Math 644 - Homework 3 - Due Friday, Sept. 28, 2012

1. (Evans, Problem 4 in Chapter 2) (A direct proof of the maximum principle.) Suppose that $U \subseteq \mathbb{R}^n$ is open and bounded and suppose that $u \in C^2(U) \cap C(\overline{U})$ is harmonic on U. By considering the functions $u_\epsilon := u + \epsilon |x|^2$, for $\epsilon > 0$, show:

$$\max_{\overline{U}} u = \max_{\partial U} u.$$

2. (Evans, Problem 5 in Chapter 2) (Subharmonic functions.)

   We say that $v \in C^2(\overline{U})$ is **subharmonic** if:

$$-\Delta v \leq 0.$$

   (a) Prove that for v subharmonic, one has:

$$v(x) \leq \fint_{B(x,r)} v(y)dy,$$

   for all $B(x,r) \subseteq U$.

   (b) Show that $\max_{\overline{U}} v = \max_{\partial U} v$.

   (c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume that u is harmonic and take $v := \phi(u)$. Prove that u is subharmonic.

   (d) Prove that $v := |\nabla u|^2$ is subharmonic whenever u is harmonic.

3. (Weyl's lemma.) We outline the proof of Weyl's lemma, which is a generalization of the Theorem we proved in class that states that all (C^2) harmonic functions are smooth, or more generally that all continuous functions which satisfy the mean value property are smooth. The claim which we prove is:

   Lemma 1. *Let $U \subseteq \mathbb{R}^n$ be open and bounded. Suppose that $u : U \to \mathbb{R}$ and $u \in L^1_{loc}(U)$. Furthermore, suppose that:*

$$\int_U u(x)\Delta\phi(x)dx = 0$$

   for all $\phi \in C_c^\infty(U)$. Then u is harmonic on U, and in particular it is smooth.

Notice that harmonic functions on U satisfy the condition from the Lemma.

If one proves this for $u \in C(U)$, this counts for the full credit. If one shows the claim for $u \in L^1_{loc}(U)$, this counts for additional extra credit.

a) Given $\epsilon > 0$, let us consider $\phi \in C_c^\infty(U)$ which are supported inside

$$U_\epsilon := \{x \in U; dist(x, \partial U) > \epsilon\}.$$

Let $u^\epsilon := u * \eta_\epsilon$, where $\eta_\epsilon := \frac{1}{\epsilon^n}\eta(\frac{\cdot}{\epsilon})$ is the mollifier constructed in class. Show that u^ϵ is harmonic on U_ϵ.

b) Show that, for all $\epsilon > 0$, one has:

$$\int_{U_\epsilon} |u^\epsilon(x)| dx \leq \int_U |u(x)| dx.$$

c) We fix an $R > 0$ and we consider $V := \overline{U_R}$. Show that, for $0 < \epsilon < \frac{R}{2}$, $|u^\epsilon|$ is bounded on V with a bound independent of ϵ.
(HINT: Recall the Mean Value Property).

d) Show that u_ϵ is equicontinuous on V.

e) Use the Arzela-Ascoli Theorem and show that we can find $v \in C(\overline{V})$ such that, up to a subsequence, u_ϵ converges to v uniformly on $\overline{V}$, as $\epsilon \to 0$. Why is v harmonic on V?

f) Recall that $u_\epsilon \to u$ on U (you are allowed to use the L^1_{loc} version of this statement, which was stated in class). Deduce that u is harmonic on U.

4. (Evans, Problem 6 from Chapter 2) (An estimate for solutions to Poisson's equation.) Suppose that U is a bounded, open subset of $\mathbb{R}^n$ and that $u \in C^2(U) \cap C(\bar{U})$ solves:

$$\begin{cases} -\Delta u = f, \text{ in } U \\ u = g, \text{ on } \partial U \end{cases} \tag{1}$$

for $f \in C(\bar{U})$, $g \in C(\partial U)$. Show that there exists a constant $C > 0$ which depends only on U such that:

$$\max_{\bar{U}} |u| \leq \max_{\partial U} |g| + C \max_{\bar{U}} |f|.$$

(HINT: Look at the function $u + \lambda |x|^2$ for an appropriate value of λ and use properties of subharmonic functions from earlier.)