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On the global existence for the Muskat problem

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Abstract. The Muskat problem models the dynamics of the interface between two incompressible immiscible fluids with different constant densities. In this work we prove three results. First we prove an $L^2(\mathbb{R})$ maximum principle, in the form of a new "log" conservation law (3) which is satisfied by the equation (1) for the interface. Our second result is a proof of global existence for unique strong solutions if the initial data is smaller than an explicitly computable constant, for instance $\|f\|_1 \le 1/5$. Previous results of this sort used a small constant $\epsilon \ll 1$ which was not explicit [7, 19, 9, 14]. Lastly, we prove a global existence result for Lipschitz continuous solutions with initial data that satisfy $\|f_0\|_{L^\infty} < \infty$ and $\|\partial_x f_0\|_{L^\infty} < 1$. We take advantage of the fact that the bound $\|\partial_x f_0\|_{L^\infty} < 1$ is propagated by solutions, which grants strong compactness properties in comparison to the log conservation law.

Keywords. Porous media, incompressible flows, fluid interface, global existence

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1. Introduction

The Muskat problem models the dynamics of an interface between two incompressible immiscible fluids with different characteristics, in porous media. The phenomenon has

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been described using the experimental Darcy law that is given in two dimensions by the following momentum equation:

$$\frac{\mu}{\kappa}u = -\nabla p - g(0, \rho).$$

Here μ is viscosity, κ permeability of the isotropic medium, u velocity, p pressure, g gravity and ρ density. Saffman and Taylor [18] related this problem to the evolution of an interface in a Hele-Shaw cell since both physical scenarios can be modeled analogously (see also [7] and reference therein). Recently, the well-posedness has been shown without surface tension in [8] (for previous work on this topic see [1], [21], [2] and [9]) using arguments that rely upon the boundedness properties of the Hilbert transforms associated to $C^{1,\gamma}$ curves. Precise estimates are obtained with arguments involving conformal mappings, the Hopf maximum principle and Harnack inequalities. The initial data have to satisfy the Rayleigh–Taylor condition initially, otherwise the problem has been shown to be ill-posed in the sense of Hadamard (see [19] and [9] for more details). With surface tension, the initial value problem becomes more regular, and instabilities do not appear [13]. The case of more than one free boundary has been treated in [11] and [14].

In this paper we consider an interface given by fluids of different constant densities ρ^i, with the same viscosity and without surface tension. The step function ρ is represented by

$$\rho(x,t) = \begin{cases} \rho^1, & x \in \Omega^1(t), \\ \rho^2, & x \in \Omega^2(t) = \mathbb{R}^2 \setminus \Omega^1(t), \end{cases}$$

for $\Omega^i(t)$ two connected regions. As the density ρ is transported by the flow

$$\rho_t + u \cdot \nabla\rho = 0,$$

the free boundary evolves with the two-dimensional velocity $u = (u_1, u_2)$. The Biot–Savart law allows one to recover u from the vorticity given by $\omega = \partial_{x_1} u_2 - \partial_{x_2} u_1$, via the integral operator

$$u(x,t) = \nabla^{\perp}\Delta^{-1}\omega(x,t).$$

Darcy's law then provides the relation $\omega = -\partial_{x_1}\rho$ where μ/κ and g are taken equal to 1 for the sake of simplicity. Then the velocity field can be obtained in terms of the density as follows:

$$u(x,t) = \mathrm{PV}\int_{\mathbb{R}^2} K(x-y)\rho(y,t)\,dy - \frac{1}{2}(0, \rho(x,t)).$$

Here the kernel K is of Calderón–Zygmund type:

$$K(x) = \frac{1}{\pi}\left(-\frac{x_1 x_2}{|x|^2}, \frac{x_1^2 - x_2^2}{2|x|^2}\right)$$

(see [20]). As a consequence of $\rho \in L^{\infty}(\mathbb{R}^2 \times \mathbb{R}^+)$ it follows that the velocity belongs to BMO. Moreover, as K is an even kernel, it has the property that the mean of K (in the principal value sense) is zero on hemispheres [4], and this yields a bound on the velocity $u(x,t)$ in terms of $C^{1,\gamma}$ norms $(0 < \gamma < 1)$ of the free boundary [11].

In order to have a well-posed problem we need to consider initially an interface parameterized as the graph of a function with the denser fluid below: $\rho^2 > \rho^1$ as in [9]. The interface is characterized as the graph of the function $(x, f(x,t))$. This characterization is preserved by the system and f satisfies

$$\begin{aligned} f_t(x,t) &= \frac{\rho^2-\rho^1}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}} d\alpha\, \frac{(\partial_x f(x,t) - \partial_x f(x-\alpha,t))\alpha}{\alpha^2 + (f(x,t) - f(x-\alpha,t))^2}, \\ f(x,0) &= f_0(x), \qquad x \in \mathbb{R}. \end{aligned} \tag{1}$$

The above equation can be linearized around the flat solution, which yields the following nonlocal partial differential equation:

$$\begin{aligned} f_t(x,t) &= -\frac{\rho^2-\rho^1}{2}\Lambda f(x,t), \\ f(\alpha,0) &= f_0(\alpha), \qquad \alpha \in \mathbb{R}, \end{aligned} \tag{2}$$

where the operator Λ is the square root of the Laplacian. This linearization shows the parabolic character of the problem in the stable case ($\rho^2 > \rho^1$), as well as the ill-posedness in the unstable case ($\rho^2 < \rho^1$). We use the term ill-posedness here to mean that some solutions do leave the H^s spaces right away even for arbitrarily small data.

The nonlinear equation (1) is ill-posed in the unstable situation and locally well-posed in H^k ($k \geq 3$) for the stable case [9]. Furthermore the stable system satisfies a maximum principle $\|f\|_{L^\infty}(t) \leq \|f\|_{L^\infty}(0)$ (see [10]); decay rates are obtained for the periodic case:

$$\|f\|_{L^\infty}(t) \leq \|f_0\|_{L^\infty} e^{-Ct},$$

and also for the case on the real line (flat at infinity):

$$\|f\|_{L^\infty}(t) \leq \frac{\|f_0\|_{L^\infty}}{1+Ct}.$$

Numerical solutions performed in [12] further indicate a regularizing effect. The decay of the slope and the curvature is observed to be stronger than the rate of decay of the maximum of the difference between f and its mean value. Thus, the irregular regions in the graph are observed to be rapidly smoothed and the flat regions are smoothly bent. It is shown analytically in [10] that, if the initial data satisfy $\|\partial_x f_0\|_{L^\infty} < 1$, then there is a maximum principle so that this derivative remains in absolute value smaller than 1.

The three main results we present in this paper are the following:

1) In Section 2, we prove that a solution of (1) formally satisfies

$$\|f\|^2_{L^2}(t) + \frac{\rho^2-\rho^1}{2\pi}\int_0^t ds \int_{\mathbb{R}} d\alpha \int_{\mathbb{R}} dx \ln\left(1 + \left(\frac{f(x,s) - f(\alpha,s)}{x-\alpha}\right)^2\right) = \|f_0\|^2_{L^2}. \tag{3}$$

Furthermore, we have the inequality

$$\int_{\mathbb{R}} dx \int_{\mathbb{R}} d\alpha \ln\left(1 + \left(\frac{f(x,s) - f(\alpha,s)}{x-\alpha}\right)^2\right) \leq C\|f\|_{L^1}(s).$$

This identity shows a major difference with the linear equation (2) where the evolution of the L^2 norm provides a gain of half derivative for $\rho^2 > \rho^1$:

$$\|f\|^2_{L^2}(t) + (\rho^2 - \rho^1)\int_0^t ds\, \|\Lambda^{1/2} f\|^2_{L^2}(s) = \|f_0\|^2_{L^2}, \tag{4}$$

or equivalently

$$\|f\|^2_{L^2}(t) + \frac{\rho^2 - \rho^1}{2\pi}\int_0^t ds \int_{\mathbb{R}} dx \int_{\mathbb{R}} d\alpha \left(\frac{f(x,s) - f(\alpha,s)}{x-\alpha}\right)^2 = \|f_0\|^2_{L^2}.$$

Notice that this linear energy balance (4) directly implies compactness, whereas compactness does not follow from the nonlinear L^2 energy (3).

2) Our second result proves global existence of unique $C([0,T]; H^3(\mathbb{R}))$ solutions if initially the norm (7) of f_0 is controlled as $\|f_0\|_1 < c_0$ where

$$\|f_0\|_1 = \int_{\mathbb{R}} d\xi\, |\xi|\, |\hat{f}_0(\xi)|.$$

Of course here $\hat{f}$ denotes the standard Fourier transform of f. There are several results of global existence for small initial data (small compared to 1 or $\epsilon \ll 1$) in several norms [7, 19, 9, 14] taking advantage of the parabolic character of the equation for small initial data. For example in [19] and [9], in order to measure the analyticity of the solution, global existence is shown when $\|f_0\|_1 \le \varepsilon$ for ε very small (compared to 1) and

$$\int_{\mathbb{R}} d\xi\, |\xi|^2 e^{b(t)|\xi|} |\hat{f}_0(\xi)| \le \varepsilon e^{b(t)}(1 + |b(t)|^{\gamma-1}), \tag{5}$$

where $0 < \gamma < 1$ and $b(t) = a - ct/2$. Here c depends on the Rayleigh–Taylor condition and $a \le ct/2$.

For the Hele-Shaw problem, [7] proves the global existence in time for small analytic perturbations of the circle, and nonlinear asymptotic stability of the steady circular solution. Also recently [14] considers the Muskat problem in a periodic geometry, and proves the well-posedness as well as the exponential stability of a certain flat equilibrium.

The key point for our result, in comparison to previous work [7, 19, 9, 14], is that the constant c_0 can be easily explicitly computed (see (8)). We have checked numerically that c_0 is not that small: it is greater than $1/5$.

3) In Section 4 we prove global in time existence of Lipschitz continuous solutions in the stable case. Being a solution of (1) is understood in its weak formulation:

$$\int_0^T dt \int_{\mathbb{R}} dx\, \eta_t(x,t) f(x,t) + \int_{\mathbb{R}} dx\, \eta(x,0) f_0(x)$$
$$= \int_0^T dt \int_{\mathbb{R}} dx\, \eta_x(x,t) \frac{\rho^2 - \rho^1}{2\pi}\, \mathrm{PV}\int_{\mathbb{R}} d\alpha\, \arctan\left(\frac{f(x,t) - f(\alpha,t)}{x-\alpha}\right) \tag{6}$$

for all $\eta \in C_c^\infty([0,T)\times\mathbb{R})$. For initial data f_0 satisfying $\|f_0\|_{L^\infty} < \infty$ and $\|\partial_x f_0\|_{L^\infty} < 1$ we prove that there exists a solution of (6) that remains in the spaces $C([0,T]\times\mathbb{R}) \cap L^\infty([0,T]; W^{1,\infty}(\mathbb{R}))$ for any $T > 0$. We point out that, because of the condition $f \in L^\infty(\mathbb{R})$, the nonlinear term in (6) has to be understood as a principal value for the integral of two functions, one in $\mathcal{H}^1$, the Hardy space, and the other in BMO [20].

Kim [16] studied viscosity solutions for the one-phase Hele-Shaw and Stefan problems. For the Muskat problem, previous results of global existence [7, 19, 9, 14] need small initial data more regular than Lipschitz. We show here that we just need $\|\partial_x f_0\|_{L^\infty} < 1$, therefore

$$\left|\frac{f_0(x) - f_0(\alpha)}{x-\alpha}\right| < 1.$$

Notice that if we consider the first order term in the Taylor series of $\ln(1+y^2)$ (absolutely convergent for $|y| < 1$), then the identity (3) becomes (4).

2. L^2 maximum principle

In this section we provide a proof of the identity (3). As we are in the stable case, we take without loss of generality $(\rho^2 - \rho^1)/(2\pi) = 1$ to simplify the exposition. The contour equation (1) can be written as follows:

$$f_t(x,t) = \mathrm{PV}\int_{\mathbb{R}} \partial_x \arctan\left(\frac{f(x,t) - f(x-\alpha,t)}{\alpha}\right) d\alpha.$$

We multiply by f, integrate over dx, and use integration by parts to observe

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|f\|_{L^2}^2(t) &= -\int_{\mathbb{R}} dx \int_{\mathbb{R}} d\alpha\, f_x(x) \arctan\left(\frac{f(x,t) - f(x-\alpha,t)}{\alpha}\right)\\ &= -\int_{\mathbb{R}} dx \int_{\mathbb{R}} d\alpha\, f_x(x) \arctan\left(\frac{f(x,t) - f(\alpha,t)}{x-\alpha}\right).\end{aligned}$$

We use the splitting

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|f\|_{L^2}^2(t) &= -\int_{\mathbb{R}}\int_{\mathbb{R}} \left(\frac{f(x,t) - f(\alpha,t)}{x-\alpha}\right) \arctan\left(\frac{f(x,t) - f(\alpha,t)}{x-\alpha}\right) dx\, d\alpha\\ &\quad -\int_{\mathbb{R}}\int_{\mathbb{R}} \left(\frac{f_x(x)(x-\alpha) - (f(x,t) - f(\alpha,t))}{x-\alpha}\right) \arctan\left(\frac{f(x,t) - f(\alpha,t)}{x-\alpha}\right) dx\, d\alpha\\ &= I_1 + I_2.\end{aligned}$$

We also use the function G defined by

$$G(x) = x \arctan x - \ln\sqrt{1+x^2} = \int_0^x dy \arctan y.$$

With these, it is easy to observe that

$$I_2 = -\int_{\mathbb{R}}\int_{\mathbb{R}} (x-\alpha)\partial_x G\Big(\frac{f(x,t)-f(\alpha,t)}{x-\alpha}\Big)\,dx\,d\alpha.$$

The identity

$$\lim_{|x|\to\infty} (x-\alpha) G\Big(\frac{f(x,t)-f(\alpha,t)}{x-\alpha}\Big) = 0$$

allows us to integrate by parts to obtain

$$\begin{aligned} I_2 &= \int_{\mathbb{R}}\int_{\mathbb{R}} G\Big(\frac{f(x,t)-f(\alpha,t)}{x-\alpha}\Big)\,dx\,d\alpha \\ &= -I_1 - \int_{\mathbb{R}}\int_{\mathbb{R}} \ln\sqrt{1+\Big(\frac{f(x,t)-f(\alpha,t)}{x-\alpha}\Big)^2}\,dx\,d\alpha. \end{aligned}$$

This equality gives

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2}(t) = -\int_{\mathbb{R}}\int_{\mathbb{R}} \ln\sqrt{1+\Big(\frac{f(x,t)-f(\alpha,t)}{x-\alpha}\Big)^2}\,dx\,d\alpha,$$

and integrating in time we get the desired identity.

The above equality indicates that for large initial data, the system is not parabolic at the level of f in L^2. We prove below the inequality

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \ln\Big(1+\Big(\frac{f(x,t)-f(\alpha,t)}{x-\alpha}\Big)^2\Big)\,dx\,d\alpha \le 4\pi\sqrt{2}\,\|f\|_{L^1}(t),$$

which shows that there is no gain of derivatives for the stable case. If the initial data are positive, then $\|f\|_{L^1}(t) \le \|f_0\|_{L^1}$ follows from [10], so that the dissipation is bounded in terms of the initial data with zero derivatives.

For the proof of the inequality, we denote by J the integral

$$J := \int_{\mathbb{R}}\int_{\mathbb{R}} \ln\Big(1+\Big(\frac{f(x)-f(x-\alpha)}{\alpha}\Big)^2\Big)\,dx\,d\alpha.$$

We now use that the function $\ln(1+y^2)$ is increasing to observe that

$$J \le \int_{\mathbb{R}}\int_{\mathbb{R}} \ln\Big(1+\frac{2|f(x)|^2}{\alpha^2}+\frac{2|f(x-\alpha)|^2}{\alpha^2}\Big)\,dx\,d\alpha.$$

The inequality $\ln(1+a^2+b^2) \le \ln(1+a^2)+\ln(1+b^2)$ yields

$$J \le \int_{\mathbb{R}}\int_{\mathbb{R}} \ln\Big(1+\frac{2|f(x)|^2}{\alpha^2}\Big)\,dx\,d\alpha + \int_{\mathbb{R}}\int_{\mathbb{R}} \ln\Big(1+\frac{2|f(x-\alpha)|^2}{\alpha^2}\Big)\,dx\,d\alpha,$$

and therefore

$$J \le 2 \int_{\mathbb{R}} \int_{\mathbb{R}} \ln\left(1 + \frac{2|f(x)|^2}{\alpha^2}\right) dx\, d\alpha = K.$$

For K it is easy to get

$$K = 2 \int_{\{x\,:\,|f(x)|\neq 0\}} dx \int_{\mathbb{R}} d\alpha\ \ln\left(1 + \frac{2|f(x)|^2}{\alpha^2}\right),$$

so that an easy integration in α provides

$$K = 4\pi\sqrt{2} \int_{\{x\,:\,|f(x)|\neq 0\}} dx\, |f(x)| = 4\pi\sqrt{2}\, \|f\|_{L^1}.$$

This concludes our discussion of the L^2 maximum principle (3) for (1).

3. A global existence result for data less than $1/5$

In this section we prove global existence of $C([0, T]; H^3(\mathbb{R}))$ small data solutions. A key point is to consider the norm

$$\|f\|_s := \int_{\mathbb{R}} d\xi\, |\xi|^s |\hat{f}(\xi)|, \qquad s \ge 1. \tag{7}$$

This norm allows us to use Fourier techniques for small initial data that give rise to a global existence result for classical solutions. The key point is that (7) appears naturally when taking the Fourier transform of the equation in our computations below (see (9) and (10)). As a result, (7) provides sharper constants than the ones that one could obtain with different norms.

Theorem 3.1. *Suppose that initially $f_0 \in H^3(\mathbb{R})$ and $\|f_0\|_1 < c_0$, where c_0 is a constant such that*

$$2 \sum_{n\ge 1} (2n+1)^{2+\delta} c_0^{2n} \le 1 \tag{8}$$

for a fixed $0 < \delta < 1/2$. Then there is a unique solution f of (1) that satisfies $f \in C([0, T]; H^3(\mathbb{R}))$ for any $T > 0$.

Remark 3.2. We compute the limit case $\delta = 0$, so that

$$2 \sum_{n\ge 1} (2n+1)^2 c_0^{2n} \le 1$$

for

$$0 \le c_0 \le \frac{1}{3}\sqrt{7 - \frac{14 \times 5^{2/3}}{\sqrt[3]{9\sqrt{39} - 38}} + 2\sqrt[3]{5(9\sqrt{39} - 38)}} \approx 0.2199617648835399.$$

In particular,

$$2 \sum_{n\ge 1} (2n+1)^{2.1} c_0^{2n} < 1$$

if say $c_0 \le 1/5$.

The remainder of this section is devoted to the proof of Theorem 3.1. The contour equation for the stable Muskat problem (1) can be written as

$$f_t(x,t) = -\rho(\Lambda f + T(f)), \tag{9}$$

where we recall that $\rho = (\rho^2 - \rho^1)/2 > 0$ and we have

$$T(f) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\partial_x f(x) - \partial_x f(x-\alpha)}{\alpha} \frac{\left(\frac{f(x)-f(x-\alpha)}{\alpha}\right)^2}{1+\left(\frac{f(x)-f(x-\alpha)}{\alpha}\right)^2}\, d\alpha. \tag{10}$$

We define

$$\Delta_\alpha f(x) := \frac{f(x) - f(x-\alpha)}{\alpha}.$$

We consider the evolution of the norm $\|f\|_1$ given in (7):

$$\begin{aligned}\frac{d}{dt}\|f\|_1(t) &= \int_{\mathbb{R}} d\xi\, |\xi|\, \mathrm{sgn}(\hat{f}(\xi))\hat{f}_t(\xi)\\ &= \rho\int_{\mathbb{R}} d\xi\, |\xi|\, \mathrm{sgn}(\hat{f}(\xi))(-|\xi|\hat{f}(\xi) - \mathcal{F}(T)(\xi)).\end{aligned}$$

We will show that the first term dominates the second if initially

$$\|f_0\|_1 < \sqrt{(4-\sqrt{13})/6}, \quad \text{where} \quad \sqrt{(4-\sqrt{13})/6} > 1/4.$$

The key point, again, is that the constant is given explicitly.

Notice that under the local existence theorem of [9], this bound will be propagated for a short time. Then we may use the Taylor expansion

$$\frac{x^2}{1+x^2} = \sum_{n=1}^{\infty}(-1)^{n+1}x^{2n},$$

to obtain

$$T(f) = \frac{-1}{\pi}\sum_{n\geq 1}(-1)^n \int_{\mathbb{R}} \partial_x(\Delta_\alpha f)(\Delta_\alpha f)^{2n}\, d\alpha. \tag{11}$$

Notice that

$$\mathcal{F}(\Delta_\alpha f) = \hat{f}(\xi)m(\xi,\alpha), \quad \mathcal{F}(\partial_x\Delta_\alpha f) = -i\xi\hat{f}(\xi)m(\xi,\alpha), \quad m(\xi,\alpha) = \frac{1-e^{-i\xi\alpha}}{\alpha}.$$

Therefore

$$\mathcal{F}(\partial_x(\Delta_\alpha f)\,(\Delta_\alpha f)^{2n}) = ((-i\xi\hat{f}m) * (\hat{f}m) * \cdots * (\hat{f}m))(\xi,\alpha),$$

with $2n$ convolutions, one with $-i\xi \hat{f}m$ and $2n-1$ with $\hat{f}m$. Using (11) we obtain

$$\begin{aligned}
\mathcal{F}(T)(\xi) &= \frac{i}{\pi}\sum_{n\geq 1}(-1)^n \int_{\mathbb{R}} d\alpha \int_{\mathbb{R}} d\xi_1 \cdots \int_{\mathbb{R}} d\xi_{2n}(\xi-\xi_1)\hat{f}(\xi-\xi_1)m(\xi-\xi_1,\alpha)\\
&\quad \times \hat{f}(\xi_1-\xi_2)m(\xi_1-\xi_2,\alpha)\cdots\hat{f}(\xi_{2n-1}-\xi_{2n})m(\xi_{2n-1}-\xi_{2n},\alpha)\hat{f}(\xi_{2n})m(\xi_{2n},\alpha)\\
&= \sum_{n\geq 1}\int_{\mathbb{R}} d\xi_1 \cdots \int_{\mathbb{R}} d\xi_{2n}\,(\xi-\xi_1)\hat{f}(\xi-\xi_1)\Big(\prod_{i=1}^{2n-1}\hat{f}(\xi_i-\xi_{i+1})\Big)\hat{f}(\xi_{2n})M_n,
\end{aligned}$$

where $M_n = M_n(\xi,\xi_1,\ldots,\xi_{2n})$ is given by

$$M_n := \frac{i}{\pi}(-1)^n \int_{\mathbb{R}} m(\xi-\xi_1,\alpha)\Big(\prod_{i=1}^{2n-1} m(\xi_i-\xi_{i+1},\alpha)\Big)m(\xi_{2n},\alpha)\,d\alpha. \tag{12}$$

Since $m(\xi,\alpha) = i\xi\int_0^1 ds\, e^{i\alpha(s-1)\xi}$ we obtain

$$M_n(\xi,\xi_1,\ldots,\xi_{2n}) = m_n(\xi,\xi_1,\ldots,\xi_{2n})\,(\xi_1-\xi_2)\cdots(\xi_{2n-1}-\xi_{2n})\xi_{2n}$$

with

$$\begin{aligned}
m_n &= \frac{i}{\pi}\int_0^1 ds_1\cdots\int_0^1 ds_{2n}\int_{\mathbb{R}} d\alpha\,\frac{1-e^{-i\alpha(\xi-\xi_1)}}{\alpha}\\
&\qquad\qquad\qquad \times \exp\Big(i\alpha\sum_{j=1}^{2n-1}(s_j-1)(\xi_j-\xi_{j+1}) + i\alpha(s_{2n}-1)\xi_{2n}\Big)\\
&= \frac{i}{\pi}\int_0^1 ds_1\cdots\int_0^1 ds_{2n}\left(\mathrm{PV}\int_{\mathbb{R}}\exp(i\alpha A)\,\frac{d\alpha}{\alpha} - \mathrm{PV}\int_{\mathbb{R}}\exp(i\alpha B)\,\frac{d\alpha}{\alpha}\right)\\
&= -\int_0^1 ds_1\cdots\int_0^1 ds_{2n}\,(\mathrm{sgn}\,A - \mathrm{sgn}\,B),
\end{aligned}$$

where

$$\begin{aligned}
A &= \sum_{j=1}^{2n-1}(s_j-1)(\xi_j-\xi_{j+1}) + (s_{2n}-1)\xi_{2n} = -\xi_1 + \sum_{j=1}^{2n} s_j\xi_j - \sum_{j=1}^{2n-1} s_j\xi_{j+1},\\
B &= -(\xi-\xi_1) + \sum_{j=1}^{2n-1}(s_j-1)(\xi_j-\xi_{j+1}) + (s_{2n}-1)\xi_{2n}\\
&= -\xi + \sum_{j=1}^{2n} s_j\xi_j - \sum_{j=1}^{2n-1} s_j\xi_{j+1}.
\end{aligned}$$

It follows that

$$\mathcal{F}(T)(\xi) = \sum_{n\geq 1} \int_{\mathbb{R}} d\xi_1 \cdots \int_{\mathbb{R}} d\xi_{2n}\, m_n(\xi, \xi_1, \ldots, \xi_{2n})\, (\xi - \xi_1)\hat{f}(\xi - \xi_1)$$
$$\times \Big(\prod_{i=1}^{2n-1} (\xi_i - \xi_{i+1})\hat{f}(\xi_i - \xi_{i+1})\Big)\xi_{2n}\hat{f}(\xi_{2n}),$$

with $|m_n(\xi, \xi_1, \ldots, \xi_{2n})| \leq 2$. We then have

$$\int_{\mathbb{R}} d\xi\, |\xi|\, |\mathcal{F}(T)(\xi)| \leq 2 \sum_{n\geq 1} \int_{\mathbb{R}} d\xi \int_{\mathbb{R}} d\xi_1 \cdots \int_{\mathbb{R}} d\xi_{2n}\, |\xi|\, |\xi - \xi_1|\, |\hat{f}(\xi - \xi_1)|$$
$$\times |\xi_1 - \xi_2|\, |\hat{f}(\xi_1 - \xi_2)| \cdots |\xi_{2n-1} - \xi_{2n}|\, |\hat{f}(\xi_{2n-1} - \xi_{2n})|\, |\xi_{2n}|\, |\hat{f}(\xi_{2n})|.$$

The inequality $|\xi| \leq |\xi - \xi_1| + |\xi_1 - \xi_2| + \cdots + |\xi_{2n-1} - \xi_{2n}| + |\xi_{2n}|$ yields

$$\int_{\mathbb{R}} d\xi\, |\xi|\, |\mathcal{F}(T)(\xi)| \leq 2 \sum_{n\geq 1} (2n+1) \Big(\int_{\mathbb{R}} d\xi\, |\xi|^2 |\hat{f}(\xi)|\Big) \Big(\int_{\mathbb{R}} d\xi\, |\xi|\, |\hat{f}(\xi)|\Big)^{2n},$$

and therefore

$$\int_{\mathbb{R}} d\xi\, |\xi|\, |\mathcal{F}(T)(\xi)| \leq \Big(\int_{\mathbb{R}} d\xi\, |\xi|^2 |\hat{f}(\xi)|\Big) 2 \sum_{n\geq 1} (2n+1) \|f\|_1^{2n}$$
$$\leq \Big(\int_{\mathbb{R}} d\xi\, |\xi|^2 |\hat{f}(\xi)|\Big) \frac{2\|f\|_1^2(3 - \|f\|_1^2)}{(1 - \|f\|_1^2)^2}.$$

Notice $2x^2(3 - x^2)/(1 - x^2)^2 < 1$ if $0 \leq x < \sqrt{(4 - \sqrt{13})/6} \approx 0.256400964$. If $\|f_0\|_1 < \sqrt{(4 - \sqrt{13})/6}$, then this inequality will continue to hold for some time so that

$$\frac{d}{dt}\|f\|_1(t) \leq 0,$$

and we conclude that $\|f\|_1(t) \leq \|f_0\|_1$ if $\|f_0\|_1 < \sqrt{(4 - \sqrt{13})/6}$.

Now we repeat the argument but with $s > 1$ in (7). Our goal is to obtain

$$\frac{d}{dt}\|f\|_{2+\delta}(t) \leq 0, \qquad 0 < \delta < 1/2. \tag{13}$$

Let us point out that

$$\|f_0\|_{2+\delta} \leq C(\|f_0\|_{L^2} + \|\partial_x^3 f_0\|_{L^2})$$

for $0 < \delta < 1/2$. Using the inequality

$$|\xi|^{2+\delta} \leq (2n+1)^{1+\delta}(|\xi - \xi_1|^{2+\delta} + |\xi_1 - \xi_2|^{2+\delta} + \cdots + |\xi_{2n-1} - \xi_{2n}|^{2+\delta} + |\xi_{2n}|^{2+\delta}),$$

we proceed as before to get

$$\int_{\mathbb{R}} |\xi|^{2+\delta} |\mathcal{F}(T)(\xi)|\, d\xi \leq \int_{\mathbb{R}} |\xi|^{3+\delta} |\hat{f}(\xi)|\, d\xi \; 2 \sum_{n\geq 1} (2n+1)^{2+\delta} \|f\|_1^{2n}.$$

In particular, taking $\|f\|_1$ small enough we find

$$\int_{\mathbb{R}} |\xi|^{2+\delta} |\mathcal{F}(T)(\xi)|\, d\xi \leq \int_{\mathbb{R}} |\xi|^{3+\delta} |\hat{f}(\xi)|\, d\xi,$$

and bound (13) therefore holds.

If $\|f\|_{C^{2,\delta}}$ remains bounded ($0 < \delta < 1$), then from previous work [9], we can deduce global existence in $C([0,T]; H^3(\mathbb{R}))$ for any $T > 0$. Note that the Hölder seminorm

$$|g|_{C^\delta} = \sup_{x,\, y\neq 0} \frac{|g(x+y) - g(x)|}{|y|^\delta}$$

is bounded by

$$\sup_{x,\, y\neq 0} \frac{|g(x+y) - g(x)|}{|y|^\delta} = \sup_{x,\, y\neq 0} \left| \frac{C}{|y|^\delta} \int_{\mathbb{R}} \hat{g}(\xi) e^{ix\xi} (e^{iy\xi} - 1)\, d\xi \right| \leq C \int_{\mathbb{R}} |\xi|^\delta |\hat{g}(\xi)|\, d\xi,$$

and therefore

$$\|f\|_{C^{2,\delta}} \leq C\left(\|f\|_{L^\infty} + \int_{\mathbb{R}} d\xi\, |\xi|\, |\hat{f}(\xi)| + \int_{\mathbb{R}} d\xi\, |\xi|^{2+\delta} |\hat{f}(\xi)| \right).$$

We conclude that the solution can be continued for all time if $\|f_0\|_1$ is initially smaller than a computable constant c_0, and $\|f_0\|_{2+\delta}$ is bounded. The constant c_0 is defined by the condition

$$2 \sum_{n\geq 1} (2n+1)^{2+\delta} c_0^{2n} \leq 1,$$

which has been numerically verified to be no smaller than say $1/5$.

4. Global existence for initial data smaller than 1

We prove now the existence of a weak solution of the system (1) which can be written as follows:

$$f_t = \frac{\rho}{\pi} \partial_x \, \mathrm{PV} \int_{\mathbb{R}} \arctan\left(\frac{f(x) - f(x-\alpha)}{\alpha} \right) d\alpha, \tag{14}$$

where $\rho = (\rho^2 - \rho^1)/2$. We first extend the sense of the contour equation with a weak formulation: for any $\eta \in C_c^\infty([0,T) \times \mathbb{R})$, a weak solution f should satisfy (6). We show here that this is the case if $\|\partial_x f_0\|_{L^\infty} < 1$. Then it follows that $\|f\|_{L^\infty}(t) \leq \|f_0\|_{L^\infty}$ and $\|\partial_x f\|_{L^\infty}(t) \leq \|\partial_x f_0\|_{L^\infty} < 1$ as in [10]. Then the solution is in fact Lipschitz continuous by Morrey's inequality. The main result we prove below is the following:

Theorem 4.1. *Suppose that* $\|f_0\|_{L^\infty} < \infty$ *and* $\|\partial_x f_0\|_{L^\infty} < 1$. *Then there exists a global in time weak solution of* (6) *that satisfies*

$$f \in C([0,T] \times \mathbb{R}) \cap L^\infty([0,T]; W^{1,\infty}(\mathbb{R})).$$

In particular f *is Lipschitz continuous.*

The rest of this section is devoted to the proof of Theorem 4.1. The first step is to prove global in time existence of classical solutions to the regularized model (15) below. This is done in Section 4.2. Prior to that, in Section 4.1 we prove some necessary a priori bounds. Then, in Section 4.3 we explain how to approximate the initial data. Section 4.4 shows how to prove the existence of solutions of (6), subject to the strong convergence established in Section 4.5.

From now on, in the next two subsections we write $f = f^\varepsilon$ for the solution to (15) for the sake of simplicity of the notation. The regularized model is given by

$$f_t(x,t) = -\varepsilon C \Lambda^{1-\varepsilon} f + \varepsilon f_{xx} + \frac{\rho}{\pi} \partial_x \, \mathrm{PV} \int_{\mathbb{R}} d\alpha \arctan(\Delta_\alpha^\varepsilon f(x)). \tag{15}$$

where $C > 0$ is a universal constant fixed below, the operator $\Lambda^{1-\varepsilon} f$ is given by the formula

$$\Lambda^{1-\varepsilon} f(x) = c_1(\varepsilon) \int_{\mathbb{R}} \frac{f(x) - f(x-\alpha)}{|\alpha|^{2-\varepsilon}} d\alpha, \tag{16}$$

with $0 < c_m \le c_1(\varepsilon) \le c_M$ for $0 \le \varepsilon \le 1/4$, and we define

$$\Delta_\alpha^\varepsilon f(x) := \frac{f(x) - f(x-\alpha)}{\phi(\alpha)},$$

with $\phi(\alpha) = \phi^\varepsilon(\alpha) = \alpha/|\alpha|^\varepsilon$ and $\varepsilon > 0$ is small enough. Initially we consider the data $f_0 \in W^{1,\infty}(\mathbb{R})$ with $\|\partial_x f_0\|_{L^\infty(\mathbb{R})} < 1$. We will explain how to approximate this initial data later on in Section 4.3.

4.1. A priori bounds

In this section we show two a priori bounds for the regularized system (15). We prove the following result:

Proposition 4.2. *Let* $f(x,t)$ *be a regular solution of the system* (15). *Then*

$$\|f\|_{L^\infty}(t) \le \|f_0\|_{L^\infty}, \qquad \|\partial_x f\|_{L^\infty}(t) \le \|\partial_x f_0\|_{L^\infty} < 1,$$

for any $t > 0$.

In order to prove the first estimate, we check the evolution of

$$M(t) = \max_x f(x,t) = f(x_t, t).$$

Then M is differentiable for almost every t and

$$M'(t) = f_t(x_t, t) = -\varepsilon C \Lambda^{1-\varepsilon} f(x_t) + \varepsilon f_{xx}(x_t) + \frac{\rho}{\pi} \partial_x \,\mathrm{PV} \int_{\mathbb{R}} \arctan(\Delta_\alpha^\varepsilon f(x_t))\, d\alpha$$

(see [5, 10] for more details regarding the differentiability of $M(t)$). Using formula (16) it is easy to check that the second and the third term above have the correct sign. We have to deal with the third one. Now

$$I(x) = \partial_x \,\mathrm{PV} \int_{\mathbb{R}} \arctan(\Delta_\alpha^\varepsilon f(x))\, d\alpha = \partial_x \,\mathrm{PV} \int_{\mathbb{R}} \arctan(\Delta_{x-\alpha}^\varepsilon f(x))\, d\alpha, \tag{17}$$

and thus

$$\begin{aligned} I(x) = {} & \partial_x f(x) \,\mathrm{PV} \int_{\mathbb{R}} \frac{\frac{1}{\phi(x-\alpha)}}{1 + (\Delta_{x-\alpha}^\varepsilon f(x))^2}\, d\alpha \\ & - (1-\varepsilon) \,\mathrm{PV} \int_{\mathbb{R}} \frac{\frac{f(x)-f(\alpha)}{|x-\alpha|^{2-\varepsilon}}}{1 + (\Delta_{x-\alpha}^\varepsilon f(x))^2}\, d\alpha. \end{aligned} \tag{18}$$

Therefore $I(x_t) \leq 0$ (since $\partial_x f(x_t) = 0$). Then $M'(t) \leq 0$ for a.e. $t \in (0, T]$ and $M(t) \leq M(0)$. Analogously, we check the evolution of

$$m(t) = \min_x f(x, t) = f(x_t, t)$$

and find $m(t) \geq m(0)$.

From (15) and (17) we have

$$f_{xt} = -\varepsilon C \Lambda^{1-\varepsilon} f_x + \varepsilon f_{xxx} + \frac{\rho}{\pi} I_x.$$

Using (18) we rewrite $I(x) = J^1(x) + J^2(x)$ where

$$\begin{aligned} J^1(x) &= \mathrm{PV} \int_{\mathbb{R}} \frac{f_x(x)(x-\alpha) - (f(x) - f(\alpha))}{|x-\alpha|^{2-\varepsilon}} \frac{1}{1 + (\Delta_{x-\alpha}^\varepsilon f(x))^2}\, d\alpha, \\ J^2(x) &= \varepsilon \,\mathrm{PV} \int_{\mathbb{R}} \frac{f(x) - f(x-\alpha)}{|\alpha|^{2-\varepsilon}} \frac{1}{1 + (\Delta_\alpha^\varepsilon f(x))^2}\, d\alpha, \end{aligned}$$

to find

$$\begin{aligned} J_x^1(x) = {} & f_{xx}(x) \,\mathrm{PV} \int_{\mathbb{R}} \frac{1}{\phi(x-\alpha)} \frac{1}{1 + (\Delta_{x-\alpha}^\varepsilon f(x))^2}\, d\alpha \\ & - (2-\varepsilon) \,\mathrm{PV} \int_{\mathbb{R}} \frac{f_x(x) - \frac{f(x)-f(\alpha)}{x-\alpha}}{|x-\alpha|^{2-\varepsilon}} \frac{1}{1 + (\Delta_{x-\alpha}^\varepsilon f(x))^2}\, d\alpha \\ & - \mathrm{PV} \int_{\mathbb{R}} \frac{f_x(x)(x-\alpha) - (f(x) - f(\alpha))}{|x-\alpha|^{2-\varepsilon}} \frac{2\Delta_{x-\alpha}^\varepsilon f(x)}{(1 + (\Delta_{x-\alpha}^\varepsilon f(x))^2)^2} \\ & \qquad \times \frac{f_x(x)(x-\alpha) - (1-\varepsilon)(f(x) - f(\alpha))}{|x-\alpha|^{2-\varepsilon}}\, d\alpha, \end{aligned}$$

and we split further $J_x^1(x) = K^1(x) + K^2(x) + K^3(x) + K^4(x)$ where

$$K^1(x) = f_{xx}(x)\,\mathrm{PV}\int_{\mathbb{R}} \frac{1}{\phi(\alpha)}\,\frac{1}{1+(\Delta_\alpha^\varepsilon f(x))^2}\,d\alpha,$$
$$K^2(x) = \varepsilon\,\mathrm{PV}\int_{\mathbb{R}} \frac{f_x(x)-\frac{f(x)-f(x-\alpha)}{\alpha}}{|\alpha|^{2-\varepsilon}}\,\frac{1}{1+(\Delta_\alpha^\varepsilon f(x))^2}\,d\alpha,$$
$$K^3(x) = -\,\mathrm{PV}\int_{\mathbb{R}} \frac{f_x(x)-\frac{f(x)-f(x-\alpha)}{\alpha}}{|\alpha|^{2-\varepsilon}}\,\frac{2}{1+(\Delta_\alpha^\varepsilon f(x))^2}\,d\alpha,$$
$$K^4(x) = -\,\mathrm{PV}\int_{\mathbb{R}} d\alpha\,\frac{f_x(x)-\frac{f(x)-f(x-\alpha)}{\alpha}}{|\alpha|^{2-\varepsilon}}\,\frac{2\Delta_\alpha^\varepsilon f(x)}{(1+(\Delta_\alpha^\varepsilon f(x))^2)^2} \times (f_x(x)|\alpha|^\varepsilon - (1-\varepsilon)\Delta_\alpha^\varepsilon f(x)).$$

For J^2 it is easy to check that

$$J_x^2(x) = \varepsilon\,\mathrm{PV}\int_{\mathbb{R}} \frac{f_x(x)-f_x(x-\alpha)}{|\alpha|^{2-\varepsilon}}\,\frac{d\alpha}{1+(\Delta_\alpha^\varepsilon f(x))^2} - \varepsilon\,\mathrm{PV}\int_{\mathbb{R}} \frac{f_x(x)-f_x(x-\alpha)}{|\alpha|^{2-\varepsilon}}\,\frac{2(\Delta_\alpha^\varepsilon f(x))^2\,d\alpha}{(1+(\Delta_\alpha^\varepsilon f(x))^2)^2}.$$

Next, as we did before, we consider $M(t) = \max_x f_x = f_x(x_t, t)$. Then $M(t)$ is differentiable (as we explained previously). It follows that

$$M'(t) = f_{xt}(x_t, t) = -\varepsilon C\Lambda^{1-\varepsilon} f_x(x_t) + \varepsilon f_{xxx}(x_t) + \frac{\rho}{\pi} I_x(x_t).$$

Now we claim that if $M(t) < 1$ then $M'(t) \le 0$ for a.e. t. We can conclude analogously for $m(t) = \min_x f_x > -1$, $m'(t) \ge 0$ for a.e. t.

We check that if $M(t) < 1$, then

$$-\varepsilon C\Lambda^{1-\varepsilon} f_x(x_t) + \varepsilon f_{xxx}(x_t) + \frac{\rho}{\pi} I_x(x_t) \le 0.$$

We can use in some cases the following formula for the operator $\Lambda^{1-\varepsilon} f_x$:

$$\Lambda^{1-\varepsilon} f_x(x) = c_2(\varepsilon)\int_{\mathbb{R}} \frac{f_x(x)-\frac{f(x)-f(x-\alpha)}{\alpha}}{|\alpha|^{2-\varepsilon}}\,d\alpha, \tag{19}$$

where $0 < c_m \le c_2(\varepsilon) \le c_M$ for $0 \le \varepsilon \le 1/4$.

We claim that

$$-\varepsilon C\Lambda^{1-\varepsilon} f_x(x_t) + \frac{\rho}{\pi}(K^2(x_t) + J_x^2(x_t)) \le 0.$$

We will show that

$$-\varepsilon\frac{C}{2}\Lambda^{1-\varepsilon} f_x(x_t) + \frac{\rho}{\pi} K^2(x_t) \le 0,$$

using (19) and that

$$-\varepsilon\frac{C}{2}\Lambda^{1-\varepsilon}f_x(x_t)+\frac{\rho}{\pi}J^2(x_t)\leq 0,$$

by (16). In fact

$$\begin{aligned}&-\varepsilon\frac{C}{2}\Lambda^{1-\varepsilon}f_x(x_t)+\frac{\rho}{\pi}K^2(x_t)\\&\qquad=-\varepsilon\,\mathrm{PV}\int_{\mathbb{R}}\frac{f_x(x_t)-\frac{f(x_t)-f(x_t-\alpha)}{\alpha}}{|\alpha|^{2-\varepsilon}}\,\frac{\frac{Cc_2(\varepsilon)}{2}(\Delta_\alpha^\varepsilon f(x_t))^2+\frac{Cc_2(\varepsilon)}{2}-\frac{\rho}{\pi}}{1+(\Delta_\alpha^\varepsilon f(x_t))^2}\,d\alpha.\end{aligned}$$

The mean value theorem gives

$$|f(x)-f(x-\alpha)|/|\alpha|\leq\|f_x\|_{L^\infty}.$$

Thus if we take $C\geq 2\rho/(c_m\pi)$ we obtain the first inequality. Also

$$\begin{aligned}&-\varepsilon\frac{C}{2}\Lambda^{1-\varepsilon}f_x(x_t)+\frac{\rho}{\pi}J^2(x_t)\\&\qquad=-\varepsilon\,\mathrm{PV}\int_{\mathbb{R}}\frac{f_x(x_t)-f_x(x_t-\alpha)}{|\alpha|^{2-\varepsilon}}\,\frac{\frac{Cc_1(\varepsilon)}{2}(\Delta_\alpha^\varepsilon f(x_t))^2+\frac{Cc_1(\varepsilon)}{2}-\frac{\rho}{\pi}}{1+(\Delta_\alpha^\varepsilon f(x_t))^2}\,d\alpha\\&\qquad\quad-\varepsilon\,\mathrm{PV}\int_{\mathbb{R}}\frac{f_x(x_t)-f_x(x_t-\alpha)}{|\alpha|^{2-\varepsilon}}\,\frac{2(\Delta_\alpha^\varepsilon f(x_t))^2\,d\alpha}{(1+(\Delta_\alpha^\varepsilon f(x_t))^2)^2}\leq 0.\end{aligned}$$

Thus the term above has the desired sign.

We find $f_{xxx}(x_t)\leq 0$ and $K^1(x_t)=0$. We still have to deal with K^3 and K^4. Considering $K^3(x_t)+K^4(x_t)$, we realize that if

$$P(\alpha)=2+2(\Delta_\alpha^\varepsilon f(x_t))^2+2(\Delta_\alpha^\varepsilon f(x_t))(f_x(x_t)|\alpha|^\varepsilon-(1-\varepsilon)\Delta_\alpha^\varepsilon f(x_t))\geq 0,$$

then we are done. We rewrite

$$P(\alpha)=2+2\varepsilon(\Delta_\alpha^\varepsilon f(x_t))^2+2(\Delta_\alpha^\varepsilon f(x_t))f_x(x_t)|\alpha|^\varepsilon,$$

and therefore we need

$$|(\Delta_\alpha^\varepsilon f(x_t))f_x(x_t)|\alpha|^\varepsilon|\leq 1.$$

This fact holds if

$$|f|_{C^{1-2\varepsilon}}=\sup_{\alpha\neq 0}\frac{|f(x_t)-f(x_t-\alpha)|}{|\alpha|^{1-2\varepsilon}}<1.$$

Now we will check that if $\|f\|_{L^\infty}\leq\|f_0\|_{L^\infty}$ and $\|f_x\|_{L^\infty}<1$ then $|f|_{C^{1-2\varepsilon}}<1$ for ε small enough uniformly. We replace 2ε by ε without loss of generality. If $\|f_0\|_{L^\infty}=0$ or $\|f_x\|_{L^\infty}=0$ then there is nothing to prove. Otherwise

$$\frac{|f(x_t)-f(x_t-\alpha)|}{|\alpha|^{1-\varepsilon}}\leq\|f_x\|_{L^\infty}\delta^\varepsilon$$

for $0 < |\alpha| \le \delta$, and

$$\frac{|f(x_t) - f(x_t - \alpha)|}{|\alpha|^{1-\varepsilon}} \le 2\frac{\|f_0\|_{L^\infty}}{\delta^{1-\varepsilon}}$$

for $|\alpha| \ge \delta$. We take $\delta^{1-\varepsilon} = 2\|f_0\|_{L^\infty}/\|f_x\|_{L^\infty}$ and therefore

$$|f|_{C^{1-\varepsilon}} \le \max\{\|f_x\|_{L^\infty}, \|f_x\|_{L^\infty}^{1-\varepsilon/(1-\varepsilon)}(2\|f_0\|_{L^\infty})^{\varepsilon/(1-\varepsilon)}\}.$$

Now it is clear that given $\|f_0\|_{L^\infty}$, if $\|f_x\|_{L^\infty} < 1$ there exists $\varepsilon_0 > 0$ such that $|f|_{C^{1-\varepsilon}} \le 1$ for any $0 \le \varepsilon \le \varepsilon_0$.

4.2. Global existence for the regularized model

In this section we use the a priori bounds to show global existence. Local existence can be easily proved using the local existence proof for the non-regularized Muskat problem (1), as in [9]. We use energy estimates and the Gronwall inequality. We have the following result.

Proposition 4.3. *Let $f(x,t)$ be a regular solution of the system* (15). *Then*

$$\|f\|_{L^2}^2(t) + \|\partial_x^3 f\|_{L^2}^2(t) \le (\|f_0\|_{L^2}^2 + \|\partial_x^3 f_0\|_{L^2}^2)\exp\left(\int_0^t C(\varepsilon)G(s)\,ds\right) \tag{20}$$

for

$$G(s) = \|f\|_{L^\infty}^4 + \|f\|_{L^\infty}^2 + \|f_x\|_{L^\infty}^4(s) + \|f_x\|_{L^\infty}^2(s) + 1$$

and any $t > 0$.

Estimate (20) allows us to find $f \in C([0,T]; H^3(\mathbb{R}))$ for any $T > 0$ by the a priori bounds.

Furthermore, as we did for (1), it follows that

$$\begin{aligned}\frac{d}{dt}\|f\|_{L^2}^2(t) = &-\frac{\rho}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1-\varepsilon}{|x-\alpha|^\varepsilon}\ln\left(1 + \left(\frac{f(x,t) - f(\alpha,t)}{\phi(x-\alpha)}\right)^2\right)dx\,d\alpha \\ &- 2C\varepsilon\|\Lambda^{(1-\varepsilon)/2} f\|_{L^2}^2(t) - 2\varepsilon\|f_x\|_{L^2}^2(t).\end{aligned}$$

Therefore $\|f\|_{L^2}(t) \le \|f_0\|_{L^2}$.

Remark 4.4. The above maximum principle for the regularized system shows that Theorem 4.1 can also be proved with

$$\|f_0\|_{L^2} < \infty \quad \text{instead of} \quad \|f_0\|_{L^\infty} < \infty.$$

We picked the version above because it is more suitable. We see that if the solution satisfies initially an L^2 bound then $f \in L^\infty([0,T]; L^2(\mathbb{R}))$.

Next, we consider the evolution of

$$\int_{\mathbb{R}} \partial_x^3 f \partial_x^3 f_t \, dx \leq -C\varepsilon \|\Lambda^{(1-\varepsilon)/2} \partial_x^3 f\|_{L^2}(t) - \varepsilon \|\partial_x^4 f_x\|_{L^2}^2 + L_1 + L_2,$$

where

$$L_1 = \frac{\rho}{\pi} \int_{\mathbb{R}} \partial_x^3 f(x) \partial_x^3 \left(\mathrm{PV} \int_{\mathbb{R}} \frac{f_x(x) - f_x(x-\alpha)}{\phi(\alpha)} \, d\alpha \right) dx,$$
$$L_2 = -\frac{\rho}{\pi} \int_{\mathbb{R}} \partial_x^3 f(x) \partial_x^3 \left(\mathrm{PV} \int_{\mathbb{R}} \frac{f_x(x) - f_x(x-\alpha)}{\phi(\alpha)} \frac{(\Delta_\alpha^\varepsilon f(x))^2 \, d\alpha}{1 + (\Delta_\alpha^\varepsilon f(x))^2} \right) dx.$$

The term $f_x(x)$ cancels out in L_1 due to the PV. An integration by parts further shows that

$$L_1 = -\frac{\rho}{\pi} C(\varepsilon) \int_{\mathbb{R}} \partial_x^3 f(x) \Lambda^{1-\varepsilon} \partial_x^3 f(x) \, dx \leq 0.$$

For L_2 one finds

$$L_2 = \frac{\rho}{\pi} \int_{\mathbb{R}} \partial_x^4 f(x) \partial_x^2 \left(\mathrm{PV} \int_{\mathbb{R}} \frac{f_x(x) - f_x(x-\alpha)}{\phi(\alpha)} \frac{(\Delta_\alpha^\varepsilon f(x))^2 \, d\alpha}{1 + (\Delta_\alpha^\varepsilon f(x))^2} \right) dx,$$

which splits as $L_2 = M_1 + M_2 + M_3$ with

$$M_1 = \frac{\rho}{\pi} \int_{\mathbb{R}} \partial_x^4 f(x) \int_{\mathbb{R}} \frac{\partial_x^3 f(x) - \partial_x^3 f(x-\alpha)}{\phi(\alpha)} \frac{(\Delta_\alpha^\varepsilon f(x))^2 \, d\alpha}{1 + (\Delta_\alpha^\varepsilon f(x))^2} \, dx,$$
$$M_2 = \frac{3\rho}{\pi} \int_{\mathbb{R}} \partial_x^4 f(x) \int_{\mathbb{R}} \frac{\partial_x^2 f(x) - \partial_x^2 f(x-\alpha)}{\phi(\alpha)} \frac{f_x(x) - f_x(x-\alpha)}{\phi(\alpha)} \times \frac{2(\Delta_\alpha^\varepsilon f(x)) \, d\alpha}{(1 + (\Delta_\alpha^\varepsilon f(x))^2)^2} \, dx,$$
$$M_3 = \frac{\rho}{\pi} \int_{\mathbb{R}} \partial_x^4 f(x) \int_{\mathbb{R}} \left(\frac{f_x(x) - f_x(x-\alpha)}{\phi(\alpha)} \right)^3 \frac{(2 - 6(\Delta_\alpha^\varepsilon f(x))^2) \, d\alpha}{(1 + (\Delta_\alpha^\varepsilon f(x))^2)^3} \, dx.$$

We will now estimate each of these terms from above.

For M_1 we proceed as follows:

$$|M_1| = \frac{\rho}{\pi} \left(\int_{|\alpha|>1} d\alpha \int_{\mathbb{R}} dx + \int_{|\alpha|<1} d\alpha \int_{\mathbb{R}} dx \right) \leq C(\varepsilon)(\|f\|_{L^\infty} + 1) \|\partial_x^3 f\|_{L^2} \|\partial_x^4 f\|_{L^2}.$$

The identity

$$\partial_x^2 f(x) - \partial_x^2 f(x-\alpha) = \int_0^1 \partial_x^3 f(x + (s-1)\alpha) \alpha \, ds$$

yields

$$
\begin{aligned}
|M_2| \le \frac{6\rho}{\pi}\int_0^1 ds \int_{|\alpha|<1} \frac{d\alpha}{|\alpha|^{1-2\varepsilon}} \int_{\mathbb{R}} dx\, |\partial_x^4 f(x)|\, |\partial_x^3 f(x+(s-1)\alpha)| \\
\times (|f_x(x)| + |f_x(x-\alpha)|) \\
+ \frac{6\rho}{\pi}\int_0^1 ds \int_{|\alpha|>1} \frac{d\alpha}{|\alpha|^{2-3\varepsilon}} \int_{\mathbb{R}} dx\, |\partial_x^4 f(x)|\, |\partial_x^3 f(x+(s-1)\alpha)| \\
\times (|f_x(x)| + |f_x(x-\alpha)|)(|f(x)| + |f(x-\alpha)|),
\end{aligned}
$$

and therefore

$$|M_2| \le C(\varepsilon)(1 + \|f\|_{L^\infty})\|\partial_x^3 f\|_{L^2}\|\partial_x^4 f\|_{L^2}\|f_x\|_{L^\infty}.$$

In M_3 we use the splitting $M_3 = N_1 + N_2$ where

$$N_1 = \frac{\rho}{\pi}\int_{|\alpha|>1} d\alpha \int_{\mathbb{R}} dx, \qquad N_2 = \int_{|\alpha|<1} d\alpha \int_{\mathbb{R}} dx,$$

and then

$$
\begin{aligned}
|N_1| &\le \frac{16\rho}{\pi}\|f_x\|^2_{L^\infty} \int_{|\alpha|>1} \frac{d\alpha}{|\alpha|^{3-3\varepsilon}} \int_{\mathbb{R}} dx\, |\partial_x^4 f(x)|(|f_x(x)| + |f_x(x-\alpha)|) \\
&\le C\|f_x\|^2_{L^\infty}\|f_x\|_{L^2}\|\partial_x^4 f\|_{L^2} \le C\|f_x\|^2_{L^\infty}(\|f\|_{L^2} + \|\partial_x^3 f\|_{L^2})\|\partial_x^4 f\|_{L^2}.
\end{aligned}
$$

To finish, the equality

$$f_x(x) - f_x(x-\alpha) = \int_0^1 \partial_x^2 f(x+(s-1)\alpha)\alpha\, ds$$

allows us to obtain (since $1/2 + 1/4 + 1/4 = 1$)

$$
\begin{aligned}
|N_2| &\le \frac{16\rho}{\pi}\|f_x\|_{L^\infty}\int_0^1 dr \int_0^1 ds \int_{|\alpha|<1} \frac{d\alpha}{|\alpha|^{1-3\varepsilon}} \int_{\mathbb{R}} dx \\
&\qquad \times |\partial_x^4 f(x)|\, |\partial_x^2 f(x+(r-1)\alpha)|\, |\partial_x^2 f(x+(s-1)\alpha)| \\
&\le C\|f_x\|_{L^\infty}\|\partial_x^4 f\|_{L^2}\|\partial_x^2 f\|^2_{L^4}.
\end{aligned}
$$

The estimate

$$\|\partial_x^2 f\|^4_{L^4} = \int_{\mathbb{R}} (\partial_x^2 f)^3 \partial_x^2 f\, dx = -3\int_{\mathbb{R}} (\partial_x^2 f)^2 \partial_x^3 f\, \partial_x f\, dx \le 3\|f_x\|_{L^\infty}\|\partial_x^2 f\|^2_{L^4}\|\partial_x^3 f\|_{L^2}$$

yields

$$|N_2| \le C\|f_x\|^2_{L^\infty}\|\partial_x^4 f\|_{L^2}\|\partial_x^3 f\|_{L^2}.$$

Using Young's inequality we obtain

$$\frac{d}{dt}\|\partial_x^3 f\|^2_{L^2} \le C(\varepsilon)(\|f\|^4_{L^\infty} + \|f\|^2_{L^\infty} + \|f_x\|^4_{L^\infty} + \|f_x\|^2_{L^\infty} + 1)(\|f\|^2_{L^2} + \|\partial_x^3 f\|^2_{L^2}),$$

and therefore the Gronwall inequality yields (20).

4.3. Approximation of the initial data

The approximation of the initial data described below is needed in order to construct a weak solution. First consider a common approximation to the identity $\zeta \in C_c^\infty(\mathbb{R})$ satisfying

$$\int_{\mathbb{R}} dx\, \zeta(x) = 1, \quad \zeta \geq 0, \quad \zeta(x) = \zeta(-x).$$

Now the standard mollifier $\zeta_\varepsilon(x) = \zeta(x/\varepsilon)/\varepsilon$ continues to satisfy the normalization condition above.

For any $f_0 \in W^{1,\infty}(\mathbb{R})$ with $\|\partial_x f_0\|_{L^\infty} < 1$, we define the initial data for the regularized system as follows:

$$f_0^\varepsilon(x) = \frac{(\zeta_\varepsilon * f_0)(x)}{1+\varepsilon x^2}.$$

Notice that $f_0^\varepsilon \in H^s(\mathbb{R})$ for any $s > 0$, and

$$\|f_0^\varepsilon\|_{L^\infty} \leq \|f_0\|_{L^\infty}.$$

More importantly, $\|\partial_x f_0^\varepsilon\|_{L^\infty} < 1$ when $\|\partial_x f_0\|_{L^\infty} < 1$ if ε is sufficiently small (here ε will generally depend upon the size of $\|f_0\|_{L^\infty}$).

In particular

$$\partial_x f_0^\varepsilon(x) = \frac{(\zeta_\varepsilon * \partial_x f_0)(x)}{1+\varepsilon x^2} - 2\varepsilon x \frac{(\zeta_\varepsilon * f_0)(x)}{(1+\varepsilon x^2)^2},$$

and clearly

$$\left| \frac{(\zeta_\varepsilon * \partial_x f_0)(x)}{1+\varepsilon x^2} \right| \leq \|(\zeta_\varepsilon * \partial_x f_0)\|_{L^\infty(\mathbb{R})} \leq \|\partial_x f_0\|_{L^\infty(\mathbb{R})}.$$

On the other hand, by splitting into $|x| \leq \varepsilon^{-2/3}$ and $|x| > \varepsilon^{-2/3}$ we have

$$2\varepsilon x(1+\varepsilon x^2)^{-2} \leq 2\max\{\varepsilon^{1/3}, \varepsilon\}.$$

On the unbounded region we have

$$x(1+\varepsilon x^2)^{-2} = (1/\sqrt{x} + \varepsilon x^{3/2})^{-2} \leq 1.$$

Thus, the desired bound follows if ε is small enough. Therefore global existence of the regularized system (15) holds for f_0^ε if ε is small enough.

Now consider the solution to the regularized system (15) with initial data given by the f_0^ε just described above. For $\varepsilon > 0$ sufficiently small, we decompose

$$\int_{\mathbb{R}} \eta(x,0) f_0^\varepsilon(x)\, dx = \int_{\mathbb{R}} \eta(x,0) \frac{(\zeta_\varepsilon * f_0)(x)}{1+\varepsilon x^2}\, dx = I_1^\varepsilon + I_2^\varepsilon,$$

where

$$I_1^\varepsilon = \int_{\mathbb{R}} \eta(x,0)(\zeta_\varepsilon * f_0)(x) \left(\frac{1}{1+\varepsilon x^2} - 1 \right) dx, \quad I_2^\varepsilon = \int_{\mathbb{R}} \eta(x,0)(\zeta_\varepsilon * f_0)(x)\, dx.$$

We apply the dominated convergence theorem to find that $I_1^\varepsilon \to 0$ as $\varepsilon \downarrow 0$. For I_2^ε we write

$$I_2^\varepsilon = \int_{\mathbb{R}} \zeta_\varepsilon * (\eta(\cdot, 0)) f_0(x)\, dx.$$

The L^1 approximation of the identity property shows that

$$I_2^\varepsilon \to \int_{\mathbb{R}} \eta(x, 0) f_0(x)\, dx.$$

Thus, it remains to check the convergence of the rest of the terms in (6).

4.4. *Weak solution*

In this section we prove that solutions of the regularized system converge to a weak solution satisfying the bounds

$$\|f\|_{L^\infty}(t) \le \|f_0\|_{L^\infty}, \qquad \|\partial_x f\|_{L^\infty}(t) \le \|\partial_x f_0\|_{L^\infty} < 1. \tag{21}$$

Given a collection of regularized solutions $\{f^\varepsilon\}$ to (15), we have the uniform (in $\varepsilon > 0$) bounds

$$\|f^\varepsilon\|_{L^\infty}(t) \le \|f_0\|_{L^\infty}, \qquad \|\partial_x f^\varepsilon\|_{L^\infty(\mathbb{R})}(t) \le 1, \qquad \varepsilon > 0. \tag{22}$$

This implies that there is a subsequence (denoted again by f^ε) such that

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}} f^\varepsilon(x,t) g(x,t)\, dx\, dt &\to \int_0^T \int_{\mathbb{R}} f(x,t) g(x,t)\, dx\, dt,\\
\int_0^T \int_{\mathbb{R}} \partial_x f^\varepsilon(x,t) g(x,t)\, dx\, dt &\to \int_0^T \int_{\mathbb{R}} \partial_x f(x,t) g(x,t)\, dx\, dt,
\end{aligned}$$

for $f \in L^\infty([0,T]; W^{1,\infty}(\mathbb{R}))$ and any $g \in L^1([0,T] \times \mathbb{R})$ by the Banach–Alaoglu theorem. This yields weak* convergence in $L^\infty([0,T]; W^{1,\infty}(\mathbb{R}))$.

We denote $B_N = [-N, N]$. Then we *claim* that there is a subsequence (denoted again by f^ε) such that

$$\|f^\varepsilon - f\|_{L^\infty([0,T]\times B_N)} \to 0 \quad \text{as } \varepsilon \to 0.$$

We will prove this in Section 4.5. Then, up to a subsequence, we infer uniform convergence of f^ε to f on compact sets. Since $f^\varepsilon \in C([0,T] \times \mathbb{R})$ we find that f is continuous.

The only thing remaining to check is that as $\varepsilon \downarrow 0$ we have

$$\begin{aligned}
&\int_0^T dt \int_{\mathbb{R}} dx\, \eta_x(x,t) \frac{\rho}{\pi} \,\mathrm{PV} \int_{\mathbb{R}} d\alpha \arctan\left(\frac{f^\varepsilon(x) - f^\varepsilon(x-\alpha)}{\phi^\varepsilon(\alpha)} \right)\\
&\qquad \to \int_0^T dt \int_{\mathbb{R}} dx\, \eta_x(x,t) \frac{\rho}{\pi} \,\mathrm{PV} \int_{\mathbb{R}} d\alpha \arctan\left(\frac{f(x) - f(x-\alpha)}{\alpha} \right),
\end{aligned}$$

where $\phi^\varepsilon(\alpha) = \alpha / |\alpha|^\varepsilon$. The other terms will converge in the usual obvious way (since they are linear).

Choose $M > 0$ so that $\operatorname{supp}(\eta) \subseteq B_M$. For any small $\delta > 0$ and any large $R \gg 1$ with $R > M + 1$, we split the integral as

$$\int_{\mathbb{R}} d\alpha = \int_{B_\delta} d\alpha + \int_{B_R - B_\delta} d\alpha + \int_{B_R^c} d\alpha.$$

We first prove that the first and last integrals separately are arbitrarily small independent of ε for $R > 0$ sufficiently large and for $\delta > 0$ sufficiently small. One finds that

$$\left| \arctan\left(\frac{f^\varepsilon(x) - f^\varepsilon(x-\alpha)}{\phi^\varepsilon(\alpha)} \right) \right| \leq \frac{\pi}{2}.$$

Here we do not need any regularity for f^ε, and we conclude that

$$\left| \int_0^T dt \int_{\mathbb{R}} dx\, \eta_x(x,t) \frac{\rho}{\pi} \operatorname{PV} \int_{B_\delta} d\alpha \arctan\left(\frac{f^\varepsilon(x) - f^\varepsilon(x-\alpha)}{\phi^\varepsilon(\alpha)} \right) \right| \leq \rho \|\eta_x\|_{L^1([0,T]\times\mathbb{R})} \delta.$$

Therefore this term can clearly be made arbitrarily small, depending upon the smallness of δ.

We now estimate the term integrated over B_R^c. We note that

$$\arctan y = \int_0^1 \frac{d}{ds}(\arctan(sy))\, ds = y \int_0^1 \frac{1}{1+s^2y^2}\, ds,$$

and therefore

$$\arctan y = y\left(1 - \int_0^1 \frac{s^2y^2}{1+s^2y^2}\, ds \right).$$

This is morally the first order Taylor expansion for arctan with remainder in integral form. From this expression we have

$$\begin{aligned} &\operatorname{PV} \int_{B_R^c} d\alpha \arctan\left(\frac{f^\varepsilon(x) - f^\varepsilon(x-\alpha)}{\phi^\varepsilon(\alpha)} \right) \\ &\quad = -H_R^\varepsilon(f^\varepsilon) - \operatorname{PV} \int_{B_R^c} d\alpha \left(\frac{f^\varepsilon(x) - f^\varepsilon(x-\alpha)}{\phi^\varepsilon(\alpha)} \right)^3 \int_0^1 \frac{s^2\, ds}{1 + s^2\left(\frac{f^\varepsilon(x) - f^\varepsilon(x-\alpha)}{\phi^\varepsilon(\alpha)} \right)^2}. \end{aligned}$$

Here H_R^ε is a (Hilbert-type) transform which has the form

$$H_R^\varepsilon(f^\varepsilon) := \operatorname{PV} \int_{B_R^c} d\alpha\, \frac{f^\varepsilon(x-\alpha)}{\phi^\varepsilon(\alpha)}.$$

The principal value is evaluated at infinity (if necessary). For the second term on the left hand side notice that the integral is over B_R^c and the principal value is not necessary. In particular, we have

$$\left| \int_{B_R^c} d\alpha \int_0^1 ds \right| \leq C \|f^\varepsilon\|_{L^\infty}^3 \int_R^\infty \frac{d\alpha}{\alpha^{3-3\varepsilon}} \leq \frac{C\|f_0\|_\infty^3}{R}.$$

This term is therefore arbitrarily small if R is chosen sufficiently large. We are going to show the same for

$$I_R := \int_{-M}^{M} dx\, \eta_x(x,t) H_R^{\varepsilon}(f^{\varepsilon}).$$

We write $I_R = J_R + K_R$ where

$$J_R := \lim_{n\to\infty} \int_{-M}^{M} dx\, \eta_x(x,t) \int_{-n}^{-R} d\alpha\, \frac{f^{\varepsilon}(x-\alpha)}{\phi^{\varepsilon}(\alpha)},$$
$$K_R := \lim_{n\to\infty} \int_{-M}^{M} dx\, \eta_x(x,t) \int_{R}^{n} d\alpha\, \frac{f^{\varepsilon}(x-\alpha)}{\phi^{\varepsilon}(\alpha)}.$$

We shall show how to control J_R; the same follows for K_R. We write

$$J_R = \lim_{n\to\infty} \int_{-M}^{M} dx\, \eta_x(x,t) \int_{x+R}^{n} d\alpha\, \frac{f^{\varepsilon}(\alpha)}{\phi^{\varepsilon}(x-\alpha)}.$$

An integration by parts yields

$$J_R = \lim_{n\to\infty} \int_{-M}^{M} dx\, \eta(x,t) \left(\frac{f(x+R)|R|^{\varepsilon}}{-R} + (1-\varepsilon) \int_{x+R}^{n} d\alpha\, \frac{f^{\varepsilon}(\alpha)}{|x-\alpha|^{2-\varepsilon}} \right).$$

Then for $\epsilon \in (0, 1/2)$ we have

$$|J_R| \le 2\|\eta\|_{L^1} \|f_0\|_{L^\infty} / R^{1/2}.$$

Since the same estimate holds for $|K_R|$, one finds that I_R is arbitrarily small if R is arbitrarily large.

It remains to prove the convergence of

$$\int_0^T \int_{B_R - B_\delta} d\alpha\, \eta_x(x,t) \arctan\left(\frac{f^{\varepsilon}(x) - f^{\varepsilon}(x-\alpha)}{\phi^{\varepsilon}(\alpha)} \right).$$

Recall that we have uniform convergence on compact sets. Let us consider

$$G^{\varepsilon} = \frac{f^{\varepsilon}(x) - f^{\varepsilon}(x-\alpha)}{\phi^{\varepsilon}(\alpha)},$$

where $x \in B_M$ and $\alpha \in B_R - B_\delta$. Since arctan is a continuous function, we have $\arctan(G^{\varepsilon}) \to \arctan(G^0)$ uniformly. Thus the integral of $\arctan(G^{\varepsilon})$ over a bounded region also converges. Hence for any $R > M + 1$ and any small $\delta > 0$ as $\varepsilon \downarrow 0$ we have

$$\int_0^T dt \int_{\mathbb{R}} dx\, \eta_x(x,t) \frac{\rho}{\pi} \int_{B_R - B_\delta} d\alpha\, \arctan\left(\frac{f^{\varepsilon}(x) - f^{\varepsilon}(x-\alpha)}{\phi^{\varepsilon}(\alpha)} \right)$$
$$\to \int_0^T dt \int_{\mathbb{R}} dx\, \eta_x(x,t) \frac{\rho}{\pi} \int_{B_R - B_\delta} d\alpha\, \arctan\left(\frac{f(x) - f(x-\alpha)}{\alpha} \right).$$

We conclude by first choosing R sufficiently large and $\delta > 0$ sufficiently small and then sending $\varepsilon \downarrow 0$. Note that R and δ will generally depend upon the size of $\|f_0\|_\infty$, but this does not affect our argument.

4.5. Strong convergence in $L^\infty([0,T]; L^\infty(B_R))$

In order to prove the strong convergence in $L^\infty([0,T]; L^\infty(B_R))$, the idea is to use the non-standard weak space $W_*^{-2,\infty}(B_R)$ which will be defined below. Crucially, we will have the uniform bounds:

$$\begin{aligned}\sup_{t\in[0,T]} \|f^\varepsilon(t)\|_{W^{1,\infty}(B_R)} &\le C\|f_0\|_{W^{1,\infty}(B_R)},\\ \sup_{t\in[0,T]} \left\|\frac{\partial f^\varepsilon}{\partial t}(t)\right\|_{W_*^{-2,\infty}(B_R)} &\le C\|f_0\|_{L^\infty(\mathbb{R})},\end{aligned} \tag{23}$$

where C does not depend on R or ε. From this we will conclude that for any finite $R > 0$ there exists a subsequence such that $f^\varepsilon \to f$ strongly in $L^\infty([0,T]; L^\infty(B_R))$.

We define the space $W_*^{-2,\infty}(B_R)$ as follows. For $v \in L^\infty(B_R)$ we consider the norm

$$\|v\|_{-2,\infty} = \sup_{\phi\in W_0^{2,1}(B_R):\, \|\phi\|_{2,1}\le 1} \left|\int_{B_R} \phi(x)v(x)\,dx\right|.$$

Here $W_0^{2,1}(B_R)$ is the usual set of functions in $W^{2,1}(B_R)$ which vanish on the boundary of B_R together with their first two weak derivatives. Now the Banach space $W_*^{-2,\infty}(B_R)$ is defined to be the completion of $L^\infty(B_R)$ with respect to the norm $\|\cdot\|_{-2,\infty}$. In general this is all we need for our convergence study. This will be explained after the proof of Lemma 4.5 below. The full space $W_*^{-2,\infty}$ may be large, but because we are going to deal with f^ε and df^ε/dt both in $L^\infty([0,T]; L^\infty(B_R))$, it is not difficult to find the norms of both functions in $W_*^{-2,\infty}(B_R)$.

These spaces are suitable because

$$W^{1,\infty}(B_R) \subset L^\infty(B_R) \subset W_*^{-2,\infty}(B_R).$$

Now the embedding $L^\infty(B_R) \subset W_*^{-2,\infty}(B_R)$ is continuous, and the embedding $W^{1,\infty}(B_R) \subset L^\infty(B_R)$ is compact by the Arzelà–Ascoli theorem.

We now proceed to discuss the convergence argument. Arguments related to Lemma 4.5 below are described for instance in [6]. However in [6] reflexive Banach spaces are used. None of the spaces used here are reflexive.

Lemma 4.5. *Consider a sequence $\{u_m\}$ in $C([0,T]\times B_R)$ that is uniformly bounded in the space $L^\infty([0,T]; W^{1,\infty}(B_R))$. Assume further that the weak derivative du_m/dt is in $L^\infty([0,T]; L^\infty(B_R))$ (not necessarily uniformly) and is uniformly bounded in $L^\infty([0,T]; W_*^{-2,\infty}(B_R))$. Finally suppose that $\partial_x u_m \in C([0,T]\times B_R)$. Then there exists a subsequence of u_m that converges strongly in $L^\infty([0,T]; L^\infty(B_R))$.*

Proof. Notice that it is enough to prove that the convergence is strong in the space $L^\infty([0,T]; W_*^{-2,\infty}(B_R))$ because of the following interpolation theorem: for any small $\eta > 0$ there exists $C_\eta > 0$ such that

$$\|u\|_{L^\infty} \le \eta\|u\|_{1,\infty} + C_\eta\|u\|_{-2,\infty}.$$

This holds for all $u \in W^{1,\infty}(B_R)$. See, for example, [6, Lemma 8.3]. Here we can replace reflexivity with the Banach–Alaoglu theorem in $W^{1,\infty}(B_R)$.

Let $t, s \in [0, T]$ be arbitrary. We have

$$u_m(t) - u_m(s) = \int_s^t d\tau \, \frac{\partial u_m}{\partial \tau}(\tau).$$

This holds rigorously in the sense that

$$\int_{B_R} u_m(t)\phi \, dx - \int_{B_R} u_m(s)\phi \, dx = \int_s^t d\tau \int_{B_R} \frac{\partial u_m}{\partial \tau}(\tau)\phi \, dx \tag{24}$$

for any $\phi \in W^{2,1}(B_R)$. Clearly,

$$\|u_m(t) - u_m(s)\|_{W_*^{-2,\infty}(B_R)} \le \sup_{\tau \in [0,T]} \left\| \frac{\partial u_m}{\partial \tau}(\tau) \right\|_{W_*^{-2,\infty}(B_R)} |t - s|,$$

and therefore

$$\|u_m(t) - u_m(s)\|_{W_*^{-2,\infty}(B_R)} \le L|t - s|, \tag{25}$$

where

$$L = \sup_{m \in \mathbb{N}} \sup_{\tau \in [0,T]} \left\| \frac{\partial u_m}{\partial \tau}(\tau) \right\|_{W_*^{-2,\infty}(B_R)}.$$

Now we consider $\{t_k\}_{k \in \mathbb{N}} = [0, T] \cap \mathbb{Q}$. We have $u_m(t_k) \in W^{1,\infty}(B_R)$ for any m and k. By the standard diagonalization argument, we can get a subsequence (still denoted by m) such that

$$u_m(t_k) \to u(t_k)$$

in $L^\infty(B_R)$ for any k as in the Arzelà–Ascoli theorem.

Consider $\epsilon > 0$. Since $[0, T]$ is compact, there exists $J \in \mathbb{N}$ such that

$$[0, T] \subset \bigcup_{j=1}^{J} \left(t_{k_j} - \frac{\epsilon}{6L}, t_{k_j} + \frac{\epsilon}{6L} \right).$$

Then there exists N_j such that for all $m_1, m_2 \ge N_j$,

$$\|u_{m_1}(t_{k_j}) - u_{m_2}(t_{k_j})\|_{W_*^{-2,\infty}(B_R)} < \epsilon/3.$$

Taking $N = \max_{j=1,\dots,J} N_j$, it is easy to check that for all $m_1, m_2 \ge N$,

$$\sup_{t \in [0,T]} \|u_{m_1}(t) - u_{m_2}(t)\|_{W_*^{-2,\infty}(B_R)} < \epsilon.$$

We find that the sequence is uniformly Cauchy in $L^\infty([0, T]; W_*^{-2,\infty}(B_R))$, and it converges strongly to an element of $L^\infty([0, T]; W_*^{-2,\infty}(B_R))$. □

Now we apply Lemma 4.5 to prove the strong convergence which was *claimed* in Section 4.4. It remains to prove that for any solution f^ε to (15) we have $\partial f^\varepsilon / \partial t \in L^\infty([0, T]; L^\infty(B_R))$ (but not uniformly) and that the second inequality in (23) holds for all sufficiently small $\varepsilon > 0$ and for any $R > 0$.

Recall that $f^\varepsilon \in C([0,T]; H^3(\mathbb{R}))$. Then in (15) the first two linear terms are bounded easily. The last term can be written as

$$NL = -\widetilde{C}(\varepsilon)\Lambda^{1-\varepsilon} f^\varepsilon - \int_{\mathbb{R}} \frac{f_x^\varepsilon(x) - f_x^\varepsilon(x-\alpha)}{\phi^\varepsilon(\alpha)} \frac{(\Delta_\alpha^\varepsilon f^\varepsilon(x))^2}{1+(\Delta_\alpha^\varepsilon f^\varepsilon(x))^2}\, d\alpha$$

for $\widetilde{C}(\varepsilon)$ a constant, and therefore

$$|NL(x,t)| \le C(\varepsilon)\|f^\varepsilon\|_{H^3}(t),$$

by Sobolev embedding.

The norm of $\partial f^\varepsilon/\partial t \in W_*^{-2,\infty}(B_R)$ is given by

$$\left\|\frac{\partial f^\varepsilon}{\partial t}(t)\right\|_{W_*^{-2,\infty}(B_R)} = \sup_{\phi\in W_0^{2,1}(B_R):\, \|\phi\|_{W^{2,1}}\le 1} \left|\int_{\mathbb{R}} dx\, \frac{\partial f^\varepsilon}{\partial t}(x,t)\phi(x)\right|.$$

Since ϕ vanishes on the boundary of B_R, we can think of $\phi(x)$ as being zero outside of the ball of radius R. Then we are allowed to integrate over the whole space $\mathbb{R}$, which is important because we want to estimate the non-local operator $\Lambda^{1-\varepsilon}$ in this norm via integration by parts. Then we have

$$I = \int_{B_R} \Lambda^{1-\varepsilon} f(x)\phi(x)\, dx = \int_{\mathbb{R}} \Lambda^{1-\varepsilon} f(x)\phi(x)\, dx = \int_{\mathbb{R}} f(x)\Lambda^{1-\varepsilon}\phi(x)\, dx,$$

and therefore

$$|I| \le \|f\|_{L^\infty}(t)\|\Lambda^{1-\varepsilon}\phi\|_{L^1}.$$

We compute

$$\Lambda^{1-\varepsilon}\phi(x) = c\int_{\mathbb{R}} \frac{\phi(x)-\phi(x-\alpha)}{|\alpha|^{2-\varepsilon}}\, d\alpha = \int_{|\alpha|>1} d\alpha + \int_{|\alpha|<1} d\alpha = J_1(x) + J_2(x),$$

thus

$$\int_{\mathbb{R}} |J_1(x)|\, dx \le \int_{|\alpha|>1} \frac{d\alpha}{|\alpha|^{2-\varepsilon}} \int_{\mathbb{R}} dx\, (|\phi(x)| + |\phi(x-\alpha)|) \le C\|\phi\|_{L^1(B_R)}.$$

It is easy to rewrite J_2 as follows:

$$J_2(x) = c\int_{|\alpha|<1} \frac{\phi(x)-\phi(x-\alpha)-\phi_x(x)\alpha}{|\alpha|^{2-\varepsilon}}\, d\alpha,$$

and therefore the identities

$$\phi(x)-\phi(x-\alpha) = \alpha\int_0^1 \phi_x(x+(s-1)\alpha)\, ds,$$

$$\phi(x)-\phi(x-\alpha)-\phi_x(x)\alpha = \alpha^2\int_0^1 (s-1)\, ds\int_0^1 dr\, \phi_{xx}(x+r(s-1)\alpha)$$

allow us to find

$$\int_{\mathbb{R}} |J_1(x)|\,dx \le \int_{|\alpha|<1}\int_0^1 ds \int_0^1 dr \int_{\mathbb{R}} dx\, |\phi_{xx}(x + r(s-1)\alpha)| \le 2\|\phi_{xx}\|_{L^1(B_R)}.$$

Now it follows from the a priori bounds that

$$\|\Lambda^{1-\varepsilon} f^\varepsilon\|_{W_*^{-2,\infty}(B_R)} + \|f^\varepsilon_{xx}\|_{W_*^{-2,\infty}(B_R)} \le C\|f^\varepsilon\|_{L^\infty(\mathbb{R})} \le C\|f_0\|_{L^\infty(\mathbb{R})}.$$

Here the constant is independent of ε and $R > 0$.

For the last term in (15), integrating by parts in the definition of the norm, we are led to estimate

$$\int_{\mathbb{R}} dx\, \partial_x\phi(x)\ \mathrm{PV}\int_{\mathbb{R}} d\alpha\, \arctan(\Delta_\alpha^\varepsilon f^\varepsilon(x)).$$

Using exactly the arguments from Section 4.4 with say $R = \delta = 1$ we have

$$\left|\int_{\mathbb{R}} dx\, \partial_x\phi(x)\,\mathrm{PV}\int_{\mathbb{R}} d\alpha\, \arctan(\Delta_\alpha^\varepsilon f(x))\right| \le C\|\phi\|_{W^{1,1}}\|f_0\|_{L^\infty(\mathbb{R})}.$$

We thus deduce (23), completing the proof.

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