlinear dynamics. For the soft potentials, such linear decay estimates can be very difficult and delicate. It thus has been a longstanding open problem to study the decay property as well as to construct global in time smooth solutions for very soft potentials with γ near -3.

Recently, a nonlinear energy method for constructing global solutions was developed by the second author to avoid using the linear decay. Indeed, by showing the linearized collision operator was always positive definite along the full nonlinear dynamics, global in time smooth solutions near Maxwellians were constructed for all cutoff soft potentials of $-3 < \gamma < 0$ [8], even for the Landau equation with Coulomb interaction [9]. However, the time decay of such solutions was left open. See [7, 10, 11, 13] for more applications of such a method.

From a completely different approach, Desvillettes and Villani [4] have recently developed a framework to study the trend to Maxwellians for *general* smooth solutions, not necessarily near any Maxwellian. As an application, their method leads to the almost exponential decay rates (i.e., faster than any given polynomial) for smooth solutions constructed earlier by the second author for all cutoff soft potentials and the Landau equation.

Inspired by such a striking result, in [12], we re-examined and improved the energy method to give a more direct proof of such almost exponential decay in the close to Maxwellian setting. We introduced a family of polynomial velocity weight functions and used some simple interpolation techniques. It is interesting to note that our decay estimate is a consequence of the weighted energy estimate for the global solution, not the other way around as in earlier methods [1, 2, 14, 15]. And it is clear from our analysis that a stronger velocity weight yields faster time decay.

It is thus very natural to try to use exponential velocity weight functions to get exponential time decay, which is the main purpose of our current investigation. The key is to show that the new energy with an exponential velocity weight is bounded for all time. In order to carry out such an energy estimate, we follow the general framework and strategy in [8], [9], [12]. However, many new analytical difficulties arise and we have to develop new techniques accordingly. The main difficulty lies in the estimates for linearized collision operators around a Maxwellians. The presence of the exponential weight factor $\exp\{\frac{q}{4}(1+|v|^2)^{\vartheta/2}\}$ in (10) requires much more precise estimates at almost all levels. In the case of a cutoff soft potential, a careful application of the Caflisch estimate (Lemma 1) is combined with the splitting trick in [8] to treat the very soft potential of $-3<\gamma\le -1$. Furthermore, we take a close look at the variables (21) in the Hilbert-Schmidt form for K to estimate the trickiest terms in Lemma 2. On the other hand, in the Landau case, an extra v factor from the derivative of the weight $\exp\{\frac{q}{4}(1+|v|^2)^{\vartheta/2}\}$ creates the most challenging difficulty to close the estimate in the same norm. We have to use different weight functions, that appeared in the norm (12), very precisely to balance between the derivative part $\sigma^{ij}\partial_i g\partial_j g$ and the no derivative part $\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}|g|^2$ in Lemma 8. Thus $\tau\le -1$ is assumed. Moreover, in Lemma 9, we have to introduce a new splitting of the linear Landau operator in the $\vartheta = 2$ case where $q<1$ is crucially used. Although we have quoted some estimates in previous papers, we have made an effort to provide complete, self-consistent proofs throughout the article. We also have made more comments between the proofs to help the readers.

The paper is organized as follows. In Section 2, we establish the estimates with the exponential weight (10) for the Boltzmann equation. In Section 3, we establish estimates with weight (10) for the Landau equation. Finally, in Section 4 we establish the crucial energy estimate uniformly for both cases. Some details are exactly the same as in [8], [9] or [12]. We will sketch these details which can be found elsewhere. Finally, we prove exponential decay in Section 5 using the global bound (15).

2. Boltzmann Estimates

In this section, we will prove the basic estimates used to obtain global existence of solutions with an exponential weight in the Boltzmann case. These estimates are similar to those in [8], but here the exponential weight which was not present earlier forces us to modify the proofs and some of the estimates. We will use the classical soft potential estimate of Caflisch [1] (Lemma 1) with v derivatives to estimate the linear operator (Lemma 2). We discuss the new features of each proof after the statement of each Lemma.

Recall K and Γ from (6) and (5). $K = K_2 - K_1$ is defined as [5, 6]:

$$[K_1 g](v) = \int_{\mathbb{R}^3\times S^2} B(\theta)|u-v|^\gamma \mu^{1/2}(u)\mu^{1/2}(v)g(u)\,du\,d\omega, \tag{17}$$

$$\begin{aligned}[K_2 g](v) = &\int_{\mathbb{R}^3\times S^2} B(\theta)|u-v|^\gamma \mu^{1/2}(u)\mu^{1/2}(u')g(v')\,du\,d\omega \\ &+ \int_{\mathbb{R}^3\times S^2} B(\theta)|u-v|^\gamma \mu^{1/2}(u)\mu^{1/2}(v')g(u')\,du\,d\omega.\end{aligned} \tag{18}$$

Consider a smooth cutoff function $0 \le \chi_m \le 1$ such that (for $m > 0$)

$$\chi_m(s) \equiv 1, \text{ for } s \ge 2m; \quad \chi_m(s) \equiv 0, \text{ for } s \le m. \tag{19}$$

Then define $\bar{\chi}_m = 1 - \chi_m$. Use χ_m to split $K_2 = K_2^{\chi} + K_2^{1-\chi}$ where

$$\begin{aligned} K_2^{\chi} g \equiv & \int_{\mathbb{R}^3 \times S^2} B(\theta)|u-v|^{\gamma} \mu^{1/2}(u) \chi_m(|u-v|) \mu^{1/2}(u') g(v') du d\omega \\ & + \int_{\mathbb{R}^3 \times S^2} B(\theta)|u-v|^{\gamma} \mu^{1/2}(u) \chi_m(|u-v|) \mu^{1/2}(v') g(u') du d\omega. \end{aligned}$$

After removing the singularity at $u = v$, following the procedure in [5,6] (see also eqns. (35) and (36) in [8]), we can write

$$K_2^{\chi} g = \int_{\mathbb{R}^3} k_2^{\chi}(v,\xi) g(v+\xi) d\xi,$$

where

$$k_2^{\chi}(v,\xi) \equiv \frac{e^{-\frac{1}{8}|\xi|^2 - \frac{1}{2}|\zeta_{\|}|^2}}{|\xi|\sqrt{\pi^3/2}} \int_{\mathbb{R}^2} \frac{\chi_m(\sqrt{|\xi|^2 + |\xi_{\perp}|^2})}{(|\xi|^2 + |\xi_{\perp}|^2)^{\frac{1-\gamma}{2}}} e^{-\frac{1}{2}|\xi_{\perp} + \zeta_{\perp}|^2} \frac{B(\theta)}{|\cos\theta|} d\xi_{\perp}. \tag{20}$$

The integration variables are $d\xi_{\perp} = d\xi_{\perp}^1 d\xi_{\perp}^2$ but $\xi_{\perp} = \xi_{\perp}^1 \xi^1 + \xi_{\perp}^2 \xi^2 \in \mathbb{R}^3$ where $\{\xi^1, \xi^2, \xi/|\xi|\}$ is an orthonormal basis for $\mathbb{R}^3$. Also

$$\zeta_{\|} = \frac{(v\cdot\xi)\xi}{|\xi|^2} + \frac{1}{2}\xi, \ \zeta_{\perp} = v - \frac{(v\cdot\xi)\xi}{|\xi|^2} = (v\cdot\xi^1)\xi^1 + (v\cdot\xi^2)\xi^2. \tag{21}$$

It should be noted that, as in [8], we have removed the symmetry of the kernel k_2^{χ} via the translation $\xi \to v + \xi$. This formulation is well suited for taking high order v-derivatives. Caflisch [1] proved Lemma 1 just below with no derivatives. In contrast, we have already removed the singularity and this allows us to extend the estimate from $-1 < \gamma < 0$ to the full range $-3 < \gamma < 0$. As in [8], we will see in Lemma 2 that the singular part of K_2, e.g. $K_2^{1-\chi}$, has stronger decay.

Lemma 1. *For any multi-index β and any $0 < s_1 < s_2 < 1$, we have*

$$|\partial_{\beta} k_2^{\chi}(v,\xi)| \le C \frac{\exp\left(-\frac{s_2}{8}|\xi|^2 - \frac{s_1}{2}|\zeta_{||}|^2\}\right)}{|\xi|(1+|v|+|\xi+v|)^{1-\gamma}}.$$

Here $C > 0$ will depend on s_1, s_2 and β.

Proof. Fix $0 < s_1 < s_2 < 1$. If $|\beta| > 0$, from (20) and (21) we have

$$\partial_{\beta} k_2^{\chi}(v,\xi) = \frac{e^{-\frac{1}{8}|\xi|^2}}{|\xi|\sqrt{\pi^3/2}} \int_{\mathbb{R}^2} \frac{\chi_m(\sqrt{|\xi|^2 + |\xi_{\perp}|^2})}{(|\xi|^2 + |\xi_{\perp}|^2)^{\frac{1-\gamma}{2}}} \partial_{\beta}\left(e^{-\frac{1}{2}|\xi_{\perp}+\zeta_{\perp}|^2 - \frac{1}{2}|\zeta_{\|}|^2}\right) \frac{B(\theta)}{|\cos\theta|} d\xi_{\perp}.$$

By a simple induction, for any $0 < q' < 1$, we have

$$\left|\partial_{\beta}\left(e^{-\frac{1}{2}|\xi_{\perp}+\zeta_{\perp}|^2 - \frac{1}{2}|\zeta_{\|}|^2}\right)\right| \le C(|\beta|, q') e^{-\frac{q'}{2}|\xi_{\perp}+\zeta_{\perp}|^2 - \frac{q'}{2}|\zeta_{\|}|^2}.$$

Further restrict $q' > s_1$. For $|\beta| \ge 0$, using the last display and (2) we have

$$|\partial_{\beta} k_2^{\chi}(v,\xi)| \le C \frac{e^{-\frac{1}{8}|\xi|^2}}{|\xi|} \int_{\mathbb{R}^2} \frac{\chi_m(\sqrt{|\xi|^2 + |\xi_{\perp}|^2})}{(|\xi|^2 + |\xi_{\perp}|^2)^{\frac{1-\gamma}{2}}} e^{-\frac{q'}{2}|\xi_{\perp}+\zeta_{\perp}|^2 - \frac{q'}{2}|\zeta_{\|}|^2} d\xi_{\perp}.$$

Change variables $\xi_\perp \to \xi_\perp - \zeta_\perp$ on the r.h.s. to obtain

$$C\frac{e^{-\frac{1}{8}|\xi|^2-\frac{q'}{2}|\zeta_\parallel|^2}}{|\xi|}\int_{\mathbb{R}^2}\frac{\chi_m(\sqrt{|\xi|^2+|\xi_\perp-\zeta_\perp|^2})}{(|\xi|^2+|\xi_\perp-\zeta_\perp|^2)^{\frac{1-\gamma}{2}}}e^{-\frac{q'}{2}|\xi_\perp|^2}d\xi_\perp.$$

Using (19), then, we have

$$|\partial_\beta k_2^\chi(v,\xi)| \le C(m)\frac{e^{-\frac{1}{8}|\xi|^2-\frac{q'}{2}|\zeta_\parallel|^2}}{|\xi|}\int_{\mathbb{R}^2}\frac{e^{-\frac{q'}{2}|\xi_\perp|^2}d\xi_\perp}{(1+|\xi|^2+|\xi_\perp-\zeta_\perp|^2)^{\frac{1-\gamma}{2}}}. \tag{22}$$

In the rest of the proof, we will refine this estimate via splitting.

Choose any $q''>0$ with $q''<s_1$ and then define $\tau_* = \sqrt{\frac{q'-s_1}{q'-q''}} < 1$. Split the integration region as follows

$$\{|\xi_\perp| > \tau_*|\zeta_\perp|\} \cup \{|\xi_\perp| \le \tau_*|\zeta_\perp|\}.$$

Further split the r.h.s. of (22) into $k_2^{\chi,1}(v,\xi)+k_2^{\chi,2}(v,\xi)$ where $k_2^{\chi,1}(v,\xi)$ is restricted to the region $\{|\xi_\perp| > \tau_*|\zeta_\perp|\}$:

$$k_2^{\chi,1}(v,\xi) \equiv C(m)\frac{e^{-\frac{1}{8}|\xi|^2-\frac{q'}{2}|\zeta_\parallel|^2}}{|\xi|}\int_{|\xi_\perp|>\tau_*|\zeta_\perp|}\frac{e^{-\frac{q'}{2}|\xi_\perp|^2}d\xi_\perp}{(1+|\xi|^2+|\xi_\perp-\zeta_\perp|^2)^{\frac{1-\gamma}{2}}}.$$

For $k_2^{\chi,1}(v,\xi)$ we will observe exponential decay. And for $k_2^{\chi,2}(v,\xi)$ we can extract from the denominator on the r.h.s. of (22) the exact decay stated in Lemma 1.

First consider $k_2^{\chi,1}(v,\xi)$. Since $\{|\xi_\perp| > \tau_*|\zeta_\perp|\}$ and $q'-q''>0$ we have

$$\begin{aligned}\left|k_2^{\chi,1}(v,\xi)\right| &\le C\frac{e^{-\frac{1}{8}|\xi|^2-\frac{q'}{2}|\zeta_\parallel|^2}}{|\xi|}\int_{\{|\xi_\perp|>\tau_*|\zeta_\perp|\}}\frac{e^{-\frac{q''}{2}|\xi_\perp|^2-\frac{q'-q''}{2}|\xi_\perp|^2}}{(1+|\xi|^2+|\xi_\perp-\zeta_\perp|^2)^{\frac{1-\gamma}{2}}}d\xi_\perp \\ &\le C\frac{e^{-\frac{1}{8}|\xi|^2-\frac{q'}{2}|\zeta_\parallel|^2}}{|\xi|}\int_{\{|\xi_\perp|>\tau_*|\zeta_\perp|\}}\frac{e^{-\frac{q''}{2}|\xi_\perp|^2-\frac{q'-s_1}{2}|\zeta_\perp|^2}}{(1+|\xi|^2+|\xi_\perp-\zeta_\perp|^2)^{\frac{1-\gamma}{2}}}d\xi_\perp.\end{aligned}$$

By (21),

$$|\zeta_\parallel|^2+|\zeta_\perp|^2 = |\zeta_\parallel+\zeta_\perp|^2 = |v+\xi/2|^2.$$

Splitting $q' = s_1 + (q'-s_1)$ we have

$$\begin{aligned}\left|k_2^{\chi,1}(v,\xi)\right| &\le \frac{C}{|\xi|}e^{-\frac{1}{8}|\xi|^2-\frac{s_1}{2}|\zeta_\parallel|^2}\int_{\{|\xi_\perp|>\tau_*|\zeta_\perp|\}}\frac{e^{-\frac{q''}{2}|\xi_\perp|^2-\frac{q'-s_1}{2}(|\zeta_\perp|^2+|\zeta_\parallel|^2)}}{(1+|\xi|^2+|\xi_\perp-\zeta_\perp|^2)^{\frac{1-\gamma}{2}}}d\xi_\perp \\ &\le \frac{C}{|\xi|}e^{-\frac{1}{8}|\xi|^2-\frac{s_1}{2}|\zeta_\parallel|^2-\frac{q'-s_1}{2}|v+\xi/2|^2}.\end{aligned}$$

This will be more than enough decay. We expand

$$\begin{aligned}|v+\xi/2|^2 &= |v|^2+\frac{1}{4}|\xi|^2+v\cdot\xi = \frac{1}{4}|v+\xi|^2+\frac{3}{4}|v|^2+\frac{1}{2}v\cdot\xi \\ &\ge \frac{1}{4}|v+\xi|^2+\frac{3}{4}|v|^2-\frac{1}{4}|v|^2-\frac{1}{4}|\xi|^2 \\ &= \frac{1}{4}|v+\xi|^2+\frac{1}{2}|v|^2-\frac{1}{4}|\xi|^2.\end{aligned} \tag{23}$$

Plug the last display into the one above it to obtain

$$\begin{aligned}\left|k_2^{\chi,1}(v,\xi)\right| &\le \frac{C}{|\xi|}e^{-\frac{1}{8}|\xi|^2-\frac{s_1}{2}|\zeta_\parallel|^2}e^{-\frac{q'-s_1}{8}|v+\xi|^2-\frac{q'-s_1}{4}|v|^2+\frac{q'-s_1}{8}|\xi|^2}\\ &= \frac{C}{|\xi|}e^{-\frac{s_1+1-q'}{8}|\xi|^2-\frac{s_1}{2}|\zeta_\parallel|^2}e^{-\frac{q'-s_1}{8}|v+\xi|^2-\frac{q'-s_1}{4}|v|^2}.\end{aligned}$$

Given s_2 with $s_1 < s_2 < 1$, we can always choose q', restricted by $s_1 < q' < 1$, such that $s_2 = s_1 + 1 - q'$. This completes the estimate for $k_2^{\chi,1}(v,\xi)$.

On $\{|\xi_\perp| \le \tau_*|\zeta_\perp|\}$, $|\zeta_\perp - \xi_\perp| \ge |\zeta_\perp| - |\xi_\perp| \ge (1-\tau_*)|\zeta_\perp|$ $(0 < \tau_* < 1)$. Hence (22) over this region is bounded as

$$\begin{aligned}\left|k_2^{\chi,2}(v,\xi)\right| &\le \frac{Ce^{-\frac{1}{8}|\xi|^2-\frac{q'}{2}|\zeta_\parallel|^2}}{|\xi|\left(1+|\xi|^2+(1-\tau_*)^2|\zeta_\perp|^2\right)^{\frac{1-\gamma}{2}}}\int_{\{|\xi_\perp|\le\tau_*|\zeta_\perp|\}} e^{-\frac{q'}{2}|\xi_\perp|^2}d\xi_\perp\\ &\le \frac{Ce^{-\frac{1}{8}|\xi|^2-\frac{s_1}{2}|\zeta_\parallel|^2}}{|\xi|\left(1+|\xi|^2+|\zeta_\perp|^2+|\zeta_\parallel|^2\right)^{\frac{1-\gamma}{2}}}\\ &= \frac{Ce^{-\frac{1}{8}|\xi|^2-\frac{s_1}{2}|\zeta_\parallel|^2}}{|\xi|\left(1+|\xi|^2+|v+\xi/2|^2\right)^{\frac{1-\gamma}{2}}},\end{aligned}$$

where we used $s_1 < q'$ to go from the first line to the second. Now plug (23) into the last display to complete the estimate. $\square$

Next we will prove the energy estimates for the linear operator (6).

Lemma 2. *Let $|\beta| > 0$, $\ell \in \mathbb{R}$, $0 \le \vartheta \le 2$ and $0 < q$. If $\vartheta = 2$ restrict $0 < q < 1$. Then $\forall \eta > 0$ $\exists C(\eta) > 0$ such that*

$$\langle w^2(\ell,\vartheta)\partial_\beta[\nu g], \partial_\beta g\rangle \ge |\partial_\beta g|^2_{\boldsymbol{\nu},\ell,\vartheta} - \eta\sum_{|\beta_1|\le|\beta|}|\partial_{\beta_1} g|^2_{\boldsymbol{\nu},\ell,\vartheta} - C(\eta)\left|\bar{\chi}_{C(\eta)} g\right|^2_\ell .$$

Furthermore, for any $|\beta| \ge 0$ we have

$$|\langle w^2(\ell,\vartheta)\partial_\beta[K g_1], g_2\rangle| \le \left\{\eta\sum_{|\beta_1|\le|\beta|}|\partial_{\beta_1} g_1|_{\boldsymbol{\nu},\ell,\vartheta} + C(\eta)\left|\bar{\chi}_{C(\eta)} g_1\right|_\ell\right\}|g_2|_{\boldsymbol{\nu},\ell,\vartheta},$$

where we are using (19).

Some parts of the proof of Lemma 2 are exactly the same as in [8]. For instance, the proof of the lower bound for $\langle w^2(\ell,\vartheta)\partial_\beta[\nu g], \partial_\beta g\rangle$ is exactly the same. But the estimate for $|\langle w^2(\ell,\vartheta)\partial_\beta[K g_1], g_2\rangle|$ requires extra care in particular because g_1 in the argument of $[K g_1]$ does not depend only on v. We therefore need to control the new exponentially growing factor of $w(\ell,\vartheta)(v)$. This requires a close look at the variables from (20) in (21). In particular, we write down the analogue of the conservation of energy (4) in this new coordinate system (29) in order absorb the exponentially growing weight. For completeness, we present all details of the proof.

Proof. The First Estimate in the Lemma.

Fix $\eta > 0$. Recall

$$\langle w^2\partial_\beta[\nu g], \partial_\beta g\rangle = \langle w^2\nu\partial_\beta g, \partial_\beta g\rangle + \sum_{0<\beta_1\le\beta} C_\beta^{\beta_1}\langle w^2\partial_{\beta_1}\nu\partial_{\beta-\beta_1} g, \partial_\beta g\rangle.$$

By Lemma 2 in [9], for $|\beta_1| > 0$,

$$|\partial_{\beta_1}\nu| \leq C(1+|v|^2)^{\frac{\gamma-1}{2}}.$$

We use this estimate (for m is chosen large enough) to obtain

$$\begin{aligned}\langle w^2\partial_{\beta_1}\nu\partial_{\beta-\beta_1}g, \partial_\beta g\rangle &= \int_{|v|\leq m} + \int_{|v|\geq m} \\ &\leq \int_{|v|\leq m} + \frac{C}{m}\left|\partial_{\beta-\beta_1}g\right|_{\boldsymbol{\nu},\ell,\vartheta}\left|\partial_\beta g\right|_{\boldsymbol{\nu},\ell,\vartheta} \\ &\leq \int_{|v|\leq m} + \frac{\eta}{2}\left|\partial_{\beta-\beta_1}g\right|_{\boldsymbol{\nu},\ell,\vartheta}\left|\partial_\beta g\right|_{\boldsymbol{\nu},\ell,\vartheta}.\end{aligned}$$

On the other hand, for such a $m > 0$ and $\beta - \beta_1 < \beta$, the first integral over $|v| \leq m$ is bounded by a compact Sobolev interpolation

$$\int_{|v|\leq m} \leq \frac{\eta}{2}\sum_{|\beta_1|=|\beta|}\left|\partial_{\beta_1}g\right|^2_{\boldsymbol{\nu},\ell} + C(\eta)\left|\bar{\chi}_{C(\eta)}g\right|^2_{\boldsymbol{\nu},\ell}. \tag{24}$$

This concludes the lower bound for $\langle w^2(\ell,\vartheta)\partial_\beta[\nu g], \partial_\beta g\rangle$.

The Second Estimate in the Lemma. The proof of the second estimate is divided into several parts. Recall $K = K_1 - K_2$.

Step 1. The Estimate for K_1.

Next consider K_1 from (17). We change variables $u \to u + v$ to obtain

$$[K_1g_1](v) = \int_{\mathbb{R}^3\times S^2} B(\theta)|u|^\gamma \mu^{1/2}(u+v)\mu^{1/2}(v)g_1(u+v)dud\omega$$

Notice that now $\cos\theta = u\cdot\omega/|u|$ so that for $|\beta| > 0$, $\partial_\beta[K_1g_1](v)$

$$= \sum_{\beta_1\leq\beta} C_\beta^{\beta_1}\int_{\mathbb{R}^3\times S^2} B(\theta)|u|^\gamma\partial_{\beta-\beta_1}\left(\mu^{1/2}(u+v)\mu^{1/2}(v)\right)\partial_{\beta_1}g_1(u+v)dud\omega.$$

For any $0 < q' < 1$ we have

$$|\partial_{\beta-\beta_1}\{\mu^{1/2}(u+v)\mu^{1/2}(v)\}| \leq C(|\beta|, q')\mu^{q'/2}(u+v)\mu^{q'/2}(v).$$

We will use this exponential decay to control half of the exponential growth in the weight. If $0 \leq \vartheta < 2$ then

$$w(\ell,\vartheta)(v)\mu^{q'/2}(v) \leq C\mu^{q'/4}(v).$$

If $\vartheta = 2$ then for given $0 < q < 1$ we choose q' so that $q < q' < 1$. And in this case

$$w(\ell,\vartheta)(v)\mu^{q'/2}(v) = (1+|v|^2)^{\tau\ell/2}e^{\frac{q}{4}}e^{\frac{q}{4}|v|^2}\mu^{q'/2}(v) \leq C\mu^{(q'-q)/4}(v).$$

Choosing $0 < q'' < \min\{|q'-q|/4, q'/4\}$, we can always write $\langle w^2(\ell,\vartheta)\partial_\beta[K_1g_1], g_2\rangle$

$$= \sum_{\beta_1\leq\beta}\int_{\mathbb{R}^3\times\mathbb{R}^3} w(\ell,\vartheta)(v)|u|^\gamma\mu^{q''}(u+v)\mu^{q''}(v)\mu_{\beta_1}(u+v,v)\partial_{\beta_1}g_1(u+v)g_2(v)dudv,$$

where $\mu_{\beta_1}(u+v,v)$ is a collection of smooth functions satisfying

$$\left|\partial^u_{\bar{\beta}}\mu_{\beta_1}(u+v,v)\right| \leq C(|\bar{\beta}|, q, q', q'').$$

Change variables $u \to u - v$ to obtain

$$\sum_{\beta_1\leq\beta}\int_{\mathbb{R}^3\times\mathbb{R}^3} w(\ell,\vartheta)(v)|u-v|^\gamma\mu^{q''}(u)\mu^{q''}(v)\mu_{\beta_1}(u,v)\partial_{\beta_1}g_1(u)g_2(v)dudv,$$

Now further split

$$\begin{aligned}\langle w^2(\ell,\vartheta)\partial_\beta[K_1 g_1], g_2\rangle &= \langle w^2(\ell,\vartheta)\partial_\beta[K_1^\chi g_1], g_2\rangle + \langle w^2(\ell,\vartheta)\partial_\beta[K_1^{1-\chi} g_1], g_2\rangle \\ &= \mathbf{K}_1^\chi + \mathbf{K}_1^{1-\chi}.\end{aligned}$$

Using (19) we have

$$\mathbf{K}_1^{1-\chi} \equiv \int w(\ell,\vartheta)(v)\bar{\chi}_m(|u-v|)|u-v|^\gamma \mu^{q''}(u)\mu^{q''}(v)\mu_{\beta_1}(u,v)\partial_{\beta_1} g_1(u) g_2(v) du dv,$$

where $\bar{\chi}_m = 1-\chi_m$ and we implicitly sum over $\beta_1 \le \beta$. Then

$$\begin{aligned}\left|\mathbf{K}_1^{1-\chi}\right| &\le C\left\{\int w^2(\ell,\vartheta)(v)\bar{\chi}_m(|u-v|)|u-v|^\gamma \mu^{q''}(u)\mu^{q''}(v)\left|g_2(v)\right|^2 du dv\right\}^{1/2} \\ &\times \sum_{\beta_1\le\beta}\left\{\int \bar{\chi}_m(|u-v|)|u-v|^\gamma \mu^{q''}(u)\mu^{q''}(v)\left|\partial_{\beta_1} g_1(u)\right|^2 du dv\right\}^{1/2}. \\ &\le C(2m)^{\frac{3+\gamma}{2}}|g_2|_{\boldsymbol{\nu},\ell,\vartheta}\sum_{\beta_1\le\beta}(2m)^{\frac{3+\gamma}{2}}|\partial_{\beta_1} g_1|_{\boldsymbol{\nu},\ell} \\ &\le \frac{\eta}{2}|g_2|_{\boldsymbol{\nu},\ell,\vartheta}\sum_{\beta_1\le\beta}|\partial_{\beta_1} g_1|_{\boldsymbol{\nu},\ell}.\end{aligned}$$

The last step follows by choosing m small enough.

Further,

$$\mathbf{K}_1^\chi \equiv \int w(\ell,\vartheta)(v)\chi_m(|u-v|)|u-v|^\gamma \mu^{q''}(u)\mu^{q''}(v)\mu_{\beta_1}(u,v)\partial_{\beta_1} g_1(u) g_2(v) du dv,$$

where we again implicitly sum over $\beta_1 \le \beta$. After an integration by parts

$$\begin{aligned}\mathbf{K}_1^\chi = \sum_{\beta_1\le\beta}(-1)^{|\beta_1|}\int w(\ell,\vartheta)(v)\partial_{\beta_1}^u\left\{\chi_m(|u-v|)|u-v|^\gamma\mu^{q''}(u)\mu_{\beta_1}(u,v)\right\} \\ \times\mu^{q''}(v) g_1(u) g_2(v) du dv.\end{aligned}$$

Since $|u-v|^\gamma$ is bounded now, from (19), choosing $m'>0$ large enough we have

$$\begin{aligned}|\mathbf{K}_1^\chi| &\le C(|\beta|,m)\int w(\ell,\vartheta)(v)\mu^{q''/2}(u)\mu^{q''/2}(v)\left|g_1(u)g_2(v)\right| du dv \\ &= \int_{|u|\le m'} + \int_{|u|>m'} \\ &\le C\int w(\ell,\vartheta)(v)\bar{\chi}_{m'}(|u|)\mu^{q''/2}(u)\mu^{q''/2}(v)\left|g_1(u)g_2(v)\right| du dv \\ &\quad + Ce^{-\frac{q''}{8}m'}\int w(\ell,\vartheta)(v)\mu^{q''/4}(u)\mu^{q''/2}(v)\left|g_1(u)g_2(v)\right| du dv \\ &\le \left\{\frac{\eta}{2}|g_1|_{\boldsymbol{\nu},\ell} + C(m')|\bar{\chi}_{m'} g_1|_{\boldsymbol{\nu},\ell}\right\}|g_2|_{\boldsymbol{\nu},\ell,\vartheta}.\end{aligned}$$

This completes the estimate for $\langle w^2(\ell,\vartheta)\partial_\beta[K_1 g_1], g_2\rangle$ and step one.

Step 2. The Estimate for K_2.

We turn to K_2 from (18). Split $K_2 = K_2^\chi + K_2^{1-\chi}$ and consider K_2^χ in (20).

Step (2a). The Estimate of $K_2^{1-\chi}$.

Now consider $K_2^{1-\chi} = K_2 - K_2^{\chi}$ which is given by

$$K_2^{1-\chi} g_1 \equiv \int_{\mathbb{R}^3 \times S^2} B(\theta)|u-v|^{\gamma} \mu^{1/2}(u) \bar{\chi}_m(|u-v|) \mu^{1/2}(u') g_1(v') du d\omega$$
$$+ \int_{\mathbb{R}^3 \times S^2} B(\theta)|u-v|^{\gamma} \mu^{1/2}(u) \bar{\chi}_m(|u-v|) \mu^{1/2}(v') g_1(u') du d\omega.$$

Here $\bar{\chi}_m = 1 - \chi_m$ and χ_m is defined in (19). (3) and $\{|u-v| \le 2m\}$ imply

$$|u'| = |v + u - v - [(u-v)\cdot\omega]\omega| \ge |v| - 2|u-v| \ge |v| - 4m,$$
$$|v'| = |v + [(u-v)\cdot\omega]\omega| \ge |v| - |u-v| \ge |v| - 2m.$$

Therefore for any $0 < q' < 1$ we have

$$\mu^{1/2}(u)\mu^{1/2}(u') + \mu^{1/2}(u)\mu^{1/2}(v') \le e^{C(q')m^2} \mu^{1/2}(u) \mu^{q'/2}(v). \tag{25}$$

This will be the key point in estimating the $K_2^{1-\chi}$ part.

First we take look at $\partial_\beta [K_2^{1-\chi} g_1]$. Change variables $u - v \to u$ to obtain

$$K_2^{1-\chi} g_1 \equiv \int_{\mathbb{R}^3 \times S^2} B(\theta)|u|^{\gamma} \mu^{1/2}(u+v) \bar{\chi}_m(|u|) \mu^{1/2}(v+u_\perp) g_1(v+u_\parallel) du d\omega$$
$$+ \int_{\mathbb{R}^3 \times S^2} B(\theta)|u|^{\gamma} \mu^{1/2}(u+v) \bar{\chi}_m(|u|) \mu^{1/2}(v+u_\parallel) g_1(v+u_\perp) du d\omega,$$

Note that $u_\parallel$ and $u_\perp$ are defined using notation from [8]:

$$u_\parallel \equiv [u\cdot\omega]\omega, \ \ u_\perp \equiv u - [u\cdot\omega]\omega. \tag{26}$$

Now derivatives will not hit the singular kernel. $\partial_\beta [K_2^{1-\chi} g_1]$ is

$$C_\beta^{\beta_1} \int_{\mathbb{R}^3 \times S^2} B(\theta)|u|^{\gamma} \bar{\chi}_m(|u|) \partial_{\beta-\beta_1} \{\mu^{1/2}(u+v)\mu^{1/2}(v+u_\perp)\} \partial_{\beta_1} g_1(v+u_\parallel) du d\omega$$
$$+ C_\beta^{\beta_1} \int_{\mathbb{R}^3 \times S^2} B(\theta)|u|^{\gamma} \bar{\chi}_m(|u|) \partial_{\beta-\beta_1} \{\mu^{1/2}(u+v)\mu^{1/2}(v+u_\parallel)\} \partial_{\beta_1} g_1(v+u_\perp) du d\omega,$$

where we implicitly sum over multi-indices $\beta_1 \le \beta$. Therefore, for any $0 < q'' < 1$, $|\partial_\beta [K_2^{1-\chi} g_1]|$ is bounded by

$$C \int_{\mathbb{R}^3 \times S^2} |u|^{\gamma} \bar{\chi}_m(|u|) \mu^{q''/2}(u+v) \mu^{q''/2}(v+u_\perp) |\partial_{\beta_1} g_1(v+u_\parallel)| du d\omega$$
$$+ C \int_{\mathbb{R}^3 \times S^2} |u|^{\gamma} \bar{\chi}_m(|u|) \mu^{q''/2}(u+v) \mu^{q''/2}(v+u_\parallel) |\partial_{\beta_1} g_1(v+u_\perp)| du d\omega.$$

We change variables $u \to u - v$ again to see that $|\partial_\beta [K_2^{1-\chi} g_1]|$ is bounded by

$$C \int_{\mathbb{R}^3 \times S^2} |u-v|^{\gamma} \bar{\chi}_m(|u-v|) \mu^{q''/2}(u) \mu^{q''/2}(v') |\partial_{\beta_1} g_1(u')| du d\omega$$
$$+ C \int_{\mathbb{R}^3 \times S^2} |u-v|^{\gamma} \bar{\chi}_m(|u-v|) \mu^{q''/2}(u) \mu^{q''/2}(u') |\partial_{\beta_1} g_1(v')| du d\omega.$$

Use (25) with $0 < q' < q''$ to say this is bounded above by

$$C \int_{\mathbb{R}^3 \times S^2} |u-v|^{\gamma} \bar{\chi}_m(|u-v|) \mu^{q''/2}(u) \mu^{q'/2}(v) \left\{ |\partial_{\beta_1} g_1(u')| + |\partial_{\beta_1} g_1(v')| \right\} du d\omega.$$

We remark that this last bound is true (and trivial) when $|\beta| = 0$ in which case $\partial_{\beta_1} = \partial_0 = 1$ by convention. Thus, $|\langle w^2(\ell,\vartheta)\partial_\beta\{K_2^{1-\chi} g_1\}, g_2\rangle|$ is

$$\leq C \int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} |u-v|^\gamma \bar{\chi}_m(|u-v|) w^2(\ell,\vartheta)(v)\mu^{q''/2}(u)\mu^{q'/2}(v)|\partial_{\beta_1} g_1(v')||g_2(v)|,$$

$$+C \int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} |u-v|^\gamma \bar{\chi}_m(|u-v|) w^2(\ell,\vartheta)(v)\mu^{q''/2}(u)\mu^{q'/2}(v)|\partial_{\beta_1} g_1(u')||g_2(v)|.$$

Here again we need to control the large exponentially growing factor $w(\ell,\vartheta)(v)$ by the strong exponential decay of the Maxwellians.

We will control this growth in cases. If $0 \leq \vartheta < 2$ then $q' > 0$ means

$$w(\ell,\vartheta)(v)\mu^{q'/2}(v) \leq C\mu^{q'/4}(v).$$

Alternatively, if $\vartheta = 2$ with $0 < q < 1$ then we can choose q' and q'' such that $q < q' < q''$. Then

$$w(\ell,\vartheta)(v)\mu^{q'/2}(v) \leq Cw(\ell,0)(v)\mu^{(q'-q)/2}(v) \leq C\mu^{(q'-q)/4}(v).$$

In either case, choose $q_1 = \min\{q''/2, q'/4, |q'-q|/4\} > 0$. Then we have the upper bound of

$$C\int |u-v|^\gamma \bar{\chi}_m(|u-v|) w(\ell,\vartheta)(v)\mu^{q_1}(u)\mu^{q_1}(v)|\partial_{\beta_1} g_1(v')||g_2(v)|dvdud\omega$$

$$+C\int |u-v|^\gamma \bar{\chi}_m(|u-v|) w(\ell,\vartheta)(v)\mu^{q_1}(u)\mu^{q_1}(v)|\partial_{\beta_1} g_1(u')||g_2(v)|dvdud\omega.$$

Further note that

$$\int |u-v|^\gamma \bar{\chi}_m(|u-v|)\mu^{q_1}(u)du \leq Cm^{3+\gamma}.$$

Apply Cauchy-Schwartz and the last display to obtain the upper bound

$$\leq Cm^{\frac{3+\gamma}{2}} \left\{\int |u-v|^\gamma \bar{\chi}_m(|u-v|)\mu^{q_1}(u)\mu^{q_1}(v)|\partial_{\beta_1} g_1(v')|^2 dvdud\omega\right\}^{1/2} |g_2|_{\boldsymbol{\nu},\ell,\vartheta}$$

$$+Cm^{\frac{3+\gamma}{2}} \left\{\int |u-v|^\gamma \bar{\chi}_m(|u-v|)\mu^{q_1}(u)\mu^{q_1}(v)|\partial_{\beta_1} g_1(u')|^2 dvdud\omega\right\}^{1/2} |g_2|_{\boldsymbol{\nu},\ell,\vartheta}.$$

Now apply the change of variables $(u,v) \to (u',v')$ using $|u-v| = |u'-v'|$ and (4) to see that $|\langle w^2(\ell,\vartheta)\partial_\beta\{K_2^{1-\chi} g_1\}, g_2\rangle|$ is bounded by

$$Cm^{3+\gamma} \left\{\int |u-v|^\gamma \bar{\chi}_m(|u-v|)\mu^{q_1}(u)\mu^{q_1}(v)|\partial_{\beta_1} g_1(v)|^2 dvdu\right\}^{1/2} |g_2|_{\boldsymbol{\nu},\ell,\vartheta}.$$

Hence,

$$|\langle w^2(\ell,\vartheta)\partial_\beta\{K_2^{1-\chi} g_1\}, g_2\rangle| \leq Cm^{\gamma+3}|g_2|_{\boldsymbol{\nu},\ell,\vartheta} \sum_{|\beta_1|\leq|\beta|} |\partial_{\beta_1} g_1|_{\boldsymbol{\nu},\ell}.$$

For $m > 0$ small enough, we have completed the estimate of $K_2^{1-\chi}$, step (2a).

Step (2b). Estimate of K_2^χ.

For some large but fixed $m' > 0$ we define the smooth cutoff function

$$\Upsilon_{m'} = \Upsilon_{m'}(v,\xi) = \chi_{m'}(\sqrt{1+|v|^2+|v+\xi|^2}), \quad \bar{\Upsilon}_{m'} = 1 - \Upsilon_{m'}(v,\xi).$$

Now split again $K_2^\chi = K_2^\Upsilon + K_2^{1-\Upsilon}$ where

$$K_2^\Upsilon g_1 = \int_{\mathbb{R}^3} \Upsilon_{m'}(v,\xi) k_2^\chi(v,\xi) g_1(v+\xi)d\xi.$$

We will estimate this term first. Taking derivatives

$$\partial_\beta[K_2^\Upsilon g_1] = \sum_{\beta_1\le\beta} C_\beta^{\beta_1}\int_{\mathbb{R}^3}\partial^v_{\beta_1}[\Upsilon_{m'}(v,\xi)k_2^\chi(v,\xi)]\partial_{\beta-\beta_1}g_1(v+\xi)d\xi$$

Using Lemma 1, with $0<s_1<s_2<1$, $|\langle w^2(\ell,\vartheta)\partial_\beta[K_2^\Upsilon g_1],g_2\rangle|$ is bounded by

$$C\sum_{\beta_1\le\beta}\int_{|v|+|v+\xi|>m'}\frac{w^2(\ell,\vartheta)(v)|\partial_{\beta-\beta_1}g_1(v+\xi)||g_2(v)|}{|\xi|(1+|v|+|v+\xi|)^{1-\gamma}}e^{-\frac{s_2}{8}|\xi|^2-\frac{s_1}{2}|\zeta_{||}|^2}d\xi dv. \tag{27}$$

By (10) we expand

$$w(\ell,\vartheta)(v)e^{-\frac{s_2}{8}|\xi|^2-\frac{s_1}{2}|\zeta_{||}|^2}=(1+|v|^2)^{\tau\ell/2}e^{\frac{q}{4}(1+|v|^2)^{\vartheta/2}}e^{-\frac{s_2}{8}|\xi|^2-\frac{s_1}{2}|\zeta_{||}|^2}. \tag{28}$$

If we can control (28) by $w(\ell,\vartheta)(v+\xi)$ times decay in other directions then we can estimate (27). To do this, we look for an analogue of (4) in the variables (21).

Using (21) we have

$$|v|^2+|v+\xi-\zeta_\perp|^2=|v+\xi|^2+|v-\zeta_\perp|^2=|v+\xi|^2+\left(v\cdot\frac{\xi}{|\xi|}\right)^2. \tag{29}$$

Since $0\le\vartheta\le 2$ we have

$$|v|^\vartheta\le\left(|v+\xi|^2+\left(v\cdot\frac{\xi}{|\xi|}\right)^2\right)^{\vartheta/2}\le|v+\xi|^\vartheta+\left|v\cdot\frac{\xi}{|\xi|}\right|^\vartheta.$$

Thus,

$$e^{\frac{q}{4}(1+|v|^2)^{\vartheta/2}}\le e^{\frac{q}{4}}e^{\frac{q}{4}|v|^\vartheta}\le e^{\frac{q}{4}}e^{\frac{q}{4}|v+\xi|^\vartheta}e^{\frac{q}{4}\left|v\cdot\frac{\xi}{|\xi|}\right|^\vartheta}. \tag{30}$$

Further, from (21) notice that

$$|\zeta_{\|}|^2=\left(\left(v\cdot\frac{\xi}{|\xi|}\right)+\frac{1}{2}|\xi|\right)^2\ge\frac{1}{2}\left(v\cdot\frac{\xi}{|\xi|}\right)^2-\frac{1}{4}|\xi|^2$$

Therefore with $0<s_1<s_2$ we have

$$e^{-\frac{s_2}{8}|\xi|^2-\frac{s_1}{2}|\zeta_{||}|^2}\le e^{-\frac{s_2-s_1}{8}|\xi|^2-\frac{s_1}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2}$$

Combine the above with (30), to obtain

$$\begin{aligned}e^{\frac{q}{4}(1+|v|^2)^{\vartheta/2}}e^{-\frac{s_2}{8}|\xi|^2-\frac{s_1}{2}|\zeta_{||}|^2}&\le e^{\frac{q}{4}}e^{\frac{q}{4}|v+\xi|^\vartheta}e^{\frac{q}{4}\left|v\cdot\frac{\xi}{|\xi|}\right|^\vartheta}e^{-\frac{s_2}{8}|\xi|^2-\frac{s_1}{2}|\zeta_{||}|^2}\\&\le Ce^{\frac{q}{4}|v+\xi|^\vartheta}e^{\frac{q}{4}\left|v\cdot\frac{\xi}{|\xi|}\right|^\vartheta}e^{-\frac{s_2-s_1}{8}|\xi|^2-\frac{s_1}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2}\\&=Ce^{\frac{q}{4}|v+\xi|^\vartheta}e^{-\frac{s_2-s_1}{8}|\xi|^2}e^{\frac{q}{4}\left|v\cdot\frac{\xi}{|\xi|}\right|^\vartheta-\frac{s_1}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2}.\end{aligned}$$

If $0\le\vartheta<2$ then

$$e^{\frac{q}{4}\left|v\cdot\frac{\xi}{|\xi|}\right|^\vartheta-\frac{s_1}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2}\le Ce^{-\frac{s_1}{8}\left(v\cdot\frac{\xi}{|\xi|}\right)^2}.$$

And if $\vartheta=2$ then $0<q<1$ and we can choose s_1 with $1>s_1>q$ so that

$$e^{\frac{q}{4}\left|v\cdot\frac{\xi}{|\xi|}\right|^2-\frac{s_1}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2}\le Ce^{-\frac{s_1-q}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2}.$$

In either case, choosing $s_3=\min\{|s_1-q|,s_1/2\}>0$ and plugging these estimates into (28), we conclude that

$$\begin{aligned}&w(\ell,\vartheta)(v)e^{-\frac{s_2}{8}|\xi|^2-\frac{s_1}{2}|\zeta_{||}|^2}\\&\le C(1+|v|^2)^{\tau\ell/2}e^{\frac{q}{4}(1+|v+\xi|^2)^{\vartheta/2}}e^{-\frac{s_2-s_1}{8}|\xi|^2-\frac{s_3}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2}.\end{aligned} \tag{31}$$

Next, we estimate $(1+|v|^2)^{\tau\ell/2}$ with $\tau<0$ and $\ell\in\mathbb{R}$. If $\ell\tau>0$ then (29) yields

$$(1+|v|^2)^{\tau\ell/2} \le \left(1+|v+\xi|^2+\left(v\cdot\frac{\xi}{|\xi|}\right)^2\right)^{\tau\ell/2}$$
$$\le C(1+|v+\xi|^2)^{\tau\ell/2}\left(1+\left(v\cdot\frac{\xi}{|\xi|}\right)^2\right)^{\tau\ell/2}.$$

Conversely if $\ell\tau\le 0$ then we split the region into

$$\{|v+\xi|>2|v|\}\cup\{|v+\xi|\le 2|v|\}.$$

On $\{|v+\xi|\le 2|v|\}$ and $\ell\tau\le 0$ then

$$(1+|v|^2)^{\tau\ell/2}\le C(1+|v+\xi|^2)^{\tau\ell/2}.$$

Alternatively, if $\{|v+\xi|>2|v|\}$ then

$$|\xi|\ge|v+\xi|-|v|>|v+\xi|/2.$$

We therefore always have

$$(1+|v|^2)^{\tau\ell/2}e^{-\frac{s_2-s_1}{8}|\xi|^2}\le e^{-\frac{s_2-s_1}{8}|\xi|^2}\le e^{-\frac{s_2-s_1}{16}|\xi|^2}e^{-\frac{s_2-s_1}{64}|v+\xi|^2}.$$

In either of these last few cases, since $s_2>s_1>q$, from (31) we have

$$w(\ell,\vartheta)(v)e^{-\frac{s_2}{8}|\xi|^2-\frac{s_1}{2}|\zeta_{||}|^2}\le C(1+|v|^2)^{\tau\ell/2}e^{\frac{q}{4}(1+|v+\xi|^2)^{\vartheta/2}}e^{-\frac{s_2-s_1}{8}|\xi|^2-\frac{s_3}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2}$$
$$\le C(1+|v+\xi|^2)^{\tau\ell/2}e^{\frac{q}{4}(1+|v+\xi|^2)^{\vartheta/2}}e^{-\frac{s_2-s_1}{16}|\xi|^2-\frac{s_3}{8}\left(v\cdot\frac{\xi}{|\xi|}\right)^2}$$
$$=Cw(\ell,\vartheta)(v+\xi)e^{-\frac{s_2-s_1}{16}|\xi|^2-\frac{s_3}{8}\left(v\cdot\frac{\xi}{|\xi|}\right)^2}.$$

Plug this into (27) to obtain the following upper bound for (27) of

$$C\int_{|v|+|v+\xi|>m'}\frac{\left(w(\ell,\vartheta)(v+\xi)|\partial_{\beta-\beta_1}g_1(v+\xi)|\right)\left(w(\ell,\vartheta)(v)|g_2(v)|\right)}{|\xi|(1+|v|+|v+\xi|)^{1-\gamma}}e^{-\frac{s_2-s_1}{16}|\xi|^2}d\xi dv,$$

where we implicitly sum over $\beta_1\le\beta$. Using Cauchy-Schwartz and translation invariance this is

$$\le\frac{C}{m'}\int\frac{\left(w(\ell,\vartheta)(v+\xi)|\partial_{\beta-\beta_1}g_1(v+\xi)|\right)\left(w(\ell,\vartheta)(v)|g_2(v)|\right)}{|\xi|(1+|v|+|v+\xi|)^{-\gamma}}e^{-\frac{s_2-s_1}{16}|\xi|^2}d\xi dv$$
$$\le\frac{C}{m'}|g_2|_{\boldsymbol{\nu},\ell,\vartheta}\int\frac{w^2(\ell,\vartheta)(v+\xi)|\partial_{\beta-\beta_1}g_1(v+\xi)|^2}{|\xi|(1+|v+\xi|)^{-\gamma}}e^{-\frac{s_2-s_1}{16}|\xi|^2}d\xi dv$$
$$\le\frac{C}{m'}\sum_{\beta_1\le\beta}|\partial_{\beta-\beta_1}g_1|_{\boldsymbol{\nu},\ell,\vartheta}|g_2|_{\boldsymbol{\nu},\ell,\vartheta}.$$

This completes the estimate for K_2^{Υ}.

We now estimate $K_2^{1-\Upsilon}$. Taking derivatives

$$\partial_\beta[K_2^{1-\Upsilon}g_1]=\sum_{\beta_1\le\beta}C_\beta^{\beta_1}\int_{\mathbb{R}^3}\partial_{\beta_1}^v[\bar{\Upsilon}_{m'}(v,\xi)k_2^\chi(v,\xi)]\partial_{\beta-\beta_1}g_1(v+\xi)d\xi$$

Also,

$$\langle w^2\partial_\beta[K_2^{1-\Upsilon}g_1],g_2\rangle=\int w^2(\ell,\vartheta)\partial_\beta[K_2^{1-\Upsilon}g_1]g_2(v)dv.$$

By Cauchy-Schwartz and the compact support of $K_2^{1-\Upsilon}$ we have

$$\left|\langle w^2\partial_\beta[K_2^{1-\Upsilon}g_1], g_2\rangle\right| \le C(m')\left\{\int_{|v|\le m'}\left(\partial_\beta[K_2^{1-\Upsilon}g_1]\right)^2 dv\right\}^{1/2}|g_2|_{\boldsymbol{\nu},\ell}.$$

With Lemma 1 we establish that $\partial_\beta[K_2^{1-\Upsilon}g_1]$ is compact from H^k to H^k. Then by the general interpolation for compact operators from H^k to H^k we have

$$\left|\langle w^2\partial_\beta[K_2^{1-\Upsilon}g_1], g_2\rangle\right| \le \left\{\frac{\eta}{4}\sum_{|\beta_1|=|\beta|}|\partial_{\beta_1}g_1|_{\boldsymbol{\nu},\ell} + C(\eta,m')\,|g_1|_{\boldsymbol{\nu},\ell}\right\}|g_2|_{\boldsymbol{\nu},\ell}.$$

This completes the estimate for $K_2^{1-\Upsilon}$ and thus for K_2^{χ}, step (2b). We have therefore finished the whole proof. $\square$

The following Corollary is used to prove existence of global solutions.

Corollary 1. *Let $|\beta|>0$, $\ell\in\mathbb{R}$, $0\le\vartheta\le 2$ and $0<q$. If $\vartheta=2$ restrict $0<q<1$. Then $\forall\eta>0$ there exists $C(\eta)>0$ such that*

$$\langle w^2\partial_\beta[Lg], \partial_\beta g\rangle \ge |\partial_\beta g|^2_{\boldsymbol{\nu},\ell,\vartheta} - \eta\sum_{|\beta_1|\le|\beta|}|\partial_{\beta_1}g|^2_{\boldsymbol{\nu},\ell,\vartheta} - C(\eta)\left|\bar{\chi}_{C(\eta)}g\right|^2_{\ell},$$

where $\bar{\chi}_{C(\eta)}$ is from (19).

The rest of section is devoted to estimates for nonlinear collision term $\Gamma[g_1,g_2]$, with $g_i(x,v)$ $(i=1,2)$. In (5), the (non-symmetric) bilinear form $\Gamma[g_1,g_2]$ in the Boltzmann case is

$$\begin{aligned}\Gamma[g_1,g_2] &= \mu^{-1/2}(v)Q[\mu^{1/2}g_1,\mu^{1/2}g_2] \equiv \Gamma_{\mathrm{gain}}[g_1,g_2]-\Gamma_{\mathrm{loss}}[g_1,g_2],\\ &= \int_{\mathbb{R}^3}|u-v|^\gamma\mu^{1/2}(u)\left[\int_{S^2}B(\theta)g_1(u')g_2(v')d\omega\right]du,\\ &\quad -g_2(v)\int_{\mathbb{R}^3}|u-v|^\gamma\mu^{1/2}(u)\left[\int_{S^2}B(\theta)d\omega\right]g_1(u)du.\end{aligned}\tag{32}$$

The change of variables $u-v\to u$ gives

$$\begin{aligned}\partial^\alpha_\beta\Gamma[g_1,g_2] &\equiv \partial^\alpha_\beta\left[\int_{\mathbb{R}^3}\int_{S^2}|u|^\gamma\mu^{1/2}(u+v)g_1(v+u_\parallel)g_2(v+u_\perp)B(\theta)dud\omega\right],\\ &\quad -\partial^\alpha_\beta\left[\int_{\mathbb{R}^3}\int_{S^2}|u|^\gamma\mu^{1/2}(u+v)g_1(v+u)g_2(v)B(\theta)dud\omega\right],\\ &\equiv \sum C_\beta^{\beta_0\beta_1\beta_2}C_\alpha^{\alpha_1\alpha_2}\Gamma^0[\partial^{\alpha_1}_{\beta_1}g_1,\partial^{\alpha_2}_{\beta_2}g_2].\end{aligned}$$

where the summation is over $\beta_0+\beta_1+\beta_2=\beta$ and $\alpha_1+\alpha_2=\alpha$. Also $u_\perp, u_\parallel$ are given by (26). By the product rule and the reverse change of variables we have

$$\begin{aligned}\Gamma^0[\partial^{\alpha_1}_{\beta_1}g_1,\partial^{\alpha_2}_{\beta_2}g_2], &\equiv \int_{\mathbb{R}^3}\int_{S^2}|u-v|^\gamma\partial_{\beta_0}[\mu^{1/2}(u)]\partial^{\alpha_1}_{\beta_1}g_1(u')\partial^{\alpha_2}_{\beta_2}g_2(v')B(\theta)\\ &\quad -\partial^{\alpha_2}_{\beta_2}g_2(v)\int_{\mathbb{R}^3}\int_{S^2}|u-v|^\gamma\partial_{\beta_0}[\mu^{1/2}(u)]\partial^{\alpha_1}_{\beta_1}g_1(u)B(\theta)\\ &\equiv \Gamma^0_{\mathrm{gain}}-\Gamma^0_{\mathrm{loss}}.\end{aligned}\tag{33}$$

With these formulas, we have the following nonlinear estimate:

Lemma 3. *Recall (33) and let* $\beta_0+\beta_1+\beta_2=\beta$, $\alpha_1+\alpha_2=\alpha$. *Say* $0\le\vartheta\le 2$, $0<q$. *If* $\vartheta=2$, *restrict* $0<q<1$. *Let* $\ell=|\beta|-l$ *with* $l\ge 0$. *If* $|\alpha_1|+|\beta_1|\le N/2$, *then*

$$|(w^2(\ell,\vartheta)\Gamma^0[\partial^{\alpha_1}_{\beta_1}g_1,\partial^{\alpha_2}_{\beta_2}g_2],\ \partial^\alpha_\beta g_3)|\le C\mathcal{E}^{1/2}_{l,\vartheta}(g_1)||\partial^{\alpha_2}_{\beta_2}g_2||_{\boldsymbol{\nu},|\beta_2|-l,\vartheta}||\partial^\alpha_\beta g_3||_{\boldsymbol{\nu},\ell,\vartheta}.$$

Alternatively, if $|\alpha_2|+|\beta_2|\le N/2$, *then*

$$|(w^2(\ell,\vartheta)\Gamma^0[\partial^{\alpha_1}_{\beta_1}g_1,\partial^{\alpha_2}_{\beta_2}g_2],\ \partial^\alpha_\beta g_3)|\le C||\partial^{\alpha_1}_{\beta_1}g_1||_{\boldsymbol{\nu},|\beta_1|-l,\vartheta}\mathcal{E}^{1/2}_{l,\vartheta}(g_2)||\partial^\alpha_\beta g_3||_{\boldsymbol{\nu},\ell,\vartheta}.$$

The proof of Lemma 3 is more or less the same as in [8]. However, small modifications are needed to facilitate the exponentially growing weight. In (37) we need to use (4) to properly distribute the exponentially growing factor in $w^2(\ell,\vartheta)(v)$.

Proof. Case 1. **The Loss Term Estimate.**

First consider the second term Γ^0_{loss}in (33). Note that

$$|\partial_{\beta_0}[\mu^{1/2}(u)]|\le Ce^{-|u|^2/8}.$$

With $|\alpha_1|+|\beta_1|\le N/2$ and $\gamma>-3$ we have

$$\begin{aligned}&\int_{\mathbb{R}^3}|u-v|^\gamma|\partial_{\beta_0}[\mu^{1/2}(u)]\partial^{\alpha_1}_{\beta_1}g_1(x,u)|du\\ &\le C\left\{\int_{\mathbb{R}^3}|u-v|^\gamma e^{-|u|^2/8}|\partial^{\alpha_1}_{\beta_1}g_1(x,u)|^2du\right\}^{1/2}\left\{\int_{\mathbb{R}^3}|u-v|^\gamma e^{-|u|^2/8}du\right\}^{1/2}\\ &\le C\sup_{x,u}\left|e^{-|u|^2/16}\partial^{\alpha_1}_{\beta_1}g_1(x,u)\right|\left\{\int_{\mathbb{R}^3}|u-v|^\gamma e^{-|u|^2/16}du\right\}\\ &\le C\mathcal{E}^{1/2}_l(g_1)[1+|v|]^\gamma.\end{aligned}$$

Since $N\ge 8$, we have used the embedding $H^4(\mathbb{T}^3\times\mathbb{R}^3)\subset L^\infty$ to argue that

$$\sup_{x,u}\left|e^{-|u|^2/16}\partial^{\alpha_1}_{\beta_1}g_1(x,u)\right|\le C\mathcal{E}^{1/2}_l(g_1). \tag{34}$$

Hence $\left|(w^2(\ell,\vartheta)\Gamma^0_{\mathrm{loss}}[\partial^{\alpha_1}_{\beta_1}g_1,\partial^{\alpha_2}_{\beta_2}g_2],\partial^\alpha_\beta g_3)\right|$ is bounded by

$$\begin{aligned}&C\mathcal{E}^{1/2}_l(g_1)\int[1+|v|]^\gamma w^2(\ell,\vartheta)(v)|\partial^{\alpha_2}_{\beta_2}g_2(v)\partial^\alpha_\beta g_3(v)|dvdx\\ &\le C\mathcal{E}^{1/2}_l(g_1)||\partial^{\alpha_2}_{\beta_2}g_2||_{\boldsymbol{\nu},\ell,\vartheta}||\partial^\alpha_\beta g_3||_{\boldsymbol{\nu},\ell,\vartheta}.\end{aligned}$$

This completes the estimate for Γ^0_{loss} when $|\alpha_1|+|\beta_1|\le N/2$.

Now consider Γ^0_{loss} with $|\alpha_2|+|\beta_2|\le N/2$. Here we split the (u,v) integration domain is split into three parts

$$\{|v-u|\le|v|/2\}\cup\ \{|v-u|\ge|v|/2,\ |v|\ge 1\}\cup\{|v-u|\ge|v|/2,\ |v|\le 1\}.$$

In the first region, $|u|$ is comparable to $|v|$ and thus we can use exponential decay in both variables to get the estimate. In the second and third regions, $|u|$ is not comparable to $|v|$ but we exploit the largeness or smallness of $|v|$ to get the estimate.

Case (1a). The Loss Term in the First Region $\{|v-u|\le|v|/2\}$.

For the first part, $\{|v-u|\le|v|/2\}$, we have

$$|u|\ge|v|-|v-u|\ge|v|/2.$$

So that

$$e^{-|u|^2/8}\le e^{-|u|^2/16}e^{-|v|^2/64}.$$

Then the integral of $w^2\Gamma^0_{\text{loss}}[\partial^{\alpha_1}_{\beta_1}g_1,\partial^{\alpha_2}_{\beta_2}g_2]\partial^\alpha_\beta g_3$ over $\{|u-v|\le |v|/2\}$ is bounded by

$$C\int |u-v|^\gamma e^{-|u|^2/16}e^{-|v|^2/64}w^2(\ell,\vartheta)(v)|\partial^{\alpha_1}_{\beta_1}g_1(u)\partial^{\alpha_2}_{\beta_2}g_2(v)\partial^\alpha_\beta g_3(v)|dudvdx$$
$$\le C\left\{\int |u-v|^\gamma e^{-|u|^2/16}e^{-|v|^2/64}|\partial^{\alpha_1}_{\beta_1}g_1(u)|^2dudvdx\right\}^{1/2}\times$$
$$\left\{\int |u-v|^\gamma e^{-|u|^2/16}e^{-|v|^2/64}w^4(\ell,\vartheta)(v)|\partial^{\alpha_2}_{\beta_2}g_2(v)|^2|\partial^\alpha_\beta g_3(v)|^2dudvdx\right\}^{1/2}.$$

Integrating over dv, the first factor is bounded by $C||\partial^{\alpha_1}_{\beta_1}g_1||_{\boldsymbol{\nu},\ell}$. Integrating first over u variables in the second factor yields an upper bound

$$C\left\{\int\left(\int |u-v|^\gamma e^{-|u|^2/16}du\right)e^{-|v|^2/64}w^4|\partial^{\alpha_2}_{\beta_2}g_2|^2|\partial^\alpha_\beta g_3|^2dvdx\right\}^{1/2}$$
$$\le C\left\{\int\int e^{-|v|^2/64}w^4(\ell,\vartheta)(v)[1+|v|]^\gamma|\partial^{\alpha_2}_{\beta_2}g_2|^2|\partial^\alpha_\beta g_3|^2dvdx\right\}^{1/2}.$$

And as in (34), since $N\ge 8$ and $|\alpha_2|+|\beta_2|\le N/2$ we have

$$\sup_{x,v} w(\ell,\vartheta)(v)e^{-|v|^2/256}|\partial^{\alpha_2}_{\beta_2}g_2(x,v)|\le C\mathcal{E}^{1/2}_{l,\vartheta}(g_2). \tag{35}$$

We thus conclude the estimate over the first region.

Case (1b). The Loss Term in the Second Region $\{|v-u|\ge |v|/2,\ |v|\ge 1\}$.

Next consider Γ^0_{loss} over the second region $\{|v-u|\ge |v|/2,\ |v|\ge 1\}$. Since $\gamma<0$, we have

$$|u-v|^\gamma\le C[1+|v|]^\gamma.$$

Then the integral of $w^2(\ell,\vartheta)\Gamma^0_{\text{loss}}[\partial^{\alpha_1}_{\beta_1}g_1,\partial^{\alpha_2}_{\beta_2}g_2]\partial^\alpha_\beta g_3$ over this region is bounded by

$$C\int [1+|v|]^\gamma e^{-|u|^2/8}w^2(\ell,\vartheta)|\partial^{\alpha_1}_{\beta_1}g_1(u)\partial^{\alpha_2}_{\beta_2}g_2(v)\partial^\alpha_\beta g_3(v)|dudvdx$$
$$\le C\int\left\{\int e^{-|u|^2/8}|\partial^{\alpha_1}_{\beta_1}g_1(u)|du\right\}\left\{\int [1+|v|]^\gamma w^2(\ell,\vartheta)|\partial^{\alpha_2}_{\beta_2}g_2(v)\partial^\alpha_\beta g_3(v)|dv\right\}dx$$
$$\le C\int |\partial^{\alpha_1}_{\beta_1}g_1|_{\boldsymbol{\nu},\ell}\left\{\int w^2|\partial^{\alpha_2}_{\beta_2}g_2|^2dv\right\}^{1/2}\left\{\int [1+|v|]^{2\gamma}w^2|\partial^\alpha_\beta g_3|^2dv\right\}^{1/2}dx.$$

Since $|\alpha_2|+|\beta_2|\le N/2$, $N\ge 8$ and $H^2(\mathbb{T}^3)\subset L^\infty(\mathbb{T}^3)$, we have

$$\sup_x\int w^2(\ell,\vartheta)(v)|\partial^{\alpha_2}_{\beta_2}g_2(x,v)|^2dv\le C\mathcal{E}_{l,\vartheta}(g_2). \tag{36}$$

Thus, by the Cauchy-Schwartz inequality, the Γ^0_{loss} term over $\{|v-u|\ge |v|/2,\ |v|\ge 1\}$ is bounded by $C||\partial^{\alpha_1}_{\beta_1}g_1||_{\boldsymbol{\nu},\ell}\mathcal{E}^{1/2}_{l,\vartheta}(g_2)||\partial^\alpha_\beta g_3||_{\boldsymbol{\nu},\ell,\vartheta}$.

Case (1c). The Loss Term in the Third Region: $\{|v-u|\ge |v|/2,\ |v|\ge 1\}$

For the last region, $\{|v-u|\ge |v|/2,\ |v|\le 1\}$, we have

$$|u-v|^\gamma\le C|v|^\gamma,\quad w(\ell,\vartheta)(v)\le C.$$

And then that the integral of $w^2(\ell,\vartheta)\Gamma^0_{\mathrm{loss}}[\partial^{\alpha_1}_{\beta_1}g_1,\partial^{\alpha_2}_{\beta_2}g_2]\partial^{\alpha}_{\beta}g_3$ over this region is bounded by

$$C\int_{\{|v-u|\geq|v|/2,\ |v|\leq 1\}}|u-v|^{\gamma}e^{-|u|^2/8}|\partial^{\alpha_1}_{\beta_1}g_1(u)\partial^{\alpha_2}_{\beta_2}g_2(v)\partial^{\alpha}_{\beta}g_3(v)|dudvdx$$

$$\leq C\int\left\{\int|u-v|^{\gamma/2}e^{-|u|^2/8}|\partial^{\alpha_1}_{\beta_1}g_1(u)|du\right\}\left\{\int_{|v|\leq 1}|v|^{\gamma/2}|\partial^{\alpha_2}_{\beta_2}g_2(v)\partial^{\alpha}_{\beta}g_3(v)|dv\right\}dx.$$

Using the Cauchy-Schwartz inequality a few times we have

$$\leq C\int\left\{\int|u-v|^{\gamma}e^{-|u|^2/8}du\right\}^{1/2}\left\{\int e^{-|u|^2/8}|\partial^{\alpha_1}_{\beta_1}g_1(u)|^2du\right\}^{1/2}$$
$$\times\left\{\int_{|v|\leq 1}|v|^{\gamma}|\partial^{\alpha_2}_{\beta_2}g_2|^2dv\right\}^{1/2}\left\{\int_{|v|\leq 1}|\partial^{\alpha}_{\beta}g_3|^2dv\right\}^{1/2}dx$$
$$\leq C\int|\partial^{\alpha_1}_{\beta_1}g_1|_{\boldsymbol{\nu},\ell}\left\{\int_{|v|\leq 1}|v|^{\gamma}|\partial^{\alpha_2}_{\beta_2}g_2|^2dv\right\}^{1/2}\left\{\int_{|v|\leq 1}|\partial^{\alpha}_{\beta}g_3|^2dv\right\}^{1/2}dx.$$

By $|\alpha_2|+|\beta_2|\leq N/2$, $\gamma>-3$ and $H^4(\mathbb{T}^3\times\mathbb{R}^3)\subset L^\infty$,

$$\int_{|v|\leq 1}|v|^{\gamma}|\partial^{\alpha_2}_{\beta_2}g_2|^2dv\leq C\sup_{|v|\leq 1,x\in\mathbb{T}^3}|\partial^{\alpha_2}_{\beta_2}g_2|^2\leq C\mathcal{E}_l(g_2).$$

Hence, by Cauchy-Schwartz, the last part is bounded by

$$C||\partial^{\alpha_1}_{\beta_1}g_1||_{\boldsymbol{\nu},\ell}\mathcal{E}_l^{1/2}(g_2)||\partial^{\alpha}_{\beta}g_3||_{\boldsymbol{\nu},\ell}.$$

This concludes the desired estimate for Γ^0_{loss}.

Case 2. **The Gain Term Estimate.**

The next step is to estimate the gain term Γ^0_{gain} in (33), for which the (u,v) integration domain is split into two parts:

$$\{|u|\geq|v|/2\}\ \cup\ \{|u|\leq|v|/2\}.$$

Case (2a) The Gain Term over $\{|u|\geq|v|/2\}$.

For the first region $\{|u|\geq|v|/2\}$,

$$e^{-|u|^2/8}\leq e^{-|u|^2/16}e^{-|v|^2/64}.$$

Then the integral of $w^2(\ell,\vartheta)\Gamma^0_{\mathrm{gain}}[\partial^{\alpha_1}_{\beta_1}g_1,\partial^{\alpha_2}_{\beta_2}g_2]\partial^{\alpha_3}_{\beta_3}g_3$ over $\{|u|\geq|v|/2\}$ is thus bounded by

$$\int_{|u|\geq|v|/2}|u-v|^{\gamma}e^{-|u|^2/16}e^{-|v|^2/64}w^2(\ell,\vartheta)(v)|\partial^{\alpha_1}_{\beta_1}g_1(u')\partial^{\alpha_2}_{\beta_2}g_2(v')\partial^{\alpha}_{\beta}g_3(v)|d\omega dudvdx$$
$$\leq C\left\{\int|u-v|^{\gamma}e^{-|u|^2/16}e^{-|v|^2/64}w^2|\partial^{\alpha_1}_{\beta_1}g_1(u')|^2|\partial^{\alpha_2}_{\beta_2}g_2(v')|^2d\omega dudvdx\right\}^{1/2}$$
$$\times\left\{\int|u-v|^{\gamma}e^{-|u|^2/16}e^{-|v|^2/64}w^2|\partial^{\alpha}_{\beta}g_3(v)|^2dudvdx\right\}^{1/2}$$
$$\leq C\left\{\int|u'-v'|^{\gamma}e^{-\frac{1}{64}(|u'|^2+|v'|^2)}w^2(v)|\partial^{\alpha_1}_{\beta_1}g_1(u')|^2|\partial^{\alpha_2}_{\beta_2}g_2(v')|^2\right\}^{1/2}||\partial^{\alpha}_{\beta}g_3||_{\boldsymbol{\nu},\ell,\vartheta}.$$

In the first factor we have used (4) and $|u'-v'| = |u-v|$. By (4),

$$e^{\frac{q}{4}(1+|v|^2)^{\vartheta/2}} \le e^{\frac{q}{4}(1+|v'|^2+|u'|^2)^{\vartheta/2}} \le e^{\frac{q}{4}(1+|v'|^2)^{\vartheta/2}} e^{\frac{q}{4}(1+|u'|^2)^{\vartheta/2}}. \tag{37}$$

Using this, (10) and (4) we have

$$e^{-\frac{1}{64}(|u'|^2+|v'|^2)} w^2(\ell,\vartheta)(v) \le e^{-\frac{1}{128}(|u'|^2+|v'|^2)} w^2(\ell,\vartheta)(v') w^2(\ell,\vartheta)(u').$$

So that the factor in braces is

$$\le C\int |u'-v'|^\gamma e^{-\frac{1}{128}(|u'|^2+|v'|^2)} w^2(u') |\partial^{\alpha_1}_{\beta_1} g_1(u')|^2 w^2(v') |\partial^{\alpha_2}_{\beta_2} g_2(v')|^2 d\omega du dv dx.$$

The change of variables $(u,v)\to(u',v')$ implies

$$= C\left\{\int |u-v|^\gamma e^{-\frac{1}{128}(|u|^2+|v|^2)} w^2(u) |\partial^{\alpha_1}_{\beta_1} g_1(u)|^2 w^2(v) |\partial^{\alpha_2}_{\beta_2} g_2(v)|^2 du dv dx\right\}^{1/2}.$$

Assume $|\alpha_1|+|\beta_1|\le N/2$. As in (35),

$$\sup_{x,u}\left\{w(\ell,\vartheta)(u) e^{-\frac{1}{256}|u|^2} |\partial^{\alpha_1}_{\beta_1} g_1(x,u)|\right\} \le C\mathcal{E}^{1/2}_{l,\vartheta}(g_1).$$

Integrate first over du to obtain

$$\begin{aligned}\int |u-v|^\gamma e^{-\frac{1}{128}|u|^2} w(\ell,\vartheta)(u) |\partial^{\alpha_1}_{\beta_1} g_1(u)|^2 du &\le C\mathcal{E}^{1/2}_{l,\vartheta}(g_1)\int |u-v|^\gamma e^{-\frac{1}{256}|u|^2} du\\ &\le C\mathcal{E}^{1/2}_{l,\vartheta}(g_1)[1+|v|]^\gamma.\end{aligned}$$

If $|\alpha_2|+|\beta_2|\le N/2$ use this last argument but switch $\partial^{\alpha_1}_{\beta_1} g_1$ with $\partial^{\alpha_2}_{\beta_2} g_2$. Then the bound for the gain term over $\{|u|\ge|v|/2\}$, if $|\alpha_1|+|\beta_1|\le N/2$, is

$$\int_{|u|\ge|v|/2} \le C\mathcal{E}^{1/2}_{l,\vartheta}(g_1)||\partial^{\alpha_2}_{\beta_2} g_2||_{\boldsymbol{\nu},\ell,\vartheta} ||\partial^{\alpha}_{\beta} g_3||_{\boldsymbol{\nu},\ell,\vartheta}.$$

And the bound for the gain term over $\{|u|\ge|v|/2\}$, if $|\alpha_2|+|\beta_2|\le N/2$, is

$$\int_{|u|\ge|v|/2} \le C||\partial^{\alpha_2}_{\beta_2} g_1||_{\boldsymbol{\nu},\ell,\vartheta}\mathcal{E}^{1/2}_{l,\vartheta}(g_2) ||\partial^{\alpha}_{\beta} g_3||_{\boldsymbol{\nu},\ell,\vartheta}.$$

This completes the estimate for the gain term over $\{|u|\ge|v|/2\}$.

Case (2b). The Gain Term over $\{|u|\le|v|/2\}$.

Now consider the gain term over $\{|u|\le|v|/2\}$. Since $|v-u|<|v|/2$ implies $|u|\ge|v|-|v-u|>|v|/2$, we obtain

$$\{|u|\le|v|/2\} = \{|u|\le|v|/2\}\cup\{|v-u|\ge|v|/2\}.$$

Further assume $|v| \le 1$, then $|u| \le 1/2$ and the gain term is bounded by

$$\int_{|v|\le 1, |u|\le |v|/2} |u-v|^\gamma e^{-|u|^2/8} w^2 |\partial^{\alpha_1}_{\beta_1} g_1(u') \partial^{\alpha_2}_{\beta_2} g_2(v') \partial^{\alpha}_{\beta} g_3(v)| d\omega du dv dx \tag{38}$$

$$\begin{aligned}
&\le C \int_{|v|\le 1} \left\{ |v|^{\gamma/2} \int_{|u|\le 1/2} |u-v|^{\gamma/2} e^{-|u|^2/8} |\partial^{\alpha_1}_{\beta_1} g_1(u') \partial^{\alpha_2}_{\beta_2} g_2(v')| d\omega du \right\} |\partial^{\alpha}_{\beta} g_3(v)| dv dx \\
&\le C \int_{|v|\le 1} \left\{ |v|^{\gamma} \int_{|u|\le 1/2} |\partial^{\alpha_1}_{\beta_1} g_1(u')|^2 |\partial^{\alpha_2}_{\beta_2} g_2(v')|^2 d\omega du \right\}^{1/2} \\
&\qquad \times \left\{ \int_{|u|\le 1/2} |u-v|^{\gamma} e^{-|u|^2/4} du \right\}^{1/2} |\partial^{\alpha}_{\beta} g_3(v)| dv dx \\
&\le C \int_{|v|\le 1} \left\{ |v|^{\gamma} \int_{|u|\le 1/2} |\partial^{\alpha_1}_{\beta_1} g_1(u')|^2 |\partial^{\alpha_2}_{\beta_2} g_2(v')|^2 d\omega du \right\}^{1/2} |\partial^{\alpha}_{\beta} g_3(v)| dv dx \\
&\le C \left\{ \int_{|v|\le 1, |u|\le 1/2} |v|^{\gamma} |\partial^{\alpha_1}_{\beta_1} g_1(u')|^2 |\partial^{\alpha_2}_{\beta_2} g_2(v')|^2 d\omega du dv dx \right\}^{1/2} \|\partial^{\alpha}_{\beta} g_3\|_{\boldsymbol{\nu},\ell}.
\end{aligned}$$

We now estimate the first factor. Since $|u| \le |v|/2$, from (3) we have

$$|u'| + |v'| \le 2[|u| + |v|] \le 3|v|.$$

Since $\gamma < 0$, this implies

$$|v|^\gamma \le 3^{-\gamma} |u'|^\gamma, \ \ |v|^\gamma \le 3^{-\gamma} |v'|^\gamma.$$

Thus,

$$\begin{aligned}
&\int_{|v|\le 1, |u|\le 1/2} |v|^{\gamma} |\partial^{\alpha_1}_{\beta_1} g_1(u')|^2 |\partial^{\alpha_2}_{\beta_2} g_2(v')|^2 d\omega du dv \\
&\le C \int_{|v'|\le 3, |u'|\le 3} \min[|v'|^\gamma, |u'|^\gamma] |\partial^{\alpha_1}_{\beta_1} g_1(u')|^2 |\partial^{\alpha_2}_{\beta_2} g_2(v')|^2 d\omega du dv dx.
\end{aligned}$$

Now change variables $(v,u) \to (v',u')$ so that the above is

$$C \int_{|v|\le 3, |u|\le 3} \min[|v|^\gamma, |u|^\gamma] |\partial^{\alpha_1}_{\beta_1} g_1(u)|^2 |\partial^{\alpha_2}_{\beta_2} g_2(v)|^2 d\omega du dv dx.$$

Assume $|\alpha_1| + |\beta_1| \le N/2$ and majorize the above by

$$\begin{aligned}
&C \int \left\{ \int_{|u|\le 3} |u|^\gamma |\partial^{\alpha_1}_{\beta_1} g_1(u)|^2 du \right\} \left\{ \int_{|v|\le 3} |\partial^{\alpha_2}_{\beta_2} g_2(v)|^2 dv \right\} dx \\
&\le C \sup_{x, |u|\le 3} |\partial^{\alpha_1}_{\beta_1} g_1(u)|^2 \|\partial^{\alpha_2}_{\beta_2} g_2\|^2_{\boldsymbol{\nu},\ell} \le C \mathcal{E}_l(g_1) \|\partial^{\alpha_2}_{\beta_2} g_2\|^2_{\boldsymbol{\nu},\ell}.
\end{aligned}$$

Alternatively, if $|\alpha_2| + |\beta_2| \le N/2$ then

$$\begin{aligned}
&C \int \left\{ \int_{|u|\le 3} |\partial^{\alpha_1}_{\beta_1} g_1(u)|^2 du \right\} \left\{ \int_{|v|\le 3} |v|^\gamma |\partial^{\alpha_2}_{\beta_2} g_2(v)|^2 dv \right\} dx \\
&\le C \|\partial^{\alpha_1}_{\beta_1} g_1\|^2_{\boldsymbol{\nu},\ell} \sup_{x, |v|\le 3} |\partial^{\alpha_2}_{\beta_2} g_2(v)|^2 \le C \|\partial^{\alpha_1}_{\beta_1} g_1\|^2_{\boldsymbol{\nu},\ell} \mathcal{E}_l(g_2).
\end{aligned}$$

Combine this upper bound with (38) to complete the estimate for the gain term over $\{|u| \le |v|/2, \ |v| \le 1\}$.

Case (2c) The Gain Term over $\{|u|\leq |v|/2,\ |v-u|\geq |v|/2, |v|\geq 1\}$.

The last case is the gain term over the region $\{|u|\leq |v|/2,\ |v-u|\geq |v|/2, |v|\geq 1\}$. The integral of $w^2(\ell,\vartheta)\Gamma^0_{\rm gain}[\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2]\partial^{\alpha}_{\beta} g_3$ over such a region is bounded by

$$
\begin{aligned}
&\int_{|u|\leq |v|/2,|v|\geq 1} |u-v|^{\gamma} e^{-|u|^2/4} w^2(\ell,\vartheta)|\partial^{\alpha_1}_{\beta_1} g_1(u')\partial^{\alpha_2}_{\beta_2} g_2(v')\partial^{\alpha}_{\beta} g_3(v)| d\omega du dv dx \\
&\leq C\int_{|u|\leq |v|/2,|v|\geq 1} [1+|v|]^{\gamma} e^{-|u|^2/4} w^2(\ell,\vartheta)|\partial^{\alpha_1}_{\beta_1} g_1(u')\partial^{\alpha_2}_{\beta_2} g_2(v')\partial^{\alpha}_{\beta} g_3(v)| d\omega du dv dx \\
&\qquad \leq C\left\{\int [1+|v|]^{\gamma} e^{-|u|^2/4} w^2(\ell,\vartheta)|\partial^{\alpha_1}_{\beta_1} g_1(u')|^2|\partial^{\alpha_2}_{\beta_2} g_2(v')|^2 d\omega du dv dx\right\}^{1/2} \\
&\qquad\qquad \times\left\{\int [1+|v|]^{\gamma} e^{-|u|^2/4} w^2(\ell,\vartheta)|\partial^{\alpha}_{\beta} g_3(v)|^2 d\omega du dv dx\right\}^{1/2} \qquad (39)\\
&\leq C\left\{\int [1+|v|]^{\gamma} w^2(\ell,\vartheta)(v)|\partial^{\alpha_1}_{\beta_1} g_1(u')|^2|\partial^{\alpha_2}_{\beta_2} g_2(v')|^2 d\omega du dv dx\right\}^{1/2} ||\partial^{\alpha}_{\beta} g_3||_{\boldsymbol{\nu},\ell,\vartheta}.
\end{aligned}
$$

We have used $|v-u|^{\gamma}\leq 4^{-\gamma}[1+|v|]^{\gamma}$ in the first inequality. If $\ell\tau<0$, use $|v|\geq 2|u|$ and (4) to establish

$$
(1+|v|^2)^{\ell\tau/2}\leq C(1+|v'|^2+|u'|^2)^{\ell\tau/2}.
$$

Conversely, if $\ell\tau\geq 0$, just use (4) to establish the same inequality. Recall (37) and $M(v)=\exp\left(\frac{q}{4}(1+|v|^2)^{\vartheta/2}\right)$. Thus,

$$
w^2(\ell,\vartheta)(v)\leq C(1+|v'|^2+|u'|^2)^{\ell\tau}M(v')M(u')
$$

Then since $\ell=|\beta|-l$ we have

$$
\begin{aligned}
w^2(\ell,\vartheta)(v) &\leq C(1+|v'|^2+|u'|^2)^{-l\tau}(1+|v'|^2+|u'|^2)^{|\beta|\tau}M(v')M(u') \\
&\leq C(1+|v'|^2)^{-l\tau}(1+|u'|^2)^{-l\tau}(1+|v'|^2+|u'|^2)^{|\beta|\tau}M(v')M(u').
\end{aligned}
$$

Assume $|\alpha_2|+|\beta_2|\leq N/2$. Using this estimate and the change of variable $(v,u)\to(v',u')$ we obtain

$$
\begin{aligned}
&\int [1+|v|]^{\gamma} w^2(\ell,\vartheta)(v)|\partial^{\alpha_1}_{\beta_1} g_1(u')|^2|\partial^{\alpha_2}_{\beta_2} g_2(v')|^2 d\omega du dv dx \\
&\leq C\int [1+|u'|]^{\gamma} w^2(|\beta_1|-l,\vartheta)(u')|\partial^{\alpha_1}_{\beta_1} g_1(u')|^2 w^2(|\beta_2|-l,\vartheta)(v')|\partial^{\alpha_2}_{\beta_2} g_2(v')|^2 \\
&= C\int [1+|u|]^{\gamma} w^2(|\beta_1|-l,\vartheta)(u)|\partial^{\alpha_1}_{\beta_1} g_1(u)|^2 w^2(|\beta_2|-l,\vartheta)(v)|\partial^{\alpha_2}_{\beta_2} g_2(v)|^2 du dv dx \\
&= C\int\left\{\int w^2(|\beta_2|-l,\vartheta)(v)|\partial^{\alpha_2}_{\beta_2} g_2(v)|^2 dv\right\} \\
&\qquad \times\left\{\int [1+|u|]^{\gamma} w^2(|\beta_1|-l,\vartheta)(u)|\partial^{\alpha_1}_{\beta_1} g_1(u)|^2 du\right\} dx.
\end{aligned}
$$

Using the embedding in (36), we see that this is bounded by

$$
\begin{aligned}
C\mathcal{E}_{l,\vartheta}(g_2)&\int [1+|u|]^{\gamma} w^2(|\beta_1|-l,\vartheta)(u)|\partial^{\alpha_1}_{\beta_1} g_1(u)|^2 du dx \\
&\leq C\mathcal{E}_{l,\vartheta}(g_2)||\partial^{\alpha_1}_{\beta_1} g_1||^2_{\boldsymbol{\nu},|\beta_1|-l,\vartheta}.
\end{aligned}
$$

Similarly if $|\alpha_1|+|\beta_1|\le N/2$ then this is bounded by

$$C||\partial^{\alpha_2}_{\beta_2} g_2||^2_{\boldsymbol{\nu},|\beta_2|-l,\vartheta}\mathcal{E}_{l,\vartheta}(g_1).$$

Combine this with (39) to complete the nonlinear estimate. □

This completes the estimates for the Boltzmann case. In Section 4 we use these to establish global existence. Then in Section 5 we prove the decay. In the next section we establish the analogous estimates for the Landau case.

3. Landau Estimates

In this section, we will prove the basic estimates used to obtain global existence of solutions with an exponential weight in the Landau case. In this case, the derivatives in the Landau operator cause extra difficulties in particular because $\partial_i w(\ell,\vartheta)$ can grow faster in v than $w(\ell,\vartheta)$. This new feature of the exponential weight (10) forces us to weaken the linear estimate with high order velocity derivatives (Lemma 8) from the analogous estimate [9, Lemma 6, p.403]. A new linear estimate with no extra derivatives (Lemma 9) is also necessary because of the exponential weight. It turns out that we need to dig up exact cancellation in order to prove this estimate in the $\vartheta=2$ case. We will again point out the new features of each proof after the statement of each lemma.

For any vector-valued function $\mathbf{g}(v)=(g_1,g_2,g_3)$, we define the projection to the vector v as

$$P_v g_i \equiv \frac{v_i}{|v|}\sum_{j=1}^3 \frac{v_j}{|v|}g_j. \tag{40}$$

Furthermore, in this section we will use the Einstein summation convention over i and j, e.g. repeated indices are always summed:

$$\sigma^i(v)=\sigma^{ij}(v)\frac{v_j}{2}=\sum_{j=1}^3\sigma^{ij}(v)\frac{v_j}{2}.$$

With this notation we have

Lemma 4. [3,9] *$\sigma^{ij}(v)$, $\sigma^i(v)$ are smooth functions such that*

$$|\partial_\beta\sigma^{ij}(v)|+|\partial_\beta\sigma^i(v)|\le C_\beta[1+|v|]^{\gamma+2-|\beta|},$$

and furthermore

$$\sigma^{ij}(v)=\lambda_1(v)\frac{v_iv_j}{|v|^2}+\lambda_2(v)\left(\delta_{ij}-\frac{v_iv_j}{|v|^2}\right). \tag{41}$$

Thus

$$\sigma^{ij}(v)g_ig_j=\lambda_1(v)\sum_{i=1}^3\{P_vg_i\}^2+\lambda_2(v)\sum_{i=1}^3\{[I-P_v]g_i\}^2. \tag{42}$$

Moreover, there are constants c_1 and $c_2>0$ such that as $|v|\to\infty$

$$\lambda_1(v)\sim c_1[1+|v|]^\gamma,\ \ \lambda_2(v)\sim c_2[1+|v|]^{\gamma+2}.$$

The estimate of σ^{ij} and σ^i with high derivatives was already established in [3]. The computation of the eigenvalues and their convergence rate was already shown in [9]. We prove the representation (41) below because we will use it in important places in later proofs and it is not formally written down in the other papers.

Proof. Recall from (9) that

$$\sigma^{ij}(v) = (2\pi)^{-3/2}\int_{\mathbb{R}^3}\left(\delta_{ij} - \frac{(v_i-u_i)(v_j-u_j)}{|v-u|^2}\right)|v-u|^{\gamma+2}e^{-|u|^2/2}du.$$

Changing variables $u \to v-u$ we have

$$\sigma^{ij}(v) = (2\pi)^{-3/2}\int_{\mathbb{R}^3}\left(\delta_{ij} - \frac{u_iu_j}{|u|^2}\right)|u|^{\gamma+2}e^{-|v-u|^2/2}du.$$

Given $v \in \mathbb{R}^3$ define $v^1 = v/|v|$ and complete an orthonormal basis $\{v^1, v^2, v^3\}$ where $v^i \cdot v^j = \delta_{ij}$. Then define the corresponding orthogonal 3×3 matrix as

$$\mathcal{O} = [v^1 \ v^2 \ v^3].$$

Applying this orthogonal transformation to the integral in σ^{ij} above we obtain

$$\sigma^{ij}(v) = (2\pi)^{-3/2}\int_{\mathbb{R}^3}\left(\delta_{ij} - \frac{(\mathcal{O}u)_i(\mathcal{O}u)_j}{|u|^2}\right)|u|^{\gamma+2}e^{-|v-\mathcal{O}u|^2/2}du.$$

Here we have used $|\mathcal{O}u| = |u|$. Also

$$(\mathcal{O}u)_i = u_1v_i^1 + u_2v_i^2 + u_3v_i^3.$$

And

$$|v-\mathcal{O}u|^2 = |v-u_1v^1-u_2v^2-u_3v^3|^2 = (|v|-u_1)^2+u_2^2+u_3^2. \tag{43}$$

We therefore have

$$\begin{aligned}\sigma^{ij}(v) &= (2\pi)^{-3/2}\int_{\mathbb{R}^3}\left(\delta_{ij} - \sum_{l,m=1}^3\frac{u_lu_m}{|u|^2}v_i^lv_j^m\right)|u|^{\gamma+2}e^{-\frac{1}{2}[(|v|-u_1)^2+u_2^2+u_3^2]}du\\ &= (2\pi)^{-3/2}\int_{\mathbb{R}^3}\left(\delta_{ij} - \sum_{m=1}^3\frac{u_m^2}{|u|^2}v_i^mv_j^m\right)|u|^{\gamma+2}e^{-\frac{1}{2}[(|v|-u_1)^2+u_2^2+u_3^2]}du,\end{aligned}$$

where we have used the odd function argument. Write $(m = 1,2,3)$

$$B_0(v) \equiv (2\pi)^{-3/2}\int_{\mathbb{R}^3}|u|^{\gamma+2}e^{-\frac{1}{2}[(|v|-u_1)^2+u_2^2+u_3^2]}du,$$
$$B_m(v) \equiv (2\pi)^{-3/2}\int_{\mathbb{R}^3}\frac{u_m^2}{|u|^2}|u|^{\gamma+2}e^{-\frac{1}{2}[(|v|-u_1)^2+u_2^2+u_3^2]}du.$$

Then by symmetry

$$B_2(v) = B_3(v) = (2\pi)^{-3/2}\int_{\mathbb{R}^3}\frac{u_2^2+u_3^2}{2|u|^2}|u|^{\gamma+2}e^{-\frac{1}{2}[(|v|-u_1)^2+u_2^2+u_3^2]}du,$$

and we have

$$\sigma^{ij}(v) = B_0(v)\delta_{ij} - B_1(v)v_j^1v_j^1 - B_2(v)(v_i^2v_j^2 + v_i^3v_j^3).$$

Define the orthogonal projections $P_j = v^j \otimes v^j$. Then we have the resolution of the identity $I = P_1+P_2+P_3$. In component form this is

$$v_i^2v_j^2 + v_i^3v_j^3 = \delta_{ij} - \frac{v_iv_j}{|v|^2}.$$

Then we have

$$\sigma^{ij}(v) = \{B_0(v)-B_1(v)\}\frac{v_iv_j}{|v|^2} + \{B_0(v)-B_2(v)\}\left(\delta_{ij} - \frac{v_iv_j}{|v|^2}\right).$$

Now the eigenvalues are $\lambda_1(v) = B_0(v) - B_1(v)$ and $\lambda_2(v) = B_0(v) - B_2(v)$ or

$$\lambda_1(v) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} |u|^{\gamma+2} \left(1 - \frac{u_1^2}{|u|^2}\right) e^{-\frac{1}{2}[(|v|-u_1)^2 + u_2^2 + u_3^2]} du,$$

and

$$\lambda_2(v) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} |u|^{\gamma+2} \left(1 - \frac{u_2^2 + u_3^2}{2|u|^2}\right) e^{-\frac{1}{2}[(|v|-u_1)^2 + u_2^2 + u_3^2]} du.$$

This completes the derivation of the spectral representation for $\sigma^{ij}(v)$. □

Next, we write bounds for the σ norm.

Lemma 5. [9, Corollary 1, p.399] *There exists $c > 0$ such that*

$$c|g|^2_{\boldsymbol{\sigma},\ell,\vartheta} \geq \left|[1+|v|]^{\frac{\gamma}{2}} \{P_v \partial_i g\}\right|^2_{\ell,\vartheta} + \left|[1+|v|]^{\frac{\gamma+2}{2}} \{[I - P_v]\partial_i g\}\right|^2_{\ell,\vartheta} + \left|[1+|v|]^{\frac{\gamma+2}{2}} g\right|^2_{\ell,\vartheta}.$$

Furthermore,

$$\frac{1}{c}|g|^2_{\boldsymbol{\sigma},\ell,\vartheta} \leq \left|[1+|v|]^{\frac{\gamma}{2}} \{P_v \partial_i g\}\right|^2_{\ell,\vartheta} + \left|[1+|v|]^{\frac{\gamma+2}{2}} \{[I - P_v]\partial_i g\}\right|^2_{\ell,\vartheta} + \left|[1+|v|]^{\frac{\gamma+2}{2}} g\right|^2_{\ell,\vartheta}.$$

The upper bound was not written down in [9], but the proof is the same. We write it down here because we will use it in the non-linear estimate. Next, the operators A, K and Γ from (5) and (8) in the Landau case are defined.

Lemma 6. [9, Lemma 1, p.395] *We have the following representations for A, K and Γ.*

$$Ag_2 = \partial_i[\sigma^{ij}\partial_j g_2] - \sigma^{ij}\frac{v_i}{2}\frac{v_j}{2} g_2 + \partial_i \sigma^i g_2 \tag{44}$$

$$\begin{aligned} Kg_1 &= -\mu^{-1/2}\partial_i\{\mu[\phi^{ij} * \{\mu^{1/2}[\partial_j g_1 + \frac{v_j}{2} g_1]\}]\} \\ &= -\mu^{-1/2}\partial_i\left\{\mu \int_{\mathbb{R}^3} \phi^{ij}(v - v')\mu^{1/2}(v')[\partial_j g_1(v') + \frac{v'_j}{2} g_1(v')]dv'\right\}, \end{aligned} \tag{45}$$

$$\begin{aligned} \Gamma[g_1, g_2] &= \partial_i[\{\phi^{ij} * [\mu^{1/2} g_1]\}\partial_j g_2] - \{\phi^{ij} * [\frac{v_i}{2}\mu^{1/2} g_1]\}\partial_j g_2 \\ &\quad -\partial_i[\{\phi^{ij} * [\mu^{1/2}\partial_j g_1]\} g_2] + \{\phi^{ij} * [\frac{v_i}{2}\mu^{1/2}\partial_j g_1]\} g_2. \end{aligned} \tag{46}$$

These representations are different in a few places by a factor of $\frac{1}{2}$ from those in [9]. The only reason for this difference is our use of a different normalization for the Maxwellian in this paper.

Proof. We only reprove A. First notice that for either fixed i or j

$$\sum_i \phi^{ij}(v)v_i = \sum_j \phi^{ij}(v)v_j = 0. \tag{47}$$

We now take the derivatives inside Ag_2

$$
\begin{aligned}
Ag_2 &= \mu^{-1/2}Q(\mu,\mu^{1/2}g_2)\\
&= \mu^{-1/2}\partial_i\{\sigma^{ij}\mu^{1/2}[\partial_j g_2 - \frac{v_j}{2}g_2]\} + \mu^{-1/2}\partial_i\{\{\phi^{ij}*[v_j\mu]\}\mu^{1/2}g_2\}\\
&= \mu^{-1/2}\partial_i\{\sigma^{ij}\mu^{1/2}[\partial_j g_2 - \frac{v_j}{2}g_2]\}\\
&\quad +\mu^{-1/2}\partial_i\{[\phi^{ij}*\mu]v_j\mu^{1/2}g_2\} \qquad\qquad \text{by (47)}\\
&= \mu^{-1/2}\partial_i\{\sigma^{ij}\mu^{1/2}[\partial_j g_2 + \frac{v_j}{2}g_2]\}\\
&= \mu^{-1/2}\partial_i\{\sigma^{ij}\mu^{1/2}\partial_j g_2\} + \mu^{-1/2}\partial_i\{\sigma^{ij}\mu^{1/2}\frac{v_j}{2}g_2\}\\
&= \partial_i[\sigma^{ij}\partial_j g_2] + \mu^{-1/2}\partial_i[\mu^{1/2}]\sigma^{ij}\partial_j g_2 + \partial_i\{\sigma^{ij}\frac{v_j}{2}g_2\} + \mu^{-1/2}\partial_i[\mu^{1/2}]\sigma^{ij}\frac{v_j}{2}g_2\\
&= \partial_i[\sigma^{ij}\partial_j g_2] - \sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g_2 + \mu^{-1/2}\partial_i[\mu^{1/2}]\sigma^{ij}\partial_j g_2 + \partial_i\{\sigma^{ij}\frac{v_j}{2}g_2\}\\
&= \partial_i[\sigma^{ij}\partial_j g_2] - \sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g_2 + \partial_i\{\sigma^{ij}\frac{v_j}{2}\}g_2,
\end{aligned}
$$

from (9), where $\partial_i[\mu^{1/2}] = \frac{v_i}{2}\mu^{1/2}$. □

In the rest of this section, we will prove estimates for A, K and Γ.

Lemma 7. *Let $0 \le \vartheta \le 2$, $\ell \in \mathbb{R}$ and $0 < q$. If $\vartheta = 2$ restrict $0 < q < 1$. Then for any $\eta > 0$, there is $0 < C = C(\eta) < \infty$ such that*

$$
|\langle w^2\partial_i\sigma^i g_1, g_2\rangle| + |\langle w^2 K g_1, g_2\rangle| \le \eta\,|g_1|_{\boldsymbol{\sigma},\ell,\vartheta}\,|g_2|_{\boldsymbol{\sigma},\ell,\vartheta} + C|g_1\bar{\chi}_C|_\ell|g_2\bar{\chi}_C|_\ell, \tag{48}
$$

where $w^2 = w^2(\ell,\vartheta)$ and $\bar{\chi}_{C(\eta)}$ is defined in (19).

The estimate for the $\partial_i\sigma^i$ term is exactly the same as in [9]. But, as in the Boltzmann case, the estimate for K needs modification because g_1 in Kg_1 does not depend on v. So we need to show that K can control one exponentially growing factor $w(\ell,\vartheta)(v)$. We remark that, although it is not used in this paper, the proof clearly shows $|\langle w^2 K g_1, g_2\rangle| \le \eta\,|g_1|_{\boldsymbol{\sigma},\ell}\,|g_2|_{\boldsymbol{\sigma},\ell,\vartheta} + C|g_1\bar{\chi}_C|_\ell|g_2\bar{\chi}_C|_\ell$.

Proof. For $m > 0$, we split

$$
\int w^2\partial_i\sigma^i g_1 g_2 = \int_{\{|v|\le m\}} + \int_{\{|v|\ge m\}}.
$$

By Lemma 4, $|\partial_i\sigma^i| \le C[1+|v|]^{\gamma+1}$. Thus, the integral over $\{|v| \le m\}$ is $\le C(m)|g_1\bar{\chi}_m|_\ell|g_2\bar{\chi}_m|_\ell$. From Lemma 5 and the Cauchy-Schwartz inequality

$$
\int_{\{|v|\ge m\}} w^2|\partial_i\sigma^i g_1 g_2|dv \le \frac{C}{m}\int w^2[1+|v|]^{\gamma+2}|g_1 g_2| \le \frac{C}{m}\,|g_1|_{\boldsymbol{\sigma},\ell,\vartheta}\,|g_2|_{\boldsymbol{\sigma},\ell,\vartheta}\,. \tag{49}
$$

This completes (48) for the $\partial_i\sigma^i$ term.

Recalling the linear operator K in (45), we have

$$
\begin{aligned}
w^2 K g_1 &= -\partial_i\{w^2\mu^{1/2}[\phi^{ij}*\{\mu^{1/2}\partial_j g_1 + \frac{v_j}{2}\mu^{1/2}g_1\}]\}\\
&\quad +\partial_i(w^2)\mu^{1/2}[\phi^{ij}*\{\mu^{1/2}\partial_j g_1 + \frac{v_j}{2}\mu^{1/2}g_1\}]\\
&\quad +w^2\frac{v_i}{2}\mu^{1/2}[\phi^{ij}*\{\mu^{1/2}\partial_j g_1 + \frac{v_j}{2}\mu^{1/2}g_1\}].
\end{aligned} \tag{50}
$$

The derivative of the weight function is

$$\partial_i(w^2(\ell,\vartheta)) = w^2(\ell,\vartheta)w_1(v)v_i. \tag{51}$$

where

$$w_1(v) = \left\{2\ell\tau(1+|v|^2)^{-1} + q\frac{\vartheta}{2}(1+|v|^2)^{\frac{\vartheta}{2}-1}\right\}. \tag{52}$$

After integrating by parts for the first term and collecting terms, we can rewrite $\langle w^2Kg_1, g_2\rangle$ as

$$\sum_{|\beta_1|,|\beta_2|\le 1}\int w^2(v)\phi^{ij}(v-v')\mu^{1/2}(v)\mu^{1/2}(v')\bar{\mu}_{\beta_1\beta_2}(v,v')\partial_{\beta_1}g_1(v')\partial_{\beta_2}g_2(v)dv'dv,$$

where $\bar{\mu}_{\beta_1\beta_2}(v,v')$ is a collection of smooth functions satisfying

$$|\nabla_v\bar{\mu}_{\beta_1\beta_2}(v,v')| + |\nabla_{v'}\bar{\mu}_{\beta_1\beta_2}(v,v')| + |\bar{\mu}_{\beta_1\beta_2}(v,v')| \le C(1+|v'|^2)^{1/2}(1+|v|^2)^{1/2}.$$

Since either $0\le\vartheta<2$ or $\vartheta=2$ and $0<q<1$, there exists $0<q'<1$ such that

$$w(\ell,\vartheta)(v)\mu^{1/2}(v) \le C\mu^{q'/2}(v) \tag{53}$$

If $0\le\vartheta<2$ choose any $0<q'<1$ and if $\vartheta=2$ choose $0<q'<1-q$. Therefore, we can rewrite $\langle w^2Kg_1, g_2\rangle$ as

$$\sum_{|\beta_1|,|\beta_2|\le 1}\int w(v)\phi^{ij}(v-v')\mu^{q'/4}(v)\mu^{1/4}(v')\mu_{\beta_1\beta_2}(v,v')\partial_{\beta_1}g_1(v')\partial_{\beta_2}g_2(v)dv'dv,$$

where $\mu_{\beta_1\beta_2}(v,v')$ is a different collection of smooth functions satisfying

$$|\nabla_v\mu_{\beta_1\beta_2}(v,v')| + |\nabla_{v'}\mu_{\beta_1\beta_2}(v,v')| + |\mu_{\beta_1\beta_2}(v,v')| \le Ce^{-\frac{q'}{16}|v|^2}e^{-\frac{1}{16}|v'|^2}.$$

We have removed an exponentially growing factor $w(\ell,\vartheta)(v)$.

Since $\phi^{ij}(v) = O(|v|^{\gamma+2}) \in L^2_{loc}(\mathbb{R}^3)$ and $\gamma\ge -3$, Fubini's Theorem implies

$$\phi^{ij}(v-v')\mu^{q'/4}(v)\mu^{1/4}(v') \in L^2(\mathbb{R}^3\times\mathbb{R}^3).$$

Therefore, for any given $m>0$, we can choose a C_c^∞ function $\psi^{ij}(v,v')$ such that

$$||\phi^{ij}(v-v')\mu^{q'/4}(v)\mu^{1/4}(v') - \psi^{ij}(v,v')||_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \le \frac{1}{m},$$
$$\mathrm{supp}\{\psi^{ij}\} \subset \{|v'|+|v|\le C(m)\} < \infty.$$

We split

$$\phi^{ij}(v-v')\mu^{q'/4}(v)\mu^{1/4}(v') = \psi^{ij} + [\phi^{ij}(v-v')\mu^{q'/4}(v)\mu^{1/4}(v') - \psi^{ij}].$$

Then

$$\langle w^2Kg_1, g_2\rangle = J_1[g_1,g_2] + J_2[g_1,g_2], \tag{54}$$

where

$$J_1 = \int w(v)\psi^{ij}(v,v')\mu_{\beta_1\beta_2}(v,v')\partial_{\beta_1}g_1(v')\partial_{\beta_2}g_2(v)dv'dv,$$

$$J_2 = \int w(v)[\phi^{ij}(v-v')\mu^{q'/4}(v)\mu^{1/4}(v') - \psi^{ij}]\mu_{\beta_1\beta_2}(v,v')\partial_{\beta_1}g_1(v')\partial_{\beta_2}g_2(v)dv'dv.$$

Above we are implicitly summing over $|\beta_1|, |\beta_2|\le 1$. We will bound each of these terms separately.

The J_2 term is bounded as

$$
\begin{aligned}
|J_2| &\le ||\phi^{ij}(v-v')\mu^{q'/4}(v)\mu^{1/4}(v') - \psi^{ij}(v,v')||_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \\
&\quad \times ||w(v)\mu_{\beta_1\beta_2}(v,v')\partial_{\beta_1}g_1(v')\partial_{\beta_2}g_2(v)||_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \\
&\le \frac{C}{m}\left|\mu^{1/16}\partial_{\beta_1}g_1\right|_0 \left|\mu^{q'/16}\partial_{\beta_2}g_2\right|_{\ell,\vartheta} \le \frac{C}{m}\left|g_1\right|_{\boldsymbol{\sigma},\ell}\left|g_2\right|_{\boldsymbol{\sigma},\ell,\vartheta}.
\end{aligned}
$$

Now for the first term J_1, integrations by parts over v and v' variables yields

$$
\begin{aligned}
|J_1| &= \left|(-1)^{\beta_1+\beta_2}\int \partial_{\beta_2}[w(v)\partial_{\beta_1}\{\psi^{ij}(v,v')\mu_{\beta_1\beta_2}(v,v')\}]g_1(v')g_2(v)\right| \\
&\le C||\psi^{ij}||_{C^2}\left\{\int_{|v|\le C(m)}|g_1|^2dv\right\}^{1/2}\left\{\int_{|v|\le C(m)}|g_2|^2dv\right\}^{1/2}.
\end{aligned}
$$

We thus conclude (48) by choosing $m>0$ large enough. □

Next, we estimate the linear terms with velocity derivatives.

Lemma 8. *Let $|\beta|>0$, $\ell\in\mathbb{R}$, $0\le\vartheta\le 2$ and $q>0$. If $\vartheta=2$ fix $0<q<1$. Then for small $\eta>0$, there exists $C(\eta)>0$ such that*

$$
|\langle w^2(\ell,\vartheta)\partial_\beta[Kg_1],g_2\rangle| \le \left\{\eta\sum_{|\bar\beta|\le|\beta|}\left|\partial_{\bar\beta}g_1\right|_{\boldsymbol{\sigma},\ell} + C(\eta)\left|\bar\chi_{C(\eta)}g_1\right|_\ell\right\}|g_2|_{\boldsymbol{\sigma},\ell,\vartheta}.
$$

Further if $\tau\le -1$ in (10) and $\ell = r-l$ where $l\ge 0$ and $r\ge|\beta|$, then

$$
-\langle w^2\partial_\beta[Ag],\partial_\beta g\rangle \ge |\partial_\beta g|^2_{\boldsymbol{\sigma},\ell,\vartheta} - \eta\sum_{|\bar\beta|=|\beta|}\left|\partial_{\bar\beta}g\right|^2_{\boldsymbol{\sigma},\ell,\vartheta} - C(\eta)\sum_{|\bar\beta|<|\beta|}\left|\partial_{\bar\beta}g\right|^2_{\boldsymbol{\sigma},|\bar\beta|-l,\vartheta},
$$

where $w^2=w^2(\ell,\vartheta)$.

Notice that the estimate involving $[Ag]$ is much weaker than the analogous estimate [9, Lemma 6, p.403] with no exponential weight. In [9], there are no derivatives in the last term on the right. The key problem here is that derivatives of the exponential weight, in particular $\partial_i(w^2(\ell,\vartheta))$, can grow faster than $w^2(\ell,\vartheta)$. Then, in some cases, we don't have enough decay to get the sharper estimate. Instead, we weaken the estimate and use lower order derivatives to extract polynomial decay from higher order weights. For the estimate involving $[Kg_1]$ the difference is the same as in the previous cases; we again show that K controls an exponentially growing factor of $w(\ell,\vartheta)(v)$. We remark that these estimates are not at all optimal. It is not hard to see that you can use a smaller norm over a compact region, in particular, for the terms with no derivatives in the $[Ag]$ estimate.

Proof. We begin with the estimate involving $\partial_\beta[Ag]$. Using Lemma 6, we have

$$
\begin{aligned}
\langle w^2\partial_\beta[Ag],\partial_\beta g\rangle &= -|\partial_\beta g|^2_{\boldsymbol{\sigma},\ell,\vartheta} - C_\beta^{\beta_1}\langle w^2\partial_{\beta_1}\sigma^{ij}\partial_{\beta-\beta_1}\partial_j g,\partial_\beta\partial_i g\rangle \\
&\quad -C_\beta^{\beta_2}\langle\partial_i(w^2)\partial_{\beta_2}\sigma^{ij}\partial_{\beta-\beta_2}\partial_j g,\partial_\beta g\rangle \\
&\quad -C_\beta^{\beta_1}\langle w^2\partial_{\beta_1}\{\sigma^{ij}v_iv_j\}\partial_{\beta-\beta_1}g,\partial_\beta g\rangle \\
&\quad +C_\beta^{\beta_2}\langle w^2\partial_{\beta_2}\partial_i\sigma^i\partial_{\beta-\beta_2}g,\partial_\beta g\rangle.
\end{aligned}
\tag{55}
$$

Here summations are over $\beta\ge\beta_1>0$ and $\beta\ge\beta_2\ge 0$. We will estimate each of these terms separately.

Case 1. The Last Two Terms.

First, we consider the last two terms in (55). We claim that

$$|w^2\partial_{\beta_2}\partial_i\sigma^i(v)| + |w^2\partial_{\beta_1}\{\sigma^{ij}v_iv_j\}| \le C[1+|v|]^{\gamma+1}w^2,$$

For the first term on the l.h.s., this follows from Lemma 4. For the estimate for the second term on the r.h.s., from (9) and (47) we have

$$\sigma^{ij}(v)v_iv_j = \int_{\mathbb{R}^3}\phi^{ij}(v-u)u_iu_j\mu(u)du.$$

Now the estimate follows from [9, Lemma 2,p.397] and $|\beta_1|>0$. Using the claim, the last two terms in (55) are bounded by

$$\begin{aligned}
&C\int w^2[1+|v|]^{\gamma+1}\{|\partial_{\beta-\beta_1}g|+|\partial_{\beta-\beta_2}g|\}|\partial_\beta g| = C\int_{|v|\le m} + C\int_{|v|\ge m}\\
&\le C\int_{|v|\le m} + \frac{C}{m}\left|[1+|v|]^{\frac{\gamma+2}{2}}\{|\partial_{\beta-\beta_1}g|+|\partial_{\beta-\beta_2}g|\}\right|_{\ell,\vartheta}\left|[1+|v|]^{\frac{\gamma+2}{2}}\partial_\beta g\right|_{\ell,\vartheta}\\
&\le C\int_{|v|\le m} + \frac{C}{m}\sum_{|\bar\beta|\le|\beta|}\left|\partial_{\bar\beta}g\right|_{\boldsymbol{\sigma},\ell,\vartheta}|\partial_\beta g|_{\boldsymbol{\sigma},\ell,\vartheta}. \qquad (56)
\end{aligned}$$

We have used Lemma 5 in the last step. For the part $|v|\le m$, for any $m'>0$, we use the compact Sobolev space interpolation and Lemma 5 to get

$$\begin{aligned}
\int_{|v|\le m} &\le \frac{1}{m'}\sum_{|\bar\beta|=|\beta|+1}\int_{|v|\le m}|\partial_{\bar\beta}g|^2 + C_{m'}\int_{|v|\le m}|g|^2\\
&\le \frac{C}{m'}\sum_{|\bar\beta|=|\beta|}\left|\partial_{\bar\beta}g\right|^2_{\boldsymbol{\sigma},\ell} + C_{m'}\left|\bar\chi_m g\right|^2_{\ell}. \qquad (57)
\end{aligned}$$

We used Lemma 5 again in the last step. This completes the estimate for the last two terms in (55).

Case 2. The Second Term.

Next, we consider the second term in (55). Since $|\beta_1|\ge 1$, we have

$$\begin{aligned}
&|\langle w^2\partial_{\beta_1}\sigma^{ij}\partial_{\beta-\beta_1}\partial_j g, \partial_\beta\partial_i g\rangle| \le C\int[1+|v|]^{\gamma+1}w^2|\partial_{\beta-\beta_1}\partial_j g\partial_\beta\partial_i g|\\
&\le C\,|\partial_\beta g|_{\boldsymbol{\sigma},\ell,\vartheta}\left\{\int[1+|v|]^{\gamma+2}w^2(\ell,\vartheta)|\partial_{\beta-\beta_1}\partial_j g|^2\right\}^{1/2}
\end{aligned}$$

Now using (61) with $\beta_1=\beta_2$, given $m'>0$ this is

$$\le C\,|\partial_\beta g|_{\boldsymbol{\sigma},\ell,\vartheta}\left\{\int[1+|v|]^{\gamma}w^2(|\beta-\beta_1|-l,\vartheta)|\partial_{\beta-\beta_1}\partial_j g|^2\right\}^{1/2} \qquad (58)$$

$$\le C\,|\partial_\beta g|_{\boldsymbol{\sigma},\ell,\vartheta}\sum_{|\bar\beta|\le|\beta|-1}\left|\partial_{\bar\beta}g\right|_{\boldsymbol{\sigma},|\bar\beta|-l,\vartheta} \le \frac{1}{m'}|\partial_\beta g|^2_{\boldsymbol{\sigma},\ell,\vartheta} + C_{m'}\sum_{|\bar\beta|\le|\beta|-1}\left|\partial_{\bar\beta}g\right|^2_{\boldsymbol{\sigma},|\bar\beta|-l,\vartheta}.$$

We have now estimated all the terms in (55). We conclude case 2 by first choosing m large enough.

Case 3. The Third Term.

Next consider the third and most delicate term in (55) when $|\beta_2|=0$. Recall $\partial_i(w^2)$ from (51); from (52) we have $|w_1(v)|\le C$ since $0\le\vartheta\le 2$. And from (41) we have

$$\sigma^{ij}(v)v_i = \lambda_1(v)v_j \tag{59}$$

Using Lemma 4 for the decay of $\lambda_1(v)$, the third term in (55) with $|\beta_2| = 0$ is

$$\begin{aligned}\left|\langle \partial_i(w^2)\sigma^{ij}\partial_\beta\partial_j g, \partial_\beta g\rangle\right| &\leq C\int w^2(\ell,\vartheta)[1+|v|]^{\gamma+1}\,|\partial_\beta\partial_j g|\,|\partial_\beta g|\,dv,\\ &= C\int \left(w(\ell,\vartheta)[1+|v|]^{\frac{\gamma}{2}}\,|\partial_\beta\partial_j g|\right)\left(w(\ell,\vartheta)[1+|v|]^{(\gamma+2)/2}\,|\partial_\beta g|\right)dv.\end{aligned}$$

Consider the second term in parenthesis. We will use the weight to extract extra polynomial decay and look at this as a term with lower order derivatives in the σ norm. Write $\partial_\beta = \partial_{\beta-e_k}\partial_k$ were e_k is an element of the standard basis. Further, from (10) with $\tau \leq -1$, write out

$$\begin{aligned}w(\ell,\vartheta) = (1+|v|^2)^{\tau\ell/2}\exp\left(\frac{q}{4}(1+|v|^2)^{\frac{\vartheta}{2}}\right) &= w(\ell-1,\vartheta)(1+|v|^2)^{\tau/2}\\ &\leq Cw(\ell-1,\vartheta)[1+|v|]^{-1}.\end{aligned}$$

Since $|e_k| = 1$ and $\ell = r - l$ with $r \geq |\beta|$, $\ell - 1 \geq |\beta| - 1 - l = |\beta - e_k| - l$. Thus,

$$w(\ell-1,\vartheta)[1+|v|]^{-1} \leq w(|\beta-e_k|-l,\vartheta)[1+|v|]^{-1}.$$

Hence,

$$w(\ell,\vartheta)[1+|v|]^{(\gamma+2)/2} \leq w(|\beta-e_k|-l,\vartheta)[1+|v|]^{\frac{\gamma}{2}}.$$

Then, for any large $m' > 0$, $\left|\langle \partial_i(w^2)\sigma^{ij}\partial_\beta\partial_j g, \partial_\beta g\rangle\right|$ is

$$\begin{aligned}&\leq C\int \left(w(\ell,\vartheta)[1+|v|]^{\frac{\gamma}{2}}\,|\partial_\beta\partial_j g|\right)\left(w(|\beta-e_k|-l,\vartheta)[1+|v|]^{\frac{\gamma}{2}}\,|\partial_{\beta-e_k}\partial_k g|\right)dv\\ &\qquad\leq C|\partial_\beta g|_{\boldsymbol{\sigma},\ell,\vartheta}|\partial_{\beta-e_k} g|_{\boldsymbol{\sigma},|\beta-e_k|-l,\vartheta}\\ &\qquad\leq \frac{1}{m'}|\partial_\beta g|^2_{\boldsymbol{\sigma},\ell,\vartheta} + C_{m'}\sum_{|\bar\beta|<|\beta|}|\partial_{\bar\beta} g|^2_{\boldsymbol{\sigma},|\bar\beta|-l,\vartheta}.\end{aligned}\tag{60}$$

This completes the estimate for the third term in (55) when $|\beta_2| = 0$.

Next consider the third term in (55) when $|\beta_2| > 0$. Since $|\beta_2| \geq 1$,

$$\left|\partial_i(w^2(\ell,\vartheta))\partial_{\beta_2}\sigma^{ij}\right| \leq Cw^2(\ell,\vartheta)[1+|v|]^{\gamma+2}$$

Notice that the order of $\partial_{\beta-\beta_2}\partial_j$ in this case is $< |\beta|$. Again we exploit the lower order derivative to gain some decay from the weight. Since $\tau \leq -1$, we split

$$\begin{aligned}w(\ell,\vartheta) = w(\ell-1+1,\vartheta) &= w(\ell-1,\vartheta)(1+|v|^2)^{\tau/2}\\ &\leq w(|\beta-\beta_2|-l,\vartheta)[1+|v|]^{-1}.\end{aligned}\tag{61}$$

In this last step we have used $\ell = r - l$, $r \geq |\beta|$ so that $r - 1 \geq |\beta - \beta_2|$ since $|\beta_2| \geq 1$. Given $m' > 0$, in this case, the third term in (55) has the upper bound

$$C \int \left|\partial_i (w^2(\ell,\vartheta))\partial_{\beta_2}\sigma^{ij}\right| \left|\partial_{\beta-\beta_2}\partial_j g \partial_\beta g\right| \leq C \int w^2 [1+|v|]^{\gamma+2} \left|\partial_{\beta-\beta_2}\partial_j g \partial_\beta g\right|$$

$$\leq C \left|\partial_\beta g\right|_{\boldsymbol{\sigma},\ell,\vartheta} \left\{ \int w^2(\ell,\vartheta)[1+|v|]^{\gamma+2} \left|\partial_{\beta-\beta_2}\partial_j g\right|^2 dv \right\}^{1/2} \tag{62}$$

$$\leq C \left|\partial_\beta g\right|_{\boldsymbol{\sigma},\ell,\vartheta} \left\{ \int w^2(|\beta-\beta_2| - l,\vartheta)[1+|v|]^{\gamma} \left|\partial_{\beta-\beta_2}\partial_j g\right|^2 dv \right\}^{1/2}$$

$$\leq C \left|\partial_\beta g\right|_{\boldsymbol{\sigma},\ell,\vartheta} \sum_{|\bar\beta| \leq |\beta|-1} \left|\partial_{\bar\beta} g\right|_{\boldsymbol{\sigma},|\bar\beta|-l,\vartheta} \leq \frac{1}{m'} \left|\partial_\beta g\right|^2_{\boldsymbol{\sigma},\ell,\vartheta} + C_{m'} \sum_{|\bar\beta| \leq |\beta|-1} \left|\partial_{\bar\beta} g\right|^2_{\boldsymbol{\sigma},|\bar\beta|-l,\vartheta}.$$

This completes the estimate for the third term in (55). By combining (56), (57), (60), (62) and (58) with m and m' chosen large enough we complete this estimate.

We now estimate $\langle w^2 \partial_\beta [K g_1], g_2\rangle$. Recalling (45), we have

$$\begin{aligned} w^2 \partial_\beta K g_1 \quad = \quad & -\partial_i [w^2 \partial_\beta \{\mu^{1/2}[\phi^{ij} * \{\mu^{1/2}\partial_j g_1 + \frac{v_j}{2}\mu^{1/2} g_1\}]\}] \\ & + \partial_i (w^2) \partial_\beta \{\mu^{1/2}[\phi^{ij} * \{\mu^{1/2}\partial_j g_1 + \frac{v_j}{2}\mu^{1/2} g_1\}]\} \\ & + w^2 \partial_\beta \{\frac{v_i}{2}\mu^{1/2}[\phi^{ij} * \{\mu^{1/2}\partial_j g_1 + \frac{v_j}{2}\mu^{1/2} g_1\}]\}. \end{aligned}$$

We take derivatives only on the factor $\{\mu^{1/2}\partial_j g_1 + v_j \mu^{1/2} g_1\}$ in the convolutions above. Upon integrating by parts for the first term, using (51) and (52) and collecting terms we can express $\langle w^2 \partial_\beta [K g_1], g_2\rangle$ as

$$\sum_{|\beta_1| \leq |\beta|+1, |\beta_2| \leq 1} \int w^2 \phi^{ij}(v-v')\mu^{1/2}(v)\mu^{1/2}(v')\bar\mu^{\beta_1\beta_2}(v,v')\partial_{\beta_1} g_1(v')\partial_{\beta_2} g_2(v) dv' dv,$$

where $\bar\mu^{\beta_1\beta_2}(v,v')$ is a collection of smooth functions which, for any $k-$th order derivatives, satisfies

$$|\nabla^k_{v,v'} \bar\mu^{\beta_1\beta_2}(v',v)| \leq C(1+|v|^2)^{|\beta|/2}(1+|v'|^2)^{|\beta|/2}.$$

Using the same argument as in (53) for the same $0 < q' < 1$ as in (53) we can rewrite $\langle w^2 \partial_\beta [K g_1], g_2\rangle$ as

$$\sum_{|\beta_1| \leq |\beta|+1, |\beta_2| \leq 1} \int w(v)\phi^{ij}(v-v')\mu^{q'/4}(v)\mu^{1/4}(v')\mu^{\beta_1\beta_2}(v,v')\partial_{\beta_1} g_1(v')\partial_{\beta_2} g_2(v) dv' dv,$$

where $\mu^{\beta_1\beta_2}(v,v')$ is a collection of smooth functions satisfying (for any $k-$th order derivatives)

$$|\nabla^k_{v,v'} \mu^{\beta_1\beta_2}(v',v)| \leq C e^{-\frac{q'}{16}|v|^2} e^{-\frac{1}{16}|v'|^2}.$$

We split $\langle w^2 \partial_\beta [K g_1], g_2\rangle$ as in (54) to get

$$\sum \int w(v)\psi^{ij}(v,v')\mu^{\beta_1\beta_2}(v,v')\partial_{\beta_1} g_1(v')\partial_{\beta_2} g_2(v) dv' dv$$

$$+ \sum \int w(v)\{\phi^{ij}(v-v')\mu^{q'/4}(v)\mu^{1/4}(v') - \psi^{ij}\}\mu^{\beta_1\beta_2}(v,v')\partial_{\beta_1} g_1(v')\partial_{\beta_2} g_2(v) dv' dv.$$

Using the estimates as for J_2 in (54) and Lemma 5 the last term is bounded by

$$\frac{C}{m}\sum_{|\bar\beta|\leq|\beta|}\left|\partial_{\bar\beta}g_1\right|_{\boldsymbol{\sigma},\ell}|g_2|_{\boldsymbol{\sigma},\ell,\vartheta}.$$

Since ψ^{ij} has compact support, integrating by parts over v' and v, the first term is equal to

$$\sum_{|\beta_1|\leq|\beta|+1,|\beta_2|\leq 1}(-1)^{|\beta_1|+|\beta_2|}\int\partial_{\beta_2}\{w(v)\partial_{\beta_1}[\psi^{ij}(v,v')\bar\mu_{\beta_1\beta_2}(v,v')]\}g_1(v')g_2(v)dv'dv.$$

Then, by Cauchy-Schwartz, this term is $\leq C(m)\left|\bar\chi_{C(m)}\mu g_1\right|_\ell\left|\bar\chi_{C(m)}g_2\right|_\ell$. And our lemma follows by first choosing $m>0$ large. □

Next, from Lemma 8 we get a general lower bound for L with high derivatives. We also prove a lower bound for L with no derivatives.

Lemma 9. *Let $0\leq\vartheta\leq 2$, $q>0$ and $l\geq 0$ with $|\beta|>0$ and $\ell=|\beta|-l$. If $\vartheta=2$ further restrict $0<q<1$. Then for $\eta>0$ small enough there exists $C(\eta)>0$ such that*

$$\langle w^2\partial_\beta[Lg],\partial_\beta g\rangle\geq|\partial_\beta g|^2_{\boldsymbol{\sigma},\ell,\vartheta}-\eta\sum_{|\beta_1|=|\beta|}|\partial_{\beta_1}g|^2_{\boldsymbol{\sigma},\ell,\vartheta}-C(\eta)\sum_{|\beta_1|<|\beta|}|\partial_{\beta_1}g|^2_{\boldsymbol{\sigma},|\beta_1|-l,\vartheta},$$

where $w^2=w^2(\ell,\vartheta)$. If $|\beta|=0$ we have

$$\langle w^2(\ell,\vartheta)[Lg],g\rangle\geq\delta_q|g|^2_{\boldsymbol{\sigma},\ell,\vartheta}-C(\eta)\left|\bar\chi_{C(\eta)}g\right|^2_\ell,$$

where $\delta_q=1-q^2-\eta>0$ for $\eta>0$ small enough.

It turns out that the lower bound for L with no extra v derivatives and an exponential weight needs a new approach. We need to use exact cancellation to make it work in the $\vartheta=2$ case.

Proof. By Lemma 8, we need only consider the case with $|\beta|=0$.

First assume $0\leq\vartheta<2$. In this case, after an integration by parts, (44) gives

$$\langle w^2Lg,g\rangle=|g|^2_{\boldsymbol{\sigma},\ell,\vartheta}+\langle\partial_i(w^2)\sigma^{ij}\partial_jg,g\rangle-\langle w^2\partial_i\sigma^ig,g\rangle-\langle w^2Kg,g\rangle.$$

By Lemma 7, the last two terms on the r.h.s. satisfy the $|\beta|=0$ estimate. Thus, we only consider $\langle\partial_i(w^2)\sigma^{ij}\partial_jg,g\rangle$. By (51) and (59) we can write

$$\partial_i(w^2(v))\sigma^{ij}(v)=w^2(v)w_1(v)\lambda_1(v)v_j.$$

From (52), $|w_1(v)|\leq C(1+|v|^2)^{\frac{\vartheta}{2}-1}$, and by Lemma 4, $|\lambda_1(v)v_j|\leq C[1+|v|]^{\gamma+1}$. Thus for any $m'>0$

$$\begin{aligned}\left|\langle\partial_i(w^2)\sigma^{ij}\partial_jg,g\rangle\right|&\leq C\int w^2(\ell,\vartheta)[1+|v|]^{\gamma+1+\frac{\vartheta}{2}-1}|\partial_jg|\,|g|\,dv\\&=C\int\left(w(\ell,\vartheta)[1+|v|]^{\frac{\gamma}{2}}|\partial_jg|\right)\left(w(\ell,\vartheta)[1+|v|]^{\frac{\gamma+2}{2}+\frac{\vartheta}{2}-1}|g|\right)\\&\leq C|g|_{\boldsymbol{\sigma},\ell,\vartheta}\left|[1+|v|]^{\frac{\gamma+2}{2}+\frac{\vartheta}{2}-1}g\right|_{\ell,\vartheta}\\&\leq\frac{1}{m'}|g|^2_{\boldsymbol{\sigma},\ell,\vartheta}+C(m')\left|[1+|v|]^{\frac{\gamma+2}{2}+\frac{\vartheta}{2}-1}g\right|^2_{\ell,\vartheta}.\end{aligned}$$

For $m>0$ further split

$$
\begin{aligned}
\left|[1+|v|]^{(\gamma+2)/2+\frac{\vartheta}{2}-1}g\right|^2_{\ell,\vartheta} &= \int_{|v|\le m} + \int_{|v|>m} \\
&\le \int_{|v|\le m} + Cm^{\vartheta-2}\int_{|v|>m} w^2[1+|v|]^{\gamma+2}|g|^2dv \\
&\le C(m)\int_{|v|\le m} w^2(\ell,0)|g|^2dv + Cm^{\vartheta-2}\,|g|^2_{\boldsymbol{\sigma},\ell,\vartheta}\,. \\
&\le C(m)|\bar{\chi}_m g|_\ell + Cm^{\vartheta-2}\,|g|^2_{\boldsymbol{\sigma},\ell,\vartheta}\,.
\end{aligned} \tag{63}
$$

We thus complete the estimate for $0\le\vartheta<2$ by choosing m and m' large.

Finally consider the case $\vartheta=2$ and $0<q<1$. We will prove this case in two steps. Split $L=-A-K$. Define $M(v)\equiv\exp\left(\frac{q}{4}\left(1+|v|^2\right)\right)$. First we will show that there is $\delta_q>0$ such that

$$
-\langle w^2(\ell,2)[Ag],g\rangle \ge \delta_q\,|Mg|^2_{\boldsymbol{\sigma},\ell,} - C(\delta_q)\left|\bar{\chi}_{C(\delta_q)}g\right|^2_\ell. \tag{64}
$$

Second we will establish

$$
|Mg|^2_{\boldsymbol{\sigma},\ell} \ge \delta_q|g|^2_{\boldsymbol{\sigma},\ell,2} - C(\delta_q)|g\bar{\chi}_{C(\delta_q)}|^2_\ell, \tag{65}
$$

where $\delta_q=1-q^2-\frac{\eta}{2}>0$ since $\eta>0$ can be chosen arbitrarily small. This will be enough to establish the case $\vartheta=2$ because the K part is controlled by Lemma 7.

We now establish (64). By (44), we obtain

$$
\begin{aligned}
-M[Ag] &= -M\partial_i\{\sigma^{ij}\partial_j g\} + M\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g - M\partial_i\sigma^i g \\
&= -\partial_i\{M\sigma^{ij}\partial_j g\} + q\sigma^{ij}\frac{v_i}{2}M\partial_j g + M\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g - M\partial_i\sigma^i g \\
&= -\partial_i\{\sigma^{ij}\partial_j[Mg]\} + q\partial_i\{\sigma^{ij}\frac{v_j}{2}Mg\} + q\sigma^{ij}\frac{v_i}{2}M\partial_j g \\
&\quad + M\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g - M\partial_i\sigma^i g \\
&= -\partial_i\{\sigma^{ij}\partial_j[Mg]\} + q\partial_i\{\sigma^i Mg\} + q\sigma^j\partial_j[Mg] \\
&\quad - q^2M\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g + M\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g - M\partial_i\sigma^i g.
\end{aligned}
$$

Notice that after an integration by parts

$$
\langle(1+|v|^2)^{\tau\ell}\{q\partial_i\{\sigma^iMg\}+q\sigma^j\partial_j[Mg]\},Mg\rangle = -q\langle\partial_i(1+|v|^2)^{\tau\ell}\sigma^iMg,Mg\rangle.
$$

Also, by (10),

$$
-\langle w^2(\ell,2)[Ag],g\rangle = -\langle(1+|v|^2)^{\tau\ell}M[Ag],Mg\rangle.
$$

We therefore have

$$
\begin{aligned}
-\langle w^2(\ell,2)[Ag],g\rangle &= \int (1+|v|^2)^{\tau\ell}\left\{\sigma^{ij}\partial_j[Mg]\partial_i[Mg]+(1-q^2)\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}[Mg]^2\right\}dv\\
&\quad -\int w^2(\ell,2)\partial_i\sigma^i g^2 dv\\
&\quad +\int \partial_i(1+|v|^2)^{\tau\ell}\left\{\sigma^{ij}\{\partial_j[Mg]\}[Mg]-q\sigma^i[Mg]^2\right\}dv\\
&\geq (1-q^2)|Mg|^2_{\sigma,\ell}-\int w^2(\ell,2)\partial_i\sigma^i g^2dv\\
&\quad +\int \partial_i(1+|v|^2)^{\tau\ell}\left\{\sigma^{ij}\{\partial_j[Mg]\}[Mg]-q\sigma^i[Mg]^2\right\}dv.
\end{aligned}
$$

By Lemma 4, $\left|\partial_i\sigma^i\right|\leq C[1+|v|]^{\gamma+1}$. Then as in (63), for $m>0$, we have

$$
\begin{aligned}
\int w^2(\ell,2)\left|\partial_i\sigma^i\right|g^2dv &= \int_{|v|\leq m}+\int_{|v|>m}\\
&\leq \int_{|v|\leq m}+\frac{C}{m}\int_{|v|>m}(1+|v|^2)^{\tau\ell}[1+|v|]^{\gamma+2}[Mg]^2dv\\
&\leq \int_{|v|\leq m}+\frac{C}{m}\left|Mg\right|^2_{\boldsymbol{\sigma},\ell}\\
&\leq C(m)\left|\bar{\chi}_m g\right|^2_\ell+\frac{\eta}{4}\left|Mg\right|^2_{\boldsymbol{\sigma},\ell},
\end{aligned}
\tag{66}
$$

where the last line follows from choosing $m>0$ large enough. We integrate by parts on the next term to obtain

$$
\int \partial_i(1+|v|^2)^{\tau\ell}\sigma^{ij}\{\partial_j[Mg]\}[Mg]dv=-\frac{1}{2}\int \partial_j\left\{\partial_i(1+|v|^2)^{\tau\ell}\sigma^{ij}\right\}[Mg]^2dv
$$

By Lemma 4 and (10),

$$
\left|\partial_i(1+|v|^2)^{\tau\ell}\sigma^i\right|+\left|\partial_j\left\{\partial_i(1+|v|^2)^{\tau\ell}\sigma^{ij}\right\}\right|\leq C(1+|v|^2)^{\tau\ell}[1+|v|]^{\gamma+1}.
$$

Thus the estimate for the final term follows from the same argument as (66). This establishes (64).

We finally establish (65). Notice that

$$
|Mg|^2_{\boldsymbol{\sigma},\ell}=\int(1+|v|^2)^{\tau\ell}\left\{\sigma^{ij}\partial_i[Mg]\partial_j[Mg]+\sigma^{ij}v_iv_j[gM]^2\right\}dv.
$$

We expand the first term in $|Mg|^2_{\boldsymbol{\sigma},\ell}$ to obtain

$$
\begin{aligned}
\int(1+|v|^2)^{\tau\ell}\sigma^{ij}\partial_i[Mg]\partial_j[Mg]dv &= \int(1+|v|^2)^{\tau\ell}M^2\sigma^{ij}\partial_i g\partial_j g dv\\
&\quad + q^2\int(1+|v|^2)^{\tau\ell}M^2\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g^2dv\\
&\quad + 2q\int(1+|v|^2)^{\tau\ell}M^2\sigma^{ij}\frac{v_i}{2}\{\partial_j g\}gdv\\
&= \int w^2(\ell,2)\left\{\sigma^{ij}\partial_i g\partial_j g+q^2\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g^2\right\}dv\\
&\quad +2q\int(1+|v|^2)^{\tau\ell}M^2\sigma^j\{\partial_j g\}gdv.
\end{aligned}
\tag{67}
$$

In the last step we used (10). We integrate by parts on the last term to obtain

$$\begin{aligned}2q\int(1+|v|^2)^{\tau\ell}M^2\sigma^j\{\partial_j g\}g dv &= q\int(1+|v|^2)^{\tau\ell}M^2\sigma^j\{\partial_j g^2\}dv\\ &= -q\int(1+|v|^2)^{\tau\ell}M^2\partial_j\sigma^j g^2 dv\\ &\quad - q\int\partial_j(1+|v|^2)^{\tau\ell}M^2\sigma^j g^2 dv\\ &\quad - 2q^2\int(1+|v|^2)^{\tau\ell}M^2\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g^2 dv.\end{aligned}$$

Since $\partial_j(1+|v|^2)^{\tau\ell} = (1+|v|^2)^{\tau\ell}\left\{2\tau\ell(1+|v|^2)^{-1}v_j\right\}$, we define the error as

$$\bar{w}(v) = q\left\{\partial_j\sigma^j + 2\tau\ell(1+|v|^2)^{-1}\sigma^j v_j\right\}.$$

Then (67) is

$$= \int w^2(\ell,2)\left\{\sigma^{ij}\partial_i g\partial_j g - q^2\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g^2\right\}dv - \int w^2(\ell,2)\bar{w}g^2 dv.$$

Adding the second term in $|Mg|^2_{\boldsymbol{\sigma},\ell}$ to both sides of the last display yields

$$|Mg|^2_{\boldsymbol{\sigma},\ell} \geq (1-q^2)|g|^2_{\boldsymbol{\sigma},\ell,2} - \int w^2(\ell,2)\bar{w}g^2 dv.$$

By Lemma 4 we have

$$|\bar{w}(v)| \leq C[1+|v|]^{\gamma+1}.$$

Thus, using (66), for any small $\eta > 0$ we have

$$\int w^2(\ell,2)|\bar{w}|g^2 dv \leq C(m)|g\bar{\chi}_m|^2_\ell + \frac{\eta}{2}|g|^2_{\boldsymbol{\sigma},\ell,2}.$$

This completes the estimate (65) and the proof. □

We thus conclude our estimates for the linear terms and finish the section by estimating the nonlinear term.

Lemma 10. *Let $|\alpha|+|\beta| \leq N$, $0 \leq \vartheta \leq 2$, $q > 0$ and $l \geq 0$ with $\ell = |\beta| - l$. If $\vartheta = 2$ restrict $0 < q < 1$. Then*

$$(68)\qquad \begin{aligned}&\langle w^2(\ell,\vartheta)\partial^\alpha_\beta\Gamma[g_1,g_2],\ \partial^\alpha_\beta g_3\rangle\\ &\leq C\sum\left\{|\partial^{\alpha_1}_{\bar{\beta}}g_1|_\ell|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\boldsymbol{\sigma},\ell,\vartheta} + |\partial^{\alpha_1}_{\bar{\beta}}g_1|_{\boldsymbol{\sigma},\ell}|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\ell,\vartheta}\right\}|\partial^\alpha_\beta g_3|_{\boldsymbol{\sigma},\ell,\vartheta},\end{aligned}$$

where summation is over $|\alpha_1|+|\beta_1| \leq N$, $\bar{\beta} \leq \beta_1 \leq \beta$.

Furthermore,

$$(69)\qquad \begin{aligned}&\left(w^2(\ell,\vartheta)\partial^\alpha_\beta\Gamma[g_1,g_2],\ \partial^\alpha_\beta g_3\right)\\ &\leq C\left\{\mathcal{E}^{1/2}_l(g_1)\mathcal{D}^{1/2}_{l,\vartheta}(g_2) + \mathcal{D}^{1/2}_l(g_1)\mathcal{E}^{1/2}_{l,\vartheta}(g_2)\right\}\|\partial^\alpha_\beta g_3\|_{\boldsymbol{\sigma},\ell,\vartheta}.\end{aligned}$$

The proof of Lemma 10 is more or less the same as in [9] save a few details. The differences mainly come from taking derivatives of the exponential weight $w(\ell,\vartheta)(v)$ which creates extra polynomial growth.

Proof. Recall $\Gamma[g_1,g_2]$ in (46). By the product rule, we expand

$$\langle w^2\partial^\alpha_\beta\Gamma[g_1,g_2],\partial^\alpha_\beta g_2\rangle=\sum C^{\alpha_1}_\alpha C^{\beta_1}_\beta\times G_{\alpha_1\beta_1},$$

where $G_{\alpha_1\beta_1}$ takes the form:

$$-\langle w^2\{\phi^{ij}*\partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1}g_1]\}\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2,\partial_i\partial^\alpha_\beta g_3\rangle \tag{70}$$

$$-\langle w^2\{\phi^{ij}*\partial_{\beta_1}[\frac{v_i}{2}\mu^{1/2}\partial^{\alpha_1}g_1]\}\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2,\partial^\alpha_\beta g_3\rangle \tag{71}$$

$$+\langle w^2\ \{\phi^{ij}*\partial_{\beta_1}[\mu^{1/2}\partial_j\partial^{\alpha_1}g_1]\}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2,\partial_i\partial^\alpha_\beta g_3\rangle \tag{72}$$

$$+\langle w^2\{\phi^{ij}*\partial_{\beta_1}[\frac{v_i}{2}\mu^{1/2}\partial_j\partial^{\alpha_1}g_1]\}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2,\partial^\alpha_\beta g_3\rangle \tag{73}$$

$$-\langle\partial_i[w^2]\{\phi^{ij}*\partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1}g_1]\}\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2,\partial^\alpha_\beta g_3\rangle \tag{74}$$

$$+\langle\partial_i[w^2]\{\phi^{ij}*\partial_{\beta_1}[\mu^{1/2}\partial_j\partial^{\alpha_1}g_1]\}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2,\partial^\alpha_\beta g_3\rangle. \tag{75}$$

The last two terms appear when we integrate by parts over the v_i variable.

We estimate the last term (75) first. Recall from (51) and (52) that $\partial_i[w^2]=w^2(v)w_1(v)v_i$ where $|w_1(v)|\le C$. By first summing over i and using (47) we can rewrite (75) as

$$\langle[w^2w_1]\{\phi^{ij}*v_i\partial_{\beta_1}[\mu^{1/2}\partial_j\partial^{\alpha_1}g_1]\}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2,\partial^\alpha_\beta g_3\rangle.$$

Since $-1\le\gamma+2<0$, $\phi^{ij}(v)\in L^2_{loc}(\mathbb{R}^3)$ and $|\partial_{\beta_1}\{\mu^{1/2}\}|\le C\mu^{1/4}$, we deduce by the Cauchy-Schwartz inequality and Lemma 5 that

$$\begin{aligned}\{\phi^{ij}*v_i\partial_{\beta_1}[\mu^{1/2}\partial_j\partial^{\alpha_1}g_1]\}\quad&\le\quad[|\phi^{ij}|^2*\mu^{1/8}]^{1/2}(v)\sum_{\bar\beta\le\beta_1}|\mu^{1/32}\partial_j\partial^{\alpha_1}_{\bar\beta}g_1|_\ell.\\&\le\quad C[1+|v|]^{\gamma+2}\sum_{\bar\beta\le\beta_1}\left|\partial^{\alpha_1}_{\bar\beta}g_1\right|_{\boldsymbol{\sigma},\ell},\end{aligned}\tag{76}$$

where we have used Lemma 2 in [9] to argue that

$$[|\phi^{ij}|^2*\mu^{1/8}]^{1/2}(v)\le C[1+|v|]^{\gamma+2}.$$

Using the above, (75) is bounded by Lemma 5 as

$$\begin{aligned}C\sum_{\bar\beta\le\beta_1}&\left|\partial^{\alpha_1}_{\bar\beta}g_1\right|_{\boldsymbol{\sigma},\ell}\int w^2(\ell,\vartheta)[1+|v|]^{\gamma+2}|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial^\alpha_\beta g_3|dv\\&\le C\sum_{\bar\beta\le\beta_1}|\partial^{\alpha_1}_{\bar\beta}g_1|_{\boldsymbol{\sigma},\ell}|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\ell,\vartheta}|[1+|v|]^{\gamma+2}\partial^\alpha_\beta g_3|_{\ell,\vartheta}\\&\le C\sum_{\bar\beta\le\beta_1}|\partial^{\alpha_1}_{\bar\beta}g_1|_{\boldsymbol{\sigma},\ell}|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\ell,\vartheta}|\partial^\alpha_\beta g_3|_{\boldsymbol{\sigma},\ell,\vartheta}.\end{aligned}$$

This completest the estimate for (75).

We now estimate (70)-(74). We decompose their double integration region $[v,v']\in\mathbb{R}^3\times\mathbb{R}^3$ into three parts:

$$\{|v|\le1\},\quad\{2|v'|\ge|v|,\ |v|\ge1\}\quad\text{and}\quad\{2|v'|\le|v|,\ |v|\ge1\}.$$

Case 1. Terms (70)-(74) *over* $\{|v|\le1\}$.

For the first part $\{|v|\leq 1\}$, recall $\phi^{ij}(v)=O(|v|^{\gamma+2})\in L^2_{loc}$. As in (76), we have

$$|\phi^{ij}*\partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1}g_1]|+|\phi^{ij}*\partial_{\beta_1}[v_i\mu^{1/2}\partial^{\alpha_1}g_1]|+|\phi^{ij}*v_i\partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1}g_1]|$$
$$\leq C[1+|v|]^{\gamma+2}\sum_{\bar\beta\leq\beta}|\partial^{\alpha_1}_{\bar\beta}g_1|_\ell,$$
$$|\phi^{ij}*\partial_{\beta_1}[\mu^{1/2}\partial_j\partial^{\alpha_1}g_1]|+|\phi^{ij}*\partial_{\beta_1}[\{v_i\mu^{1/2}\}\partial_j\partial^{\alpha_1}g_1]|$$
$$\leq C[1+|v|]^{\gamma+2}\sum_{\bar\beta\leq\beta}|\partial^{\alpha_1}_{\bar\beta}g_1|_{\boldsymbol{\sigma},\ell}.$$

Hence their corresponding integrands over the region $\{|v|\leq 1\}$ are bounded by

$$C\sum\left|\partial^{\alpha_1}_{\bar\beta}g_1\right|_\ell[1+|v|]^{\gamma+2}|\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|\left[|\partial_i\partial^\alpha_\beta g_3|+|\partial^\alpha_\beta g_3|\right]$$
$$+C\sum\left|\partial^{\alpha_1}_{\bar\beta}g_1\right|_{\boldsymbol{\sigma},\ell}[1+|v|]^{\gamma+2}|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|\left[|\partial_i\partial^\alpha_\beta g_3|+|\partial^\alpha_\beta g_3|\right],$$

whose $v-$integral over $\{|v|\leq 1\}$ is clearly bounded by right hand side of (68). We thus conclude the first part of $\{|v|\leq 1\}$ for (70)-(74).

Case 2. Terms (70)-(74) *over* $\{2|v'|\geq|v|,|v|\geq 1\}$.

For the second part $\{2|v'|\geq|v|,|v|\geq 1\}$, we have

$$|\partial_{\beta_1}\mu^{1/2}(v')|+|\partial_{\beta_1}\{v'_j\mu^{1/2}(v')\}|\leq C\mu^{1/8}(v')\mu^{1/32}(v).$$

Thus, by the same type of estimates as in (76), using the region, we have

$$|\phi^{ij}*\partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1}g_1]|+|\phi^{ij}*\partial_{\beta_1}[v_i\mu^{1/2}\partial^{\alpha_1}g_1]|+|\phi^{ij}*v_i\partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1}g_1]|$$
$$\leq C[1+|v|]^{\gamma+2}\mu^{1/64}(v)\sum_{\bar\beta\leq\beta}|\partial^{\alpha_1}_{\bar\beta}g_1|_\ell,$$
$$|\phi^{ij}*\partial_{\beta_1}[\mu^{1/2}\partial_j\partial^{\alpha_1}g_1]|+|\phi^{ij}*\partial_{\beta_1}[\{v_i\mu^{1/2}\}\partial_j\partial^{\alpha_1}g_1]|$$
$$\leq C[1+|v|]^{\gamma+2}\mu^{1/64}(v)\sum_{\bar\beta\leq\beta}|\partial^{\alpha_1}_{\bar\beta}g_1|_{\boldsymbol{\sigma},\ell}.$$

And then the $v-$ integrands in (70) to (74) over this region are bounded by

$$C\sum\left|\partial^{\alpha_1}_{\bar\beta}g_1\right|_\ell w^2(\ell,\vartheta)[1+|v|]^{\gamma+2}\mu^{1/64}(v)|\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|\left[|\partial_i\partial^\alpha_\beta g_3|+|\partial^\alpha_\beta g_3|\right]$$
$$+C\sum\left|\partial^{\alpha_1}_{\bar\beta}g_1\right|_{\boldsymbol{\sigma},\ell}w^2(\ell,\vartheta)[1+|v|]^{\gamma+2}\mu^{1/64}(v)|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|\left[|\partial_i\partial^\alpha_\beta g_3|+|\partial^\alpha_\beta g_3|\right],$$

By Lemma 5, its $v-$integral is bounded by the right hand side of (68) because of the fast decaying factor $\mu^{1/64}(v)$. We thus conclude the estimate for the second region $\{2|v'|\geq|v|,|v|\geq 1\ \}$ for (70) to (74).

Case 3. Terms (70)-(74) *over* $\{2|v'|\leq|v|,|v|\geq 1\}$.

We finally consider the third part of $\{2|v'|\leq|v|,|v|\geq 1\}$, for which we shall estimate each term from (70) to (74). The key is to taylor expand $\phi^{ij}(v-v')$. To estimate (70) over the this region we expand $\phi^{ij}(v-v')$ to get

$$\phi^{ij}(v-v')=\phi^{ij}(v)-\sum_k\partial_k\phi^{ij}(v)v'_k+\frac{1}{2}\sum_{k,l}\partial_{kl}\phi^{ij}(\bar v)v'_kv'_l. \tag{77}$$

where $\bar v$ is between v and $v-v'$. We plug (77) into the integrand of (70). From (40), (41) and (47), we can decompose $\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2$ and $\partial_i\partial^\alpha_\beta g_3$ into their P_v parts as well as $I-P_v$ parts. For the first term in the expansion (77) we have

$$\sum_{ij}\phi^{ij}(v)\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial_i\partial^{\alpha}_{\beta}g_3(v)$$
$$=\sum_{ij}\phi^{ij}(v)\{[I-P_v]\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\}\{[I-P_v]\partial_i\partial^{\alpha}_{\beta}g_3(v)\}.$$

Here we have used (47) so that sum of terms with either $P_v\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2$ or $P_v\partial_i\partial^{\alpha}_{\beta}g_3$ vanishes. The absolute value of this is bounded by

$$C[1+|v|]^{\gamma+2}|[I-P_v]\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)|\times|[I-P_v]\partial_i\partial^{\alpha}_{\beta}g_3(v)|. \tag{78}$$

For the second term in the expansion (77), by taking a k derivative of

$$\sum_{i,j}\phi^{ij}(v)v_iv_j=0$$

we have

$$\sum_{i,j}\partial_k\phi^{ij}(v)v_iv_j=-2\sum_j\phi^{kj}(v)v_j=0.$$

Therefore

$$\sum_{i,j}\partial_k\phi^{ij}(v)P_v\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2P_v\partial_i\partial^{\alpha}_{\beta}g_3=0.$$

Splitting $\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2$ and $\partial_i\partial^{\alpha}_{\beta}g_3$ into their P_v and $I-P_v$ parts yields

$$\sum_{i,j}\partial_k\phi^{ij}(v)\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial_i\partial^{\alpha}_{\beta}g_3(v)$$
$$=\sum_{i,j}\partial_k\phi^{ij}(v)\{[P_v\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2][I-P_v]\partial_i\partial^{\alpha}_{\beta}g_3+[I-P_v]\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2[P_v\partial_i\partial^{\alpha}_{\beta}g_3]\}$$
$$+\sum_{i,j}\partial_k\phi^{ij}(v)[I-P_v]\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2[I-P_v]\partial_i\partial^{\alpha}_{\beta}g_3.$$

Notice that $|\partial_k\phi^{ij}(v)|\le C[1+|v|]^{\gamma+1}$ for $|v|\ge 1$, we therefore majorize the above by

$$\begin{aligned}&C[1+|v|]^{\gamma/2}\{|P_v\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|+|P_v\partial_i\partial^{\alpha}_{\beta}g_3|\}\\ &\times[1+|v|]^{(\gamma+2)/2}\{|[I-P_v]\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|+|[I-P_v]\partial_i\partial^{\alpha}_{\beta}g_3|\}\\ &+C[1+|v|]^{\gamma+1}|[I-P_v]\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2||[I-P_v]\partial_i\partial^{\alpha}_{\beta}g_3|.\end{aligned} \tag{79}$$

Next, we estimate third term in (77). Using the region we have

$$\frac{1}{2}|v|\le|v|-|v'|\le|\bar{v}|\le|v'|+|v|\le\frac{3}{2}|v|. \tag{80}$$

Thus

$$|\partial_{kl}\phi^{ij}(\bar{v})|\le C[1+|v|]^{\gamma},$$

and

$$\left|\partial_{kl}\phi^{ij}(\bar{v})\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial_i\partial^{\alpha}_{\beta}g_3(v)\right|\le C[1+|v|]^{\gamma}|\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial_i\partial^{\alpha}_{\beta}g_3|. \tag{81}$$

Combining (77), (78), (79) and (81) we obtain the estimate

$$
\begin{aligned}
&\left|\sum_{i,j}\phi^{ij}(v-v')\partial_i\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial_i\partial^{\alpha}_{\beta}g_3\right|\\
&\leq C[1+|v'|]^2\left|\sum_{i,j}\phi^{ij}(v)\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial_i\partial^{\alpha}_{\beta}g_3(v)\right|\\
&+C[1+|v'|]^2\left|\sum_{i,j}\partial_k\phi^{ij}(v)\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial_i\partial^{\alpha}_{\beta}g_3(v)\right|\\
&+C[1+|v'|]^2\sum_{i,j}\left|\partial_{kl}\phi^{ij}(\bar v)\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial_i\partial^{\alpha}_{\beta}g_3(v)\right|\\
&\leq C[1+|v'|]^2\{\sigma^{ij}\partial_i\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\}^{1/2}\{\sigma^{ij}\partial_i\partial^{\alpha}_{\beta}g_3\partial_j\partial^{\alpha}_{\beta}g_3\}^{1/2},
\end{aligned}
$$

where we have used (42) in the last line. The v integrand over $\{2|v'|\leq|v|,\,|v|\geq 1\}$ in (70) is thus bounded by

$$
\begin{aligned}
&w^2\int[1+|v'|]^2\mu^{1/4}(v')|\partial^{\alpha_1}_{\bar\beta}g_1(v')|dv'\\
&\times\{\sigma^{ij}\partial_i\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\}^{1/2}\{\sigma^{ij}\partial_i\partial^{\alpha}_{\beta}g_3\partial_j\partial^{\alpha}_{\beta}g_3\}^{1/2}\\
&\leq C|\partial^{\alpha}_{\bar\beta}g_1|_{\ell}\{w^2\sigma^{ij}\partial_i\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\}^{1/2}\{w^2\sigma^{ij}\partial_i\partial^{\alpha}_{\beta}g_3\partial_j\partial^{\alpha}_{\beta}g_3\}^{1/2}.
\end{aligned}
$$

Its further integration over v is bounded by the right hand side of (68).

We now consider the second term (71). We again expand $\phi^{ij}(v-v')$ as

$$
\phi^{ij}(v-v')=\phi^{ij}(v)-\sum_k\partial_k\phi^{ij}(\bar v)v'_k, \tag{82}
$$

with $\bar v$ between v and $v-v'$. Since $\sum_j\phi^{ij}(v)v_j=0$ we obtain as before

$$
\begin{aligned}
&\sum_j\phi^{ij}(v)\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial^{\alpha}_{\beta}g_3(v)=\sum_j\phi^{ij}(v)\{I-P_v\}\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\times\partial^{\alpha}_{\beta}g_3(v)\\
&\leq C[1+|v|]^{\gamma+2}|\{I-P_v\}\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)||\partial^{\alpha}_{\beta}g_3(v)| \qquad (83)\\
&\leq C|[1+|v|]^{\frac{\gamma+2}{2}}\{I-P_v\}\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)||[1+|v|]^{\frac{\gamma+2}{2}}\partial^{\alpha}_{\beta}g_3(v)|\\
&\leq C\{\sigma^{ij}\partial_i\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\}^{1/2}\{\sigma^{ij}v_iv_j|\partial^{\alpha}_{\beta}g_3|^2\}^{1/2},
\end{aligned}
$$

where we have used (42). From (80), $|\partial_k\phi^{ij}(\bar v)|\leq C[1+|v|]^{\gamma+1}$. Hence

$$
\begin{aligned}
&|\partial_k\phi^{ij}(\bar v)\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial^{\alpha}_{\beta}g_3(v)| \qquad (84)\\
&\leq C[1+|v|]^{\gamma+1}\{|\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)|\}\{|\partial^{\alpha}_{\beta}g_3(v)|\}\\
&\leq C\{[1+|v|]^{\gamma/2}|\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)|\}\{[1+|v|]^{\frac{\gamma+2}{2}}|\partial^{\alpha}_{\beta}g_3(v)|\}\\
&\leq C\{\sigma^{ij}\partial_i\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\}^{1/2}\{\sigma^{ij}v_iv_j|\partial^{\alpha}_{\beta}g_3|^2\}^{1/2}.
\end{aligned}
$$

From (83) and (84), we thus conclude

$$\left|\sum_{ij}\phi^{ij}(v-v')\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial^\alpha_\beta g_3(v)\right|$$
$$\leq C[1+|v'|]\{\sigma^{ij}\partial_i\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\}^{1/2}\{\sigma^{ij}v_iv_j|\partial^\alpha_\beta g_3|^2\}^{1/2}.$$

We thus conclude that the v integrand in (71) can be majorized by

$$C\sum\int[1+|v'|]\mu^{1/4}(v')|\partial^{\alpha_1}_{\bar\beta}g_1(v')|dv'$$
$$\times w^2\{\sigma^{ij}\partial_i\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\}^{1/2}\{\sigma^{ij}\partial_i\partial^\alpha_\beta g_3\partial_j\partial^\alpha_\beta g_3\}^{1/2}$$
$$\leq C\sum|\partial^{\alpha_1}_{\bar\beta}g_1|_\ell\{w^2\sigma^{ij}\partial_i\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\}^{1/2}\{w^2\sigma^{ij}\partial_i\partial^\alpha_\beta g_3\partial_j\partial^\alpha_\beta g_3\}^{1/2}.$$

Further integration over v shows that this bounded by the right hand side of (68).

We now consider the third term (72) over $\{2|v'|\leq|v|,|v|\geq 1\}$. We use an integration by parts inside the convolution to split (72) into two parts

$$\phi^{ij}*\partial_{\beta_1}[\mu^{1/2}\partial_j\partial^{\alpha_1}g_1]=\partial_j\phi^{ij}*\partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1}g_1]-\phi^{ij}*\partial_{\beta_1}[\partial_j\mu^{1/2}\ \partial^{\alpha_1}g_1]. \tag{85}$$

Recall expansion (82) and decompose

$$\partial_i\partial^\alpha_\beta g_3=P_v\partial_i\partial^\alpha_\beta g_3+[I-P_v]\partial_i\partial^\alpha_\beta g_3.$$

By similar estimates to (83) and (84), the second part of (72) can be estimated as

$$\int_{\{|v|\geq1,2|v'|\leq|v|\}}|w^2\ \phi^{ij}(v-v')\partial_{\beta_1}[\partial_j\mu^{1/2}(v')\partial^{\alpha_1}g_1(v')]\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial_i\partial^\alpha_\beta g_3(v)|$$
$$=\int_{|v|\geq1,2|v'|\leq|v|}|w^2[\phi^{ij}(v)-\partial_k\phi^{ij}(\bar v)v'_k]\partial_{\beta_1}[\partial_j\mu^{1/2}(v')\partial^{\alpha_1}g_1(v')]\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial_i\partial^\alpha_\beta g_3|$$
$$\leq C\left|\partial^{\alpha_1}_{\bar\beta}g_1\right|_\ell\left|[1+|v|]^{\frac{\gamma+2}{2}}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\right|_{\ell,\vartheta}\left|w^\vartheta[1+|v|]^{\frac{\gamma+2}{2}}\{I-P_v\}\partial_i\partial^\alpha_\beta g_3\right|_{\ell,\vartheta}$$
$$+C\left|\partial^{\alpha_1}_{\bar\beta}g_1\right|_\ell\left|[1+|v|]^{\frac{\gamma+2}{2}}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\right|_{\ell,\vartheta}\left|[1+|v|]^{\gamma/2}\partial_i\partial^\alpha_\beta g_3\right|_{\ell,\vartheta}.$$

By Lemma 5, this is bounded by the right hand side of (68).

For the first part of (72) by (85) notice that our integration region implies

$$|\partial_j\phi^{ij}(v-v')|\leq C[1+|v|]^{\gamma+1}.$$

We thus have

$$\int_{\{|v|\geq1,2|v'|\leq|v|\}}w^2\ |\partial_j\phi^{ij}(v-v')|\partial_{\beta_1}[\mu^{1/2}(v')\partial^{\alpha_1}g_1(v')]\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2(v)\partial_i\partial^\alpha_\beta g_3(v)$$
$$\leq C\sum_{\bar\beta\leq\beta_1}\left|\partial^{\alpha_1}_{\bar\beta}g_1\right|_\ell\left|[1+|v|]^{\frac{\gamma+2}{2}}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\right|_{\ell,\vartheta}\left|[1+|v|]^{\gamma/2}\partial_j\partial^\alpha_\beta g_3\right|_{\ell,\vartheta},$$

which is bounded by the right hand side of (68) by Lemma 5.

Next consider (73) over $\{2|v'|\leq|v|,\ |v|\geq1\}$. We split (73) as in (85)

$$\phi^{ij}*\partial_{\beta_1}[v_i\mu^{1/2}\partial_j\partial^{\alpha_1}g_1]=\partial_j\phi^{ij}*\partial_{\beta_1}[v_i\mu^{1/2}\partial^{\alpha_1}g_1]-\phi^{ij}*\partial_{\beta_1}[\partial_j\{v_i\mu^{1/2}\}\ \partial^{\alpha_1}g_1].$$

Since $|\phi^{ij}(v-v')| \leq C[1+|v|]^{\gamma+2}$, and $|\partial_j \phi^{ij}(v-v')| \leq C[1+|v|]^{\gamma+1}$, (73) is bounded by

$$\int w^2[1+|v|]^{\gamma+1}|\partial_{\beta_1}[v_i'\mu^{1/2}(v')\partial^{\alpha_1}g_1(v')]\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial^\alpha_\beta g_3|dv'dv$$
$$+\int w^2[1+|v|]^{\gamma+2}|\partial_{\beta_1}[\partial_j\{v_i'\mu^{1/2}(v')\}\partial^{\alpha_1}g_1(v')]\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial^\alpha_\beta g_3|dv'dv$$
$$\leq C\sum_{\bar\beta\leq\beta_1}\left|\partial^{\alpha_1}_{\bar\beta}g_1\right|_\ell\left|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\right|_{\boldsymbol{\sigma},\ell,\vartheta}\left|\partial^\alpha_\beta g_3\right|_{\boldsymbol{\sigma},\ell,\vartheta}.$$

We thus conclude the estimate for (73).

Finally, consider the term (74) over $\{2|v'|\leq|v|,\ |v|\geq 1\}$. First sum over v_i so that (74) is given by

$$\langle [w^2w_1]\{\phi^{ij}*v_i\partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1}g_1]\}\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2,\partial^\alpha_\beta g_3\rangle$$

We again expand $\phi^{ij}(v-v')$ as in (82). By (47), we have the estimates (83) and (84). Plugging (83) and (84) into (82), we thus conclude that the v integrand in (74) can be majorized by

$$C\sum\int[1+|v'|]^2\mu^{1/4}(v')|\partial^{\alpha_1}_{\bar\beta}g_1(v')|dv'$$
$$\times w^2\{\sigma^{ij}\partial_i\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\}^{1/2}\{\sigma^{ij}v_iv_j|\partial^\alpha_\beta g_3|^2\}^{1/2}$$
$$\leq C\sum|\partial^{\alpha_1}_{\bar\beta}g_1|_\ell\{w^2\sigma^{ij}\partial_i\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\}^{1/2}\{w^2\sigma^{ij}v_iv_j|\partial^\alpha_\beta g_3|^2\}^{1/2}.$$

By further integrating over v, this bounded by the right hand side of (68). And thus, the proof of (68) is complete.

The proof of (69) now follows from the Sobolev embedding $H^2(\mathbb{T}^3)\subset L^\infty(\mathbb{T}^3)$ and (68). Without loss of generality, assume $|\alpha_1|+|\bar\beta|\leq N/2$ in (68). Then

$$\left(\sup_x|\partial^{\alpha_1}_{\bar\beta}g_1(x)|_\ell\right)|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\boldsymbol{\sigma},\ell,\vartheta}+\left(\sup_x|\partial^{\alpha_1}_{\bar\beta}g_1(x)|_{\boldsymbol{\sigma},\ell}\right)|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\ell,\vartheta}$$
$$\leq\left(\sum\|\partial^{\alpha'}_{\beta'}g_1\|_\ell\right)|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\boldsymbol{\sigma},\ell,\vartheta}+\left(\sum\|\partial^{\alpha'}_{\beta'}g_1\|_{\boldsymbol{\sigma},\ell}\right)|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\ell,\vartheta},$$

where the summation is over $|\alpha'|+|\beta'|\leq\frac{N}{2}+2\leq N$. We deduce (69) by integrating (68) over $\mathbb{T}^3$ and using the this computation. $\square$

4. Energy Estimate and Global Existence

In this section we will prove the energy estimate which is a crucial step in constructing global solutions. By now, it is standard to prove local existence of small solutions using the estimates either in Section 2 for the Boltzmann case or Section 3 for the Landau case:

Theorem 3. *For any sufficiently small $M^*>0$, $T^*>0$ with $T^*\leq\frac{M^*}{2}$ and*

$$\frac{1}{2}\sum_{|\alpha|+|\beta|\leq N}||\partial^\alpha_\beta f_0||^2_{|\beta|-l,\vartheta}\leq\frac{M^*}{2},$$

there is a unique classical solution $f(t,x,v)$ to (5) in either the Boltzmann or the Landau case in $[0,T^*)\times\mathbb{T}^3\times\mathbb{R}^3$ such that

$$\sup_{0\le t\le T^*}\left\{\frac{1}{2}\sum_{|\alpha|+|\beta|\le N}||\partial^\alpha_\beta f||^2_{|\beta|-l,\vartheta}(t)+\int_0^t\mathcal{D}_{l,\vartheta}(f)(s)ds\right\}\le M^*,$$

and $\frac{1}{2}\sum_{|\alpha|+|\beta|\le N}||\partial^\alpha_\beta f||^2_{|\beta|-l,\vartheta}(t)+\int_0^t\mathcal{D}_{l,\vartheta}(f)(s)ds$ is continuous over $[0,T^*)$.

Next, we define some notation. For fixed $N\ge 8$, $0\le m\le N$ and $\vartheta,q,l\ge 0$, a modified instant energy functional satisfies

$$\frac{1}{C}\mathcal{E}^m_{l,\vartheta}(g)(t)\le\sum_{|\beta|\le m,|\alpha|+|\beta|\le N}||\partial^\alpha_\beta g(t)||^2_{|\beta|-l,\vartheta}\le C\mathcal{E}^m_{l,\vartheta}(g)(t). \tag{86}$$

Similarly the modified dissipation rate is given by

$$\mathcal{D}^m_{l,\vartheta}(g)(t)\equiv\sum_{|\beta|\le m,|\alpha|+|\beta|\le N}||\partial^\alpha_\beta g(t)||^2_{\boldsymbol{D},|\beta|-l,\vartheta}. \tag{87}$$

Note that, $\mathcal{E}^N_{l,\vartheta}(g)(t)=\mathcal{E}_{l,\vartheta}(g)(t)$ and $\mathcal{D}^N_{l,\vartheta}(g)(t)=\mathcal{D}_{l,\vartheta}(g)(t)$. And as before, we will write $\mathcal{E}^m_{l,0}(g)(t)=\mathcal{E}^m_l(g)(t)$ and $\mathcal{D}^m_{l,0}(g)(t)=\mathcal{D}^m_l(g)(t)$. Now we are ready to state a result from equation (4.5) in [12] using this new notation:

Lemma 11. *Let $f(t,x,v)$ be a classical solution to (5) satisfying (13) in either the Boltzmann or the Landau case. In the Boltzmann case assume $\tau\le\gamma$ but in the Landau case assume $\tau\le -1$ in (10). For any $l\ge 0$, there exists M_l, $\delta_l=\delta_l(M_l)>0$ such that if*

$$\frac{1}{2}\sum_{|\alpha|+|\beta|\le N}||\partial^\alpha_\beta f||^2_{|\beta|-l}(t)\le M_l, \tag{88}$$

then for any $0\le m\le N$ we have an instant energy functional such that

$$\frac{d}{dt}\mathcal{E}^m_l(f)(t)+\mathcal{D}^m_l(f)(t)\le C\sqrt{\mathcal{E}_l(f)(t)}\mathcal{D}_l(f)(t). \tag{89}$$

We will bootstrap this energy estimate without an exponential weight ($\vartheta=0$) to Corollary 1 in the Boltzmann case and Lemma 9 on the Landau case to obtain the following general energy estimate.

Lemma 12. *Fix $N\ge 8$, $0<\vartheta\le 2$, $q>0$ and $l\ge 0$. If $\vartheta=2$ let $0<q<1$. In the Boltzmann case assume $\tau\le\gamma$ but in the Landau case assume $\tau\le -1$ in (10). Let $f(t,x,v)$ be a classical solution to (5) satisfying (13) and (88) in either the Boltzmann or the Landau case. For any given $0\le m\le N$ there is a modified instant energy functional such that*

$$\frac{d}{dt}\mathcal{E}^m_{l,\vartheta}(f)(t)+\mathcal{D}^m_{l,\vartheta}(f)(t)\le C\mathcal{E}^{1/2}_{l,\vartheta}(f)(t)\mathcal{D}_{l,\vartheta}(f)(t). \tag{90}$$

Proof. We use an induction over m, the order of the $v-$derivatives. For $m=0$, by taking the pure ∂^α derivatives of (5) we obtain

$$\{\partial_t+v\cdot\nabla_x\}\partial^\alpha f+L\{\partial^\alpha f\}=\sum_{\alpha_1\le\alpha}C^{\alpha_1}_\alpha\Gamma(\partial^{\alpha_1}f,\partial^{\alpha-\alpha_1}f) \tag{91}$$

Multiply $w^2(-l,\vartheta)\partial^\alpha f$ with (91), integrate over $\mathbb{T}^3\times\mathbb{R}^3$ and sum over $|\alpha|\le N$ to deduce the following for some constant $C>0$,

$$\sum_{|\alpha|\le N}\left\{\frac{1}{2}\frac{d}{dt}\|\partial^\alpha f(t)\|_{-l,\vartheta}^2+\big(w^2(-l,\vartheta)L\{\partial^\alpha f(t)\},\partial^\alpha f(t)\big)\right\}$$
$$\le C\mathcal{E}_{l,\vartheta}^{1/2}(f)(t)\mathcal{D}_{l,\vartheta}(f)(t). \tag{92}$$

We have used Lemma 3 in the Boltzmann case and (69) in the Landau case to bound the r.h.s. of (91). Notice that Lemma 2 (in the Boltzmann case) implies

$$\begin{aligned}\big(w^2(-l,\vartheta)L\{\partial^\alpha f(t)\},\partial^\alpha f(t)\big)&=\|\partial^\alpha f(t)\|_{\boldsymbol{\nu},-l,\vartheta}^2-\big(w^2(-l,\vartheta)K\{\partial^\alpha f(t)\},\partial^\alpha f(t)\big)\\&\ge\frac{1}{2}\|\partial^\alpha f(t)\|_{\boldsymbol{\nu},-l,\vartheta}^2-C\|\partial^\alpha f(t)\|_{\boldsymbol{\nu},-l}^2,\end{aligned}$$

where $C>0$ is a large constant. In the Landau case, Lemma 9 gives

$$\begin{aligned}\big(w^2(-l,\vartheta)L\{\partial^\alpha f(t)\},\partial^\alpha f(t)\big)&\ge\delta_q\|\partial^\alpha f(t)\|_{\boldsymbol{\sigma},-l,\vartheta}^2-C\|\bar{\chi}_C\partial^\alpha f(t)\|_{-l}^2\\&\ge\delta_q\|\partial^\alpha f(t)\|_{\boldsymbol{\sigma},-l,\vartheta}^2-C\|\partial^\alpha f(t)\|_{\boldsymbol{\sigma},-l}^2.\end{aligned}$$

Without loss of generality assume $0<\delta_q<\frac{1}{2}$. Then in either case, plugging these into (92) we have

$$\sum_{|\alpha|\le N}\left\{\frac{1}{2}\frac{d}{dt}\|\partial^\alpha f(t)\|_{-l,\vartheta}^2+\frac{1}{2}\|\partial^\alpha f(t)\|_{\boldsymbol{D},-l,\vartheta}^2-C\|\partial^\alpha f(t)\|_{\boldsymbol{D},-l}^2\right\}$$
$$\le C\sqrt{\mathcal{E}_{l,\vartheta}(f)(t)}\mathcal{D}_{l,\vartheta}(f)(t).$$

Add to this inequality to (89) with $m=0$, possibly multiplied by a large constant, to obtain (90) with $m=0$.

Now assume the Lemma is valid for some fixed $m>0$. For $|\beta|=m+1$, taking ∂_β^α of (5) we obtain

$$\{\partial_t+v\cdot\nabla_x\}\,\partial_\beta^\alpha f+\partial_\beta\{L\partial^\alpha f\}$$
$$=-\sum_{|\beta_1|=1}C_\beta^{\beta_1}\partial_{\beta_1}v\cdot\nabla_x\partial_{\beta-\beta_1}^\alpha f+\sum C_\alpha^{\alpha_1}\partial_\beta\Gamma(\partial^{\alpha_1}f,\partial^{\alpha-\alpha_1}f). \tag{93}$$

We take the inner product of (93) with $w^2(|\beta|-l,\vartheta)\partial_\beta^\alpha f$ over $\mathbb{T}^3\times\mathbb{R}^3$. The first inner product on the left is equal to $\frac{1}{2}\frac{d}{dt}||\partial_\beta^\alpha f(t)||_{|\beta|-l,\vartheta}^2$. From either Corollary 1 in the Boltzmann case or Lemma 9 in the Landau case, we deduce that the inner product of $\partial_\beta\{L\partial^\alpha f\}$ is bounded from below as

$$\sum_{|\beta|=m+1}\big(w^2\partial_\beta\{L\partial^\alpha f\},\partial_\beta^\alpha f\big)\ge\frac{1}{2}\sum_{|\beta|=m+1}||\partial_\beta^\alpha f||_{\boldsymbol{D},|\beta|-l,\vartheta}^2-C\sum_{|\bar{\beta}|\le m}||\partial_{\bar{\beta}}^\alpha f||_{\boldsymbol{D},|\bar{\beta}|-l,\vartheta}^2.$$

Since $|\beta_1|=1$, as in [8] the streaming term on the r.h.s. of (93) is bounded by

$$\begin{aligned}(w^2(|\beta|-l,\vartheta)\{\partial_{\beta_1}v_j\}\partial_{x_j}\partial_{\beta-\beta_1}^\alpha f,\partial_\beta^\alpha f)&\le\int w^2(|\beta|-l,\vartheta)|\partial_{x_j}\partial_{\beta-\beta_1}^\alpha f\partial_\beta^\alpha f|dxdv\\&\le\big\|w(|\beta|+1/2-l,\vartheta)\partial_\beta^\alpha f\big\|\,\big\|w(1/2+\{|\beta|-1\}-l,\vartheta)\partial_{x_j}\partial_{\beta-\beta_1}^\alpha f\big\|\\&\le\eta\left\|\partial_\beta^\alpha f\right\|_{\boldsymbol{D},|\beta|-l,\vartheta}^2+C_\eta\left\|\partial_{x_j}\partial_{\beta-\beta_1}^\alpha f\right\|_{\boldsymbol{D},|\beta-\beta_1|-l,\vartheta}^2.\end{aligned}$$

Further, by Lemma 3 in the Boltzmann case and Lemma 10 in the Landau case, the inner product involving Γ on the r.h.s. of (93) is $\le C\mathcal{E}_{l,\vartheta}^{1/2}(f)(t)\mathcal{D}_{l,\vartheta}(f)(t)$.

Collect terms and sum over $|\beta| = m+1, |\alpha| + |\beta| \le N$ to obtain

$$\sum_{|\beta|=m+1,|\alpha|+|\beta|\le N} \left\{ \frac{1}{2}\frac{d}{dt}||\partial^\alpha_\beta f(t)||^2_{|\beta|-l,\vartheta} + \left(\frac{1}{2} - W\eta\right) \left\|\partial^\alpha_\beta f\right\|^2_{\boldsymbol{D},|\beta|-l,\vartheta} \right\}$$
$$\le \sum_{|\beta|=m+1,|\alpha|+|\beta|\le N} C \left(\sum_{|\beta_1|=1} \left\|\partial_{x_j}\partial^\alpha_{\beta-\beta_1} f\right\|^2_{\boldsymbol{D},|\beta-\beta_1|-l,\vartheta} + \sum_{|\bar\beta|\le m} ||\partial^\alpha_{\bar\beta} f||^2_{\boldsymbol{D},|\beta|-l,\vartheta} \right)$$
$$+ C Z_{m+1} \mathcal{E}^{1/2}_{l,\vartheta}(f)(t)\mathcal{D}_{l,\vartheta}(f)(t).$$

Here $Z_{m+1} = \sum_{|\beta|=m+1,|\alpha|+|\beta|\le N} 1$ and $W = \sum_{|\beta_1|=1} C^{\beta_1}_\beta$. Choosing $\eta > 0$ such that $\frac{1}{2} - W\eta = \frac{1}{4} > 0$ we get

$$\sum_{|\beta|=m+1,|\alpha|+|\beta|\le N} \left\{ \frac{1}{2}\frac{d}{dt}||\partial^\alpha_\beta f(t)||^2_{|\beta|-l,\vartheta} + \frac{1}{4} \left\|\partial^\alpha_\beta f\right\|^2_{\boldsymbol{D},|\beta|-l,\vartheta} \right\}$$
$$\le \tilde{C} \sum_{|\beta|\le m,|\alpha|+|\beta|\le N} ||\partial^\alpha_\beta f||^2_{\boldsymbol{D},|\beta|-l,\vartheta} + C Z_{m+1} \sqrt{\mathcal{E}_{l,\vartheta}(f)(t)}\mathcal{D}_{l,\vartheta}(f)(t).$$

Choose $\bar{A}_{m+1}$ such that $\bar{A}_{m+1} - \tilde{C} \ge 1$. Now multiply (90) for $|\beta| \le m$ by $\bar{A}_{m+1}$ and add it to the display above to obtain (90) for $|\beta| \le m+1$. We thus conclude the energy estimate. □

With Lemma 12, we can prove existence of global in time solutions with an exponential weight using exactly the same argument as in the last Section of [10].

5. Proof of Exponential Decay

In this section we prove exponential decay using the differential inequality (14) and the uniform bound (15) with $\vartheta > 0$. The main difficulty in establishing decay from (14) is rooted in the fact that the dissipation $\mathcal{D}_{l,\vartheta}(f)(t)$ is in general weaker than the instant energy $\mathcal{E}_{l,\vartheta}(f)(t)$. As in the work of Caflisch [1], the key point is to split $\mathcal{E}_l(f)(t)$ into a time dependent low velocity part

$$E = \{|v| \le \rho t^{p'}\},$$

and it's complementary high velocity part $E^c = \{|v| > \rho t^{p'}\}$, where $p' > 0$ and $\rho > 0$ will be chosen at the end of the proof.

First consider the Boltzmann case. Let $\mathcal{E}^{low}_l(f)(t)$ be the instant energy restricted to E. Then from (16), for $t > 0$, we have

$$\mathcal{D}_l(f)(t) \ge C\rho^\gamma t^{\gamma p'} \mathcal{E}^{low}_l(f)(t). \tag{94}$$

Plugging this into the the differential inequality (14) we obtain

$$\frac{d}{dt}\mathcal{E}_l(f)(t) + C\rho^\gamma t^{\gamma p'} \mathcal{E}^{low}_l(f)(t) \le 0.$$

Letting $\mathcal{E}^{high}_l(f)(t) = \mathcal{E}_l(f)(t) - \mathcal{E}^{low}_l(f)(t)$ we have

$$\frac{d}{dt}\mathcal{E}_l(f)(t) + C\rho^\gamma t^{\gamma p'} \mathcal{E}_l(f)(t) \le C\rho^\gamma t^{\gamma p'} \mathcal{E}^{high}_l(f)(t).$$

Define $\lambda = C\rho^\gamma/p$ where for now $p = \gamma p' + 1$ and $p' > 0$ is arbitrary. Then

$$\frac{d}{dt}\mathcal{E}_l(f)(t) + \lambda p t^{p-1} \mathcal{E}_l(f)(t) \le \lambda p t^{p-1} \mathcal{E}^{high}_l(f)(t).$$

Equivalently

$$\frac{d}{dt}\left(e^{\lambda t^p}\mathcal{E}_l(f)(t)\right) \leq \lambda p t^{p-1} e^{\lambda t^p}\mathcal{E}_l^{high}(f)(t).$$

The integrated form is

$$\mathcal{E}_l(f)(t) \leq e^{-\lambda t^p}\mathcal{E}_l(f_0) + \lambda p e^{-\lambda t^p}\int_0^t s^{p-1}e^{\lambda s^p}\mathcal{E}_l^{high}(f)(s)ds.$$

Above $p > 0$ or equivalently $\gamma p' > -1$ is assumed to guarantee the integral on the r.h.s. is finite. Since $\mathcal{E}_l^{high}(f)(s)$ is on $E^c = \{|v| > \rho s^{p'}\}$

$$\mathcal{E}_l^{high}(f)(s) = \mathcal{E}_{l,0}^{high}(f)(s) \leq Ce^{-\frac{q}{2}\rho s^{\vartheta p'}}\mathcal{E}_{l,\vartheta}^{high}(f)(s).$$

In the last display we have used the region and

$$1 \leq \exp\left(\frac{q}{2}(1+|v|^2)^{\frac{\vartheta}{2}}\right)e^{-\frac{q}{2}|v|^\vartheta} \leq \exp\left(\frac{q}{2}(1+|v|^2)^{\frac{\vartheta}{2}}\right)e^{-\frac{q}{2}\rho s^{\vartheta p'}}.$$

Hence (15) implies

$$\mathcal{E}_l(f)(t) \leq e^{-\lambda t^p}\left(\mathcal{E}_{l,0}(f_0) + \lambda p \mathcal{E}_{l,\vartheta}(f_0)\int_0^t s^{p-1}e^{\lambda s^p - \frac{q}{2}\rho s^{\vartheta p'}}ds\right).$$

The biggest exponent p that we can allow with this splitting is $p = \vartheta p'$; since also $p = \gamma p' + 1$ we have $p' = \frac{1}{\vartheta - \gamma}$ so that

$$p = \frac{\gamma}{\vartheta - \gamma} + 1 = \frac{\vartheta}{\vartheta - \gamma}.$$

Further choose $\rho > 0$ large enough so that $\lambda = C\rho^\gamma / p < \frac{q}{2}\rho$ ($\gamma < 0$) and hence

$$\int_0^\infty s^{p-1}e^{\lambda s^p - \frac{q}{2}\rho s^p}ds < +\infty.$$

This completes the proof of decay in the Boltzmann case.

For the proof of decay in the Landau case, instead of (94), we use Lemma 5 to see that

$$\mathcal{D}_l(f)(t) \geq C\rho^{2+\gamma}t^{(2+\gamma)p'}\mathcal{E}_l^{low}(f)(t).$$

And the rest of the proof is exactly the same. But we find, in this case, that $p = \frac{\vartheta}{\vartheta - (2+\gamma)}$. **Q.E.D.**

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