ALMOST EXPONENTIAL DECAY NEAR MAXWELLIAN

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Abstract. By direct interpolation of a family of smooth energy estimates for solutions near Maxwellian equilibrium and in a periodic box to several Boltzmann type equations in [7–9, 11], we show convergence to Maxwellian with any polynomial rate in time. Our results not only resolve the important open problem for both the Vlasov-Maxwell-Boltzmann system and the relativistic Landau-Maxwell system for charged particles, but also lead to a simpler alternative proof of recent decay results [6] for soft potentials as well as the Coulombic interaction, with precise decay rate depending on the initial conditions.

1. Introduction

There are two motivations of the current study. Recently, a new nonlinear energy method has been developed to construct global-in-time solutions *near* Maxwellian for various types of Boltzmann equations. One of the highlights of such a program is the construction of global-in-time solutions for the Boltzmann equation in the presence of a self-consistent electromagnetic field. Unfortunately, the question of determining a possible time-decay rate for the electromagnetic field has been left open. For near Maxwellian periodic solutions to the Boltzmann equation with no electromagnetic field, Ukai [13] obtained exponential convergence in the case of a cutoff hard potential. Caflisch [1,2], for cutoff soft potentials with $\gamma > -1$, obtained a convergence rate like $O(e^{-\lambda t^\beta})$ for $\lambda > 0$ and $0 < \beta < 1$. In the whole space, also for cutoff soft potentials with $\gamma > -1$, Ukai and Asano [14] obtained the rate $O(t^{-\alpha})$ with $0 < \alpha < 1$. However the decay for very soft potentials had been open.

Desvillettes and Villani have recently undertaken an impressive program to study the time-decay rate to a Maxwellian of large data solutions to Boltzmann type equations. Even though their assumptions *in general* are a priori and impossible to verify at present time, their method does lead to an almost exponential decay rate for the soft potentials and the Landau equation for solutions close to a Maxwellian. This surprising new decay result relies crucially on recent energy estimates in [7, 8] as well as other extensive and delicate work [3–6, 10, 12, 15]. To obtain time decay for these models with very 'weak' collision effects has been a very challenging open problem even for solutions near a Maxwellian, therefore it is of great interest to find simpler proofs for such decay.

The objective of this article is to give a direct proof of an almost exponential decay rate for solutions near Maxwellian to the Vlasov-Maxwell-Boltzmann system, the relativistic Landau-Maxwell system, the Boltzmann equation with a soft potential as well as the Landau equation. The common difficulty of all such problems is the fact that in the energy estimates [7–9, 11] the instant energy functional

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at each time is stronger than the dissipation rate. It is thus impossible to use a Gronwall type of argument to get the decay rate in time. Our main observation is that *a family* of energy estimates, not just one, have been derived in [7–9, 11]. Even though the instant energy is stronger for a fixed family, it is possible to be bounded by a fractional power of the dissipation rate via simple interpolations with stronger energy norms of another family of estimates. Only algebraic decay is possible due to such interpolations, but more regular initial data grants faster decay. In comparison to [6], our proofs are much simpler and our decay rates are more precise.

For both Vlasov-Maxwell-Boltzmann and relativistic Landau-Maxwell, such a family of energy norms are exactly the same as in [9, 11] with higher and higher derivatives. The interpolation in this case is between Sobolev norms.

On the other hand, for both soft potentials and Landau equations, such a family of energy norms depends on higher and higher powers of velocity weights and the interpolation in this case is between the different weight powers. Although these are different from the original norms used in [7, 8], new energy estimates can be derived by the improved method in [9, 11] with minimum modifications.

We remark that all the constants in general are not explicit, except for the polynomial decay exponent. This is due to the fact that the lower bound for the linearized collision operator is not obtained constructively.

2. **Vlasov-Maxwell-Boltzmann**

Dynamics of charged dilute particles (e.g., electrons and ions) are described by the Vlasov-Maxwell-Boltzmann system:

$$\begin{aligned} &\partial_t F_+ + v\cdot\nabla_x F_+ + (E+v\times B)\cdot\nabla_v F_+ = Q(F_+,F_+) + Q(F_+,F_-),\\ &\partial_t F_- + v\cdot\nabla_x F_- - (E+v\times B)\cdot\nabla_v F_- = Q(F_-,F_+) + Q(F_-,F_-), \end{aligned} \tag{2.1}$$

with initial data $F_\pm(0,x,v) = F_{0,\pm}(x,v)$. For notational simplicity we have set all the physical constants to be unity, see [9] for more background. Here $F_\pm(t,x,v)\geq 0$ are the spatially periodic number density functions for the ions (+) and electrons (-) respectively at time $t\geq 0$, position $x=(x_1,x_2,x_3)\in[-\pi,\pi]^3=\mathbb{T}^3$ and velocity $v=(v_1,v_2,v_3)\in\mathbb{R}^3$. As a model problem, the collision between particles is given by the standard Boltzmann collision operator with hard-sphere interaction Q in [9].

The self-consistent, spatially periodic electromagnetic field $[E(t,x),B(t,x)]$ in (2.1) is coupled with $F_\pm(t,x,v)$ through the Maxwell system:

$$\partial_t E - \nabla_x\times B = -\mathcal{J},\ \partial_t B + \nabla_x\times E = 0, \tag{2.2}$$

with constraints $\nabla_x\cdot B = 0$, $\nabla_x\cdot E=\rho$ and initial data $E(0,x)=E_0(x)$, $B(0,x) = B_0(x)$. The coupling comes through the charge density ρ and the current density $\mathcal{J}$ which are given by

$$\rho = \int_{\mathbb{R}^3}\{F_+-F_-\}dv,\ \mathcal{J}=\int_{\mathbb{R}^3} v\{F_+-F_-\}dv. \tag{2.3}$$

We consider the perturbation $F_\pm = \mu+\sqrt{\mu}f_\pm\geq 0$ where the normalized Maxwellian is $\mu=\mu(v)=e^{-|v|^2}$.

Notation: Let $||\cdot||$ be the standard norm in either $L^2(\mathbb{T}^3\times\mathbb{R}^3)$ or $L^2(\mathbb{T}^3)$. We also define $||\cdot||_\nu$ to be the following weighted L^2 norm

$$||g||_\nu^2 \equiv \int_{\mathbb{T}^3\times\mathbb{R}^3} \nu(v)|g(x,v)|^2 dxdv,$$

where $\nu(v)$ is the collision frequency for hard-sphere interactions, $\nu(v)\sim C|v|$ as $|v|$ tends to ∞. Let the multi-indices α and β be $\alpha=[\alpha^0,\alpha^1,\alpha^2,\alpha^3]$, $\beta=[\beta^1,\beta^2,\beta^3]$ with length $|\alpha|$ and $|\beta|$ respectively. Then let $\partial^\alpha_\beta \equiv \partial_t^{\alpha^0}\partial_{x_1}^{\alpha^1}\partial_{x_2}^{\alpha^2}\partial_{x_3}^{\alpha^3}\partial_{v_1}^{\beta^1}\partial_{v_2}^{\beta^2}\partial_{v_3}^{\beta^3}$, which includes *temporal* derivatives. Define

$$|||f|||_N^2(t) \equiv \sum_{|\alpha|+|\beta|\le N} ||\partial^\alpha_\beta f_+(t)||^2 + \sum_{|\alpha|+|\beta|\le N} ||\partial^\alpha_\beta f_-(t)||^2, \tag{2.4}$$

$$|||[E,B]|||_N^2(t) \equiv \sum_{|\alpha|\le N} ||\partial^\alpha E(t)||^2 + \sum_{|\alpha|\le N} ||\partial^\alpha B(t)||^2.$$

Given initial datum $[f_0(x,p),E_0(x),B_0(x)]$, we define

$$\mathcal{E}_N(0) = \frac{1}{2}|||f_0|||_N^2 + |||[E_0,B_0]|||_N^2, \tag{2.5}$$

where the temporal derivatives of $[f_0,E_0,B_0]$ are defined naturally through equations (2.1) and (2.2). Further define the the *dissipation rate* as

$$|||f|||_{N,\nu}^2(t) \equiv \sum_{|\alpha|+|\beta|\le N} ||\partial^\alpha_\beta f_+||_\nu^2 + \sum_{|\alpha|+|\beta|\le N} ||\partial^\alpha_\beta f_-||_\nu^2.$$

We study the decay of solutions near $F_\pm \equiv \mu$, $E\equiv 0$, and $B\equiv \bar{B} =$ a constant.

Theorem 2.1. *Fix integers $N\ge 4$ and $k\ge 1$. Choose initial data $[f_{0,\pm},E_0,B_0]$ which satisfies the assumptions of Theorem 1 in* [9] *for $N+k$ (therefore also for N) including for $\epsilon_{N+k}>0$ small enough*

$$\mathcal{E}_{N+k}(0)\le \epsilon_{N+k},$$

and let $[F_\pm = \mu+\sqrt{\mu}f_\pm, E, B]$ be the unique solution to (2.1) and (2.2) with (2.3) from Theorem 1 of [9]*. Then there exists $C_{N,k}>0$ such that*

$$|||f|||_N^2(t) + |||[E,B-\bar{B}]|||_N^2(t) \le C_{N,k}\mathcal{E}_{N+k}(0)\left\{1+\frac{t}{k}\right\}^{-k},$$

where $\bar{B} = \frac{1}{|\mathbb{T}^3|}\int_{\mathbb{T}^3} B(t,x)dx$ is constant in time [9].

Proof. On the last page of [9], we see the following inequality (for $C_{|\beta|}>0$, $\delta_N>0$)

$$\frac{d}{dt}y_N(t) + \frac{\delta_N}{2}|||f|||_{\nu,N}^2(t)\le 0.$$

where the modified instant energy is

$$y_N(t) \equiv \frac{1}{2}\sum_{|\alpha|+|\beta|\le N} C_{|\beta|}||\partial^\alpha_\beta f(t)||^2 + |||[E,B-\bar{B}]|||_N^2(t).$$

To obtain a decay rate in t is to bound the instant energy $y_N(t)$ in terms of the dissipation rate $|||f|||_{\nu,N}^2(t)$. Since the collision frequency $\nu(v)$ is bounded from below in our hard-sphere interaction, part of $y_N(t)$ is clearly bounded by the dissipation rate because

$$\frac{1}{2}\sum_{|\alpha|+|\beta|\le N} C_{|\beta|}||\partial_\beta^\alpha f(t)||^2 \le C|||f|||_{\nu,N}^2(t). \tag{2.6}$$

On the other hand, to estimate the field component $[E, B-\bar{B}]$ of $y_N(t)$, Lemma 9 in [9] only implies

$$|||[E, B-\bar{B}]|||_{N-1}^2(t) \le C|||f|||_{\nu,N}^2(t). \tag{2.7}$$

No estimates for the highest N-th derivatives of E and B are available. Hence for fixed N, $|||f(t)|||_{\nu,N}^2$ is weaker than the total instant energy $y_N(t)$.

The key is to use interpolation to get a lower bound for the N-th order derivatives $\partial^\alpha E$ and $\partial^\alpha B$ with the bound of higher energy $|||f|||_{N+k}^2(t)$, for some k large. Let $\partial^\alpha = \partial_t^{\alpha_0}\partial_x^{\alpha'}$ and $|\alpha| = N$. For purely spatial derivaitves with $\alpha_0 = 0$ and $|\alpha'| = N$, a simple interpolation between spatial Sobolev spaces H_x^{N-1} and H_x^{N+k} yields

$$\begin{aligned}
||\partial^\alpha E||^2 + ||\partial^\alpha B||^2 &= ||\partial_x^{\alpha'} E||^2 + ||\partial_x^{\alpha'}\{B-\bar{B}\}||^2 \\
&\le C||[E, B-\bar{B}]||_{H_x^{N-1}}^{2k/(k+1)} \times ||[E, B-\bar{B}]||_{H_x^{N+k}}^{2/(k+1)} \\
&\le C||[E, B-\bar{B}]||_{N-1}^{2k/(k+1)}||[E, B-\bar{B}]||_{N+k}^{2/(k+1)}.
\end{aligned}$$

By our assumption and Theorem 1 in [9]

$$||[E, B-\bar{B}]||_{N+k}^2(t) \le C_{N+k}\mathcal{E}_{N+k}(0) < \infty.$$

We thus conclude by (2.7) that

$$\begin{aligned}
||\partial^\alpha E||^2 + ||\partial^\alpha B||^2 &\le C\left(C_{N+k}\mathcal{E}_{N+k}(0)\right)^{1/(k+1)}||[E, B-\bar{B}]||_{N-1}^{2k/(k+1)} \\
&\le C\left(\mathcal{E}_{N+k}(0)\right)^{1/(k+1)}|||f|||_{\nu,N}^{2k/(k+1)}.
\end{aligned} \tag{2.8}$$

Now for the general case with $\alpha_0 \neq 0$, we take $\partial_t^{\alpha_0-1}\partial_x^{\alpha'}$ of the Maxwell system (2.2) to get

$$\begin{aligned}
\partial_t^{\alpha_0}\partial_x^{\alpha'} E &= \partial_t^{\alpha_0-1}\partial_x^{\alpha'}\nabla_x \times B - \int \sqrt{\mu}\,\partial_t^{\alpha_0-1}\partial_x^{\alpha'}\{f_+ - f_-\}dv, \\
\partial_t^{\alpha_0}\partial_x^{\alpha'} B &= -\partial_t^{\alpha_0-1}\partial_x^{\alpha'}\nabla_x \times E.
\end{aligned}$$

Since $||\int\sqrt{\mu}\,\partial_t^{\alpha_0-1}\partial_x^{\alpha'}\{f_+ - f_-\}dv||_{L^2(\mathbb{T}^3)}^2$ is clearly bounded by $C|||f(t)|||_N^2$ or equivalently $Cy_N(t)$, which is bounded by

$$C\left(\mathcal{E}_{N+k}(0)\right)^{1/(k+1)}|||f|||_{\nu,N}^{2k/(k+1)}.$$

We therefore deduce that (2.8) is valid for general α with $\alpha_0 \ge 1$ via a simple induction of α_0, starting from $\alpha_0 = 1$. Moreover, for the full dissipation rate

$$\frac{\delta_N}{2}|||f|||_{\nu,N}^2 \ge C_{N,k}\left(\mathcal{E}_{N+k}(0)\right)^{-1/k} y_N^{\{k+1\}/k}.$$

It follows that

$$\frac{dy_N}{dt} + C_{N,k}\left(\mathcal{E}_{N+k}(0)\right)^{-1/k} y_N^{\{k+1\}/k} \le 0, \tag{2.9}$$

and $y_N'(t)(y_N(t))^{-1-1/k} \le -C_{N,k}\left(\mathcal{E}_{N+k}(0)\right)^{-1/k}$. Integrating this over $[0,t]$, we have

$$k\{y_N(0)\}^{-1/k} - k\{y_N(t)\}^{-1/k} \le -\left(\mathcal{E}_{N+k}(0)\right)^{-1/k} C_{N,k}t.$$

Hence

$$\{y_N(t)\}^{-1/k} \geq t\frac{C_{N,k}}{k}\left(\mathcal{E}_{N+k}(0)\right)^{-1/k} + \{y_N(0)\}^{-1/k},$$

since we can assume $y_N(0) \leq C_{N,k}\mathcal{E}_{N+k}(0)$, our theorem thus follows. □

3. **Relativistic Landau-Maxwell System**

The relativistic Landau-Maxwell system is the most fundamental and complete model for describing the dynamics of a dilute collisional cold plasma in which particles interact through Coulombic collisions and through their self-consistent electromagnetic field. For notational simplicity, we consider the following normalized Landau-Maxwell system

$$\begin{aligned} &\partial_t F_+ + \frac{p}{p_0}\cdot\nabla_x F_+ + \left(E + \frac{p}{p_0}\times B\right)\cdot\nabla_p F_+ = \mathcal{C}(F_+,F_+) + \mathcal{C}(F_+,F_-) \\ &\partial_t F_- + \frac{p}{p_0}\cdot\nabla_x F_- - \left(E + \frac{p}{p_0}\times B\right)\cdot\nabla_p F_- = \mathcal{C}(F_-,F_-) + \mathcal{C}(F_-,F_+), \end{aligned} \tag{3.1}$$

with initial condition $F_\pm(0,x,p) = F_{0,\pm}(x,p)$. Here $F_\pm(t,x,p) \geq 0$ are the spatially periodic number density functions for ions (+) and electrons (−) at time $t \geq 0$, position $x = (x_1,x_2,x_3) \in \mathbb{T}^3$ and momentum $p = (p_1,p_2,p_3) \in \mathbb{R}^3$. The energy of a particle is given by $p_0 = \sqrt{1+|p|^2}$.

To completely describe a dilute plasma, the electromagnetic field $E(t,x)$ and $B(t,x)$ is coupled with $F_\pm(t,x,p)$ through the celebrated Maxwell system (2.2) where the charge density ρ and the current density $\mathcal{J}$ are given by

$$\rho = \int_{\mathbb{R}^3}\{F_+ - F_-\}\,dp,\ \ \mathcal{J} = \int_{\mathbb{R}^3}\frac{p}{p_0}\{F_+ - F_-\}dp. \tag{3.2}$$

The collision between particles is modelled by the relativistic Landau collision operator $\mathcal{C}$, and its normalized form is given by

$$\mathcal{C}(g,h)(p) \equiv \nabla_p\cdot\left\{\int_{\mathbb{R}^3}\Phi(P,P')\{\nabla_p g(p)h(p') - g(p)\nabla_{p'}h(p')\}dp'\right\},$$

where the four-vectors are $P = (p_0, p_{1,}p_{2,}p_3)$ and $P' = (p_0', p_1', p_2', p_3')$. Morover, $\Phi(P,P') \equiv \Lambda(P,P')S(P,P')$ with $\Lambda(P,P') \equiv \frac{(P\cdot P')^2}{p_0 p_0'}\left\{(P\cdot P')^2 - 1\right\}^{-3/2}$ and

$$S(P,P') \equiv \left\{(P\cdot P')^2 - 1\right\}I_3 + \{(P\cdot P') - 1\}(p\otimes p') - (p-p')\otimes(p-p').$$

Here the Lorentz inner product is $P\cdot P' = p_0p_0' - p\cdot p'$. Let $\sigma^{ij}(p) = \int \Phi^{ij}(P,P')J(p')dp'$, where $J(p) \equiv e^{-p_0}$ is a relativistic Maxwellian. We define

$$||g||_\sigma^2 \equiv \sum_{i,j}\int_{\mathbb{T}^3\times\mathbb{R}^3}\left\{2\sigma^{ij}\partial_{p_j}g\partial_{p_i}g + \frac{\sigma^{ij}}{2}\frac{p_i}{p_0}\frac{p_j}{p_0}g^2\right\}dxdp.$$

See [11] for more details. We write $F_\pm \equiv J + \sqrt{J}f_\pm$ and study the decay rate for $f_\pm$, E and B. Define $|||f|||_N^2(t)$ and $|||[E,B]|||_N^2(t)$ as in (2.4) with $\partial_\beta^\alpha \equiv \partial_t^{\alpha^0}\partial_{x_1}^{\alpha^1}\partial_{x_2}^{\alpha^2}\partial_{x_3}^{\alpha^3}\partial_{p_1}^{\beta^1}\partial_{p_2}^{\beta^2}\partial_{p_3}^{\beta^3}$ and $dv = dp$. Then the initial data is again measured by (2.5), but the *dissipation rate* is given by

$$|||f|||_{N,\sigma}^2(t) \equiv \sum_{|\alpha|+|\beta|\leq N}||\partial_\beta^\alpha f_+||_\sigma^2 + \sum_{|\alpha|+|\beta|\leq N}||\partial_\beta^\alpha f_-||_\sigma^2.$$

We study the decay of solutions near $F_{\pm} \equiv J(p)$, $E \equiv 0$ and $B \equiv \bar{B} =$ a constant.

Theorem 3.1. *Fix $N \geq 4$, $k \geq 1$ and choose initial data $[F_{0,\pm}, E_0, B_0]$ which satisfies the assumptions of Theorem 1 in* [11] *for $N+k$ (therefore also for N) including*

$$\mathcal{E}_{N+k}(0) \leq \epsilon_{N+k}, \ \epsilon_{N+k} > 0.$$

Let $[F_{\pm} = J + \sqrt{J} f_{\pm}, E, B]$ be the unique solution to (3.1) and (2.2) with (3.2) from Theorem 1 of [11]*, then there exists $C_{N,k} > 0$ such that*

$$|||f|||^2_N(t) + |||[E, B - \bar{B}]|||^2_N(t) \leq C_{N,k}\mathcal{E}_{N+k}(0)\left\{1 + \frac{t}{k}\right\}^{-k},$$

where $\bar{B} = \frac{1}{|\mathbb{T}^3|}\int_{\mathbb{T}^3} B(t,x)dx$ is constant in time [11].

The proof is exactly as in the case for the Vlasov-Maxwell-Boltzmann system because of two facts: $C||g||_\sigma \geq ||g||$ (Lemma 2 in [11]) and from Lemma 12 in [11]

$$|||[E, B]|||_{N-1}(t) \leq C|||f|||_{N,\sigma}(t).$$

4. **Soft Potentials**

The Boltzmann equation with a soft potential is

$$\partial_t F + v \cdot \nabla_x F = Q(F, F), \ F(0, x, v) = F_0(x, v),$$

where $F(t, x, v)$ is the spatially periodic distribution function for the particles at time $t \geq 0$, position $x = (x_1, x_2, x_3) \in \mathbb{T}^3$ and velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. The l.h.s. of this equation models the transport of particles and the operator on the r.h.s. models the effect of collisions on the transport:

$$Q(F, F) \equiv \int_{\mathbb{R}^3 \times S^2} |u - v|^\gamma B(\theta)\{F(u')F(v') - F(u)F(v)\}dud\omega.$$

The exponent in the collision operator is $\gamma = 1 - \frac{4}{s}$ with $1 < s < 4$ so that $-3 < \gamma < 0$ (soft potentials), and $B(\theta)$ satisfies the Grad angular cutoff assumption: $0 < B(\theta) \leq C|\cos\theta|$.

Notation: We define the collision frequency $\nu(v) \equiv \int |v - u|^\gamma \mu(u)B(\theta)dud\omega$, and

$$||g||^2_\nu = \int_{\mathbb{T}^3 \times \mathbb{R}^3} \nu(v)g^2 dxdv.$$

Notice that $\nu(v) \sim C|v|^\gamma$ as $|v| \to \infty$. Define a weight function in v by $w = w_\gamma(v) = [1 + |v|]^\gamma$ with $-3 < \gamma < 0$ and denote a weighted L^2 norm as

$$||g||^2_\ell \equiv \int_{\mathbb{T}^3 \times \mathbb{R}^3} w^{2\ell}|g|^2 dxdv, \ ||g||^2_{\nu,\ell} \equiv \int_{\mathbb{T}^3 \times \mathbb{R}^3} w^{2\ell}\nu g^2 dxdv.$$

Let $\partial^\alpha_\beta \equiv \partial^{\alpha^0}_t \partial^{\alpha^1}_{x_1}\partial^{\alpha^2}_{x_2}\partial^{\alpha^3}_{x_3}\partial^{\beta^1}_{v_1}\partial^{\beta^2}_{v_2}\partial^{\beta^3}_{v_3}$ which includes temporal derivatives. For fixed $N \geq 4$, we define the family of norms depending on ℓ as

$$|||f|||^2_\ell(t) \equiv \sum_{|\alpha|+|\beta| \leq N} ||\partial^\alpha_\beta f(t)||^2_{|\beta|-\ell},$$

$$|||f|||^2_{\nu,\ell}(t) \equiv \sum_{|\alpha|+|\beta| \leq N} ||\partial^\alpha_\beta f(t)||^2_{\nu,|\beta|-\ell}.$$

Since $\gamma < 0$, these norms include higher powers of v for larger $\ell \geq 0$. We denote the perturbation f such that $F(t,x,v) = \mu(v) + \sqrt{\mu(v)} f(t,x,v)$. Then the initial value problem can be written as

$$[\partial_t + v \cdot \nabla_x] f + Lf = \Gamma[f,f], \tag{4.1}$$

where L is the linear part of the collision operator Q and Γ is the non-linear part. Define the *instant energy* as $\mathcal{E}_\ell(f(t)) \equiv \frac{1}{2}|||f|||_\ell^2(t)$, with its initial counterpart $\mathcal{E}_\ell(0) \equiv \frac{1}{2}|||f_0|||_\ell^2$. At $t = 0$ we define the temporal derivatives naturally through (4.1).

Theorem 4.1. *Fix $-3 < \gamma < 0$ and fix integers $N \geq 8$, $\ell \geq 0, k \geq 0$. Choose initial data $F_0(x,v)$ which satisfies the assumptions of Theorem 1 in* [7]. *Furthermore, there are constants $C_\ell > 0$ and $\epsilon_\ell > 0$, such that if $\mathcal{E}_\ell(0) < \epsilon_\ell$ then there exists a unique global solution with $F(t,x,v) = \mu + \mu^{1/2} f(t,x,v) \geq 0$, where equivalently f satisfies (4.1) and*

$$\sup_{0 \leq s \leq \infty} \mathcal{E}_\ell(f(s)) \leq C_\ell \mathcal{E}_\ell(0). \tag{4.2}$$

Moreover if, $\mathcal{E}_{\ell+k}(0) < \epsilon_{\ell+k}$, then the unique global solution to the Boltzmann equation (4.1) satisfies

$$|||f|||_\ell^2(t) \leq C_{\ell,k} \mathcal{E}_{\ell+k}(0) \left(1 + \frac{t}{k}\right)^{-k}.$$

Remark 4.2. *The existence part of both this theorem and the next were proven for $\ell = 0$ and with no temporal derivatives in* [7] *and* [8]. *But the method was improved in* [9]. *Since the proof is very similar to those in* [7–9, 11], *we will only sketch it.*

Proof. 1. BASIC ESTIMATES: Our first goal is to show all the basic estimates in [7] are valid, with little changes in the proofs, for new norms with (harmless) temporal derivatives and an additional powers of the weight factor w. In what follows, we shall only pin point these minor chanegs. Lemma 1 in [7] holds for any $\theta < 0$ with the same proof except we replace Eq. (19) by

$$w^{2\theta}(v) \leq w^{2\theta}(|v| + |u|) \leq C w^{2\theta}(|v'| + |u'|) \leq C w^{2\theta}(v') w^{2\theta}(u').$$

One factor on the r.h.s. is then controlled by either $\sqrt{\mu(v')}$ or $\sqrt{\mu(u')}$ in Eq. (18) of [7]. Lemma 2 in [7] holds for $\theta < 0$ as well and the only difference is in Eqs. (48)-(50) of [7] we should use $w^{2\theta}(v) \leq C w^{2\theta}(v + u_{||}) w^{2\theta}(u_{||})$ instead of $w^{2\theta}(v) \leq 1$ (which is used for the case $\theta \geq 0$). Not only Lemma 4 in [7] is valid for $\theta < 0$ with the same proof, it is also valid for the case $\beta = 0$ as well. In fact, for $\theta < 0$, by Lemma 2 in [7], we split $K = K_c + K_s$ where $\left|\left(w^{2\theta} K_s \partial^\alpha f, \partial^\alpha f\right)\right| \leq \frac{1}{2}\|\partial^\alpha f\|_{\nu,\theta}^2$ by a trivial extension of Eq. (21) in [7] to the case $\theta < 0$. Finally, using the definition of K_c in Eqs. (24) and (40) as well as the Cauchy-Schwartz inequality we have $\left|\left(w^{-2\theta} K_c \partial^\alpha f, \partial^\alpha f\right)\right| \leq C\|\partial^\alpha f\|_\nu^2$ for some constant $C > 0$ which completes the case for $\beta = 0$ in Lemma 4 in [7]. Furthermore, by splitting $L = \nu + K$, we deduce that for some $C > 0$ and $\ell \geq 0$,

$$\left(w^{-2\ell} L \partial^\alpha f, \partial^\alpha f\right) \geq \frac{1}{2}\|\partial^\alpha f\|_{\nu,-\ell}^2 - C\|\partial^\alpha f\|_\nu^2. \tag{4.3}$$

Theorem 3 in [7] is valid for any weight function $w^{2\theta - 2\ell}$ instead of $w^{2\theta}$ with the same proof. Now Lemma 7 and the local well-posedness Theorem 4 in [7] are valid with the new norms with the same proof.

2. POSITIVITY OF L : Instead of Theorem 2 in [7], we show the following instantaneous positivity for the linearized Boltzmann operator L**:** Fix $\ell \geq 0$. Let $f(t,x,v)$ be a (local) classical solution to the Boltzmann equation with a soft potential with the new norm. There exists M_0 and $\delta_0 = \delta_0(M_0) > 0$ such that if $\mathcal{E}_\ell(f(t)) \leq M_0$, then

$$\sum_{|\alpha|\leq N} \langle L\partial^\alpha f(t), \partial^\alpha f(t)\rangle \geq \delta_0 |||\partial^\alpha f(t)|||_\nu^2. \tag{4.4}$$

The proof of (4.4) follows exactly as in section 4 in [9] in a much simplier fashion with *no* electromagnetic fields everywhere. Both Lemma 5 and Lemma 6 in [9] holds with $E = B = 0$. Lemma 7 (with $J = 0$) in [9] holds with new L for soft potentials and with $||\partial^\gamma f||$ replaced by $||\partial^\gamma f||_\nu$, thanks to Lemma 1 in [7]. Lemma 8 (with $E = B = 0$) in [9] is valid for new Γ for the soft potentials with $||\partial^\gamma f||$ replaced by $||\partial^\gamma f||_\nu$, thanks to Lemma 6 in [7]. The rest of the proof is the same (much simpler!) as the proof of Theorem 3 starting at p.620 in [9].

3. ENERGY ESTIMATE. Fix an integer $\ell \geq 0$. Let f be the unique local solution which satisfies the small amplitude assumption $\mathcal{E}_\ell(f(t)) \leq M_0$. We now show that for any given $0 \leq m \leq N$, $|\beta| \leq m$, there are constants $C_{|\beta|,\bar{\ell}} > 0$, $C^*_{m,\ell} > 0$ and $\delta_{m,\ell} > 0$ such that

$$\begin{aligned} \sum_{|\beta|\leq m, 0\leq \bar{\ell}\leq \ell, |\alpha|+|\beta|\leq N} & \left(\frac{1}{2}\frac{d}{dt} C_{|\beta|,\bar{\ell}} ||\partial^\alpha_\beta f(t)||^2_{|\beta|-\bar{\ell}} + \delta_{m,\ell} ||\partial^\alpha_\beta f(t)||^2_{\nu,|\beta|-\bar{\ell}} \right) \\ & \leq C^*_{m,\ell}\sqrt{\mathcal{E}_\ell(t)} |||f|||^2_{\nu,\ell}(t). \end{aligned} \tag{4.5}$$

We derive (4.5) in three steps.

We first consider pure spatial and temporal derivatives. Taking pure ∂^α derivatives of (4.1) we obtain

$$[\partial_t + v\cdot\nabla_x]\partial^\alpha f + L\partial^\alpha f = \partial^\alpha \Gamma[f,f]. \tag{4.6}$$

Take the inner product of this with $\partial^\alpha f$. We apply (4.4) and Theorem 3 in [7] with the new norms to obtain for some constant $C > 0$,

$$\sum_{|\gamma|\leq N} \left(\frac{1}{2}\frac{d}{dt} ||\partial^\alpha f(t)||^2 + \delta_0 ||\partial^\alpha f(t)||^2_\nu \right) \leq C\sqrt{\mathcal{E}_0(t)} |||f|||^2_\nu(t).$$

Next, we consider pure spatial and temporal derivatives with weights. We prove the result for $\ell > 0$ by a simple induction starting at $\ell = 0$, assume (4.5) is valid for $0 \leq \ell' < \ell$. Take the inner product of (4.6) with $w^{-2\ell}\partial^\alpha f$. Then for some constant $C > 0$, we obtain the bound

$$\sum_{|\alpha|\leq N} \left(\frac{1}{2}\frac{d}{dt} ||\partial^\alpha f(t)||^2_{-\ell} + \left(w^{-2\ell} L\partial^\alpha f, \partial^\alpha f \right) \right) \leq C\sqrt{\mathcal{E}_\ell(t)} |||f|||^2_{\nu,\ell}(t).$$

Plug in the lower bound (4.3) and add the cases $0 \leq \ell' < \ell$ multiplied by a large constant to establish (4.5) for $|\beta| = 0$ and $\ell \geq 0$.

Finally we prove the general case by an induction over the order of v derivatives. Assume (4.5) is valid for $|\beta| = m$. For $|\beta| = m+1$, we take ∂^α_β of (4.1) to obtain

$$[\partial_t + v\cdot\nabla_x]\partial^\alpha_\beta f + \partial_\beta L\partial^\alpha f = -\sum_{|\beta_1|=1} \binom{\beta}{\beta_1} \partial_{\beta_1} v\cdot\nabla_x \partial^\alpha_{\beta-\beta_1} f + \partial^\alpha_\beta \Gamma[f,f]. \tag{4.7}$$

Take the inner product of this with $w^{2|\beta|-2\ell}\partial^\alpha f$. Using Lemma 4 of [7] with $\theta = 2|\beta| - 2\ell$ we get

$$\left(w^{2|\beta|-2\ell}\partial_\beta\{L\partial^\alpha f\}, \partial_\beta^\alpha f\right) \geq ||\partial_\beta^\alpha f||^2_{\nu,|\beta|-\ell} - \eta \sum_{|\bar\beta|\leq|\beta|} ||\partial_{\bar\beta}^\alpha f||^2_{\nu,|\bar\beta|-\ell} - C_\eta||\partial^\alpha f||^2_{\nu,-\ell}.$$

Using Cauchy's inequality for $\eta > 0$, since $|\beta_1| = 1$ we have

$$\left|\left(w^{2|\beta|-2\ell}\partial_{\beta_1} v\cdot\nabla_x\partial^\alpha_{\beta-\beta_1} f, \partial^\alpha_\beta f\right)\right| \leq \eta||\partial^\alpha_\beta f(t)||^2_{\nu,|\beta|-\ell} + C_\eta||\nabla_x\partial^\alpha_{\beta-\beta_1} f||^2_{\nu,|\beta-\beta_1|-\ell},$$

which is found on p.336 of [7]. We use these two inequalities to conclude (4.5) as well as global existence and (4.2) exactly as in section 5 in [9, 11].

5. DECAY RATE: We let $m = N$ in (4.5) and define

$$y_\ell(t) \equiv \sum_{0\leq\bar\ell\leq\ell, |\alpha|+|\beta|\leq N} C_{|\beta|,\bar\ell}||\partial^\alpha_\beta f(t)||^2_{|\beta|-\bar\ell}.$$

Notice that $y_\ell(t)$ is equivalent to $|||f|||^2_\ell(t)$. By (4.2) for sufficiently small $\mathcal{E}_\ell(0)$, there exists $C > 0$ such that

$$\frac{d}{dt}y_\ell(t) + C|||f|||^2_{\nu,\ell}(t) \leq 0.$$

Clearly for fixed ℓ, the dissipation rate $|||f|||^2_{\nu,\ell}$ is weaker than the instant energy $y_\ell(t)$ since $\|f\|_{1/2}$ is equivalent to $\|f\|_\nu$. We shall interpolate with the stronger norm $y_{\ell+k}(t)$. Fix $\ell, k \geq 0$. A simple intepolation for $w^{-2\ell}$ between the weight functions $w^{-2\ell+2}$ and $w^{-2\ell-2k}$ yields

$$|||f|||_\ell \leq |||f|||^{k/(k+1)}_{\ell-1}|||f|||^{1/(k+1)}_{\ell+k} \leq C|||f|||^{k/(k+1)}_{\ell-1} \times \left(C_{\ell+k}\mathcal{E}_{\ell+k}(0)\right)^{1/2(k+1)}.$$

But from the definition of $w(v)$ and the fact $\nu(v) \leq Cw(v)$ we have

$$|||f|||^2_{\ell-1/2} \leq C|||f|||^2_{\nu,\ell}.$$

We thus have

$$y_\ell \leq C|||f|||^2_\ell \leq C\mathcal{E}^{1/(k+1)}_{\ell+k}(0)|||f|||^{2k/(k+1)}_{\ell-1} \leq C\mathcal{E}^{1/(k+1)}_{\ell+k}(0)|||f|||^{2k/k+1}_{\nu,\ell},$$

and for some $C_{\ell,k} > 0$,

$$\frac{d}{dt}y_\ell + C_{\ell,k}\mathcal{E}^{-1/k}_{\ell+k}(0)y_\ell^{\{k+1\}/k} \leq 0.$$

Now the theorem follows from the analysis of (2.9). □

5. **The Classical Landau Equation**

The Landau equation has the same form as the Boltzmann equation except

$$Q(F,F) = \nabla_v\cdot\left\{\int_{\mathbb{R}^3}\phi(v-v')[F(v')\nabla_v F(v) - F(v)\nabla_{v'}F(v')]dv'\right\},$$

where $\phi^{ij}(v) = \frac{1}{|v|}\left\{\delta_{ij} - \frac{v_iv_j}{|v|^2}\right\}$. Throught out this section we use the weight function

$$w_1(v) \equiv \{1 + |v|^2\}^{-1/2}.$$

In the classical case, we define $\sigma^{ij}(v) \equiv \int \phi^{ij}(v-v')\mu(v')dv'$ and

$$||g||^2_{\sigma,\theta} \equiv \sum_{i,j}\int_{\mathbb{T}^3\times\mathbb{R}^3} w^{2\theta}\sigma^{ij}\{\partial_{v_i}g\partial_{v_j}g + v_iv_jg^2\}dxdv.$$

We again define ∂^α_β including temporal derivativess. Fix $N \geq 4$ and $\ell \geq 0$ and define $|||f|||^2_\ell(t) \equiv \sum_{|\alpha|+|\beta|\leq N} ||w^{2(|\beta|-\ell)}\partial^\alpha_\beta f(t)||^2$. The Landau dissipation rate is

$$|||f|||^2_{\sigma,\ell}(t) \equiv \sum_{|\alpha|+|\beta|\leq N} ||\partial^\alpha_\beta f(t)||^2_{\sigma,|\beta|-\ell},$$

The standard perturbation $f(t,x,v)$ to μ is $F(t,x,v) = \mu(v) + \sqrt{\mu(v)}f(t,x,v)$. The instant energy is $\mathcal{E}_\ell(f(t)) \equiv \frac{1}{2}|||f|||^2_\ell(t)$ with its initial counterpart $\mathcal{E}_\ell(0) \equiv \frac{1}{2}|||f_0|||^2_\ell$.

Theorem 5.1. *Fix integers $N \geq 8$ and $\ell \geq 0$. Assume that $F_0(x,v) = \mu + \mu^{1/2}f_0(x,v)$ satisfies the assumptions of Theorem 1 in* [8]*. There are constants $C_\ell > 0$ and $\epsilon_\ell > 0$ such that if $\mathcal{E}_\ell(f_0) < \epsilon_\ell$ then there exists a unique global solution $F(t,x,v)$ to the classical Landau equation such that $F(t,x,v) = \mu + \mu^{1/2}f(t,x,v) \geq 0$, and*

$$\sup_{0\leq s\leq\infty} \mathcal{E}_\ell(f(s)) \leq C_\ell \mathcal{E}_\ell(0).$$

Furthermore if $\mathcal{E}_{\ell+k}(f_0) < \epsilon_{\ell+k}$ for some integer $k > 0$ then the unique global solution $f(t,x,v)$ satisfies

$$|||f|||^2_\ell(t) \leq C_{\ell,k}\mathcal{E}_{\ell+k}(0)\left(1+\frac{t}{k}\right)^{-k}.$$

Proof. We follow the same procedure as in the case for soft potentials. First, we check that basic estimates in [8] are valid with little changes in the proofs for new norms with (harmless) temporal derivatives as well as new weights $w = w_1^{-2\ell}$. We first notice that Lemma 6 in [8] is valid for $|\beta| = 0$ as well. The result for K is Lemma 5 in [8]. Therefore we focus on A. Lemma 3 in [8] implies

$$|\partial_{v_j}\{\partial_{v_i}w_1^{-2\ell}\sigma^{ij}\}| + |\partial_{v_j}\{w_1^{-2\ell}\partial_{v_i}\sigma^{ij}\}| \leq Cw_1^{2\ell}[1+|v|]^{-2}.$$

Now both in second and the fourth term in Eq. (31) in [8] with $\beta = 0$, we write $\partial_j f f = \frac{1}{2}\partial_j f^2$ and integate by parts to get a upper bound of

$$C\sum_{i,j}\int_{\mathbb{T}^3\times\mathbb{R}^3} w_1^{-2\ell}[1+|v|]^{-2}|f|^2\,dxdv \tag{5.1}$$

Since $||f||_{\sigma,-\ell}$ dominates $||f||_{-\ell+1/2}$, (5.1) is therefore bounded by $\eta||f||_{\sigma,-\ell}$ for large v for any small number η. This concludes the proof of Lemma 6 in [8] for the new norm. All the other Lemmas in sections 1-3 are valid for the new norms with identical proofs, which lead to local well-posedness Theorem 4 in [8] for the new norm.

Now we follow exactly the proof for the soft potential, simply replacing $|||\cdot|||_\nu$ by the corresponding Landau dissipation rate $|||\cdot|||_\sigma$ and the linearized Boltzmann operator by the linearized Landau operator. To establish the positivity (4.4) with Landau dissipation, we notice that $Lg = -\Gamma(g,\sqrt{\mu}) - \Gamma(\sqrt{\mu},g)$. By Lemma 7 in [8] for the new norms, we deduce that both Lemma 7 and Lemma 8 in [9] for Landau dissipation rate $||\partial^\gamma f||^2_\sigma$, instead of $||\partial^\gamma f||^2$. The rest of the proof is identical. □

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