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UNIFORM L^1-STABILITY OF THE RELATIVISTIC BOLTZMANN EQUATION NEAR VACUUM

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Abstract. We present the uniform L^1-stability estimate for the relativistic Boltzmann equation near vacuum. For this, we explicitly construct a relativistic counterpart of the nonlinear functional which is a linear combination of L^1-distance and a collision potential. This functional measures the L^1-distance between two continuous mild solutions. When the initial data is sufficiently small and decays exponentially fast, we show that the functional satisfies the uniform stability estimate leading to the uniform L^1-stability estimate with respect to initial data.

1. **Introduction.** The purpose of this paper is to address the uniform L^1-stability estimate for the continuous mild solutions to the relativistic Boltzmann equation (in short RBE) near vacuum, which means that the solution operator is Lipschitz continuous with respect to initial data. Consider the ensemble of collisional relativistic particles whose relativistic velocities is not negligible compared to the speed of light (assumed to be unity). In this case, the statistical description of the ensemble is described by the one-particle distribution function $f = f(x, p, t)$ at the position $x \in \mathbb{R}^3$, the momentum $p \in \mathbb{R}^3$ at time t:

$$\partial_t f + v(p) \cdot \nabla_x f = Q(f, f), \quad x, p \in \mathbb{R}^3,\ t \in \mathbb{R}_+, \tag{1.1}$$

subject to initial datum:

$$f(x, p, 0) = f^{in}(x, p), \tag{1.2}$$

where the mass of particles is assumed to be unity and $Q(f, f)$ denotes the nonlocal collision operator, and $v(p)$ is the relativistic velocity corresponding to momentum p:

$$v(p) := \frac{p}{\sqrt{1 + |p|^2}}.$$

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Note that if $|p| \ll 1$, the relativistic velocity corresponding to the momentum p can be approximated by the momentum p itself, i.e., $v(p) \approx p$. Hence the above equation (1.1) formally reduces to the classical Boltzmann equation, as the speed of light goes to infinity.

We next briefly review the existence theory for the equation (1.1); The RBE (1.1) was first used by Lichnerowicz and Marrot [21] in 1940 and Bichteler [4] explained that the general RBE has a local solution if the differential cross-section is bounded and the initial distribution has good assumptions (see [2, 3, 4, 5] for further references). For the global weak solutions, Dudyński and Ekiel-Jeżewska [7] extended the theory of Diperna-Lions renormalized solutions to the relativistic Boltzmann equation in 1992 [9]. In contrast, when the initial data is small perturbation of vacuum, the global existence of mild solutions is still far from complete understanding. Recently there were two breakthroughs in this direction by allowing a rather restrictive assumptions on the cross sections in the collision kernel. The first result is due to Glassey in [13], where he obtained the global existence of unique mild solutions to the RBE with initial data near the vacuum under the some restrictive cross-section and initial data. Glassey's solution may not have a finite mass. Recently Strain [23] established a global existence result near vacuum with a cut-off cross-section assumption. Strain's result is more close to the Illner-Shinbrot's framework [19], where small and decaying mild solutions are considered. In particular, Strain's solutions have finite masses, hence it is well suited to the L^1-stability framework.

The novelty of this paper is two-fold: First we present a generalized Glimm's type collision potential whose classical versions were studied in [14, 15, 17]. When the cross section of the collision kernel is bounded, similar functional was proposed in [16] for the purpose of scattering estimates. Our functional improves that of [16] for the unbounded cross section. Second, we prove the uniform L^1-stability for the continuous mild solutions. Our strategy for this is parallel to the corresponding one for the classical Boltzmann equation. Following the idea of the collision potential, we explicitly construct a weighted L^1-distance functional $\mathcal{H}(t)$ which is a linear combination of a usual L^1-distance and collision potential. This nonlinear functional approach was first introduced by the first author for the classical Boltzmann equation. We derive a Gronwall's inequality for the functional and using the detailed pointwise estimate of the gain part of the collision operator, we obtain the uniform stability estimate.

The rest of this paper is organized as follows. In Section 2 and 3, we simply review the RBE, our main assumptions for the result and we present a pointwise estimate for the gain operator. In Section 4, we study a Glimm type collision potential measuring all possible future collisions between relativistic particles. In Section 5, we study the uniform L^1-stability under our main assumptions. We present a nonlinear functional which is equivalent to the L^1-distance between two mild solutions to the relativistic Boltzmann equation, and we study the time-decay properties this functional. Finally Section 6 is devoted to the brief summary of our results.

Notations: We use simplified notations for local and global norms: For any measurable functions $h = h(x,p,t)$ and $g = g(x,t)$ defined on $\mathbb{R}^3_x \times \mathbb{R}^3_p \times \mathbb{R}_t$ and $\mathbb{R}^3_p \times \mathbb{R}_t$ respectively, for $1 \le q \le \infty$,

$$||h(t)||_{L^q_{x,p}} = ||h(\cdot,\cdot,t)||_{L^q(\mathbb{R}^3_x \times \mathbb{R}^3_p)}, \quad ||h(x,t)||_{L^q_p} = ||h(x,\cdot,t)||_{L^q(\mathbb{R}^3_p)},$$

$$||h(t)||_{L^\infty_x(L^q_p)} = \sup_{x\in\mathbb{R}^3} ||h(x,\cdot,t)||_{L^q(\mathbb{R}^3_p)}, \quad ||g(t)||_{L^q_x} = ||g(\cdot,t)||_{L^q(\mathbb{R}^3_x)}.$$

2. **Discussion of a framework and main results.** In this section, we briefly review the relativistic Boltzmann equation and discuss Strain's cut-off assumptions for the collision kernel. We also present the stability framework and main results of the paper.

2.1. **The relativistic Boltzmann equation.** In the absence of external forces, the one-particle distribution function f satisfies

$$\begin{aligned} \partial_t f + v(p)\cdot\nabla_x f &= Q(f,f), \quad x,p\in\mathbb{R}^3, \quad t>0,\\ f(x,p,0) &= f^{in}(x,p). \end{aligned} \tag{2.1}$$

Let (p,p_*) and (p',p'_*) be the pairs of pre-collisional and post-collisional velocities respectively, and they satisfy conservations of momentum and energy:

$$p+p_* = p'+p'_*, \qquad p_0+p_{*0} = p'_0+p'_{*0}, \qquad p_0 := \sqrt{1+|p|^2}. \tag{2.2}$$

We now introduce two quantities g and s appearing the microscopic collision process:

$$\begin{aligned} s &:= 2(\sqrt{1+|p_*|^2}\sqrt{1+|p|^2} - p_*\cdot p + 1),\\ g^2 &:= 2(\sqrt{1+|p_*|^2}\sqrt{1+|p|^2} - p_*\cdot p - 1) = s-4. \end{aligned} \tag{2.3}$$

Then the microscopic conservation laws (2.2) yield the following collision transformation [24]:

$$\begin{aligned} p' &= \frac{p+p_*}{2} + \frac{g}{2}\Big[\theta + \Big(\frac{p_0+p_{*0}}{\sqrt{s}} - 1\Big)(p+p_*)\frac{(p+p_*)\cdot\theta}{|p+p_*|^2}\Big], \quad \theta\in\mathbb{S}^2,\\ p'_* &= \frac{p+p_*}{2} - \frac{g}{2}\Big[\theta + \Big(\frac{p_0+p_{*0}}{\sqrt{s}} - 1\Big)(p+p_*)\frac{(p+p_*)\cdot\theta}{|p+p_*|^2}\Big], \end{aligned}$$

and the useful invariant quantities [23]:

$$\begin{aligned} &p'_{*0}|x+t(v(p)-v(p'_*))|^2 + p'_0|x+t(v(p)-v(p'))|^2\\ &\qquad = p_0|x|^2 + p_{*0}|x+t(v(p)-v(p_*))|^2 + \frac{t^2}{p_0} + \frac{t^2}{p_{*0}} - \frac{t^2}{p'_0} - \frac{t^2}{p'_{*0}}, \quad \text{and} \end{aligned} \tag{2.4}$$

$$p_0p_{*0} - p\cdot p_* = p'_0p'_{*0} - p'_*\cdot p'. \tag{2.5}$$

where $a\cdot b$ is the standard Euclidean inner product in $\mathbb{R}^3$.

The relativistic collision operator $Q(f,f)$ takes the form of

$$Q(f,f)(x,p,t) := \int_{\mathbb{R}^3\times\mathbb{S}^2} \mathbf{p}_M\sigma(g,\theta)(f'f'_* - ff_*)dp_*d\theta, \tag{2.6}$$

where $\sigma=\sigma(g,\theta)$ and $\mathbf{p}_M$ denote the scattering cross section and the Møller velocity respectively. In (2.6), we used the handy notations:

$$f = f(x,p,t), \quad f_* = f(x,p_*,t), \quad f' = f(x,p',t) \quad \text{and} \quad f'_* = f(x,p'_*,t).$$

The Møller velocity $\mathbf{p}_M$ is defined as

$$\mathbf{p}_M := \sqrt{|v(p)-v(p_*)|^2 - |v(p)\times v(p_*)|^2}.$$

This implies

$$\mathbf{p}_M^2 = \frac{s(s-4)}{4p_0^2p_{*0}^2} \quad \text{and} \quad \mathbf{p}_M \le |v(p)-v(p_*)|. \tag{2.7}$$

We now introduce Shinbrot's notation representing a function evaluated along the particle trajectory:

$$f^\sharp(x,p,t) := f(x+tv(p),p,t),$$
$$Q^\sharp(f,f)(x,p,t) := Q(f,f)(x+tv(p),p,t), \quad x,p \in \mathbb{R}^3,\ t \in \mathbb{R}.$$

Then along the particle trajectory $(x+sv(p),p,s)$, we can get a mild form

$$f^\sharp(x,p,t) = f^{in}(x,p) + \int_0^t Q^\sharp(f,f)(x,p,s)ds, t \geq 0. \tag{2.8}$$

The definition of mild and classical solutions can be stated as follows.

Definition 2.1. *(i) Let T be a given positive number. A nonnegative function $f \in C([0,T);\ L^1_+(\mathbb{R}^3 \times \mathbb{R}^3))$ is a mild solution of (2.1) with a nonnegative initial datum f^{in} if and only if for all $t \in [0,T)$ and a.e $(x,p) \in \mathbb{R}^3 \times \mathbb{R}^3$, f satisfies the integral equation (2.8) pointwise.*

(ii) A function $f = f(x,p,t) \in C^1(\mathbb{R}^3 \times \mathbb{R}^3 \times [0,T))$ is a classical solution of (2.1) with a nonnegative initial datum f^{in} if and only if f satisfies the equation (2.1) pointwise.

2.2. **Cut-off assumptions on collision kernel.** Throughout the paper we will assume that the differential cross section in (2.6) satisfies the growth/decay estimates

$$\begin{aligned} &0 \leq \sigma(g,\theta) \leq \Big\{A_1\Big[1+\Big(\frac{g}{1+g}\Big)^{\alpha_1}\Big] + A_2 g^{-\gamma}\Big\}\tilde{\sigma}(\theta), \\ &\tilde{\sigma}(\theta) \leq \sigma_1 < \infty, \quad 0 \leq \gamma < 3, \end{aligned} \tag{2.9}$$

where A_1, A_2 and α_1 are non-negative constants satisfying $(A_1,A_2) \neq (0,0)$ and it is assumed that $\sigma(g,\theta)$ is not identically zero.

We next discuss the cut-off assumption for the collision kernel which was introduced in [23]. For a given positive constant $B > 0$ and a number $0 \leq a < 1$ and $t > 0$, we define

$$h \;=\; h(x,p,p_*,t) \;:=\; \frac{B}{t^2} + a\frac{p_{*0}|x+t(v(p)-v(p_*))|^2}{t^2} > 0,$$

and the cut-off set in impact angle $\mathbb{S}^2$:

$$\mathcal{B} := \Big\{\theta \in \mathbb{S}^2 : \frac{1}{p_0} + \frac{1}{p_{*0}} - \frac{1}{p_0'} - \frac{1}{p_{*0}'} \geq -h\Big\}. \tag{2.10}$$

We now redefine a new cross section $\hat{\sigma}$ by

$$\hat{\sigma}(\theta,p,p_*) := \sigma(g,\theta)\mathbf{1}_{\mathcal{B}}(\theta). \tag{2.11}$$

Note that [8] includes a physical discussion of general assumptions about the collision kernel. Recently, one of the authors proved the global existence for unique mild solutions to the relativistic Boltzmann equation (2.1) with the above collision kernel assumption (2.11) as follows.

We define a relativistic local Maxwellian $M_{\alpha,\beta}$:

$$M_{\alpha,\beta}(x,p) := \exp\Big(-\alpha p_0|x|^2 - \beta p_0\Big), \qquad \alpha,\beta > 0,$$

and recall Strain's existence result in the following theorem.

Theorem 2.1 ([23])**.** *Suppose the nonnegative initial datum f^{in} satisfies*

$$f^{in} \in C(\mathbb{R}^3_x \times \mathbb{R}^3_p), \quad \frac{f^{in}(x,p)}{M_{\alpha,\beta}(x,p)} \le b.$$

Then there exists b_0 such that if $b \le b_0$, the equation (2.1) with the angular cut-off collision operator (2.6), (2.9), (2.11) has the unique global mild solution f satisfying

$$\frac{f^\sharp(x,p,t)}{M_{\alpha,\beta}(x,p)} \le \mathcal{O}(1)b.$$

Above $C(\mathbb{R}^3_x \times \mathbb{R}^3_p)$ is of course the space of continuous functions on $\mathbb{R}^3_x \times \mathbb{R}^3_p$.

2.3. **A framework for uniform stability.** In this part, we discuss a stability framework and present our result regarding the uniform stability.

We introduce a set $\mathcal{S}(\alpha,\beta,\delta)$ as follows: For some positive constants α,β and δ,

$$\mathcal{S}(\alpha,\beta,\delta) := \{f \in C(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+) \ : \ |||f||| := \sup_{x,p,t} |f^\sharp(x,p,t)| e^{\alpha p_0 |x|^2} e^{\beta p_0} < \delta\},$$

where the parameters are assumed to satisfy the following stability conditions $(\mathcal{A})$:

$$\begin{aligned} &0 < \alpha < \beta \quad \text{and} \quad 0 < \delta < \sqrt{\frac{\alpha}{2\pi}} \frac{1}{16\pi\sigma_1 \mathcal{C}}, \\ &\mathcal{C} := \sup_{p \in \mathbb{R}^3} \int_{\mathbb{R}^3} (2A_1 + A_2 g^{-\gamma}) e^{-(\beta-\alpha)|p_*|} dp_* > 0, \end{aligned} \tag{2.12}$$

where $\mathcal{C}$ is a finite positive constant (see Remark 3.1).

Note that we can consider the unique global mild solution in $\mathcal{S}(\alpha,\beta,\delta)$ to (2.1) corresponding to a given small initial datum f^{in} by Theorem 2.1.

The main result of this paper is as follows.

Theorem 2.2. *Let f and $\bar{f}$ be two global mild solutions in $\mathcal{S}(\alpha,\beta,\delta)$ to (2.1) corresponding to initial data f^{in} and $\bar{f}^{in}$ respectively. If $0 < \gamma < 2$, then the uniform L^1-stability estimate holds:*

$$||f(t) - \bar{f}(t)||_{L^1_{x,p}} \le G||f^{in} - \bar{f}^{in}||_{L^1_{x,p}},$$

where G is a positive constant independent of t.

Remark 2.1. *(i) For our main result, we need a pointwise estimate for the gain operator $Q_+(f,f)$ (see Proposition 3.1) and its integrability in t. Proposition 3.1 implies*

$$||Q_+(f,f)||_{L^\infty_{x,p}} \le C(1+t)^{\gamma-3},$$

which yields that the time-integrability of $Q_+(f,f)$ is guaranteed only for $\gamma \in (0,2)$.
(ii) The existence and stability issues for the non-relativistic Boltzmann equation near vacuum have been studied in [1, 14, 15, 17, 19, 20, 22, 26], and for the corresponding issues near a global Maxwellian, we refer to [6, 12, 11, 17, 18, 23, 24, 25, 27].

3. **Approximate RBEs and basic estimates.** In this section, we present a mollification procedure for the continuous mild solutions given by R. M. Strain [23] and provide several basic estimates which be used in later sections. We first discuss a mollification procedure of mild solutions to get approximate relativistic Boltzmann equations, and then apply the nonlinear functional approach to them. When the mollification parameter $\varepsilon \to 0$, we recover the time-decay estimates along the continuous mild solutions (see [17] for the corresponding results for the non-relativistic Boltzmann equation). We basically follows the presentation given in [17].

Let $\phi \in C_0^\infty(\mathbb{R}^3)$ and ϕ_ε be a standard mollifier and a corresponding rescaled mollifier respectively, i.e.,

$$0 \le \phi \le 1, \quad \mathrm{supp}(\phi) \subset \mathbf{B}_1(0), \quad \phi \in C^\infty(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} \phi(x)dx = 1,$$
$$\phi_\varepsilon(x) := \frac{1}{\varepsilon^3}\phi\Big(\frac{x}{\varepsilon}\Big), \quad \mathrm{supp}(\phi_\varepsilon) \subset \mathbf{B}_\varepsilon(0), \quad x \in \mathbb{R}^3,\ \varepsilon > 0,$$

where $\mathbf{B}_r(0)$ is the open ball of radius r which is centered at the origin. Since we only need smoothness in the spatial variable of the nonlinear functionals, we mollify f and $Q(f,f)$ in x only and denote them by

$$f_\varepsilon := f *_x \phi_\varepsilon \quad \text{and} \quad Q_\varepsilon(f,f) := Q(f,f) *_x \phi_\varepsilon,$$

where $*_x$ denotes the convolution with respect to the x-variable. Since the mollification commutes with $\sharp$ in the sense that $(f_\varepsilon)^\sharp = (f^\sharp)_\varepsilon$, we use the notation $f_\varepsilon^\sharp$ to denote either $(f_\varepsilon)^\sharp$ or $(f^\sharp)_\varepsilon$. We now mollify our equation (2.1) with ϕ_ε to find the approximate relativistic Boltzmann equation

$$\begin{aligned} \partial_t f_\varepsilon + v(p)\cdot\nabla_x f_\varepsilon &= Q(f_\varepsilon, f_\varepsilon) + P(f, f_\varepsilon), \quad x, p \in \mathbb{R}^3,\ t > 0, \\ f_\varepsilon(x,p,0) &= f^{in}(x,p) *_x \phi_\varepsilon, \end{aligned} \tag{3.1}$$

where

$$P(f, f_\varepsilon) := Q_\varepsilon(f,f) - Q(f_\varepsilon, f_\varepsilon).$$

In the next lemma, we study the pointwise bound of $f^\sharp$.

Lemma 3.1. *Suppose that $f \in \mathcal{S}(\alpha,\beta,\delta)$ and $0 < \varepsilon < 1$. Then we have*

$$f_\varepsilon^\sharp(x,p,t) \le \delta \left(M_{\alpha,\beta}\right)_\varepsilon(x,p) \quad \textit{and} \quad \left(M_{\alpha,\beta}\right)_\varepsilon(x,p) \le e^{-\frac{\alpha}{2}p_0|x|^2-(\beta-\alpha)p_0}.$$

Proof. (i) Note that $|||f||| < \delta$ implies

$$f^\sharp(x,p,t) \le \delta M_{\alpha,\beta}(x,p), \quad \text{for all } x, p \in \mathbb{R}^3.$$

Then the first inequality is clearly satisfied.
(ii) By definition of $M_{\alpha,\beta}$, we have

$$\begin{aligned} e^{\frac{1}{2}\alpha p_0|x|^2+\beta p_0}\left(M_{\alpha,\beta}\right)_\varepsilon(x,p) &= e^{\frac{1}{2}\alpha p_0|x|^2+\beta p_0}\int_{\mathbb{R}^3} M_{\alpha,\beta}(x-y,p)\phi_\varepsilon(y)dy \\ &= e^{\frac{1}{2}\alpha p_0|x|^2+\beta p_0}\int_{\mathbf{B}_\varepsilon(0)} e^{-\alpha p_0|x-y|^2-\beta p_0}\phi_\varepsilon(y)dy \\ &\le \int_{\mathbf{B}_\varepsilon(0)} e^{\alpha p_0|y|^2}\phi_\varepsilon(y)dy \\ &\le e^{\alpha p_0\varepsilon^2}\int_{\mathbf{B}_\varepsilon(0)} \phi_\varepsilon(y)dy \\ &\le e^{\alpha p_0}, \end{aligned}$$

where we used $0 < \varepsilon < 1$. Therefore we have

$$\left(M_{\alpha,\beta}\right)_\varepsilon(x,p) \le e^{-\frac{\alpha}{2}p_0|x|^2-(\beta-\alpha)p_0}.$$

This concludes the proof. □

3.1. **Basic estimates.** In this part, we present a pointwise estimate for the gain operator. This pointwise estimate will be crucially used in the uniform L^1-stability estimate in Section 5.

Recall the gain and loss terms in the collision operator (2.6):

$$\begin{aligned} Q_+(x,p,t) &:= \int_{\mathbb{R}^3\times\mathbb{S}^2} \mathbf{p}_M \sigma(g,\theta) f' f'_* d\theta dp_*, \\ Q_-(x,p,t) &:= \int_{\mathbb{R}^3\times\mathbb{S}^2} \mathbf{p}_M \sigma(g,\theta) f f_* d\theta dp_*. \end{aligned}$$

Before we present the pointwise estimate for Q_+, we study a series of elementary estimates.

Lemma 3.2. *There exists a positive constant $C_1 > 0$ such that for $0 \le \gamma < 3$ we have*

$$\sup_{p\in\mathbb{R}^3} \int_{\mathbb{R}^3} \Big(\frac{\sqrt{p_0 p_{*0}}}{|p-p_*|}\Big)^\gamma e^{-\beta|p_*|} dp_* \le C_1 < \infty. \tag{3.2}$$

Proof. For a given $p \in \mathbb{R}^3$, we divide the momentum space into two regions:

$$\mathcal{R}_1 := \Big\{ q \in \mathbb{R}^3 : |q| \le \frac{|p|+1}{2} \Big\} \quad \text{and} \quad \mathcal{R}_2 := \Big\{ q \in \mathbb{R}^3 : |q| > \frac{|p|+1}{2} \Big\}.$$

- Case 1 ($p_* \in p + \mathcal{R}_1$): Note that

$$p_* \in p+\mathcal{R}_1 \quad \Longrightarrow \quad |p_*| - |p| \le |p_* - p| \le \frac{|p|+1}{2}, \quad \text{i.e.} \quad |p_*| \le \frac{3|p|+1}{2}.$$

We use the above estimate for p_* to get

$$\begin{aligned} &\int_{p_*-p\in\mathcal{R}_1} \Big(\frac{\sqrt{p_0 p_{*0}}}{|p-p_*|}\Big)^\gamma e^{-\beta|p_*|} dp_* \\ &\quad \le \Big(1+|p|^2\Big)^{\frac{\gamma}{4}} \Big[1 + \Big(\frac{3|p|+1}{2}\Big)^2\Big]^{\frac{\gamma}{4}} \int_{q\in\mathcal{R}_1} \frac{e^{-\beta|p-q|}}{|q|^\gamma} dq \\ &\quad \le \Big(1+|p|^2\Big)^{\frac{\gamma}{4}} \Big[1 + \Big(\frac{3|p|+1}{2}\Big)^2\Big]^{\frac{\gamma}{4}} e^{-\beta|p|} \int_{q\in\mathcal{R}_1} \frac{e^{\beta|q|}}{|q|^\gamma} dq \\ &\quad \le \Big(1+|p|^2\Big)^{\frac{\gamma}{4}} \Big[1 + \Big(\frac{3|p|+1}{2}\Big)^2\Big]^{\frac{\gamma}{4}} e^{-\beta(|p|-1)/2} \int_{q\in\mathcal{R}_1} \frac{1}{|q|^\gamma} dq \\ &\quad = \frac{4\pi}{3-\gamma} \Big(1+|p|^2\Big)^{\frac{\gamma}{4}} \Big[1 + \Big(\frac{3|p|+1}{2}\Big)^2\Big]^{\frac{\gamma}{4}} e^{-\beta(|p|-1)/2} \Big(\frac{|p|+1}{2}\Big)^{3-\gamma} \\ &\quad =: \tilde{C}_1(p), \end{aligned}$$

where we have used the change of variable $p_* = p - q$.

- Case 2 ($p_* \in p + \mathcal{R}_2$): Similar to Case 1, we obtain

$$\begin{aligned} \int_{p_*-p\in\mathcal{R}_2} \Big(\frac{\sqrt{p_0 p_{*0}}}{|p-p_*|}\Big)^\gamma e^{-\beta|p_*|} dp_* &\le \frac{2^\gamma (1+|p|^2)^{\gamma/4}}{(|p|+1)^\gamma} \int_{p_*-p\in\mathcal{R}_2} \Big(1+|p_*|^2\Big)^{\frac{\gamma}{4}} e^{-\beta|p_*|} dp_* \\ &\le \frac{2^\gamma (1+|p|^2)^{\gamma/4}}{(|p|+1)^\gamma} \int_{\mathbb{R}^3} \Big(1+|p_*|^2\Big)^{\frac{\gamma}{4}} e^{-\beta|p_*|} dp_* \\ &=: \tilde{C}_2(p). \end{aligned}$$

We now combine Case 1 and Case 2 to find the desired estimate:

$$\int_{\mathbb{R}^3} \Big(\frac{\sqrt{p_0 p_{*0}}}{|p-p_*|}\Big)^\gamma e^{-\beta|p_*|} dp_* \le \tilde{C}_1(p) + \tilde{C}_2(p) \quad \text{for any } p \in \mathbb{R}^3.$$

Since $\tilde{C}_1(p) + \tilde{C}_2(p)$ is continuous function of p and

$$\tilde{C}_1(0) + \tilde{C}_2(0) < \infty \quad \text{and} \quad \lim_{|p|\to\infty} \Big(\tilde{C}_1(p) + \tilde{C}_2(p)\Big) = 0,$$

we can see that

$$\int_{\mathbb{R}^3} \Big(\frac{\sqrt{p_0 p_{*0}}}{|p-p_*|}\Big)^\gamma e^{-\beta|p_*|} dp_* \le \sup_{p\in\mathbb{R}^3} \Big(\tilde{C}_1(p) + \tilde{C}_2(p)\Big) =: C_1 < \infty.$$

This is the desired estimate. □

Lemma 3.3. *For given $p, x \in \mathbb{R}^3$ and $t > 0$, let $\Phi_p : \mathbb{R}^3 \to \mathbb{R}^3$ be a differentiable function defined by*

$$\Phi_p(p_*) := x + t(v(p) - v(p_*)).$$

Then we have

$$\Big|\frac{\partial \Phi_p(p_*)}{\partial p_*}\Big| = \frac{t^3}{p_{*0}^5}.$$

Proof. By direct estimate, we have

$$\frac{\partial \Phi_p(p_*)}{\partial p_*} = -t\, \frac{\partial v(p_*)}{\partial p_*} = -t \begin{pmatrix} \frac{1}{p_{*0}} - \frac{(p_*^1)^2}{p_{*0}^3} & -\frac{p_*^1 p_*^2}{p_{*0}^3} & -\frac{p_*^1 p_*^3}{p_{*0}^3} \\ -\frac{p_*^1 p_*^2}{p_{*0}^3} & \frac{1}{p_{*0}} - \frac{(p_*^2)^2}{p_{*0}^3} & -\frac{p_*^2 p_*^3}{p_{*0}^3} \\ -\frac{p_*^1 p_*^3}{p_{*0}^3} & -\frac{p_*^2 p_*^3}{p_{*0}^3} & \frac{1}{p_{*0}} - \frac{(p_*^3)^2}{p_{*0}^3} \end{pmatrix}.$$

This yields

$$\frac{1}{t^3}\Big|\frac{\partial \Phi_p(p_*)}{\partial p_*}\Big| = \frac{1}{p_{*0}^3} - \frac{|p_*|^2}{p_{*0}^5} = \frac{1}{p_{*0}^5}.$$

This concludes the proof. □

Proposition 3.1. *Suppose that f is a global mild solution of (2.1) in $\mathcal{S}(\alpha, \beta, \delta)$ with initial datum f^{in}. Then there exist positive constants $\eta_1, \eta_2 > 0$ such that $0 < \eta_1 < \alpha, 0 < \eta_2 < \beta$ and*

$$Q_+^\sharp(f,f)(x,p,t) \le C_2 \Big(\frac{1}{1+t}\Big)^{3-\gamma} e^{-(\alpha-\eta_1)|x|^2} e^{-(\beta-\eta_2)p_0}, \quad (x,p) \in \mathbb{R}^6,$$

where $C_2 > 0$ is a positive constant which is independent of x, p, t and f.

Proof. First, we observe the cross-section σ in the gain term from (2.9). Since we know a pointwise estimate about g (see [23]), i.e.,

$$\frac{|p-p_*|}{\sqrt{p_0 p_{*0}}} \le g \le |p - p_*| \quad \text{for any } p, p_* \in \mathbb{R}^3,$$

we can obtain an upper bound of σ as follows:

$$0 \le \sigma(g,\theta) \le \sigma_1 \Big(2A_1 + A_2 \Big(\frac{\sqrt{p_0 p_{*0}}}{|p-p_*|}\Big)^\gamma\Big).$$

Then this yields

$$
\begin{aligned}
& Q_+^\sharp(f,f)(x,p,t) \\
\le\ & \iint_{\mathbb{R}^3\times\mathbb{S}^2} \mathbf{p}_M \sigma(g,\theta)|f(x+tv(p),p',t)||f(x+tv(p),p'_*,t)|d\theta dp_* \\
\le\ & \delta^2 \iint_{\mathbb{R}^3\times\mathbb{S}^2} \mathbf{p}_M \sigma(g,\theta) e^{-\beta(p'_0+p'_{*0})-\alpha(p'_0|x+t(v(p)-v(p'))|^2+p'_{*0}|x+t(v(p)-v(p'_*))|^2)} d\theta dp_* \\
\le\ & 4\pi\delta^2\sigma_1 A_2 e^{-\beta p_0-\alpha p_0|x|^2+\alpha B} \\
& \times \int_{\mathbb{R}^3} |v(p)-v(p_*)| \Big(\frac{\sqrt{p_0 p_{*0}}}{|p-p_*|}\Big)^\gamma e^{-\beta p_{*0}-\alpha(1-a)p_{*0}|x+t(v(p)-v(p_*))|^2} dp_* \\
& +8\pi\delta^2\sigma_1 A_1 e^{-\beta p_0-\alpha p_0|x|^2+\alpha B}\int_{\mathbb{R}^3}|v(p)-v(p_*)|e^{-\beta p_{*0}-\alpha(1-a)p_{*0}|x+t(v(p)-v(p_*))|^2}dp_* \\
=:\ & \mathcal{I}_1+\mathcal{I}_2,
\end{aligned}
$$

where we used the useful equality (2.4) and the cut-off (2.11). We separate the estimate into two cases:

$$
\text{Either} \quad t \ \le\ 1 \quad \text{or } t\ >\ 1.
$$

• Case 1 ($t>1$): For the estimate of $\mathcal{I}_1$, we will divide into two parts:

$$
\begin{aligned}
\mathcal{I}_1 \ &=\ 4\pi\delta^2\sigma_1 A_2 e^{-\beta p_0-\alpha p_0|x|^2+\alpha B} \\
&\qquad \times \int_{\mathbb{R}^3} |v(p)-v(p_*)| \Big(\frac{\sqrt{p_0 p_{*0}}}{|p-p_*|}\Big)^\gamma e^{-\beta p_{*0}-\alpha(1-a)p_{*0}|x+t(v(p)-v(p_*))|^2} dp_* \\
&=\ 4\pi\delta^2\sigma_1 A_2 e^{-\beta p_0-\alpha p_0|x|^2+\alpha B}\Big(\int_{|p-p_*|<\frac1t}\cdots\ dp_* + \int_{|p-p_*|\ge\frac1t}\cdots\ dp_*\Big) \\
&=\ 4\pi\delta^2\sigma_1 A_2 e^{-\beta p_0-\alpha p_0|x|^2+\alpha B}\Big(\mathcal{I}_{11}+\mathcal{I}_{12}\Big).
\end{aligned}
$$

◇ (Estimate of $\mathcal{I}_{11}$): We first use

$$
e^{-\alpha(1-a)p_{*0}|x+t(v(p)-v(p_*))|^2} \le 1, \quad \text{and} \quad |p_*| \le |p| + \frac{1}{t} \le |p|+1,
$$

to obtain

$$
\begin{aligned}
\mathcal{I}_{11} \ &\le\ \int_{|p-p_*|<\frac1t} \Big(|v(p)|+|v(p_*)|\Big)\Big(\frac{\sqrt{p_0p_{*0}}}{|p-p_*|}\Big)^\gamma e^{-\beta p_{*0}}dp_* \\
&\le\ 2(1+|p|^2)^{\frac{\gamma}{4}}(1+(1+|p|)^2)^{\frac{\gamma}{4}} \int_{|p-p_*|<\frac1t}\Big(\frac{1}{|p-p_*|}\Big)^\gamma e^{-\beta|p_*|}dp_*.
\end{aligned}
$$

We set $q\ :=\ p_*-p$ and use $|p_*|=|q+p|\ge|p|-|q|$ to see

$$
\begin{aligned}
\mathcal{I}_{11} \ &\le\ 2(1+|p|^2)^{\frac{\gamma}{4}}(1+(1+|p|)^2)^{\frac{\gamma}{4}} e^{-\beta|p|}\int_{|q|<\frac1t}\Big(\frac{1}{|q|}\Big)^\gamma e^{\beta|q|}dq \\
&\le\ 2(1+|p|^2)^{\frac{\gamma}{4}}(1+(1+|p|)^2)^{\frac{\gamma}{4}} e^{-\beta|p|}e^{\beta\frac1t}\int_0^{\frac1t}\Big(\frac1s\Big)^\gamma 4\pi s^2 ds \\
&=\ \frac{8\pi e^\beta}{3-\gamma}(1+|p|^2)^{\frac{\gamma}{4}}(1+(1+|p|)^2)^{\frac{\gamma}{4}} e^{-\beta|p|}\Big(\frac1t\Big)^{3-\gamma} \\
&\le\ \frac{16\pi e^\beta}{3-\gamma}\Big(\frac{1}{t+1}\Big)^{3-\gamma}\tilde{C}_3(\gamma),
\end{aligned}
$$

where

$$
\tilde{C}_3(\gamma) := \sup_p\big\{(1+|p|^2)^{\frac{\gamma}{4}}(1+(1+|p|)^2)^{\frac{\gamma}{4}}e^{-\beta|p|}\big\}.
$$

$\diamond$ (Estimate of $\mathcal{I}_{12}$): We set

$$\begin{aligned}\tilde{C}_4(\gamma,\beta) &:= \sup_{p_*}\Big\{p_{*0}^{5+\frac{\gamma}{2}}\exp(-\beta p_{*0})\Big\} < \infty,\\ \tilde{C}_5(\alpha,a) &:= \int_{\mathbb{R}^3}|q|e^{-\alpha(1-a)|q|^2}dq < \infty.\end{aligned}$$

We use the change of variable from Lemma 3.3 to obtain

$$\begin{aligned}\mathcal{I}_{12} \le& t^\gamma\left(\sqrt{p_0}\right)^\gamma \tilde{C}_4(\gamma,\beta)\int_{|p-p_*|>\frac{1}{t}}|v(p)-v(p_*)|p_{*0}^{-5}e^{-\alpha(1-a))|x+t(v(p)-v(p_*))|^2}dp_*\\ \le& t^\gamma\left(\sqrt{p_0}\right)^\gamma \tilde{C}_4(\gamma,\beta)\int_{\mathcal{B}}\frac{|q-x|}{t}p_{*0}^{-5}e^{-\alpha(1-a)|q|^2}\frac{p_{*0}^5}{t^3}dq\\ \le& \tilde{C}_4(\gamma,\beta)t^{\gamma-4}\left(\sqrt{p_0}\right)^\gamma\int_{\mathcal{B}}\Big(|q|+|x|\Big)e^{-\alpha(1-a)|q|^2}dq\\ \le& 2\tilde{C}_4(\gamma,\beta)(1+t)^{\gamma-4}\left(\sqrt{p_0}\right)^\gamma\Big[\tilde{C}_5(\alpha,a)+|x|\big(\frac{\pi}{\alpha(1-a)}\big)^{\frac{3}{2}}\Big].\end{aligned}$$

Note that the region $\mathcal{B}$ in (2.10) results from the change of variable in Lemma 3.3. Here we also used the following inequality:

$$\begin{aligned}p_{*0}^5\Big(\frac{\sqrt{p_0p_{*0}}}{|p-p_*|}\Big)^\gamma e^{-\beta p_{*0}} <& t^\gamma p_{*0}^5(p_0p_{*0})^{\frac{\gamma}{2}}e^{-\beta p_{*0}}\\ =& t^\gamma p_0^{\frac{\gamma}{2}}\sup_{p_*\in\mathbb{R}^3}\{p_{*0}^{5+\frac{\gamma}{2}}e^{-\beta p_{*0}}\} \le t^\gamma p_0^\gamma\tilde{C}_4(\gamma,\beta).\end{aligned}$$

We choose $0<\eta_1<\beta$ and $0<\eta_2<\alpha$ so that

$$p_0^\gamma e^{-\beta p_0} \le \tilde{C}_6 e^{-(\beta-\eta_1)p_0}, \quad |x|e^{-\alpha|x|^2} \le \tilde{C}_6 e^{-(\alpha-\eta_2)|x|^2} \quad \text{for all } p,x\in\mathbb{R}^3,$$

where $\tilde{C}_6>0$ is some positive constant. We now combine above estimates to get

$$\begin{aligned}\mathcal{I}_1 \le& \frac{64\pi^2}{3-\gamma}\delta^2\sigma_1A_2e^{\alpha B+\beta}e^{-\beta p_0-\alpha p_0|x|^2}\tilde{C}_3(\gamma)\Big(\frac{1}{1+t}\Big)^{3-\gamma}\\ &+4\pi\delta^2\sigma_1A_2e^{\alpha B}C_1(\gamma,\beta)\tilde{C}_6e^{-(\beta-\eta_1)p_0}\\ &\times\Big[C_2(\alpha,a)e^{-\alpha|x|^2}+\Big(\frac{\pi}{\alpha(1-a)}\Big)^{\frac{3}{2}}\tilde{C}_6e^{-(\alpha-\eta_2)|x|^2}\Big]\Big(\frac{1}{1+t}\Big)^{4-\gamma}.\end{aligned}$$

And we also use the change of variable from Lemma 3.3 in following estimate:

$$\begin{aligned}&\mathcal{I}_2\\ \le& 8\pi\delta^2\sigma_1A_1e^{\alpha B}e^{-\beta p_0-\alpha p_0|x|^2}\int_{\mathbb{R}^3}|v(p)-v(p_*)|\frac{1}{p_{*0}^5}\tilde{C}_4(0,\beta)e^{-\alpha(1-a)|x+t(v(p)-v(p_*))|^2}dp_*\\ \le& 8\pi\delta^2\sigma_1A_1e^{\alpha B}e^{-\beta p_0-\alpha p_0|x|^2}\frac{1}{t^4}\int_{\mathcal{B}}|q-x|\tilde{C}_4(0,\beta)e^{-\alpha(1-a)|q|^2}dq\\ \le& 16\pi\delta^2\sigma_1A_1\tilde{C}_4(0,\beta)e^{\alpha B}e^{-\beta p_0-\alpha p_0|x|^2}\frac{1}{(1+t)^4}\Big(\tilde{C}_5(\alpha,a)+|x|\int_{\mathbb{R}^3}e^{-\alpha(1-a)|q|^2}dq\Big).\end{aligned}$$

• Case 2 ($t<1$): In this case we use

$$e^{-\alpha(1-a)p_{*0}|x+t(v(p)-v(p_*))|^2} \le 1$$

to obtain

$$\begin{aligned} Q_+^\sharp(f,f)(x,p,t) &\le 16\pi\delta^2\sigma_1 A_1 e^{-\beta p_0-\alpha p_0|x|^2}\int_{\mathbb{R}^3} e^{-\beta p_{*0}}dp_* \\ &\quad + 8\pi\delta^2\sigma_1 A_2 e^{-\beta p_0-\alpha p_0|x|^2}\int_{\mathbb{R}^3}\Big(\frac{\sqrt{p_0p_{*0}}}{|p-p_*|}\Big)^\gamma e^{-\beta p_{*0}}dp_* \\ &\le 8\pi\delta^2\sigma_1 e^{-\beta p_0-\alpha p_0|x|^2}\Big[4A_1\Big(\frac{\pi}{\beta}\Big)^{\frac32}+A_2C_1\Big] \\ &\le 64\pi\delta^2\sigma_1 e^{-\beta p_0-\alpha p_0|x|^2}\Big(\frac{1}{1+t}\Big)^{3-\gamma}\Big[4A_1(\frac{\pi}{\beta})^{\frac32}+A_2C_1\Big], \end{aligned}$$

where we used the above Lemma 3.2 and also the cut-off (2.11). We combine two cases, we can find a positive constant $C_2>0$ so that

$$Q_+^\sharp(f,f)(x,p,t)\le C_2\Big(\frac{1}{1+t}\Big)^{3-\gamma}e^{-(\alpha-\eta_1)|x|^2}e^{-(\beta-\eta_2)p_0},\quad (x,p)\in\mathbb{R}^6,$$

where $C_2>0$ is independence of x,p,t and f. ☐

Corollary 3.1. *Suppose that f is a global mild solution of (2.1) in $\mathcal{S}(\alpha,\beta,\delta)$ with initial datum f^{in}. Then for $0<\varepsilon<1$ we have*

$$Q_\varepsilon^\sharp(f,f)(x,p,t)\le C_2e^{\alpha-\eta_1}\Big(\frac{1}{1+t}\Big)^{3-\gamma}e^{-(\alpha-\eta_1)|x|^2/2}e^{-(\beta-\eta_2)p_0},\quad (x,p)\in\mathbb{R}^6,$$

where η_1,η_2 and C_2 are positive constant in Proposition 3.1 .

Proof. Since Q_+ and Q_- are positive for all x,p and t, from Proposition 3.1 it can be shown in the same manner as the proof of Lemma 3.1. ☐

Remark 3.1. *From the same method of Lemma 3.2, we can also know that $\mathcal{C}$ in the additional assumption $(\mathcal{A})$ from (2.12) is integrable. i.e.,*

$$\mathcal{C}=\sup_{p\in\mathbb{R}^3}\int_{\mathbb{R}^3}(2A_1+A_2g^{-\gamma})e^{-(\beta-\alpha)|p_*|}dp_*<\infty.$$

4. **A Glimm type interaction potential.** In this section, we present the relativistic counterpart for the classical collision potential in [14] and study its time-evolution along the continuous mild solutions to (1.1).

4.1. **Interaction potential.** The preliminary form of relativistic Glimm type potential was introduced in [16] for a bounded σ in the collision kernel (2.6). Before we present its explicit form, we briefly motivate a collision potential as follows. For the moment, we assume that the particle trajectory is a straight line and the one-particle distribution along the particle trajectory is invariant, i.e., we assume that f follows the relativistic free transport equation:

$$\partial_t f+v(p)\cdot\nabla_x f=0.$$

Of course, these assumptions are oversimplified, but since we are interested in small amplitude solutions near vacuum, the variation of amplitude in f along the particle trajectory will be small. Hence the constancy assumption along the particle trajectory will give us the first-order picture. Under this oversimplified setting, we are interested in the following questions:

- What are the pair of particles to collide with in future ?
- Once we identify the pair of colliding particles, how to quantify the total future collisions ?

Note that the probability of collisions in $\mathbb{R}^3$ between two randomly chosen particles following microscopic conservation laws (2.2) will be zero. Hence it is not an easy job to identify colliding pairs. To circumvent this difficulty, we reduce a degree of difficulty as follows. We consider a batch of particles (which is called test particles) issued from a phase-space point $(x,p) \in \mathbb{R}^6$ at time $s = 0$. Then these test particles will be located at $(x + tv(p), p)$ at the present time $s = t$. We now think of the particles at the present phase-space $\mathbb{R}^6$ to collide with these test particles in future. For this, we employ the method of backward characteristics as follows. For a future time $s = t + \tau,\ \tau > 0$, we know the phase-space position of the test particles, i.e., $(x + tv(p) + \tau(v(p) - v(p_*)), p_*)$. If some field particles with momentum p_* collide with this test particles with momentum p at time $s = t+\tau$, then the current spatial location of these field particles at current time $s = t$ will be determined by the backward characteristics, i.e.,

$$\underbrace{x + (t+\tau)v(p)}_{\text{physical location at } s = t+\tau} \qquad -\tau v(p_*) = \underbrace{x + tv(p) + \tau(v(p) - v(p_*))}_{\text{physical location at } s = t}.$$

Therefore a number of field particles measured by $f(x+tv(p)+\tau(v(p)-v(p_*)), p_*, t)$ will collide with test-particles $f(x + tv(p), p, t)$ at time $s = t + \tau$. In this collision process, we measure the strength of the collision impact by

$$\omega(g)|v(p) - v(p_*)|, \qquad \text{where} \quad \omega(g) := 2A_1 + A_2 g^{-\gamma},$$

where A_1 and A_2 are constants appearing in (2.9).

Then following Glimm's spirit [13], we define the total amount of impact to be collided with the particles $f^\sharp(x,p,t) := f(x + tv(p), p, t)$ as

$$f^\sharp(x,p,t)\Big[\int_{\mathbb{R}^3}\int_0^\infty \omega(g)|v(p) - v(p_*)| f^\sharp(x + (t+\tau)(v(p) - v(p_*)), p_*, t)\, d\tau dp_*\Big] dpdx.$$

Based on this intuitive idea, we define the interaction potential and its production rate as follows.

$$\begin{aligned}
\mathcal{D}(f(t)) &:= \int_{\mathbb{R}^6} f^\sharp(x,p,t)\Big[\int_{\mathbb{R}^3}\int_0^\infty \omega(g)|v(p) - v(p_*)| \\
&\qquad\qquad \times\ f^\sharp(x + (t+\tau)(v(p) - v(p_*)), p_*, t)\, d\tau dp_*\Big] dpdx, \\
\Lambda(f(t)) &:= \int_{\mathbb{R}^9} \omega(g)|v(p) - v(p_*)| \\
&\qquad\qquad \times\ f^\sharp(x,p,t) f^\sharp(x + t(v(p) - v(p_*)), p_*, t)\, dp_* dpdx.
\end{aligned} \tag{4.1}$$

4.2. **Time-decay estimates of the functionals.** Since we are dealing with continuous mild solution, we cannot differentiate the above functionals directly. Hence we first estimate the above functionals along the smooth mollifiers and then passing to the limit of $\varepsilon \to 0$, we recover the estimate for the original mild solutions.

Let f_ε be the corresponding mollification of f. Then as in (4.1) we can similarly define $\mathcal{D}(f_\varepsilon(t))$ and $\Lambda(f_\varepsilon(t))$. Note that from (2.7) and (2.9) we have

$$\begin{aligned}
&\int_{\mathbb{R}^6} Q_-^\sharp(f_\varepsilon, f_\varepsilon)(x,p,t)\, dpdx \\
&\qquad = \int_{\mathbb{R}^9 \times \mathbf{S}^2} \mathbf{p}_M \sigma(g,\theta) f_\varepsilon^\sharp(t,x,p) f_\varepsilon^\sharp(x + t(v(p) - v(p_*)), p_*, t)\, d\theta dp_* dpdx \\
&\qquad \le 4\pi\sigma_1 \Lambda(f_\varepsilon(t)).
\end{aligned}$$

On the other hand, the relation $\int Q^\sharp_+(f_\varepsilon, f_\varepsilon)dpdx = \int Q^\sharp_-(f_\varepsilon, f_\varepsilon)dpdx$ implies

$$\int_{\mathbb{R}^6} Q^\sharp_+(f_\varepsilon, f_\varepsilon)(x,p,t)dpdx \;\le\; 4\pi\sigma_1\Lambda(f_\varepsilon(t)).$$

The next lemma establishes that the collision potential is well-defined in $\mathcal{S}(\alpha,\beta,\delta)$.

Lemma 4.1. *Suppose f is a mild solution in $\mathcal{S}(\alpha,\beta,\delta)$ and f_ε is the corresponding mollification of f. If $0<\alpha<\beta$ and $0<\varepsilon<1$, then we have for any $p\in\mathbb{R}^3$*

$$\int_{\mathbb{R}^3}\int_0^\infty \omega(g)|v(p)-v(p_*)|f^\sharp_\varepsilon(x+(t+\tau)(v(p)-v(p_*)),p_*,t)d\tau dp_* \le \delta\sqrt{\frac{2\pi}{\alpha}}\mathcal{C}.$$

Here $\mathcal{C}$ is defined in (2.12).

Proof. Since $f^\sharp_\varepsilon(x,p,t)\le\delta\left(M_{\alpha,\beta}\right)_\varepsilon(x,p)$ from Lemma 3.1, we have

$$f^\sharp_\varepsilon(x+(t+\tau)(v(p)-v(p_*)),p_*,t) \le \delta e^{-(\alpha/2)p_{*0}|x+(t+\tau)(v(p)-v(p_*))|^2}e^{-(\beta-\alpha)\sqrt{1+|p_*|^2}}.$$

We use the above estimate to see that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_0^\infty \omega(g)|v(p)-v(p_*)|f^\sharp_\varepsilon(x+(t+\tau)(v(p)-v(p_*)),p_*,t)d\tau dp_*\\
&\le \delta\int_{\mathbb{R}^3}\omega(g)|v(p)-v(p_*)|e^{-(\beta-\alpha)\sqrt{1+|p_*|^2}}\int_0^\infty e^{-\alpha|x+(t+\tau)(v(p)-v(p_*))|^2/2}d\tau dp_*\\
&\le \delta\sqrt{\frac{2\pi}{\alpha}}\int_{\mathbb{R}^3}\omega(g)e^{-(\beta-\alpha)|p_*|}dp_*\\
&= \delta\sqrt{\frac{2\pi}{\alpha}}\mathcal{C},
\end{aligned}$$

where we used the known estimate

$$\int_0^\infty e^{-\alpha|x+(t+\tau)(v(p)-v(p_*))|^2}d\tau \le \sqrt{\frac{\pi}{\alpha}}\frac{1}{|v(p)-v(p_*)|}.$$

This completes the proof. □

Remark 4.1. *Note that the collision potential $\mathcal{D}(f(t))$ is well-defined for L^1-functions:*

$$|\mathcal{D}(f(t))| \le \delta\sigma_1\sqrt{\frac{\pi}{\alpha}}C_4||f^{in}||_{L^1_{x,p}},$$

which holds for example under the conditions of Lemma 4.1. Here the constant C_4 is defined as follows.

$$C_4 := 2A_1\int_{\mathbb{R}^3}e^{-\beta|p_*|}dp_* + A_2C_1 < \infty.$$

We next study the behavior of time-phase space integral of the perturbation $P(f,f_\varepsilon)$ as $\varepsilon\to0$. For the convenience of notation, we define

$$\begin{aligned}
P(f,f_\varepsilon) &= Q_\varepsilon(f,f) - Q(f_\varepsilon,f_\varepsilon),\\
P_+(f,f_\varepsilon) &= Q_{+\varepsilon}(f,f) - Q_+(f_\varepsilon,f_\varepsilon),\\
P_-(f,f_\varepsilon) &= Q_{-\varepsilon}(f,f) - Q_-(f_\varepsilon,f_\varepsilon).
\end{aligned}$$

Here of course Q_ε, $Q_{+\varepsilon}$ and $Q_{-\varepsilon}$ are simply the mollified versions of Q_ε, Q_+ and Q_- respectively.

Lemma 4.2. *Suppose that the additional assumption* $(\mathcal{A})$ *from* (2.12) *is satisfied. Let* f *and* f_ε *be a mild solution in* $\mathcal{S}(\alpha,\beta,\delta)$ *to* (2.1) *with initial datum* f^{in} *and its corresponding mollification respectively. Then we have*

$$\lim_{\varepsilon\to 0+}\int_0^t\int_{\mathbb{R}^6}|P(f,f_\varepsilon)(x,p,s)|dpdxds=0,\quad t\in\mathbb{R}_+.$$

Proof. Since

$$\int_0^t\int_{\mathbb{R}^6}|P(f,f_\varepsilon)|dpdxds\le\int_0^t\int_{\mathbb{R}^6}|P_+(f,f_\varepsilon)|dpdxds+\int_0^t\int_{\mathbb{R}^6}|P_-(f,f_\varepsilon)|dpdxds,$$

it suffices to show that the integrand $P_\pm(f,f_\varepsilon)$ goes to zero pointwise as $\varepsilon\to 0$, and it is dominated by some L^1-function to apply the Lebesgue Dominated Convergence Theorem(LDCT).

- Case 1: Note that

$$\int_0^t\int_{\mathbb{R}^6}|P_-^\sharp(f,f_\varepsilon)|dpdxds\le\int_0^t\int_{\mathbb{R}^9\times\mathbb{S}^2}\mathbf{p}_M\sigma_1\omega(g)|(f^\sharp f_*^\sharp)_\varepsilon-f_\varepsilon^\sharp f_{*\varepsilon}^\sharp|d\theta dp_*dpdxds$$

Therefore we claim:

$$\mathbf{p}_M\sigma_1\omega(g)|(f^\sharp f_*^\sharp)_\varepsilon-f_\varepsilon^\sharp f_{*\varepsilon}^\sharp|\quad\text{satisfies the conditions of LDCT.}$$

The proof of claim:

- (Pointwise converegence): By the property of a convolution, it is easy to see that

$$\mathbf{p}_M\sigma_1\omega(g)|(f^\sharp f_*^\sharp)_\varepsilon-f_\varepsilon^\sharp f_{*\varepsilon}^\sharp|\to 0\text{ as }\varepsilon\to 0\quad\text{a.e.}$$

- (Finding a dominating L^1-function): From Lemma 3.1, we obtain for $0<\varepsilon<1$,

$$\begin{aligned}&|(f^\sharp f_*^\sharp)_\varepsilon-f_\varepsilon^\sharp f_{*\varepsilon}^\sharp|\\
\le&\delta^2\Big[\int_{\mathbf{B}_\varepsilon(0)}M_{\alpha,\beta}(x-y,p)M_{\alpha,\beta}(x-y,p_*)\phi_\varepsilon(y)dy+(M_{\alpha,\beta})_\varepsilon(x,p)\,(M_{\alpha,\beta})_\varepsilon(x,p_*)\Big]\\
\le&\delta^2\Big[e^{-\beta(p_0+p_{*0})}\int_{\mathbf{B}_\varepsilon(0)}e^{-\alpha|x-y|^2(p_0+p_{*0})}\phi_\varepsilon(y)dy+e^{-(\alpha/2)|x|^2(p_0+p_{*0})}e^{-(\beta-\alpha)(p_0+p_{*0})}\Big]\\
\le&\delta^2\Big[e^{-\beta(p_0+p_{*0})}e^{-(\alpha/2)|x|^2(p_0+p_{*0})}e^{\alpha(p_0+p_{*0})}+e^{-(\alpha/2)|x|^2(p_0+p_{*0})}e^{-(\beta-\alpha)(p_0+p_{*0})}\Big]\\
\le&2\delta^2e^{-(\alpha/2)|x|^2(p_0+p_{*0})}e^{-(\beta-\alpha)(p_0+p_{*0})}.\end{aligned}\tag{4.2}$$

This yields that $\sigma_1\omega(g)\times$(4.2) is integrable on $S^2\times\mathbb{R}^9\times[0,t]$. Therefore, by LDCT we have

$$\lim_{\varepsilon\to 0+}\int_0^t\int_{\mathbb{R}^6}|P_-(f,f_\varepsilon)|dpdxds=0.$$

- Case 2: Similar to Case 1, we also obtain

$$\lim_{\varepsilon\to 0+}\int_0^t\int_{\mathbb{R}^6}|P_+(f,f_\varepsilon)|dpdxds=0.$$

Finally we combine Case 1 and Case 2 to obtain the desired result. □

Proposition 4.1. *Let* f *be a global mild solution in* $\mathcal{S}(\alpha,\beta,\delta)$ *with corresponding initial datum* f^{in}. *If* $0<\alpha<\beta$ *and*

$$0<\delta<\frac{1}{8\pi\sigma_1\mathcal{C}}\sqrt{\frac{\alpha}{2\pi}},\tag{4.3}$$

then the collision potential $\mathcal{D}(f(t))$ satisfies

$$\mathcal{D}(f(t)) + \tilde{G}\int_0^t \Lambda(f(s))ds \le \mathcal{D}(f(0)), \qquad t \ge 0, \tag{4.4}$$

where $\tilde{G}$ is a positive constant independent of t.

Proof. Let f_ε be a corresponding mollification of f. It follows from (3.1) that

$$\partial_t f^\sharp_\varepsilon(x,p,t) = Q^\sharp(f_\varepsilon, f_\varepsilon)(x,p,t) + P^\sharp(f, f_\varepsilon)(x,p,t), \tag{4.5}$$

$$\begin{aligned}&\partial_t[f^\sharp_\varepsilon(x+(t+\tau)(v(p)-v(p_*)),p_*,t)]\\ &\quad = \partial_\tau[f^\sharp_\varepsilon(x+(t+\tau)(v(p)-v(p_*)),p_*,t)]\\ &\qquad + \Big(Q^\sharp(f_\varepsilon,f_\varepsilon)+P^\sharp(f,f_\varepsilon)\Big)(x+(t+\tau)(v(p)-v(p_*)),p_*,t).\end{aligned} \tag{4.6}$$

We use the above relation to see

$$\begin{aligned}&\partial_t\Big[f^\sharp_\varepsilon(x,p,t)f^\sharp_\varepsilon(x+(t+\tau)(v(p)-v(p_*)),p_*,t)\Big]\\ &\quad = \partial_\tau\Big[f^\sharp_\varepsilon(x,p,t)f^\sharp_\varepsilon(x+(t+\tau)(v(p)-v(p_*)),p_*,t)\Big]\\ &\qquad + \Big[Q^\sharp(f_\varepsilon,f_\varepsilon)+P^\sharp(f,f_\varepsilon)\Big](x,p,t)f^\sharp(x+(t+\tau)(v(p)-v(p_*)),p_*,t)\\ &\qquad + f^\sharp(x,p,t)\Big[Q^\sharp(f_\varepsilon,f_\varepsilon)+P^\sharp(f,f_\varepsilon)\Big](x+(t+\tau)(v(p)-v(p_*)),p_*,t).\end{aligned} \tag{4.7}$$

We multiply (4.7) by $\omega(g)|v(p)-v(p_*)|$ and integrate over $\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}_+$ with respect to (x,p,p_*,τ) to get

$$\begin{aligned}&\frac{d\mathcal{D}(f_\varepsilon(t))}{dt}\\ &= -\int_{\mathbb{R}^9}\omega(g)|v(p)-v(p_*)|\times f^\sharp_\varepsilon(x,p,t)f^\sharp_\varepsilon(x+t(v(p)-v(p_*)),p_*,t)dp_*dpdx\\ &\quad + \int_{\mathbb{R}^9\times\mathbb{R}_+}\omega(g)|v(p)-v(p_*)|\Big(Q^\sharp(f_\varepsilon,f_\varepsilon)+P^\sharp(f,f_\varepsilon)\Big)(x,p,t)\\ &\quad \times f^\sharp_\varepsilon(x+(t+\tau)(v(p)-v(p_*)),p_*,t)d\tau dp_*dpdx\\ &\quad + \int_{\mathbb{R}^9\times\mathbb{R}_+}\omega(g)|v(p)-v(p_*)|\\ &\quad \times\Big(Q^\sharp(f_\varepsilon,f_\varepsilon)+P^\sharp(f,f_\varepsilon)\Big)(x+(t+\tau)(v(p)-v(p_*)),p_*,t)f^\sharp_\varepsilon(x,p,t)d\tau dp_*dpdx\\ &:= -\Lambda(f_\varepsilon(t))+\mathcal{I}_1(t)+\mathcal{I}_2(t).\end{aligned} \tag{4.8}$$

By the change of variable $(x+(t+\tau)(v(p)-v(p_*))\to\bar{x})$ and the $p\leftrightarrow p_*$ symmetry, we have $\mathcal{I}_1=\mathcal{I}_2$. We use Lemma 4.1 to estimate $\mathcal{I}_1$

$$\begin{aligned}\mathcal{I}_1(t) &\le \int_{\mathbb{R}^6}\Big(Q^\sharp_+(f_\varepsilon,f_\varepsilon)+P^\sharp(f,f_\varepsilon)\Big)(x,p,t)\int_{\mathbb{R}^3\times\mathbb{R}_+}\omega(g)|v(p)-v(p_*)|\\ &\qquad \times f^\sharp_\varepsilon(x+(t+\tau)(v(p)-v(p_*)),p_*,t)d\tau dp_*dpdx\\ &\le \delta\sqrt{\frac{2\pi}{\alpha}}\mathcal{C}\Big(4\pi\sigma_1\Lambda(f_\varepsilon(t))+\int_{\mathbb{R}^6}P^\sharp(f,f_\varepsilon)(x,p,t)dxdp\Big).\end{aligned} \tag{4.9}$$

Here we recall $\mathcal{C}$ from Remark 3.1. In (4.8), we use (4.9) and $\mathcal{I}_1=\mathcal{I}_2$ to see

$$\frac{d\mathcal{D}(f_\varepsilon(t))}{dt} \le -\tilde{G}\Lambda(f_\varepsilon(t)) + 2\delta\sqrt{\frac{2\pi}{\alpha}}\mathcal{C}\int_{\mathbb{R}^6}P^\sharp(f,f_\varepsilon)(x,p,t)dxdp, \tag{4.10}$$

where we used simple notation

$$\tilde{G} := 1 - 8\pi\sigma_1\delta\sqrt{\frac{2\pi}{\alpha}}\mathcal{C}.$$

On the other hand, it follows from the assumption (4.3) that

$$\tilde{G} > 0$$

,and we integrate (4.10) to get

$$\mathcal{D}(f_\varepsilon(t)) + \tilde{G}\int_0^t \Lambda(f_\varepsilon(s))ds \quad \leq \quad \mathcal{D}(f_\varepsilon(0)) + \int_0^t \int_{\mathbb{R}^6} P^\sharp(f, f_\varepsilon)(x,p,s)dxdpds.$$

We let $\varepsilon \to 0$ and use Lemma 4.2 to obtain

$$\mathcal{D}(f(t)) + \tilde{G}\int_0^t \Lambda(f(s))ds \;\leq\; \mathcal{D}(f(0)).$$

□

Remark 4.2. *(i) Proposition 4.1 yields*

$$\int_0^\infty \int_{\mathbb{R}^6} |Q^\sharp(f,f)|(x,p,t)dxdvdt \leq 8\pi\sigma_1 \int_0^\infty \Lambda(t)dt \leq \frac{8\pi\sigma_1}{\tilde{G}}\mathcal{D}(f^{in}),$$

which satisfies the scattering condition in [16].
(ii) A similar Glimm type functional without the factor $\mathbf{p}_M\sigma(g,\theta)$ has been employed in [16] for a bounded σ.
(iii) In fact, the L^1-completeness can be proved without the smallness condition of initial datum in [23] using the pointwise estimate of the collision operator.

5. **Uniform L^1-stability estimates.** In this section, we provide the relativistic version of the nonlinear functional which is equivalent to the L^1-distance between two mild solutions, and also estimate time-decay of the functional.

5.1. **Nonlinear functional.** Let f and $\bar{f}$ be two mild solutions in $\mathcal{S}(\alpha,\beta,\delta)$ to (2.1) corresponding to initial data f^{in} and $\bar{f}^{in}$ respectively and let f_ε and $\bar{f}_\varepsilon$ be their corresponding mollifications of f and $\bar{f}$ respectively.

Recall that the difference $|f_\varepsilon - \bar{f}_\varepsilon|^\sharp(x,p,t)$ satisfies a differential inequality:

$$\frac{d}{dt}(|f_\varepsilon - \bar{f}_\varepsilon|^\sharp(x,p,t)) \leq \mathcal{R}^\sharp(f_\varepsilon,\bar{f}_\varepsilon)(x,p,t) + |P^\sharp(f,f_\varepsilon) - P^\sharp(\bar{f},\bar{f}_\varepsilon)|(x,p,t), \quad (5.1)$$

where

$$\begin{aligned}
&\mathcal{R}(f_\varepsilon,\bar{f}_\varepsilon)(x+tv(p),p,t) \\
&\quad := \frac{\sigma_1}{2}\int_{\mathbb{R}^3\times\mathbb{S}^2} \omega(g)\ \mathbf{p}_M \times\ \Big\{|f_\varepsilon - \bar{f}_\varepsilon|(x+tv(p),p',t)\big(f_\varepsilon+\bar{f}_\varepsilon\big)(x+tv(p),p'_*,t) \\
&\qquad +\ |f_\varepsilon - \bar{f}_\varepsilon|(x+tv(p),p'_*,t)\big(f_\varepsilon+\bar{f}_\varepsilon\big)(x+tv(p),p',t) \\
&\qquad +\ |f_\varepsilon - \bar{f}_\varepsilon|(x+tv(p),p,t)\big(f_\varepsilon+\bar{f}_\varepsilon\big)(x+tv(p),p_*,t) \\
&\qquad +\ |f_\varepsilon - \bar{f}_\varepsilon|(x+tv(p),p_*,t)\big(f_\varepsilon+\bar{f}_\varepsilon\big)(x+tv(p),p,t)\ \Big\}\ d\theta dp_*.
\end{aligned} \quad (5.2)$$

We now introduce the nonlinear functional $\mathcal{H}^\epsilon(t)$ as follows:

$$\mathcal{H}^\varepsilon(t) := \int_{\mathbb{R}^6} |f_\varepsilon - \bar{f}_\varepsilon|^\sharp(x,p,t)\Big[1 + K\int_{\mathbb{R}^3\times\mathbb{R}_+} \mathbf{p}_M\omega(g) \\ \times(f_\varepsilon + \bar{f}_\varepsilon)^\sharp(x + (t+\tau)(v(p) - v(p_*)), p_*, t)dp_*d\tau\Big]dxdp,$$

where K is a positive constant to be determined later.

It follows from Lemma 4.1 that

$$\int_{\mathbb{R}^3\times\mathbb{R}_+} \mathbf{p}_M\omega(g)(f_\varepsilon + \bar{f}_\varepsilon)^\sharp(x + (t+\tau)(v(p) - v(p_*)), p_*, t)dp_*d\tau\Big] \le 2\delta\sqrt{\frac{2\pi}{\alpha}}\mathcal{C}.$$

Hence we have the equivalence between $\mathcal{H}^\varepsilon$ and $||f_\varepsilon(t) - \bar{f}_\varepsilon(t)||_{L^1_{x,p}}$:

$$||f_\varepsilon(t) - \bar{f}_\varepsilon(t)||_{L^1_{x,p}} \le \mathcal{H}^\varepsilon(t) \le \Big(1 + 2\delta\sqrt{\frac{2\pi}{\alpha}}\mathcal{C}K\Big)||f_\varepsilon(t) - \bar{f}_\varepsilon(t)||_{L^1_{x,p}}.$$

Letting $\varepsilon \to 0$, it follows from the property of convolution that

$$||f(t) - \bar{f}(t)||_{L^1_{x,p}} \le \mathcal{H}(t) \le \Big(1 + 2\delta\sqrt{\frac{2\pi}{\alpha}}\mathcal{C}K\Big)||f(t) - \bar{f}(t)||_{L^1_{x,p}}, \tag{5.3}$$

where $\mathcal{H}(t)$ is a similar functional corresponding to f and $\bar{f}$. In the following proposition, we study time-evolution of the nonlinear functional $\mathcal{H}(t)$.

Proposition 5.1. *Suppose the parameters α and β satisfy*

$$0 < \alpha < \beta$$

and let $f, \bar{f} \in \mathcal{S}(\alpha,\beta,\delta)$ be mild solutions and f_ε and $\bar{f}_\epsilon$ be their mollifications satisfying the equation (3.1). Then for $0 < \varepsilon < 1$ we have

$$\begin{aligned} \frac{d}{dt}\mathcal{H}^\varepsilon(t) \quad &\le \quad -C_5\Lambda_d^\varepsilon(t) + \mathcal{O}(1)\big(\frac{1}{1+t}\big)^{3-\gamma}\mathcal{H}^\varepsilon(t) \\ &\quad + \mathcal{O}(1)\int_{\mathbb{R}^6} |P(f,f_\varepsilon) - P(\bar{f},\bar{f}_\varepsilon)|^\sharp(x,p,t)dxdp, \end{aligned}$$

where C_5 is a constant independent of t, and Λ_d^ε is an interaction production rate:

$$\Lambda_d^\varepsilon(t) := \int_{\mathbb{R}^6} |f_\varepsilon - \bar{f}_\varepsilon|^\sharp(x,p,t) \\ \times\Big\{\int_{\mathbb{R}^3} \omega(g)\mathbf{p}_M(f_\varepsilon + \bar{f}_\varepsilon)^\sharp(x + t(v(p) - v(p_*)), p_*, t)dp_*\Big\}dxdp,$$

and we similarly define $\Lambda_d(t)$ as corresponding to f and $\bar{f}$.

Proof. We can obtain by a direct computation that

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{H}^{\varepsilon}(t)\\
&\le -K\int_{\mathbb{R}^9}|f_\varepsilon-\bar f_\varepsilon|^\sharp(x,p,t)\omega(g)\mathbf{p}_M\big(f_\varepsilon+\bar f_\varepsilon\big)^\sharp(x+t(v(p)-v(p_*)),p_*,t)dp_*dxdp\\
&\quad+(1+2\sqrt{\frac{2\pi}{\alpha}}\mathcal{C}K\delta)\int_{\mathbb{R}^6}\mathcal{R}^\sharp(f_\varepsilon,\bar f_\varepsilon)(x,p,t)+|P(f,f_\varepsilon)-P(\bar f,\bar f_\varepsilon)|^\sharp(x,p,t)dpdx\\
&\quad+K\int_{\mathbb{R}^6}|f_\varepsilon-\bar f_\varepsilon|^\sharp(x,p,t)\int_{\mathbb{R}^3\times\mathbb{R}_+}\omega(g)\mathbf{p}_M\\
&\quad\times\Big(Q^\sharp(f_\varepsilon,f_\varepsilon)+Q^\sharp(\bar f_\varepsilon,\bar f_\varepsilon)\Big)(x+(t+\tau)(v(p)-v(p_*)),p_*,t)d\tau dp_*dxdp\\
&\quad+K\int_{\mathbb{R}^6}|f_\varepsilon-\bar f_\varepsilon|^\sharp(x,p,t)\int_{\mathbb{R}^3\times\mathbb{R}_+}\omega(g)\mathbf{p}_M\\
&\quad\times\Big(P^\sharp(f,f_\varepsilon)+P^\sharp(\bar f,\bar f_\varepsilon)\Big)(x+(t+\tau)(v(p)-v(p_*)),p_*,t)d\tau dp_*dxdp\\
&=:-K\Lambda_d^\varepsilon(t)+(1+2\sqrt{\frac{2\pi}{\alpha}}\mathcal{C}K\delta)\int_{\mathbb{R}^6}\mathcal{R}^\sharp(f_\varepsilon,\bar f_\varepsilon)(x,p,t)dxdp\\
&\quad+(1+2\sqrt{\frac{2\pi}{\alpha}}\mathcal{C}K\delta)\int_{\mathbb{R}^6}|P(f,f_\varepsilon)-P(\bar f,\bar f_\varepsilon)|^\sharp(x,p,t)dxdp+K\mathcal{J}_1(t)+K\mathcal{J}_2(t).
\end{aligned}
$$

We now claim:

$$\int_{\mathbb{R}^6}\mathcal{R}^\sharp(f_\varepsilon,\bar f_\varepsilon)(x,p,t)\le 8\pi\sigma_1\Lambda_d^\varepsilon(t).$$

(*The proof of claim*): We first note that the mapping $(p',p'_*)\to(p,p_*)$ has the following "effective" Jacobian (see [24] for more details) as

$$\Big|\frac{\partial(p,p_*)}{\partial(p',p'_*)}\Big|=\frac{p_0p_{*0}}{p'_0p'_{*0}},$$

and we also consider

$$s=2(p_0p_{*0}-p\cdot p_*+1)\;=\;2(p'_0p'_{*0}-p'\cdot p'_*+1)=s'$$

by using (2.5). This yields

$$g=\sqrt{s-4}=\sqrt{s'-4}=g'\quad\text{and}\quad \mathbf{p}_M\frac{p_0p_{*0}}{p'_0p'_{*0}}=\frac{\sqrt{s(s-4)}}{2p_0p_{*0}}\frac{p_0p_{*0}}{p'_0p'_{*0}}=\mathbf{p}'_M.$$

Then, we use a linear change of variable in x to get

$$
\begin{aligned}
&\int_{\mathbb{R}^9\times\mathbb{S}^2}\omega(g)\mathbf{p}_M|f_\varepsilon-\bar f_\varepsilon|(x+tv(p),p',t)\big(f_\varepsilon+\bar f_\varepsilon\big)(x+tv(p),p'_*,t)d\theta dp_*dpdx\\
=\;&\int_{\mathbb{R}^9\times\mathbb{S}^2}\omega(g')\mathbf{p}'_M|f_\varepsilon-\bar f_\varepsilon|(x+tv(p),p',t)\big(f_\varepsilon+\bar f_\varepsilon\big)(x+tv(p),p'_*,t)d\theta dp'_*dp'dx\\
=\;&\int_{\mathbb{R}^9\times\mathbb{S}^2}\omega(g)\mathbf{p}_M|f_\varepsilon-\bar f_\varepsilon|(x+tv(p'),p,t)\big(f_\varepsilon+\bar f_\varepsilon\big)(x+tv(p'),p_*,t)d\theta dp_*dpdx\\
=\;&\int_{\mathbb{R}^9\times\mathbb{S}^2}\omega(g)\mathbf{p}_M|f_\varepsilon-\bar f_\varepsilon|(x+tv(p),p,t)\big(f_\varepsilon+\bar f_\varepsilon\big)(x+tv(p),p_*,t)d\theta dp_*dpdx\\
\le\;&4\pi\Lambda_d^\varepsilon(t).
\end{aligned}
$$

We next apply similar method to other terms of $\mathcal{R}^\sharp(f_\varepsilon,\bar{f}_\varepsilon)(x,p,t)$ to obtain

$$\int_{\mathbb{R}^6}\mathcal{R}^\sharp(f_\varepsilon,\bar{f}_\varepsilon)(x,p,t)\le 8\pi\sigma_1\Lambda_d^\varepsilon(t).$$

This completes the proof of claim.

We next estimate $\mathcal{J}_1$ and $\mathcal{J}_2$. Since

$$Q^\sharp(f_\epsilon,f_\epsilon)+P^\sharp(f,f_\epsilon)=Q_\epsilon^\sharp(f,f),$$

we have

$$\begin{aligned}
&\mathcal{J}_1+\mathcal{J}_2\\
&\le\int_{\mathbb{R}^6}|f_\varepsilon-\bar{f}_\varepsilon|^\sharp(x,p,t)\int_{\mathbb{R}^3\times\mathbb{R}_+}\omega(g)\mathbf{p}_M\\
&\quad\times\Big(Q_\varepsilon^\sharp(f,f)+Q_\varepsilon^\sharp(\bar{f},\bar{f})\Big)(x+(t+\tau)(v(p)-v(p_*)),p_*,t)d\tau dp_*dxdp\\
&\le\frac{2C_2e^{\alpha-\eta_1}}{(1+t)^{3-\gamma}}\int_{\mathbb{R}^6}|f_\varepsilon-\bar{f}_\varepsilon|^\sharp(x,p,t)\\
&\quad\times\Big\{\int_{\mathbb{R}^3\times\mathbb{R}_+}\omega(g)|v(p)-v(p_*)|e^{-(\alpha-\eta_1)|x+(t+\tau)(v(p)-v(p_*))|^2/2}e^{-(\beta-\eta_2)p_{*0}}d\tau dp_*\Big\}dxdp\\
&\le 2e^{\alpha-\eta_1}C_2\sqrt{\frac{2\pi}{\alpha-\eta_1}}\Big(\frac{1}{1+t}\Big)^{3-\gamma}\Big(\sup_{p\in\mathbb{R}^3}\int_{\mathbb{R}^3}\omega(g)e^{-(\beta-\eta_2)p_{*0}}dp_*\Big)\int_{\mathbb{R}^6}|f_\varepsilon-\bar{f}_\varepsilon|^\sharp(x,p,t)dxdp\\
&\le\frac{\mathcal{O}(1)}{(1+t)^{3-\gamma}}\mathcal{H}^\varepsilon(t),
\end{aligned}$$

where we used Corollary 3.1. We set

$$C_5:=K-8\pi\sigma_1\Big(1+2\sqrt{\frac{2\pi}{\alpha}}\mathcal{C}K\delta\Big).$$

This yields

$$\begin{aligned}
&\frac{d}{dt}\mathcal{H}^\varepsilon(t)\\
&\le -C_5\Lambda_d^\varepsilon(t)+\mathcal{O}(1)\Big(\frac{1}{1+t}\Big)^{3-\gamma}\mathcal{H}^\varepsilon(t)+\ \mathcal{O}(1)\int_{\mathbb{R}^6}|P(f,f_\varepsilon)-P(\bar{f},\bar{f}_\varepsilon)|^\sharp(x,p,t)dxdp.
\end{aligned}$$

This completes the proof. $\square$

We now return to the proof of Theorem 2.2.

5.2. **The proof of Theorem 2.2.** Since δ satisfies the additional assumption $(\mathcal{A})$ in (2.12) and $0<\gamma<2$, we can choose $K>0$ large enough to ensure that C_5 is positive and by Proposition 5.1 we obtain

$$\begin{aligned}
&\mathcal{H}^\varepsilon(t)+C_5\int_0^t\Lambda_d^\varepsilon(s)ds\\
&\le\mathcal{O}(1)e^{\int_0^t(1+s)^{-3+\gamma}ds}\mathcal{H}^\varepsilon(0)+\mathcal{O}(1)\int_0^t\int_{\mathbb{R}^6}|P(f,f_\varepsilon)-P(\bar{f},\bar{f}_\varepsilon)|^\sharp(x,p,s)dxdpds.
\end{aligned}$$

Then we send $\varepsilon \to 0$ to find

$$\begin{aligned}\mathcal{H}(t) + C_5 \int_0^t \Lambda_d(s) ds \leq & \mathcal{O}(1) e^{\int_0^t (1+s)^{-3+\gamma} ds} \mathcal{H}(0) \\ \leq & \mathcal{O}(1) e^{\int_0^\infty (1+s)^{-3+\gamma} ds} \mathcal{H}(0) =: C_6 \mathcal{H}(0).\end{aligned}$$

where we used Lemma 4.2.

From the positivity of the C_5, we have

$$\mathcal{H}(t) \leq C_6 \mathcal{H}(0).$$

Then the L^1 stability of mild solutions can be done as follows

$$||f(t) - \bar{f}(t)||_{L^1} \leq \mathcal{H}(t) \leq C_6 \mathcal{H}(0) \leq C_6 (1 + 2\delta \sqrt{\frac{2\pi}{\alpha}} \mathcal{C} K) ||f^{in} - \bar{f}^{in}||_{L^1},$$

where we used the relation (5.3). We set

$$G := C_6 \left(1 + 2\delta \sqrt{\frac{2\pi}{\alpha}} \mathcal{C} K \right)$$

to obtain the desired result. **Q.E.D.**

6. **Conclusion.** We provided a construction of a Glimm type interaction potential and the uniform L^1-stability estimate of a global mild solution to relativistic Boltzmann equation with some angular cutoff assumption in collision kernel near vacuum using a nonlinear functional approach. As in classical case [15], we explicitly constructed a relativistic analogue of a nonlinear functional which is equivalent to the L^1-distance between two mild solutions. Thanks to the uniform stability estimate for the nonlinear functional and pointwise estimate of the gain collision operator, we obtain the uniform L^1-stability of the continuous mild solutions to the relativistic Boltzmann equation. Our uniform L^1-stability estimate illustrates that the solution operator for the relativistic Boltzmann equation is Lipschitz continuous with respect to L^1-topology.

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