Some Applications of an Energy Method in Collisional Kinetic Theory

by

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Abstract of "Some Applications of an Energy Method in Collisional Kinetic Theory" by Robert Mills Strain III, Ph.D., Brown University, May 2005

The collisional Kinetic Equations we study are all of the form

$$\partial_t F + v \cdot \nabla_x F + V(t,x) \cdot \nabla_v F = Q(F,F).$$

Here $F = F(t,x,v)$ is a probabilistic density function (of time $t \geq 0$, space $x \in \Omega$ and velocity $v \in \mathbb{R}^3$) for a particle taken chosen randomly from a gas or plasma. $V(t,x)$ is a field term which usually represents Maxwell's theory of electricity and magnetism, sometimes this term is neglected. $Q(F,F)$ is the collision operator which models the interaction between colliding particles. We consider both the Boltzmann and Landau collision operators.

We prove existence, uniqueness and regularity of close to equilibrium solutions to the relativistic Landau-Maxwell system in the first part of this thesis. Our main tool is an energy method.

In the second part, we prove arbitrarily high polynomial time decay rates to equilibrium for four kinetic equations. These are cutoff soft potential Boltzmann and Landau equations, but also the Vlasov-Maxwell-Boltzmann system and the relativistic Landau-Maxwell system. The main technique used here is interpolation.

In the third part, we prove exponential decay for the cutoff soft potential Boltzmann and Landau equations. The main point here is to show that exponential decay of the initial data is propagated by a solution.

In the fourth and final part of this thesis, we write down a few important calculations in the relativistic Boltzmann theory which are scattered around the literature. We also calculate a few Lorentz transformations which maybe useful in relativistic transport theory. We use these calculations to comment about extending the results in this thesis to the relativistic Boltzmann equation.

CHAPTER 1

Introduction

In this thesis, we study some mathematical properties of a class of nonlinear partial differential equations called "kinetic equations" which arise out of the Kinetic Theory of Gases and Plasmas. We are primarily concerned with stability theorems, e.g. results on existence, uniqueness and decay rates of solutions which are initially close to steady state.

1.0.1. Brief Introduction to Kinetic Theory. A rarefied gas is a large collection of electrically neutral particles (around 10^{23}) in which collisions form a small part of each particles lifetime. A plasma is a large collection of fast-moving charged particles. The large number of particles makes it difficult to attempt a physical description of a gas or plasma using Classical Mechanics.

Boltzmann (1872) instead decided to posit the existence of a probability density function $F(t,x,v)$ which roughly speaking tells you the average number of particles a time t, position x and velocity v. He proposed a partial differential equation which governs the time evolution of F. Boltzmann's Equation is the foundation of kinetic theory [11, 12, 15, 30, 50, 67, 68]. The density function $F(t,x,v)$ contains a great deal of information, it can be a practical tool for determining detailed properties of dilute gases and plasmas as well as calculating many important physical quantities.

Kinetic Theory has been among the most active areas in the mathematical study of nonlinear partial differential equations over the last few decades–spectacular progress has been made yet still more is expected. The main goal of this research is to study the existence, uniqueness, regularity and asymptotic properties of solutions F. And the major result thus far is the global renormalized weak solutions of Diperna & Lions [23] to the Boltzmann equation with large initial data.

Recently, Yan Guo has developed a flexible nonlinear energy method [39] used to construct global in time smooth solutions to the "hard sphere" Vlasov-Maxwell-Boltzmann system for initial data close to Maxwellian equilibrium. This system is a generalization of the Boltzmann equation which models two species of particles (electrons and ions) interacting and includes electromagnetic effects. It is a system of ten partial differential equations, two coupled Boltzmann equations that represent electrons and ions interacting also coupled with Maxwell's equations for electricity and magnetism. See [35–40] for more applications of such a method. There is also related work near vacuum [41]. In this Ph.D thesis, we apply this energy method a few important problems in kinetic theory.

1.1. Precise statements of the results

In this doctoral thesis, I develop new techniques to study existence, uniqueness, regularity and asymptotic convergence rates of several nonlinear partial differential equations from collisional Kinetic Theory. In Section 1.1.1, we discuss the relativistic Landau-Maxwell system. In Section 1.1.2, we discuss the asymptotic decay of four kinetic equations. And in Section 1.1.3, we discuss exponential decay for soft potential Boltzmann and Landau equations.

1.1.1. Relativistic Landau-Maxwell System. Dilute hot plasmas appear commonly in important physical problems such as Nuclear fusion and Tokamaks. The relativistic Landau-Maxwell system [50] is the most fundamental and complete model for describing the dynamics of a dilute collisional plasma in which particles interact through binary Coulombic collisions and through their self-consistent electromagnetic field. It is widely accepted as the most complete model for describing the dynamics of a dilute collisional fully ionized plasma.

Yan Guo and I proved existence of global in time classical solutions [62] with initial data near the relativistic Maxwellian. To the best of our knowledge, this is the first existence proof for the relativistic Landau-Maxwell system. In 2000, Lemou [47] studied the linearized relativistic Landau equation with no electromagnetic field. In the non-relativistic situation there are a few results for classical solutions to kinetic equations with the Landau collision operator [17, 38, 72].

We considered the following coupled relativistic Landau-Maxwell system (two equations):

$$\partial_t F_\pm + \frac{p}{p_0} \cdot \nabla_x F_\pm \pm \left(E + \frac{p}{p_0} \times B \right) \cdot \nabla_p F_\pm = \mathcal{C}(F_\pm, F_\pm) + \mathcal{C}(F_\pm, F_\mp)$$

Here $F_\pm(t, x, p) \geq 0$ are the spatially periodic number density functions for ions $(+)$ and electrons $(-)$ at time $t \geq 0$, position $x = (x_1, x_2, x_3) \in \mathbb{T}^3 = [-\pi, \pi]^3$ and momentum $p = (p_1, p_2, p_3) \in \mathbb{R}^3$. The energy of a particle is given by $p_0 = \sqrt{1 + |p|^2}$. The electromagnetic field, $E(t, x)$ and $B(t, x)$, is coupled with $F_\pm(t, x, p)$ through the celebrated Maxwell system:

$$\partial_t E - \nabla_x \times B = -4\pi \int_{\mathbb{R}^3} \frac{p}{p_0} \{F_+ - F_-\} \, dp, \ \partial_t B + \nabla_x \times E = 0,$$

with constraints $\nabla_x \cdot B = 0$, $\nabla_x \cdot E = 4\pi \int_{\mathbb{R}^3} \{F_+ - F_-\} \, dp$ and spatially periodic initial conditions.

The relativistic Landau collision operator is defined by

$$\mathcal{C}(G, H) \equiv \nabla_p \cdot \int_{\mathbb{R}^3} \Phi(p, q) \{\nabla_p G(t, x, p) H(t, x, q) - G(t, x, p) \nabla_q H(t, x, q)\} \, dq,$$

where the 3×3 matrix $\Phi(p, q)$ is a complicated expression [47, 50, 62].

Steady state solutions are $[F_\pm(t, x, p), E(t, x), B(t, x)] = [e^{-p_0}, 0, \bar{B}]$, where $\bar{B}$ is a constant which is determined initially. We consider the standard perturbation around the relativistic Maxwellian $F_\pm(t, x, p) = e^{-p_0} + e^{-\frac{1}{2}p_0} f_\pm(t, x, p)$. Define the high order energy norm for a solution to the relativistic Landau-Maxwell system as

$$\mathcal{E}_N(t) \equiv \frac{1}{2} |||[f_+, f_-]|||_N^2(t) + |||[E, B]|||_N^2(t) + \int_0^t |||[f_+, f_-]|||_{\sigma, N}^2(s) ds,$$

where $|||\cdot|||_N$ represents an L^2 sobolev norm in (x, p) (or just in x in the case of $[E, B]$) but also includes t derivatives. $|||\cdot|||_N$ contains (t, x, p) derivatives of all orders $\leq N$. The dissipation $|||\cdot|||_{\sigma, N}$ represents the same norm, but with a momentum weight and one extra p-derivative. At $t = 0$ the temporal derivatives are defined naturally through the equations.

THEOREM 1.1. [62]. *Fix* $N \geq 4$. *Let* $F_{0,\pm}(x,p) = e^{-p_0} + e^{-\frac{1}{2}p_0} f_{0,\pm}(x,p) \geq 0$. *Assume that initially* $[F_{0,\pm}, E_0, B_0]$ *has the same mass, total momentum and total energy as the steady state* $[e^{-p_0}, 0, \bar{B}]$.

$\exists C > 0, \epsilon > 0$ *such that if* $\mathcal{E}_N(0) \leq \epsilon$ *then there exists a unique global in time solution* $[F_\pm(t,x,p), E(t,x), B(t,x)]$ *to the relativistic Landau-Maxwell system. Moreover,*

$$F_\pm(t,x,p) = e^{-p_0} + e^{-\frac{1}{2}p_0} f_\pm(t,x,p) \geq 0$$

and $\sup_{0 \leq s \leq \infty} \mathcal{E}_N(s) \leq C\mathcal{E}_N(0)$.

The proof of this theorem uses the energy method from [39]. However, severe new mathematical difficulties were present in terms of the complexity of the relativistic Landau Collision kernel $\Phi(p,q)$ and the momentum derivatives in the collision operator $\mathcal{C}$. One important question left open in this result and in the case of the Vlasov-Maxwell-Boltzmann system [39] is the question of a precise time decay rate to equilibrium.

1.1.2. Almost Exponential Decay Near Maxwellian. The key difficulty in obtaining time decay for the relativistic Landau-Maxwell system is contained in the following differential inequality ($\delta_N > 0$):

$$\frac{dy_N}{dt}(t) + \frac{\delta_N}{2} |||[f_+, f_-]|||^2_{\sigma,N}(t) \leq 0,$$

where the instant energy, $y_N(t)$, is equivalent to

$$\frac{1}{2}|||[f_+, f_-]|||^2_N(t) + |||[E, B - \bar{B}]|||^2_N(t).$$

The difficulty lies in the fact that the instant energy functional at each time is stronger than the dissipation rate $|||[f_+, f_-]|||^2_{\sigma,N}(t)$:

$$|||[f_+, f_-]|||^2_N(t) + |||[E, B - \bar{B}]|||^2_{N-1}(t) \leq C|||[f_+, f_-]|||^2_{\sigma,N}(t).$$

No estimates for the N-th order derivatives of E and B are available. It is therefore difficult to imagine being able to use a Gronwall type of argument to get the time decay rate. Our main observation is that *a family* of energy estimates have been derived in [62]. The instant energy is stronger for fixed N, but it is possible to be bounded by a fractional power of the dissipation rate via simple interpolations with stronger energy norms. Only algebraic decay is possible due to such interpolations, but more regular initial data grants faster decay.

THEOREM 1.2. [60]. *Fix $N \geq 4$, $k \geq 1$ and choose initial data $[F_{0,\pm}, E_0, B_0]$ which satisfies the assumptions of the previous Theorem for both N and $N+k$. Then there exists $C_{N,k} > 0$ such that*

$$|||[f_+, f_-]|||_N^2(t) + |||[E, B - \bar{B}]|||_N^2(t) \leq C_{N,k}\mathcal{E}_{N+k}(0)\left(1 + \frac{t}{k}\right)^{-k}.$$

And the same decay holds for the Vlasov-Maxwell-Boltzmann system under similar conditions.

On a related note, Desvillettes and Villani have recently undertaken an impressive program to study the time-decay rate to Maxwellian of large data solutions to soft potential Boltzmann type equations. Even though their assumptions *in general* are a priori and impossible to verify at the present time, their method does lead to an almost exponential decay rate for the soft potential Boltzmann and Landau equations for solutions close to a Maxwellian. This surprising new decay result relies crucially on recent energy estimates in [37, 38] as well as other extensive and delicate work [18–21, 53, 64, 69]. To obtain time decay for these models with very 'weak' collision effects has been a very challenging open problem even for solutions near a Maxwellian, therefore it is of great interest to find simpler proofs. Using interpolations with higher velocity weights, Yan Guo and I gave a much simpler proof of almost exponential decay for the Boltzmann and Landau equations.

The Boltzmann equation is written as

$$\partial_t F + v \cdot \nabla_x F = Q(F, F), \;\; F(0, x, v) = F_0(x, v).$$

Above $F(t, x, v)$ is the spatially periodic density function for particles at time $t \geq 0$, position $x = (x_1, x_2, x_3) \in \mathbb{T}^3$ and velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. The collision operator is

$$Q(F,F) \equiv \int_{\mathbb{R}^3} du \int_{S^2} d\omega B(\theta)|u-v|^\gamma \{F(t,x,u')F(t,x,v') - F(t,x,u)F(t,x,v)\}.$$

Here $\theta = \theta(u,v,\omega)$ is the scattering angle and $[u',v'] = [u'(u,v,\omega), v'(u,v,\omega)]$ are the post-collisional velocities. The exponent in the collision operator is $-3 < \gamma < 0$ (soft potentials) and we assume $B(\theta)$ satisfies the Grad angular cutoff assumption: $0 < B(\theta) \leq C|\cos\theta|$. The Landau Equation is obtained in the so-called "grazing collision" limit of the Boltzmann equation [50].

We write the perturbation from Maxwellian, $\mu = e^{-|v|^2}$, as

$$F(t,x,v) = \mu(v) + \sqrt{\mu(v)} f(t,x,v).$$

Then define the high order energy norm

$$\mathcal{E}_\ell(t) \equiv \frac{1}{2}|||f|||_\ell^2(t) + \int_0^t |||f|||_{\nu,\ell}^2(s)ds, \tag{1.1}$$

where $|||f|||_\ell^2(t)$ is an instantaneous norm in time and an energy norm in the space and velocity variables which includes derivatives in all variables (t,x,v) and of all orders $\leq N$ where $N \geq 8$. Here ℓ denotes a polynomial velocity weight like $(1+|v|^2)^{-\gamma\ell/2}$. At $t=0$ we define the temporal derivatives naturally through the equation. $|||\cdot|||_{\nu,\ell}$ is the same norm as $|||\cdot|||_\ell$, but with another dissipation weight.

Theorem 1.3. [60]. *Fix $-3 < \gamma < 0$; fix integers $\ell \geq 0$ and $k > 0$. Let*

$$F_0(x,v) = \mu + \sqrt{\mu} f_0(x,v) \geq 0.$$

Assume the initial data $F_0(x,v)$ has the same mass, momentum and energy as the steady state μ.

$\exists C_\ell > 0, \epsilon_\ell > 0$ such that if $\mathcal{E}_\ell(0) < \epsilon_\ell$ then there exists a unique global in time solution to the soft potential Boltzmann equation

$$F(t,x,v) = \mu + \mu^{1/2} f(t,x,v) \geq 0$$

with $\sup_{0\leq s\leq\infty} \mathcal{E}_\ell(f(s)) \leq C_\ell \mathcal{E}_\ell(0)$.

Moreover, if $\mathcal{E}_{\ell+k}(0) < \epsilon_{\ell+k}$, then the unique global solution satisfies

$$|||f|||_{\ell}^{2}(t) \leq C_{\ell,k}\mathcal{E}_{\ell+k}(0)\left(1+\frac{t}{k}\right)^{-k}.$$

The same result holds for the Landau Equation.

We remark that the existence for both the Boltzmann and Landau case were proven in [37, 38] at $\ell = 0$.

1.1.3. Exponential Decay Near Maxwellian. These "almost exponential" decay results suggest strongly that under some conditions exponential decay can be achieved. Indeed, in the case of the Boltzmann equation with a soft potential Caflisch [8, 9] got decay of the form $\exp(-\lambda t^{p})$ for some $\lambda > 0$ and $0 < p < 1$ but only for $-1 < \gamma < 0$. Caflisch uses an exponential weight of the for $e^{q|v|^2}$ in his norms for $0 < q < \frac{1}{4}$. Yan Guo and I extend Caflisch's result to the soft potential Boltzmann equation for the full range $-3 < \gamma < 0$ and to the Landau equation using an exponential weight like $e^{q|v|^{\vartheta}}$ for $0 \leq \vartheta \leq 2$.

CHAPTER 2

Stability of the Relativistic Maxwellian in a Collisional Plasma

Abstract. The relativistic Landau-Maxwell system is the most fundamental and complete model for describing the dynamics of a dilute collisional plasma in which particles interact through Coulombic collisions and through their self-consistent electromagnetic field. We construct the first global in time classical solutions. Our solutions are constructed in a periodic box and near the relativistic Maxwellian, the Jüttner solution. This result has already appeared in a modified form as [62].

2.1. Collisional Plasma

A dilute hot plasma is a collection of fast moving charged particles [42]. Such plasmas appear commonly in such important physical problems as in Nuclear fusion and Tokamaks. Landau, in 1936, introduced the kinetic equation used to model a dilute plasma in which particles interact through binary Coulombic collisions. Landau did not, however, incorporate Einstein's theory of special relativity into his model. When particle velocities are close to the speed of light, denoted by c, relativistic effects become important. The relativistic version of Landau's equation was proposed by Budker and Beliaev in 1956 [3–5]. It is widely accepted as the most complete model for describing the dynamics of a dilute collisional fully ionized plasma.

The relativistic Landau-Maxwell system is given by

$$\begin{aligned}\partial_t F_+ + c\frac{p}{p_0^+}\cdot\nabla_x F_+ + e_+\left(E+\frac{p}{p_0^+}\times B\right)\cdot\nabla_p F_+ &= \mathcal{C}(F_+,F_+)+\mathcal{C}(F_+,F_-)\\ \partial_t F_- + c\frac{p}{p_0^-}\cdot\nabla_x F_- - e_-\left(E+\frac{p}{p_0^-}\times B\right)\cdot\nabla_p F_- &= \mathcal{C}(F_-,F_-)+\mathcal{C}(F_-,F_+)\end{aligned}$$

with initial condition $F_\pm(0,x,p) = F_{0,\pm}(x,p)$. Here $F_\pm(t,x,p) \geq 0$ are the spatially periodic number density functions for ions (+) and electrons (−), at time $t \geq 0$, position $x = (x_1, x_2, x_3) \in \mathbb{T}^3 \equiv [-\pi,\pi]^3$ and momentum $p = (p_1, p_2, p_3) \in \mathbb{R}^3$. The constants $\pm e_\pm$ and $m_\pm$ are the magnitude of the particles charges and rest masses respectively. The energy of a particle is given by $p_0^\pm = \sqrt{(m_\pm c)^2 + |p|^2}$.

The l.h.s. of the relativistic Landau-Maxwell system models the transport of the particle density functions and the r.h.s. models the effect of collisions between particles on the transport. The heuristic derivation of this equation is

total derivative along particle trajectories = rate of change due to collisions,

where the total derivative of $F_\pm$ is given by Newton's laws

$$\begin{aligned} \dot{x} &= \text{the relativistic velocity} = \frac{p}{\sqrt{m_\pm + |p|^2/c}}, \\ \dot{p} &= \text{the Lorentzian force} = \pm e_\pm \left(E + \frac{p}{p_0^\pm} \times B \right). \end{aligned}$$

The collision between particles is modelled by the relativistic Landau collision operator $\mathcal{C}$ in (2.1) and [3, 4, 50] (sometimes called the relativistic Fokker-Plank-Landau collision operator).

To completely describe a dilute plasma, the electromagnetic field $E(t,x)$ and $B(t,x)$ is generated by the plasma, coupled with $F_\pm(t,x,p)$ through the celebrated Maxwell system:

$$\begin{aligned} \partial_t E - c\nabla_x \times B &= -4\pi\mathcal{J} = -4\pi \int_{\mathbb{R}^3} \left\{ e_+ \frac{p}{p_0^+} F_+ - e_- \frac{p}{p_0^-} F_- \right\} dp, \\ \partial_t B + c\nabla_x \times E &= 0, \end{aligned}$$

with constraints

$$\nabla_x \cdot B = 0,\ \nabla_x \cdot E = 4\pi\rho = 4\pi \int_{\mathbb{R}^3} \{ e_+ F_+ - e_- F_- \} \, dp,$$

and initial conditions $E(0,x) = E_0(x)$ and $B(0,x) = B_0(x)$. The charge density and current density due to all particles are denoted ρ and $\mathcal{J}$ respectively.

We define relativistic four vectors as $P_+ = (p_0^+, p) = (p_0^+, p_1, p_2, p_3)$ and $Q_- = (q_0^-, q)$. Let $g_+(p)$, $h_-(p)$ be two number density functions for two types of particles, then the Landau collision operator is defined by

$$\mathcal{C}(g_+, h_-)(p) \equiv \nabla_p \cdot \int_{\mathbb{R}^3} \Phi(P_+, Q_-) \left\{ \nabla_p g_+(p) h_-(q) - g_+(p) \nabla_q h_-(q) \right\} dq. \tag{2.1}$$

The ordering of the $+, -$ in the kernel $\Phi(P_+, Q_-)$ corresponds to the order of the functions in argument of the collision operator $\mathcal{C}(g_+, h_-)(p)$. The collision kernel is given by the 3×3 non-negative matrix

$$\Phi(P_+, Q_-) \equiv \frac{2\pi}{c} e_+ e_- L_{+,-} \left(\frac{p_0^+}{m_+ c} \frac{q_0^-}{m_- c} \right)^{-1} \Lambda(P_+, Q_-) S(P_+, Q_-),$$

where $L_{+,-}$ is the Couloumb logarithm for $+-$ interactions. The Lorentz inner product with signature $(+ - --)$ is given by

$$P_+ \cdot Q_- = p_0^+ q_0^- - p \cdot q.$$

We distinguish between the standard inner product and the Lorentz inner product of relativistic four-vectors by using capital letters P_+ and Q_- to denote the four-vectors. Then, for the convenience of future analysis, we define

$$\begin{aligned}
\Lambda &\equiv \left(\frac{P_+}{m_+ c} \cdot \frac{Q_-}{m_- c} \right)^2 \left\{ \left(\frac{P_+}{m_+ c} \cdot \frac{Q_-}{m_- c} \right)^2 - 1 \right\}^{-3/2}, \\
S &\equiv \left\{ \left(\frac{P_+}{m_+ c} \cdot \frac{Q_-}{m_- c} \right)^2 - 1 \right\} I_3 - \left(\frac{p}{m_+ c} - \frac{q}{m_- c} \right) \otimes \left(\frac{p}{m_+ c} - \frac{q}{m_- c} \right) \\
&\quad + \left\{ \left(\frac{P_+}{m_+ c} \cdot \frac{Q_-}{m_- c} \right) - 1 \right\} \left(\frac{p}{m_+ c} \otimes \frac{q}{m_- c} + \frac{q}{m_- c} \otimes \frac{p}{m_+ c} \right).
\end{aligned}$$

This kernel is the relativtistic counterpart of the celebrated classical (non-relativistic) Landau collision operator.

It is well known that the collision kernel Φ is a non-negative matrix satisfying

$$\sum_{i=1}^{3} \Phi^{ij}(P_+, Q_-) \left(\frac{q_i}{q_0^-} - \frac{p_i}{p_0^+} \right) = \sum_{j=1}^{3} \Phi^{ij}(P_+, Q_-) \left(\frac{q_j}{q_0^-} - \frac{p_j}{p_0^+} \right) = 0, \tag{2.2}$$

and [47, 50]

$$\sum_{i,j} \Phi^{ij}(P_+, Q_-) w_i w_j > 0 \text{ if } w \neq d\left(\frac{p}{p_0^+} - \frac{q}{q_0^-}\right) \ \forall d \in \mathbb{R}.$$

The same is true for each other sign configuration $((+,+),(-,+),(-,-))$. This property represents the physical assumption that so-called "grazing collisions" dominate, e.g. the change in "momentum of the colliding particles is perpendicular to their relative velocity" [50] [p. 170]. This is also the key property used to derive the conservation laws and the entropy dissipation below.

It formally follows from (2.2) that for number density functions $g_+(p)$, $h_-(p)$

$$\int_{\mathbb{R}^3} \left\{ \begin{pmatrix} 1 \\ p \\ p_0^+ \end{pmatrix} \mathcal{C}(h_+, g_-)(p) + \begin{pmatrix} 1 \\ p \\ p_0^- \end{pmatrix} \mathcal{C}(g_-, h_+)(p) \right\} dp = 0.$$

The same property holds for other sign configurations. By integrating the relativistic Landau-Maxwell system and plugging in this identity, we deduce the local conservation of mass

$$\int_{\mathbb{R}^3} m_\pm \left\{ \partial_t + c\frac{p}{p_0^\pm} \cdot \nabla_x \right\} F_\pm(t,x) dp = 0,$$

the local conservation of total momentum (both kinetic and electromagnetic)

$$\begin{aligned} \int_{\mathbb{R}^3} p \left(m_+ \left\{ \partial_t + c\frac{p}{p_0^+} \cdot \nabla_x \right\} F_+(t,x) + m_- \left\{ \partial_t + c\frac{p}{p_0^-} \cdot \nabla_x \right\} F_-(t,x) \right) dp \\ + \frac{1}{4\pi}\partial_t \left(E(t) \times B(t) \right) \\ = \frac{1}{4\pi} \left\{ (B \cdot \nabla_x) B + (E \cdot \nabla_x + \nabla_x \cdot E) E - \frac{1}{2}\nabla_x \left(|B|^2 + |E|^2 \right) \right\}, \end{aligned}$$

and the local conservation of total energy (both kinetic and electromagnetic)

$$\begin{aligned} \int_{\mathbb{R}^3} \left(m_+ p_0^+ \left\{ \partial_t + c\frac{p}{p_0^+} \cdot \nabla_x \right\} F_+(t,x) + m_- p_0^- \left\{ \partial_t + c\frac{p}{p_0^-} \cdot \nabla_x \right\} F_-(t,x) \right) dp \\ + \frac{1}{8\pi}\partial_t \left(|E(t)|^2 + |B(t)|^2 \right) = \frac{1}{4\pi} \left\{ E \cdot (\nabla_x \times B) - B \cdot (\nabla_x \times E) \right\}. \end{aligned}$$

Integration over the periodic box $\mathbb{T}^3$ yields the conservation of mass, total momentum and total energy for solutions as

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^3\times\mathbb{R}^3} m_+F_+(t) = \frac{d}{dt}\int_{\mathbb{T}^3\times\mathbb{R}^3} m_-F_-(t) = 0,\\
&\frac{d}{dt}\left\{\int_{\mathbb{T}^3\times\mathbb{R}^3} p(m_+F_+(t)+m_-F_-(t)) + \frac{1}{4\pi}\int_{\mathbb{T}^3} E(t)\times B(t)\right\} = 0,\\
&\frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3} (m_+p_0^+F_+(t)+m_-p_0^-F_-(t)) + \frac{1}{8\pi}\int_{\mathbb{T}^3} |E(t)|^2+|B(t)|^2\right\} = 0.
\end{aligned}
$$

The entropy of the relativistic Landau-Maxwell system is defined as

$$
\mathcal{H}(t) \equiv \int_{\mathbb{T}^3\times\mathbb{R}^3} \{F_+(t,x,p)\ln F_+(t,x,p) + F_-(t,x,p)\ln F_-(t,x,p)\}\, dxdp \geq 0.
$$

Boltzmann's famous H-Theorem for the relativistic Landau-Maxwell system is

$$
\frac{d}{dt}\mathcal{H}(t) \leq 0,
$$

e.g. the entropy of solutions is non-increasing as time passes.

The global relativistic Maxwellian (a.k.a. the Jüttner solution) is given by

$$
J_\pm(p) = \frac{\exp\left(-cp_0^\pm/(k_BT)\right)}{4\pi e_\pm m_\pm^2 ck_BTK_2(m_\pm c^2/(k_BT))},
$$

where $K_2(\cdot)$ is the Bessel function $K_2(z) \equiv \frac{z^2}{3}\int_1^\infty e^{-zt}(t^2-1)^{3/2}dt$, T is the temperature and k_B is Boltzmann's constant. From the Maxwell system and the periodic boundary condition of $E(t,x)$, we see that $\frac{d}{dt}\int_{\mathbb{T}^3} B(t,x)dx \equiv 0$. We thus have a constant $\bar{B}$ such that

$$
\frac{1}{|\mathbb{T}^3|}\int_{\mathbb{T}^3} B(t,x)dx = \bar{B}. \tag{2.3}
$$

Let $[\cdot,\cdot]$ denote a column vector. We then have the following steady state solution to the relativisitic Landau-Maxwell system

$$
[F_\pm(t,x,p), E(t,x), B(t,x)] = [J_\pm, 0, \bar{B}],
$$

which minnimizes the entropy ($\mathcal{H}(t) = 0$).

It is our purpose to study the effects of collisions in a hot plasma and to construct global in time classical solutions for the relativistic Landau-Maxwell system with initial data close to the relativistic Maxwellian (Theorem 2.1). Our construction implies the asymptotic stability of the relativistic Maxwellian, which is suggested by the H-Theorem.

2.2. Main Results

We define the standard perturbation $f_\pm(t,x,p)$ to $J_\pm$ as

$$F_\pm \equiv J_\pm + \sqrt{J}_\pm f_\pm.$$

We will plug this perturbation into the Landau-Maxwell system of equations to derive a perturbed Landau-Maxwell system for $f_\pm(t,x,p)$, $E(t,x)$ and $B(t,x)$. The two Landau-Maxwell equations for the perturbation $f = [f_+, f_-]$ take the form

$$\left\{\partial_t + c\frac{p}{p_0^\pm}\cdot\nabla_x \pm e_\pm\left(E + \frac{p}{p_0^\pm}\times B\right)\cdot\nabla_p\right\} f_\pm \mp \frac{e_\pm c}{k_B T}\left\{E\cdot\frac{p}{p_0^\pm}\right\}\sqrt{J}_\pm + L_\pm f$$

$$= \pm\frac{e_\pm c}{2k_B T}\left\{E\cdot\frac{p}{p_0^\pm}\right\} f_\pm + \Gamma_\pm(f,f), \tag{2.4}$$

with $f(0,x,p) = f_0(x,p) = [f_{0,+}(x,p), f_{0,-}(x,p)]$. The linear operator $L_\pm f$ defined in (2.21) and the non-linear operator $\Gamma_\pm(f,f)$ defined in (2.23) are derived from an expansion of the Landau collision operator (2.1). The coupled Maxwell system takes the form

$$\partial_t E - c\nabla_x\times B = -4\pi\mathcal{J} = -4\pi\int_{\mathbb{R}^3}\left\{e_+\frac{p}{p_0^+}\sqrt{J}_+ f_+ - e_-\frac{p}{p_0^-}\sqrt{J}_- f_-\right\}dp,$$

$$\partial_t B + c\nabla_x\times E = 0, \tag{2.5}$$

with constraints

$$\nabla_x\cdot E = 4\pi\rho = 4\pi\int_{\mathbb{R}^3}\left\{e_+\sqrt{J}_+ f_+ - e_-\sqrt{J}_- f_-\right\}dp, \quad \nabla_x\cdot B = 0, \tag{2.6}$$

with $E(0,x) = E_0(x)$, $B(0,x) = B_0(x)$. In computing the charge ρ, we have used the normalization $\int_{\mathbb{R}^3} J_\pm(p)dp = \frac{1}{e_\pm}$.

Notation: For notational simplicity, we shall use $\langle\cdot,\cdot\rangle$ to denote the standard L^2 inner product in $\mathbb{R}^3$ and $(\cdot,\cdot)$ to denote the standard L^2 inner product in $\mathbb{T}^3\times\mathbb{R}^3$. We define the collision frequency as the 3×3 matrix

$$\sigma^{ij}_{\pm,\mp}(p) \equiv \int \Phi^{ij}(P_\pm, Q_\mp) J_\mp(q)dq. \tag{2.7}$$

These four weights (corresponding to signatures $(+,+),(+,-),(-,+),(-,-)$) are used to measure the dissipation of the relativistic Landau collision term. Unless otherwise stated $g=[g_+,g_-]$ and $h=[h_+,h_-]$ are functions which map $\{t\geq 0\}\times\mathbb{T}^3\times\mathbb{R}^3\to\mathbb{R}^2$. We define

$$
\begin{aligned}
\langle g,h\rangle_\sigma \ \equiv\ & \int_{\mathbb{R}^3}\left\{\left(\sigma^{ij}_{+,+}+\sigma^{ij}_{+,-}\right)\partial_{p_j}g_+\partial_{p_i}h_+ +\left(\sigma^{ij}_{-,-}+\sigma^{ij}_{-,+}\right)\partial_{p_j}g_-\partial_{p_i}h_-\right\}dp, \\
& +\frac{1}{4}\int_{\mathbb{R}^3}\left(\sigma^{ij}_{+,+}+\sigma^{ij}_{+,-}\right)\frac{p_i}{p_0^+}\frac{p_j}{p_0^+}g_+h_+dp \\
& +\frac{1}{4}\int_{\mathbb{R}^3}\left(\sigma^{ij}_{-,-}+\sigma^{ij}_{-,+}\right)\frac{p_i}{p_0^-}\frac{p_j}{p_0^-}g_-h_-dp,
\end{aligned}
\tag{2.8}
$$

where in (2.8) and the rest of the paper we use the Einstein convention of implicitly summing over $i,j\in\{1,2,3\}$ (unless otherwise stated). This complicated inner product is motivated by following splitting, which is a crucial element of the energy method used in this paper (Lemma 2.6 and Lemma 2.8):

$$\langle Lg,h\rangle=\langle [L_+g,L_-g],h\rangle=\langle g,h\rangle_\sigma+\text{a “compact” term}.$$

We will also use the corresponding L^2 norms

$$|g|_\sigma^2\equiv\langle g,g\rangle_\sigma,\ \ \|g\|_\sigma^2\equiv(g,g)_\sigma\equiv\int_{\mathbb{T}^3}\langle g,g\rangle_\sigma dx.$$

We use $|\cdot|_2$ to denote the L^2 norm in $\mathbb{R}^3$ and $\|\cdot\|$ to denote the L^2 norm in either $\mathbb{T}^3\times\mathbb{R}^3$ or $\mathbb{T}^3$ (depending on whether the function depends on both (x,p) or only on x). Let the multi-indices γ and β be $\gamma=[\gamma^0,\gamma^1,\gamma^2,\gamma^3]$, $\beta=[\beta^1,\beta^2,\beta^3]$. We use the following notation for a high order derivative

$$\partial^\gamma_\beta\equiv\partial_t^{\gamma^0}\partial_{x_1}^{\gamma^1}\partial_{x_2}^{\gamma^2}\partial_{x_3}^{\gamma^3}\partial_{p_1}^{\beta^1}\partial_{p_2}^{\beta^2}\partial_{p_3}^{\beta^3}.$$

If each component of β is not greater than that of $\bar{\beta}$'s, we denote by $\beta \leq \bar{\beta}$; $\beta < \bar{\beta}$ means $\beta \leq \bar{\beta}$, and $|\beta| < |\bar{\beta}|$. We also denote $\begin{pmatrix} \beta \\ \bar{\beta} \end{pmatrix}$ by $C_{\beta}^{\bar{\beta}}$. Let

$$
\begin{aligned}
|||f|||^2(t) &\equiv \sum_{|\gamma|+|\beta|\leq N} ||\partial_\beta^\gamma f(t)||^2, \\
|||f|||_\sigma^2(t) &\equiv \sum_{|\gamma|+|\beta|\leq N} ||\partial_\beta^\gamma f(t)||_\sigma^2, \\
|||[E,B]|||^2(t) &\equiv \sum_{|\gamma|\leq N} ||[\partial^\gamma E(t), \partial^\gamma B(t)]||^2.
\end{aligned}
$$

It is important to note that our norms include the temporal derivatives. For a function independent of t, we use the same notation but we drop the (t). The above norms and their associated spaces are used throughout the paper for arbitrary functions.

We further define the high order energy norm for a solution $f(t,x,p)$, $E(t,x)$ and $B(t,x)$ to the relativistic Landau-Maxwell system (2.4) and (2.5) as

$$
\mathcal{E}(t) \equiv \frac{1}{2}|||f|||^2(t) + |||[E,B]|||^2(t) + \int_0^t |||f|||_\sigma^2(s)ds. \tag{2.9}
$$

Given initial datum $[f_0(x,p), E_0(x), B_0(x)]$, we define

$$
\mathcal{E}(0) = \frac{1}{2}|||f_0|||^2 + |||[E_0, B_0]|||^2,
$$

where the temporal derivatives of $[f_0, E_0, B_0]$ are defined naturally through equations (2.4) and (2.5). The high order energy norm is consistent at $t = 0$ for a smooth solution and $\mathcal{E}(t)$ is continuous (Theorem 2.6).

Assume that initially $[F_0, E_0, B_0]$ has the same mass, total momentum and total energy as the steady state $[J_\pm, 0, \bar{B}]$, then we can rewrite the conservation laws in terms of the perturbation $[f, E, B]$:

$$
\int_{\mathbb{T}^3\times\mathbb{R}^3} m_+ f_+(t)\sqrt{J}_+ \equiv \int_{\mathbb{T}^3\times\mathbb{R}^3} m_- f_-(t)\sqrt{J}_- \equiv 0, \tag{2.10}
$$

$$
\int_{\mathbb{T}^3\times\mathbb{R}^3} p\left\{m_+ f_+(t)\sqrt{J}_+ + m_- f_-(t)\sqrt{J}_-\right\} \equiv -\frac{1}{4\pi}\int_{\mathbb{T}^3} E(t)\times B(t), \tag{2.11}
$$

$$
\int_{\mathbb{T}^3\times\mathbb{R}^3} m_+ p_0^+ f_+(t)\sqrt{J}_+ + m_- p_0^- f_-(t)\sqrt{J}_- \equiv -\frac{1}{8\pi}\int_{\mathbb{T}^3} |E(t)|^2 + |B(t)-\bar{B}|^2. \tag{2.12}
$$

We have used (2.3) for the normalized energy conservation (2.12).

The effect of this restriction is to guarantee that a solution can only converge to the specific relativistic Maxwellian that we perturb away from (if the solution converges to a relativistic Maxwellian). The value of the steady state $\bar{B}$ is also defined by the initial conditions (2.3).

We are now ready to state our main results:

THEOREM 2.1. *Fix N, the total number of derivatives in (2.9), with $N \geq 4$. Assume that $[f_0, E_0, B_0]$ satisfies the conservation laws (2.10), (2.11), (2.12) and the constraint (2.6) initially. Let*

$$F_{0,\pm}(x,p) = J_\pm + \sqrt{J}_\pm f_{0,\pm}(x,p) \geq 0.$$

There exist $C_0 > 0$ and $M > 0$ such that if

$$\mathcal{E}(0) \leq M,$$

then there exists a unique global solution $[f(t,x,p), E(t,x), B(t,x)]$ to the perturbed Landau-Maxwell system (2.4), (2.5) with (2.6). Moreover,

$$F_\pm(t,x,p) = J_\pm + \sqrt{J}_\pm f_\pm(t,x,p) \geq 0$$

solves the relativistic Landau-Maxwell system and

$$\sup_{0 \leq s \leq \infty} \mathcal{E}(s) \leq C_0 \mathcal{E}(0).$$

Remarks:

- These solutions are C^1, and in fact C^k, for N large enough.
- Since $\int_0^\infty |||f|||_\sigma^2(t)dt < +\infty$, $f(t,x,p)$ gains one momentum derivative over it's initial data and $|||f|||_\sigma^2(t) \to 0$ in a certain sense.
- Further, Lemma 2.5 together with Lemma 2.13 imply that

  $$\sum_{|\gamma| \leq N-1} \left\{ ||\partial^\gamma E(t)|| + ||\partial^\gamma \{B(t) - \bar{B}\}|| \right\} \leq C \sum_{|\gamma| \leq N} ||\partial^\gamma f(t)||_\sigma.$$

  Therefore, except for the highest order derivatives, the field also converges.
- It is an interesting open question to determine the asymptotic behavior of the highest order derivatives of the electromagnetic field.

Recently, global in time solutions to the related classical Vlasov-Maxwell-Boltzmann equation were constructed by the second author in [39]. The Boltzmann equation is a widely accepted model for binary interactions in a dilute gas, however it fails to hold for a dilute plasma in which grazing collisions dominate.

The following classical Landau collision operator (with normalized constants) was designed to model such a plasma:

$$\mathcal{C}_{cl}(F_-, F_+) \equiv \nabla_v \cdot \left\{ \int_{\mathbf{R}^3} \phi(v-v') \left\{ \nabla_v F_-(v) F_+(v') - F_-(v) \nabla_{v'} F_+(v') \right\} dv' \right\}.$$

The non-negative 3×3 matrix is

$$\phi^{ij}(v) = \left\{ \delta_{ij} - \frac{v_i v_j}{|v|^2} \right\} \frac{1}{|v|}. \tag{2.13}$$

Unfortunately, because of the crucial hard sphere assumption, the construction in [39] fails to apply to a non-relativistic Coulombic plasma interacting with it's electromagnetic field. The key problem is that the classical Landau collision operator, which was studied in detail in [38], offers weak dissipation of the form $\int_{\mathbb{R}^3} (1+|v|)^{-1} |f|^2 dv$. The global existence argument in Section 2.6 (from [39]) does not work because of this weak dissipation. Further, the unbounded velocity v, which is inconsistent with Einstein's theory of special relativity, in particular makes it impossible to control a nonlinear term like $\{E \cdot v\} f_\pm$ in the classical theory.

On the other hand, in the relativistic case our key observation is that the corresponding nonlinear term $c\left\{E \cdot p/p_0^\pm\right\} f_\pm$ can be easily controlled by the dissipation because $|cp/p_0^\pm| \leq c$ and the dissipation in the relativisitc Landau operator is $\int_{\mathbb{R}^3} |f|^2 dp$ (Lemma 2.5 and Lemou [47]).

However, it is well-known that the relativity effect can produce severe mathematical difficulties. Even for the related pure relativistic Boltzmann equation, global smooth solutions were only constructed in [32, 33].

The first new difficulty is due to the complexity of the relativistic Landau collision kernel $\Phi(P_+, Q_-)$. Since

$$\frac{P_+}{m_+ c} \cdot \frac{Q_-}{m_- c} - 1 \sim \frac{1}{2c} \left| \frac{p}{m_+} - \frac{q}{m_-} \right|^2 \text{ when } \frac{P_+}{m_+ c} \approx \frac{Q_-}{m_- c},$$

the kernel in (2.1) has a first order singularity. Hence it can not absorb many derivatives in high order estimates (Lemma 2.7 and Theorem 2.4). The same issue exists for the classical Landau kernel $\phi(v-v')$, but the obvious symmetry makes it easy to express v derivatives of ϕ in terms of v' derivatives. It is then possible to integrate by parts and move derivatives off the singular kernel in the estimates of high order derivatives. On the contrary, no apparent symmetry exists beween p and q in the relativistic case. We overcome this severe difficulty with the splitting

$$\partial_{p_j}\Phi^{ij}(P_+,Q_-) = -\frac{q_0^-}{p_0^+}\partial_{q_j}\Phi^{ij}(P_+,Q_-) + \left(\partial_{p_j} + \frac{q_0^-}{p_0^+}\partial_{q_j}\right)\Phi^{ij}(P_+,Q_-),$$

where the operator $\left(\partial_{p_j} + \frac{q_0^-}{p_0^+}\partial_{q_j}\right)$ does not increase the order of the singularity mainly because

$$\left(\partial_{p_j} + \frac{q_0^-}{p_0^+}\partial_{q_j}\right) P_+ \cdot Q_- = 0.$$

This splitting is crucial for performing the integration by parts in all of our estimates (Lemma 2.2 and Theorem 2.3). We believe that such an integration by parts technique should shed new light on the study of the relativistic Boltzmann equation.

As in [38,39], another key point in our construction is to show that the linearized collision operator L is in fact coercive for solutions of small amplitude to the full nonlinear system (2.4), (2.5) and (2.6):

Theorem 2.2. *Let $[f(t,x,p), E(t,x), B(t,x)]$ be a classical solution to (2.4) and (2.5) satisfying (2.6), (2.10), (2.11) and (2.12). There exists M_0, $\delta_0 = \delta_0(M_0) > 0$ such that if*

$$\sum_{|\gamma|\le N}\left\{\frac{1}{2}||\partial^\gamma f(t)||^2 + ||\partial^\gamma E(t)||^2 + ||\partial^\gamma B(t)||^2\right\} \le M_0, \tag{2.14}$$

then

$$\sum_{|\gamma|\le N}(L\partial^\gamma f(t), \partial^\gamma f(t)) \ge \delta_0 \sum_{|\gamma|\le N}||\partial^\gamma f(t)||_\sigma^2.$$

Theorem 2.2 is proven through a careful study of the macroscopic equations (2.98) - (2.102). These macroscopic equations come from a careful study of solutions f to the perturbed relativistic Landau-Maxwell system (2.4), (2.5) with (2.6) projected onto the null space $\mathcal{N}$ of the linearized collision operator $L = [L_+, L_-]$ defined in (2.21).

As expected from the H-theorem, L is non-negative and for every fixed (t,x) the null space of L is given by the six dimensional space $(1 \le i \le 3)$

$$\mathcal{N} \equiv \text{span}\{[\sqrt{J_+}, 0],\ [0, \sqrt{J_-}],\ [p_i\sqrt{J_+}, p_i\sqrt{J_-}],\ [p_0^+\sqrt{J_+}, p_0^-\sqrt{J_-}]\}. \tag{2.15}$$

This is shown in Lemma 2.1. We define the orthogonal projection from $L^2(\mathbb{R}^3_p)$ onto the null space $\mathcal{N}$ by $\mathbf{P}$. We then decompose $f(t,x,p)$ as

$$f = \mathbf{P}f + \{\mathbf{I} - \mathbf{P}\}f.$$

We call $\mathbf{P}f = [\mathbf{P}_+ f, \mathbf{P}_- f] \in \mathbb{R}^2$ the hydrodynamic part of f and $\{\mathbf{I} - \mathbf{P}\}f = [\{\mathbf{I} - \mathbf{P}\}_+ f, \{\mathbf{I} - \mathbf{P}\}_- f]$ is called the microscopic part. By separating its linear and nonlinear part, and using $L_\pm\{\mathbf{P}f\} = 0$, we can express the hydrodynamic part of f through the microscopic part up to a higher order term $h(f)$:

$$\left\{\partial_t + c\frac{p}{p_0^\pm}\cdot\nabla_x\right\}\mathbf{P}_\pm f \mp \frac{e_\pm c}{k_B T}\left\{E\cdot\frac{p}{p_0^\pm}\right\}\sqrt{J_\pm} = l_\pm(\{\mathbf{I} - \mathbf{P}\}f) + h_\pm(f), \tag{2.16}$$

where

$$l_\pm(\{\mathbf{I} - \mathbf{P}\}f) \equiv -\left\{\partial_t + \frac{p}{p_0^\pm}\cdot\nabla_x\right\}\{\mathbf{I} - \mathbf{P}\}_\pm f + L_\pm\left\{\{\mathbf{I} - \mathbf{P}\}f\right\}, \tag{2.17}$$

$$\begin{aligned} h_\pm(f) \equiv\ & \mp e_\pm\left(E + \frac{p}{p_0^\pm}\times B\right)\cdot\nabla_p f_\pm \\ & \pm\frac{e_\pm c}{2k_B T}\left\{E\cdot\frac{p}{p_0^\pm}\right\}f_\pm + \Gamma_\pm(f,f). \end{aligned} \tag{2.18}$$

We further expand $\mathbf{P}_\pm f$ as a linear combination of the basis in (2.15)

$$\mathbf{P}_\pm f \equiv \left\{a_\pm(t,x) + \sum_{j=1}^3 b_j(t,x)p_j + c(t,x)p_0^\pm\right\}\sqrt{J_\pm}. \tag{2.19}$$

A precise definition of these coefficients will be given in (2.94). The relativistic system of macroscopic equations (2.98) - (2.102) are obtained by plugging (2.19) into (2.16).

These macroscopic equations for the coefficients in (2.19) enable us to show that there exists a constant $C > 0$ such that solutions to (2.4) which satisfy the smallness constraint (2.14) (for $M_0 > 0$ small enough) will also satisfy

$$\sum_{|\gamma|\le N} \{||\partial^\gamma a_\pm|| + ||\partial^\gamma b|| + ||\partial^\gamma c||\} \le C(M_0) \sum_{|\gamma|\le N} ||\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f(t)||_\sigma. \tag{2.20}$$

This implies Theorem 2.2 since $\|\mathbf{P}f\|_\sigma$ is trivially bounded above by the l.h.s. (Proposition 2.2) and L is coercive with respect to $\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f(t)$ (Lemma 2.8).

Since our smallness assumption (2.14) involves no momentum derivatives, in proving (2.20) the presence of momentum derivatives in the collision operator (2.1) causes another serious mathematical difficulty. We develop a new estimate (Theorem 2.5) which involves purely spatial derivatives of the linear term (2.21) and the nonlinear term (2.23) to overcome this difficulty.

To the best of the authors' knowledge, until now there were no known solutions for the relativistic Landau-Maxwell system. However in 2000, Lemou [47] studied the linearized relativistic Landau equation with no electromagnetic field. We will use one of his findings (Lemma 2.5) in the present work.

For the classical Landau equation, the 1990's have seen the first solutions. In 1994, Zhan [72] proved local existence and uniqueness of classical solutions to the Landau-Poisson equation ($B \equiv 0$) with Coulomb potential and a smallness assumption on the initial data at infinity. In the same year, Zhan [73] proved local existence of weak solutions to the Landau-Maxwell equation with Coulomb potential and large initial data.

On the other hand, in the absence of an electromagnetic field we have the following results. In 2000, Desvillettes and Villani [17] proved global existence and uniqueness of classical solutions for the spatially homogeneous Landau equation for hard potentials and a large class of initial data. In 2002, the second author [38] constructed global in time classical solutions near Maxwellian for a general Landau equation (both hard and soft potentials) in a periodic box based on a nonlinear energy method.

Our paper is organized as follows. In section 2.3 we establish linear and nonlinear estimates for the relativistic Landau collision operator. In section 2.4 we construct local in time solutions to the relativistic Landau-Maxwell system. In section 2.5 we prove Theorem 2.2. And in section 2.6 we extend the solutions to $T = \infty$.

Remark 2.1. *It turns out that the presence of the physical constants do not cause essential mathematical difficulties. Therefore, for notational simplicity, after the proof of Lemma 2.1 we will normalize all constants in the relativistic Landau-Maxwell system (2.4), (2.5) with (2.6) and in all related quantities to be one.*

2.3. The Relativistic Landau Operator

Our main results in this section include the crucial Theorem 2.3, which allows us to express p derivatives of $\Phi(P,Q)$ in terms of q derivatives of $\Phi(P,Q)$. This is vital for establishing the estimates found at the end of the section (Lemma 2.7, Theorem 2.4 and Theorem 2.5). Other important results include the equivalence of the norm $|\cdot|_\sigma$ with the standard Sobolev space norm for H^1 (Lemma 2.5) and a weak formulation of compactness for K which is enough to prove coercivity for L away from the null space $\mathcal{N}$ (Lemma 2.8). We also compute the sum of second order derivatives of the Landau kernel (Lemma 2.3).

We first introduce some notation. Using (2.2), we observe that quadratic collision operator (2.1) satisfies

$$\mathcal{C}(J_+, J_+) = \mathcal{C}(J_+, J_-) = \mathcal{C}(J_-, J_+) = \mathcal{C}(J_-, J_-) = 0.$$

Therefore, the linearized collision operator Lg is defined by

$$Lg = [L_+g, L_-g], \ L_\pm g \equiv -A_\pm g - K_\pm g, \tag{2.21}$$

where

$$\begin{aligned}
A_+g &\equiv J_+^{-1/2}\mathcal{C}(\sqrt{J}_+g_+, J_+) + J_+^{-1/2}\mathcal{C}(\sqrt{J}_+g_+, J_-),\\
A_-g &\equiv J_-^{-1/2}\mathcal{C}(\sqrt{J}_-g_-, J_-) + J_-^{-1/2}\mathcal{C}(\sqrt{J}_-g_-, J_+),\\
K_+g &\equiv J_+^{-1/2}\mathcal{C}(J_+, \sqrt{J}_+g_+) + J_+^{-1/2}\mathcal{C}(J_+, \sqrt{J}_-g_-),\\
K_-g &\equiv J_-^{-1/2}\mathcal{C}(J_-, \sqrt{J}_-g_-) + J_-^{-1/2}\mathcal{C}(J_-, \sqrt{J}_+g_+).
\end{aligned} \tag{2.22}$$

And the nonlinear part of the collision operator (2.1) is defined by

$$\Gamma(g,h) = [\Gamma_+(g,h), \Gamma_-(g,h)],$$

where

$$\begin{aligned} \Gamma_+(g,h) &\equiv J_+^{-1/2}\mathcal{C}(\sqrt{J}_+ g_+, \sqrt{J}_+ h_+) + J_+^{-1/2}\mathcal{C}(\sqrt{J}_+ g_+, \sqrt{J}_- h_-), \\ \Gamma_-(g,h) &\equiv J_-^{-1/2}\mathcal{C}(\sqrt{J}_- g_-, \sqrt{J}_- h_-) + J_-^{-1/2}\mathcal{C}(\sqrt{J}_- g_-, \sqrt{J}_+ h_+). \end{aligned} \tag{2.23}$$

We will next derive the null space (2.15) of the linear operator in the presence of all the physical constants.

Lemma 2.1. $\langle Lg, h\rangle = \langle Lh, g\rangle$, $\langle Lg, g\rangle \geq 0$. *And* $Lg = 0$ *if and only if* $g = \mathbf{P}g$.

Proof. From (2.21) we split $\langle Lg, h\rangle$, with $Lg = [L_+ g, L_- g]$, as

$$\begin{aligned} &-\int_{\mathbb{R}^3} \frac{h_+}{\sqrt{J_+}}\{\mathcal{C}(\sqrt{J_+}g_+, J_+) + \mathcal{C}(J_+, \sqrt{J_+}g_+)\}dp \\ &-\int_{\mathbb{R}^3} \left\{\frac{h_+}{\sqrt{J_+}}\{\mathcal{C}(\sqrt{J_+}g_+, J_-) + \mathcal{C}(J_+, \sqrt{J_-}g_-)\}\right\} dp \\ &-\int_{\mathbb{R}^3} \left\{\frac{h_-}{\sqrt{J_-}}\{\mathcal{C}(\sqrt{J_-}g_-, J_+) + \mathcal{C}(J_-, \sqrt{J_+}g_+)\}\right\} dp \\ &-\int_{\mathbb{R}^3} \frac{h_-}{\sqrt{J_-}}\{\mathcal{C}(\sqrt{J_-}g_-, J_-) + \mathcal{C}(J_-, \sqrt{J_-}g_-)\}dp. \end{aligned} \tag{2.24}$$

We use the fact that $\partial_{q_i} J_-(q) = -\frac{c}{k_B T}\frac{q_i}{q_0^-}J_-(q)$ and $\partial_{p_i} J_+^{1/2}(p) = -\frac{c}{k_B T}\frac{p_i}{p_0^+}J_+(p)$ as well as the null space of Φ in (2.2) to show that

$$\begin{aligned} &\mathcal{C}(J_+^{1/2}g_+, J_-) \\ = \ &\partial_{p_i}\int_{\mathbb{R}^3} \Phi^{ij}(P_+, Q_-)J_-(q)J_+^{1/2}(p)\left\{\left(\frac{q_i}{q_0^-} - \frac{p_i}{2p_0^+}\right)g_+(p) + \partial_{p_j}g_+(p)\right\}dq \\ = \ &\partial_{p_i}\int_{\mathbb{R}^3} \Phi^{ij}(P_+, Q_-)J_-(q)J_+^{1/2}(p)\left\{\frac{p_i}{2p_0^+}g_+(p) + \partial_{p_j}g_+(p)\right\}dq \\ = \ &\partial_{p_i}\int_{\mathbb{R}^3} \Phi^{ij}(P_+, Q_-)J_-(q)J_+(p)\partial_{p_j}(J_+^{-1/2}g_+(p))dq \end{aligned}$$

And similarly

$$
\begin{aligned}
& \mathcal{C}(J_-, J_+^{1/2} g_+) \\
= \quad & -\partial_{p_i} \int_{\mathbb{R}^3} \Phi^{ij}(P_-, Q_+) J_-(p) J_+^{1/2}(q) \left\{ \left(\frac{p_i}{p_0^-} - \frac{q_i}{2q_0^+} \right) g_+(q) + \partial_{q_j} g_+(q) \right\} dq \\
= \quad & -\partial_{p_i} \int_{\mathbb{R}^3} \Phi^{ij}(P_-, Q_+) J_-(p) J_+^{1/2}(q) \left\{ \frac{q_i}{2q_0^+} g_+(q) + \partial_{q_j} g_+(q) \right\} dq \\
(2.25) = \quad & -\partial_{p_i} \int_{\mathbb{R}^3} \Phi^{ij}(P_-, Q_+) J_-(p) J_+(q) \partial_{q_j} (J_+^{-1/2} g_+(q)) dq.
\end{aligned}
$$

Similar expressions hold by exchanging the + terms and the − terms in the appropriate places. For the first term in (2.24), we integrate by parts over p variables on the first line, then relabel the variables switching p and q on the second line and finally adding them up on the last line to obtain

$$
\begin{aligned}
= \quad & \iint \Phi^{ij}(P_+, Q_+) J_+(p) J_+(q) \partial_{p_i} (h_+ J_+^{-1/2}(p)) \\
& \times \{ \partial_{p_j} (g_+ J_+^{-1/2}(p)) - \partial_{q_j} (g_+ J_+^{-1/2}(q)) \} dp dq \\
= \quad & \iint \Phi^{ij}(P_+, Q_+) J_+(p) J_+(q) \partial_{q_i} (h_+ J_+^{-1/2}(q)) \\
& \times \{ \partial_{q_j} (g_+ J_+^{-1/2}(q)) - \partial_{p_j} (g_+ J_+^{-1/2}(p)) \} dp dq \\
= \quad & \frac{1}{2} \iint \Phi^{ij}(P_+, Q_+) J_+(p) J_+(q) \{ \partial_{p_i} (h_+ J_+^{-1/2}(p)) - \partial_{q_i} (h_+ J_+^{-1/2}(q)) \} \\
& \times \{ \partial_{p_j} (g_+ J_+^{-1/2}(p)) - \partial_{q_j} (g_+ J_+^{-1/2}(q)) \} dp dq.
\end{aligned}
$$

By (2.2) the first term in (2.24) is symmetric and ≥ 0 if $h = g$. The fourth term can be treated similarly (with + replaced by − everywhere. We combine the second and third terms in (2.24); again we integrate by parts over p variables to compute

$$
\begin{aligned}
= \quad & \iint \Phi^{ij}(P_+, Q_-) J_+(p) J_-(q) \partial_{p_i} (h_+ J_+^{-1/2}(p)) \\
& \times \{ \partial_{p_j} (g_+ J_+^{-1/2}(p)) - \partial_{q_j} (g_- J_-^{-1/2}(q)) \} dp dq \\
& + \iint \Phi^{ij}(P_-, Q_+) J_-(p) J_+(q) \partial_{p_i} (h_- J_-^{-1/2}(p)) \\
& \times \{ \partial_{p_j} (g_- J_-^{-1/2}(p)) - \partial_{q_j} (g_+ J_+^{-1/2}(q)) \} dp dq.
\end{aligned}
$$

We switch the role of p and q in the second term to obtain

$$
\begin{aligned}
= \quad & \iint \Phi^{ij}(P_+,Q_-)J_+(p)J_-(q)\{\partial_{p_i}(h_+J_+^{-1/2}(p)) - \partial_{q_i}(h_-J_-^{-1/2}(q))\} \\
& \times\{\partial_{p_j}(g_+J_+^{-1/2}(p)) - \partial_{q_j}(g_-J_-^{-1/2}(q))\}dpdq.
\end{aligned}
$$

Again by (2.2) this piece of the operator is symmetric and ≥ 0 if $g = h$. We therefore conclude that L is a non-negative symmetric operator.

We will now determine the null space (2.15) of the linear operator. Assume $Lg = 0$. From $\langle Lg, g\rangle = 0$ we deduce, by (2.2), that there are scalar functions $\zeta_l(p,q)$ $(l = \pm)$ such that

$$
\partial_{p_i}(g_lJ_l^{-1/2}(p)) - \partial_{q_i}(g_lJ_l^{-1/2}(q)) \equiv \zeta_l(p,q)\left(\frac{p_i}{p_0^l} - \frac{q_i}{q_0^l}\right),\ i \in \{1,2,3\}.
$$

Setting $q = 0$, $\partial_{p_i}(g_lJ_l^{-1/2}(p)) = \zeta_l(p,0)\frac{p_i}{p_0^l} + b_{li}$. By replacing p by q and subtracting we obtain

$$
\begin{aligned}
\partial_{p_i}(g_lJ_l^{-1/2}(p)) - \partial_{q_i}(g_lJ_l^{-1/2}(q)) &= \zeta_l(p,0)\frac{p_i}{p_0^l} - \zeta_l(q,0)\frac{q_i}{q_0^l} \\
&= \zeta_l(p,0)\left(\frac{p_i}{p_0^l} - \frac{q_i}{q_0^l}\right) + (\zeta_l(p,0) - \zeta_l(q,0))\frac{q_i}{q_0^l}.
\end{aligned}
$$

We deduce, again by (2.2), that $\zeta_l(p,0) - \zeta_l(q,0) = 0$ and therefore that $\zeta_l(p,0) \equiv c_l$ (a constant). We integrate $\partial_{p_i}(g_lJ_l^{-1/2}(p)) = c_l\frac{p_i}{p_0^l} + b_{li}$ to obtain

$$
g_l = \{a_l^g + \sum_{i=1}^{3} b_{li}^g p_j + c_l^g p_0^l\}J_l^{1/2}.
$$

Here a_l^g, b_{lj}^g and c_l^g are constants with respect to p (but could be functions of t and x). Moreover, we deduce from the middle terms in (2.24) as well as (2.2) that

$$
\partial_{p_i}(g_+J_+^{-1/2}(p)) - \partial_{q_i}(g_-J_-^{-1/2}(q)) \equiv \tilde{\zeta}(p,q)\left(\frac{p_i}{p_0^+} - \frac{q_i}{q_0^-}\right).
$$

Therefore $b_{+i}^g - b_{-i}^g + c_+^g\frac{p_i}{p_0^+} - c_-^g\frac{q_i}{q_0^-} = \tilde{\zeta}(p,q)\left(\frac{p_i}{p_0} - \frac{q_i}{q_0}\right)$. We conclude

$$
\begin{aligned}
b_{+i}^g &\equiv b_{-i}^g,\ i = 1,2,3; \\
c_+^g &\equiv c_-^g.
\end{aligned}
$$

That means $g(t,x,p) \in \mathcal{N}$ as in (2.15), so that $g = \mathbf{P}g$. Conversely, $L\{\mathbf{P}g\} = 0$ by a direct calculation. $\square$

For notational simplicity, as in Remark 2.1, we will normalize all the constants to be one. Accordingly, we write $p_0 = \sqrt{1+|p|^2}$, $P = (p_0, p)$, and the collision kernel $\Phi(P,Q)$ takes the form

$$\Phi(P,Q) \equiv \frac{\Lambda(P,Q)}{p_0 q_0} S(P,Q), \tag{2.26}$$

where

$$\begin{aligned}
\Lambda &\equiv (P\cdot Q)^2 \left\{(P\cdot Q)^2 - 1\right\}^{-3/2}, \\
S &\equiv \left\{(P\cdot Q)^2 - 1\right\} I_3 - (p-q)\otimes(p-q) + \{(P\cdot Q) - 1\}\,(p\otimes q + q\otimes p)\,.
\end{aligned}$$

We normalize the relativistic Maxwellian as

$$J(p) \equiv J_+(p) = J_-(p) = e^{-p_0}.$$

We further normalize the collision freqency

$$\sigma^{ij}_{\pm,\mp}(p) = \sigma^{ij}(p) = \int \Phi^{ij}(P,Q) J(q)\, dq, \tag{2.27}$$

and the inner product $\langle \cdot,\cdot\rangle_\sigma$ takes the form

$$\begin{aligned}
\langle g,h\rangle_\sigma &\equiv 2\int_{\mathbb{R}^3} \sigma^{ij}\left\{\partial_{p_j} g_+ \partial_{p_i} h_+ + \partial_{p_j} g_- \partial_{p_i} h_-\right\} dp, \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^3} \sigma^{ij} \frac{p_i}{p_0}\frac{p_j}{p_0}\left\{g_+ h_+ dp + g_- h_-\right\} dp.
\end{aligned} \tag{2.28}$$

The norms are, as before, naturally built from this normalized inner product.

The normalized vector-valued Landau-Maxwell equation for the perturbation f in (2.4) now takes the form

$$\begin{aligned}
&\left\{\partial_t + \frac{p}{p_0}\cdot\nabla_x + \xi\left(E + \frac{p}{p_0}\times B\right)\cdot\nabla_p\right\} f - \left\{E\cdot\frac{p}{p_0}\right\}\sqrt{J}\xi_1 + Lf \\
&= \frac{\xi}{2}\left\{E\cdot\frac{p}{p_0}\right\} f + \Gamma(f,f).
\end{aligned} \tag{2.29}$$

with $f(0,x,v) = f_0(x,v)$, $\xi_1 = [1,-1]$, and the 2×2 matrix ξ is diag$(1,-1)$. Further, the normalized Maxwell system in (2.5) and (2.6) takes the form

$$\partial_t E - \nabla_x \times B = -\mathcal{J} = -\int_{\mathbb{R}^3} \frac{p}{p_0}\sqrt{J}(f_+ - f_-)dp, \quad \partial_t B + \nabla_x \times E = 0, \tag{2.30}$$

$$\nabla_x \cdot E = \rho = \int_{\mathbb{R}^3} \sqrt{J}(f_+ - f_-)dp, \quad \nabla_x \cdot B = 0, \tag{2.31}$$

with $E(0,x) = E_0(x),\ B(0,x) = B_0(x)$.

We have a basic (but useful) inequality taken from Glassey & Strauss [32].

PROPOSITION 2.1. *Let $p, q \in \mathbb{R}^3$ with $P = (p_0, p)$ and $Q = (q_0, q)$ then*

$$\frac{|p-q|^2 + |p\times q|^2}{2p_0 q_0} \leq P\cdot Q - 1 \leq \frac{1}{2}|p-q|^2. \tag{2.32}$$

This will inequality will be used many times for estimating high order derivatives of the the collision kernel.

Notice that

$$\begin{aligned}\left(\partial_{p_i} + \frac{q_0}{p_0}\partial_{q_i}\right) P\cdot Q &= \left(\partial_{p_i} + \frac{q_0}{p_0}\partial_{q_i}\right)(p_0 q_0 - p\cdot q) \\ &= \frac{p_i}{p_0}q_0 - q_i + \frac{q_0}{p_0}\left(\frac{q_i}{q_0}p_0 - p_i\right) = 0.\end{aligned}$$

This is the key observation which allows us do analysis on the relativistic Landau Operator (Lemma 2.2). We define the following relativistic differential operator

$$\mathbf{\Theta}_\alpha(p,q) \equiv \left(\partial_{p_3} + \frac{q_0}{p_0}\partial_{q_3}\right)^{\alpha^3}\left(\partial_{p_2} + \frac{q_0}{p_0}\partial_{q_2}\right)^{\alpha^2}\left(\partial_{p_1} + \frac{q_0}{p_0}\partial_{q_1}\right)^{\alpha^1}. \tag{2.33}$$

Unless otherwise stated, we omit the p, q dependence and write $\mathbf{\Theta}_\alpha = \mathbf{\Theta}_\alpha(p,q)$. Note that the three terms in $\mathbf{\Theta}_\alpha$ do not commute (and we choose this order for no special reason).

We will use the following splitting many times in the rest of this section,

$$\mathcal{A} = \{|p-q| + |p\times q| \geq [|p|+1]/2\},\ \mathcal{B} = \{|p-q| + |p\times q| \leq [|p|+1]/2\}. \tag{2.34}$$

The set $\mathcal{A}$ is designed to be away from the first order singularity in collision kernel $\Phi(P,Q)$ (Proposition 2.1). And the set $\mathcal{B}$ contains a $\Phi(P,Q)$ singularity ((2.26) and

(2.32)) but we will exploit the fact that we can compare the size of p and q. We now develop crucial estimates for $\mathbf{\Theta}_\alpha \Phi(P,Q)$:

LEMMA 2.2. *For any multi-index α, the Lorentz inner product of P and Q is in the null space of $\mathbf{\Theta}_\alpha$,*

$$\mathbf{\Theta}_\alpha (P \cdot Q) = 0.$$

Further, recalling (2.26), for p and q on the set $\mathcal{A}$ we have the estimate

$$|\mathbf{\Theta}_\alpha (p,q)\Phi(P,Q)| \leq C p_0^{-|\alpha|} q_0^6. \tag{2.35}$$

And on $\mathcal{B}$,

$$\frac{1}{6} q_0 \leq p_0 \leq 6 q_0. \tag{2.36}$$

Using this inequality, we have the following estimate on $\mathcal{B}$

$$|\mathbf{\Theta}_\alpha (p,q)\Phi(P,Q)| \leq C q_0^7 p_0^{-|\alpha|} |p-q|^{-1}. \tag{2.37}$$

PROOF. Let e_i $(i = 1,2,3)$ be an element of the standard basis in $\mathbb{R}^3$. We have seen that $\mathbf{\Theta}_{e_i}(P \cdot Q) = 0$. And the general case follows from a simple induction over $|\alpha|$.

By (2.26) and (2.33), we can now write

$$\mathbf{\Theta}_\alpha (p,q)\Phi^{ij}(P,Q) = \Lambda(P,Q)\mathbf{\Theta}_\alpha (p,q) \left(\frac{S^{ij}(p,q)}{p_0 q_0} \right),$$

where

$$\begin{aligned} \mathbf{\Theta}_\alpha \left(\frac{S^{ij}(p,q)}{p_0 q_0} \right) &= \left\{ (P\cdot Q)^2 - 1 \right\} \mathbf{\Theta}_\alpha \left\{ \delta_{ij}/(p_0 q_0) \right\} \\ &+ (P\cdot Q - 1) \mathbf{\Theta}_\alpha \left\{ (p_i q_j + p_j q_i)/(p_0 q_0) \right\} \\ &- \mathbf{\Theta}_\alpha \left\{ (p_i - q_i)(p_j - q_j)/(p_0 q_0) \right\}. \end{aligned} \tag{2.38}$$

We will break up this expression and estimate the different pieces.

Using (2.33), the following estimates are straight forward

$$|\mathbf{\Theta}_\alpha \left\{ \delta_{ij}/(p_0 q_0) \right\}| \leq C q_0^{-1} p_0^{-1-|\alpha|}, \tag{2.39}$$

$$|\mathbf{\Theta}_\alpha \left\{ (p_i q_j + p_j q_i)/(p_0 q_0) \right\}| \leq C p_0^{-|\alpha|}. \tag{2.40}$$

On the other hand, we *claim* that

$$|\boldsymbol{\Theta}_\alpha \{(p_i - q_i)(p_j - q_j)/(p_0 q_0)\}| \leq C \frac{|p-q|^2}{p_0 q_0} p_0^{-|\alpha|}. \tag{2.41}$$

This last estimate is not so trivial because only a lower order estimate of $|p-q|$ is expected after applying even a first order derivative like $\boldsymbol{\Theta}_{e_i}$. The key observation is that

$$\left(\partial_{p_i} + \frac{q_0}{p_0}\partial_{q_i}\right)(p_i - q_i)(p_j - q_j) = \left(1 - \frac{q_0}{p_0}\right)(p_j - q_j),$$

and the r.h.s. is again second order. Therefore the operator $\boldsymbol{\Theta}_\alpha$ can maintain the order of the cancellation.

Proof of claim: To prove (2.41), it is sufficient to show that for any multi-index α and any $i, j, k, l \in \{1, 2, 3\}$ there exists a smooth function $G_{kl}^{\alpha,ij}(p,q)$ satisfying

$$\boldsymbol{\Theta}_\alpha \{(p_i - q_i)(p_j - q_j)/(p_0 q_0)\} = \sum_{k,l=1}^{3} (p_k - q_k)(p_l - q_l) G_{kl}^{\alpha,ij}(p,q) \tag{2.42}$$

as well as the decay

$$\left|\partial_{\nu_1}\partial^q_{\nu_2} G_{kl}^{\alpha,ij}(p,q)\right| \leq C q_0^{-1-|\nu_2|} p_0^{-1-|\alpha|-|\nu_1|}, \tag{2.43}$$

which holds for any multi-indices ν_1, ν_2. We prove (2.42) with (2.43) by a simple induction over $|\alpha|$.

If $|\alpha| = 0$, we define

$$G_{kl}^{0,ij}(p,q) = \frac{\delta_{ki}\delta_{lj}}{q_0 p_0}.$$

The decay (2.43) for $G_{kl}^{0,ij}(p,q)$ is straight forward to check. And (2.42) holds trivially for $|\alpha| = 0$.

Assume the (2.42) with (2.43) holds for $|\alpha| \leq n$. To conclude the proof, let $|\alpha'| = n+1$ and write $\boldsymbol{\Theta}_{\alpha'} = \boldsymbol{\Theta}_{e_m}\boldsymbol{\Theta}_\alpha$ for some multi-index α with

$$m = \max\{j : (\alpha')^j > 0\}.$$

This specification of m is needed because of our chosen ordering of the three differential operators in (2.33), which don't commute. Recalling (2.33),

$$\mathbf{\Theta}_{e_m}(p_k - q_k) = \delta_{km}\left(1 - \frac{q_0}{p_0}\right).$$

From the induction assumption and the last display, we have

$$\begin{aligned}
&\mathbf{\Theta}_{\alpha'}\left\{(p_i - q_i)(p_j - q_j)/(p_0 q_0)\right\} \\
= \ &\mathbf{\Theta}_{e_m}\sum_{k,l=1}^{3}(p_k - q_k)(p_l - q_l)G_{kl}^{\alpha,ij}(p,q), \\
= \ &\left(1 - \frac{q_0}{p_0}\right)\sum_{k=1}^{3}(p_k - q_k)\left\{G_{mk}^{\alpha,ij}(p,q) + G_{km}^{\alpha,ij}(p,q)\right\} \\
&+ \sum_{k,l=1}^{3}(p_k - q_k)(p_l - q_l)\mathbf{\Theta}_{e_m}G_{kl}^{\alpha,ij}(p,q).
\end{aligned}$$

We compute

$$1 - \frac{q_0}{p_0} = \frac{p_0 - q_0}{p_0} = \frac{p_0^2 - q_0^2}{p_0(p_0 + q_0)} = \frac{(p - q)\cdot(p + q)}{p_0(p_0 + q_0)} = \frac{\sum_l (p_l - q_l)(p_l + q_l)}{p_0(p_0 + q_0)}.$$

We plug this display into the one above it to obtain (2.42) for α' with the new coefficients

$$G_{kl}^{\alpha',ij}(p,q) \equiv \mathbf{\Theta}_{e_m}G_{kl}^{\alpha,ij}(p,q) + \frac{\left\{G_{mk}^{\alpha,ij}(p,q) + G_{km}^{\alpha,ij}(p,q)\right\}(p_l + q_l)}{p_0(p_0 + q_0)}.$$

We check that $G_{kl}^{\alpha',ij}(p,q)$ satisfies (2.43) using the Leibnitz differentiation formula as well as the induction assumption (2.43). This establishes the claim (2.41).

With the estimates (2.39), (2.40) and (2.41) in hand, we return to establishing (2.35) and (2.37). We plug the estimates (2.39), (2.40) and (2.41) into $\mathbf{\Theta}_\alpha\Phi^{ij}(P,Q)$ from (2.38) to obtain that

$$\begin{aligned}
\left|\mathbf{\Theta}_\alpha\Phi^{ij}(P,Q)\right| \ \leq \ &Cp_0^{-|\alpha|}(P\cdot Q)^2\left\{(P\cdot Q)^2 - 1\right\}^{-3/2}\frac{(P\cdot Q)^2 - 1}{p_0 q_0} \\
&+Cp_0^{-|\alpha|}(P\cdot Q)^2\left\{(P\cdot Q)^2 - 1\right\}^{-3/2}(P\cdot Q - 1) \\
&+Cp_0^{-|\alpha|}(P\cdot Q)^2\left\{(P\cdot Q)^2 - 1\right\}^{-3/2}\frac{|p - q|^2}{p_0 q_0}.
\end{aligned} \tag{2.44}$$

We will use this estimate twice to get (2.35) and (2.37).

We first establish (2.35). On the set $\mathcal{A}$ we have

$$2|p-q|^2 + 2|p\times q|^2 \geq (|p-q| + |p\times q|)^2 \geq \frac{1}{4}p_0^2 + \frac{|p|}{2} \geq \frac{1}{4}p_0^2.$$

From (2.32) and the last display we have

$$P\cdot Q + 1 \geq P\cdot Q - 1 \geq \frac{1}{16}\frac{p_0}{q_0}.$$

From the Cauchy-Schwartz inequality we also have

$$0 \leq P\cdot Q - 1 \leq P\cdot Q \leq p_0 q_0 + |p\cdot q| \leq 2p_0 q_0.$$

We plug these last two inequalities (one at a time) into (2.44) to obtain

$$\begin{aligned}
\left|\boldsymbol{\Theta}_\alpha \Phi^{ij}(p,q)\right| &\leq C(P\cdot Q)^2\left\{(P\cdot Q)^2 - 1\right\}^{-3/2} p_0^{-|\alpha|}\left\{\frac{p_0^2q_0^2 + p_0^2q_0^2 + p_0^2q_0^2}{p_0q_0}\right\}\\
&\leq C(p_0q_0)^2\left\{(P\cdot Q)^2 - 1\right\}^{-3/2} p_0^{-|\alpha|}p_0q_0\\
&\leq C(p_0q_0)^3\left\{P\cdot Q - 1\right\}^{-3} p_0^{-|\alpha|}\\
&\leq C(p_0q_0)^3\left(\frac{p_0}{q_0}\right)^{-3} p_0^{-|\alpha|}.
\end{aligned}$$

We move on to establishing (2.36). If $|p|\leq 1$, then $p_0 \leq 2 \leq 2q_0$. Assume $|p|\geq 1$, using $\mathcal{B}$ we compute

$$q_0 \geq |q| \geq |p| - |p-q| \geq \frac{1}{2}|p| - \frac{1}{2} \geq \frac{1}{4}p_0 - \frac{1}{2}.$$

Therefore, $p_0 \leq 6q_0$ on $\mathcal{B}$. For the other half of (2.36),

$$q_0 \leq p_0 + |p-q| \leq \frac{3}{2}p_0 + \frac{1}{2} \leq 2p_0.$$

We move on to establishing (2.37). On the set $\mathcal{B}$ we have a first order singularity. Also (2.32) tells us

$$\frac{|p-q|^2}{2p_0q_0} \leq P\cdot Q - 1 \leq \frac{1}{2}|p-q|^2.$$

We plug this into (2.44) to observe that on $\mathcal{B}$ we have

$$
\begin{aligned}
\left|\boldsymbol{\Theta}_\alpha \Phi^{ij}(p,q)\right| &\leq C(P\cdot Q)^2\left\{(P\cdot Q)^2-1\right\}^{-3/2} p_0^{-|\alpha|} \\
&\quad \times\left\{\frac{(P\cdot Q+1)|p-q|^2+|p-q|^2+|p-q|^2 q_0 p_0}{q_0 p_0}\right\} \\
&\leq C(P\cdot Q)^2\left\{(P\cdot Q)^2-1\right\}^{-3/2} p_0^{-|\alpha|}|p-q|^2 \\
&\leq C(p_0 q_0)^2\left\{(P\cdot Q)^2-1\right\}^{-3/2} p_0^{-|\alpha|}|p-q|^2 \\
&\leq C(p_0 q_0)^2 (p_0 q_0)^{3/2}|p-q|^{-1}\left\{P\cdot Q+1\right\}^{-3/2} p_0^{-|\alpha|} \\
&\leq C(p_0 q_0)^{7/2}|p-q|^{-1} p_0^{-|\alpha|}.
\end{aligned}
$$

We achieve the last inequality because (2.32) says $P\cdot Q\geq 1$. $\square$

Next, let $\mu(p,q)$ be an arbitrary smooth scalar function which decay's rapidly at infinity. We consider the following integral

$$\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)J^{1/2}(q)\mu(p,q)dq.$$

Both the linear term L and the nonlinear term Γ are of this form (Lemma 2.6). We develop a new integration by parts technique.

Theorem 2.3. *Given $|\beta|>0$, we have*

$$
\begin{aligned}
&\partial_\beta\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)J^{1/2}(q)\mu(p,q)dq \\
&= \sum_{\beta_1+\beta_2+\beta_3\leq\beta}\int_{\mathbb{R}^3}\boldsymbol{\Theta}_{\beta_1}\Phi^{ij}(P,Q)J^{1/2}(q)\partial^q_{\beta_2}\partial_{\beta_3}\mu(p,q)\varphi^{\beta}_{\beta_1,\beta_2,\beta_3}(p,q)dq
\end{aligned}
\tag{2.45}
$$

where $\varphi^{\beta}_{\beta_1,\beta_2,\beta_3}(p,q)$ is a smooth function which satisfies

$$\left|\partial^q_{\nu_1}\partial_{\nu_2}\varphi^{\beta}_{\beta_1,\beta_2,\beta_3}(p,q)\right|\leq C q_0^{|\beta|-|\nu_1|}p_0^{|\beta_1|+|\beta_3|-|\beta|-|\nu_2|}, \tag{2.46}$$

for all multi-indices ν_1 and ν_2.

Proof. We prove (2.45) by an induction over the number of derivatives $|\beta|$. Assume $\beta=e_i$ $(i=1,2,3)$. We write

$$\partial_{p_i}=-\frac{q_0}{p_0}\partial_{q_i}+\left(\partial_{p_i}+\frac{q_0}{p_0}\partial_{q_i}\right)=-\frac{q_0}{p_0}\partial_{q_i}+\boldsymbol{\Theta}_{e_i} \tag{2.47}$$

Instead of hitting $\Phi^{ij}(P,Q)$ with ∂_{p_i}, we apply the r.h.s. term above and integrate by parts over $-\frac{q_0}{p_0}\partial_{q_i}$ to obtain

$$\begin{aligned}
&\partial_{p_i}\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)J^{1/2}(q)\mu(p,q)dq\\
=\ &\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)J^{1/2}(q)\partial_{p_i}\mu(p,q)dq\\
&+\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)J^{1/2}(q)\frac{q_0}{p_0}\partial_{q_i}\mu(p,q)dq\\
&+\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)J^{1/2}(q)\left(\frac{q_i}{q_0p_0}-\frac{q_i}{2p_0}\right)\mu(p,q)dq\\
&+\int_{\mathbb{R}^3}\mathbf{\Theta}_{e_i}\Phi^{ij}(P,Q)J^{1/2}(q)\mu(p,q)dq.
\end{aligned}$$

We can write the above in the form (2.45) with the coefficients given by

$$\phi^{e_i}_{0,0,0}(p,q)=\frac{q_i}{q_0p_0}-\frac{q_i}{2p_0},\ \ \phi^{e_i}_{e_i,0,0}(p,q)=1,\ \ \phi^{e_i}_{0,e_i,0}(p,q)=\frac{q_0}{p_0},\ \ \phi^{e_i}_{0,0,e_i}(p,q)=1. \tag{2.48}$$

And define the rest of the coefficients to be zero. Note that these coefficients satisfy the decay (2.46). This establishes the first step in the induction.

Assume the result holds for all $|\beta|\le n$. Fix an arbitrary β' such that $|\beta'|=n+1$ and write $\partial_{\beta'}=\partial_{p_m}\partial_\beta$ for some multi-index β and

$$m=\max\{j:(\beta')^j>0\}.$$

This specification of m is needed because of our chosen ordering of the three differential operators in (2.33), which don't commute.

By the induction assumption

$$\begin{aligned}
&\partial_{\beta'}\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)J^{1/2}(q)\mu(p,q)dq\\
=\ &\sum_{\bar\beta_1+\bar\beta_2+\bar\beta_3\le\beta}\partial_{p_m}\int_{\mathbb{R}^3}\mathbf{\Theta}_{\bar\beta_1}\Phi^{ij}(P,Q)J^{1/2}(q)\partial^q_{\bar\beta_2}\partial_{\bar\beta_3}\mu(p,q)\varphi^{\beta}_{\bar\beta_1,\bar\beta_2,\bar\beta_3}(p,q)dq
\end{aligned}$$

We approach applying the last derivative the same as the $|\beta| = 1$ case above. We obtain

$$= \sum \int_{\mathbb{R}^3} \mathbf{\Theta}_{\bar{\beta}_1} \Phi^{ij}(P,Q) J^{1/2}(q) \varphi^{\beta}_{\bar{\beta}_1,\bar{\beta}_2,\bar{\beta}_3}(p,q) \partial_{p_m} \partial^q_{\bar{\beta}_2} \partial_{\bar{\beta}_3} \mu(p,q) dq \tag{2.49}$$

$$+ \sum \int_{\mathbb{R}^3} \mathbf{\Theta}_{\bar{\beta}_1} \Phi^{ij}(P,Q) J^{1/2}(q) \varphi^{\beta}_{\bar{\beta}_1,\bar{\beta}_2,\bar{\beta}_3}(p,q) \frac{q_0}{p_0} \partial_{q_m} \partial^q_{\bar{\beta}_2} \partial_{\bar{\beta}_3} \mu(p,q) dq \tag{2.50}$$

$$+ \sum \int_{\mathbb{R}^3} \mathbf{\Theta}_{e_m} \mathbf{\Theta}_{\bar{\beta}_1} \Phi^{ij}(P,Q) J^{1/2}(q) \partial^q_{\bar{\beta}_2} \partial_{\bar{\beta}_3} \mu(p,q) \varphi^{\beta}_{\bar{\beta}_1,\bar{\beta}_2,\bar{\beta}_3}(p,q) dq \tag{2.51}$$

$$\begin{aligned} &+ \sum \int_{\mathbb{R}^3} \mathbf{\Theta}_{\bar{\beta}_1} \Phi^{ij}(P,Q) J^{1/2}(q) \partial^q_{\bar{\beta}_2} \partial_{\bar{\beta}_3} \mu(p,q) \\ &\times \left(\partial_{p_m} + \frac{q_0}{p_0} \partial_{q_m} + \frac{q_m}{p_0 q_0} - \frac{q_m}{2p_0} \right) \varphi^{\beta}_{\bar{\beta}_1,\bar{\beta}_2,\bar{\beta}_3}(p,q) dq, \end{aligned} \tag{2.52}$$

where the unspecified summations above are over $\bar{\beta}_1 + \bar{\beta}_2 + \bar{\beta}_3 \leq \beta$. We collect all the terms above with the same order of differentiation to obtain

$$= \sum_{\beta_1+\beta_2+\beta_3 \leq \beta'} \int_{\mathbb{R}^3} \mathbf{\Theta}_{\beta_1} \Phi^{ij}(P,Q) J^{1/2}(q) \partial^q_{\beta_2} \partial_{\beta_3} \mu(p,q) \varphi^{\beta'}_{\beta_1,\beta_2,\beta_3}(p,q) dq,$$

where the functions $\varphi^{\beta'}_{\beta_1,\beta_2,\beta_3}(p,q)$ are defined naturally as the coefficient in front of each term of the form $\mathbf{\Theta}_{\beta_1} \Phi^{ij}(P,Q) J^{1/2}(q) \partial^q_{\beta_2} \partial_{\beta_3} \mu(p,q)$ and we recall that $\beta' = \beta + e_m$.

We check (2.46) by comparing the decay with the order of differentiation in each of the four terms (2.49-2.52). For (2.49), the order of differentiation is

$$\beta_1 = \bar{\beta}_1, \ \beta_2 = \bar{\beta}_2, \ \beta_3 = \bar{\beta}_3 + e_m.$$

And by the induction assumption,

$$\begin{aligned} \left| \partial^q_{\nu_1} \partial_{\nu_2} \varphi^{\beta}_{\bar{\beta}_1,\bar{\beta}_2,\bar{\beta}_3}(p,q) \right| &\leq C q_0^{|\beta|-|\nu_1|} p_0^{|\bar{\beta}_1|+|\bar{\beta}_3|-|\beta|-|\nu_2|}, \\ &\leq C q_0^{|\beta+e_m|-|\nu_1|} p_0^{|\beta_1|+|\bar{\beta}_3+e_m|-|\beta+e_i|-|\nu_2|} \\ &= C q_0^{|\beta'|-|\nu_1|} p_0^{|\beta_1|+|\beta_3|-|\beta'|-|\nu_2|}. \end{aligned}$$

This establishes (2.46) for (2.49).

For (2.50), the order of differentiation is

$$\beta_1 = \bar{\beta}_1, \ \beta_2 = \bar{\beta}_2 + e_m, \ \beta_3 = \bar{\beta}_3.$$

And by the induction assumption as well as the Leibnitz rule,

$$\begin{aligned}\left|\partial^q_{\nu_1}\partial_{\nu_2}\left(\frac{q_0}{p_0}\varphi^\beta_{\bar\beta_1,\bar\beta_2,\bar\beta_3}(p,q)\right)\right| &\leq C q_0^{|\beta|+1-|\nu_1|} p_0^{|\bar\beta_1|+|\bar\beta_3|-|\beta|-1-|\nu_2|}, \\ &= C q_0^{|\beta'|-|\nu_1|} p_0^{|\beta_1|+|\beta_3|-|\beta'|-|\nu_2|}.\end{aligned}$$

This establishes (2.46) for (2.50).

For (2.51), the order of differentiation is

$$\beta_1 = \bar\beta_1 + e_m,\ \beta_2 = \bar\beta_2,\ \beta_3 = \bar\beta_3.$$

And by the induction assumption,

$$\begin{aligned}\left|\partial^q_{\nu_1}\partial_{\nu_2}\varphi^\beta_{\bar\beta_1,\bar\beta_2,\bar\beta_3}(p,q)\right| &\leq C q_0^{|\beta|-|\nu_1|} p_0^{|\bar\beta_1|+|\bar\beta_3|-|\beta|-|\nu_2|}, \\ &\leq C q_0^{|\beta'|-|\nu_1|} p_0^{|\beta_1|+|\beta_3|-|\beta'|-|\nu_2|}.\end{aligned}$$

This establishes (2.46) for (2.51).

For (2.52), the order of differentiation is

$$\beta_1 = \bar\beta_1,\ \beta_2 = \bar\beta_2,\ \beta_3 = \bar\beta_3.$$

And by the induction assumption as well as the Leibnitz rule,

$$\begin{aligned}&\left|\partial^q_{\nu_1}\partial_{\nu_2}\left\{\left(\partial_{p_m} + \frac{q_0}{p_0}\partial_{q_m} + \frac{q_m}{p_0 q_0} - \frac{q_m}{2p_0}\right)\varphi^\beta_{\bar\beta_1,\bar\beta_2,\bar\beta_3}(p,q)\right\}\right| \\ &\leq C q_0^{|\beta|+1-|\nu_1|} p_0^{|\bar\beta_1|+|\bar\beta_3|-|\beta|-1-|\nu_2|}, \\ &= C q_0^{|\beta'|-|\nu_1|} p_0^{|\beta_1|+|\beta_3|-|\beta'|-|\nu_2|}.\end{aligned}$$

This establishes (2.46) for (2.52) and therefore for all of the coefficients. □

Next, we compute derivatives of the collision kernel in (2.26) which will be important for showing that solutions $F_\pm$ to the relativistic Landau-Maxwell system are positive.

Lemma 2.3. *We compute a sum of first derivatives in q of (2.26) as*

$$\sum_j \partial_{q_j}\Phi^{ij}(P,Q) = 2\frac{\Lambda(P,Q)}{p_0 q_0}\left(P\cdot Q p_i - q_i\right). \tag{2.53}$$

This term has a second order singularity at $p = q$. We further compute a sum of (2.53) over first derivatives in p as

$$\sum_{i,j} \partial_{p_i} \partial_{q_j} \Phi^{ij}(P,Q) = 4\frac{P \cdot Q}{p_0 q_0} \left\{ (P \cdot Q)^2 - 1 \right\}^{-1/2} \geq 0. \tag{2.54}$$

This term has a first order singularity.

This result is quite different from the classical theory, it is straightforward compute the derivative of the classical kernel in (2.13) as

$$\sum_{i,j} \partial_{v_i} \partial_{v'_j} \phi^{ij}(v - v') = 0.$$

On the contrary, the proof of Lemma 2.3 is quite technical.

Proof. Throught this proof, we temporarily suspend our use of the Einstein summation convention. Differentiating (2.26), we have

$$\begin{aligned}
\partial_{q_j} \Phi^{ij}(P,Q) \quad \equiv \quad & \frac{\partial_{q_j} \Lambda(P,Q)}{p_0 q_0} S^{ij}(P,Q) \\
& + \frac{\Lambda(P,Q)}{p_0 q_0} \left(\partial_{q_j} S^{ij}(P,Q) - \frac{q_j}{q_0^2} S^{ij}(P,Q) \right).
\end{aligned}$$

And

$$\begin{aligned}
\partial_{q_j} \Lambda(P,Q) \quad = \quad & 2(P \cdot Q) \left\{ (P \cdot Q)^2 - 1 \right\}^{-3/2} \left(\frac{q_j}{q_0} p_0 - p_j \right) \\
& -3(P \cdot Q)^3 \left\{ (P \cdot Q)^2 - 1 \right\}^{-5/2} \left(\frac{q_j}{q_0} p_0 - p_j \right).
\end{aligned}$$

Since (2.2) implies $\sum_j S^{ij}(P,Q) \left(\frac{q_j}{q_0} p_0 - p_j \right) = 0$, we conclude

$$\sum_j \frac{\partial_{q_j} \Lambda(P,Q) S^{ij}(P,Q)}{p_0 q_0} = 0.$$

Therefore it remains to evaluate the r.h.s. of

$$\sum_j \partial_{q_j} \Phi^{ij}(P,Q) = \frac{\Lambda(P,Q)}{p_0 q_0} \sum_j \left(\partial_{q_j} S^{ij}(P,Q) - \frac{q_j}{q_0^2} S^{ij}(P,Q) \right). \tag{2.55}$$

We take a derivative of S^{ij} in (2.26) as

$$\begin{aligned}
\partial_{q_j} S^{ij} &= 2\,(P\cdot Q)\left(\frac{q_j}{q_0}p_0 - p_j\right)\delta_{ij} + \left(\frac{q_j}{q_0}p_0 - p_j\right)(p_i q_j + q_i p_j) \\
&\quad + \{P\cdot Q - 1\}\,(p_i + \delta_{ij}p_j) + (1+\delta_{ij})\,(p_i - q_i) \\
&= 2\,(P\cdot Q)\left(\frac{q_j}{q_0}p_0 - p_j\right)\delta_{ij} + \left(q_j^2\frac{p_0}{q_0}p_i + p_j q_j \frac{p_0}{q_0}q_i - p_i p_j q_j - q_i p_j^2\right) \\
&\quad + P\cdot Q\,(1+\delta_{ij})\,p_i - (1+\delta_{ij})\,q_i.
\end{aligned}$$

Next, sum this expression over j to obtain

$$\begin{aligned}
\sum_j \partial_{q_j} S^{ij} &= 2\,(P\cdot Q)\left(\frac{q_i}{q_0}p_0 - p_i\right) + \frac{p_0}{q_0}\left(|q|^2 p_i + p\cdot q q_i\right) \\
&\quad - p\cdot q p_i - |p|^2 q_i + 4P\cdot Q p_i - 4q_i.
\end{aligned}$$

We collect terms which are coefficients of p_i and q_i respectively

$$\begin{aligned}
\sum_j \partial_{q_j} S^{ij} &= q_i\left\{2\,(P\cdot Q)\frac{p_0}{q_0} + p\cdot q\frac{p_0}{q_0} - |p|^2 - 4\right\} \\
&\quad + p_i\left\{-2P\cdot Q + |q|^2\frac{p_0}{q_0} - p\cdot q + 4P\cdot Q\right\} \\
&= q_i\left(\frac{p_0}{q_0}P\cdot Q - 3\right) + p_i\left(3P\cdot Q - \frac{p_0}{q_0}\right), \qquad (2.56)
\end{aligned}$$

where the last line follows from plugging $|p|^2 = p_0^2 - 1 = p_0 q_0 \frac{p_0}{q_0} - 1$ into the first line and plugging

$$|q|^2\frac{p_0}{q_0} = q_0^2\frac{p_0}{q_0} - \frac{p_0}{q_0} = p_0 q_0 - \frac{p_0}{q_0}$$

into the second line.

Turning to the computation of $-\frac{1}{q_0^2}\sum_j q_j S^{ij}$, we plug (2.26) into the following

$$\begin{aligned}
\sum_j q_j S^{ij}(P,Q) &= \left\{(P\cdot Q)^2 - 1\right\} q_i - (p_i - q_i)\, q\cdot(p-q) \\
&\quad + \{(P\cdot Q) - 1\}\left(p\cdot q q_i + |q|^2 p_i\right)
\end{aligned}$$

We collect terms which are coefficients of p_i and q_i respectively to obtain

$$
\begin{aligned}
&= q_i\left((P\cdot Q)^2 - 1 + q\cdot(p-q) + p\cdot q\,\{P\cdot Q - 1\}\right) \\
&\quad + p_i\left(-q\cdot(p-q) + |q|^2\{P\cdot Q - 1\}\right) \\
&= q_i\left((P\cdot Q)^2 - 1 - |q|^2 + p\cdot q\,(P\cdot Q)\right) + p_i\left(-p\cdot q + |q|^2 P\cdot Q\right) \\
&= q_i\left(|p|^2 q_0^2 - p_0 q_0 p\cdot q\right) + p_i\left(|q|^2 P\cdot Q - p\cdot q\right) \\
&= q_i\left(p_0^2 q_0^2 - p_0 q_0 p\cdot q - q_0^2\right) + p_i\left(q_0^2 P\cdot Q - P\cdot Q - p\cdot q\right) \\
&= p_0 q_0 q_i\left(P\cdot Q - \frac{q_0}{p_0}\right) + q_0^2 p_i\left(P\cdot Q - \frac{p_0}{q_0}\right).
\end{aligned}
$$

Divide this expression by q_0^2 to conclude

$$
\sum_j \frac{q_j}{q_0^2} S^{ij} = q_i\left(\frac{p_0}{q_0} P\cdot Q - 1\right) + p_i\left(P\cdot Q - \frac{p_0}{q_0}\right). \tag{2.57}
$$

This and (2.56) are very symmetric expressions.

We combine (2.56) and (2.57) to obtain

$$
\begin{aligned}
\sum_j\left(\partial_{q_j} S^{ij} - \frac{q_j}{q_0^2} S^{ij}\right) &= q_i\left(\frac{p_0}{q_0} P\cdot Q - 3\right) + p_i\left(3P\cdot Q - \frac{p_0}{q_0}\right) \\
&\quad - q_i\left(\frac{p_0}{q_0} P\cdot Q - 1\right) - p_i\left(P\cdot Q - \frac{p_0}{q_0}\right) \\
&= 2\left(P\cdot Q p_i - q_i\right).
\end{aligned}
$$

We note that this term has a first order cancellation at $p = q$. We plug this last display into (2.55) to obtain (2.53).

We differentiate (2.53) to obtain

$$
\begin{aligned}
\sum_i \partial_{p_i} \sum_j \partial_{q_j} \Phi^{ij}(P,Q) &= 2\sum_i \frac{\partial_{p_i}\Lambda(P,Q)}{p_0 q_0}\left(P\cdot Q p_i - q_i\right) \\
&\quad + 2\frac{\Lambda(P,Q)}{p_0 q_0}\sum_i\left(\partial_{p_i} - \frac{p_i}{p_0^2}\right)\left(P\cdot Q p_i - q_i\right).
\end{aligned} \tag{2.58}
$$

And we can write the derivative of Λ as

$$
\partial_{p_i}\Lambda(P,Q) = -(P\cdot Q)\left\{(P\cdot Q)^2 - 1\right\}^{-5/2}\left(\frac{p_i}{p_0} q_0 - q_i\right)\left((P\cdot Q)^2 + 2\right).
$$

We compute

$$\begin{aligned}\sum_i \left(\frac{p_i}{p_0}q_0 - q_i\right)((P\cdot Q)p_i - q_i) &= \sum_i \left(\frac{q_0}{p_0}(P\cdot Q)p_i^2 - \frac{q_0}{p_0}p_i q_i - p_i q_i (P\cdot Q) + q_i^2\right) \\ &= \frac{q_0}{p_0}(P\cdot Q)|p|^2 - \frac{q_0}{p_0}p\cdot q - p\cdot q(P\cdot Q) + |q|^2.\end{aligned}$$

We further add and subtract $\frac{q_0}{p_0}(P\cdot Q)$ to obtain

$$\begin{aligned}&= p_0q_0(P\cdot Q) - \frac{q_0}{p_0}(P\cdot Q) - \frac{q_0}{p_0}p\cdot q - p\cdot q(P\cdot Q) + |q|^2 \\ &= p_0q_0(P\cdot Q) - q_0^2 - p\cdot q(P\cdot Q) + |q|^2 \\ &= p_0q_0(P\cdot Q) - p\cdot q(P\cdot Q) - 1 \\ &= (P\cdot Q)^2 - 1.\end{aligned}$$

We conclude that

$$\sum_i \frac{\partial_{p_i}\Lambda(P,Q)}{p_0q_0}(P\cdot Qp_i - q_i) = -\frac{P\cdot Q}{p_0q_0}\frac{((P\cdot Q)^2+2)}{\left\{(P\cdot Q)^2-1\right\}^{3/2}}. \tag{2.59}$$

This term has a third order singularity. We will find that the second term in (2.58) also has a third order singularity, but there is second order cancellation between the two terms in (2.58).

We now evaluate the sum in second term in (2.58) as

$$\begin{aligned}\sum_i \left(\partial_{p_i} - \frac{p_i}{p_0^2}\right)(P\cdot Qp_i - q_i) &= \sum_i \left(P\cdot Q + p_i\left(p_i\frac{q_0}{p_0} - q_i\right) - P\cdot Q\frac{p_i^2}{p_0^2} + \frac{p_iq_i}{p_0^2}\right) \\ &= 3P\cdot Q + |p|^2\frac{q_0}{p_0} - p\cdot q - P\cdot Q\frac{|p|^2}{p_0^2} + \frac{p\cdot q}{p_0^2}.\end{aligned}$$

We add and subtract $\frac{q_0}{p_0}$ as well as $\frac{P\cdot Q}{p_0^2}$ to obtain

$$\begin{aligned}&= 3P\cdot Q - \frac{q_0}{p_0} + p_0q_0 - p\cdot q - P\cdot Q + \frac{P\cdot Q}{p_0^2} + \frac{p\cdot q}{p_0^2} \\ &= 3P\cdot Q - \frac{q_0}{p_0} + \frac{p_0q_0}{p_0^2} = 3P\cdot Q.\end{aligned}$$

Therefore, plugging in (2.26), we obtain

$$\frac{\Lambda(P,Q)}{p_0q_0}\sum_i\left(\partial_{p_i} - \frac{p_i^2}{p_0^2}\right)(P\cdot Qp_i - q_i) = 3\frac{(P\cdot Q)^3}{p_0q_0}\left\{(P\cdot Q)^2-1\right\}^{-3/2}.$$

Further plugging this and (2.59) into (2.58) we obtain (2.54). $\square$

In the following Lemma, we will use Lemma 2.3 to obtain a simplified expression for part of the collision operator (2.1) which will be used to prove the positvity of our solutions $F_\pm$ to the relativistic Landau-Maxwell system.

LEMMA 2.4. *Given a smooth scalar function $G(q)$ which decays rapidly at infinity, we have*

$$
\begin{aligned}
-\partial_{p_i}\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)\partial_{q_j}G(q)dq &= 4\int_{\mathbb{R}^3}\frac{P\cdot Q}{p_0q_0}\left\{(P\cdot Q)^2-1\right\}^{-1/2}G(q)dq \\
&\quad +\kappa(p)G(p),
\end{aligned}
$$

where $\kappa(p)=2^{7/2}\pi p_0\int_0^\pi\left(1+|p|^2\sin^2\theta\right)^{-3/2}\sin\theta d\theta$.

PROOF. We write out ∂_{p_i} as in (2.47) to observe

$$
\begin{aligned}
-\partial_{p_i}\int\Phi^{ij}(P,Q)\partial_{q_j}G(q)dq &= -\int\left\{-\frac{q_0}{p_0}\partial_{q_i}+\mathbf{\Theta}_{e_i}\right\}\Phi^{ij}(P,Q)\partial_{q_j}G(q)dq \\
&= -\int\mathbf{\Theta}_{e_i}\Phi^{ij}(P,Q)\partial_{q_j}G(q)dq \\
&\quad +\int\frac{q_0}{p_0}\partial_{q_i}\Phi^{ij}(P,Q)\partial_{q_j}G(q)dq.
\end{aligned}
$$

We split these integrals into $|p-q|\le\epsilon$ and $|p-q|>\epsilon$ for $\epsilon>0$. We note that the integrals over $|p-q|\le\epsilon$ converge to zero as $\epsilon\downarrow 0$. We will eventually send $\epsilon\downarrow 0$, so we focus on the region $|p-q|>\epsilon$. We rewrite

$$
\begin{aligned}
\frac{q_0}{p_0}\partial_{q_i}\Phi^{ij}(P,Q)\partial_{q_j}G(q) &= \partial_{q_j}\left\{\frac{q_0}{p_0}\partial_{q_i}\Phi^{ij}(P,Q)G(q)\right\} \\
&\quad -\partial_{q_j}\left\{\frac{q_0}{p_0}\partial_{q_i}\Phi^{ij}(P,Q)\right\}G(q).
\end{aligned}
$$

After an integration by parts, the integrals over $|p-q|>\epsilon$ are

$$
\begin{aligned}
&= \int_{|p-q|>\epsilon}\partial_{q_j}\left\{\mathbf{\Theta}_{e_i}\Phi^{ij}(P,Q)\right\}G(q)dq \\
&\quad -\int_{|p-q|>\epsilon}\partial_{q_j}\left\{\frac{q_0}{p_0}\partial_{q_i}\Phi^{ij}(P,Q)\right\}G(q)dq \\
&\quad +\int_{|p-q|>\epsilon}\partial_{q_j}\left\{\frac{q_0}{p_0}\partial_{q_i}\Phi^{ij}(P,Q)G(q)\right\}dq.
\end{aligned}
$$

By the definition of $\boldsymbol{\Theta}_{e_i}$ in (2.33), this is

$$= \int_{|p-q|>\epsilon} \partial_{q_j}\partial_{p_i}\Phi^{ij}(P,Q)G(q)dq + \int_{|p-q|>\epsilon} \partial_{q_j}\left\{\frac{q_0}{p_0}\partial_{q_i}\Phi^{ij}(P,Q)G(q)\right\}dq.$$

We plug (2.54) into the first term above to obtain the first term on the r.h.s. of this Lemma as $\epsilon \downarrow 0$. For the second term above, we apply the divergence theorem to obtain

$$\int_{|p-q|>\epsilon} \partial_{q_j}\left\{\frac{q_0}{p_0}\partial_{q_i}\Phi^{ij}(P,Q)G(q)\right\}dq = \int_{|p-q|=\epsilon} \frac{q_0}{p_0}\partial_{q_i}\Phi^{ij}(P,Q)\frac{p_j-q_j}{|p-q|}G(q)dS,$$

where dS is given below. By a Taylor expansion, $P\cdot Q = 1 + O(|p-q|^2)$. Using this and (2.53) we have

$$\frac{q_0}{p_0}\partial_{q_i}\Phi^{ij}(P,Q)\frac{p_j-q_j}{|p-q|} = 2\frac{\Lambda(P,Q)}{p_0^2}|p-q| + O(|p-q|^{-1}).$$

And the integral over $|p-q|=\epsilon$ which includes the terms in $O(|p-q|^{-1})$ goes to zero as $\epsilon\downarrow 0$. We focus on the main part

$$2p_0^{-2}\int_{|p-q|=\epsilon} \Lambda(P,Q)|p-q|G(q)dS.$$

We multiply and divide by $p_0q_0 + p\cdot q + 1$ to observe that

$$P\cdot Q - 1 = \frac{|p-q|^2 + |p\times q|^2}{p_0q_0 + p\cdot q + 1}.$$

This and (2.26) imply

$$\Lambda = (P\cdot Q)^2\left(\frac{p_0q_0 + p\cdot q + 1}{p_0q_0 - p\cdot q + 1}\right)^{3/2}\left(|p-q|^2 + |p\times q|^2\right)^{-3/2}.$$

We change variables as $q\to p-q$ so that the integrand becomes $\Lambda|q|G(p-q)$ and we define $\bar{q}_0 = \sqrt{1+|p-q|^2}$ so that after the change of variables

$$\Lambda = (p_0\bar{q}_0 - |p|^2 + p\cdot q)^2\left(\frac{p_0\bar{q}_0 + |p|^2 - p\cdot q + 1}{p_0\bar{q}_0 - |p|^2 + p\cdot q + 1}\right)^{3/2}\left(|q|^2 + |p\times q|^2\right)^{-3/2}.$$

We choose the angular integration over $|q|=\epsilon$ in such a way that $p\cdot q = |p||q|\cos\theta$ and $dS = \epsilon^2\sin\theta d\theta d\phi = \epsilon^2 d\omega$ with $0\le\theta\le\pi$, $0\le\phi\le 2\pi$. Note that as $\epsilon\downarrow 0$ (on $|q|=\epsilon$)

$$\epsilon^3\Lambda \to 2^{3/2}\left(1+|p|^2\sin^2\theta\right)^{-3/2}p_0^3.$$

Hence, as $\epsilon \downarrow 0$,

$$2p_0^{-2}\int_{|q|=\epsilon} \Lambda|q|G(p-q)dS \;=\; 2p_0^{-2}\int_{S^2} \epsilon^3\Lambda G(p-\epsilon\omega)d\omega \to \kappa(p)G(p),$$

with κ defined in the statement of this Lemma. $\square$

LEMMA 2.5. *There exists $C>0$, such that*

$$\frac{1}{C}\left\{|\nabla_p g|_2^2 + |g|_2^2\right\} \le |g|_\sigma^2 \le C\left\{|\nabla_p g|_2^2 + |g|_2^2\right\}. \tag{2.60}$$

Further, $\sigma^{ij}(p)$ is a smooth function satisfying

$$\left|\partial_\beta \sigma^{ij}(p)\right| \le C p_0^{-|\beta|}. \tag{2.61}$$

PROOF. The spectrum of $\sigma^{ij}(p)$, (2.27), consists of a simple eigenvalue $\lambda_1(p)>0$ associated with the vector p and a double eigenvalue $\lambda_2(p)>0$ associated with $p^\perp$; there are constants $c_1, c_2 > 0$ such that, as $|p|\to\infty$, $\lambda_1(p)\to c_1$, $\lambda_2(p)\to c_2$. In Lemou [47] there is a full discussion of these eigenvalues. This is enough to prove (2.60); see [38] for more details on a similar argument.

We move on to (2.61). We combine (2.27) and (2.45) (with $\mu(p,q) = J^{1/2}(q)$) to obtain

$$\begin{aligned}
\partial_\beta \sigma^{ij}(p) &= \partial_\beta \int_{\mathbb{R}^3} \Phi^{ij}(P,Q)J(q)dq \\
&= \sum_{\beta_1+\beta_2\le\beta} \int_{\mathbb{R}^3} \mathbf{\Theta}_{\beta_1}\Phi(P,Q)J^{1/2}(q)\partial^q_{\beta_2}J^{1/2}(q)\varphi^{\beta}_{\beta_1,\beta_2,0}(p,q)dq.
\end{aligned}$$

By (2.46) then

$$\left|\partial_\beta \sigma^{ij}(p)\right| \le C\sum_{\beta_1+\beta_2\le\beta} p_0^{|\beta_1|-|\beta|}\int \left|\mathbf{\Theta}_{\beta_1}\Phi(P,Q)\right| J^{1/2}(q)dq.$$

Recall (2.34), we split this integration into the sets $\mathcal{A}$, $\mathcal{B}$. We plug in the estimate (2.35) to get

$$p_0^{|\beta_1|-|\beta|}\int_{\mathcal{A}} \left|\mathbf{\Theta}_{\beta_1}\Phi(P,Q)\right| J^{1/2}(q)dq \le Cp_0^{|\beta_1|-|\beta|}p_0^{-|\beta_1|} = Cp_0^{-|\beta|}.$$

On $\mathcal{B}$ we have a first order singularity but q is larger than p, in fact we use (2.36) to get exponential decay in p over this region. With (2.37) we obtain

$$\int_{\mathcal{B}} |\mathbf{\Theta}_{\beta_1}\Phi(P,Q)|\, J^{1/2}(q)dq \leq CJ^{1/16}(p)\int |p-q|^{-1}\, J^{1/4}(q)dq.$$

We can now deduce (2.61). $\square$

We now write the Landau Operators A, K, Γ in a new form which will be used throughout the rest of the paper.

LEMMA 2.6. *We have the following representations for $A, K, \Gamma \in \mathbb{R}^2$, which are defined in (2.22) and (2.23),*

$$\begin{aligned} Ag &= 2J^{-1/2}\partial_{p_i}\{J^{1/2}\sigma^{ij}(\partial_{p_j}g + \frac{p_j}{2p_0}g)\} \\ &= 2\partial_{p_i}(\sigma^{ij}\partial_{p_j}g) - \frac{1}{2}\sigma^{ij}\frac{p_i}{p_0}\frac{p_j}{p_0}g + \partial_{p_i}\left\{\sigma^{ij}\frac{p_j}{p_0}\right\}g, \end{aligned} \tag{2.62}$$

$$\begin{aligned} Kg &= -J(p)^{-1/2}\partial_{p_i}\left\{J(p)\int_{\mathbb{R}^3}\Phi^{ij}J^{1/2}(q)\partial_{q_j}(g(q)\cdot[1,1])dq\right\}[1,1] \\ &\quad -J(p)^{-1/2}\partial_{p_i}\left\{J(p)\int_{\mathbb{R}^3}\Phi^{ij}J^{1/2}(q)\frac{q_j}{2q_0}(g(q)\cdot[1,1])dq\right\}[1,1], \end{aligned} \tag{2.63}$$

where $\Phi^{ij} = \Phi^{ij}(P,Q)$. Further

$$\Gamma(g,h) = [\Gamma_+(g,h), \Gamma_-(g,h)], \tag{2.64}$$

where

$$\begin{aligned} \Gamma_\pm(g,h) &= \left(\partial_{p_i} - \frac{p_i}{2p_0}\right)\int \Phi^{ij}J^{1/2}(q)\partial_{p_j}g_\pm(p)\left(h(q)\cdot[1,1]\right)dq, \\ &\quad -\left(\partial_{p_i} - \frac{p_i}{2p_0}\right)\int \Phi^{ij}J^{1/2}(q)g_\pm(p)\partial_{q_j}\left(h(q)\cdot[1,1]\right)dq. \end{aligned}$$

PROOF. For (2.62) it suffices to consider $2J(p)^{-1/2}\mathcal{C}(J^{1/2}g_\pm, J)$:

$$\begin{aligned} &\equiv 2J(p)^{-1/2}\partial_{p_i}\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)\left\{\partial_{p_j}\left(J^{1/2}g_\pm(p)\right)J(q) - \left(J^{1/2}g_\pm(p)\right)\partial_{q_j}J(q)\right\}dq \\ &= 2J(p)^{-1/2}\partial_{p_i}\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)J(q)J(p)^{1/2}\left\{\partial_{p_j}g_\pm + \left(\frac{q_j}{q_0} - \frac{p_j}{2p_0}\right)g_\pm\right\}dq \\ &= 2J(p)^{-1/2}\partial_{p_i}\left\{\sigma^{ij}(p)J(p)^{1/2}\left(\partial_{p_j}g_\pm + \frac{p_j}{2p_0}g_\pm\right)\right\}. \end{aligned}$$

Above, we have used the null space of Φ in (2.2). Below, we move some derivatives inside and cancel out one term.

$$
\begin{aligned}
&= 2\partial_{p_i}\left\{\sigma^{ij}(p)\left(\partial_{p_j}g_\pm+\frac{p_j}{2p_0}g_\pm\right)\right\}-\frac{p_i}{p_0}\left\{\sigma^{ij}(p)\left(\partial_{p_j}g_\pm+\frac{p_j}{2p_0}g_\pm\right)\right\}\\
&= 2\partial_{p_i}\left\{\sigma^{ij}(p)\partial_{p_j}g_\pm\right\}+\partial_{p_i}\left\{\sigma^{ij}(p)\frac{p_j}{p_0}\right\}g_\pm-\frac{1}{2}\sigma^{ij}(p)\frac{p_i}{p_0}\frac{p_j}{p_0}g_\pm.
\end{aligned}
$$

For K simply plug (2.25) with normalized constants into (2.22). For Γ, we use the null condition (2.2) to compute $J(p)^{-1/2}\mathcal{C}(\sqrt{J}g_+,\sqrt{J}h_-)$

$$
\begin{aligned}
&= J(p)^{-1/2}\partial_{p_i}\int\Phi^{ij}(P,Q)J^{1/2}(q)J^{1/2}(p)\left\{h_-(q)\partial_{p_j}g_+(p)-\partial_{q_j}h_-(q)g_+(p)\right\}dq\\
&\quad +J(p)^{-1/2}\partial_{p_i}\int\Phi^{ij}(P,Q)J^{1/2}(q)J^{1/2}(p)\left\{\frac{q_j}{2q_0}-\frac{p_j}{2p_0}\right\}h_-(q)g_+(p)dq\\
&= J(p)^{-1/2}\partial_{p_i}\int\Phi^{ij}(P,Q)J^{1/2}(q)J^{1/2}(p)\left\{h_-(q)\partial_{p_j}g_+(p)-\partial_{q_j}h_-(q)g_+(p)\right\}dq\\
&= \left(\partial_{p_i}-\frac{p_i}{2p_0}\right)\int\Phi^{ij}(P,Q)J^{1/2}(q)\left\{h_-(q)\partial_{p_j}g_+(p)-\partial_{q_j}h_-(q)g_+(p)\right\}dq.
\end{aligned}
$$

Plug four of these type calculations into (2.23) to obtain (2.64). □

We will use these expressions just proven to get the estimates below.

LEMMA 2.7. *Let $|\beta|>0$. For any small $\eta>0$, there exists $C_\eta>0$ such that*

$$
-\langle\partial_\beta\{Ag\},\partial_\beta g\rangle \;\geq\; |\partial_\beta g|_\sigma^2-\eta\sum_{|\alpha|\leq|\beta|}|\partial_\alpha g|_\sigma^2-C_\eta|g|_2^2, \tag{2.65}
$$

$$
|\langle\partial_\beta\{Kg\},\partial_\beta h\rangle| \;\leq\; \left\{\eta\sum_{|\bar\beta|\leq|\beta|}\left|\partial_{\bar\beta}g\right|_\sigma+C_\eta\left|g\right|_2\right\}\left|\partial_\beta h\right|_\sigma. \tag{2.66}
$$

PROOF. We will prove (2.65) for a real valued function g to make the notation less cumbersome, although the result follows trivially for $g=[g_+,g_-]$. We write out the inner product in (2.65) using (2.62) to achieve

$$
\begin{aligned}
\langle \partial_\beta\{Ag\}, \partial_\beta g\rangle &= \int_{\mathbb{R}^3} \partial_\beta \left(\partial_{p_i}\left\{\sigma^{ij}\frac{p_i}{p_0}\right\} g\right) \partial_\beta g dp. \\
&\quad - \int_{\mathbb{R}^3} \left\{ \frac{1}{2}\partial_\beta \left(\sigma^{ij}\frac{p_i}{p_0}\frac{p_j}{p_0} g\right)\partial_\beta g + 2\partial_\beta[\sigma^{ij}\partial_{p_j} g]\partial_{p_i}\partial_\beta g \right\} dp \\
&= -|\partial_\beta g|_\sigma^2 + \sum_{\alpha\le\beta} C_\beta^\alpha \int_{\mathbb{R}^3} \partial_{\beta-\alpha}\partial_{p_i}\left\{\sigma^{ij}\frac{p_i}{p_0}\right\}\partial_\alpha g \partial_\beta g dp \\
&\quad - \sum_{\alpha<\beta} C_\beta^\alpha \int_{\mathbb{R}^3} 2\partial_{\beta-\alpha}\sigma^{ij}\partial_\alpha\partial_{p_j} g \partial_{p_i}\partial_\beta g dp \\
&\quad - \sum_{\alpha<\beta} C_\beta^\alpha \int_{\mathbb{R}^3} \frac{1}{2}\partial_{\beta-\alpha}\left(\sigma^{ij}\frac{p_i}{p_0}\frac{p_j}{p_0}\right)\partial_\alpha g \partial_\beta g dp
\end{aligned} \tag{2.67}
$$

Since $\alpha<\beta$ in the last two terms below, (2.61) gives us the following estimate

$$
\left|\partial_{\beta-\alpha}\partial_{p_i}\left\{\sigma^{ij}\frac{p_i}{p_0}\right\}\right| + \left|\partial_{\beta-\alpha}\sigma^{ij}\right| + \left|\partial_{\beta-\alpha}\left(\sigma^{ij}\frac{p_i}{p_0}\frac{p_j}{p_0}\right)\right| \le C p_0^{-1}
$$

We bound the second and fourth terms in (2.67) by

$$
C\sum_{\alpha\le\beta}\int_{\mathbb{R}^3} p_0^{-1}\left|\partial_\alpha g \partial_\beta g\right| dp = \int_{|p|\le m} + \int_{|p|>m}
$$

On the unbounded part we use Cauchy-Schwartz

$$
\sum_{\alpha\le\beta}\int_{|p|>m} p_0^{-1}\left|\partial_\alpha g\partial_\beta g\right| dp \le \frac{C}{m}|\partial_\beta g|_\sigma \sum_{\alpha\le\beta}|\partial_\alpha g|_\sigma \le \frac{C}{m}\sum_{\alpha\le\beta}|\partial_\alpha g|_\sigma^2
$$

On the compact part we use the compact interpolation of Sobolev-spaces

$$
\begin{aligned}
\int_{|p|\le m}\sum_{\alpha\le\beta}\left|\partial_\alpha g\partial_\beta g\right| dp &\le \int_{|p|\le m}\sum_{\alpha\le\beta}\left|\partial_\alpha g\right|^2 + \left|\partial_\beta g\right|^2 dp \\
&\le \eta' \sum_{|\alpha|=|\beta|+1}\int_{|p|\le m}\left|\partial_\alpha g\right|^2 dp + C_{\eta'}\int_{|p|\le m}|g|^2 dp \\
&\le \eta\sum_{|\alpha|\le|\beta|}|\partial_\alpha g|_\sigma^2 + C_\eta |g|_2^2
\end{aligned}
$$

For the third term in (2.67), we split into two cases, first suppose $|\alpha|<|\beta|-1$ and integrate by parts on ∂_{p_i} to obtain

$$
\sum_{|\alpha|<|\beta|-1}\int_{\mathbb{R}^3} 2\left(\partial_{\beta-\alpha}\partial_{p_i}\sigma^{ij}\partial_\alpha\partial_{p_j} g + \partial_{\beta-\alpha}\sigma^{ij}\partial_{p_i}\partial_\alpha\partial_{p_j} g\right)\partial_\beta g dp
$$

We bound this term by

$$
\begin{aligned}
C\sum_{|\alpha|<|\beta|}\int_{\mathbb{R}^3} p_0^{-1}\left|\partial_\alpha\partial_{p_j}g\partial_\beta g\right|dp &= \int_{|p|\le m}+\int_{|p|>m} \\
&\le \int_{|p|\le m}+\frac{C}{m}|\partial_\beta g|_\sigma\sum_{|\alpha|<|\beta|}|\partial_\alpha g|_\sigma \\
&\le \int_{|p|\le m}+\frac{C}{m}\sum_{|\alpha|\le|\beta|}|\partial_\alpha g|_\sigma^2
\end{aligned}
$$

By the compact interpolation of Sobolev spaces

$$
\sum_{|\alpha|<|\beta|}\int_{|p|\le m}\left|\partial_\alpha\partial_{p_j}g\partial_\beta g\right|dp\le\eta\sum_{|\alpha|\le|\beta|}|\partial_\alpha g|_\sigma^2+C_\eta|g|_2^2
$$

Finally, if $|\alpha|=|\beta|-1$ for the third term in (2.67), we integrate by parts and use symmetry

$$
\begin{aligned}
2\int_{\mathbb{R}^3}\partial_{\beta-\alpha}\sigma^{ij}\partial_\alpha\partial_{p_j}g\partial_{p_i}\partial_\beta g dp &= 2\int_{\mathbb{R}^3}\partial_{\beta-\alpha}\sigma^{ij}\partial_\alpha\partial_{p_j}g\partial_{\beta-\alpha}\partial_\alpha\partial_{p_i}g dp \\
&= -\int_{\mathbb{R}^3}\partial^2_{\beta-\alpha}\sigma^{ij}\partial_\alpha\partial_{p_j}g\partial_\alpha\partial_{p_i}g dp.
\end{aligned}
$$

Because the order of the derivatives on g is now $=|\beta|$, we can again use the compact interpolation of Sobolev spaces and (2.61) to get the same bounds as for the last case $|\alpha|<|\beta|-1$. We obtain

$$
-\langle\partial_\beta\{Ag\},\partial_\beta g\rangle\ge|\partial_\beta g|_\sigma^2-\left(\eta+\frac{C}{m}\right)\sum_{|\alpha|\le|\beta|}|\partial_\alpha g|_\sigma^2-C_\eta|g|_2^2,
$$

This completes the estimate (2.65).

We now consider $\langle\partial_\beta\{Kg\},\partial_\beta h\rangle$ and (2.66). Recalling (2.63), we use (2.45) with $\mu(p,q)=\left\{\partial_{q_j}+\frac{q_j}{2q_0}\right\}(g(q)\cdot[1,1])$, the Leibnitz formula as well as an integration by parts to express $\langle\partial_\beta\{Kg\},\partial_\beta h\rangle$ as

$$
\begin{aligned}
\sum\iint \mathbf{\Theta}_{\alpha_1}\Phi^{ij}(P,Q)\sqrt{J(p)J(q)}\partial^q_{\alpha_2}\left\{\partial_{q_j}+\frac{q_j}{2q_0}\right\}(g(q)\cdot[1,1]) \\
\times\partial_{\beta_1}(h(p)\cdot[1,1])\bar{\varphi}^{\beta,\beta_1}_{\alpha_1,\alpha_2,0}(p,q)dqdp,
\end{aligned}
$$

where the sum is $\alpha_1+\alpha_2\le\beta,|\beta|\le|\beta_1|\le|\beta|+1$. And $\bar{\varphi}^{\beta,\beta_1}_{\alpha_1,\alpha_2,0}(p,q)$ is a collection of the inessential terms, it satisfies the decay estimate (2.46) independent of the value of β_1. Therefore we can further express $\langle \partial_\beta \{Kg\}, \partial_\beta h\rangle$ as

$$\sum \iint \bar{\mu}_{\beta_1\beta_2}(p,q) J(p)^{1/4} J(q)^{1/4} \partial_{\beta_1}(h(p)\cdot[1,1])\partial^q_{\beta_2}(g(q)\cdot[1,1])dqdp$$

where the sum is over $|\beta_2| \leq |\beta|+1, |\beta| \leq |\beta_1| \leq |\beta|+1$. And, using (2.35) and (2.37), we see that $\bar{\mu}_{\beta_1\beta_2}(p,q)$ is a collection of L^2 functions. Therefore, as in (2.80), we split $\langle \partial_\beta \{Kg\}, \partial_\beta h\rangle$ to get

$$\sum \iint \psi_{ij}(p,q) J(p)^{1/4} J(q)^{1/4} \partial_{\beta_1}(h(p)\cdot[1,1])\partial^q_{\beta_2}(g(q)\cdot[1,1])dqdp$$
$$+\sum \iint \{\bar{\mu}_{\beta_1\beta_2} - \psi_{ij}\} J(p)^{1/4} J(q)^{1/4} \partial_{\beta_1}(h(p)\cdot[1,1])\partial^q_{\beta_2}(g(q)\cdot[1,1])dqdp.$$

In the same fashion as J_2 in (2.80), for any $m>0$, we estimate the second term above by

$$\frac{C}{m}\left|\partial_\beta h\right|_\sigma \sum_{|\bar{\beta}|\leq|\beta|}\left|\partial_{\bar{\beta}} g\right|_\sigma .$$

Since $\psi_{ij}(p,q)$ is a smooth function with compact support, we integrate by parts over q repeatedly to bound the first term by

$$\sum\left|\iint \partial_{\beta_1}\{\psi_{ij}(p,q)J(q)^{1/4}\}J(p)^{1/4}(g(q)\cdot[1,1])\partial_{\beta_1}(h(p)\cdot[1,1])dqdp\right|$$
$$\leq C(m)\sum\iint J(q)^{1/8}J(p)^{1/4}\left|(g(q)\cdot[1,1])\partial_{\beta_1}(h(p)\cdot[1,1])\right|dqdp$$
$$\leq C(m)\left|g\right|_2\sum\left|\partial_{\beta_1}h\right|_2 \leq C(m)\left|g\right|_2\left|\partial_\beta h\right|_\sigma .$$

where the final sum above is over $|\beta| \leq |\beta_1| \leq |\beta|+1$. And we have used (2.60) to get the last inequality. We conclude our lemma by choosing m large. □

We now estimate the nonlinear term $\Gamma(f,g)$:

Theorem 2.4. *Let $|\gamma|+|\beta|\leq N$, then*

$$\left|\langle \partial^\gamma_\beta \Gamma(f,g),\ \partial^\gamma_\beta h\rangle\right| \leq C\sum\{|\partial^{\gamma_1}_{\beta_3} f|_2 |\partial^{\gamma-\gamma_1}_{\beta_2} g|_\sigma + |\partial^{\gamma_1}_{\beta_3} f|_\sigma |\partial^{\gamma-\gamma_1}_{\beta_2} g|_2\}|\partial^\gamma_\beta h|_\sigma, \tag{2.68}$$

where the summation is over $\gamma_1\leq\gamma$, $\beta_2+\beta_3\leq\beta$. Further

$$\left|(\partial^\gamma_\beta \Gamma(f,g),\ \partial^\gamma_\beta h)\right| \ \leq\ C\|\partial^\gamma_\beta h\|_\sigma\left\{|||g|||_\sigma |||f||| + |||f|||_\sigma |||g|||\right\}.$$

PROOF. Notice $\partial^\gamma \Gamma(f,g) = \sum_{\gamma_1 \le \gamma} C_\gamma^{\gamma_1} \Gamma(\partial^{\gamma_1} f, \partial^{\gamma-\gamma_1} g)$; thus it suffices to only consider the p derivatives. From (2.64), using (2.45), we can write $\partial_\beta \Gamma(f,g)$ as

$$
\begin{aligned}
&\partial_{p_i} \int \mathbf{\Theta}_{\beta_1} \Phi^{ij}(P,Q) \sqrt{J(q)} \partial_{\beta_2}^q \partial_{\beta_3} \left\{ \partial_{p_j} f_l(p) g_k(q) \right\} \varphi^\beta_{\beta_1,\beta_2,\beta_3}(p,q) dq \\
&-\partial_{p_i} \int \mathbf{\Theta}_{\beta_1} \Phi^{ij}(P,Q) \sqrt{J(q)} \partial_{\beta_2}^q \partial_{\beta_3} \left\{ f_l(p) \partial_{q_j} g_k(q) \right\} \varphi^\beta_{\beta_1,\beta_2,\beta_3}(p,q) dq \\
&+ \int \mathbf{\Theta}_{\beta_1} \Phi^{ij}(P,Q) \sqrt{J(q)} \partial_{\beta_2}^q \partial_{\beta_3} \left\{ f_l(p) \partial_{q_j} g_k(q) \frac{p_i}{2p_0} \right\} \varphi^\beta_{\beta_1,\beta_2,\beta_3}(p,q) dq \\
&- \int \mathbf{\Theta}_{\beta_1} \Phi^{ij}(P,Q) \sqrt{J(q)} \partial_{\beta_2}^q \partial_{\beta_3} \left\{ \partial_{p_j} f_l(p) g_k(q) \frac{p_i}{2p_0} \right\} \varphi^\beta_{\beta_1,\beta_2,\beta_3}(p,q) dq.
\end{aligned}
$$

Above, we implicitly sum over $i,j \in \{1,2,3\}$, $\beta_1+\beta_2+\beta_3 \le \beta$ and $k \in \{+,-\}$. And $l \in \{+,-\}$. Recall $\varphi^\beta_{\beta_1,\beta_2,\beta_3}(p,q)$ from Theorem 2.3. Further (2.46) implies

$$
\left| \varphi^\beta_{\beta_1,\beta_2,\beta_3}(p,q) \right| \le C q_0^{|\beta|}.
$$

We use the above two displays and integrate by parts over ∂_{p_i} in the first two terms, to bound $|\langle \partial_\beta \Gamma(f,g), \partial_\beta h \rangle|$ above by

$$
C \iint J^{1/4}(q) \left| \mathbf{\Theta}_{\beta_1} \Phi^{ij}(P,Q) \partial^q_{\beta_2} g_k(q) \partial_{\beta_3} \partial_{p_j} f_l(p) \partial_{p_i} \partial_\beta h_l(p) \right| dqdp \tag{2.69}
$$

$$
+C \iint J^{1/4}(q) \left| \mathbf{\Theta}_{\beta_1} \Phi^{ij}(P,Q) \partial^q_{\beta_2} \partial_{q_j} g_k(q) \partial_{\beta_3} f_l(p) \partial_{p_i} \partial_\beta h_l(p) \right| dqdp \tag{2.70}
$$

$$
\begin{aligned}
&+C \iint J^{1/4}(q) \left| \mathbf{\Theta}_{\beta_1} \Phi^{ij}(P,Q) \partial^q_{\beta_2} \partial_{q_j} g_k(q) \partial_{\beta_3} f_l(p) \partial_\beta h_l(p) \right| dqdp \\
&+C \iint J^{1/4}(q) \left| \mathbf{\Theta}_{\beta_1} \Phi^{ij}(P,Q) \partial^q_{\beta_2} g_k(q) \partial_{\beta_3} \partial_{p_j} f_l(p) \partial_\beta h_l(p) \right| dqdp.
\end{aligned}
$$

In the above expressions, we add the summation over $l \in \{+,-\}$. It suffices to estimate (2.69) and (2.70); the other two terms are similar. While estimating these terms we will repeatedly use the equivalence of the norms in (2.60) without mention. We start with (2.69).

We next split (2.69) into the two regions, $\mathcal{A}$ and $\mathcal{B}$, defined in (2.34). We then use (2.37) and the Cauchy-Schwartz inequality to estimate (2.69) over $\mathcal{B}$

$$
\begin{aligned}
&\iint_{\mathcal{B}} J^{1/4}(q) \left|\mathbf{\Theta}_{\beta_1}\Phi^{ij}(P,Q)\partial^q_{\beta_2}g_k(q)\partial_{\beta_3}\partial_{p_j}f_l(p)\partial_{p_i}\partial_\beta h_l(p)\right| dqdp \\
&\leq C \iint_{\mathcal{B}} |p-q|^{-1}J^{1/8}(q)\left|\partial^q_{\beta_2}g_k(q)\partial_{\beta_3}\partial_{p_j}f_l(p)\partial_{p_i}\partial_\beta h_l(p)\right| dqdp \\
&\leq C|\partial_\beta h|_\sigma \left(\int_{\mathbb{R}^3}\left|\partial_{\beta_3}\partial_{p_j}f_l(p)\right|^2\left(\int_{\mathcal{B}}|p-q|^{-1}J^{1/8}(q)\left|\partial^q_{\beta_2}g_k(q)\right|dq\right)^2 dp\right)^{1/2} \\
&\leq C|\partial_\beta h|_\sigma|\partial_{\beta_2}g|_2\left(\int_{\mathbb{R}^3}\left|\partial_{\beta_3}\partial_{p_j}f_l(p)\right|^2\left(\int_{\mathcal{B}}|p-q|^{-2}J^{1/4}(q)dq\right)dp\right)^{1/2} \\
&\leq C|\partial_\beta h|_\sigma|\partial_{\beta_2}g|_2|\partial_{\beta_3}f|_\sigma,
\end{aligned}
$$

where use (2.36) to say that $\int_{\mathcal{B}}|p-q|^{-2}J^{1/4}(q)dq \leq C$. This completes the estimate of (2.69) over $\mathcal{B}$.

We estimate (2.69) over $\mathcal{A}$ using (2.35)

$$
\begin{aligned}
&\iint_{\mathcal{A}} J^{1/4}(q)\left|\mathbf{\Theta}_{\beta_1}\Phi^{ij}(P,Q)\partial^q_{\beta_2}g_k(q)\partial_{\beta_3}\partial_{p_j}f_l(p)\partial_{p_i}\partial_\beta h_l(p)\right| dqdp \\
&\leq C\iint J^{1/8}(q)\left|\partial^q_{\beta_2}g_k(q)\partial_{\beta_3}\partial_{p_j}f_l(p)\partial_{p_i}\partial_\beta h_l(p)\right| dqdp \\
&\leq C|\partial_\beta h|_\sigma|\partial_{\beta_2}g|_2|\partial_{\beta_3}f|_\sigma.
\end{aligned}
$$

This completes the estimate for (2.69).

Note that the difference between (2.70) and (2.69) is that ∂_{q_j} hits g in (2.70) whereas ∂_{p_j} hit f in (2.69). This difference means we will need to use the norm $|\cdot|_\sigma$ to estimate g but we are able to use the smaller norm $|\cdot|_2$ to estimate f. This is in contrast to the estimate for (2.69) where the opposite situation held. The main point is that $|\cdot|_\sigma$ includes first order p derivatives.

Using the same splitting over $\mathcal{A}$ and $\mathcal{B}$ as well as the same type calculation, (2.70) is bounded by $|\partial_\beta h|_\sigma|\partial_{\beta_2}g|_\sigma|\partial_{\beta_3}f|_2$. This completes the estimate for (2.70).

The proof of (2.69) follows from the Sobolev embedding: $H^2(\mathbb{T}^3) \subset L^\infty(\mathbb{T}^3)$. Without loss of generality, assume $|\gamma_1| \leq N/2$. This Sobolev embedding grants us that

$$\left(\sup_x |\partial_{\beta_2}^{\gamma_1} f(x)|_2\right) |\partial_{\beta_3}^{\gamma-\gamma_1} g(x)|_\sigma + \left(\sup_x |\partial_{\beta_2}^{\gamma_1} f(x)|_\sigma\right) |\partial_{\beta_3}^{\gamma-\gamma_1} g(x)|_2$$
$$\leq \left(\sum \|\partial_{\beta_2}^{\bar{\gamma}} f\|\right) |\partial_{\beta_3}^{\gamma-\gamma_1} g(x)|_\sigma + \left(\sum \|\partial_{\beta_2}^{\bar{\gamma}} f\|_\sigma\right) |\partial_{\beta_3}^{\gamma-\gamma_1} g(x)|_2,$$

where summation is over $|\bar{\gamma}| \leq |\gamma_1| + 2 \leq N$ since $N \geq 4$. We conclude (2.69) by integrating (2.68) further over $\mathbb{T}^3$. □

We next prove the important estimates which are needed to prove Theorem 2.2 in Section 2.5.

THEOREM 2.5. *Let $|\gamma| \leq N$. let $g(x,p)$ be a smooth vector valued $L^2(\mathbb{T}^3_x \times \mathbb{R}^3_p; \mathbb{R}^2)$ function and $h(p)$ a smooth vector valued $L^2(\mathbb{R}^3_p; \mathbb{R}^2)$ function, we have*

$$\|\langle \partial^\gamma \Gamma(g,g), h\rangle\| \leq C \sum_{|\beta|\leq 2} |\partial_\beta h|_2 \sum_{|\gamma|\leq N} \|\partial^\gamma g\| \sum_{|\gamma|\leq N} \|\partial^\gamma g\|_\sigma \tag{2.71}$$

Moreover,

$$\|\langle L\partial^\gamma g, h\rangle\| \leq C\|\partial^\gamma g\| \sum_{|\beta|\leq 2} |\partial_\beta h|_2 \tag{2.72}$$

PROOF. We begin with the linear term. By Lemma 2.6, (2.21) and two integrations by parts $\langle L\partial^\gamma g, h\rangle$ is given by

$$\int \left\{ -\partial^\gamma g \cdot \partial_{p_j} \left(\partial_{p_i} h(p) 2\sigma^{ij}\right) + \frac{1}{2}\sigma^{ij} \frac{p_i}{p_0}\frac{p_j}{p_0} \partial^\gamma g \cdot h(p) \right\} dp$$
$$- \int \left\{ \partial_{p_i} \left\{ \sigma^{ij} \frac{p_i}{p_0} \right\} \partial^\gamma g \cdot h(p) \right\} dp$$
$$- \iint \frac{q_j}{2q_0} \Phi^{ij}(P,Q)\sqrt{J(q)} J(p) \partial^\gamma g_l(q) \partial_{p_i} \left\{ J(p)^{-1/2} h_k(p)\right\} dq dp,$$
$$+ \iint \partial_{q_j} \left\{ \Phi^{ij}(P,Q)\sqrt{J(q)}\right\} J(p)\partial^\gamma g_l(q) \partial_{p_i} \left\{ J(p)^{-1/2} h_k(p)\right\} dq dp$$

we implicitly sum over $i,j \in \{1,2,3\}$ and $k,l \in \{+,-\}$. Using cauchy's inequality, $\|\langle L\partial^\gamma g, h\rangle\|^2$ is bounded by

$$2\int\left(\int\left\{-\partial^\gamma g\cdot\partial_{p_j}\left(\partial_{p_i}h(p)2\sigma^{ij}\right)+\frac{1}{2}\sigma^{ij}\frac{p_i}{p_0}\frac{p_j}{p_0}\partial^\gamma g\cdot h(p)\right\}dp\right)^2 dx$$

$$2\int\left(\int\left\{\partial_{p_i}\left\{\sigma^{ij}\frac{p_i}{p_0}\right\}\partial^\gamma g\cdot h(p)\right\}dp\right)^2 dx \tag{2.73}$$

$$+2\int\left(\iint\frac{q_j}{2q_0}\Phi^{ij}(P,Q)\sqrt{J(q)}J(p)\partial^\gamma g_l(q)\partial_{p_i}\left\{J(p)^{-1/2}h_k(p)\right\}dqdp\right)^2 dx$$

plus

$$2\int\left(\iint\partial_{q_j}\left\{\Phi^{ij}(P,Q)\sqrt{J(q)}\right\}J(p)\partial^\gamma g_l(q)\partial_{p_i}\left\{J(p)^{-1/2}h_k(p)\right\}dqdp.\right)^2 dx \tag{2.74}$$

We have split (2.73) and (2.74) because we will estimate each one separately.

By the Cauchy-Schwartz inequality as well as (2.61), and that $h(p)$ is not a function of x, the first and second lines of (2.73) are bounded by

$$C\sum_{|\beta|\le 2}|\partial_\beta h|_2^2\int|\partial^\gamma g(x)|_2^2dx = C\|\partial^\gamma g\|^2\sum_{|\beta|\le 2}|\partial_\beta h|_2^2$$

This completes (2.72) for the first and second lines of (2.73).

By the Cauchy-Schwartz inequality over dp the third line of (2.73) is bounded by

$$C\sum_{|\beta|\le 1}|\partial_\beta h|_2^2\int\left(\int J^{1/2}(q)\left|\partial^\gamma g_l(q)\right|\left\{\int\Phi^{ij}(P,Q)^2J^{1/2}(p)dp\right\}^{1/2}dq\right)^2 dx$$

Recall again the splitting in (2.34), apply (2.35) and (2.37) (with $\alpha_1 = 0$) to obtain

$$\begin{aligned}\int\Phi^{ij}(P,Q)^2J^{1/2}(p)dp &= \int_{\mathcal{A}}+\int_{\mathcal{B}}\\ &\le Cq_0^{12}+q_0^{14}\int_{\mathcal{B}}|p-q|^{-2}J^{1/4}(p)dp\\ &\le Cq_0^{14}\end{aligned}$$

Using the Cauchy-Schwartz inequality again, this implies that the third line of (2.73) is bounded by $C\|\partial^\gamma g\|^2\sum_{|\beta|\le 1}|\partial_\beta h|_2^2$.

To establish (2.72) it remains to estimate (2.74). From (2.33), we write

$$\partial_{q_j} = -\frac{p_0}{q_0}\partial_{p_j}+\left(\partial_{q_j}+\frac{p_0}{q_0}\partial_{p_j}\right) = -\frac{p_0}{q_0}\partial_{p_j}+\mathbf{\Theta}_{e_j}(q,p)$$

where e_j is an element of the standard basis in $\mathbb{R}^3$. For the rest of this proof, we write $\mathbf{\Theta}_{e_j} = \mathbf{\Theta}_{e_j}(q,p)$ for notational simplicity (although the reader should note that it is the opposite of the shorthand we were using previously). Further,

$$\begin{aligned}\partial_{q_j}\left\{\Phi^{ij}(P,Q)J^{1/2}(q)\right\} &= -\Phi^{ij}(P,Q)\frac{q_j}{2q_0}\sqrt{J(q)}\\ &\quad -\sqrt{J(q)}\frac{p_0}{q_0}\partial_{p_j}\Phi^{ij}(P,Q)+\sqrt{J(q)}\mathbf{\Theta}_{e_j}\Phi^{ij}(P,Q).\end{aligned}$$

We plug the above into (2.74) and integrate by parts for the middle term in (2.74) to bound (2.74) by

$$\begin{aligned}&C\int\left(\iint\sqrt{J(q)}\sqrt{J(p)}\left|\Phi^{ij}(P,Q)\partial^\gamma g_l(q)\right|\left|\partial_\beta h_k(p)\right|dqdp\right)^2 dx\\ &+C\int\left(\iint\sqrt{J(q)}\sqrt{J(p)}\left|\mathbf{\Theta}_{e_j}\Phi^{ij}(P,Q)\partial^\gamma g_l(q)\right|\left|\partial_\beta h_k(p)\right|dqdp\right)^2 dx.\end{aligned}\tag{2.75}$$

Above, we add the implicit summation over $|\beta|\le 2$. By the same estimates as for (2.73), the first term in (2.75) is $\le C\|\partial^\gamma g\|^2\sum_{|\beta|\le 2}|\partial_\beta h|_2^2$.

To establish (2.72), it remains to estimate the second term in (2.75). To this end, we note that $\mathbf{\Theta}_{e_j}(q,p)P\cdot Q=0$. Further, $\mathbf{\Theta}_{e_j}(q,p)\Phi^{ij}(P,Q)$ satisfies the estimates

$$\begin{aligned}|\mathbf{\Theta}_{e_j}(q,p)\Phi(P,Q)| &\le Cq_0^6 q_0^{-1}\text{ on }\mathcal{A},\\ |\mathbf{\Theta}_{e_j}(q,p)\Phi(P,Q)| &\le Cq_0^7 q_0^{-1}|p-q|^{-1}\text{ on }\mathcal{B},\end{aligned}$$

where we recall the sets from (2.34). This can be shown directly, by repeating the proof of (2.35) and (2.37) using the operator $\mathbf{\Theta}_{e_j}(q,p)$ in place of the operator $\mathbf{\Theta}_{e_j}(p,q)$. Plugging these estimates into the second term of (2.75), we can show that it is bounded by $C\|\partial^\gamma g\|^2\sum_{|\beta|\le 2}|\partial_\beta h|_2^2$ using the same estimates as used for (2.73). This completes the estimate (2.72).

We turn to the estimate for the non-linear term (2.71). This estimate employs the same idea as for the linear term (2.72), i.e. to move the momentum derivatives around so that we can get an upper bound in terms of at least one $\|\cdot\|$ norm.

The inner product $\langle \partial^\gamma \Gamma(g,g), h\rangle$, using (2.64) and an integration by parts, is equal to

$$-C_\gamma^{\gamma_1} \iint \Phi^{ij} \sqrt{J(q)} \partial_{p_j} \partial^{\gamma_1} g_l(p) \partial^{\gamma-\gamma_1} g_k(q) \left(\partial_{p_i} + \frac{p_i}{2p_0} \right) h_l(p) dqdp,$$
$$+C_\gamma^{\gamma_1} \iint \Phi^{ij} \sqrt{J(q)} \partial^{\gamma_1} g_l(p) \partial_{q_j} \partial^{\gamma-\gamma_1} g_k(q) \left(\partial_{p_i} + \frac{p_i}{2p_0} \right) h_l(p) dqdp.$$

Above we implicitly sum is over $i, j \in \{1,2,3\}$, $\gamma_1 \le \gamma$ and $l, k \in \{+,-\}$.

Assume, without loss of generality, that $|\gamma_1| \le N/2$ (and $N \ge 4$). We then integrate by parts with respect to ∂_{p_j} for the first term above. After this integration by parts, we obtain a term like $\partial_{p_j} \Phi^{ij}$. For this term we write it as $\partial_{p_j} \Phi^{ij} = -\frac{q_0}{p_0} \partial_{q_j} \Phi^{ij} + \mathbf{\Theta}_{e_j}(p,q) \Phi^{ij}$, where we used the notation (2.33). We then integrate by parts again for this term with respect to ∂_{q_j}. The result is bounded above by

$$C \iint J^{1/4}(q) \left| \mathbf{\Theta}_{e_j}(p,q) \Phi^{ij} \right| \left| \partial^{\gamma_1} g_l(p) \partial^{\gamma-\gamma_1} g_k(q) \right| \left| \partial_\beta h_l(p) \right| dqdp,$$
$$+C \iint \sqrt{J(q)} \left| \Phi^{ij}(P,Q) \right| \left| \partial^{\gamma_1} g_l(p) \partial^q_{\beta_1} \partial^{\gamma-\gamma_1} g_k(q) \right| \left| \partial_\beta h_l(p) \right| dqdp,$$

where we sum over everything from the last display as well as over $|\beta_1| \le 1$ and $|\beta| \le 2$. The main point is that we took ∂_{p_j} off of the function on which we want to have an L^2 estimate (the function with less spatial derivatives). We use the same procedure used to estimate (2.69) to obtain the upper bound (for both terms above)

$$C \sum_{|\beta| \le 2} |\partial_\beta h|_2 \sum_{|\gamma_1| \le N/2} |\partial^{\gamma_1} g|_2 |\partial^{\gamma-\gamma_1} g|_\sigma$$

Therefore $\left| \langle \partial^\gamma \Gamma(g,g), h\rangle \right|^2 \le C \sum_{|\beta|\le 2} |\partial_\beta h|_2^2 \sum_{|\gamma_1| \le N/2} |\partial^{\gamma_1} g|_2^2 |\partial^{\gamma-\gamma_1} g|_\sigma^2$. Further integrating over $\mathbb{T}^3$ we get

$$\| \langle \partial^\gamma \Gamma(g,g), h\rangle \|^2 \le C \sum_{|\beta|\le 2} |\partial_\beta h|_2^2 \sum_{|\gamma_1| \le N/2} \int |\partial^{\gamma_1} g|_2^2 |\partial^{\gamma-\gamma_1} g|_\sigma^2 dx.$$

We establish (2.71) by using the Sobolev embedding: $H^2(\mathbb{T}^3) \subset L^\infty(\mathbb{T}^3)$. $\square$

We end this section with a proof that L is coercive away from it's null space $\mathcal{N}$.

Lemma 2.8. *For any $m > 1$, there is $0 < C(m) < \infty$ such that*

$$|\langle \partial_{p_i} \left\{ \sigma^{ij} \frac{p_j}{p_0} \right\} g, h\rangle| + |\langle Kg, h\rangle|$$

$$\leq \frac{C}{m} |g|_\sigma |h|_\sigma + C(m) \left\{ \int_{|p|\leq C(m)} |g|^2 dp \right\}^{1/2} \left\{ \int_{|p|\leq C(m)} |h|^2 dp \right\}^{1/2}. \tag{2.76}$$

Moreover, there is $\delta > 0$, such that

$$\langle Lg, g\rangle \geq \delta |\left(\mathbf{I} - \mathbf{P}\right) g|_\sigma^2. \tag{2.77}$$

Proof. We first prove (2.76). We split

$$\int \partial_{p_i} \left\{ \sigma^{ij} \frac{p_j}{p_0} \right\} \{g_+ h_+ + g_- h_-\} \, dp = \int_{\{|p|\leq m\}} + \int_{\{|p|\geq m\}}. \tag{2.78}$$

By (2.61)

$$\left| \partial_{p_i} \left\{ \sigma^{ij} \frac{p_j}{p_0} \right\} \right| \leq C p_0^{-1}.$$

So the first integral in (2.78) is bounded by the right hand side of (2.76). From the Cauchy-Schwartz inequality and (2.60) we obtain

$$\int_{\{|p|\geq m\}} \leq \frac{C}{m} \int |g||h| dp \leq \frac{C}{m} |g|_\sigma \ |h|_\sigma \ . \tag{2.79}$$

Consider the linear operator K in (2.63). After an integration by parts we can write

$$\langle Kg, h\rangle = \sum \iint \Phi^{ij} J^{1/4}(p) J^{1/4}(q) \Psi_{\alpha_1\alpha_2}(p,q) \partial_{\alpha_1} g_k(q) \partial_{\alpha_2} h_l(p) dq dp,$$

where $\Phi^{ij} = \Phi^{ij}(P,Q)$ and the sum is over $i, j \in \{1,2,3\}$, $|\alpha_1| \leq 1$, $|\alpha_2| \leq 1$ and $k, l \in \{+,-\}$. Also, $\Psi_{\alpha_1\alpha_2}(p,q)$ is a collection of smooth functions, in which we collect all the inessential terms, that satisfies

$$|\nabla \Psi_{\alpha_1\alpha_2}(p,q)| + |\Psi_{\alpha_1\alpha_2}(p,q)| \leq C J^{1/8}(p) J^{1/8}(q).$$

From (2.26) as well as Proposition 2.1,

$$\Phi^{ij}(P,Q) J^{1/4}(p) J^{1/4}(q) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3).$$

Therefore, for any given $m > 0$, we can choose a C_c^∞ function $\psi_{ij}(p,q)$ such that

$$||\Phi^{ij} J^{1/4}(p) J^{1/4}(q) - \psi_{ij}||_{L^2(\mathbb{R}^3_p \times \mathbb{R}^3_q)} \leq \tfrac{1}{m},$$

$$\text{supp}\{\psi_{ij}\} \subset \{|p| + |q| \leq C(m)\}, \ C(m) < \infty.$$

We split

$$\Phi^{ij} J^{1/4}(p) J^{1/4}(q) = \psi_{ij} + \{\Phi^{ij} J^{1/4}(p) J^{1/4}(q) - \psi_{ij}\}$$

and

$$\langle Kg, h \rangle = J_1(g,h) + J_2(g,h), \tag{2.80}$$

with

$$\begin{aligned} J_1 &= \sum \iint \psi_{ij}(p,q) \Psi_{\alpha_1\alpha_2}(p,q) \partial_{\alpha_1} g_k(q) \partial_{\alpha_2} h_l(p) dq dp, \\ J_2 &= \sum \iint \{\Phi^{ij} J^{1/4}(p) J^{1/4}(q) - \psi_{ij}\} \Psi_{\alpha_1\alpha_2}(p,q) \partial_{\alpha_1} g_k(q) \partial_{\alpha_2} h_l(p) dq dp. \end{aligned}$$

The second term J_2 is bounded in absolute value by

$$\begin{aligned} &||\Phi^{ij} J^{1/4}(p) J^{1/4}(q) - \psi_{ij}||_{L^2(\mathbb{R}^3_p \times \mathbb{R}^3_q)} ||\Psi_{\alpha_1\alpha_2} \partial_{\alpha_1} g_k(q) \partial_{\alpha_2} h_l(p)||_{L^2(\mathbb{R}^3_p \times \mathbb{R}^3_q)} \\ &\qquad \leq \frac{C}{m} \left| J^{1/8} \partial_{\alpha_1} g_k \right|_2 \left| J^{1/8} \partial_{\alpha_2} h_l \right|_2 \leq \frac{C}{m} |g|_\sigma |h|_\sigma , \end{aligned}$$

where we have used the equivalence of the norms (2.60). For J_1, an integration by parts over p and q yields

$$\begin{aligned} J_1 &= \sum (-1)^{\alpha_1+\alpha_2} \iint \partial_{\alpha_2} \partial^q_{\alpha_1} \{\psi_{ij}(p,q) \Psi_{\alpha_1\alpha_2}(p,q)\} g_k(q) h_l(p) dq dp \\ &\leq C ||\psi_{ij}||_{C^2} \left\{ \int_{|p| \leq C(m)} |g|^2 dp \right\}^{1/2} \left\{ \int_{|p| \leq C(m)} |h|^2 dp \right\}^{1/2}. \end{aligned} \tag{2.81}$$

This concludes (2.76).

We use the method of contradiction to prove (2.77). The converse grants us a sequence of normalized functions $g^n(p) = [g^n_+(p), g^n_-(p)]$ such that $|g^n|_\sigma \equiv 1$ and

$$\int_{\mathbb{R}^3} g^n J^{1/2} dp = \int_{\mathbb{R}^3} p_j g^n J^{1/2} dp = \int_{\mathbb{R}^3} g^n p_0 J^{1/2} dp = 0, \tag{2.82}$$

$$\langle L g^n, g^n \rangle = -\langle A g^n, g^n \rangle - \langle K g^n, g^n \rangle \leq 1/n. \tag{2.83}$$

We denote the weak limit, with respect to the inner product $\langle \cdot, \cdot \rangle_\sigma$, of g^n (up to a subsequence) by g^0. Lower semi-continuity of the weak limit implies $|g^0|_\sigma \leq 1$. From (2.62), (2.63) and (2.21) we have

$$\langle L g^n, g^n \rangle = |g^n|^2_\sigma - \langle \partial_{p_i} \left\{ \sigma^{ij} \frac{p_j}{p_0} \right\} g^n, g^n \rangle - \langle K g^n, g^n \rangle.$$

We *claim* that

$$\lim_{n\to\infty}\langle\partial_{p_i}\left\{\sigma^{ij}\frac{p_j}{p_0}\right\}g^n,g^n\rangle=\langle\partial_{p_i}\left\{\sigma^{ij}\frac{p_j}{p_0}\right\}g^0,g^0\rangle,\ \lim_{n\to\infty}\langle Kg^n,g^n\rangle\to\langle Kg^0,g^0\rangle.$$

For any given $m>0$, since $\partial_{p_i}g^n$ are bounded in $L^2\{|p|\le m\}$ from $|g^n|_\sigma=1$ and (2.60), the Rellich theorem implies

$$\int_{\{|p|\le m\}}\partial_{p_i}\left\{\sigma^{ij}\frac{p_j}{p_0}\right\}(g^n)^2\to\int_{\{|p|\le m\}}\partial_{p_i}\left\{\sigma^{ij}\frac{p_j}{p_0}\right\}(g^0)^2.$$

On the other hand, by (2.79) with $g=h=g^n$, the integral over $\{|p|\ge m\}$ is bounded by $O(1/m)$. By first choosing m sufficiently large and then sending $n\to\infty$, we conclude $\langle\partial_{p_i}\{\sigma^{ij}p_j/p_0\}g^n,g^n\rangle\to\langle\partial_{p_i}\{\sigma^{ij}p_j/p_0\}g^0,g^0\rangle$.

We split $\langle Kg^n,g^n\rangle$ into J_1 and J_2 as in (2.80), then $J_2(g^n,g^n)\le\frac{C}{m}$. In the same manner as for (2.81), we obtain

$$\left|J_1(g^n,g^n)-J_1(g^0,g^0)\right|\le C(m)\left\{\int_{|p|\le C(m)}|g^n-g^0|^2dp\right\}^{1/2}$$

Then the Rellich theorem implies, up to a subsequence, $J_1(g^n,g^n)\to J_1(g^0,g^0)$. Again by first choosing m large and then letting $n\to\infty$, we conclude that $\langle Kg^n,g^n\rangle\to\langle Kg_0,g_0\rangle$.

Letting $n\to\infty$ in (2.83), we have shown that

$$0=1-\langle\partial_{p_i}\{\sigma^{ij}p_j/p_0\}g^0,g^0\rangle-\langle Kg^0,g^0\rangle.$$

Equivalently

$$0=\left(1-|g^0|_\sigma^2\right)+\langle Lg^0,g^0\rangle.$$

Since both terms are non-negative, $|g^0|_\sigma^2=1$ and $\langle Lg^0,g^0\rangle=0$. By Lemma 2.1, $g^0=\mathbf{P}g^0$. On the other hand, letting $n\to\infty$ in (2.82) we deduce that $g^0=(\mathbf{I}-\mathbf{P})\,g^0$ or $g^0\equiv 0$; this contradicts $|g^0|_\sigma^2=1$. $\square$

2.4. Local Solutions

We now construct a unique local-in time solution to the relativistic Landau-Maxwell system with normalized constants (2.29) and (2.30), with constraint (2.31).

THEOREM 2.6. *There exist* $M_0 > 0$ *and* $T^* > 0$ *such that if* $T^* \leq M_0/2$ *and*

$$\mathcal{E}(0) \leq M_0/2,$$

then there exists a unique solution $[f(t,x,p), E(t,x), B(t,x)]$ *to the relativistic Landau-Maxwell system (2.29) and (2.30) with constraint (2.31) in* $[0, T^*) \times \mathbb{T}^3 \times \mathbb{R}^3$ *such that*

$$\sup_{0 \leq t \leq T^*} \mathcal{E}(t) \leq M_0.$$

The high order energy norm $\mathcal{E}(t)$ *is continuous over* $[0, T^*)$*. If*

$$F_0(x,p) = J + J^{1/2} f_0 \geq 0,$$

then $F(t,x,p) = J + J^{1/2} f(t,x,p) \geq 0$*. Furthermore, the conservation laws (2.10), (2.11), (2.12) hold for all* $0 < t < T^*$ *if they are valid initially at* $t = 0$*.*

We consider the following iterating sequence ($n \geq 0$) for solving the relativistic Landau-Maxwell system for the perturbation (2.29) with normalized constants (Remark 2.1):

$$\begin{aligned}
&\left\{\partial_t + \frac{p}{p_0}\cdot\nabla_x + \xi\left(E^n + \frac{p}{p_0}\times B^n\right)\cdot\nabla_p - A - \frac{\xi}{2}\left(E^n\cdot\frac{p}{p_0}\right)\right\} f^{n+1} \\
&= \xi_1\left\{E^n\cdot\frac{p}{p_0}\right\}\sqrt{J} + Kf^n + \Gamma(f^{n+1}, f^n) \\
&+\sqrt{J}\left(f^{n+1} - f^n\right)\partial_{p_i}\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)\partial_{q_i}\left(\sqrt{J(q)}f^n(q)\cdot[1,1]\right)dq \\
&+\sqrt{J}\left(f^{n+1} - f^n\right)\partial_{p_i}\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)\partial_{q_i}J(q)dq, \\
&\partial_t E^n - \nabla_x\times B^n = -\mathcal{J}^n = -\int_{\mathbb{R}^3}\frac{p}{p_0}\sqrt{J}\{f_+^n - f_-^n\}dp, \\
&\partial_t B^n + \nabla_x\times E^n = 0, \\
&\nabla_x\cdot E^n = \rho^n = \int_{\mathbb{R}^3}\sqrt{J}\{f_+^n - f_-^n\}dp,\ \nabla_x\cdot B^n = 0.
\end{aligned} \tag{2.84}$$

Above $\xi_1 = [1, -1]$, and the 2×2 matrix ξ is $\mathrm{diag}(1,-1)$. We start the iteration with

$$f^0(t,x,p) = [f_+^0(t,x,p), f_-^0(t,x,p)] \equiv [f_{0,+}(x,p), f_{0,-}(x,p)].$$

Then solve for $[E^0(t,x), B^0(t,x)]$ through the Maxwell system with initial datum $[E_0(x), B_0(x)]$. We then iteratively solve for

$$f^{n+1}(t,x,p) = [f_+^{n+1}(t,x,p), f_-^{n+1}(t,x,p)],\ \ E^{n+1}(t,x),\ \ B^{n+1}(t,x)$$

with initial datum $[f_{0,\pm}(x,p), E_0(x), B_0(x)]$.

It is standard from the linear theory to verify that the sequence $[f^n, E^n, B^n]$ is well-defined for all $n \geq 0$. Our goal is to get an uniform in n estimate for the energy $\mathcal{E}_n(t) \equiv \mathcal{E}(f^n, E^n, B^n)(t)$.

Lemma 2.9. *There exists $M_0 > 0$ and $T^* > 0$ such that if $T^* \leq \frac{M_0}{2}$ and*

$$\mathcal{E}(0) \leq M_0/2$$

then $\sup_{0\leq t\leq T^*} \mathcal{E}_n(t) \leq M_0$ *implies* $\sup_{0\leq t\leq T^*} \mathcal{E}_{n+1}(t) \leq M_0$.

Proof. Assume $|\gamma| + |\beta| \leq N$ and take ∂_β^γ derivatives of (2.84), we obtain:

$$\begin{aligned}
&\left\{\partial_t + \frac{p}{p_0}\cdot\nabla_x + \xi\left(E^n + \frac{p}{p_0}\times B^n\right)\cdot\nabla_p\right\}\partial_\beta^\gamma f^{n+1} \\
&-\partial_\beta\{A\partial^\gamma f^{n+1}\} - \xi_1\partial^\gamma E^n\cdot\partial_\beta\left\{\frac{p}{p_0}J^{1/2}\right\} \\
&= -\sum_{\beta_1\neq 0} C_\beta^{\beta_1}\partial_{\beta_1}\left(\frac{p}{p_0}\right)\cdot\nabla_x\partial_{\beta-\beta_1}^\gamma f^{n+1} \\
&+\sum C_\beta^{\beta_1}\frac{\xi}{2}\left\{\partial^{\gamma_1}E^n\cdot\partial_{\beta_1}\left(\frac{p}{p_0}\right)\right\}\partial_{\beta-\beta_1}^{\gamma-\gamma_1} f^{n+1} \\
&-\xi\sum_{\gamma_1\neq 0} C_\gamma^{\gamma_1}\partial^{\gamma_1}E^n\cdot\nabla_p\partial_\beta^{\gamma-\gamma_1} f^{n+1} \\
&+\xi\sum_{(\gamma_1,\beta_1)\neq(0,0)} C_\gamma^{\gamma_1}C_\beta^{\beta_1}\partial_{\beta_1}\left(\frac{p}{p_0}\right)\times\partial^{\gamma_1}B^n\cdot\nabla_p\partial_{\beta-\beta_1}^{\gamma-\gamma_1} f^{n+1} \\
&+\partial_\beta\{K\partial^\gamma f^n\} + \partial_\beta^\gamma\Gamma(f^{n+1}, f^n) \\
&+\sum C_\beta^{\beta_1}C_\gamma^{\gamma_1}\partial_{\beta-\beta_1}\left\{\sqrt{J}\partial^{\gamma-\gamma_1}\left(f^{n+1}-f^n\right)\right\} \\
&\times\partial_{\beta_1}\partial_{p_i}\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)\partial_{q_i}\left(\sqrt{J(q)}\partial^{\gamma_1}f^n(q)\cdot[1,1]\right)dq \\
&+\sum C_\beta^{\beta_1}\partial_{\beta-\beta_1}\left\{\sqrt{J}\partial^\gamma\left(f^{n+1}-f^n\right)\right\}\partial_{\beta_1}\partial_{p_i}\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)\partial_{q_i}J(q)dq.
\end{aligned} \tag{2.85}$$

We take the inner product of (2.85) with $\partial_\beta^\gamma f^{n+1}$ over $\mathbb{T}^3 \times \mathbb{R}^3$ and estimate this inner product term by term.

Using (2.65), the inner product of the first two terms on the l.h.s of (2.85) are bounded from below by

$$\frac{1}{2}\frac{d}{dt}||\partial_\beta^\gamma f^{n+1}(t)||^2 + ||\partial_\beta^\gamma f^{n+1}(t)||_\sigma^2 - \eta|||f^{n+1}(t)|||_\sigma^2 - C_\eta\|\partial^\gamma f^{n+1}(t)\|^2.$$

For the third term on l.h.s. of (2.85) we separate two cases. If $\beta \neq 0$, its inner product is bounded by

$$\left|\left(\partial^\gamma E^n \cdot \partial_\beta\{p\sqrt{J}/p_0\}\xi_1, \partial_\beta^\gamma f^{n+1}\right)\right| \leq C||\partial^\gamma E^n||\ |||f^{n+1}|||. \tag{2.86}$$

If $\beta = 0$, we have a pure temporal and spatial derivative $\partial^\gamma = \partial_t^{\gamma^0}\partial_{x_1}^{\gamma^1}\partial_{x_2}^{\gamma^2}\partial_{x_3}^{\gamma^3}$. We first split this term as

$$\begin{aligned} -\partial^\gamma E^n \cdot \left(\frac{p}{p_0}J^{1/2}\right)\xi_1 \ &\equiv\ -\partial^\gamma E^{n+1} \cdot \left(\frac{p}{p_0}J^{1/2}\right)\xi_1 \\ &\quad -\{\partial^\gamma E^n - \partial^\gamma E^{n+1}\} \cdot \left(\frac{p}{p_0}J^{1/2}\right)\xi_1. \end{aligned} \tag{2.87}$$

From the Maxwell system (2.30) and an integration by parts the inner product of the first part is

$$\begin{aligned} &-\left(\partial^\gamma E^{n+1} \cdot \{p\sqrt{J}/p_0\}\xi_1, \partial^\gamma f^{n+1}\right) \\ =\ &-\iint \partial^\gamma E^{n+1} \cdot \left(\frac{p}{p_0}\sqrt{J}\right)\{\partial^\gamma f_+^{n+1} - \partial^\gamma f_-^{n+1}\}dpdx \\ =\ &-\int \partial^\gamma E^{n+1} \cdot \partial^\gamma \mathcal{J}^{n+1}dx \\ =\ &\frac{1}{2}\frac{d}{dt}\left(||\partial^\gamma E^{n+1}(t)||^2 + ||\partial^\gamma B^{n+1}(t)||^2\right). \end{aligned} \tag{2.88}$$

And the inner product of second part in (2.87) is bounded by

$$C\{|||E^n||| + |||E^{n+1}|||\}|||f^{n+1}|||.$$

We now turn to the r.h.s. of (2.85). The first inner product is bounded by ($|\beta_1| \geq 1$) $C|||f^{n+1}|||^2$. The second, third and fourth inner products on r.h.s. of (2.85) can be bounded by a collection of terms of the same form

$$C\sum\int_{\mathbb{T}^3}\{|\partial^{\gamma_1}E^n|+|\partial^{\gamma_1}B^n|\}\left(\int_{\mathbb{R}^3}|\partial^{\gamma-\gamma_1}_{\beta-\beta_1}f^{n+1}\partial^{\gamma}_{\beta}f^{n+1}|dp\right)dx \tag{2.89}$$
$$+C\sum_{(\gamma_1,\beta_1)\neq(0,0)}\int_{\mathbb{T}^3}\{|\partial^{\gamma_1}E^n|+|\partial^{\gamma_1}B^n|\}\left(\int_{\mathbb{R}^3}|\nabla_p\partial^{\gamma-\gamma_1}_{\beta-\beta_1}f^{n+1}\partial^{\gamma}_{\beta}f^{n+1}|dp\right)dx$$

where the sums are over $\gamma_1\le\gamma$, and $\beta_1\le\beta$. From the Sobolev embedding $H^2(\mathbb{T}^3)\subset L^\infty(\mathbb{T}^3)$ we have

$$\sup_x\left\{\int_{\mathbb{R}^3}|g(x,q)|^2dq\right\}\le\int_{\mathbb{R}^3}\sup_x|g(x,q)|^2dq\le C\sum_{|\gamma|\le 2}||\partial^\gamma g||^2. \tag{2.90}$$

We take the L^∞ norm in x of the one of first two factors in (2.89) depending on whether $|\gamma_1|\le N/2$ (take the first term) or $|\gamma_1|>N/2$ (take the second term). Since $N\ge 4$, by (2.90) and (2.60) we can majorize (2.89) by

$$C\{|||E^n|||+|||B^n|||\}|||f^{n+1}|||^2\le C\{|||E^n|||+|||B^n|||\}|||f^{n+1}|||^2_\sigma. \tag{2.91}$$

We take (2.66), use Cauchy's inequality with η and integrate over $\mathbb{T}^3$ to obtain

$$\left(\partial_\beta[K\partial^\gamma f^n],\partial^\gamma_\beta f^{n+1}\right)\le\eta|||f^n|||^2_\sigma+\eta\left\|\partial^\gamma_\beta f^{n+1}\right\|^2_\sigma+C_\eta\left\|\partial^\gamma f^n\right\|^2.$$

For the nonlinear term we use Theorem 2.4 to obtain

$$\begin{aligned}(\partial^\gamma_\beta\Gamma(f^n_, f^{n+1}),\partial^\gamma_\beta f^{n+1}) \;\le\; & C|||f^n(t)||||||f^{n+1}(t)|||_\sigma||\partial^\gamma_\beta f^{n+1}(t)||_\sigma\\ & +C|||f^n(t)|||_\sigma|||f^{n+1}(t)|||||\partial^\gamma_\beta f^{n+1}(t)||_\sigma,\end{aligned}$$

We turn our attention to the inner product of the second to last term in (2.85). We integrate by parts over ∂_{p_i} and apply Theorem 2.3 to the dq integral differentiated by ∂_{β_1}. Then this term is bounded by

$$\int\left|\mathbf{\Theta}_{\bar\beta_1}\Phi^{ij}(P,Q)\right|J^{1/4}(q)\left|\partial_{\alpha_2}\partial^{\gamma_1}_{\bar\beta_2}f^n_k(q)\right|\left|\partial_{\alpha_2}\left(\partial^{\gamma-\gamma_1}_{\bar\beta_3}f^m_l(p)\partial^\gamma_\beta f^{n+1}_l(p)\right)\right|dpdqdx,$$

where we sum over $m\in\{n,n+1\}$, $\bar\beta_1+\bar\beta_1+\bar\beta_3\le\beta$, $i,j\in\{1,2,3\}$, $k,l\in\{+,-\}$, $|\alpha_1|\le 1$ $|\alpha_2|\le 1$ and $\gamma_1\le\gamma$. We remark that a few of these sum's are over estimates used to simplify the presentation. This term is always of the form of one of the four terms in (2.69)-(2.70) up to the location of one p derivative. Therefore, as in the proof of Theorem 2.4, this term is bounded above by

$$C\left(|||f^{n+1}|||_\sigma|||f^n|||_\sigma|||f^n||| + |||f^n|||_\sigma^2|||f^{n+1}|||\right)$$
$$+C\left(|||f^{n+1}|||_\sigma^2|||f^n||| + |||f^{n+1}|||_\sigma|||f^n|||_\sigma|||f^{n+1}|||\right).$$

where the sum is over $|\gamma_i| + |\bar{\beta}_i| \leq N$, $\bar{\beta}_1 + \bar{\beta}_2 \leq \beta$. For the inner product of the last term in (2.85). The null space in (2.2) implies

$$\int_{\mathbb{R}^3} \Phi^{ij}(P,Q)\partial_{q_i}J(q)dq = \int_{\mathbb{R}^3} \Phi^{ij}(P,Q)\frac{q_i}{q_0}J(q)dq = \sigma^{ij}\frac{p_i}{p_0}.$$

Therefore (2.61) applies to the derivatives of the dq integral. Therefore, the inner product of the last term in (2.85) is bounded by

$$\int \left|\partial^\gamma_{\bar{\beta}} f^m_k(p)\partial^\gamma_\beta f^{n+1}_k(p)\right| dpdx,$$

where we sum over $m \in \{n, n+1\}$, $|\bar{\beta}| \leq |\beta|$ and $k \in \{+,-\}$. This term is bounded above by

$$C\left(|||f^{n+1}|||^2 + |||f^n|||(t)|||f^{n+1}|||\right) \leq C|||f^{n+1}|||^2 + C|||f^n|||^2.$$

By collecting all the above estimates, we obtain the following bound for our iteration

$$\frac{1}{2}\frac{d}{dt}\left(||\partial^\gamma_\beta f^{n+1}(t)||^2 + ||\partial^\gamma E^{n+1}(t)||^2 + ||\partial^\gamma B^{n+1}(t)||^2\right) + ||\partial^\gamma_\beta f^{n+1}(t)||^2_\sigma$$
$$\leq \eta|||f^{n+1}(t)|||^2_\sigma + C_\eta\|\partial^\gamma f^{n+1}(t)\|^2 + C\{|||E^n||| + |||E^{n+1}|||\}|||f^{n+1}|||$$
$$+C|||f^{n+1}(t)|||^2 + C\{|||E^n||| + |||B^n|||\}|||f^{n+1}|||^2_\sigma$$
$$+\eta|||f^n|||^2_\sigma + \eta\left\|\partial^\gamma_\beta f^{n+1}\right\|^2_\sigma + C_\eta\left\|\partial^\gamma f^n\right\|^2$$
$$+C\left\{|||f^{n+1}|||_\sigma|||f^n||| + |||f^n|||_\sigma|||f^{n+1}|||\right\}|||f^{n+1}|||_\sigma$$
$$+C\left(|||f^{n+1}|||_\sigma|||f^n|||_\sigma|||f^n||| + |||f^n|||^2_\sigma|||f^{n+1}|||\right)$$
$$+C|||f^{n+1}|||^2 + C|||f^n|||^2.$$

Summing over $|\gamma|+|\beta|\le N$ and choosing $\eta\le\frac{1}{4}$ we have

$$
\begin{aligned}
(2.92)\qquad \mathcal{E}'_{n+1}(t)\le\ & C\{\mathcal{E}_{n+1}(t)+\mathcal{E}_n(t)+\mathcal{E}_n^{1/2}(t)|||f^{n+1}|||_\sigma^2(t)\\
&+\mathcal{E}_n^{1/2}(t)|||f^n|||_\sigma(t)|||f^{n+1}|||_\sigma(t)+C\mathcal{E}_{n+1}^{1/2}(t)|||f^n|||_\sigma^2(t)\\
&+\frac{1}{4C}|||f^n|||_\sigma^2+|||f^n|||_\sigma(t)\cdot|||f^{n+1}|||(t)\cdot|||f^{n+1}|||_\sigma(t)\}.
\end{aligned}
$$

By the induction assumption, we have

$$
\begin{aligned}
\frac{1}{2}|||f^n|||^2(t)+|||E^n|||^2(t)+|||B^n|||^2(t)+\int_0^t|||f^n|||_\sigma^2(s)ds\\
=\mathcal{E}_n(t)\le\sup_{0\le s\le t}\mathcal{E}_n(s)\le M_0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_0^t|||f^n|||_\sigma(s)\cdot|||f^{n+1}|||(s)\cdot|||f^{n+1}|||_\sigma(s)ds\\
\le\ &\sup_{0\le s\le t}|||f^{n+1}|||(s)\left\{\int_0^t|||f^n|||_\sigma^2(s)\right\}^{1/2}\left\{\int_0^t|||f^{n+1}|||_\sigma^2(s)\right\}^{1/2}\\
\le\ &\sqrt{M_0}\sup_{0\le s\le t}\mathcal{E}_{n+1}(s).
\end{aligned}
$$

Upon further integrating (2.92) over $[0,t]$ we deduce

$$
\begin{aligned}
\mathcal{E}_{n+1}(t)\ \le\ & \mathcal{E}_{n+1}(0)+C\left(t\sup_{0\le s\le t}\mathcal{E}_{n+1}(s)+M_0t+\sqrt{M_0}\mathcal{E}_{n+1}(t)\right)\\
&+\frac{M_0}{4}+CM_0\sup_{0\le s\le t}\mathcal{E}_{n+1}^{1/2}(s)+C\sqrt{M_0}\sup_{0\le s\le t}\mathcal{E}_{n+1}(s),
\end{aligned}
$$

and we will use the inequality

$$
M_0\sup_{0\le s\le t}\mathcal{E}_{n+1}(s)\le M_0^{3/2}+\sqrt{M_0}\sup_{0\le s\le t}\mathcal{E}_{n+1}(s).
$$

From the initial conditions ($n\ge 0$)

$$
\begin{aligned}
f_0^{n+1}\ &\equiv\ f^{n+1}(0,x,p)=f_0(x,p)\\
E_0^{n+1}\ &\equiv\ E^{n+1}(0,x)=E_0(x)\\
B_0^{n+1}\ &\equiv\ B^{n+1}(0,x)=B_0(x),
\end{aligned}
$$

we deduce that

$$\partial_\beta^\gamma f_0^{n+1} = \partial_\beta^\gamma f_0,\ \partial^\gamma E_0^{n+1} = \partial^\gamma E_0,\ \partial^\gamma B_0^{n+1} = \partial^\gamma B_0$$

by a simple induction over the number of temporal derivatives, where the temporal derivatives are defined naturally through (2.84). Hence

$$\mathcal{E}_{n+1}(0) = \mathcal{E}_{n+1}([f_0^{n+1}, E_0^{n+1}, B_0^{n+1}]) \equiv \mathcal{E}([f_0, E_0, B_0]) \leq M_0/2.$$

It follows that for $t \leq T^*$,

$$\begin{aligned}(1 - CT^* - CM_0^{1/2}) \sup_{0\leq t\leq T^*} \mathcal{E}_{n+1}(t) &\leq \mathcal{E}_{n+1}(0) + CM_0T^* + CM_0^{3/2} + \frac{M_0}{4} \\ &\leq \frac{3}{4}M_0 + CM_0\left(T^* + \sqrt{M_0}\right).\end{aligned}$$

We therefore conclude Lemma 2.9 if $T^* \leq \frac{M_0}{2}$ and M_0 is small. $\square$

In order to complete the proof of Theorem 2.6, we take $n \to \infty$, and obtain a solution f from Lemma 2.9. Now for uniqueness, we assume that there is another solution $[g, E_g, B_g]$, such that $\sup_{0\leq s\leq T^*} \mathcal{E}(g(s)) \leq M_0$ with $f(0, x, p) = g(0, x, p)$, $E_f(0, x) = E_g(0, x)$ and $B_f(0, x) = B_g(0, x)$. The difference $[f - g, E_f - E_g, B_f - B_g]$ satisfies

$$\begin{aligned}&\left\{\partial_t + \frac{p}{p_0}\cdot\nabla_x + \xi\left(E_f + \frac{p}{p_0}\times B_f\right)\cdot\nabla_p - A\right\}(f-g) - (E_f - E_g)\cdot\frac{p}{p_0}\sqrt{J}\xi_1 \\ &-\xi\left\{E_f - E_g + \frac{p}{p_0}\times(B_f - B_g)\right\}\nabla_p g + K(f-g) \\ &+\xi\left\{E_f\cdot\frac{p}{p_0}\right\}(f-g) + \xi\left\{(E_f - E_g)\cdot\frac{p}{p_0}\right\}g + \Gamma(f-g, f) + \Gamma(g, f-g); \\ &\partial_t(E_f - E_g) - \nabla_x\times(B_f - B_g) = -\int\frac{p}{p_0}\sqrt{J}\{(f-g)\cdot\xi_1\}, \\ &\nabla_x\cdot(E_f - E_g) = \int\sqrt{J}\{(f-g)\cdot\xi_1\}, \\ &\partial_t(B_f - B_g) + \nabla_x\times(E_f - E_g) = 0, \quad \nabla_x\cdot(B_f - B_g) = 0.\end{aligned} \tag{2.93}$$

By using the Cauchy-Schwarz inequality in the $p-$integration, and applying (2.90) for $\sup_x \int |\nabla_p g|^2 dp$, we deduce (for $N \geq 4$)

$$\begin{aligned} & \left| \left(u\{E_f - E_g + \frac{p}{p_0} \times (B_f - B_g)\} \cdot \nabla_p g, f - g \right) \right| \\ \leq\ & C \left\{ \sum_{|\gamma| \leq 2} ||\partial^\gamma g||_\sigma \right\} \{||E_f - E_g|| + ||B_f - B_g||\} ||f - g||_\sigma \\ \leq\ & C \left\{ \sum_{|\gamma| \leq 2} ||\partial^\gamma g||_\sigma^2 \right\} \{||E_f - E_g||^2 + ||B_f - B_g||^2\} + \frac{1}{4} ||f - g||_\sigma^2 \\ \leq\ & C |||\partial^\gamma g|||^2 \{||E_f - E_g||^2 + ||B_f - B_g||^2\} + \frac{1}{4} ||f - g||_\sigma^2 \\ \leq\ & C M_0 \{||E_f - E_g||^2 + ||B_f - B_g||^2\} + \frac{1}{4} ||f - g||_\sigma^2. \end{aligned}$$

Similarly, we use the Sobolev embedding theorem as well as elementary inequalities to estimate the terms below

$$\begin{aligned} \left| \left(E_f \cdot \frac{p}{p_0} (f - g), f - g \right) \right| &\leq C \sqrt{M_0} ||f - g||^2 \\ \left| \left(u\{E_f - E_g\} \cdot \frac{p}{p_0} g, f - g \right) \right| &\leq C ||f - g||_\sigma ||E_f - E_g|| \sum_{|\gamma| \leq 2} ||\partial^\gamma g||_\sigma \\ &\leq \frac{1}{4} ||f - g||_\sigma^2 + C M_0 ||E_f - E_g||^2. \end{aligned}$$

By Theorem 2.4 as well as (2.14),

$$\begin{aligned} & |(\Gamma(f - g, f) + \Gamma(g, f - g), f - g)| \\ & \leq C \{\|f - g\| \|f\|_\sigma + \|f - g\|_\sigma \|f\| + \|f - g\| \|g\|_\sigma + \|f - g\|_\sigma \|g\|\} \|f - g\|_\sigma \\ & = C \{\|f\| + \|g\|\} \|f - g\|_\sigma^2 + C \{\|f\|_\sigma + \|g\|_\sigma\} \|f - g\| \|f - g\|_\sigma \\ & \leq C \sqrt{M_0} \|f - g\|_\sigma^2 + C \left\{ \|f\|_\sigma^2 + \|g\|_\sigma^2 \right\} \|f - g\|^2 + \frac{1}{4} \|f - g\|_\sigma^2 \\ & \leq C \sqrt{M_0} \|f - g\|_\sigma^2 + C M_0 \|f - g\|^2 + \frac{1}{4} \|f - g\|_\sigma^2 \end{aligned}$$

From the Maxwell system in (2.93), we deduce from (2.88) that

$$- \left(2(E_f - E_g) \cdot (p\sqrt{J}/p_0)\xi_1, f - g \right) = \frac{d}{dt} \{||E_f - E_g||^2 + ||B_f - B_g||^2\}.$$

By taking the inner product of (2.93) with $f-g$, and collecting above estimates as well as plugging in the K and A estimates from Lemma 2.8, we have

$$
\begin{aligned}
& \frac{d}{dt}\left\{\frac{1}{2}||f-g||^2+||E_f-E_g||^2+||B_f-B_g||^2\right\}+||f-g||_\sigma^2 \\
\leq\ & C\{M_0+\sqrt{M_0}+1\}\{||f-g||^2+||E_f-E_g||^2+||B_f-B_g||^2\} \\
& +\left(\frac{C}{m}+C\sqrt{M_0}+\frac{3}{4}\right)||f-g||_\sigma^2.
\end{aligned}
$$

If we choose m and M_0 so that $\frac{C}{m}+C\sqrt{M_0}<\frac{1}{4}$ then the last term on the r.h.s. can be absorbed by $||f-g||_\sigma^2$ from the right. We deduce $f(t)\equiv g(t)$ from the Gronwall inequality.

To show the continuity of $\mathcal{E}(f(t))$ with respect to t, we have from (2.92) that as $t\to s$

$$
|\mathcal{E}(t)-\mathcal{E}(s)|\leq CM_0(t-s)+C\left(\sup_{s\leq\tau\leq t}\mathcal{E}^{1/2}(\tau)+1\right)\int_s^t|||f|||_\sigma^2(\tau)d\tau\to 0.
$$

For the positivity of $F=J+J^{1/2}f$, since f^n solves (2.84), we see that $F^n=J+J^{1/2}f^n$ solves the iterating sequence ($n\geq 0$):

$$
\left\{\partial_t+\frac{p}{p_0}\cdot\nabla_x+\xi\left(E^n+\frac{p}{p_0}\times B^n\right)\cdot\nabla_p\right\}F^{n+1}=\mathcal{C}^{mod}(F^{n+1},F^n)
$$

together with the coupled Maxwell system:

$$
\begin{aligned}
& \partial_t E^n-\nabla_x\times B^n=-\mathcal{J}^n=-\int_{\mathbb{R}^3}\frac{p}{p_0}\{F_+^n-F_-^n\}dp, \\
& \partial_t B^n+\nabla_x\times E^n=0, \quad \nabla_x\cdot B^n=0, \\
& \nabla_x\cdot E^n=\rho^n=\int_{\mathbb{R}^3}\{F_+^n-F_-^n\}dp.
\end{aligned}
$$

And, as in (2.84), the first step in the iteration is given through the initial data

$$
\begin{aligned}
F^0(t,x,p) &= [F_+^0(t,x,p),F_-^0(t,x,p)]=[F_{0,+}(x,p),F_{0,-}(x,p)] \\
&= [J+J^{1/2}f_{0,+}(x,p),J+J^{1/2}f_{0,-}(x,p)].
\end{aligned}
$$

Above we have used the modification $\mathcal{C}^{mod} = [\mathcal{C}^{mod}_+, \mathcal{C}^{mod}_-]$ where

$$\begin{aligned}
\mathcal{C}^{mod}_\pm(F^{n+1}, F^n) &= \partial_{p_i}\partial_{p_j}F^{n+1}_\pm(p)\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)\left(F^n_+ + F^n_-\right)dq \\
&\quad +\partial_{p_j}F^{n+1}_\pm(p)\int_{\mathbb{R}^3}\partial_{p_i}\Phi^{ij}(P,Q)\left(F^n_+ + F^n_-\right)dq \\
&\quad -\partial_{p_i}F^{n+1}_\pm(p)\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)\partial_{q_j}\left(F^n_+ + F^n_-\right)dq \\
&\quad -F^n_\pm(p)\partial_{p_i}\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)\partial_{q_j}\left(F^n_+ + F^n_-\right)dq.
\end{aligned}$$

Since $F^0(t,x,p) \geq 0$ Lemma 2.4, the elliptic structure of this iteration and the maximum principle imply that $F^{n+1}(t,x,p) \geq 0$ if $F^n(t,x,p) \geq 0$. This implies $F(t,x,p) \geq 0$.

Finally, since $\mathcal{E}(t) < +\infty$, $[f, \partial^2 E, \partial^2 B]$ is bounded and continuous. By $F = J + J^{1/2}f$, it is straightforward to verify that classical mass, total mometum and total energy conservations hold for such solutions constructed. We thus conclude Theorem 2.6.

2.5. Positivity of the Linearized Landau Operator

We establish the positivity of the linear operator L for any small amplitude solution $[f(t,x,p), E(t,x), B(t,x)]$ to the full relativistic Landau-Maxwell system (2.29) and (2.30). Recall the orthogonal projection $\mathbf{P}f$ with coefficients $a_\pm, b$ and c in (2.19). For solutions to the nonlinear system, Lemmas 2.11 and 2.12 are devoted to basic estimates for the linear and nonlinear parts in the macroscopic equations. We make the crucial observation in Lemma 2.13 that the electromagnetic field roughly speaking is bounded by $||f||_\sigma(t)$ at any moment t. Then based on Lemma 2.10, we finally establish Theorem 2.2 by a careful study of macroscopic equations coupled with the Maxwell system.

We begin with a formal definition of the orthogonal projection $\mathbf{P}$. Define

$$\rho_0 = \int_{\mathbb{R}^3} J(p)dp, \;\; \rho_i = \int_{\mathbb{R}^3} p_i^2 J(p)dp \;\; (i = 1,2,3),$$

$$\rho_4 = \int_{\mathbb{R}^3} |p|^2 J(p)dp, \;\; \rho_5 = \int_{\mathbb{R}^3} p_0 J(p)dp.$$

We can write an orthonormal basis for $\mathcal{N}$ in (2.15) with normalized constants as

$$\epsilon_1^* = \rho_0^{-1/2}[J^{1/2}, 0],\ \epsilon_2^* = \rho_0^{-1/2}[0, J^{1/2}],$$

$$\epsilon_{i+2}^* = (2\rho_i)^{-1/2}[p_i J^{1/2}, p_i J^{1/2}]\ (i = 1, 2, 3),\ \epsilon_6^* = c_6 \left([p_0, p_0] - \frac{\rho_5}{\rho_0}[1, 1] \right) J^{1/2}$$

where $c_6^{-2} = 2(\rho_0 + \rho_4) - 2\frac{\rho_5^2}{\rho_0}$. Now consider $\mathbf{P}f$, $f = [f_+, f_-]$, we define the coefficients in (2.19) so that $\mathbf{P}$ is an orthogonal projection:

$$a_+ \equiv \rho_0^{-1/2}\langle f, \epsilon_1^* \rangle - \tfrac{\rho_5}{\rho_0}c,\ a_- \equiv \rho_0^{-1/2}\langle f, \epsilon_2^* \rangle - \tfrac{\rho_5}{\rho_0}c,$$

$$b_j \equiv (2\rho_j)^{-1/2}\langle f, \epsilon_{j+2}^* \rangle,\ c \equiv c_6 \langle f, \epsilon_6^* \rangle. \tag{2.94}$$

Proposition 2.2. *Let $\partial^\gamma = \partial_t^{\gamma_0}\partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}\partial_{x_3}^{\gamma_3}$. There exists $C > 1$ such that*

$$\frac{1}{C}||\partial^\gamma \mathbf{P}f||_\sigma^2 \le ||\partial^\gamma a_\pm||^2 + ||\partial^\gamma b||^2 + ||\partial^\gamma c||^2 \le C||\partial^\gamma \mathbf{P}f||^2.$$

For the rest of the section, we concentrate on a solution $[f, E, B]$ to the nonlinear relativistic Landau-Maxwell system.

Lemma 2.10. *Let $[f(t,x,p), E(t,x), B(t,x)]$ be the solution constructed in Theorem 2.6 to (2.29) and (2.30), which satisfies (2.31), (2.10), (2.11) and (2.12). Then we have*

$$\frac{2}{3}\rho_4 \int_{\mathbb{T}^3} b(t,x) \ = \ \int_{\mathbb{T}^3} B(t,x) \times E(t,x), \tag{2.95}$$

$$\left| \int_{\mathbb{T}^3} a_+(t,x) \right| + \left| \int_{\mathbb{T}^3} a_-(t,x) \right| + \left| \int_{\mathbb{T}^3} c(t,x) \right| \ \le \ C\left(||E||^2 + ||B - \bar{B}||^2 \right), \tag{2.96}$$

where $a = [a_+, a_-]$, $b = [b_1, b_2, b_3]$, c are defined in (2.94).

Proof. We use the conservation of mass, momentum and energy. For fixed (t,x), notice that (2.94) implies

$$\int p\{f_+ + f_-\}\sqrt{J}dp = \frac{2}{3}b(t,x)\int |p|^2 J dp.$$

Hence (2.95) follows from momentum conservation (2.11) with normalized constants.

On the other hand, for fixed (t,x), (2.94) implies

$$\int f_\pm \sqrt{J} dp = \rho_0 a_\pm(t,x) + \rho_5 c(t,x),$$
$$\int p_0\{f_+ + f_-\}\sqrt{J} dp = \rho_5\{a_+(t,x) + a_-(t,x)\} + 2(\rho_0+\rho_4)c(t,x),$$

Upon further integration over $\mathbb{T}^3$, we deduce from the mass conservation (2.10) that $\int_{\mathbb{T}^3} a_+ = \int_{\mathbb{T}^3} a_- = -\frac{\rho_5}{\rho_0}\int_{\mathbb{T}^3} c$. From the reduced energy conservation (2.12),

$$-\int_{\mathbb{T}^3}\{|E(t)|^2 + |B(t)-\bar{B}|^2\} = 2\left(\rho_0+\rho_4-\frac{\rho_5^2}{\rho_0}\right)\int_{\mathbb{T}^3} c.$$

By the sharp form of Holder's inequality, $(\rho_0+\rho_4)\rho_0 - \rho_5^2 > 0$.

□

We now derive the macroscopic equations for $\mathbf{P}f$'s coefficients $a_\pm, b$ and c. Recalling equation (2.16) with (2.17) and (2.18) with normalized constants in (2.104) and (2.105), we further use (2.19) to expand entries of l.h.s. of (2.16) as

$$\left\{\partial^0 a_\pm + \frac{p_j}{p_0}\left\{\partial^j a_\pm \mp E_j\right\} + \frac{p_j p_i}{p_0}\partial^i b_j + p_j\left\{\partial^0 b_j + \partial^j c\right\} + p_0\partial^0 c\right\} J^{1/2}(p)$$

where $\partial^0 = \partial_t$ and $\partial^j = \partial_{x_j}$. For fixed (t,x), this is an expansion of l.h.s. of (2.16) with respect to the basis of $(1 \le i, j \le 3)$

$$[\sqrt{J},0], [0,\sqrt{J}],\ [p_j\sqrt{J}/p_0, 0],\ [0, p_j\sqrt{J}/p_0],$$
$$[p_j\sqrt{J}, p_j\sqrt{J}],\ [p_jp_i\sqrt{J}/p_0, p_jp_i\sqrt{J}/p_0],\ [p_0\sqrt{J}, p_0\sqrt{J}] \tag{2.97}$$

Expanding the r.h.s. of (2.16) with respect to the same basis (2.97) and comparing coefficients on both sides, we obtain the important macroscopic equations for $a(t,x) = [a_+(t,x), a_-(t,x)]$, $b_i(t,x)$ and $c(t,x)$:

$$\partial^0 c = l_c + h_c, \tag{2.98}$$

$$\partial^i c + \partial^0 b_i = l_i + h_i, \tag{2.99}$$

$$(1-\delta_{ij})\partial^i b_j + \partial^j b_i = l_{ij} + h_{ij}, \tag{2.100}$$

$$\partial^i a_\pm \mp E_i = l_{ai}^\pm + h_{ai}^\pm, \tag{2.101}$$

$$\partial^0 a_\pm = l_a^\pm + h_a^\pm. \tag{2.102}$$

Here $l_c(t,x), l_i(t,x),\ l_{ij}(t,x),\ l^{\pm}_{ai}(t,x)$ and $l^{\pm}_{a}(t,x)$ are the corresponding coefficients of such an expansion of the linear term $l(\{\mathbf{I}-\mathbf{P}\}f)$, and $h_c(t,x), h_i(t,x),\ h_{ij}(t,x)$, $h^{\pm}_{ai}(t,x)$ and $h^{\pm}_{a}(t,x)$ are the corresponding coefficients of the same expansion of the higher order term $h(f)$.

From (2.19) and (2.94) we see that

$$\begin{aligned}\int [p\sqrt{J}/p_0, -p\sqrt{J}/p_0]\cdot \mathbf{P}f dp &= 0,\\ \int [\sqrt{J}, -\sqrt{J}]\cdot f dp &= \rho_0\{a_+ - a_-\}.\end{aligned}$$

We plug this into the coupled maxwell system, (2.30) and (2.31), to obtain

$$\begin{aligned}&\partial_t E - \nabla_x \times B = -\mathcal{J} = \int_{\mathbb{R}^3} [p\sqrt{J}/p_0, -p\sqrt{J}/p_0]\cdot \{\mathbf{I}-\mathbf{P}\}f dp,\\ &\partial_t B + \nabla_x\times E = 0,\ \nabla_x\cdot E = \rho_0\{a_+ - a_-\},\ \nabla_x\cdot B = 0.\end{aligned} \tag{2.103}$$

We rewrite the terms (2.17) and (2.18) in (2.16) with normalized constants as

$$l(\{\mathbf{I}-\mathbf{P}\}f) \equiv -\left\{\partial_t + \frac{p}{p_0}\cdot\nabla_x + L\right\}\{\mathbf{I}-\mathbf{P}\}f, \tag{2.104}$$

$$h(f) \equiv -\xi\left(E + \frac{p}{p_0}\times B\right)\cdot\nabla_p f + \frac{\xi}{2}\left\{E\cdot\frac{p}{p_0}\right\} f + \Gamma(f,f). \tag{2.105}$$

Next, we estimate these terms.

Lemma 2.11. *For any* $1\le i,j\le 3$,

$$\sum_{|\gamma|\le N-1} ||\partial^\gamma l_c|| + ||\partial^\gamma l_i|| + ||\partial^\gamma l_{ij}|| + ||\partial^\gamma l^{\pm}_{ai}|| + ||\partial^\gamma l^{\pm}_{a}|| + ||\partial^\gamma \mathcal{J}|| \le C\sum_{|\gamma|\le N}\|\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f\|.$$

Proof. Let $\{\epsilon_n(p)\}$ represent the basis in (2.97). For fixed (t,x), we can use the Gram-Schmidt procedure to argue that the terms $l_c(t,x)$, $l_i(t,x)$, $l_{ij}(t,x)$, $l^{\pm}_{ai}(t,x)$ and $l^{\pm}_{a}(t,x)$ are of the form

$$\sum_{n=1}^{18} \bar{c}_n \langle l(\{\mathbf{I}-\mathbf{P}\}f), \epsilon_n\rangle,$$

where c_n are constants which do not depend on on f. Let $|\gamma|\le N-1$. By (2.104)

$$\int \partial^\gamma l(\{\mathbf{I}-\mathbf{P}\}f)\cdot\epsilon_n(p)dp = -\int\left\{\partial_t + \frac{p}{p_0}\cdot\nabla_x + L\right\}\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f(p)\cdot\epsilon_n(p)dp.$$

We estimate the first two terms,

$$
\begin{aligned}
&\|\int\{\partial_t+\frac{p}{p_0}\cdot\nabla_x\}(\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f)\cdot\epsilon_n dp\|^2\\
&\le 2\int|\epsilon_n|dp\times\int_{\mathbb{T}^3\times\mathbb{R}^3}|\epsilon_n(p)|(|\{\mathbf{I}-\mathbf{P}\}\partial^0\partial^\gamma f|^2+|\{\mathbf{I}-\mathbf{P}\}\nabla_x\partial^\gamma f|^2)dpdx\\
&\le C\left(||\{\mathbf{I}-\mathbf{P}\}\partial^0\partial^\gamma f||^2+||\{\mathbf{I}-\mathbf{P}\}\nabla_x\partial^\gamma f||^2\right).
\end{aligned}
$$

Similarly, we have

$$
||\partial^\gamma\mathcal{J}|| \;=\; ||\int_{\mathbb{R}^3}[-p\sqrt{J}/p_0,p\sqrt{J}/p_0]\cdot\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f dp||\le C||\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f||.
$$

Using (2.72) we can estimate the last term

$$
\|\langle L\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f,\epsilon_n\rangle\|\le C\|\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f\|.
$$

Indeed (2.72) was designed to estimate this term. □

We now estimate coefficients of the higher order term $h(f)$.

LEMMA 2.12. *Let (2.14) be valid for some $M_0>0$. Then*

$$
\sum_{|\gamma|\le N}\{||\partial^\gamma h_c||+||\partial^\gamma h_i||+||\partial^\gamma h_{ij}||+||\partial^\gamma h_{ai}^\pm||+||\partial^\gamma h_a^\pm||\}\le C\sqrt{M_0}\sum_{|\gamma|\le N}||\partial^\gamma f||_\sigma.
$$

PROOF. Let $|\gamma|\le N$, recall that $\{\epsilon_n(p)\}$ represents the basis in (2.97). Notice that $\partial^\gamma h_c,\partial^\gamma h_i,\partial^\gamma h_{ij}$, $\partial^\gamma h_{ai}^\pm$ and $\partial^\gamma h_a^\pm$ are again of the form

$$
\sum_{n=1}^{18}\tilde{c}_n\langle\partial^\gamma h(f),\epsilon_n\rangle.
$$

It again suffices to estimate $\langle\partial^\gamma h(f),\epsilon_n\rangle$. For the first term of $h(f)$ in (2.105), we use an integration by parts over the p variables to get

$$
\begin{aligned}
&-\int\partial^\gamma\{\xi(E+\frac{p}{p_0}\times B)\cdot\nabla_p f)\}\cdot\epsilon_n(p)dp\\
=&\;-\sum C_\gamma^{\gamma_1}\int\nabla_p\cdot\{\xi(\partial^{\gamma_1}E+\frac{p}{p_0}\times\partial^{\gamma_1}B)\partial^{\gamma-\gamma_1}f\}\cdot\epsilon_n(p)dp\\
=&\;\sum C_\gamma^{\gamma_1}\int\xi(\partial^{\gamma_1}E+\frac{p}{p_0}\times\partial^{\gamma_1}B)\partial^{\gamma-\gamma_1}f\cdot\nabla_p\epsilon_n(p)dp\\
\le&\;C\sum\{|\partial^{\gamma_1}E|+|\partial^{\gamma_1}B|\}\left\{\int|\partial^{\gamma-\gamma_1}f|^2dp\right\}^{1/2}.
\end{aligned}
$$

The last estimate holds because $\nabla_p \epsilon_n(p)$ has exponential decay. Take the square of the above, whose further integration over $\mathbb{T}^3$ is bounded by

$$C\int_{\mathbb{T}^3}\{|\partial^{\gamma_1}E|+|\partial^{\gamma_1}B|\}^2\left\{\int|\partial^{\gamma-\gamma_1}f|^2dp\right\}dx. \tag{2.106}$$

If $|\gamma_1|\le N/2$, by $H^2(\mathbb{T}^3)\subset L^\infty(\mathbb{T}^3)$ and the small amplitude assumption (2.14), we have

$$\sup_x\{|\partial^{\gamma_1}E|+|\partial^{\gamma_1}B|\}\le C\sum_{|\gamma|\le N}\{||\partial^\gamma E(t)||+||\partial^\gamma B(t)||\}\le C\sqrt{M_0}.$$

If $|\gamma_1|\ge N/2$ then $\int_{\mathbb{T}^3}\{|\partial^{\gamma_1}E|+|\partial^{\gamma_1}B|\}^2dx\le M_0$ and, by (2.90),

$$\sup_x\left\{\int|\partial^{\gamma-\gamma_1}f|^2dp\right\}\le C\sum_{|\gamma|\le N}||\partial^\gamma f(t)||^2.$$

We thus conclude that (2.106) is bounded by $C\sqrt{M_0}\sum_{|\gamma|\le N}||\partial^\gamma f||$.

The second term of $h(f)$ in (2.105) is easily treated by the same argument, for

$$\begin{aligned}&\int\frac{\xi}{2}\partial^\gamma\{(E\cdot\frac{p}{p_0})f\}\cdot\epsilon_n(p)dp\\ =\;&\sum C_\gamma^{\gamma_1}\int\{\frac{\xi}{2}(\partial^{\gamma_1}E\cdot\frac{p}{p_0})\partial^{\gamma-\gamma_1}f\}\cdot\epsilon_n(p)dp\\ \le\;&C\sum|\partial^{\gamma_1}E|\left\{\int|\partial^{\gamma-\gamma_1}f|^2dp\right\}^{1/2}.\end{aligned}$$

For the third term of $h(f)$ in (2.105) we apply (2.71):

$$\|\langle\partial^\gamma\Gamma(f,f),\epsilon_n\rangle\|\le C\sum_{|\gamma|\le N}||\partial^\gamma f(t)||\sum_{|\gamma|\le N}||\partial^\gamma f(t)||_\sigma\le C\sqrt{M_0}\sum_{|\gamma|\le N}||\partial^\gamma f||_\sigma.$$

We designed (2.71) to estimate this term. □

Next we estimate the electromagnetic field $[E(t,x),B(t,x)]$ in terms of $f(t,x,p)$ through the macroscopic equation (2.101) and the Maxwell system (2.103).

LEMMA 2.13. *Let $[f(t,x,p),E(t,x),B(t,x)]$ be the solution to (2.29), (2.30) and (2.31) constructed in Theorem 2.6. Let the small amplitude assumption (2.14) be valid for some $M_0>0$. Then there is a constant $C>0$ such that*

$$\sum_{|\gamma|\le N-1}\{||\partial^\gamma E(t)||+||\partial^\gamma\{B(t)-\bar{B}\}||\}\le C\sum_{|\gamma|\le N}\left(||\partial^\gamma f(t)||+\sqrt{M_0}||\partial^\gamma f(t)||_\sigma\right).$$

PROOF. We first use the plus part of the macroscopic equation (2.101) to estimate the electric field $E(t,x)$:

$$-\partial^\gamma E_i = \partial^\gamma l_{ai}^+ + \partial^\gamma h_{ai}^+ - \partial^\gamma \partial^i a_+.$$

Proposition 2.2 says $||\partial^\gamma\partial^i a_+|| \leq C||\mathbf{P}\partial^\gamma\partial^i f||$. Applying Lemmas' 2.11 and 2.12 to $\partial^\gamma l_{ai}^+$ and $\partial^\gamma h_{ai}^+$ respectively, we deduce that for $|\gamma| \leq N-1$,

$$||\partial^\gamma E|| \leq C \sum_{|\gamma'|\leq N} \left(||\partial^{\gamma'} f(t)|| + \sqrt{M_0}||\partial^{\gamma'} f(t)||_\sigma \right). \tag{2.107}$$

We next estimate the magnetic field $B(t,x)$. Let $|\gamma| \leq N-2$. Taking ∂^γ to the Maxwell system (2.103) we obtain

$$\nabla_x \times \partial^\gamma B = \partial^\gamma \mathcal{J} + \partial_t \partial^\gamma E,\ \nabla_x \cdot \partial^\gamma B = 0.$$

Lemma 2.11, (2.107) as well as $\int |\nabla \times \partial^\gamma B|^2 + (\nabla\cdot\partial^\gamma B)^2 dx = \int \sum_{i,j} (\partial_{x_i}\partial^\gamma B_j)^2 dx$ imply

$$||\nabla\partial^\gamma B|| \leq C\{||\partial^\gamma \mathcal{J}|| + ||\partial_t\partial^\gamma E||\} \leq C \sum_{|\gamma'|\leq N} \left(||\partial^{\gamma'} f(t)|| + \sqrt{M_0}||\partial^{\gamma'} f(t)||_\sigma \right)$$

By $\partial_t\partial^\gamma B + \nabla\times\partial^\gamma E = 0$, $||\partial_t\partial^\gamma B|| \leq ||\nabla\times\partial^\gamma E||$. Finally, by the Poincaré inequality $||B-\bar{B}|| \leq C||\nabla B||$, we therefore conclude our Lemma. □

We now prove the crucial positivity of L for a small solution $[f(t,x,p), E(t,x), B(t,x)]$ to the relativistic Landau-Maxwell system. The conservation laws (2.10), (2.11) and (2.12) play an important role.

Proof of Theorem 2.2. From (2.77) we have

$$(L\partial^\gamma f, \partial^\gamma f) \geq \delta||\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f||_\sigma^2.$$

By Proposition 2.2, we need only establish (2.20). The rest of the proof is devoted to establishing

$$\begin{aligned} &\sum_{|\gamma|\leq N} \{||\partial^\gamma a_\pm|| + ||\partial^\gamma b|| + ||\partial^\gamma c||\} \\ &\leq C \sum_{|\gamma|\leq N} ||\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f(t)|| + C\sqrt{M_0} \sum_{|\gamma|\leq N} ||\partial^\gamma f(t)||_\sigma, \end{aligned} \tag{2.108}$$

This is sufficient to prove the upper bound in (2.20) because the second term on the r.h.s. can be neglected for M_0 small:

$$\begin{aligned}\sum_{|\gamma|\leq N}||\partial^\gamma f(t)||_\sigma &\leq \sum_{|\gamma|\leq N}||\mathbf{P}\partial^\gamma f(t)||_\sigma+\sum_{|\gamma|\leq N}||\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f(t)||_\sigma\\ &\leq C\sum_{|\gamma|\leq N}\left(||\partial^\gamma a_\pm||+||\partial^\gamma b||+||\partial^\gamma c||\right)+\sum_{|\gamma|\leq N}||\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f(t)||_\sigma.\end{aligned}$$

We will estimate each of the terms $a_\pm, b$ and c in (2.108) one at a time.

We first estimate $\nabla\partial^\gamma b$. Let $|\gamma|\leq N-1$. From (2.100)

$$\begin{aligned}\Delta\partial^\gamma b_j+\partial^j(\nabla\cdot\partial^\gamma b) &= \sum_i \partial^i\left(\partial^\gamma\partial^i b_j+\partial^\gamma\partial^j b_i\right)\\ &= \sum_i \partial^i\partial^\gamma\left(l_{ij}+h_{ij}\right)\left(1+\delta_{ij}\right)\end{aligned}$$

Multiplying with $\partial^\gamma b_j$ and summing over j yields:

$$\begin{aligned}&\int_{\mathbb{T}^3}\left\{(\nabla\cdot\partial^\gamma b)^2+\sum_{i,j}\left(\partial^i\partial^\gamma b_j\right)^2\right\}dx\\ &=\sum_{i,j}\int_{\mathbb{T}^3}\left(\partial^\gamma l_{ij}+\partial^\gamma h_{ij}\right)(1+\delta_{ij})\partial^i\partial^\gamma b_j dx.\end{aligned}$$

Therefore

$$\sum_{i,j}||\partial^i\partial^\gamma b_j||^2\leq C\left(\sum_{i,j}||\partial^i\partial^\gamma b_j||\right)\sum\{||\partial^\gamma l_{ij}||+||\partial^\gamma h_{ij}||\},$$

which implies, using $\left(\sum_{i,j}||\partial^i\partial^\gamma b_j||\right)^2\leq C\sum_{i,j}||\partial^i\partial^\gamma b_j||^2$, that

$$\sum_{i,j}||\partial^i\partial^\gamma b_j||\leq C\sum\{||\partial^\gamma l_{ij}||+||\partial^\gamma h_{ij}||\}. \tag{2.109}$$

This is bounded by the r.h.s. of (2.108) by Lemmas 2.11 and 2.12. We estimate purely temporal derivatives of $b_i(t,x)$ with $\gamma=[\gamma^0,0,0,0]$ and $0<\gamma^0\leq N-1$. From (2.98) and (2.99), we have

$$\begin{aligned}\partial^0\partial^\gamma b_i &= \partial^\gamma l_i+\partial^\gamma h_i-\partial^i\partial^\gamma c\\ &= \partial^\gamma l_i+\partial^\gamma h_i-\partial^{\gamma'}\partial^0 c\\ &= \partial^\gamma l_i+\partial^\gamma h_i-\partial^{\gamma'}l_c-\partial^{\gamma'}h_c,\end{aligned}$$

where $|\gamma'| = \gamma^0$. Therefore,

$$\|\partial^0\partial^\gamma b_i\| \leq C\left(\|\partial^\gamma l_i\| + \|\partial^\gamma h_i\| + \|\partial^{\gamma'} l_c\| + \|\partial^{\gamma'} h_c\|\right).$$

By Lemmas 2.11 and 2.12, this is bounded by the r.h.s. of (2.108). Next, assume $0 \leq \gamma^0 \leq 1$. We use the Poincaré inequality and (2.95) to obtain

$$\begin{aligned}
||\partial_t^{\gamma^0} b_i|| &\leq C\left\{||\nabla\partial_t^{\gamma^0} b_i|| + \left|\partial_t^{\gamma^0}\int b_i(t,x)dx\right|\right\} \\
&= C\left\{||\nabla\partial_t^{\gamma^0} b_i|| + \left|\partial_t^{\gamma^0}\int E\times B dx\right|\right\}.
\end{aligned}$$

By (2.109), it suffices to estimate the last term above. From Lemma 2.13 and the assumption (2.14), with $M_0 \leq 1$, the last term is bounded by

$$\begin{aligned}
&||\partial_t^{\gamma^0} B||\cdot||E|| + ||B||\cdot||\partial_t^{\gamma^0} E|| \\
&\leq \sqrt{M_0}C\sum_{|\gamma|\leq N}\left(||\partial^\gamma f(t)|| + \sqrt{M_0}||\partial^\gamma f(t)||_\sigma\right) \\
&\leq C\sqrt{M_0}\sum_{|\gamma|\leq N}||\partial^\gamma f(t)||_\sigma.
\end{aligned}$$

We thus conclude the case for b.

Now for $c(t,x)$, from (2.98) and (2.99),

$$\begin{aligned}
||\partial^0\partial^\gamma c|| &\leq C\{||\partial^\gamma l_c|| + ||\partial^\gamma h_c||\}, \\
||\nabla\partial^\gamma c|| &\leq C||\partial^0\partial^\gamma b_i|| + ||\partial^\gamma l_i|| + ||\partial^\gamma h_i||.
\end{aligned}$$

Thus, for $|\gamma| \leq N-1$, both $||\partial^0\partial^\gamma c||$ and $||\nabla\partial^\gamma c||$ are bounded by the r.h.s. of (2.108) by the above argument for b and Lemmas 2.11 and 2.12. Next, to estimate $c(t,x)$ itself, from the Poincaré inequality and Lemma 2.10

$$\begin{aligned}
||c|| &\leq C\left\{||\nabla c|| + \left|\int c dx\right|\right\} \\
&\leq C\{||\nabla c|| + ||E||^2 + ||B-\bar{B}||^2\}.
\end{aligned}$$

Notice that from (2.3) and Jensen's inequality $|\bar{B}| \leq \|B\|$. Using this, Lemma 2.13 and (2.14), with $M_0 \leq 1$, imply

$$||E||^2 + ||B - \bar{B}||^2 \leq ||E||^2 + C||B - \bar{B}||(||B|| + ||\bar{B}||) \leq C\sqrt{M_0} \sum_{|\gamma| \leq N} ||\partial^\gamma f(t)||_\sigma.$$

We thus complete the estimate for $c(t,x)$ in (2.108).

Now we consider $a(t,x) = [a_+(t,x), a_-(t,x)]$. By (2.102),

$$||\partial_t \partial^\gamma a_\pm|| \leq C\left\{||\partial^\gamma l_a^\pm|| + ||\partial^\gamma h_a^\pm||\right\}.$$

We now use Lemma 2.11 and 2.12, for $|\gamma| \leq N-1$, to say that $||\partial_t \partial^\gamma a||$ is bounded by the r.h.s. of (2.108). We now turn to purely spatial derivatives of $a(t,x)$. Let $|\gamma| \leq N-1$ and $\gamma = [0, \gamma_1, \gamma_2, \gamma_3] \neq 0$. By taking ∂^i of (2.101) and summing over i we get

$$-\Delta \partial^\gamma a_\pm \pm \nabla \cdot \partial^\gamma E = -\sum_i \partial^i \partial^\gamma \{l_{ai}^\pm + h_{ai}^\pm\}. \tag{2.110}$$

But from the Maxwell system in (2.103),

$$\nabla \cdot \partial^\gamma E = \rho_0(\partial^\gamma a_+ - \partial^\gamma a_-).$$

Multiply (2.110) with $\partial^\gamma a_\pm$ so that the $\pm$ terms are the same and integrate over $\mathbb{T}^3$. By adding the $\pm$ terms together we have

$$\begin{gathered}||\nabla \partial^\gamma a_+||^2 + ||\nabla \partial^\gamma a_-||^2 + \rho_0||\partial^\gamma a_+ - \partial^\gamma a_-||^2 \\ \leq C\{||\nabla \partial^\gamma a_+|| + ||\nabla \partial^\gamma a_-||\} \sum_\pm ||\partial^\gamma \{l_{bi}^\pm + h_{bi}^\pm\}||.\end{gathered}$$

Therefore, $||\nabla \partial^\gamma a_+|| + ||\nabla \partial^\gamma a_-|| \leq \sum_\pm ||\partial^\gamma \{l_{bi}^\pm + h_{bi}^\pm\}||$. Since γ is purely spatial, this is bounded by the r.h.s of (2.108) because of Lemmas 2.11 and 2.12. Furthermore, by the Poincaré inequality and Lemma 2.10, a itself is bounded by

$$\begin{aligned}||a_\pm|| &\leq C||\nabla a_\pm|| + C\left|\int a_\pm dx\right| \\ &\leq C||\nabla a_\pm|| + C\{||E||^2 + ||B - \bar{B}||^2\},\end{aligned}$$

which is bounded by the r.h.s. of (2.108) by the same argument as for c. We thus complete the estimate for $a(t,x)$ and our theorem follows.

2.6. Global Solutions

In this section we establish Theorem 2.1. We first derive a refined energy estimate for the relativistic Landau-Maxwell system.

LEMMA 2.14. *Let $[f(t,x,p),E(t,x),B(t,x)]$ be the unique solution constructed in Theorem 2.6 which also satisfies the conservation laws (2.10), (2.11) and (2.12). Let the small amplitude assumption (2.14) be valid. For any given $0\le m\le N$, $|\beta|\le m$, there are constants $C_{|\beta|}>0$, $C^*_m>0$ and $\delta_m>0$ such that*

$$\begin{aligned}\sum_{|\beta|\le m,|\gamma|+|\beta|\le N}&\frac{1}{2}\frac{d}{dt}C_{|\beta|}||\partial^\gamma_\beta f(t)||^2+\frac{1}{2}\frac{d}{dt}|||[E,B]|||^2(t)\\ +\sum_{|\beta|\le m,|\gamma|+|\beta|\le N}&\delta_m||\partial^\gamma_\beta f(t)||^2_\sigma\le C^*_m\sqrt{\mathcal{E}(t)}|||f|||^2_\sigma(t).\end{aligned}\tag{2.111}$$

PROOF. We use an induction over m, the order of the $p-$derivatives. For $m=0$, by taking the pure ∂^γ derivatives of (2.29), we obtain:

$$\begin{aligned}\left\{\partial_t+\frac{p}{p_0}\cdot\nabla_x+\xi\left(E+\frac{p}{p_0}\times B\right)\cdot\nabla_p\right\}\partial^\gamma f&\\ -\left\{\partial^\gamma E\cdot\frac{p}{p_0}\right\}\sqrt{J}\xi_1+L\{\partial^\gamma f\}&\\ =-\sum_{\gamma_1\neq 0}C^{\gamma_1}_\gamma\xi\left(\partial^{\gamma_1}E+\frac{p}{p_0}\times\partial^{\gamma_1}B\right)\cdot\nabla_p\partial^{\gamma-\gamma_1}f&\\ +\sum_{\gamma_1\le\gamma}C^{\gamma_1}_\gamma\left\{\frac{\xi}{2}\left\{\partial^{\gamma_1}E\cdot\frac{p}{p_0}\right\}\partial^{\gamma-\gamma_1}f+\Gamma(\partial^{\gamma_1}f,\partial^{\gamma-\gamma_1}f)\right\}&\end{aligned}\tag{2.112}$$

Using the same argument as (2.88),

$$-\langle\partial^\gamma E\cdot\{p\sqrt{J}/p_0\}\xi_1,\partial^\gamma f\rangle=\frac{1}{2}\frac{d}{dt}\left\{||\partial^\gamma E(t)||^2+||\partial^\gamma B(t)||^2\right\}.$$

Take the inner product of $\partial^\gamma f$ with (2.112), sum over $|\gamma|\le N$ and apply Theorem 2.2 to $L\{\partial^\gamma f\}$ to deduce the following for some constant $C>0$,

$$\begin{aligned}\sum_{|\gamma|\le N}\frac{1}{2}\frac{d}{dt}&\left(||\partial^\gamma f(t)||^2+||\partial^\gamma E(t)||^2+||\partial^\gamma B(t)||^2\right)+\delta_0\sum_{|\gamma|\le N}||\partial^\gamma f(t)||^2_\sigma\\ &\le C\{|||f|||(t)+|||[E,B]|||(t)\}|||f|||^2_\sigma(t)\le C\sqrt{\mathcal{E}(t)}|||f|||^2_\sigma(t).\end{aligned}$$

We have used estimates (2.89-2.91) and Theorem 2.4 to bound the r.h.s. of (2.112). This concludes the case for $m = 0$ with $C_0 = 1$ and $C_0^* = C$.

Now assume the Lemma is valid for m. For $|\beta| = m+1$, taking $\partial_\beta^\gamma (\beta \neq 0)$ of (2.29), we obtain:

$$\begin{aligned}
&\left\{\partial_t + \frac{p}{p_0}\cdot\nabla_x + \xi\left(E + \frac{p}{p_0}\times B\right)\cdot\nabla_p\right\}\partial_\beta^\gamma f - \partial^\gamma E\cdot\partial_\beta\left\{\frac{p}{p_0}\sqrt{J}\right\}\xi_1 \\
&+\partial_\beta\{L\partial^\gamma f\} + \sum_{\beta_1\neq 0} C_\beta^{\beta_1}\partial_{\beta_1}\left(\frac{p}{p_0}\right)\cdot\nabla_x\partial_{\beta-\beta_1}^\gamma f \qquad (2.113)\\
= &\sum C_\gamma^{\gamma_1}C_\beta^{\beta_1}\frac{\xi}{2}\left\{\partial^{\gamma_1}E\cdot\partial_{\beta_1}\left(\frac{p}{p_0}\right)\right\}\partial_{\beta-\beta_1}^{\gamma-\gamma_1}f - \sum_{\gamma_1\neq 0} C_\gamma^{\gamma_1}\xi\partial^{\gamma_1}E\cdot\nabla_p\partial_\beta^{\gamma-\gamma_1}f \\
&- \sum_{(\gamma_1,\beta_1)\neq(0,0)} C_\gamma^{\gamma_1}C_\beta^{\beta_1}\xi\partial_{\beta_1}\left(\frac{p}{p_0}\right)\times\partial^{\gamma_1}B\cdot\nabla_p\partial_{\beta-\beta_1}^{\gamma-\gamma_1}f \\
&+\sum C_\gamma^{\gamma_1}\partial_\beta\Gamma(\partial^{\gamma_1}f,\partial^{\gamma-\gamma_1}f).
\end{aligned}$$

We take the inner product of (2.113) over $\mathbb{T}^3\times\mathbb{R}^3$ with $\partial_\beta^\gamma f$. The first inner product on the left is equal to $\frac{1}{2}\frac{d}{dt}||\partial_\beta^\gamma f(t)||^2$. Now $|\gamma| \leq N-1$ (since $|\beta| = m+1 > 0$), Lemma 2.13, (2.60) and $M_0 \leq 1$ imply (after an integration by parts) that the second inner product on l.h.s. is bounded by

$$\langle\partial^\gamma E\cdot\partial_\beta\{p\sqrt{J}/p_0\}\xi_1, \partial_\beta^\gamma f\rangle \leq C||\partial^\gamma E||\cdot||\partial^\gamma f|| \leq C||\partial^\gamma f||\sum_{|\gamma'|\leq N}||\partial^{\gamma'}f||_\sigma.$$

From Lemma 2.7 and Cauchy's inequality we deduce that, for any $\eta > 0$, the inner product of third term on l.h.s. is bounded from below as

$$\left(\partial_\beta\{L\partial^\gamma f\}, \partial_\beta^\gamma f\right) \geq ||\partial_\beta^\gamma f||_\sigma^2 - \eta\sum_{|\bar\beta|\leq|\beta|}||\partial_{\bar\beta}^\gamma f||_\sigma^2 - C_\eta||\partial^\gamma f||^2.$$

Using Cauchy's inequality again, the the inner product of the last term on l.h.s. of (2.113) is bounded by

$$\eta||\partial_\beta^\gamma f(t)||^2 + C_\eta\sum_{|\beta_1|\geq 1}||\nabla_x\partial_{\beta-\beta_1}^\gamma f||^2.$$

By the same estimates, (2.89-2.91) and Theorem 2.4, all the inner products in r.h.s. of (2.113) are bounded by $C\sqrt{\mathcal{E}(t)}|||f|||_\sigma^2(t)$. Collecting terms and summing over

$|\beta| = m+1$ and $|\gamma|+|\beta| \le N$, we split the highest order p-derivatives from the lower order derivatives to obtain

$$
\begin{aligned}
&\sum_{|\beta|=m+1,|\gamma|+|\beta|\le N} \left\{ \frac{1}{2}\frac{d}{dt}||\partial^\gamma_\beta f(t)||^2 + ||\partial^\gamma_\beta f(t)||^2_\sigma \right\} \\
\le\ & \sum_{|\beta|=m+1,|\gamma|+|\beta|\le N} \left\{ \sum_{|\beta|=m+1} 2\eta ||\partial^\gamma_\beta f(t)||^2_\sigma + C\sqrt{\mathcal{E}(t)}|||f|||^2_\sigma(t) \right\} \\
& + \sum_{|\beta|=m+1,|\gamma|+|\beta|\le N} (C+2C_\eta) \sum_{|\beta|\le m,|\gamma|+|\beta|\le N} ||\partial^\gamma_\beta f(t)||^2_\sigma \\
\le\ & Z_{m+1} \left\{ \sum_{|\beta|=m+1,|\gamma|+|\beta|\le N} 2\eta ||\partial^\gamma_\beta f(t)||^2_\sigma + C\sqrt{\mathcal{E}(t)}|||f|||^2_\sigma(t) \right\} \\
& + Z_{m+1}(C+2C_\eta) \sum_{|\beta|\le m,|\gamma|+|\beta|\le N} ||\partial^\gamma_\beta f(t)||^2_\sigma .
\end{aligned}
$$

Here Z_{m+1} denotes the number of all possible (γ,β) such that $|\beta| \le m+1$, $|\gamma|+|\beta| \le N$. By choosing $\eta = \frac{1}{4Z_{m+1}}$, and absorbing the first term on the r.h.s. by the second term on the left, we have, for some constant $C(Z_{m+1})$,

$$
\begin{aligned}
&\sum_{|\beta|=m+1,|\gamma|+|\beta|\le N} \left\{ \frac{1}{2}\frac{d}{dt}||\partial^\gamma_\beta f(t)||^2 + \frac{1}{2}||\partial^\gamma_\beta f(t)||^2_\sigma \right\} \\
\le\ & C(Z_{m+1}) \left\{ \sum_{|\beta|\le m,|\gamma|+|\beta|\le N} ||\partial^\gamma_\beta f(t)||^2_\sigma + \sqrt{\mathcal{E}(t)}|||f|||^2_\sigma(t) \right\}. \qquad (2.114)
\end{aligned}
$$

We may assume $C(Z_{m+1}) \ge 1$. We multiply (2.114) by $\frac{\delta_m}{2C(Z_{m+1})}$ and add it to (2.111) for $|\beta| \le m$ to get

$$
\begin{aligned}
&\sum_{|\beta|=m+1,|\gamma|+|\beta|\le N} \left\{ \frac{\delta_m}{4C(Z_{m+1})}\frac{d}{dt}||\partial^\gamma_\beta f(t)||^2 + \frac{\delta_m}{4C(Z_{m+1})}||\partial^\gamma_\beta f(t)||^2_\sigma \right\} \\
& + \sum_{|\beta|\le m,|\gamma|+|\beta|\le N} \frac{1}{2}\frac{d}{dt}\left(C_{|\beta|}||\partial^\gamma_\beta f(t)||^2 + ||\partial^\gamma E(t)||^2 + ||\partial^\gamma B(t)||^2 \right) \\
& + \sum_{|\beta|\le m,|\gamma|+|\beta|\le N} \delta_m ||\partial^\gamma_\beta f(t)||^2_\sigma \\
\le\ & \frac{\delta_m}{2} \sum_{|\beta|\le m,|\gamma|+|\beta|\le N} ||\partial^\gamma_\beta f(t)||^2_\sigma + \left\{ C^*_m + \frac{\delta_m}{2} \right\} \sqrt{\mathcal{E}(t)}|||f|||^2_\sigma(t).
\end{aligned}
$$

Absorb the first term on the right by the last term on the left. We conclude our lemma by choosing

$$C_{m+1} = \frac{\delta_m}{4C(Z_{m+1})},\ \delta_{m+1} = \frac{\delta_m}{4C(Z_{m+1})} \leq \frac{\delta_m}{2},\ C^*_{m+1} = C^*_m + \frac{\delta_m}{2}.$$

Nothing that $C(Z_{m+1}) > C(Z_m)$ and $\delta_m < \delta_{m-1}$. $\square$

We are ready to construct global in time solutions to the relativistic Landau-Maxwell system (2.29) and (2.30).

Proof of Theorem 2.1: We first fix $M_0 \leq 1$ such that both Theorems 2.2 and 2.6 are valid. For such an M_0, we let $m = N$ in (2.111), and define

$$y(t) \equiv \sum_{|\gamma|+|\beta|\leq N} C_{|\beta|}||\partial^\gamma_\beta f(t)||^2 + |||[E,B]|||^2(t).$$

We choose a constant $C_1 > 1$ such that for any $t \geq 0$,

$$\frac{1}{C_1}\left\{y(t) + \frac{\delta_N}{2}\int_0^t |||f|||^2_\sigma(s)ds\right\} \leq \mathcal{E}(t)$$

$$\mathcal{E}(t) \leq C_1\left\{y(t) + \frac{\delta_N}{2}\int_0^t |||f|||^2_\sigma(s)ds\right\}.$$

Recall constant C^*_N in (2.111). We define

$$M \equiv \min\left\{\frac{\delta_N^2}{8C_N^{*2}C_1^2}, \frac{M_0}{2C_1^2}\right\},$$

and choose initial data so that $\mathcal{E}(0) \leq M < M_0$. From Theorem 2.6, we may denote $T > 0$ so that

$$T = \sup_t\{t : \mathcal{E}(t) \leq 2C_1^2 M\} > 0.$$

Notice that, for $0 \leq t \leq T$, $\mathcal{E}(t) \leq 2C_1^2 M \leq M_0$ so that the small amplitude assumption (2.14) is valid. We now apply Lemma 2.14 and the definitions of M and T, with $0 \leq t \leq T$, to get

$$\begin{aligned} & y'(t) + \delta_N|||f|||^2_\sigma(t) \\ \leq\ & C^*_N\sqrt{\mathcal{E}(t)}|||f|||^2_\sigma(t) \leq C^*_N C_1\sqrt{2M}|||f|||^2_\sigma(t) \\ \leq\ & \frac{\delta_N}{2}|||f|||^2_\sigma(t). \end{aligned}$$

Therefore, an integration in t over $0 \le t \le s < T$ yields

$$
\begin{aligned}
\mathcal{E}(s) &\le C_1 \left\{ y(s) + \frac{\delta_N}{2} \int_0^s |||f|||_\sigma^2(\tau) d\tau \right\} \le C_1 y(0) \\
&\le C_1^2 \mathcal{E}(0) \\
&\le C_1^2 M < 2C_1^2 M.
\end{aligned}
\tag{2.115}
$$

Since $\mathcal{E}(s)$ is continuous in s, this implies $\mathcal{E}(T) \le C_1^2 M$ if $T < \infty$. This implies $T = \infty$. Furthermore, such a global solution satisfies $\mathcal{E}(t) \le C_1^2 \mathcal{E}(0)$ for all $t \ge 0$ from (2.115). **Q.E.D.**

CHAPTER 3

Almost Exponential Decay near Maxwellian

Abstract. By direct interpolation of a family of smooth energy estimates for solutions near Maxwellian equilibrium and in a periodic box to several Boltzmann type equations in [37–39, 62], we show convergence to Maxwellian with any polynomial rate in time. Our results not only resolve the important open problem for both the Vlasov-Maxwell-Boltzmann system and the relativistic Landau-Maxwell system for charged particles, but also lead to a simpler alternative proof of recent decay results [21] for soft potentials as well as the Coulombic interaction, with precise decay rate depending on the initial conditions. This result will appear in a modified form as [60].

3.1. Introduction

There are two motivations of the current study. Recently, a new nonlinear energy method has been developed to construct global-in-time solutions *near* Maxwellian for various types of Boltzmann equations. One of the highlights of such a program is the construction of global-in-time solutions for the Boltzmann equation in the presence of a self-consistent electromagnetic field. Unfortunately, the question of determining a possible time-decay rate for the electromagnetic field has been left open. For near Maxwellian periodic solutions to the Boltzmann equation with no electromagnetic field, Ukai [65] obtained exponential convergence in the case of a cutoff hard potential. Caflisch [8, 9], for cutoff soft potentials with $\gamma > -1$, obtained a convergence rate like $O(e^{-\lambda t^{\beta}})$ for $\lambda > 0$ and $0 < \beta < 1$. In the whole space, also for cutoff soft potentials with $\gamma > -1$, Ukai and Asano [66] obtained the rate $O(t^{-\alpha})$ with $0 < \alpha < 1$. However the decay for very soft potentials had been open.

Desvillettes and Villani have recently undertaken an impressive program to study the time-decay rate to a Maxwellian of large data solutions to Boltzmann type equations. Even though their assumptions *in general* are a priori and impossible to verify at present time, their method does lead to an almost exponential decay rate for the soft potentials and the Landau equation for solutions close to a Maxwellian. This surprising new decay result relies crucially on recent energy estimates in [37, 38] as well as other extensive and delicate work [18–21, 53, 64, 69]. To obtain time decay for these models with very 'weak' collision effects has been a very challenging open problem even for solutions near a Maxwellian, therefore it is of great interest to find simpler proofs for such decay.

The objective of this article is to give a direct proof of an almost exponential decay rate for solutions near Maxwellian to the Vlasov-Maxwell-Boltzmann system, the relativistic Landau-Maxwell system, the Boltzmann equation with a soft potential as well as the Landau equation. The common difficulty of all such problems is the fact that in the energy estimates [37–39, 62] the instant energy functional at each time is stronger than the dissipation rate. It is thus impossible to use a Gronwall type of argument to get the decay rate in time. Our main observation is that *a family* of energy estimates, not just one, have been derived in [37–39, 62]. Even though the instant energy is stronger for a fixed family, it is possible to be bounded by a fractional power of the dissipation rate via simple interpolations with stronger energy norms of another family of estimates. Only algebraic decay is possible due to such interpolations, but more regular initial data grants faster decay. In comparison to [21], our proofs are much simpler and our decay rates are more precise.

For both Vlasov-Maxwell-Boltzmann and relativistic Landau-Maxwell, such a family of energy norms are exactly the same as in [39, 62] with higher and higher derivatives. The interpolation in this case is between Sobolev norms.

On the other hand, for both soft potentials and Landau equations, such a family of energy norms depends on higher and higher powers of velocity weights and the interpolation in this case is between the different weight powers. Although these are different from the original norms used in [37,38], new energy estimates can be derived by the improved method in [39,62] with minimum modifications.

We remark that all the constants in general are not explicit, except for the polynomial decay exponent. This is due to the fact that the lower bound for the linearized collision operator is not obtained constructively.

3.2. Vlasov-Maxwell-Boltzmann

Dynamics of charged dilute particles (e.g., electrons and ions) are described by the Vlasov-Maxwell-Boltzmann system:

$$\begin{aligned}&\partial_t F_+ + v\cdot\nabla_x F_+ + (E+v\times B)\cdot\nabla_v F_+ = Q(F_+,F_+)+Q(F_+,F_-),\\ &\partial_t F_- + v\cdot\nabla_x F_- - (E+v\times B)\cdot\nabla_v F_- = Q(F_-,F_+)+Q(F_-,F_-),\end{aligned}\tag{3.1}$$

with initial data $F_\pm(0,x,v) = F_{0,\pm}(x,v)$. For notational simplicity we have set all the physical constants to be unity, see [39] for more background. Here $F_\pm(t,x,v)\geq 0$ are the spatially periodic number density functions for the ions (+) and electrons (-) respectively at time $t\geq 0$, position $x=(x_1,x_2,x_3)\in[-\pi,\pi]^3=\mathbb{T}^3$ and velocity $v=(v_1,v_2,v_3)\in\mathbb{R}^3$. As a model problem, the collision between particles is given by the standard Boltzmann collision operator with hard-sphere interaction Q in [39].

The self-consistent, spatially periodic electromagnetic field $[E(t,x),B(t,x)]$ in (3.1) is coupled with $F_\pm(t,x,v)$ through the Maxwell system:

$$\partial_t E - \nabla_x\times B = -\mathcal{J},\ \ \partial_t B + \nabla_x\times E = 0,\tag{3.2}$$

with constraints $\nabla_x\cdot B=0$, $\nabla_x\cdot E=\rho$ and initial data $E(0,x)=E_0(x)$, $B(0,x)=B_0(x)$. The coupling comes through the charge density ρ and the current density $\mathcal{J}$ which are given by

$$\rho=\int_{\mathbb{R}^3}\{F_+-F_-\}dv,\ \ \mathcal{J}=\int_{\mathbb{R}^3}v\{F_+-F_-\}dv.\tag{3.3}$$

We consider the perturbation $F_\pm=\mu+\sqrt{\mu}f_\pm\geq 0$ where the normalized Maxwellian is $\mu=\mu(v)=e^{-|v|^2}$.

Notation: Let $||\cdot||$ be the standard norm in either $L^2(\mathbb{T}^3\times\mathbb{R}^3)$ or $L^2(\mathbb{T}^3)$. We also define $||\cdot||_\nu$ to be the following weighted L^2 norm

$$||g||_\nu^2 \equiv \int_{\mathbb{T}^3\times\mathbb{R}^3} \nu(v)|g(x,v)|^2 dxdv,$$

where $\nu(v)$ is the collision frequency for hard-sphere interactions, $\nu(v)\sim C|v|$ as $|v|$ tends to ∞. Let the multi-indices α and β be $\alpha=[\alpha^0,\alpha^1,\alpha^2,\alpha^3]$, $\beta=[\beta^1,\beta^2,\beta^3]$ with length $|\alpha|$ and $|\beta|$ respectively. Then let $\partial^\alpha_\beta \equiv \partial_t^{\alpha^0}\partial_{x_1}^{\alpha^1}\partial_{x_2}^{\alpha^2}\partial_{x_3}^{\alpha^3}\partial_{v_1}^{\beta^1}\partial_{v_2}^{\beta^2}\partial_{v_3}^{\beta^3}$, which includes *temporal* derivatives. Define

$$|||f|||_N^2(t) \equiv \sum_{|\alpha|+|\beta|\le N} ||\partial^\alpha_\beta f_+(t)||^2 + \sum_{|\alpha|+|\beta|\le N} ||\partial^\alpha_\beta f_-(t)||^2, \tag{3.4}$$

$$|||[E,B]|||_N^2(t) \equiv \sum_{|\alpha|\le N} ||\partial^\alpha E(t)||^2 + \sum_{|\alpha|\le N} ||\partial^\alpha B(t)||^2.$$

Given initial datum $[f_0(x,p),E_0(x),B_0(x)]$, we define

$$\mathcal{E}_N(0) = \frac{1}{2}|||f_0|||_N^2 + |||[E_0,B_0]|||_N^2, \tag{3.5}$$

where the temporal derivatives of $[f_0,E_0,B_0]$ are defined naturally through equations (3.1) and (3.2). Further define the the *dissipation rate* as

$$|||f|||_{N,\nu}^2(t) \equiv \sum_{|\alpha|+|\beta|\le N} ||\partial^\alpha_\beta f_+||_\nu^2 + \sum_{|\alpha|+|\beta|\le N} ||\partial^\alpha_\beta f_-||_\nu^2.$$

We study the decay of solutions near $F_\pm\equiv\mu$, $E\equiv 0$, and $B\equiv\bar{B}=$ a constant.

THEOREM 3.1. *Fix integers $N\ge 4$ and $k\ge 1$. Choose initial data $[f_{0,\pm},E_0,B_0]$ which satisfies the assumptions of Theorem 1 in* [39] *for $N+k$ (therefore also for N) including for $\epsilon_{N+k}>0$ small enough*

$$\mathcal{E}_{N+k}(0)\le\epsilon_{N+k},$$

and let $[F_\pm=\mu+\sqrt{\mu}f_\pm,E,B]$ be the unique solution to (3.1) and (3.2) with (3.3) from Theorem 1 of [39]. *Then there exists $C_{N,k}>0$ such that*

$$|||f|||_N^2(t) + |||[E,B-\bar{B}]|||_N^2(t) \le C_{N,k}\mathcal{E}_{N+k}(0)\left\{1+\frac{t}{k}\right\}^{-k},$$

where $\bar{B}=\frac{1}{|\mathbb{T}^3|}\int_{\mathbb{T}^3}B(t,x)dx$ is constant in time [39].

PROOF. On the last page of [39], we see the following inequality (for $C_{|\beta|} > 0$, $\delta_N > 0$)

$$\frac{d}{dt}y_N(t) + \frac{\delta_N}{2}|||f|||^2_{\nu,N}(t) \le 0.$$

where the modified instant energy is

$$y_N(t) \equiv \frac{1}{2}\sum_{|\alpha|+|\beta|\le N} C_{|\beta|}||\partial^\alpha_\beta f(t)||^2 + |||[E, B - \bar{B}]|||^2_N(t).$$

To obtain a decay rate in t is to bound the instant energy $y_N(t)$ in terms of the dissipation rate $|||f|||^2_{\nu,N}(t)$. Since the collision frequency $\nu(v)$ is bounded from below in our hard-sphere interaction, part of $y_N(t)$ is clearly bounded by the dissipation rate because

$$\frac{1}{2}\sum_{|\alpha|+|\beta|\le N} C_{|\beta|}||\partial^\alpha_\beta f(t)||^2 \le C|||f|||^2_{\nu,N}(t). \tag{3.6}$$

On the other hand, to estimate the field component $[E, B - \bar{B}]$ of $y_N(t)$, Lemma 9 in [39] only implies

$$|||[E, B - \bar{B}]|||^2_{N-1}(t) \le C|||f|||^2_{\nu,N}(t). \tag{3.7}$$

No estimates for the highest N-th derivatives of E and B are available. Hence for fixed N, $|||f(t)|||^2_{\nu,N}$ is weaker than the total instant energy $y_N(t)$.

The key is to use interpolation to get a lower bound for the N-th order derivatives $\partial^\alpha E$ and $\partial^\alpha B$ with the bound of higher energy $|||f|||^2_{N+k}(t)$, for some k large. Let $\partial^\alpha = \partial_t^{\alpha_0}\partial_x^{\alpha'}$ and $|\alpha| = N$. For purely spatial derivaitves with $\alpha_0 = 0$ and $|\alpha'| = N$, a simple interpolation between spatial Sobolev spaces H^{N-1}_x and H^{N+k}_x yields

$$\begin{aligned} ||\partial^\alpha E||^2 + ||\partial^\alpha B||^2 &= ||\partial^{\alpha'}_x E||^2 + ||\partial^{\alpha'}_x\{B - \bar{B}\}||^2 \\ &\le C||[E, B - \bar{B}]||^{2k/(k+1)}_{H^{N-1}_x} \times ||[E, B - \bar{B}]||^{2/(k+1)}_{H^{N+k}_x} \\ &\le C||[E, B - \bar{B}]||^{2k/(k+1)}_{N-1}||[E, B - \bar{B}]||^{2/(k+1)}_{N+k}. \end{aligned}$$

By our assumption and Theorem 1 in [39]

$$||[E, B - \bar{B}]||^2_{N+k}(t) \le C_{N+k}\mathcal{E}_{N+k}(0) < \infty.$$

We thus conclude by (3.7) that

$$
\begin{aligned}
||\partial^\alpha E||^2 + ||\partial^\alpha B||^2 &\leq C\left(C_{N+k}\mathcal{E}_{N+k}(0)\right)^{1/(k+1)} |||[E, B-\bar{B}]|||_{N-1}^{2k/(k+1)} \\
&\leq C\left(\mathcal{E}_{N+k}(0)\right)^{1/(k+1)} |||f|||_{\nu,N}^{2k/(k+1)}.
\end{aligned} \tag{3.8}
$$

Now for the general case with $\alpha_0 \neq 0$, we take $\partial_t^{\alpha_0-1}\partial_x^{\alpha'}$ of the Maxwell system (3.2) to get

$$
\begin{aligned}
&\partial_t^{\alpha_0}\partial_x^{\alpha'} E = \partial_t^{\alpha_0-1}\partial_x^{\alpha'}\nabla_x \times B - \int \sqrt{\mu}\partial_t^{\alpha_0-1}\partial_x^{\alpha'}\{f_+ - f_-\}dv, \\
&\partial_t^{\alpha_0}\partial_x^{\alpha'} B = -\partial_t^{\alpha_0-1}\partial_x^{\alpha'}\nabla_x \times E.
\end{aligned}
$$

Since $|| \int \sqrt{\mu}\partial_t^{\alpha_0-1}\partial_x^{\alpha'}\{f_+ - f_-\}dv||^2_{L^2(\mathbb{T}^3)}$ is clearly bounded by $C|||f(t)|||_N^2$ or equivalently $Cy_N(t)$, which is bounded by

$$
C\left(\mathcal{E}_{N+k}(0)\right)^{1/(k+1)} |||f|||_{\nu,N}^{2k/(k+1)}.
$$

We therefore deduce that (3.8) is valid for general α with $\alpha_0 \geq 1$ via a simple induction of α_0, starting from $\alpha_0 = 1$. Moreover, for the full dissipation rate

$$
\frac{\delta_N}{2}|||f|||_{\nu,N}^2 \geq C_{N,k}\left(\mathcal{E}_{N+k}(0)\right)^{-1/k} y_N^{\{k+1\}/k}.
$$

It follows that

$$
\frac{dy_N}{dt} + C_{N,k}\left(\mathcal{E}_{N+k}(0)\right)^{-1/k} y_N^{\{k+1\}/k} \leq 0, \tag{3.9}
$$

and $y_N'(t)(y_N(t))^{-1-1/k} \leq -C_{N,k}\left(\mathcal{E}_{N+k}(0)\right)^{-1/k}$. Integrating this over $[0,t]$, we have

$$
k\{y_N(0)\}^{-1/k} - k\{y_N(t)\}^{-1/k} \leq -\left(\mathcal{E}_{N+k}(0)\right)^{-1/k} C_{N,k}t.
$$

Hence

$$
\{y_N(t)\}^{-1/k} \geq t\frac{C_{N,k}}{k}\left(\mathcal{E}_{N+k}(0)\right)^{-1/k} + \{y_N(0)\}^{-1/k},
$$

since we can assume $y_N(0) \leq C_{N,k}\mathcal{E}_{N+k}(0)$, our theorem thus follows. $\square$

3.3. Relativistic Landau-Maxwell System

The relativistic Landau-Maxwell system is the most fundamental and complete model for describing the dynamics of a dilute collisional cold plasma in which particles interact through Coulombic collisions and through their self-consistent electromagnetic field. For notational simplicity, we consider the following normalized Landau-Maxwell system

$$
\begin{aligned}
&\partial_t F_+ + \frac{p}{p_0}\cdot\nabla_x F_+ + \left(E + \frac{p}{p_0}\times B\right)\cdot\nabla_p F_+ = \mathcal{C}(F_+,F_+) + \mathcal{C}(F_+,F_-)\\
&\partial_t F_- + \frac{p}{p_0}\cdot\nabla_x F_- - \left(E + \frac{p}{p_0}\times B\right)\cdot\nabla_p F_- = \mathcal{C}(F_-,F_-) + \mathcal{C}(F_-,F_+),
\end{aligned}
\tag{3.10}
$$

with initial condition $F_\pm(0,x,p) = F_{0,\pm}(x,p)$. Here $F_\pm(t,x,p)\geq 0$ are the spatially periodic number density functions for ions $(+)$ and electrons $(-)$ at time $t\geq 0$, position $x=(x_1,x_2,x_3)\in\mathbb{T}^3$ and momentum $p=(p_1,p_2,p_3)\in\mathbb{R}^3$. The energy of a particle is given by $p_0=\sqrt{1+|p|^2}$.

To completely describe a dilute plasma, the electromagnetic field $E(t,x)$ and $B(t,x)$ is coupled with $F_\pm(t,x,p)$ through the celebrated Maxwell system (3.2) where the charge density ρ and the current density $\mathcal{J}$ are given by

$$
\rho = \int_{\mathbb{R}^3}\{F_+ - F_-\}\,dp,\ \ \mathcal{J} = \int_{\mathbb{R}^3}\frac{p}{p_0}\{F_+ - F_-\}dp.
\tag{3.11}
$$

The collision between particles is modelled by the relativistic Landau collision operator $\mathcal{C}$, and its normalized form is given by

$$
\mathcal{C}(g,h)(p) \equiv \nabla_p\cdot\left\{\int_{\mathbb{R}^3}\Phi(P,P')\{\nabla_p g(p)h(p') - g(p)\nabla_{p'}h(p')\}dp'\right\},
$$

where the four-vectors are $P=(p_0,p_1,p_2,p_3)$ and $P'=(p_0',p_1',p_2',p_3')$. Morover, $\Phi(P,P')\equiv\Lambda(P,P')S(P,P')$ with $\Lambda(P,P')\equiv\frac{(P\cdot P')^2}{p_0p_0'}\left\{(P\cdot P')^2-1\right\}^{-3/2}$ and

$$
S(P,P')\equiv\left\{(P\cdot P')^2-1\right\}I_3 + \{(P\cdot P')-1\}(p\otimes p') - (p-p')\otimes(p-p').
$$

Here the Lorentz inner product is $P\cdot P' = p_0 p_0' - p\cdot p'$. Let $\sigma^{ij}(p) = \int \Phi^{ij}(P,P')J(p')dp'$, where $J(p) \equiv e^{-p_0}$ is a relativistic Maxwellian. We define

$$||g||_\sigma^2 \equiv \sum_{i,j} \int_{\mathbb{T}^3\times\mathbb{R}^3} \left\{ 2\sigma^{ij}\partial_{p_j} g \partial_{p_i} g + \frac{\sigma^{ij}}{2}\frac{p_i}{p_0}\frac{p_j}{p_0} g^2 \right\} dxdp.$$

See [62] for more details. We write $F_\pm \equiv J + \sqrt{J} f_\pm$ and study the decay rate for $f_\pm$, E and B. Define $|||f|||_N^2(t)$ and $|||[E,B]|||_N^2(t)$ as in (3.4) with $\partial_\beta^\alpha \equiv \partial_t^{\alpha^0}\partial_{x_1}^{\alpha^1}\partial_{x_2}^{\alpha^2}\partial_{x_3}^{\alpha^3}\partial_{p_1}^{\beta^1}\partial_{p_2}^{\beta^2}\partial_{p_3}^{\beta^3}$ and $dv = dp$. Then the initial data is again measured by (3.5), but the *dissipation rate* is given by

$$|||f|||_{N,\sigma}^2(t) \equiv \sum_{|\alpha|+|\beta|\le N} ||\partial_\beta^\alpha f_+||_\sigma^2 + \sum_{|\alpha|+|\beta|\le N} ||\partial_\beta^\alpha f_-||_\sigma^2.$$

We study the decay of solutions near $F_\pm \equiv J(p)$, $E \equiv 0$ and $B \equiv \bar{B} =$ a constant.

THEOREM 3.2. *Fix $N \ge 4$, $k \ge 1$ and choose initial data $[F_{0,\pm}, E_0, B_0]$ which satisfies the assumptions of Theorem 1 in* [62] *for $N+k$ (therefore also for N) including*

$$\mathcal{E}_{N+k}(0) \le \epsilon_{N+k},\ \ \epsilon_{N+k} > 0.$$

Let $[F_\pm = J + \sqrt{J} f_\pm, E, B]$ be the unique solution to (3.10) and (3.2) with (3.11) from Theorem 1 of [62]*, then there exists $C_{N,k} > 0$ such that*

$$|||f|||_N^2(t) + |||[E, B-\bar{B}]|||_N^2(t) \le C_{N,k}\mathcal{E}_{N+k}(0)\left\{1 + \frac{t}{k}\right\}^{-k},$$

where $\bar{B} = \frac{1}{|\mathbb{T}^3|}\int_{\mathbb{T}^3} B(t,x)dx$ is constant in time [62].

The proof is exactly as in the case for the Vlasov-Maxwell-Boltzmann system because of two facts: $C||g||_\sigma \ge ||g||$ (Lemma 2 in [62]) and from Lemma 12 in [62]

$$|||[E,B]|||_{N-1}(t) \le C|||f|||_{N,\sigma}(t).$$

3.4. Soft Potentials

The Boltzmann equation with a soft potential is

$$\partial_t F + v\cdot\nabla_x F = Q(F,F),\ \ F(0,x,v) = F_0(x,v),$$

where $F(t,x,v)$ is the spatially periodic distribution function for the particles at time $t \geq 0$, position $x = (x_1, x_2, x_3) \in \mathbb{T}^3$ and velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. The l.h.s. of this equation models the transport of particles and the operator on the r.h.s. models the effect of collisions on the transport:

$$Q(F,F) \equiv \int_{\mathbb{R}^3 \times S^2} |u-v|^\gamma B(\theta)\{F(u')F(v') - F(u)F(v)\} du d\omega.$$

The exponent in the collision operator is $\gamma = 1 - \frac{4}{s}$ with $1 < s < 4$ so that $-3 < \gamma < 0$ (soft potentials), and $B(\theta)$ satisfies the Grad angular cutoff assumption: $0 < B(\theta) \leq C|\cos\theta|$.

Notation: We define the collision frequency $\nu(v) \equiv \int |v-u|^\gamma \mu(u) B(\theta) du d\omega$, and

$$||g||_\nu^2 = \int_{\mathbb{T}^3 \times \mathbb{R}^3} \nu(v) g^2 dx dv.$$

Notice that $\nu(v) \sim C|v|^\gamma$ as $|v| \to \infty$. Define a weight function in v by $w = w_\gamma(v) = [1+|v|]^\gamma$ with $-3 < \gamma < 0$ and denote a weighted L^2 norm as

$$||g||_\ell^2 \equiv \int_{\mathbb{T}^3 \times \mathbb{R}^3} w^{2\ell} |g|^2 dx dv, \ ||g||_{\nu,\ell}^2 \equiv \int_{\mathbb{T}^3 \times \mathbb{R}^3} w^{2\ell} \nu g^2 dx dv.$$

Let $\partial_\beta^\alpha \equiv \partial_t^{\alpha^0} \partial_{x_1}^{\alpha^1} \partial_{x_2}^{\alpha^2} \partial_{x_3}^{\alpha^3} \partial_{v_1}^{\beta^1} \partial_{v_2}^{\beta^2} \partial_{v_3}^{\beta^3}$ which includes temporal derivatives. For fixed $N \geq 4$, we define the family of norms depending on ℓ as

$$|||f|||_\ell^2(t) \equiv \sum_{|\alpha|+|\beta| \leq N} ||\partial_\beta^\alpha f(t)||_{|\beta|-\ell}^2,$$
$$|||f|||_{\nu,\ell}^2(t) \equiv \sum_{|\alpha|+|\beta| \leq N} ||\partial_\beta^\alpha f(t)||_{\nu,|\beta|-\ell}^2.$$

Since $\gamma < 0$, these norms include higher powers of v for larger $\ell \geq 0$. We denote the perturbation f such that $F(t,x,v) = \mu(v) + \sqrt{\mu(v)} f(t,x,v)$. Then the initial value problem can be written as

$$[\partial_t + v \cdot \nabla_x] f + Lf = \Gamma[f,f], \tag{3.12}$$

where L is the linear part of the collision operator Q and Γ is the non-linear part. Define the *instant energy* as $\mathcal{E}_\ell(f(t)) \equiv \frac{1}{2}|||f|||_\ell^2(t)$, with its initial counterpart $\mathcal{E}_\ell(0) \equiv \frac{1}{2}|||f_0|||_\ell^2$. At $t = 0$ we define the temporal derivatives naturally through (3.12).

THEOREM 3.3. *Fix $-3 < \gamma < 0$ and fix integers $N \geq 8$, $\ell \geq 0, k \geq 0$. Choose initial data $F_0(x,v)$ which satisfies the assumptions of Theorem 1 in* [37]. *Furthermore, there are constants $C_\ell > 0$ and $\epsilon_\ell > 0$, such that if $\mathcal{E}_\ell(0) < \epsilon_\ell$ then there exists a unique global solution with $F(t,x,v) = \mu + \mu^{1/2} f(t,x,v) \geq 0$, where equivalently f satisfies (3.12) and*

$$\sup_{0 \leq s \leq \infty} \mathcal{E}_\ell(f(s)) \leq C_\ell \mathcal{E}_\ell(0). \tag{3.13}$$

Moreover if, $\mathcal{E}_{\ell+k}(0) < \epsilon_{\ell+k}$, then the unique global solution to the Boltzmann equation (3.12) satisfies

$$|||f|||_\ell^2(t) \leq C_{\ell,k} \mathcal{E}_{\ell+k}(0) \left(1 + \frac{t}{k}\right)^{-k}.$$

REMARK 3.1. *The existence part of both this theorem and the next were proven for $\ell = 0$ and with no temporal derivatives in* [37] *and* [38]. *But the method was improved in* [39]. *Since the proof is very similar to those in* [37–39, 62], *we will only sketch it.*

PROOF. *1. BASIC ESTIMATES:* Our first goal is to show all the basic estimates in [37] are valid, with little changes in the proofs, for new norms with (harmless) temporal derivatives and an additional powers of the weight factor w. In what follows, we shall only pin point these minor chanegs. Lemma 1 in [37] holds for any $\theta < 0$ with the same proof except we replace Eq. (19) by

$$w^{2\theta}(v) \leq w^{2\theta}(|v| + |u|) \leq C w^{2\theta}(|v'| + |u'|) \leq C w^{2\theta}(v') w^{2\theta}(u').$$

One factor on the r.h.s. is then controlled by either $\sqrt{\mu(v')}$ or $\sqrt{\mu(u')}$ in Eq. (18) of [37]. Lemma 2 in [37] holds for $\theta < 0$ as well and the only difference is in Eqs. (48)-(50) of [37] we should use $w^{2\theta}(v) \leq C w^{2\theta}(v + u_{||}) w^{2\theta}(u_{||})$ instead of $w^{2\theta}(v) \leq 1$ (which is used for the case $\theta \geq 0$). Not only Lemma 4 in [37] is valid for $\theta < 0$ with the same proof, it is also valid for the case $\beta = 0$ as well. In fact, for $\theta < 0$, by Lemma 2 in [37], we split $K = K_c + K_s$ where $\left|\left(w^{2\theta} K_s \partial^\alpha f, \partial^\alpha f\right)\right| \leq \frac{1}{2} \|\partial^\alpha f\|_{\nu,\theta}^2$ by a trivial extension of Eq. (21) in [37] to the case $\theta < 0$. Finally, using the definition of K_c in Eqs. (24) and (40) as well as the Cauchy-Schwartz inequality we have $\left|\left(w^{-2\theta} K_c \partial^\alpha f, \partial^\alpha f\right)\right| \leq C \|\partial^\alpha f\|_\nu^2$ for some constant $C > 0$ which completes the case for $\beta = 0$ in Lemma 4 in [37]. Furthermore, by splitting $L = \nu + K$, we deduce that for some $C > 0$ and $\ell \geq 0$,

$$\left(w^{-2\ell} L\partial^\alpha f, \partial^\alpha f\right) \geq \frac{1}{2}\|\partial^\alpha f\|^2_{\nu,-\ell} - C\|\partial^\alpha f\|^2_\nu. \tag{3.14}$$

Theorem 3 in [37] is valid for any weight function $w^{2\theta-2\ell}$ instead of $w^{2\theta}$ with the same proof. Now Lemma 7 and the local well-posedness Theorem 4 in [37] are valid with the new norms with the same proof.

2. POSITIVITY OF L : Instead of Theorem 2 in [37], we show the following instantaneous positivity for the linearized Boltzmann operator L**:** Fix $\ell \geq 0$. Let $f(t, x, v)$ be a (local) classical solution to the Boltzmann equation with a soft potential with the new norm. There exists M_0 and $\delta_0 = \delta_0(M_0) > 0$ such that if $\mathcal{E}_\ell(f(t)) \leq M_0$, then

$$\sum_{|\alpha|\leq N} \langle L\partial^\alpha f(t), \partial^\alpha f(t)\rangle \geq \delta_0 \sum_{|\alpha|\leq N} \|\partial^\alpha f(t)\|^2_\nu. \tag{3.15}$$

The proof of (3.15) follows exactly as in section 4 in [39] in a much simplier fashion with *no* electromagnetic fields everywhere. Both Lemma 5 and Lemma 6 in [39] holds with $E = B = 0$. Lemma 7 (with $J = 0$) in [39] holds with new L for soft potentials and with $||\partial^\gamma f||$ replaced by $||\partial^\gamma f||_\nu$, thanks to Lemma 1 in [37]. Lemma 8 (with $E = B = 0$) in [39] is valid for new Γ for the soft potentials with $||\partial^\gamma f||$ replaced by $||\partial^\gamma f||_\nu$, thanks to Lemma 6 in [37]. The rest of the proof is the same (much simpler!) as the proof of Theorem 3 starting at p.620 in [39].

Remark. We remark that the assumption $\mathcal{E}_\ell(f(t)) \leq M_0$ in this section can be reduced to not include any velocity derivatives. However this requires a slight improvement of estimates found in [37]. For instance, one can prove that for $|\alpha_1| + |\alpha_2| \leq N$ and $\chi_1(v)$ a smooth function such that $\sum_{|\beta|\leq 2} |\partial_\beta \chi_1| \leq C\mu^{1/4}(v)$, then

$$\|\langle \Gamma[\partial^{\alpha_1} g, \partial^{\alpha_2} g]\chi_1\rangle\| \leq C \sum_{|\alpha|\leq N} ||\partial^\alpha g||_\nu \sum_{|\alpha|\leq N} ||\partial^\alpha g||.$$

Further, if $|\alpha| \leq N$, $\|\langle L[\partial^\alpha g], \chi_1\rangle\| \leq C||\partial^\alpha g||$.

3. ENERGY ESTIMATE. Fix an integer $\ell \geq 0$. Let f be the unique local solution which satisfies the small amplitude assumption $\mathcal{E}_\ell(f(t)) \leq M_0$. We now show that for any given $0 \leq m \leq N$, $|\beta| \leq m$, there are constants $C_{|\beta|,\bar{\ell}} > 0$, $C^*_{m,\ell} > 0$ and $\delta_{m,\ell} > 0$ such that

$$\sum_{|\beta|\leq m, 0\leq \bar{\ell}\leq \ell, |\alpha|+|\beta|\leq N} \left(\frac{1}{2}\frac{d}{dt} C_{|\beta|,\bar{\ell}} ||\partial^\alpha_\beta f(t)||^2_{|\beta|-\bar{\ell}} + \delta_{m,\ell} ||\partial^\alpha_\beta f(t)||^2_{\nu,|\beta|-\bar{\ell}} \right)$$

$$\leq C^*_{m,\ell} \sqrt{\mathcal{E}_\ell(t)} |||f|||^2_{\nu,\ell}(t). \tag{3.16}$$

We derive (3.16) in three steps.

We first consider pure spatial and temporal derivatives. Taking pure ∂^α derivatives of (3.12) we obtain

$$[\partial_t + v \cdot \nabla_x]\partial^\alpha f + L\partial^\alpha f = \partial^\alpha \Gamma[f, f]. \tag{3.17}$$

Take the inner product of this with $\partial^\alpha f$. We apply (3.15) and Theorem 3 in [37] with the new norms to obtain for some constant $C > 0$,

$$\sum_{|\gamma|\leq N} \left(\frac{1}{2}\frac{d}{dt} ||\partial^\alpha f(t)||^2 + \delta_0 ||\partial^\alpha f(t)||^2_\nu \right) \leq C\sqrt{\mathcal{E}_0(t)} |||f|||^2_\nu(t).$$

Next, we consider pure spatial and temporal derivatives with weights. We prove the result for $\ell > 0$ by a simple induction starting at $\ell = 0$, assume (3.16) is valid for $0 \leq \ell' < \ell$. Take the inner product of (3.17) with $w^{-2\ell}\partial^\alpha f$. Then for some constant $C > 0$, we obtain the bound

$$\sum_{|\alpha|\leq N} \left(\frac{1}{2}\frac{d}{dt} ||\partial^\alpha f(t)||^2_{-\ell} + \left(w^{-2\ell} L\partial^\alpha f, \partial^\alpha f \right) \right) \leq C\sqrt{\mathcal{E}_\ell(t)} |||f|||^2_{\nu,\ell}(t).$$

Plug in the lower bound (3.14) and add the cases $0 \leq \ell' < \ell$ multiplied by a large constant to establish (3.16) for $|\beta| = 0$ and $\ell \geq 0$.

Finally we prove the general case by an induction over the order of v derivatives. Assume (3.16) is valid for $|\beta| = m$. For $|\beta| = m + 1$, we take ∂^α_β of (3.12) to obtain

$$[\partial_t + v \cdot \nabla_x]\partial^\alpha_\beta f + \partial_\beta L\partial^\alpha f = -\sum_{|\beta_1|=1} \binom{\beta}{\beta_1} \partial_{\beta_1} v \cdot \nabla_x \partial^\alpha_{\beta-\beta_1} f + \partial^\alpha_\beta \Gamma[f, f]. \tag{3.18}$$

Take the inner product of this with $w^{2|\beta|-2\ell}\partial^\alpha f$. Using Lemma 4 of [37] with $\theta = 2|\beta|-2\ell$ we get

$$\left(w^{2|\beta|-2\ell}\partial_\beta\{L\partial^\alpha f\}, \partial^\alpha_\beta f\right) \geq ||\partial^\alpha_\beta f||^2_{\nu,|\beta|-\ell} - \eta \sum_{|\bar{\beta}|\leq|\beta|} ||\partial^\alpha_{\bar{\beta}} f||^2_{\nu,|\bar{\beta}|-\ell} - C_\eta ||\partial^\alpha f||^2_{\nu,-\ell}.$$

Using Cauchy's inequality for $\eta > 0$, since $|\beta_1| = 1$ we have

$$\left|\left(w^{2|\beta|-2\ell}\partial_{\beta_1} v \cdot \nabla_x \partial^\alpha_{\beta-\beta_1} f, \partial^\alpha_\beta f\right)\right| \leq \eta ||\partial^\alpha_\beta f(t)||^2_{\nu,|\beta|-\ell} + C_\eta ||\nabla_x \partial^\alpha_{\beta-\beta_1} f||^2_{\nu,|\beta-\beta_1|-\ell},$$

which is found on p.336 of [37]. We use these two inequalities to conclude (3.16) as well as global existence and (3.13) exactly as in section 5 in [39, 62].

5. DECAY RATE: We let $m = N$ in (3.16) and define

$$y_\ell(t) \equiv \sum_{0\leq\bar{\ell}\leq\ell, |\alpha|+|\beta|\leq N} C_{|\beta|,\bar{\ell}} ||\partial^\alpha_\beta f(t)||^2_{|\beta|-\bar{\ell}}.$$

Notice that $y_\ell(t)$ is equivalent to $|||f|||^2_\ell(t)$. By (3.13) for sufficiently small $\mathcal{E}_\ell(0)$, there exists $C > 0$ such that

$$\frac{d}{dt} y_\ell(t) + C|||f|||^2_{\nu,\ell}(t) \leq 0.$$

Clearly for fixed ℓ, the dissipation rate $|||f|||^2_{\nu,\ell}$ is weaker than the instant energy $y_\ell(t)$ since $\|f\|_{1/2}$ is equivalent to $\|f\|_\nu$. We shall interpolate with the stronger norm $y_{\ell+k}(t)$. Fix $\ell, k \geq 0$. A simple intepolation for $w^{-2\ell}$ between the weight functions $w^{-2\ell+2}$ and $w^{-2\ell-2k}$ yields

$$|||f|||_\ell \leq |||f|||^{k/(k+1)}_{\ell-1} |||f|||^{1/(k+1)}_{\ell+k} \leq C|||f|||^{k/(k+1)}_{\ell-1} \times \left(C_{\ell+k}\mathcal{E}_{\ell+k}(0)\right)^{1/2(k+1)}.$$

But from the definition of $w(v)$ and the fact $\nu(v) \leq Cw(v)$ we have

$$|||f|||^2_{\ell-1/2} \leq C|||f|||^2_{\nu,\ell}.$$

We thus have

$$y_\ell \leq C|||f|||^2_\ell \leq C\mathcal{E}^{1/(k+1)}_{\ell+k}(0)|||f|||^{2k/(k+1)}_{\ell-1} \leq C\mathcal{E}^{1/(k+1)}_{\ell+k}(0)|||f|||^{2k/k+1}_{\nu,\ell},$$

and for some $C_{\ell,k} > 0$,

$$\frac{d}{dt} y_\ell + C_{\ell,k}\mathcal{E}^{-1/k}_{\ell+k}(0) y_\ell^{\{k+1\}/k} \leq 0.$$

Now the theorem follows from the analysis of (3.9). $\square$

3.5. The Classical Landau Equation

The Landau equation has the same form as the Boltzmann equation except

$$Q(F,F) = \nabla_v \cdot \left\{ \int_{\mathbb{R}^3} \phi(v-v')[F(v')\nabla_v F(v) - F(v)\nabla_{v'}F(v')]dv' \right\},$$

where $\phi^{ij}(v) = \frac{1}{|v|}\left\{\delta_{ij} - \frac{v_i v_j}{|v|^2}\right\}$. Throught out this section we use the weight function

$$w_1(v) \equiv \{1+|v|^2\}^{-1/2}.$$

In the classical case, we define $\sigma^{ij}(v) \equiv \int \phi^{ij}(v-v')\mu(v')dv'$ and

$$||g||_{\sigma,\theta}^2 \equiv \sum_{i,j} \int_{\mathbb{T}^3\times\mathbb{R}^3} w^{2\theta}\sigma^{ij}\{\partial_{v_i} g \partial_{v_j} g + v_i v_j g^2\}dxdv.$$

We again define ∂_β^α including temporal derivativess. Fix $N \geq 4$ and $\ell \geq 0$ and define $|||f|||_\ell^2(t) \equiv \sum_{|\alpha|+|\beta|\leq N} ||w^{2(|\beta|-\ell)}\partial_\beta^\alpha f(t)||^2$. The Landau dissipation rate is

$$|||f|||_{\sigma,\ell}^2(t) \equiv \sum_{|\alpha|+|\beta|\leq N} ||\partial_\beta^\alpha f(t)||_{\sigma,|\beta|-\ell}^2,$$

The standard perturbation $f(t,x,v)$ to μ is $F(t,x,v) = \mu(v) + \sqrt{\mu(v)} f(t,x,v)$. The instant energy is $\mathcal{E}_\ell(f(t)) \equiv \frac{1}{2}|||f|||_\ell^2(t)$ with its initial counterpart $\mathcal{E}_\ell(0) \equiv \frac{1}{2}|||f_0|||_\ell^2$.

Theorem 3.4. *Fix integers $N \geq 8$ and $\ell \geq 0$. Assume that $F_0(x,v) = \mu + \mu^{1/2} f_0(x,v)$ satisfies the assumptions of Theorem 1 in* [38]. *There are constants $C_\ell > 0$ and $\epsilon_\ell > 0$ such that if $\mathcal{E}_\ell(f_0) < \epsilon_\ell$ then there exists a unique global solution $F(t,x,v)$ to the classical Landau equation such that $F(t,x,v) = \mu + \mu^{1/2} f(t,x,v) \geq 0$, and*

$$\sup_{0\leq s\leq\infty} \mathcal{E}_\ell(f(s)) \leq C_\ell \mathcal{E}_\ell(0).$$

Furthermore if $\mathcal{E}_{\ell+k}(f_0) < \epsilon_{\ell+k}$ for some integer $k > 0$ then the unique global solution $f(t,x,v)$ satisfies

$$|||f|||_\ell^2(t) \leq C_{\ell,k}\mathcal{E}_{\ell+k}(0)\left(1+\frac{t}{k}\right)^{-k}.$$

Proof. We follow the same procedure as in the case for soft potentials. First, we check that basic estimates in [38] are valid with little changes in the proofs for new norms with (harmless) temporal derivatives as well as new weights $w = w_1^{-2\ell}$. We first notice that Lemma 6 in [38] is valid for $|\beta| = 0$ as well. The result for K is Lemma 5 in [38]. Therefore we focus on A. Lemma 3 in [38] implies

$$|\partial_{v_j}\{\partial_{v_i} w_1^{-2\ell}\sigma^{ij}\}| + |\partial_{v_j}\{w_1^{-2\ell}\partial_{v_i}\sigma^{ij}\}| \leq C w_1^{2\ell}[1+|v|]^{-2}.$$

Now both in second and the fourth term in Eq. (31) in [38] with $\beta = 0$, we write $\partial_j f f = \frac{1}{2}\partial_j f^2$ and integate by parts to get a upper bound of

$$C\sum_{i,j}\int_{\mathbb{T}^3\times\mathbb{R}^3} w_1^{-2\ell}[1+|v|]^{-2}\,|f|^2\,dxdv \tag{3.19}$$

Since $||f||_{\sigma,-\ell}$ dominates $||f||_{-\ell+1/2}$, (3.19) is therefore bounded by $\eta||f||_{\sigma,-\ell}$ for large v for any small number η. This concludes the proof of Lemma 6 in [38] for the new norm. All the other Lemmas in sections 1-3 are valid for the new norms with identical proofs, which lead to local well-posedness Theorem 4 in [38] for the new norm.

Now we follow exactly the proof for the soft potential, simply replacing $|||\cdot|||_\nu$ by the corresponding Landau dissipation rate $|||\cdot|||_\sigma$ and the linearized Boltzmann operator by the linearized Landau operator. To establish the positivity (3.15) with Landau dissipation, we notice that $Lg = -\Gamma(g,\sqrt{\mu}) - \Gamma(\sqrt{\mu}, g)$. By Lemma 7 in [38] for the new norms, we deduce that both Lemma 7 and Lemma 8 in [39] for Landau dissipation rate $||\partial^\gamma f||_\sigma^2$, instead of $||\partial^\gamma f||^2$. The rest of the proof is identical. $\square$

CHAPTER 4

Exponential Decay for Soft Potentials near Maxwellian

Abstract. Consider both soft potentials with angular cutoff and Landau collision kernels in the Boltzmann theory inside a periodic box. We prove that any smooth perturbation near a given Maxwellian approaches to zero at the rate of $e^{-\lambda t^p}$ for some $\lambda > 0$ and $0 < p < 1$. Our method is based on an unified energy estimate with appropriate exponential velocity weight. Our results extend the classical Caflisch result [9] to the case of very soft potential and Coulomb interactions, and also improve the recent "almost exponential" decay results by [21, 60]. A version of this result will appear as [61].

4.1. Introduction

4.2. Introduction

In this article, we are concerned with soft potentials and Landau collision kernels in the Boltzmann theory for dynamics of dilute particles in a periodic box. Recall the Boltzmann equation as

$$\partial_t F + v \cdot \nabla_x F = Q(F, F), \; F(0, x, v) = F_0(x, v), \tag{4.1}$$

where $F(t, x, v)$ is the spatially periodic distribution function for the particles at time $t \geq 0$, position $x = (x_1, x_2, x_3) \in \mathbb{T}^3$ and velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. The l.h.s. of this equation models the transport of particles and the operator on the r.h.s. models the effect of collisions on the transport:

$$Q(F, G) \equiv \int_{\mathbb{R}^3 \times S^2} |u - v|^\gamma B(\theta)\{F(u')G(v') - F(u)G(v)\} du d\omega.$$

Here $F(u) = F(t, x, u)$ etc. The exponent is $\gamma = 1 - \frac{4}{s}$ with $1 < s < 4$; we assume

$$-3 < \gamma < 0,$$

(soft potentials) and $B(\theta)$ satisfies the Grad angular cutoff assumption:

$$0 < B(\theta) \leq C|\cos\theta|. \tag{4.2}$$

Moreover, the post-collisional velocities satisfy

$$v' = v + [(u - v) \cdot \omega]\omega, \qquad u' = u - [(u - v) \cdot \omega]\omega, \tag{4.3}$$

$$|u|^2 + |v|^2 = |u'|^2 + |v'|^2, \tag{4.4}$$

And θ is defined by $\cos\theta = [u - v] \cdot \omega/|u - v|$.

On the other hand, the Landau equation is formally obtained in a singular limit of the Boltzmann equation. It can also be written as (4.1) but the collision operator is given by

$$\begin{aligned} Q(F, G) &= \nabla_v \cdot \left\{ \int_{\mathbb{R}^3} \phi(v - u)[F(u)\nabla_v G(v) - G(v)\nabla_u F(u)]du \right\} \\ &= \sum_{i,j=1}^{3} \partial_i \int_{\mathbb{R}^3} \phi^{ij}(v - u)[F(u)\partial_j G(v) - G(v)\partial_j F(u)]du, \end{aligned}$$

where $\partial_i = \partial_{v_i}$ etc. The non-negative matrix ϕ is given by

$$\phi^{ij}(v) = \left\{ \delta_{ij} - \frac{v_i v_j}{|v|^2} \right\} |v|^{2+\gamma}.$$

We assume soft potentials, which means $-3 \leq \gamma < -2$. The original Landau collision operator with Coulombic interactions corresponds to $\gamma = -3$.

Denote the steady state Maxwellian by

$$\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}.$$

We perturb around the Maxwellian as

$$F(t, x, v) = \mu(v) + \sqrt{\mu(v)} f(t, x, v).$$

Then the initial value problem (4.1) can be rewritten as

$$[\partial_t + v \cdot \nabla_x]f + Lf = \Gamma[f, f], \;\; f(0, x, v) = f_0(x, v), \tag{4.5}$$

where L is the linear part of the collision operator, Q, and Γ is the non-linear part.

For the Boltzmann equation, the standard linear operator [30] is

$$Lg = \nu(v)g - Kg, \tag{4.6}$$

where the collision frequency is

$$\nu(v) = \int B(\theta)|v-u|^{\gamma}\mu(u)dud\omega. \tag{4.7}$$

The operators K and Γ, in the Boltzmann case, are defined in (4.17) and (4.46).

For the Landau equation, the linear operator [38] is

$$Lg = -Ag - Kg, \tag{4.8}$$

with A, K and Γ defined in (4.44), (4.45) and (4.46). The Landau collision frequency is

$$\sigma^{ij}(v) = \phi^{ij} * \mu = \int_{\mathbb{R}^3} \phi^{ij}(v-u)\mu(u)du. \tag{4.9}$$

We remark that $\sigma^{ij}(v)$ is a positive self-adjoint matrix [16].

Notation: Let $\langle\cdot,\cdot\rangle$ denote the standard $L^2(\mathbb{R}^3)$ inner product. We also use $(\cdot,\cdot)$ to denote the standard $L^2(\mathbb{T}^3\times\mathbb{R}^3)$ inner product with corresponding L^2 norm $\|\cdot\|$. Define a weight function in v by

$$w = w(\ell,\vartheta)(v) \equiv (1+|v|^2)^{\tau\ell/2}\exp\left(\frac{q}{4}(1+|v|^2)^{\frac{\vartheta}{2}}\right). \tag{4.10}$$

Here $\tau < 0$, $\ell \in \mathbb{R}$, $0 < q$ and $0 \le \vartheta \le 2$. Denote weighted L^2 norms as

$$|g|^2_{\ell,\vartheta} \equiv \int_{\mathbb{R}^3} w^2(\ell,\vartheta)|g|^2dv, \;\; ||g||^2_{\ell,\vartheta} \equiv \int_{\mathbb{T}^3} |g|^2_{\ell,\vartheta}dx.$$

For the Boltzmann equation, define the weighted dissipation norm as

$$|g|^2_{\boldsymbol{\nu},\ell,\vartheta} \equiv \int_{\mathbb{R}^3} w^2(\ell,\vartheta)\nu(v)|g(v)|^2dv, \tag{4.11}$$
$$||g||^2_{\boldsymbol{\nu},\ell,\vartheta} \equiv \int_{\mathbb{T}^3} |g|^2_{\boldsymbol{\nu},\ell,\vartheta}dx.$$

For the Landau equation, define the weighted dissipation norm as

$$
|g|^2_{\boldsymbol{\sigma},\ell,\vartheta} \equiv \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} w^2(\ell,\vartheta) \left\{ \sigma^{ij}\partial_i g \partial_j g + \sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}|g|^2 \right\} dv, \tag{4.12}
$$

$$
||g||^2_{\boldsymbol{\sigma},\ell,\vartheta} \equiv \int_{\mathbb{T}^3} |g|^2_{\boldsymbol{\sigma},\ell,\vartheta} dx.
$$

Since our proof of decay does not depend upon any properties which are specific to either dissipation norm, sometimes we unify the notation as $||g||_{\boldsymbol{D},\ell,\vartheta}$, which denotes either $||g||_{\boldsymbol{\nu},\ell,\vartheta}$ or $||g||_{\boldsymbol{\sigma},\ell,\vartheta}$. If $\vartheta = 0$ then we drop the index, e.g. $||g||_{\boldsymbol{D},\ell,0} = ||g||_{\boldsymbol{D},\ell}$ and the same for the other norms.

Next define a high order derivative

$$
\partial^{\alpha}_{\beta} \equiv \partial^{\alpha^0}_{t}\partial^{\alpha^1}_{x_1}\partial^{\alpha^2}_{x_2}\partial^{\alpha^3}_{x_3}\partial^{\beta^1}_{v_1}\partial^{\beta^2}_{v_2}\partial^{\beta^3}_{v_3}
$$

where $\alpha = [\alpha^0, \alpha^1, \alpha^2, \alpha^3]$ is the multi-index related to the space-time derivative and $\beta = [\beta^1, \beta^2, \beta^3]$ is the multi-index related to the velocity derivatives. If each component of β is not greater than that of β_1's, we denote by $\beta \leq \beta_1$; $\beta < \beta_1$ means $\beta \leq \beta_1$ and $|\beta| < |\beta_1|$. We also denote $\begin{pmatrix} \beta \\ \beta_1 \end{pmatrix}$ by $C^{\beta_1}_{\beta}$.

Fix $N \geq 8$ and $l \geq 0$. An "**Instant Energy functional**" satisfies

$$
\frac{1}{C}\mathcal{E}_{l,\vartheta}(g)(t) \leq \sum_{|\alpha|+|\beta|\leq N} ||\partial^{\alpha}_{\beta} g(t)||^2_{|\beta|-l,\vartheta} \leq C\mathcal{E}_{l,\vartheta}(g)(t).
$$

If g is independent of $t \geq 0$, then the temporal derivatives are defined through equation (4.5). Further, the "**Dissipation Rate**" is given by

$$
\mathcal{D}_{l,\vartheta}(g)(t) \equiv \sum_{|\alpha|+|\beta|\leq N} ||\partial^{\alpha}_{\beta} g(t)||^2_{\boldsymbol{D},|\beta|-l,\vartheta}.
$$

We will also write $\mathcal{E}_{l,0}(g)(t) = \mathcal{E}_l(g)(t)$ and $\mathcal{D}_{l,0}(g)(t) = \mathcal{D}_l(g)(t)$. We note from (4.10) that for $l > 0$ these norms contain a polynomial factor $(1 + |v|^2)^{-\tau l/2}$. The weight factor $(1 + |v|^2)^{\tau|\beta|/2}$ (dependant on the number of velocity derivatives) is designed to control the streaming term $v \cdot \nabla_x f$.

If initially $F_0(x,v) = \mu(v) + \sqrt{\mu(v)} f_0(x,v)$ has the same mass, momentum and total energy as the Maxwellian μ, then formally for any $t \geq 0$ we have

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} f(t)\mu^{1/2} = \int_{\mathbb{T}^3\times\mathbb{R}^3} v_i f(t)\mu^{1/2} = \int_{\mathbb{T}^3\times\mathbb{R}^3} |v|^2 f(t)\mu^{1/2} = 0. \tag{4.13}$$

We are now ready to state the main result.

THEOREM 4.1. *Let $N \geq 8$, $l \geq 0$, $0 \leq \vartheta \leq 2$ and $0 < q$. If $\vartheta = 2$ then further assume $q < 1$. Choose initial data $F_0(x,v) = \mu(v) + \sqrt{\mu(v)} f_0(x,v)$ such that $f_0(x,v)$ satisfies (4.13). In* (4.10)*, for the Boltzmann case assume $\tau \leq \gamma$ and for the Landau case assume $\tau \leq -1$.*

Then there exists an instant energy functional $\mathcal{E}_{l,\vartheta}(f)(t)$ such that if $\mathcal{E}_{l,\vartheta}(f_0)$ is sufficiently small, then the unique global solution to (4.1) in both the Boltzmann case and the Landau case satisfies

$$\frac{d}{dt}\mathcal{E}_{l,\vartheta}(f)(t) + \mathcal{D}_{l,\vartheta}(f)(t) \leq 0. \tag{4.14}$$

In particular,

$$\sup_{0\leq s\leq\infty} \mathcal{E}_{l,\vartheta}(f)(s) \leq \mathcal{E}_{l,\vartheta}(f_0). \tag{4.15}$$

Moreover, if $\vartheta > 0$, then there exists $\lambda > 0$ such that

$$\mathcal{E}_l(f)(t) \leq Ce^{-\lambda t^p}\mathcal{E}_{l,\vartheta}(f_0),$$

where in the Boltzmann case

$$p = p(\vartheta,\gamma) = \frac{\vartheta}{\vartheta - \gamma},$$

and in the Landau case

$$p = p(\vartheta,\gamma) = \frac{\vartheta}{\vartheta - (2+\gamma)}.$$

In the second authors papers [37, 38], (4.14) was established with $\tau = \gamma$ in the Boltzmann case, $\tau = 2 + \gamma$ in the Landau case and $\vartheta = l = 0$. We extended (4.14) to the case $l \geq 0$ in [60]. There we used (4.14) and (4.15) to establish the following theorem via direct interpolation for $\vartheta = 0$.

THEOREM 4.2. *Assume everything from Theorem 4.1 and fix $k > 0$. In addition, if $\mathcal{E}_{l+k,\vartheta}(f_0)$ is sufficiently small then*

$$\mathcal{E}_{l,\vartheta}(f)(t) \leq C_{l,k} \left(1 + \frac{t}{k}\right)^{-k} \mathcal{E}_{l+k,\vartheta}(f_0).$$

For $\vartheta > 0$, the proof of Theorem 4.2 is exactly the same as in [60]. Also in [60], for the Landau case we assumed Coulomb interactions ($\gamma = -3$). Still the proof of Theorem 4.2 in the Landau case for $\tau \leq -1$ and $-3 \leq \gamma < -2$ is exactly the same. However, we remark that for $\tau < 2 + \gamma$ the interpolations used to prove Theorem 4.2 are not optimal. The polynomial decay rate can be improved as τ grows smaller than γ by using tighter interpolations. But this is somewhat superficial because as τ grows smaller we are implicitly using a larger weight in our norms.

The main difficulty in proving any kind of decay for soft potentials is caused by the lack of a spectral gap for the linear operator (4.6) and (4.8). In the Boltzmann case, the dominant part of the linear operator (4.6) is of the form

$$\frac{1}{C}(1 + |v|^2)^{\gamma/2} \leq \nu(v) \leq C(1 + |v|^2)^{\gamma/2}, \quad C > 0. \tag{4.16}$$

From another point of view, at high velocities the dissipation is much weaker than the instant energy. However, Theorem 4.1 and Theorem 4.2 show that given explicit control over $f(t, x, v)$ at high velocities, no matter how weak, we can obtain a precise decay rate. On the other hand, we believe it very unlikely that existence of solutions can be established in this setting with a weight bigger than (4.10) with $\vartheta = 2$. From this point of view, Theorem 4.1 and Theorem 4.2 together form a rather satisfactory theory of convergence rates to Maxwellian for soft potentials and Landau operators, in a close to equilibrium context.

The constants in our estimates are certainly not optimal or explicit in all cases. However, $p = \frac{\vartheta}{\vartheta - \gamma}$ comes from the following simplification of the Boltzmann equation [8]:

$$\partial_t f(t, |v|) + |v|^\gamma f(t, |v|) = 0, \quad -3 < \gamma < 0, \quad |v| > 0.$$

Consider initial data with rapid decay as required by our norms

$$f(0,|v|) = e^{-c|v|^{\vartheta}},\ c>0,\ 0<\vartheta\leq 2.$$

Then the solution to this system is exactly

$$f(t,|v|) = e^{-c|v|^{\vartheta}-t|v|^{\gamma}}$$

By splitting into $\{|v|\geq t^{p/\vartheta}\}$ and $\{|v|<t^{p/\vartheta}\}$ we have

$$c_0 e^{-c_1 t^p} \leq \int_{|v|>0} |f(t,|v|)|^2 d|v| \leq c_2 e^{-c_3 t^p},$$

with $c_i>0$ $(i=0,1,2,3)$.

The study of trend to Maxwellians is important in kinetic theory both from physical and mathematical standpoints. In a periodic box, it was Ukai [65] who obtained exponential convergence (with $p=1$), and hence constructed the first global in time solutions in the spatially inhomogeneous Boltzmann theory. He treated the case of a cutoff hard potential. In 1980, Caflisch [8, 9] established exponential decay (with the same $p(2,\gamma)$) as well as global in time solutions for the Boltzmann equation with potentials which are not too soft $(-1<\gamma<0)$. About the same time, in the whole space setting, also for cutoff soft potentials with $\gamma>-1$, Ukai and Asano [66] obtained the rate $O(t^{-\alpha})$ with $0<\alpha<1$; their optimal case in $\mathbb{R}^3$ yields $\alpha=3/4$. In these early investigations, a sufficiently fast time decay of the linearized Boltzmann equation around a Maxwellian played the crucial role in bootstrapping to the full nonlinear dynamics. For the soft potentials, such linear decay estimates can be very difficult and delicate. It thus has been a longstanding open problem to study the decay property as well as to construct global in time smooth solutions for very soft potentials with γ near -3.

Recently, a nonlinear energy method for constructing global solutions was developed by the second author to avoid using the linear decay. Indeed, by showing the linearized collision operator was always positive definite along the full nonlinear dynamics, global in time smooth solutions near Maxwellians were constructed for all cutoff soft potentials of $-3<\gamma<0$ [37], even for the Landau equation with

Coulomb interaction [38]. However, the time decay of such solutions was left open. See [36, 39, 40, 62] for more applications of such a method.

From a completely different approach, Desvillettes and Villani [21] have recently developed a framework to study the trend to Maxwellians for *general* smooth solutions, not necessarily near any Maxwellian. As an application, their method leads to the almost exponential decay rates (i.e., faster than any given polynomial) for smooth solutions constructed earlier by the second author for all cutoff soft potentials and the Landau equation.

Inspired by such a striking result, in [60], we re-examined and improved the energy method to give a more direct proof of such almost exponential decay in the close to Maxwellian setting. We introduced a family of polynomial velocity weight functions and used some simple interpolation techniques. It is interesting to note that our decay estimate is a consequence of the weighted energy estimate for the global solution, not the other way around as in earlier methods [8, 9, 65, 66]. And it is clear from our analysis that a stronger velocity weight yields faster time decay.

It is thus very natural to try to use exponential velocity weight functions to get exponential time decay, which is the main purpose of our current investigation. The key is to show that the new energy with an exponential velocity weight is bounded for all time. In order to carry out such an energy estimate, we follow the general framework and strategy in [37], [38], [60]. However, many new analytical difficulties arise and we have to develop new techniques accordingly. The main difficulty lies in the estimates for linearized collision operators around a Maxwellians. The presence of the exponential weight factor $\exp\{\frac{q}{4}(1+|v|^2)^{\vartheta/2}\}$ in (4.10) requires much more precise estimates at almost all levels. In the case of a cutoff soft potential, a careful application of the Caflisch estimate (Lemma 4.1) is combined with the splitting trick in [37] to treat the very soft potential of $-3<\gamma\leq -1$. Furthermore, we take a close look at the variables (4.21) in the Hilbert-Schmidt form for K to estimate the trickiest terms in Lemma 4.2. On the other hand, in the Landau case, an extra v factor from the derivative of the weight $\exp\{\frac{q}{4}(1+|v|^2)^{\vartheta/2}\}$ creates the most challenging difficulty to close the estimate in the same norm. We have to use different weight functions, that appeared in the norm (4.12), very precisely to balance between the derivative part $\sigma^{ij}\partial_i g\partial_j g$ and the no derivative part $\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}|g|^2$ in Lemma 4.8. Thus $\tau \leq -1$ is assumed. Moreover, in Lemma 4.9, we have to introduce a new splitting of the linear Landau operator in the $\vartheta = 2$ case where $q < 1$ is crucially used. Although we have quoted some estimates in previous papers, we have made an effort to provide complete, self-consistent proofs throughout the article. We also have made more comments between the proofs to help the readers.

The paper is organized as follows. In Section 4.3, we establish the estimates with the exponential weight (4.10) for the Boltzmann equation. In Section 4.4, we establish estimates with weight (4.10) for the Landau equation. Finally, in Section 4.5 we establish the crucial energy estimate uniformly for both cases. Some details are exactly the same as in [37], [38] or [60]. We will sketch these details which can be found elsewhere. Finally, we prove exponential decay in Section 4.6 using the global bound (4.15).

4.3. Boltzmann Estimates

In this section, we will prove the basic estimates used to obtain global existence of solutions with an exponential weight in the Boltzmann case. These estimates are similar to those in [37], but here the exponential weight which was not present earlier forces us to modify the proofs and some of the estimates. We will use the classical soft potential estimate of Caflisch [8] (Lemma 4.1) with v derivatives to estimate the linear operator (Lemma 4.2). We discuss the new features of each proof after the statement of each Lemma.

Recall K and Γ from (4.6) and (4.5). $K = K_2 - K_1$ is defined as [30, 34]:

$$[K_1 g](v) = \int_{\mathbb{R}^3\times S^2} B(\theta)|u-v|^{\gamma}\mu^{1/2}(u)\mu^{1/2}(v)g(u)dud\omega, \tag{4.17}$$

$$\begin{aligned}[K_2 g](v) = &\int_{\mathbb{R}^3\times S^2} B(\theta)|u-v|^{\gamma}\mu^{1/2}(u)\mu^{1/2}(u')g(v')dud\omega \\ &+\int_{\mathbb{R}^3\times S^2} B(\theta)|u-v|^{\gamma}\mu^{1/2}(u)\mu^{1/2}(v')g(u')dud\omega.\end{aligned} \tag{4.18}$$

Consider a smooth cutoff function $0 \le \chi_m \le 1$ such that (for $m > 0$)

$$\chi_m(s) \equiv 1, \text{ for } s \ge 2m; \quad \chi_m(s) \equiv 0, \text{ for } s \le m. \tag{4.19}$$

Then define $\bar{\chi}_m = 1 - \chi_m$. Use χ_m to split $K_2 = K_2^{\chi} + K_2^{1-\chi}$ where

$$\begin{aligned}K_2^{\chi} g \equiv &\int_{\mathbb{R}^3\times S^2} B(\theta)|u-v|^{\gamma}\mu^{1/2}(u)\chi_m(|u-v|)\mu^{1/2}(u')g(v')dud\omega \\ &+\int_{\mathbb{R}^3\times S^2} B(\theta)|u-v|^{\gamma}\mu^{1/2}(u)\chi_m(|u-v|)\mu^{1/2}(v')g(u')dud\omega.\end{aligned}$$

After removing the singularity at $u = v$, following the procedure in [30, 34] (see also eqns. (35) and (36) in [37]), we can write

$$K_2^{\chi} g = \int_{\mathbb{R}^3} k_2^{\chi}(v,\xi)g(v+\xi)d\xi,$$

where

$$k_2^{\chi}(v,\xi) \equiv \frac{e^{-\frac{1}{8}|\xi|^2-\frac{1}{2}|\zeta_{\|}|^2}}{|\xi|\sqrt{\pi^3/2}}\int_{\mathbb{R}^2}\frac{\chi_m(\sqrt{|\xi|^2+|\xi_\perp|^2})}{(|\xi|^2+|\xi_\perp|^2)^{\frac{1-\gamma}{2}}}e^{-\frac{1}{2}|\xi_\perp+\zeta_\perp|^2}\frac{B(\theta)}{|\cos\theta|}d\xi_\perp. \tag{4.20}$$

The integration variables are $d\xi_\perp = d\xi_\perp^1 d\xi_\perp^2$ but $\xi_\perp = \xi_\perp^1\xi^1 + \xi_\perp^2\xi^2 \in \mathbb{R}^3$ where $\{\xi^1, \xi^2, \xi/|\xi|\}$ is an orthonormal basis for $\mathbb{R}^3$. Also

$$\zeta_{\|} = \frac{(v\cdot\xi)\xi}{|\xi|^2} + \frac{1}{2}\xi, \ \zeta_\perp = v - \frac{(v\cdot\xi)\xi}{|\xi|^2} = (v\cdot\xi^1)\xi^1 + (v\cdot\xi^2)\xi^2. \tag{4.21}$$

It should be noted that, as in [37], we have removed the symmetry of the kernel k_2^{χ} via the translation $\xi \to v + \xi$. This formulation is well suited for taking high order v-derivatives. Caflisch [8] proved Lemma 4.1 just below with no derivatives. In contrast, we have already removed the singularity and this allows us to extend the estimate from $-1 < \gamma < 0$ to the full range $-3 < \gamma < 0$. As in [37], we will see in Lemma 4.2 that the singular part of K_2, e.g. $K_2^{1-\chi}$, has stronger decay.

LEMMA 4.1. *For any multi-index β and any $0 < s_1 < s_2 < 1$, we have*

$$|\partial_\beta k_2^\chi(v,\xi)| \leq C \frac{\exp\left(-\frac{s_2}{8}|\xi|^2 - \frac{s_1}{2}|\zeta_{\|}|^2\}\right)}{|\xi|(1+|v|+|\xi+v|)^{1-\gamma}}.$$

Here $C > 0$ will depend on s_1, s_2 and β.

PROOF. Fix $0 < s_1 < s_2 < 1$. If $|\beta| > 0$, from (4.20) and (4.21) we have

$$\partial_\beta k_2^\chi(v,\xi) = \frac{e^{-\frac{1}{8}|\xi|^2}}{|\xi|\sqrt{\pi^3/2}} \int_{\mathbb{R}^2} \frac{\chi_m(\sqrt{|\xi|^2+|\xi_\perp|^2})}{(|\xi|^2+|\xi_\perp|^2)^{\frac{1-\gamma}{2}}} \partial_\beta \left(e^{-\frac{1}{2}|\xi_\perp+\zeta_\perp|^2-\frac{1}{2}|\zeta_{\|}|^2}\right) \frac{B(\theta)}{|\cos\theta|} d\xi_\perp.$$

By a simple induction, for any $0 < q' < 1$, we have

$$\left|\partial_\beta \left(e^{-\frac{1}{2}|\xi_\perp+\zeta_\perp|^2-\frac{1}{2}|\zeta_{\|}|^2}\right)\right| \leq C(|\beta|,q') e^{-\frac{q'}{2}|\xi_\perp+\zeta_\perp|^2-\frac{q'}{2}|\zeta_{\|}|^2}.$$

Further restrict $q' > s_1$. For $|\beta| \geq 0$, using the last display and (4.2) we have

$$|\partial_\beta k_2^\chi(v,\xi)| \leq C \frac{e^{-\frac{1}{8}|\xi|^2}}{|\xi|} \int_{\mathbb{R}^2} \frac{\chi_m(\sqrt{|\xi|^2+|\xi_\perp|^2})}{(|\xi|^2+|\xi_\perp|^2)^{\frac{1-\gamma}{2}}} e^{-\frac{q'}{2}|\xi_\perp+\zeta_\perp|^2-\frac{q'}{2}|\zeta_{\|}|^2} d\xi_\perp.$$

Change variables $\xi_\perp \to \xi_\perp - \zeta_\perp$ on the r.h.s. to obtain

$$C \frac{e^{-\frac{1}{8}|\xi|^2-\frac{q'}{2}|\zeta_{\|}|^2}}{|\xi|} \int_{\mathbb{R}^2} \frac{\chi_m(\sqrt{|\xi|^2+|\xi_\perp-\zeta_\perp|^2})}{(|\xi|^2+|\xi_\perp-\zeta_\perp|^2)^{\frac{1-\gamma}{2}}} e^{-\frac{q'}{2}|\xi_\perp|^2} d\xi_\perp.$$

Using (4.19), then, we have

$$|\partial_\beta k_2^\chi(v,\xi)| \leq C(m) \frac{e^{-\frac{1}{8}|\xi|^2-\frac{q'}{2}|\zeta_{\|}|^2}}{|\xi|} \int_{\mathbb{R}^2} \frac{e^{-\frac{q'}{2}|\xi_\perp|^2} d\xi_\perp}{(1+|\xi|^2+|\xi_\perp-\zeta_\perp|^2)^{\frac{1-\gamma}{2}}}. \tag{4.22}$$

In the rest of the proof, we will refine this estimate via splitting.

Choose any $q'' > 0$ with $q'' < s_1$ and then define $\tau_* = \sqrt{\frac{q'-s_1}{q'-q''}} < 1$. Split the integration region as follows

$$\{|\xi_\perp| > \tau_*|\zeta_\perp|\} \cup \{|\xi_\perp| \leq \tau_*|\zeta_\perp|\}.$$

Further split the r.h.s. of (4.22) into $k_2^{\chi,1}(v,\xi) + k_2^{\chi,2}(v,\xi)$ where $k_2^{\chi,1}(v,\xi)$ is restricted to the region $\{|\xi_\perp| > \tau_*|\zeta_\perp|\}$:

$$k_2^{\chi,1}(v,\xi) \equiv C(m) \frac{e^{-\frac{1}{8}|\xi|^2-\frac{q'}{2}|\zeta_{\|}|^2}}{|\xi|} \int_{|\xi_\perp|>\tau_*|\zeta_\perp|} \frac{e^{-\frac{q'}{2}|\xi_\perp|^2} d\xi_\perp}{(1+|\xi|^2+|\xi_\perp-\zeta_\perp|^2)^{\frac{1-\gamma}{2}}}.$$

For $k_2^{\chi,1}(v,\xi)$ we will observe exponential decay. And for $k_2^{\chi,2}(v,\xi)$ we can extract from the denominator on the r.h.s. of (4.22) the exact decay stated in Lemma 4.1.

First consider $k_2^{\chi,1}(v,\xi)$. Since $\{|\xi_\perp| > \tau_*|\zeta_\perp|\}$ and $q' - q'' > 0$ we have

$$\begin{aligned}
\left|k_2^{\chi,1}(v,\xi)\right| &\leq C\frac{e^{-\frac{1}{8}|\xi|^2-\frac{q'}{2}|\zeta_\parallel|^2}}{|\xi|}\int_{\{|\xi_\perp|>\tau_*|\zeta_\perp|\}}\frac{e^{-\frac{q''}{2}|\xi_\perp|^2-\frac{q'-q''}{2}|\xi_\perp|^2}}{(1+|\xi|^2+|\xi_\perp-\zeta_\perp|^2)^{\frac{1-\gamma}{2}}}d\xi_\perp \\
&\leq C\frac{e^{-\frac{1}{8}|\xi|^2-\frac{q'}{2}|\zeta_\parallel|^2}}{|\xi|}\int_{\{|\xi_\perp|>\tau_*|\zeta_\perp|\}}\frac{e^{-\frac{q''}{2}|\xi_\perp|^2-\frac{q'-s_1}{2}|\zeta_\perp|^2}}{(1+|\xi|^2+|\xi_\perp-\zeta_\perp|^2)^{\frac{1-\gamma}{2}}}d\xi_\perp.
\end{aligned}$$

By (4.21),

$$|\zeta_\parallel|^2 + |\zeta_\perp|^2 = |\zeta_\parallel + \zeta_\perp|^2 = |v+\xi/2|^2.$$

Splitting $q' = s_1 + (q'-s_1)$ we have

$$\begin{aligned}
\left|k_2^{\chi,1}(v,\xi)\right| &\leq \frac{C}{|\xi|}e^{-\frac{1}{8}|\xi|^2-\frac{s_1}{2}|\zeta_\parallel|^2}\int_{\{|\xi_\perp|>\tau_*|\zeta_\perp|\}}\frac{e^{-\frac{q''}{2}|\xi_\perp|^2-\frac{q'-s_1}{2}(|\zeta_\perp|^2+|\zeta_\parallel|^2)}}{(1+|\xi|^2+|\xi_\perp-\zeta_\perp|^2)^{\frac{1-\gamma}{2}}}d\xi_\perp \\
&\leq \frac{C}{|\xi|}e^{-\frac{1}{8}|\xi|^2-\frac{s_1}{2}|\zeta_\parallel|^2-\frac{q'-s_1}{2}|v+\xi/2|^2}.
\end{aligned}$$

This will be more than enough decay. We expand

$$\begin{aligned}
|v+\xi/2|^2 &= |v|^2+\frac{1}{4}|\xi|^2+v\cdot\xi = \frac{1}{4}|v+\xi|^2+\frac{3}{4}|v|^2+\frac{1}{2}v\cdot\xi \\
&\geq \frac{1}{4}|v+\xi|^2+\frac{3}{4}|v|^2-\frac{1}{4}|v|^2-\frac{1}{4}|\xi|^2 \qquad (4.23)\\
&= \frac{1}{4}|v+\xi|^2+\frac{1}{2}|v|^2-\frac{1}{4}|\xi|^2.
\end{aligned}$$

Plug the last display into the one above it to obtain

$$\begin{aligned}
\left|k_2^{\chi,1}(v,\xi)\right| &\leq \frac{C}{|\xi|}e^{-\frac{1}{8}|\xi|^2-\frac{s_1}{2}|\zeta_\parallel|^2}e^{-\frac{q'-s_1}{8}|v+\xi|^2-\frac{q'-s_1}{4}|v|^2+\frac{q'-s_1}{8}|\xi|^2} \\
&= \frac{C}{|\xi|}e^{-\frac{s_1+1-q'}{8}|\xi|^2-\frac{s_1}{2}|\zeta_\parallel|^2}e^{-\frac{q'-s_1}{8}|v+\xi|^2-\frac{q'-s_1}{4}|v|^2}.
\end{aligned}$$

Given s_2 with $s_1 < s_2 < 1$, we can always choose q', restricted by $s_1 < q' < 1$, such that $s_2 = s_1 + 1 - q'$. This completes the estimate for $k_2^{\chi,1}(v,\xi)$.

On $\{|\xi_\perp| \leq \tau_*|\zeta_\perp|\}$, $|\zeta_\perp - \xi_\perp| \geq |\zeta_\perp| - |\xi_\perp| \geq (1-\tau_*)|\zeta_\perp|$ $(0 < \tau_* < 1)$. Hence (4.22) over this region is bounded as

$$\begin{aligned}
\left|k_2^{\chi,2}(v,\xi)\right| &\leq \frac{Ce^{-\frac{1}{8}|\xi|^2 - \frac{q'}{2}|\zeta_\|\|^2}}{|\xi|\left(1+|\xi|^2+(1-\tau_*)^2|\zeta_\perp|^2\right)^{\frac{1-\gamma}{2}}} \int_{\{|\xi_\perp|\leq\tau_*|\zeta_\perp|\}} e^{-\frac{q'}{2}|\xi_\perp|^2} d\xi_\perp \\
&\leq \frac{Ce^{-\frac{1}{8}|\xi|^2 - \frac{s_1}{2}|\zeta_\|\|^2}}{|\xi|\left(1+|\xi|^2+|\zeta_\perp|^2+|\zeta_\|\|^2\right)^{\frac{1-\gamma}{2}}} \\
&= \frac{Ce^{-\frac{1}{8}|\xi|^2 - \frac{s_1}{2}|\zeta_\|\|^2}}{|\xi|\left(1+|\xi|^2+|v+\xi/2|^2\right)^{\frac{1-\gamma}{2}}},
\end{aligned}$$

where we used $s_1 < q'$ to go from the first line to the second. Now plug (4.23) into the last display to complete the estimate. $\square$

Next we will prove the energy estimates for the linear operator (4.6).

LEMMA 4.2. *Let $|\beta| > 0$, $\ell \in \mathbb{R}$, $0 \leq \vartheta \leq 2$ and $0 < q$. If $\vartheta = 2$ restrict $0 < q < 1$. Then $\forall \eta > 0$ $\exists C(\eta) > 0$ such that*

$$\langle w^2(\ell,\vartheta)\partial_\beta[\nu g], \partial_\beta g\rangle \geq |\partial_\beta g|^2_{\boldsymbol{\nu},\ell,\vartheta} - \eta \sum_{|\beta_1|\leq|\beta|} |\partial_{\beta_1} g|^2_{\boldsymbol{\nu},\ell,\vartheta} - C(\eta)\left|\bar{\chi}_{C(\eta)} g\right|^2_\ell .$$

Furthermore, for any $|\beta| \geq 0$ we have

$$|\langle w^2(\ell,\vartheta)\partial_\beta[K g_1], g_2\rangle| \leq \left\{\eta \sum_{|\beta_1|\leq|\beta|} |\partial_{\beta_1} g_1|_{\boldsymbol{\nu},\ell,\vartheta} + C(\eta)\left|\bar{\chi}_{C(\eta)} g_1\right|_\ell\right\} |g_2|_{\boldsymbol{\nu},\ell,\vartheta},$$

where we are using (4.19).

Some parts of the proof of Lemma 4.2 are exactly the same as in [37]. For instance, the proof of the lower bound for $\langle w^2(\ell,\vartheta)\partial_\beta[\nu g], \partial_\beta g\rangle$ is exactly the same. But the estimate for $|\langle w^2(\ell,\vartheta)\partial_\beta[K g_1], g_2\rangle|$ requires extra care in particular because g_1 in the argument of $[K g_1]$ does not depend only on v. We therefore need to control the new exponentially growing factor of $w(\ell,\vartheta)(v)$. This requires a close look at the variables from (4.20) in (4.21). In particular, we write down the analogue of the conservation of energy (4.4) in this new coordinate system (4.29) in order absorb the exponentially growing weight. For completeness, we present all details of the proof.

Proof. *The First Estimate in the Lemma.*

Fix $\eta > 0$. Recall

$$\langle w^2 \partial_\beta[\nu g], \partial_\beta g\rangle = \langle w^2 \nu \partial_\beta g, \partial_\beta g\rangle + \sum_{0<\beta_1\leq\beta} C_\beta^{\beta_1} \langle w^2 \partial_{\beta_1}\nu \partial_{\beta-\beta_1} g, \partial_\beta g\rangle.$$

By Lemma 2 in [38], for $|\beta_1| > 0$,

$$|\partial_{\beta_1}\nu| \leq C(1+|v|^2)^{\frac{\gamma-1}{2}}.$$

We use this estimate (for m is chosen large enough) to obtain

$$\begin{aligned} \langle w^2 \partial_{\beta_1}\nu \partial_{\beta-\beta_1} g, \partial_\beta g\rangle &= \int_{|v|\leq m} + \int_{|v|\geq m} \\ &\leq \int_{|v|\leq m} + \frac{C}{m} \left|\partial_{\beta-\beta_1} g\right|_{\boldsymbol{\nu},\ell,\vartheta} \left|\partial_\beta g\right|_{\boldsymbol{\nu},\ell,\vartheta} \\ &\leq \int_{|v|\leq m} + \frac{\eta}{2} \left|\partial_{\beta-\beta_1} g\right|_{\boldsymbol{\nu},\ell,\vartheta} \left|\partial_\beta g\right|_{\boldsymbol{\nu},\ell,\vartheta}. \end{aligned}$$

On the other hand, for such a $m > 0$ and $\beta - \beta_1 < \beta$, the first integral over $|v| \leq m$ is bounded by a compact Sobolev interpolation

$$\int_{|v|\leq m} \leq \frac{\eta}{2} \sum_{|\beta_1|=|\beta|} \left|\partial_{\beta_1} g\right|^2_{\boldsymbol{\nu},\ell} + C(\eta) \left|\bar{\chi}_{C(\eta)} g\right|^2_{\boldsymbol{\nu},\ell}. \tag{4.24}$$

This concludes the lower bound for $\langle w^2(\ell,\vartheta)\partial_\beta[\nu g], \partial_\beta g\rangle$.

The Second Estimate in the Lemma. The proof of the second estimate is divided into several parts. Recall $K = K_1 - K_2$.

Step 1. The Estimate for K_1.

Next consider K_1 from (4.17). We change variables $u \to u + v$ to obtain

$$[K_1 g_1](v) = \int_{\mathbb{R}^3\times S^2} B(\theta)|u|^\gamma \mu^{1/2}(u+v)\mu^{1/2}(v) g_1(u+v)\,du\,d\omega$$

Notice that now $\cos\theta = u\cdot\omega/|u|$ so that for $|\beta| > 0$, $\partial_\beta[K_1 g_1](v)$

$$= \sum_{\beta_1\leq\beta} C_\beta^{\beta_1} \int_{\mathbb{R}^3\times S^2} B(\theta)|u|^\gamma \partial_{\beta-\beta_1}\left(\mu^{1/2}(u+v)\mu^{1/2}(v)\right) \partial_{\beta_1} g_1(u+v)\,du\,d\omega.$$

For any $0 < q' < 1$ we have

$$|\partial_{\beta-\beta_1}\{\mu^{1/2}(u+v)\mu^{1/2}(v)\}| \leq C(|\beta|, q')\mu^{q'/2}(u+v)\mu^{q'/2}(v).$$

We will use this exponential decay to control half of the exponential growth in the weight. If $0 \le \vartheta < 2$ then

$$w(\ell,\vartheta)(v)\mu^{q'/2}(v) \le C\mu^{q'/4}(v).$$

If $\vartheta = 2$ then for given $0 < q < 1$ we choose q' so that $q < q' < 1$. And in this case

$$w(\ell,\vartheta)(v)\mu^{q'/2}(v) = (1+|v|^2)^{\tau\ell/2} e^{\frac{q}{4}} e^{\frac{q}{4}|v|^2} \mu^{q'/2}(v) \le C\mu^{(q'-q)/4}(v).$$

Choosing $0 < q'' < \min\{|q'-q|/4, q'/4\}$, we can always write $\langle w^2(\ell,\vartheta)\partial_\beta[K_1 g_1], g_2\rangle$

$$= \sum_{\beta_1 \le \beta} \int_{\mathbb{R}^3\times\mathbb{R}^3} w(\ell,\vartheta)(v)|u|^\gamma \mu^{q''}(u+v)\mu^{q''}(v)\mu_{\beta_1}(u+v,v)\partial_{\beta_1} g_1(u+v) g_2(v)dudv,$$

where $\mu_{\beta_1}(u+v,v)$ is a collection of smooth functions satisfying

$$\left|\partial^u_{\bar\beta}\mu_{\beta_1}(u+v,v)\right| \le C(|\bar\beta|, q, q', q'').$$

Change variables $u \to u - v$ to obtain

$$\sum_{\beta_1 \le \beta} \int_{\mathbb{R}^3\times\mathbb{R}^3} w(\ell,\vartheta)(v)|u-v|^\gamma \mu^{q''}(u)\mu^{q''}(v)\mu_{\beta_1}(u,v)\partial_{\beta_1} g_1(u) g_2(v)dudv,$$

Now further split

$$\begin{aligned}\langle w^2(\ell,\vartheta)\partial_\beta[K_1 g_1], g_2\rangle &= \langle w^2(\ell,\vartheta)\partial_\beta[K_1^\chi g_1], g_2\rangle + \langle w^2(\ell,\vartheta)\partial_\beta[K_1^{1-\chi} g_1], g_2\rangle \\ &= \mathbf{K}_1^\chi + \mathbf{K}_1^{1-\chi}.\end{aligned}$$

Using (4.19) we have

$$\mathbf{K}_1^{1-\chi} \equiv \int w(\ell,\vartheta)(v)\bar\chi_m(|u-v|)|u-v|^\gamma \mu^{q''}(u)\mu^{q''}(v)\mu_{\beta_1}(u,v)\partial_{\beta_1} g_1(u) g_2(v)dudv,$$

where $\bar\chi_m = 1 - \chi_m$ and we implicitly sum over $\beta_1 \le \beta$. Then

$$\begin{aligned}\left|\mathbf{K}_1^{1-\chi}\right| &\le C\left\{\int w^2(\ell,\vartheta)(v)\bar\chi_m(|u-v|)|u-v|^\gamma \mu^{q''}(u)\mu^{q''}(v)\left|g_2(v)\right|^2 dudv\right\}^{1/2} \\ &\quad\times \sum_{\beta_1\le\beta}\left\{\int \bar\chi_m(|u-v|)|u-v|^\gamma \mu^{q''}(u)\mu^{q''}(v)\left|\partial_{\beta_1} g_1(u)\right|^2 dudv\right\}^{1/2}. \\ &\le C(2m)^{\frac{3+\gamma}{2}}|g_2|_{\boldsymbol{\nu},\ell,\vartheta}\sum_{\beta_1\le\beta}(2m)^{\frac{3+\gamma}{2}}|\partial_{\beta_1} g_1|_{\boldsymbol{\nu},\ell} \\ &\le \frac{\eta}{2}|g_2|_{\boldsymbol{\nu},\ell,\vartheta}\sum_{\beta_1\le\beta}|\partial_{\beta_1} g_1|_{\boldsymbol{\nu},\ell}.\end{aligned}$$

The last step follows by choosing m small enough.

Further,

$$\mathbf{K}_1^\chi \equiv \int w(\ell,\vartheta)(v)\chi_m(|u-v|)|u-v|^\gamma \mu^{q''}(u)\mu^{q''}(v)\mu_{\beta_1}(u,v)\partial_{\beta_1}g_1(u)g_2(v)dudv,$$

where we again implicitly sum over $\beta_1 \leq \beta$. After an integration by parts

$$\mathbf{K}_1^\chi = \sum_{\beta_1\leq\beta}(-1)^{|\beta_1|}\int w(\ell,\vartheta)(v)\partial_{\beta_1}^u\left\{\chi_m(|u-v|)|u-v|^\gamma\mu^{q''}(u)\mu_{\beta_1}(u,v)\right\}$$
$$\times\mu^{q''}(v)g_1(u)g_2(v)dudv.$$

Since $|u-v|^\gamma$ is bounded now, from (4.19), choosing $m'>0$ large enough we have

$$\begin{aligned}
|\mathbf{K}_1^\chi| &\leq C(|\beta|,m)\int w(\ell,\vartheta)(v)\mu^{q''/2}(u)\mu^{q''/2}(v)\left|g_1(u)g_2(v)\right|dudv \\
&= \int_{|u|\leq m'}+\int_{|u|>m'} \\
&\leq C\int w(\ell,\vartheta)(v)\bar{\chi}_{m'}(|u|)\mu^{q''/2}(u)\mu^{q''/2}(v)\left|g_1(u)g_2(v)\right|dudv \\
&\quad + Ce^{-\frac{q''}{8}m'}\int w(\ell,\vartheta)(v)\mu^{q''/4}(u)\mu^{q''/2}(v)\left|g_1(u)g_2(v)\right|dudv \\
&\leq \left\{\frac{\eta}{2}|g_1|_{\boldsymbol{\nu},\ell}+C(m')|\bar{\chi}_{m'}g_1|_{\boldsymbol{\nu},\ell}\right\}|g_2|_{\boldsymbol{\nu},\ell,\vartheta}.
\end{aligned}$$

This completes the estimate for $\langle w^2(\ell,\vartheta)\partial_\beta[K_1g_1],g_2\rangle$ and step one.

Step 2. The Estimate for K_2.

We turn to K_2 from (4.18). Split $K_2 = K_2^\chi + K_2^{1-\chi}$ and consider K_2^χ in (4.20).

Step (2a). The Estimate of $K_2^{1-\chi}$.

Now consider $K_2^{1-\chi} = K_2 - K_2^\chi$ which is given by

$$\begin{aligned}
K_2^{1-\chi}g_1 \equiv &\int_{\mathbb{R}^3\times S^2} B(\theta)|u-v|^\gamma\mu^{1/2}(u)\bar{\chi}_m(|u-v|)\mu^{1/2}(u')g_1(v')dud\omega \\
&+\int_{\mathbb{R}^3\times S^2} B(\theta)|u-v|^\gamma\mu^{1/2}(u)\bar{\chi}_m(|u-v|)\mu^{1/2}(v')g_1(u')dud\omega.
\end{aligned}$$

Here $\bar{\chi}_m = 1-\chi_m$ and χ_m is defined in (4.19). (4.3) and $\{|u-v|\leq 2m\}$ imply

$$|u'| = |v+u-v-[(u-v)\cdot\omega]\omega| \geq |v|-2|u-v| \geq |v|-4m,$$

$$|v'| = |v+[(u-v)\cdot\omega]\omega| \geq |v|-|u-v| \geq |v|-2m.$$

Therefore for any $0 < q' < 1$ we have

$$\mu^{1/2}(u)\mu^{1/2}(u') + \mu^{1/2}(u)\mu^{1/2}(v') \leq e^{C(q')m^2}\mu^{1/2}(u)\mu^{q'/2}(v). \tag{4.25}$$

This will be the key point in estimating the $K_2^{1-\chi}$ part.

First we take look at $\partial_\beta[K_2^{1-\chi}g_1]$. Change variables $u - v \to u$ to obtain

$$\begin{aligned} K_2^{1-\chi}g_1 \equiv &\int_{\mathbb{R}^3\times S^2} B(\theta)|u|^\gamma\mu^{1/2}(u+v)\bar{\chi}_m(|u|)\mu^{1/2}(v+u_\perp)g_1(v+u_\parallel)dud\omega \\ &+ \int_{\mathbb{R}^3\times S^2} B(\theta)|u|^\gamma\mu^{1/2}(u+v)\bar{\chi}_m(|u|)\mu^{1/2}(v+u_\parallel)g_1(v+u_\perp)dud\omega, \end{aligned}$$

Note that $u_\parallel$ and $u_\perp$ are defined using notation from [37]:

$$u_\parallel \equiv [u\cdot\omega]\omega, \;\; u_\perp \equiv u - [u\cdot\omega]\omega. \tag{4.26}$$

Now derivatives will not hit the singular kernel. $\partial_\beta[K_2^{1-\chi}g_1]$ is

$$\begin{aligned} &C_\beta^{\beta_1}\int_{\mathbb{R}^3\times S^2} B(\theta)|u|^\gamma\bar{\chi}_m(|u|)\partial_{\beta-\beta_1}\{\mu^{1/2}(u+v)\mu^{1/2}(v+u_\perp)\}\partial_{\beta_1}g_1(v+u_\parallel)dud\omega \\ &+C_\beta^{\beta_1}\int_{\mathbb{R}^3\times S^2} B(\theta)|u|^\gamma\bar{\chi}_m(|u|)\partial_{\beta-\beta_1}\{\mu^{1/2}(u+v)\mu^{1/2}(v+u_\parallel)\}\partial_{\beta_1}g_1(v+u_\perp)dud\omega, \end{aligned}$$

where we implicitly sum over multi-indices $\beta_1 \leq \beta$. Therefore, for any $0 < q'' < 1$, $|\partial_\beta[K_2^{1-\chi}g_1]|$ is bounded by

$$\begin{aligned} &C\int_{\mathbb{R}^3\times S^2} |u|^\gamma\bar{\chi}_m(|u|)\mu^{q''/2}(u+v)\mu^{q''/2}(v+u_\perp)|\partial_{\beta_1}g_1(v+u_\parallel)|dud\omega \\ &+C\int_{\mathbb{R}^3\times S^2} |u|^\gamma\bar{\chi}_m(|u|)\mu^{q''/2}(u+v)\mu^{q''/2}(v+u_\parallel)|\partial_{\beta_1}g_1(v+u_\perp)|dud\omega. \end{aligned}$$

We change variables $u \to u - v$ again to see that $|\partial_\beta[K_2^{1-\chi}g_1]|$ is bounded by

$$\begin{aligned} &C\int_{\mathbb{R}^3\times S^2} |u-v|^\gamma\bar{\chi}_m(|u-v|)\mu^{q''/2}(u)\mu^{q''/2}(v')|\partial_{\beta_1}g_1(u')|dud\omega \\ &+C\int_{\mathbb{R}^3\times S^2} |u-v|^\gamma\bar{\chi}_m(|u-v|)\mu^{q''/2}(u)\mu^{q''/2}(u')|\partial_{\beta_1}g_1(v')|dud\omega. \end{aligned}$$

Use (4.25) with $0 < q' < q''$ to say this is bounded above by

$$C\int_{\mathbb{R}^3\times S^2} |u-v|^\gamma\bar{\chi}_m(|u-v|)\mu^{q''/2}(u)\mu^{q'/2}(v)\left\{|\partial_{\beta_1}g_1(u')| + |\partial_{\beta_1}g_1(v')|\right\}dud\omega.$$

We remark that this last bound is true (and trivial) when $|\beta| = 0$ in which case $\partial_{\beta_1} = \partial_0 = 1$ by convention. Thus, $|\langle w^2(\ell,\vartheta)\partial_\beta\{K_2^{1-\chi}g_1\}, g_2\rangle|$ is

$$\leq C\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} |u-v|^\gamma \bar{\chi}_m(|u-v|) w^2(\ell,\vartheta)(v)\mu^{q''/2}(u)\mu^{q'/2}(v)|\partial_{\beta_1} g_1(v')||g_2(v)|,$$
$$+C\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^2} |u-v|^\gamma \bar{\chi}_m(|u-v|) w^2(\ell,\vartheta)(v)\mu^{q''/2}(u)\mu^{q'/2}(v)|\partial_{\beta_1} g_1(u')||g_2(v)|.$$

Here again we need to control the large exponentially growing factor $w(\ell,\vartheta)(v)$ by the strong exponential decay of the Maxwellians.

We will control this growth in cases. If $0 \leq \vartheta < 2$ then $q' > 0$ means

$$w(\ell,\vartheta)(v)\mu^{q'/2}(v) \leq C\mu^{q'/4}(v).$$

Alternatively, if $\vartheta = 2$ with $0 < q < 1$ then we can choose q' and q'' such that $q < q' < q''$. Then

$$w(\ell,\vartheta)(v)\mu^{q'/2}(v) \leq Cw(\ell,0)(v)\mu^{(q'-q)/2}(v) \leq C\mu^{(q'-q)/4}(v).$$

In either case, choose $q_1 = \min\{q''/2, q'/4, |q'-q|/4\} > 0$. Then we have the upper bound of

$$C\int |u-v|^\gamma \bar{\chi}_m(|u-v|) w(\ell,\vartheta)(v)\mu^{q_1}(u)\mu^{q_1}(v)|\partial_{\beta_1} g_1(v')||g_2(v)|dvdud\omega$$
$$+C\int |u-v|^\gamma \bar{\chi}_m(|u-v|) w(\ell,\vartheta)(v)\mu^{q_1}(u)\mu^{q_1}(v)|\partial_{\beta_1} g_1(u')||g_2(v)|dvdud\omega.$$

Further note that

$$\int |u-v|^\gamma \bar{\chi}_m(|u-v|)\mu^{q_1}(u)du \leq Cm^{3+\gamma}.$$

Apply Cauchy-Schwartz and the last display to obtain the upper bound

$$\leq Cm^{\frac{3+\gamma}{2}}\left\{\int |u-v|^\gamma \bar{\chi}_m(|u-v|)\mu^{q_1}(u)\mu^{q_1}(v)|\partial_{\beta_1} g_1(v')|^2 dvdud\omega\right\}^{1/2} |g_2|_{\boldsymbol{\nu},\ell,\vartheta}$$
$$+Cm^{\frac{3+\gamma}{2}}\left\{\int |u-v|^\gamma \bar{\chi}_m(|u-v|)\mu^{q_1}(u)\mu^{q_1}(v)|\partial_{\beta_1} g_1(u')|^2 dvdud\omega\right\}^{1/2} |g_2|_{\boldsymbol{\nu},\ell,\vartheta}.$$

Now apply the change of variables $(u,v) \to (u',v')$ using $|u-v| = |u'-v'|$ and (4.4) to see that $|\langle w^2(\ell,\vartheta)\partial_\beta\{K_2^{1-\chi}g_1\}, g_2\rangle|$ is bounded by

$$Cm^{3+\gamma}\left\{\int |u-v|^\gamma \bar{\chi}_m(|u-v|)\mu^{q_1}(u)\mu^{q_1}(v)|\partial_{\beta_1}g_1(v)|^2 dvdu\right\}^{1/2} |g_2|_{\boldsymbol{\nu},\ell,\vartheta}.$$

Hence,

$$|\langle w^2(\ell,\vartheta)\partial_\beta\{K_2^{1-\chi}g_1\}, g_2\rangle| \leq Cm^{\gamma+3}|g_2|_{\boldsymbol{\nu},\ell,\vartheta}\sum_{|\beta_1|\leq|\beta|}|\partial_{\beta_1}g_1|_{\boldsymbol{\nu},\ell}.$$

For $m>0$ small enough, we have completed the estimate of $K_2^{1-\chi}$, step (2a).

Step (2b). Estimate of K_2^χ.

For some large but fixed $m'>0$ we define the smooth cutoff function

$$\Upsilon_{m'} = \Upsilon_{m'}(v,\xi) = \chi_{m'}(\sqrt{1+|v|^2+|v+\xi|^2}), \quad \bar{\Upsilon}_{m'} = 1-\Upsilon_{m'}(v,\xi).$$

Now split again $K_2^\chi = K_2^\Upsilon + K_2^{1-\Upsilon}$ where

$$K_2^\Upsilon g_1 = \int_{\mathbb{R}^3}\Upsilon_{m'}(v,\xi)k_2^\chi(v,\xi)g_1(v+\xi)d\xi.$$

We will estimate this term first. Taking derivatives

$$\partial_\beta[K_2^\Upsilon g_1] = \sum_{\beta_1\leq\beta}C_\beta^{\beta_1}\int_{\mathbb{R}^3}\partial_{\beta_1}^v[\Upsilon_{m'}(v,\xi)k_2^\chi(v,\xi)]\partial_{\beta-\beta_1}g_1(v+\xi)d\xi$$

Using Lemma 4.1, with $0<s_1<s_2<1$, $|\langle w^2(\ell,\vartheta)\partial_\beta[K_2^\Upsilon g_1], g_2\rangle|$ is bounded by

$$C\sum_{\beta_1\leq\beta}\int_{|v|+|v+\xi|>m'}\frac{w^2(\ell,\vartheta)(v)|\partial_{\beta-\beta_1}g_1(v+\xi)||g_2(v)|}{|\xi|(1+|v|+|v+\xi|)^{1-\gamma}}e^{-\frac{s_2}{8}|\xi|^2-\frac{s_1}{2}|\zeta_{||}|^2}d\xi dv. \tag{4.27}$$

By (4.10) we expand

$$w(\ell,\vartheta)(v)e^{-\frac{s_2}{8}|\xi|^2-\frac{s_1}{2}|\zeta_{||}|^2} = (1+|v|^2)^{\tau\ell/2}e^{\frac{q}{4}(1+|v|^2)^{\vartheta/2}}e^{-\frac{s_2}{8}|\xi|^2-\frac{s_1}{2}|\zeta_{||}|^2}. \tag{4.28}$$

If we can control (4.28) by $w(\ell,\vartheta)(v+\xi)$ times decay in other directions then we can estimate (4.27). To do this, we look for an analogue of (4.4) in the variables (4.21).

Using (4.21) we have

$$|v|^2+|v+\xi-\zeta_\perp|^2 = |v+\xi|^2+|v-\zeta_\perp|^2 = |v+\xi|^2+\left(v\cdot\frac{\xi}{|\xi|}\right)^2. \tag{4.29}$$

Since $0 \le \vartheta \le 2$ we have

$$|v|^\vartheta \le \left(|v+\xi|^2 + \left(v \cdot \frac{\xi}{|\xi|}\right)^2\right)^{\vartheta/2} \le |v+\xi|^\vartheta + \left|v \cdot \frac{\xi}{|\xi|}\right|^\vartheta.$$

Thus,

$$e^{\frac{q}{4}(1+|v|^2)^{\vartheta/2}} \le e^{\frac{q}{4}} e^{\frac{q}{4}|v|^\vartheta} \le e^{\frac{q}{4}} e^{\frac{q}{4}|v+\xi|^\vartheta} e^{\frac{q}{4}\left|v\cdot\frac{\xi}{|\xi|}\right|^\vartheta}. \tag{4.30}$$

Further, from (4.21) notice that

$$|\zeta_{||}|^2 = \left(\left(v \cdot \frac{\xi}{|\xi|}\right) + \frac{1}{2}|\xi|\right)^2 \ge \frac{1}{2}\left(v \cdot \frac{\xi}{|\xi|}\right)^2 - \frac{1}{4}|\xi|^2$$

Therefore with $0 < s_1 < s_2$ we have

$$e^{-\frac{s_2}{8}|\xi|^2 - \frac{s_1}{2}|\zeta_{||}|^2} \le e^{-\frac{s_2 - s_1}{8}|\xi|^2 - \frac{s_1}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2}$$

Combine the above with (4.30), to obtain

$$\begin{aligned}
e^{\frac{q}{4}(1+|v|^2)^{\vartheta/2}} e^{-\frac{s_2}{8}|\xi|^2 - \frac{s_1}{2}|\zeta_{||}|^2} &\le e^{\frac{q}{4}} e^{\frac{q}{4}|v+\xi|^\vartheta} e^{\frac{q}{4}\left|v\cdot\frac{\xi}{|\xi|}\right|^\vartheta} e^{-\frac{s_2}{8}|\xi|^2 - \frac{s_1}{2}|\zeta_{||}|^2} \\
&\le C e^{\frac{q}{4}|v+\xi|^\vartheta} e^{\frac{q}{4}\left|v\cdot\frac{\xi}{|\xi|}\right|^\vartheta} e^{-\frac{s_2 - s_1}{8}|\xi|^2 - \frac{s_1}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2} \\
&= C e^{\frac{q}{4}|v+\xi|^\vartheta} e^{-\frac{s_2 - s_1}{8}|\xi|^2} e^{\frac{q}{4}\left|v\cdot\frac{\xi}{|\xi|}\right|^\vartheta - \frac{s_1}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2}.
\end{aligned}$$

If $0 \le \vartheta < 2$ then

$$e^{\frac{q}{4}\left|v\cdot\frac{\xi}{|\xi|}\right|^\vartheta - \frac{s_1}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2} \le C e^{-\frac{s_1}{8}\left(v\cdot\frac{\xi}{|\xi|}\right)^2}.$$

And if $\vartheta = 2$ then $0 < q < 1$ and we can choose s_1 with $1 > s_1 > q$ so that

$$e^{\frac{q}{4}\left|v\cdot\frac{\xi}{|\xi|}\right|^2 - \frac{s_1}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2} \le C e^{-\frac{s_1 - q}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2}.$$

In either case, choosing $s_3 = \min\{|s_1 - q|, s_1/2\} > 0$ and plugging these estimates into (4.28), we conclude that

$$\begin{aligned}
&w(\ell,\vartheta)(v) e^{-\frac{s_2}{8}|\xi|^2 - \frac{s_1}{2}|\zeta_{||}|^2} \\
&\le C(1+|v|^2)^{\tau\ell/2} e^{\frac{q}{4}(1+|v+\xi|^2)^{\vartheta/2}} e^{-\frac{s_2 - s_1}{8}|\xi|^2 - \frac{s_3}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2}.
\end{aligned} \tag{4.31}$$

Next, we estimate $(1+|v|^2)^{\tau\ell/2}$ with $\tau < 0$ and $\ell \in \mathbb{R}$. If $\ell\tau > 0$ then (4.29) yields

$$
\begin{aligned}
(1+|v|^2)^{\tau\ell/2} &\leq \left(1+|v+\xi|^2+\left(v\cdot\frac{\xi}{|\xi|}\right)^2\right)^{\tau\ell/2} \\
&\leq C(1+|v+\xi|^2)^{\tau\ell/2}\left(1+\left(v\cdot\frac{\xi}{|\xi|}\right)^2\right)^{\tau\ell/2}.
\end{aligned}
$$

Conversely if $\ell\tau \leq 0$ then we split the region into

$$
\{|v+\xi| > 2|v|\} \cup \{|v+\xi| \leq 2|v|\}.
$$

On $\{|v+\xi| \leq 2|v|\}$ and $\ell\tau \leq 0$ then

$$
(1+|v|^2)^{\tau\ell/2} \leq C(1+|v+\xi|^2)^{\tau\ell/2}.
$$

Alternatively, if $\{|v+\xi| > 2|v|\}$ then

$$
|\xi| \geq |v+\xi| - |v| > |v+\xi|/2.
$$

We therefore always have

$$
(1+|v|^2)^{\tau\ell/2} e^{-\frac{s_2-s_1}{8}|\xi|^2} \leq e^{-\frac{s_2-s_1}{8}|\xi|^2} \leq e^{-\frac{s_2-s_1}{16}|\xi|^2} e^{-\frac{s_2-s_1}{64}|v+\xi|^2}.
$$

In either of these last few cases, since $s_2 > s_1 > q$, from (4.31) we have

$$
\begin{aligned}
w(\ell,\vartheta)(v)e^{-\frac{s_2}{8}|\xi|^2-\frac{s_1}{2}|\zeta_{||}|^2} &\leq C(1+|v|^2)^{\tau\ell/2} e^{\frac{q}{4}(1+|v+\xi|^2)^{\vartheta/2}} e^{-\frac{s_2-s_1}{8}|\xi|^2-\frac{s_3}{4}\left(v\cdot\frac{\xi}{|\xi|}\right)^2} \\
&\leq C(1+|v+\xi|^2)^{\tau\ell/2} e^{\frac{q}{4}(1+|v+\xi|^2)^{\vartheta/2}} e^{-\frac{s_2-s_1}{16}|\xi|^2-\frac{s_3}{8}\left(v\cdot\frac{\xi}{|\xi|}\right)^2} \\
&= Cw(\ell,\vartheta)(v+\xi)e^{-\frac{s_2-s_1}{16}|\xi|^2-\frac{s_3}{8}\left(v\cdot\frac{\xi}{|\xi|}\right)^2}.
\end{aligned}
$$

Plug this into (4.27) to obtain the following upper bound for (4.27) of

$$
C\int_{|v|+|v+\xi|>m'} \frac{(w(\ell,\vartheta)(v+\xi)|\partial_{\beta-\beta_1}g_1(v+\xi)|)\,(w(\ell,\vartheta)(v)|g_2(v)|)}{|\xi|(1+|v|+|v+\xi|)^{1-\gamma}} e^{-\frac{s_2-s_1}{16}|\xi|^2} d\xi dv,
$$

where we implicitly sum over $\beta_1 \leq \beta$. Using Cauchy-Schwartz and translation invariance this is

$$\leq \frac{C}{m'} \int \frac{\left(w(\ell,\vartheta)(v+\xi)|\partial_{\beta-\beta_1} g_1(v+\xi)|\right)\left(w(\ell,\vartheta)(v)|g_2(v)|\right)}{|\xi|(1+|v|+|v+\xi|)^{-\gamma}} e^{-\frac{s_2-s_1}{16}|\xi|^2} d\xi dv$$
$$\leq \frac{C}{m'} |g_2|_{\boldsymbol{\nu},\ell,\vartheta} \int \frac{w^2(\ell,\vartheta)(v+\xi)|\partial_{\beta-\beta_1} g_1(v+\xi)|^2}{|\xi|(1+|v+\xi|)^{-\gamma}} e^{-\frac{s_2-s_1}{16}|\xi|^2} d\xi dv$$
$$\leq \frac{C}{m'} \sum_{\beta_1 \leq \beta} |\partial_{\beta-\beta_1} g_1|_{\boldsymbol{\nu},\ell,\vartheta} |g_2|_{\boldsymbol{\nu},\ell,\vartheta}.$$

This completes the estimate for K_2^{Υ}.

We now estimate $K_2^{1-\Upsilon}$. Taking derivatives

$$\partial_\beta [K_2^{1-\Upsilon} g_1] = \sum_{\beta_1 \leq \beta} C_\beta^{\beta_1} \int_{\mathbb{R}^3} \partial_{\beta_1}^v [\bar{\Upsilon}_{m'}(v,\xi) k_2^\chi(v,\xi)] \partial_{\beta-\beta_1} g_1(v+\xi) d\xi$$

Also,

$$\langle w^2 \partial_\beta [K_2^{1-\Upsilon} g_1], g_2 \rangle = \int w^2(\ell,\vartheta) \partial_\beta [K_2^{1-\Upsilon} g_1] g_2(v) dv.$$

By Cauchy-Schwartz and the compact support of $K_2^{1-\Upsilon}$ we have

$$\left| \langle w^2 \partial_\beta [K_2^{1-\Upsilon} g_1], g_2 \rangle \right| \leq C(m') \left\{ \int_{|v| \leq m'} \left(\partial_\beta [K_2^{1-\Upsilon} g_1] \right)^2 dv \right\}^{1/2} |g_2|_{\boldsymbol{\nu},\ell}.$$

With Lemma 4.1 we establish that $\partial_\beta [K_2^{1-\Upsilon} g_1]$ is compact from H^k to H^k. Then by the general interpolation for compact operators from H^k to H^k we have

$$\left| \langle w^2 \partial_\beta [K_2^{1-\Upsilon} g_1], g_2 \rangle \right| \leq \left\{ \frac{\eta}{4} \sum_{|\beta_1| = |\beta|} |\partial_{\beta_1} g_1|_{\boldsymbol{\nu},\ell} + C(\eta, m') \left| g_1 \right|_{\boldsymbol{\nu},\ell} \right\} |g_2|_{\boldsymbol{\nu},\ell}.$$

This completes the estimate for $K_2^{1-\Upsilon}$ and thus for K_2^χ, step (2b). We have therefore finished the whole proof. □

The following Corollary is used to prove existence of global solutions.

Corollary 4.1. *Let $|\beta| > 0$, $\ell \in \mathbb{R}$, $0 \leq \vartheta \leq 2$ and $0 < q$. If $\vartheta = 2$ restrict $0 < q < 1$. Then $\forall \eta > 0$ there exists $C(\eta) > 0$ such that*

$$\langle w^2 \partial_\beta [Lg], \partial_\beta g \rangle \geq |\partial_\beta g|^2_{\boldsymbol{\nu},\ell,\vartheta} - \eta \sum_{|\beta_1| \leq |\beta|} |\partial_{\beta_1} g|^2_{\boldsymbol{\nu},\ell,\vartheta} - C(\eta) \left| \bar{\chi}_{C(\eta)} g \right|_\ell^2,$$

where $\bar{\chi}_{C(\eta)}$ is from (4.19).

The rest of section is devoted to estimates for nonlinear collision term $\Gamma[g_1,g_2]$, with $g_i(x,v)$ $(i=1,2)$. In (4.5), the (non-symmetric) bilinear form $\Gamma[g_1,g_2]$ in the Boltzmann case is

$$
\begin{aligned}
\Gamma[g_1,g_2] &= \mu^{-1/2}(v)Q[\mu^{1/2}g_1,\mu^{1/2}g_2] \equiv \Gamma_{\mathrm{gain}}[g_1,g_2]-\Gamma_{\mathrm{loss}}[g_1,g_2],\\
&= \int_{\mathbb{R}^3}|u-v|^{\gamma}\mu^{1/2}(u)\left[\int_{S^2}B(\theta)g_1(u')g_2(v')d\omega\right]du,\\
&\quad -g_2(v)\int_{\mathbb{R}^3}|u-v|^{\gamma}\mu^{1/2}(u)\left[\int_{S^2}B(\theta)d\omega\right]g_1(u)du.
\end{aligned}\tag{4.32}
$$

The change of variables $u-v\to u$ gives

$$
\begin{aligned}
\partial^{\alpha}_{\beta}\Gamma[g_1,g_2] &\equiv \partial^{\alpha}_{\beta}\left[\int_{\mathbb{R}^3}\int_{S^2}|u|^{\gamma}\mu^{1/2}(u+v)g_1(v+u_{\parallel})g_2(v+u_{\perp})B(\theta)dud\omega\right],\\
&\quad -\partial^{\alpha}_{\beta}\left[\int_{\mathbb{R}^3}\int_{S^2}|u|^{\gamma}\mu^{1/2}(u+v)g_1(v+u)g_2(v)B(\theta)dud\omega\right],\\
&\equiv \sum C^{\beta_0\beta_1\beta_2}_{\beta}C^{\alpha_1\alpha_2}_{\alpha}\Gamma^0[\partial^{\alpha_1}_{\beta_1}g_1,\partial^{\alpha_2}_{\beta_2}g_2].
\end{aligned}
$$

where the summation is over $\beta_0+\beta_1+\beta_2=\beta$ and $\alpha_1+\alpha_2=\alpha$. Also $u_{\perp},u_{\parallel}$ are given by (4.26). By the product rule and the reverse change of variables we have

$$
\begin{aligned}
\Gamma^0[\partial^{\alpha_1}_{\beta_1}g_1,\partial^{\alpha_2}_{\beta_2}g_2], &\equiv \int_{\mathbb{R}^3}\int_{S^2}|u-v|^{\gamma}\partial_{\beta_0}[\mu^{1/2}(u)]\partial^{\alpha_1}_{\beta_1}g_1(u')\partial^{\alpha_2}_{\beta_2}g_2(v')B(\theta)\\
&\quad -\partial^{\alpha_2}_{\beta_2}g_2(v)\int_{\mathbb{R}^3}\int_{S^2}|u-v|^{\gamma}\partial_{\beta_0}[\mu^{1/2}(u)]\partial^{\alpha_1}_{\beta_1}g_1(u)B(\theta)\\
&\equiv \Gamma^0_{\mathrm{gain}}-\Gamma^0_{\mathrm{loss}}.
\end{aligned}\tag{4.33}
$$

With these formulas, we have the following nonlinear estimate:

Lemma 4.3. *Recall (4.33) and let* $\beta_0+\beta_1+\beta_2=\beta$, $\alpha_1+\alpha_2=\alpha$. *Say* $0\le\vartheta\le 2$, $0<q$. *If* $\vartheta=2$, *restrict* $0<q<1$. *Let* $\ell=|\beta|-l$ *with* $l\ge 0$. *If* $|\alpha_1|+|\beta_1|\le N/2$, *then*

$$
|(w^2(\ell,\vartheta)\Gamma^0[\partial^{\alpha_1}_{\beta_1}g_1,\partial^{\alpha_2}_{\beta_2}g_2],\ \partial^{\alpha}_{\beta}g_3)|\le C\mathcal{E}^{1/2}_{l,\vartheta}(g_1)||\partial^{\alpha_2}_{\beta_2}g_2||_{\boldsymbol{\nu},|\beta_2|-l,\vartheta}||\partial^{\alpha}_{\beta}g_3||_{\boldsymbol{\nu},\ell,\vartheta}.
$$

Alternatively, if $|\alpha_2|+|\beta_2|\le N/2$, *then*

$$
|(w^2(\ell,\vartheta)\Gamma^0[\partial^{\alpha_1}_{\beta_1}g_1,\partial^{\alpha_2}_{\beta_2}g_2],\ \partial^{\alpha}_{\beta}g_3)|\le C||\partial^{\alpha_1}_{\beta_1}g_1||_{\boldsymbol{\nu},|\beta_1|-l,\vartheta}\mathcal{E}^{1/2}_{l,\vartheta}(g_2)||\partial^{\alpha}_{\beta}g_3||_{\boldsymbol{\nu},\ell,\vartheta}.
$$

The proof of Lemma 4.3 is more or less the same as in [37]. However, small modifications are needed to facilitate the exponentially growing weight. In (4.37) we need to use (4.4) to properly distribute the exponentially growing factor in $w^2(\ell,\vartheta)(v)$.

PROOF. Case 1. **The Loss Term Estimate.**

First consider the second term Γ^0_{loss}in (4.33). Note that

$$|\partial_{\beta_0}[\mu^{1/2}(u)]| \leq Ce^{-|u|^2/8}.$$

With $|\alpha_1|+|\beta_1| \leq N/2$ and $\gamma > -3$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} |u-v|^\gamma |\partial_{\beta_0}[\mu^{1/2}(u)]\partial^{\alpha_1}_{\beta_1} g_1(x,u)|du \\
&\leq C\left\{\int_{\mathbb{R}^3} |u-v|^\gamma e^{-|u|^2/8}|\partial^{\alpha_1}_{\beta_1} g_1(x,u)|^2 du\right\}^{1/2}\left\{\int_{\mathbb{R}^3} |u-v|^\gamma e^{-|u|^2/8} du\right\}^{1/2} \\
&\leq C\sup_{x,u}\left|e^{-|u|^2/16}\partial^{\alpha_1}_{\beta_1} g_1(x,u)\right|\left\{\int_{\mathbb{R}^3} |u-v|^\gamma e^{-|u|^2/16} du\right\} \\
&\leq C\mathcal{E}^{1/2}_l(g_1)[1+|v|]^\gamma.
\end{aligned}$$

Since $N \geq 8$, we have used the embedding $H^4(\mathbb{T}^3\times\mathbb{R}^3) \subset L^\infty$ to argue that

$$\sup_{x,u}\left|e^{-|u|^2/16}\partial^{\alpha_1}_{\beta_1} g_1(x,u)\right| \leq C\mathcal{E}^{1/2}_l(g_1). \tag{4.34}$$

Hence $\left|(w^2(\ell,\vartheta)\Gamma^0_{\text{loss}}[\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2], \partial^{\alpha}_{\beta} g_3)\right|$ is bounded by

$$\begin{aligned}
&C\mathcal{E}^{1/2}_l(g_1)\int [1+|v|]^\gamma w^2(\ell,\vartheta)(v)|\partial^{\alpha_2}_{\beta_2} g_2(v)\partial^{\alpha}_{\beta} g_3(v)|dvdx \\
&\leq C\mathcal{E}^{1/2}_l(g_1)||\partial^{\alpha_2}_{\beta_2} g_2||_{\boldsymbol{\nu},\ell,\vartheta}||\partial^{\alpha}_{\beta} g_3||_{\boldsymbol{\nu},\ell,\vartheta}.
\end{aligned}$$

This completes the estimate for Γ^0_{loss} when $|\alpha_1|+|\beta_1| \leq N/2$.

Now consider Γ^0_{loss} with $|\alpha_2|+|\beta_2| \leq N/2$. Here we split the (u,v) integration domain is split into three parts

$$\{|v-u| \leq |v|/2\} \cup \ \{|v-u| \geq |v|/2,\ |v| \geq 1\} \cup \{|v-u| \geq |v|/2,\ |v| \leq 1\}.$$

In the first region, $|u|$ is comparable to $|v|$ and thus we can use exponential decay in both variables to get the estimate. In the second and third regions, $|u|$ is not comparable to $|v|$ but we exploit the largeness or smallness of $|v|$ to get the estimate.

Case (1a). The Loss Term in the First Region $\{|v-u| \le |v|/2\}$.

For the first part, $\{|v-u| \le |v|/2\}$, we have

$$|u| \ge |v| - |v-u| \ge |v|/2.$$

So that

$$e^{-|u|^2/8} \le e^{-|u|^2/16} e^{-|v|^2/64}.$$

Then the integral of $w^2 \Gamma^0_{\text{loss}}[\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2] \partial^\alpha_\beta g_3$ over $\{|u-v| \le |v|/2\}$ is bounded by

$$\begin{aligned} C \int |u-v|^\gamma e^{-|u|^2/16} e^{-|v|^2/64} w^2(\ell,\vartheta)(v) |\partial^{\alpha_1}_{\beta_1} g_1(u) \partial^{\alpha_2}_{\beta_2} g_2(v) \partial^\alpha_\beta g_3(v)| du dv dx \\ \le C \left\{ \int |u-v|^\gamma e^{-|u|^2/16} e^{-|v|^2/64} |\partial^{\alpha_1}_{\beta_1} g_1(u)|^2 du dv dx \right\}^{1/2} \times \\ \left\{ \int |u-v|^\gamma e^{-|u|^2/16} e^{-|v|^2/64} w^4(\ell,\vartheta)(v) |\partial^{\alpha_2}_{\beta_2} g_2(v)|^2 |\partial^\alpha_\beta g_3(v)|^2 du dv dx \right\}^{1/2}. \end{aligned}$$

Integrating over dv, the first factor is bounded by $C||\partial^{\alpha_1}_{\beta_1} g_1||_{\boldsymbol{\nu},\ell}$. Integrating first over u variables in the second factor yields an upper bound

$$\begin{aligned} C \left\{ \int \left(\int |u-v|^\gamma e^{-|u|^2/16} du \right) e^{-|v|^2/64} w^4 |\partial^{\alpha_2}_{\beta_2} g_2|^2 |\partial^\alpha_\beta g_3|^2 dv dx \right\}^{1/2} \\ \le C \left\{ \int \int e^{-|v|^2/64} w^4(\ell,\vartheta)(v) [1+|v|]^\gamma |\partial^{\alpha_2}_{\beta_2} g_2|^2 |\partial^\alpha_\beta g_3|^2 dv dx \right\}^{1/2}. \end{aligned}$$

And as in (4.34), since $N \ge 8$ and $|\alpha_2| + |\beta_2| \le N/2$ we have

$$\sup_{x,v} w(\ell,\vartheta)(v) e^{-|v|^2/256} |\partial^{\alpha_2}_{\beta_2} g_2(x,v)| \le C \mathcal{E}^{1/2}_{l,\vartheta}(g_2). \tag{4.35}$$

We thus conclude the estimate over the first region.

Case (1b). The Loss Term in the Second Region $\{|v-u| \ge |v|/2,\ |v| \ge 1\}$.

Next consider Γ^0_{loss} over the second region $\{|v-u| \ge |v|/2,\ |v| \ge 1\}$. Since $\gamma < 0$, we have

$$|u-v|^\gamma \le C[1+|v|]^\gamma.$$

Then the integral of $w^2(\ell,\vartheta)\Gamma^0_{\text{loss}}[\partial^{\alpha_1}_{\beta_1}g_1,\partial^{\alpha_2}_{\beta_2}g_2]\partial^{\alpha}_{\beta}g_3$ over this region is bounded by

$$
\begin{aligned}
&C\int [1+|v|]^{\gamma}e^{-|u|^2/8}w^2(\ell,\vartheta)|\partial^{\alpha_1}_{\beta_1}g_1(u)\partial^{\alpha_2}_{\beta_2}g_2(v)\partial^{\alpha}_{\beta}g_3(v)|dudvdx \\
&\leq C\int\left\{\int e^{-|u|^2/8}|\partial^{\alpha_1}_{\beta_1}g_1(u)|du\right\}\left\{\int [1+|v|]^{\gamma}w^2(\ell,\vartheta)|\partial^{\alpha_2}_{\beta_2}g_2(v)\partial^{\alpha}_{\beta}g_3(v)|dv\right\}dx \\
&\leq C\int |\partial^{\alpha_1}_{\beta_1}g_1|_{\boldsymbol{\nu},\ell}\left\{\int w^2|\partial^{\alpha_2}_{\beta_2}g_2|^2dv\right\}^{1/2}\left\{\int [1+|v|]^{2\gamma}w^2|\partial^{\alpha}_{\beta}g_3|^2dv\right\}^{1/2}dx.
\end{aligned}
$$

Since $|\alpha_2|+|\beta_2|\leq N/2$, $N\geq 8$ and $H^2(\mathbb{T}^3)\subset L^\infty(\mathbb{T}^3)$, we have

$$
\sup_x\int w^2(\ell,\vartheta)(v)|\partial^{\alpha_2}_{\beta_2}g_2(x,v)|^2dv\leq C\mathcal{E}_{l,\vartheta}(g_2). \tag{4.36}
$$

Thus, by the Cauchy-Schwartz inequality, the Γ^0_{loss} term over $\{|v-u|\geq|v|/2,\ |v|\geq 1\}$ is bounded by $C||\partial^{\alpha_1}_{\beta_1}g_1||_{\boldsymbol{\nu},\ell}\mathcal{E}^{1/2}_{l,\vartheta}(g_2)||\partial^{\alpha}_{\beta}g_3||_{\boldsymbol{\nu},\ell,\vartheta}$.

Case (1c). The Loss Term in the Third Region: $\{|v-u|\geq|v|/2,\ |v|\geq 1\}$

For the last region, $\{|v-u|\geq|v|/2,\ |v|\leq 1\}$, we have

$$
|u-v|^{\gamma}\leq C|v|^{\gamma},\quad w(\ell,\vartheta)(v)\leq C.
$$

And then that the integral of $w^2(\ell,\vartheta)\Gamma^0_{\text{loss}}[\partial^{\alpha_1}_{\beta_1}g_1,\partial^{\alpha_2}_{\beta_2}g_2]\partial^{\alpha}_{\beta}g_3$ over this region is bounded by

$$
\begin{aligned}
&C\int_{\{|v-u|\geq|v|/2,\ |v|\leq 1\}}|u-v|^{\gamma}e^{-|u|^2/8}|\partial^{\alpha_1}_{\beta_1}g_1(u)\partial^{\alpha_2}_{\beta_2}g_2(v)\partial^{\alpha}_{\beta}g_3(v)|dudvdx \\
&\leq C\int\left\{\int |u-v|^{\gamma/2}e^{-|u|^2/8}|\partial^{\alpha_1}_{\beta_1}g_1(u)|du\right\}\left\{\int_{|v|\leq 1}|v|^{\gamma/2}|\partial^{\alpha_2}_{\beta_2}g_2(v)\partial^{\alpha}_{\beta}g_3(v)|dv\right\}dx.
\end{aligned}
$$

Using the Cauchy-Schwartz inequality a few times we have

$$
\begin{aligned}
&\leq C\int\left\{\int |u-v|^{\gamma}e^{-|u|^2/8}du\right\}^{1/2}\left\{\int e^{-|u|^2/8}|\partial^{\alpha_1}_{\beta_1}g_1(u)|^2du\right\}^{1/2} \\
&\qquad\times\left\{\int_{|v|\leq 1}|v|^{\gamma}|\partial^{\alpha_2}_{\beta_2}g_2|^2dv\right\}^{1/2}\left\{\int_{|v|\leq 1}|\partial^{\alpha}_{\beta}g_3|^2dv\right\}^{1/2}dx \\
&\leq C\int |\partial^{\alpha_1}_{\beta_1}g_1|_{\boldsymbol{\nu},\ell}\left\{\int_{|v|\leq 1}|v|^{\gamma}|\partial^{\alpha_2}_{\beta_2}g_2|^2dv\right\}^{1/2}\left\{\int_{|v|\leq 1}|\partial^{\alpha}_{\beta}g_3|^2dv\right\}^{1/2}dx.
\end{aligned}
$$

By $|\alpha_2|+|\beta_2|\leq N/2$, $\gamma>-3$ and $H^4(\mathbb{T}^3\times\mathbb{R}^3)\subset L^\infty$,

$$
\int_{|v|\leq 1}|v|^{\gamma}|\partial^{\alpha_2}_{\beta_2}g_2|^2dv\leq C\sup_{|v|\leq 1,x\in\mathbb{T}^3}|\partial^{\alpha_2}_{\beta_2}g_2|^2\leq C\mathcal{E}_l(g_2).
$$

Hence, by Cauchy-Schwartz, the last part is bounded by

$$C||\partial_{\beta_1}^{\alpha_1} g_1||_{\boldsymbol{\nu},\ell}\mathcal{E}_l^{1/2}(g_2)||\partial_\beta^\alpha g_3||_{\boldsymbol{\nu},\ell}.$$

This concludes the desired estimate for Γ^0_{loss}.

Case 2. **The Gain Term Estimate.**

The next step is to estimate the gain term Γ^0_{gain} in (4.33), for which the (u,v) integration domain is split into two parts:

$$\{|u|\geq |v|/2\} \ \cup \ \{|u|\leq |v|/2\}.$$

Case (2a) The Gain Term over $\{|u|\geq |v|/2\}$.

For the first region $\{|u|\geq |v|/2\}$,

$$e^{-|u|^2/8}\leq e^{-|u|^2/16}e^{-|v|^2/64}.$$

Then the integral of $w^2(\ell,\vartheta)\Gamma^0_{\mathrm{gain}}[\partial^{\alpha_1}_{\beta_1}g_1,\partial^{\alpha_2}_{\beta_2}g_2]\partial^{\alpha_3}_{\beta_3}g_3$ over $\{|u|\geq |v|/2\}$ is thus bounded by

$$\begin{aligned}
&\int_{|u|\geq|v|/2}|u-v|^\gamma e^{-|u|^2/16}e^{-|v|^2/64}w^2(\ell,\vartheta)(v)|\partial^{\alpha_1}_{\beta_1}g_1(u')\partial^{\alpha_2}_{\beta_2}g_2(v')\partial^\alpha_\beta g_3(v)|d\omega dudvdx\\
&\quad\leq C\left\{\int |u-v|^\gamma e^{-|u|^2/16}e^{-|v|^2/64}w^2|\partial^{\alpha_1}_{\beta_1}g_1(u')|^2|\partial^{\alpha_2}_{\beta_2}g_2(v')|^2d\omega dudvdx\right\}^{1/2}\\
&\qquad\qquad\times\left\{\int |u-v|^\gamma e^{-|u|^2/16}e^{-|v|^2/64}w^2|\partial^\alpha_\beta g_3(v)|^2dudvdx\right\}^{1/2}\\
&\quad\leq C\left\{\int |u'-v'|^\gamma e^{-\frac{1}{64}(|u'|^2+|v'|^2)}w^2(v)|\partial^{\alpha_1}_{\beta_1}g_1(u')|^2|\partial^{\alpha_2}_{\beta_2}g_2(v')|^2\right\}^{1/2}||\partial^\alpha_\beta g_3||_{\boldsymbol{\nu},\ell,\vartheta}.
\end{aligned}$$

In the first factor we have used (4.4) and $|u'-v'|=|u-v|$. By (4.4),

$$e^{\frac{q}{4}(1+|v|^2)^{\vartheta/2}}\leq e^{\frac{q}{4}(1+|v'|^2+|u'|^2)^{\vartheta/2}}\leq e^{\frac{q}{4}(1+|v'|^2)^{\vartheta/2}}e^{\frac{q}{4}(1+|u'|^2)^{\vartheta/2}}. \tag{4.37}$$

Using this, (4.10) and (4.4) we have

$$e^{-\frac{1}{64}(|u'|^2+|v'|^2)}w^2(\ell,\vartheta)(v)\leq e^{-\frac{1}{128}(|u'|^2+|v'|^2)}w^2(\ell,\vartheta)(v')w^2(\ell,\vartheta)(u').$$

So that the factor in braces is

$$\leq C\int |u'-v'|^\gamma e^{-\frac{1}{128}(|u'|^2+|v'|^2)}w^2(u')|\partial^{\alpha_1}_{\beta_1}g_1(u')|^2w^2(v')|\partial^{\alpha_2}_{\beta_2}g_2(v')|^2d\omega dudvdx.$$

The change of variables $(u,v)\to(u',v')$ implies

$$= C\left\{\int |u-v|^{\gamma} e^{-\frac{1}{128}(|u|^2+|v|^2)} w^2(u)|\partial_{\beta_1}^{\alpha_1} g_1(u)|^2 w^2(v)|\partial_{\beta_2}^{\alpha_2} g_2(v)|^2 dudvdx\right\}^{1/2}.$$

Assume $|\alpha_1|+|\beta_1|\le N/2$. As in (4.35),

$$\sup_{x,u}\left\{w(\ell,\vartheta)(u)e^{-\frac{1}{256}|u|^2}|\partial_{\beta_1}^{\alpha_1} g_1(x,u)|\right\}\le C\mathcal{E}_{l,\vartheta}^{1/2}(g_1).$$

Integrate first over du to obtain

$$\begin{aligned}\int |u-v|^{\gamma}e^{-\frac{1}{128}|u|^2}w(\ell,\vartheta)(u)|\partial_{\beta_1}^{\alpha_1} g_1(u)|^2 du &\le C\mathcal{E}_{l,\vartheta}^{1/2}(g_1)\int |u-v|^{\gamma}e^{-\frac{1}{256}|u|^2}du\\ &\le C\mathcal{E}_{l,\vartheta}^{1/2}(g_1)[1+|v|]^{\gamma}.\end{aligned}$$

If $|\alpha_2|+|\beta_2|\le N/2$ use this last argument but switch $\partial_{\beta_1}^{\alpha_1} g_1$ with $\partial_{\beta_2}^{\alpha_2} g_2$. Then the bound for the gain term over $\{|u|\ge |v|/2\}$, if $|\alpha_1|+|\beta_1|\le N/2$, is

$$\int_{|u|\ge|v|/2}\le C\mathcal{E}_{l,\vartheta}^{1/2}(g_1)||\partial_{\beta_2}^{\alpha_2} g_2||_{\boldsymbol{\nu},\ell,\vartheta}||\partial_{\beta}^{\alpha} g_3||_{\boldsymbol{\nu},\ell,\vartheta}.$$

And the bound for the gain term over $\{|u|\ge |v|/2\}$, if $|\alpha_2|+|\beta_2|\le N/2$, is

$$\int_{|u|\ge|v|/2}\le C||\partial_{\beta_2}^{\alpha_2} g_1||_{\boldsymbol{\nu},\ell,\vartheta}\mathcal{E}_{l,\vartheta}^{1/2}(g_2)||\partial_{\beta}^{\alpha} g_3||_{\boldsymbol{\nu},\ell,\vartheta}.$$

This completes the estimate for the gain term over $\{|u|\ge |v|/2\}$.

Case (2b). The Gain Term over $\{|u|\le |v|/2\}$.

Now consider the gain term over $\{|u|\le |v|/2\}$. Since $|v-u|<|v|/2$ implies $|u|\ge |v|-|v-u|>|v|/2$, we obtain

$$\{|u|\le |v|/2\}=\{|u|\le |v|/2\}\cup\{|v-u|\ge |v|/2\}.$$

Further assume $|v| \leq 1$, then $|u| \leq 1/2$ and the gain term is bounded by

$$\int_{|v|\leq 1, |u|\leq |v|/2} |u-v|^{\gamma} e^{-|u|^2/8} w^2 |\partial_{\beta_1}^{\alpha_1} g_1(u') \partial_{\beta_2}^{\alpha_2} g_2(v') \partial_{\beta}^{\alpha} g_3(v)| d\omega du dv dx \tag{4.38}$$

$$\leq C \int_{|v|\leq 1} \left\{ |v|^{\gamma/2} \int_{|u|\leq 1/2} |u-v|^{\gamma/2} e^{-|u|^2/8} |\partial_{\beta_1}^{\alpha_1} g_1(u') \partial_{\beta_2}^{\alpha_2} g_2(v')| d\omega du \right\} |\partial_{\beta}^{\alpha} g_3(v)| dv dx$$

$$\leq C \int_{|v|\leq 1} \left\{ |v|^{\gamma} \int_{|u|\leq 1/2} |\partial_{\beta_1}^{\alpha_1} g_1(u')|^2 |\partial_{\beta_2}^{\alpha_2} g_2(v')|^2 d\omega du \right\}^{1/2}$$

$$\times \left\{ \int_{|u|\leq 1/2} |u-v|^{\gamma} e^{-|u|^2/4} du \right\}^{1/2} |\partial_{\beta}^{\alpha} g_3(v)| dv dx$$

$$\leq C \int_{|v|\leq 1} \left\{ |v|^{\gamma} \int_{|u|\leq 1/2} |\partial_{\beta_1}^{\alpha_1} g_1(u')|^2 |\partial_{\beta_2}^{\alpha_2} g_2(v')|^2 d\omega du \right\}^{1/2} |\partial_{\beta}^{\alpha} g_3(v)| dv dx$$

$$\leq C \left\{ \int_{|v|\leq 1, |u|\leq 1/2} |v|^{\gamma} |\partial_{\beta_1}^{\alpha_1} g_1(u')|^2 |\partial_{\beta_2}^{\alpha_2} g_2(v')|^2 d\omega du dv dx \right\}^{1/2} \|\partial_{\beta}^{\alpha} g_3\|_{\boldsymbol{\nu},\ell}.$$

We now estimate the first factor. Since $|u| \leq |v|/2$, from (4.3) we have

$$|u'| + |v'| \leq 2[|u| + |v|] \leq 3|v|.$$

Since $\gamma < 0$, this implies

$$|v|^{\gamma} \leq 3^{-\gamma} |u'|^{\gamma}, \ |v|^{\gamma} \leq 3^{-\gamma} |v'|^{\gamma}.$$

Thus,

$$\int_{|v|\leq 1, |u|\leq 1/2} |v|^{\gamma} |\partial_{\beta_1}^{\alpha_1} g_1(u')|^2 |\partial_{\beta_2}^{\alpha_2} g_2(v')|^2 d\omega du dv$$

$$\leq C \int_{|v'|\leq 3, |u'|\leq 3} \min[|v'|^{\gamma}, |u'|^{\gamma}] |\partial_{\beta_1}^{\alpha_1} g_1(u')|^2 |\partial_{\beta_2}^{\alpha_2} g_2(v')|^2 d\omega du dv dx.$$

Now change variables $(v, u) \to (v', u')$ so that the above is

$$C \int_{|v|\leq 3, |u|\leq 3} \min[|v|^{\gamma}, |u|^{\gamma}] |\partial_{\beta_1}^{\alpha_1} g_1(u)|^2 |\partial_{\beta_2}^{\alpha_2} g_2(v)|^2 d\omega du dv dx.$$

Assume $|\alpha_1| + |\beta_1| \leq N/2$ and majorize the above by

$$C \int \left\{ \int_{|u|\leq 3} |u|^{\gamma} |\partial_{\beta_1}^{\alpha_1} g_1(u)|^2 du \right\} \left\{ \int_{|v|\leq 3} |\partial_{\beta_2}^{\alpha_2} g_2(v)|^2 dv \right\} dx$$

$$\leq C \sup_{x, |u|\leq 3} |\partial_{\beta_1}^{\alpha_1} g_1(u)|^2 \|\partial_{\beta_2}^{\alpha_2} g_2\|_{\boldsymbol{\nu},\ell}^2 \leq C \mathcal{E}_l(g_1) \|\partial_{\beta_2}^{\alpha_2} g_2\|_{\boldsymbol{\nu},\ell}^2.$$

Alternatively, if $|\alpha_2| + |\beta_2| \leq N/2$ then

$$C \int \left\{ \int_{|u|\leq 3} |\partial^{\alpha_1}_{\beta_1} g_1(u)|^2 du \right\} \left\{ \int_{|v|\leq 3} |v|^\gamma |\partial^{\alpha_2}_{\beta_2} g_2(v)|^2 dv \right\} dx$$

$$\leq C \|\partial^{\alpha_1}_{\beta_1} g_1\|^2_{\boldsymbol{\nu},\ell} \sup_{x,|v|\leq 3} |\partial^{\alpha_2}_{\beta_2} g_2(v)|^2 \leq C \|\partial^{\alpha_1}_{\beta_1} g_1\|^2_{\boldsymbol{\nu},\ell} \mathcal{E}_l(g_2).$$

Combine this upper bound with (4.38) to complete the estimate for the gain term over $\{|u| \leq |v|/2, \ |v| \leq 1\}$.

Case (2c) The Gain Term over $\{|u| \leq |v|/2, \ |v-u| \geq |v|/2, |v| \geq 1\}$.

The last case is the gain term over the region $\{|u| \leq |v|/2, \ |v-u| \geq |v|/2, |v| \geq 1\}$. The integral of $w^2(\ell,\vartheta)\Gamma^0_{\text{gain}}[\partial^{\alpha_1}_{\beta_1} g_1, \partial^{\alpha_2}_{\beta_2} g_2]\partial^\alpha_\beta g_3$ over such a region is bounded by

$$\int_{|u|\leq|v|/2,|v|\geq 1} |u-v|^\gamma e^{-|u|^2/4} w^2(\ell,\vartheta) |\partial^{\alpha_1}_{\beta_1} g_1(u') \partial^{\alpha_2}_{\beta_2} g_2(v') \partial^\alpha_\beta g_3(v)| d\omega du dv dx$$

$$\leq C \int_{|u|\leq|v|/2,|v|\geq 1} [1+|v|]^\gamma e^{-|u|^2/4} w^2(\ell,\vartheta) |\partial^{\alpha_1}_{\beta_1} g_1(u') \partial^{\alpha_2}_{\beta_2} g_2(v') \partial^\alpha_\beta g_3(v)| d\omega du dv dx$$

$$\leq C \left\{ \int [1+|v|]^\gamma e^{-|u|^2/4} w^2(\ell,\vartheta) |\partial^{\alpha_1}_{\beta_1} g_1(u')|^2 |\partial^{\alpha_2}_{\beta_2} g_2(v')|^2 d\omega du dv dx \right\}^{1/2}$$

$$\times \left\{ \int [1+|v|]^\gamma e^{-|u|^2/4} w^2(\ell,\vartheta) |\partial^\alpha_\beta g_3(v)|^2 d\omega du dv dx \right\}^{1/2} \tag{4.39}$$

$$\leq C \left\{ \int [1+|v|]^\gamma w^2(\ell,\vartheta)(v) |\partial^{\alpha_1}_{\beta_1} g_1(u')|^2 |\partial^{\alpha_2}_{\beta_2} g_2(v')|^2 d\omega du dv dx \right\}^{1/2} ||\partial^\alpha_\beta g_3||_{\boldsymbol{\nu},\ell,\vartheta}.$$

We have used $|v-u|^\gamma \leq 4^{-\gamma}[1+|v|]^\gamma$ in the first inequality. If $\ell\tau < 0$, use $|v| \geq 2|u|$ and (4.4) to establish

$$(1+|v|^2)^{\ell\tau/2} \leq C(1+|v'|^2+|u'|^2)^{\ell\tau/2}.$$

Conversely, if $\ell\tau \geq 0$, just use (4.4) to establish the same inequality. Recall (4.37) and $M(v) = \exp\left(\frac{q}{4}(1+|v|^2)^{\vartheta/2}\right)$. Thus,

$$w^2(\ell,\vartheta)(v) \leq C(1+|v'|^2+|u'|^2)^{\ell\tau} M(v')M(u')$$

Then since $\ell = |\beta| - l$ we have

$$w^2(\ell,\vartheta)(v) \leq C(1+|v'|^2+|u'|^2)^{-l\tau}(1+|v'|^2+|u'|^2)^{|\beta|\tau} M(v')M(u')$$

$$\leq C(1+|v'|^2)^{-l\tau}(1+|u'|^2)^{-l\tau}(1+|v'|^2+|u'|^2)^{|\beta|\tau} M(v')M(u').$$

Assume $|\alpha_2|+|\beta_2|\le N/2$. Using this estimate and the change of variable $(v,u)\to(v',u')$ we obtain

$$\int [1+|v|]^\gamma w^2(\ell,\vartheta)(v)|\partial^{\alpha_1}_{\beta_1}g_1(u')|^2|\partial^{\alpha_2}_{\beta_2}g_2(v')|^2 d\omega du dv dx$$

$$\le C\int [1+|u'|]^\gamma w^2(|\beta_1|-l,\vartheta)(u')|\partial^{\alpha_1}_{\beta_1}g_1(u')|^2 w^2(|\beta_2|-l,\vartheta)(v')|\partial^{\alpha_2}_{\beta_2}g_2(v')|^2$$

$$= C\int [1+|u|]^\gamma w^2(|\beta_1|-l,\vartheta)(u)|\partial^{\alpha_1}_{\beta_1}g_1(u)|^2 w^2(|\beta_2|-l,\vartheta)(v)|\partial^{\alpha_2}_{\beta_2}g_2(v)|^2 du dv dx$$

$$= C\int \left\{\int w^2(|\beta_2|-l,\vartheta)(v)|\partial^{\alpha_2}_{\beta_2}g_2(v)|^2 dv\right\}$$

$$\times\left\{\int [1+|u|]^\gamma w^2(|\beta_1|-l,\vartheta)(u)|\partial^{\alpha_1}_{\beta_1}g_1(u)|^2 du\right\} dx.$$

Using the embedding in (4.36), we see that this is bounded by

$$C\mathcal{E}_{l,\vartheta}(g_2)\int [1+|u|]^\gamma w^2(|\beta_1|-l,\vartheta)(u)|\partial^{\alpha_1}_{\beta_1}g_1(u)|^2 du dx$$

$$\le C\mathcal{E}_{l,\vartheta}(g_2)||\partial^{\alpha_1}_{\beta_1}g_1||^2_{\boldsymbol{\nu},|\beta_1|-l,\vartheta}.$$

Similarly if $|\alpha_1|+|\beta_1|\le N/2$ then this is bounded by

$$C||\partial^{\alpha_2}_{\beta_2}g_2||^2_{\boldsymbol{\nu},|\beta_2|-l,\vartheta}\mathcal{E}_{l,\vartheta}(g_1).$$

Combine this with (4.39) to complete the nonlinear estimate. □

This completes the estimates for the Boltzmann case. In Section 4.5 we use these to establish global existence. Then in Section 4.6 we prove the decay. In the next section we establish the analogous estimates for the Landau case.

4.4. Landau Estimates

In this section, we will prove the basic estimates used to obtain global existence of solutions with an exponential weight in the Landau case. In this case, the derivatives in the Landau operator cause extra difficulties in particular because $\partial_i w(\ell,\vartheta)$ can grow faster in v than $w(\ell,\vartheta)$. This new feature of the exponential weight (4.10) forces us to weaken the linear estimate with high order velocity derivatives (Lemma 4.8) from the analogous estimate [38, Lemma 6, p.403]. A new linear estimate with no extra derivatives (Lemma 4.9) is also necessary because of the exponential weight.

It turns out that we need to dig up exact cancellation in order to prove this estimate in the $\vartheta = 2$ case. We will again point out the new features of each proof after the statement of each lemma.

For any vector-valued function $\mathbf{g}(v) = (g_1, g_2, g_3)$, we define the projection to the vector v as

$$P_v g_i \equiv \frac{v_i}{|v|} \sum_{j=1}^{3} \frac{v_j}{|v|} g_j. \tag{4.40}$$

Furthermore, in this section we will use the Einstein summation convention over i and j, e.g. repeated indices are always summed:

$$\sigma^i(v) = \sigma^{ij}(v)\frac{v_j}{2} = \sum_{j=1}^{3} \sigma^{ij}(v)\frac{v_j}{2}.$$

With this notation we have

LEMMA 4.4. [16,38] *$\sigma^{ij}(v)$, $\sigma^i(v)$ are smooth functions such that*

$$|\partial_\beta \sigma^{ij}(v)| + |\partial_\beta \sigma^i(v)| \leq C_\beta [1 + |v|]^{\gamma+2-|\beta|},$$

and furthermore

$$\sigma^{ij}(v) = \lambda_1(v)\frac{v_i v_j}{|v|^2} + \lambda_2(v)\left(\delta_{ij} - \frac{v_i v_j}{|v|^2}\right). \tag{4.41}$$

Thus

$$\sigma^{ij}(v) g_i g_j = \lambda_1(v) \sum_{i=1}^{3} \{P_v g_i\}^2 + \lambda_2(v) \sum_{i=1}^{3} \{[I - P_v] g_i\}^2. \tag{4.42}$$

Moreover, there are constants c_1 and $c_2 > 0$ such that as $|v| \to \infty$

$$\lambda_1(v) \sim c_1 [1 + |v|]^\gamma, \ \ \lambda_2(v) \sim c_2 [1 + |v|]^{\gamma+2}.$$

The estimate of σ^{ij} and σ^i with high derivatives was already established in [16]. The computation of the eigenvalues and their convergence rate was already shown in [38]. We prove the representation (4.41) below because we will use it in important places in later proofs and it is not formally written down in the other papers.

Proof. Recall from (4.9) that

$$\sigma^{ij}(v) = (2\pi)^{-3/2}\int_{\mathbb{R}^3}\left(\delta_{ij} - \frac{(v_i-u_i)(v_j-u_j)}{|v-u|^2}\right)|v-u|^{\gamma+2}e^{-|u|^2/2}du.$$

Changing variables $u \to v-u$ we have

$$\sigma^{ij}(v) = (2\pi)^{-3/2}\int_{\mathbb{R}^3}\left(\delta_{ij} - \frac{u_iu_j}{|u|^2}\right)|u|^{\gamma+2}e^{-|v-u|^2/2}du.$$

Given $v \in \mathbb{R}^3$ define $v^1 = v/|v|$ and complete an orthonormal basis $\{v^1, v^2, v^3\}$ where $v^i \cdot v^j = \delta_{ij}$. Then define the corresponding orthogonal 3×3 matrix as

$$\mathcal{O} = [v^1\ v^2\ v^3].$$

Applying this orthogonal transformation to the integral in σ^{ij} above we obtain

$$\sigma^{ij}(v) = (2\pi)^{-3/2}\int_{\mathbb{R}^3}\left(\delta_{ij} - \frac{(\mathcal{O}u)_i(\mathcal{O}u)_j}{|u|^2}\right)|u|^{\gamma+2}e^{-|v-\mathcal{O}u|^2/2}du.$$

Here we have used $|\mathcal{O}u| = |u|$. Also

$$(\mathcal{O}u)_i = u_1v_i^1 + u_2v_i^2 + u_3v_i^3.$$

And

$$|v-\mathcal{O}u|^2 = |v-u_1v^1-u_2v^2-u_3v^3|^2 = (|v|-u_1)^2+u_2^2+u_3^2. \tag{4.43}$$

We therefore have

$$\begin{aligned}\sigma^{ij}(v) &= (2\pi)^{-3/2}\int_{\mathbb{R}^3}\left(\delta_{ij} - \sum_{l,m=1}^{3}\frac{u_lu_m}{|u|^2}v_i^lv_j^m\right)|u|^{\gamma+2}e^{-\frac{1}{2}[(|v|-u_1)^2+u_2^2+u_3^2]}du\\ &= (2\pi)^{-3/2}\int_{\mathbb{R}^3}\left(\delta_{ij} - \sum_{m=1}^{3}\frac{u_m^2}{|u|^2}v_i^mv_j^m\right)|u|^{\gamma+2}e^{-\frac{1}{2}[(|v|-u_1)^2+u_2^2+u_3^2]}du,\end{aligned}$$

where we have used the odd function argument. Write ($m=1,2,3$)

$$\begin{aligned}B_0(v) &\equiv (2\pi)^{-3/2}\int_{\mathbb{R}^3}|u|^{\gamma+2}e^{-\frac{1}{2}[(|v|-u_1)^2+u_2^2+u_3^2]}du,\\ B_m(v) &\equiv (2\pi)^{-3/2}\int_{\mathbb{R}^3}\frac{u_m^2}{|u|^2}|u|^{\gamma+2}e^{-\frac{1}{2}[(|v|-u_1)^2+u_2^2+u_3^2]}du.\end{aligned}$$

Then by symmetry

$$B_2(v) = B_3(v) = (2\pi)^{-3/2}\int_{\mathbb{R}^3}\frac{u_2^2+u_3^2}{2|u|^2}|u|^{\gamma+2}e^{-\frac{1}{2}[(|v|-u_1)^2+u_2^2+u_3^2]}du,$$

and we have

$$\sigma^{ij}(v) = B_0(v)\delta_{ij} - B_1(v)v_j^1 v_j^1 - B_2(v)(v_i^2 v_j^2 + v_i^3 v_j^3).$$

Define the orthogonal projections $P_j = v^j \otimes v^j$. Then we have the resolution of the identity $I = P_1 + P_2 + P_3$. In component form this is

$$v_i^2 v_j^2 + v_i^3 v_j^3 = \delta_{ij} - \frac{v_i v_j}{|v|^2}.$$

Then we have

$$\sigma^{ij}(v) = \{B_0(v) - B_1(v)\}\frac{v_i v_j}{|v|^2} + \{B_0(v) - B_2(v)\}\left(\delta_{ij} - \frac{v_i v_j}{|v|^2}\right).$$

Now the eigenvalues are $\lambda_1(v) = B_0(v) - B_1(v)$ and $\lambda_2(v) = B_0(v) - B_2(v)$ or

$$\lambda_1(v) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} |u|^{\gamma+2}\left(1 - \frac{u_1^2}{|u|^2}\right) e^{-\frac{1}{2}[(|v|-u_1)^2 + u_2^2 + u_3^2]}du,$$

and

$$\lambda_2(v) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} |u|^{\gamma+2}\left(1 - \frac{u_2^2 + u_3^2}{2|u|^2}\right) e^{-\frac{1}{2}[(|v|-u_1)^2 + u_2^2 + u_3^2]}du.$$

This completes the derivation of the spectral representation for $\sigma^{ij}(v)$. $\square$

Next, we write bounds for the σ norm.

LEMMA 4.5. [38, Corollary 1, p.399] *There exists $c > 0$ such that*

$$c|g|^2_{\boldsymbol{\sigma},\ell,\vartheta} \geq \left|[1+|v|]^{\frac{\gamma}{2}}\{P_v\partial_i g\}\right|^2_{\ell,\vartheta} + \left|[1+|v|]^{\frac{\gamma+2}{2}}\{[I-P_v]\partial_i g\}\right|^2_{\ell,\vartheta} + \left|[1+|v|]^{\frac{\gamma+2}{2}} g\right|^2_{\ell,\vartheta}.$$

Furthermore,

$$\frac{1}{c}|g|^2_{\boldsymbol{\sigma},\ell,\vartheta} \leq \left|[1+|v|]^{\frac{\gamma}{2}}\{P_v\partial_i g\}\right|^2_{\ell,\vartheta} + \left|[1+|v|]^{\frac{\gamma+2}{2}}\{[I-P_v]\partial_i g\}\right|^2_{\ell,\vartheta} + \left|[1+|v|]^{\frac{\gamma+2}{2}} g\right|^2_{\ell,\vartheta}.$$

The upper bound was not written down in [38], but the proof is the same. We write it down here because we will use it in the non-linear estimate. Next, the operators A, K and Γ from (4.5) and (4.8) in the Landau case are defined.

LEMMA 4.6. [38, Lemma 1, p.395] *We have the following representations for* A, K *and* Γ.

$$
\begin{aligned}
(4.44)\qquad Ag_2 &= \partial_i[\sigma^{ij}\partial_j g_2] - \sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g_2 + \partial_i\sigma^i g_2 \\
(4.45)\qquad Kg_1 &= -\mu^{-1/2}\partial_i\{\mu[\phi^{ij} * \{\mu^{1/2}[\partial_j g_1 + \frac{v_j}{2}g_1]\}]\} \\
&= -\mu^{-1/2}\partial_i\left\{\mu\int_{\mathbb{R}^3}\phi^{ij}(v-v')\mu^{1/2}(v')[\partial_j g_1(v') + \frac{v_j'}{2}g_1(v')]dv'\right\}, \\
\Gamma[g_1,g_2] &= \partial_i[\{\phi^{ij} * [\mu^{1/2}g_1]\}\partial_j g_2] - \{\phi^{ij} * [\frac{v_i}{2}\mu^{1/2}g_1]\}\partial_j g_2 \\
(4.46)\qquad &\quad -\partial_i[\{\phi^{ij} * [\mu^{1/2}\partial_j g_1]\}g_2] + \{\phi^{ij} * [\frac{v_i}{2}\mu^{1/2}\partial_j g_1]\}g_2.
\end{aligned}
$$

These representations are different in a few places by a factor of $\frac{1}{2}$ from those in [38]. The only reason for this difference is our use of a different normalization for the Maxwellian in this paper.

PROOF. We only reprove A. First notice that for either fixed i or j

$$
(4.47)\qquad \sum_i \phi^{ij}(v)v_i = \sum_j \phi^{ij}(v)v_j = 0.
$$

We now take the derivatives inside Ag_2

$$
\begin{aligned}
Ag_2 &= \mu^{-1/2}Q(\mu,\mu^{1/2}g_2) \\
&= \mu^{-1/2}\partial_i\{\sigma^{ij}\mu^{1/2}[\partial_j g_2 - \frac{v_j}{2}g_2]\} + \mu^{-1/2}\partial_i\{\{\phi^{ij} * [v_j\mu]\}\mu^{1/2}g_2\} \\
&= \mu^{-1/2}\partial_i\{\sigma^{ij}\mu^{1/2}[\partial_j g_2 - \frac{v_j}{2}g_2]\} \\
&\quad +\mu^{-1/2}\partial_i\{[\phi^{ij} * \mu]v_j\mu^{1/2}g_2\} \qquad\qquad \text{by (4.47)} \\
&= \mu^{-1/2}\partial_i\{\sigma^{ij}\mu^{1/2}[\partial_j g_2 + \frac{v_j}{2}g_2]\} \\
&= \mu^{-1/2}\partial_i\{\sigma^{ij}\mu^{1/2}\partial_j g_2\} + \mu^{-1/2}\partial_i\{\sigma^{ij}\mu^{1/2}\frac{v_j}{2}g_2\} \\
&= \partial_i[\sigma^{ij}\partial_j g_2] + \mu^{-1/2}\partial_i[\mu^{1/2}]\sigma^{ij}\partial_j g_2 + \partial_i\{\sigma^{ij}\frac{v_j}{2}g_2\} + \mu^{-1/2}\partial_i[\mu^{1/2}]\sigma^{ij}\frac{v_j}{2}g_2 \\
&= \partial_i[\sigma^{ij}\partial_j g_2] - \sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g_2 + \mu^{-1/2}\partial_i[\mu^{1/2}]\sigma^{ij}\partial_j g_2 + \partial_i\{\sigma^{ij}\frac{v_j}{2}g_2\} \\
&= \partial_i[\sigma^{ij}\partial_j g_2] - \sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g_2 + \partial_i\{\sigma^{ij}\frac{v_j}{2}\}g_2,
\end{aligned}
$$

from (4.9), where $\partial_i[\mu^{1/2}] = \frac{v_i}{2}\mu^{1/2}$. ∎

In the rest of this section, we will prove estimates for A, K and Γ.

LEMMA 4.7. *Let* $0 \le \vartheta \le 2$, $\ell \in \mathbb{R}$ *and* $0 < q$. *If* $\vartheta = 2$ *restrict* $0 < q < 1$. *Then for any* $\eta > 0$, *there is* $0 < C = C(\eta) < \infty$ *such that*

$$|\langle w^2 \partial_i \sigma^i g_1, g_2 \rangle| + |\langle w^2 K g_1, g_2 \rangle| \le \eta \, |g_1|_{\boldsymbol{\sigma},\ell,\vartheta} \, |g_2|_{\boldsymbol{\sigma},\ell,\vartheta} + C |g_1 \bar{\chi}_C|_\ell |g_2 \bar{\chi}_C|_\ell, \tag{4.48}$$

where $w^2 = w^2(\ell, \vartheta)$ *and* $\bar{\chi}_{C(\eta)}$ *is defined in* (4.19).

The estimate for the $\partial_i \sigma^i$ term is exactly the same as in [38]. But, as in the Boltzmann case, the estimate for K needs modification because g_1 in Kg_1 does not depend on v. So we need to show that K can control one exponentially growing factor $w(\ell,\vartheta)(v)$. We remark that, although it is not used in this paper, the proof clearly shows $|\langle w^2 K g_1, g_2 \rangle| \le \eta \, |g_1|_{\boldsymbol{\sigma},\ell} \, |g_2|_{\boldsymbol{\sigma},\ell,\vartheta} + C |g_1 \bar{\chi}_C|_\ell |g_2 \bar{\chi}_C|_\ell$.

PROOF. For $m > 0$, we split

$$\int w^2 \partial_i \sigma^i g_1 g_2 = \int_{\{|v| \le m\}} + \int_{\{|v| \ge m\}}.$$

By Lemma 4.4, $|\partial_i \sigma^i| \le C[1 + |v|]^{\gamma+1}$. Thus, the integral over $\{|v| \le m\}$ is $\le C(m) |g_1 \bar{\chi}_m|_\ell |g_2 \bar{\chi}_m|_\ell$. From Lemma 4.5 and the Cauchy-Schwartz inequality

$$\int_{\{|v| \ge m\}} w^2 |\partial_i \sigma^i g_1 g_2| dv \le \frac{C}{m} \int w^2 [1 + |v|]^{\gamma+2} |g_1 g_2| \le \frac{C}{m} \, |g_1|_{\boldsymbol{\sigma},\ell,\vartheta} \, |g_2|_{\boldsymbol{\sigma},\ell,\vartheta}. \tag{4.49}$$

This completes (4.48) for the $\partial_i \sigma^i$ term.

Recalling the linear operator K in (4.45), we have

$$\begin{aligned} w^2 K g_1 \;=\; & -\partial_i \{ w^2 \mu^{1/2} [\phi^{ij} * \{\mu^{1/2} \partial_j g_1 + \frac{v_j}{2} \mu^{1/2} g_1\}]\} \\ & + \partial_i (w^2) \mu^{1/2} [\phi^{ij} * \{\mu^{1/2} \partial_j g_1 + \frac{v_j}{2} \mu^{1/2} g_1\}] \\ & + w^2 \frac{v_i}{2} \mu^{1/2} [\phi^{ij} * \{\mu^{1/2} \partial_j g_1 + \frac{v_j}{2} \mu^{1/2} g_1\}]. \end{aligned} \tag{4.50}$$

The derivative of the weight function is

$$\partial_i (w^2(\ell,\vartheta)) = w^2(\ell,\vartheta) w_1(v) v_i. \tag{4.51}$$

where

$$w_1(v) = \left\{ 2\ell\tau(1+|v|^2)^{-1} + q\frac{\vartheta}{2}(1+|v|^2)^{\frac{\vartheta}{2}-1} \right\}. \tag{4.52}$$

After integrating by parts for the first term and collecting terms, we can rewrite $\langle w^2 K g_1, g_2 \rangle$ as

$$\sum_{|\beta_1|,|\beta_2|\leq 1} \int w^2(v)\phi^{ij}(v-v')\mu^{1/2}(v)\mu^{1/2}(v')\bar{\mu}_{\beta_1\beta_2}(v,v')\partial_{\beta_1}g_1(v')\partial_{\beta_2}g_2(v)dv'dv,$$

where $\bar{\mu}_{\beta_1\beta_2}(v,v')$ is a collection of smooth functions satisfying

$$|\nabla_v \bar{\mu}_{\beta_1\beta_2}(v,v')| + |\nabla_{v'}\bar{\mu}_{\beta_1\beta_2}(v,v')| + |\bar{\mu}_{\beta_1\beta_2}(v,v')| \leq C(1+|v'|^2)^{1/2}(1+|v|^2)^{1/2}.$$

Since either $0 \leq \vartheta < 2$ or $\vartheta = 2$ and $0 < q < 1$, there exists $0 < q' < 1$ such that

$$w(\ell,\vartheta)(v)\mu^{1/2}(v) \leq C\mu^{q'/2}(v) \tag{4.53}$$

If $0 \leq \vartheta < 2$ choose any $0 < q' < 1$ and if $\vartheta = 2$ choose $0 < q' < 1-q$. Therefore, we can rewrite $\langle w^2 K g_1, g_2 \rangle$ as

$$\sum_{|\beta_1|,|\beta_2|\leq 1} \int w(v)\phi^{ij}(v-v')\mu^{q'/4}(v)\mu^{1/4}(v')\mu_{\beta_1\beta_2}(v,v')\partial_{\beta_1}g_1(v')\partial_{\beta_2}g_2(v)dv'dv,$$

where $\mu_{\beta_1\beta_2}(v,v')$ is a different collection of smooth functions satisfying

$$|\nabla_v \mu_{\beta_1\beta_2}(v,v')| + |\nabla_{v'}\mu_{\beta_1\beta_2}(v,v')| + |\mu_{\beta_1\beta_2}(v,v')| \leq Ce^{-\frac{q'}{16}|v|^2}e^{-\frac{1}{16}|v'|^2}.$$

We have removed an exponentially growing factor $w(\ell,\vartheta)(v)$.

Since $\phi^{ij}(v) = O(|v|^{\gamma+2}) \in L^2_{loc}(\mathbb{R}^3)$ and $\gamma \geq -3$, Fubini's Theorem implies

$$\phi^{ij}(v-v')\mu^{q'/4}(v)\mu^{1/4}(v') \in L^2(\mathbb{R}^3\times\mathbb{R}^3).$$

Therefore, for any given $m > 0$, we can choose a C_c^∞ function $\psi^{ij}(v,v')$ such that

$$||\phi^{ij}(v-v')\mu^{q'/4}(v)\mu^{1/4}(v') - \psi^{ij}(v,v')||_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \leq \frac{1}{m},$$

$$\text{supp}\{\psi^{ij}\} \subset \{|v'|+|v| \leq C(m)\} < \infty.$$

We split

$$\phi^{ij}(v-v')\mu^{q'/4}(v)\mu^{1/4}(v') = \psi^{ij} + [\phi^{ij}(v-v')\mu^{q'/4}(v)\mu^{1/4}(v') - \psi^{ij}].$$

Then

$$(4.54) \qquad \langle w^2 K g_1, g_2\rangle = J_1[g_1,g_2] + J_2[g_1,g_2],$$

where

$$J_1 = \int w(v)\psi^{ij}(v,v')\mu_{\beta_1\beta_2}(v,v')\partial_{\beta_1}g_1(v')\partial_{\beta_2}g_2(v)dv'dv,$$

$$J_2 = \int w(v)[\phi^{ij}(v-v')\mu^{q'/4}(v)\mu^{1/4}(v') - \psi^{ij}]\mu_{\beta_1\beta_2}(v,v')\partial_{\beta_1}g_1(v')\partial_{\beta_2}g_2(v)dv'dv.$$

Above we are implicitly summing over $|\beta_1|, |\beta_2| \le 1$. We will bound each of these terms separately.

The J_2 term is bounded as

$$\begin{aligned}
|J_2| &\le ||\phi^{ij}(v-v')\mu^{q'/4}(v)\mu^{1/4}(v') - \psi^{ij}(v,v')||_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \\
&\quad \times ||w(v)\mu_{\beta_1\beta_2}(v,v')\partial_{\beta_1}g_1(v')\partial_{\beta_2}g_2(v)||_{L^2(\mathbb{R}^3\times\mathbb{R}^3)} \\
&\le \frac{C}{m}\left|\mu^{1/16}\partial_{\beta_1}g_1\right|_0 \left|\mu^{q'/16}\partial_{\beta_2}g_2\right|_{\ell,\vartheta} \le \frac{C}{m}\left|g_1\right|_{\boldsymbol{\sigma},\ell}\left|g_2\right|_{\boldsymbol{\sigma},\ell,\vartheta}.
\end{aligned}$$

Now for the first term J_1, integrations by parts over v and v' variables yields

$$\begin{aligned}
|J_1| &= \left|(-1)^{\beta_1+\beta_2}\int \partial_{\beta_2}[w(v)\partial_{\beta_1}\{\psi^{ij}(v,v')\mu_{\beta_1\beta_2}(v,v')\}]g_1(v')g_2(v)\right| \\
&\le C||\psi^{ij}||_{C^2}\left\{\int_{|v|\le C(m)}|g_1|^2dv\right\}^{1/2}\left\{\int_{|v|\le C(m)}|g_2|^2dv\right\}^{1/2}.
\end{aligned}$$

We thus conclude (4.48) by choosing $m>0$ large enough. □

Next, we estimate the linear terms with velocity derivatives.

LEMMA 4.8. *Let $|\beta|>0$, $\ell\in\mathbb{R}$, $0\le\vartheta\le 2$ and $q>0$. If $\vartheta=2$ fix $0<q<1$. Then for small $\eta>0$, there exists $C(\eta)>0$ such that*

$$|\langle w^2(\ell,\vartheta)\partial_\beta[Kg_1],g_2\rangle| \le \left\{\eta\sum_{|\bar\beta|\le|\beta|}\left|\partial_{\bar\beta}g_1\right|_{\boldsymbol{\sigma},\ell} + C(\eta)\left|\bar\chi_{C(\eta)}g_1\right|_\ell\right\}\left|g_2\right|_{\boldsymbol{\sigma},\ell,\vartheta}.$$

Further if $\tau\le -1$ in (4.10) and $\ell = r-l$ where $l\ge 0$ and $r\ge|\beta|$, then

$$-\langle w^2\partial_\beta[Ag],\partial_\beta g\rangle \ge \left|\partial_\beta g\right|^2_{\boldsymbol{\sigma},\ell,\vartheta} - \eta\sum_{|\bar\beta|=|\beta|}\left|\partial_{\bar\beta}g\right|^2_{\boldsymbol{\sigma},\ell,\vartheta} - C(\eta)\sum_{|\bar\beta|<|\beta|}\left|\partial_{\bar\beta}g\right|^2_{\boldsymbol{\sigma},|\bar\beta|-l,\vartheta},$$

where $w^2 = w^2(\ell,\vartheta)$.

Notice that the estimate involving $[Ag]$ is much weaker than the analogous estimate [38, Lemma 6, p.403] with no exponential weight. In [38], there are no derivatives in the last term on the right. The key problem here is that derivatives of the exponential weight, in particular $\partial_i(w^2(\ell,\vartheta))$, can grow faster than $w^2(\ell,\vartheta)$. Then, in some cases, we don't have enough decay to get the sharper estimate. Instead, we weaken the estimate and use lower order derivatives to extract polynomial decay from higher order weights. For the estimate involving $[Kg_1]$ the difference is the same as in the previous cases; we again show that K controls an exponentially growing factor of $w(\ell,\vartheta)(v)$. We remark that these estimates are not at all optimal. It is not hard to see that you can use a smaller norm over a compact region, in particular, for the terms with no derivatives in the $[Ag]$ estimate.

Proof. We begin with the estimate involving $\partial_\beta[Ag]$. Using Lemma 4.6, we have

$$
\begin{aligned}
\langle w^2\partial_\beta[Ag],\partial_\beta g\rangle \;=\; & -|\partial_\beta g|^2_{\boldsymbol{\sigma},\ell,\vartheta} - C_\beta^{\beta_1}\langle w^2\partial_{\beta_1}\sigma^{ij}\partial_{\beta-\beta_1}\partial_j g,\partial_\beta\partial_i g\rangle \\
& -C_\beta^{\beta_2}\langle \partial_i(w^2)\partial_{\beta_2}\sigma^{ij}\partial_{\beta-\beta_2}\partial_j g,\partial_\beta g\rangle \\
& -C_\beta^{\beta_1}\langle w^2\partial_{\beta_1}\{\sigma^{ij}v_iv_j\}\partial_{\beta-\beta_1}g,\partial_\beta g\rangle \\
& +C_\beta^{\beta_2}\langle w^2\partial_{\beta_2}\partial_i\sigma^i\partial_{\beta-\beta_2}g,\partial_\beta g\rangle.
\end{aligned}
\tag{4.55}
$$

Here summations are over $\beta\geq\beta_1>0$ and $\beta\geq\beta_2\geq 0$. We will estimate each of these terms separately.

Case 1. The Last Two Terms.

First, we consider the last two terms in (4.55). We claim that

$$
|w^2\partial_{\beta_2}\partial_i\sigma^i(v)| + |w^2\partial_{\beta_1}\{\sigma^{ij}v_iv_j\}| \leq C[1+|v|]^{\gamma+1}w^2,
$$

For the first term on the l.h.s., this follows from Lemma 4.4. For the estimate for the second term on the r.h.s., from (4.9) and (4.47) we have

$$
\sigma^{ij}(v)v_iv_j = \int_{\mathbb{R}^3}\phi^{ij}(v-u)u_iu_j\mu(u)du.
$$

Now the estimate follows from [38, Lemma 2,p.397] and $|\beta_1| > 0$. Using the claim, the last two terms in (4.55) are bounded by

$$C\int w^2[1+|v|]^{\gamma+1}\{|\partial_{\beta-\beta_1}g| + |\partial_{\beta-\beta_2}g|\}|\partial_\beta g| = C\int_{|v|\le m} + C\int_{|v|\ge m}$$

$$\le C\int_{|v|\le m} + \frac{C}{m}\left|[1+|v|]^{\frac{\gamma+2}{2}}\{|\partial_{\beta-\beta_1}g| + |\partial_{\beta-\beta_2}g|\}\right|_{\ell,\vartheta}\left|[1+|v|]^{\frac{\gamma+2}{2}}\partial_\beta g\right|_{\ell,\vartheta}$$

$$\le C\int_{|v|\le m} + \frac{C}{m}\sum_{|\bar\beta|\le|\beta|}|\partial_{\bar\beta}g|_{\boldsymbol{\sigma},\ell,\vartheta}\,|\partial_\beta g|_{\boldsymbol{\sigma},\ell,\vartheta}\,. \tag{4.56}$$

We have used Lemma 4.5 in the last step. For the part $|v|\le m$, for any $m'>0$, we use the compact Sobolev space interpolation and Lemma 4.5 to get

$$\begin{aligned}\int_{|v|\le m} &\le \frac{1}{m'}\sum_{|\bar\beta|=|\beta|+1}\int_{|v|\le m}|\partial_{\bar\beta}g|^2 + C_{m'}\int_{|v|\le m}|g|^2\\ &\le \frac{C}{m'}\sum_{|\bar\beta|=|\beta|}|\partial_{\bar\beta}g|^2_{\boldsymbol{\sigma},\ell} + C_{m'}\,|\bar\chi_m g|^2_\ell\,. \qquad (4.57)\end{aligned}$$

We used Lemma 4.5 again in the last step. This completes the estimate for the last two terms in (4.55).

Case 2. The Second Term.

Next, we consider the second term in (4.55). Since $|\beta_1|\ge 1$, we have

$$|\langle w^2\partial_{\beta_1}\sigma^{ij}\partial_{\beta-\beta_1}\partial_j g, \partial_\beta\partial_i g\rangle| \le C\int[1+|v|]^{\gamma+1}w^2|\partial_{\beta-\beta_1}\partial_j g\partial_\beta\partial_i g|$$

$$\le C\,|\partial_\beta g|_{\boldsymbol{\sigma},\ell,\vartheta}\left\{\int[1+|v|]^{\gamma+2}w^2(\ell,\vartheta)|\partial_{\beta-\beta_1}\partial_j g|^2\right\}^{1/2}$$

Now using (4.61) with $\beta_1=\beta_2$, given $m'>0$ this is

$$\le C\,|\partial_\beta g|_{\boldsymbol{\sigma},\ell,\vartheta}\left\{\int[1+|v|]^{\gamma}w^2(|\beta-\beta_1|-l,\vartheta)|\partial_{\beta-\beta_1}\partial_j g|^2\right\}^{1/2} \tag{4.58}$$

$$\le C\,|\partial_\beta g|_{\boldsymbol{\sigma},\ell,\vartheta}\sum_{|\bar\beta|\le|\beta|-1}|\partial_{\bar\beta}g|_{\boldsymbol{\sigma},|\bar\beta|-l,\vartheta} \le \frac{1}{m'}|\partial_\beta g|^2_{\boldsymbol{\sigma},\ell,\vartheta} + C_{m'}\sum_{|\bar\beta|\le|\beta|-1}|\partial_{\bar\beta}g|^2_{\boldsymbol{\sigma},|\bar\beta|-l,\vartheta}\,.$$

We have now estimated all the terms in (4.55). We conclude case 2 by first choosing m large enough.

Case 3. The Third Term.

Next consider the third and most delicate term in (4.55) when $|\beta_2| = 0$. Recall $\partial_i(w^2)$ from (4.51); from (4.52) we have $|w_1(v)| \leq C$ since $0 \leq \vartheta \leq 2$. And from (4.41) we have

$$\sigma^{ij}(v)v_i = \lambda_1(v)v_j \tag{4.59}$$

Using Lemma 4.4 for the decay of $\lambda_1(v)$, the third term in (4.55) with $|\beta_2| = 0$ is

$$\begin{aligned}\left|\langle \partial_i(w^2)\sigma^{ij}\partial_\beta\partial_j g, \partial_\beta g\rangle\right| &\leq C\int w^2(\ell,\vartheta)[1+|v|]^{\gamma+1}\left|\partial_\beta\partial_j g\right|\left|\partial_\beta g\right| dv, \\ &= C\int \left(w(\ell,\vartheta)[1+|v|]^{\frac{\gamma}{2}}\left|\partial_\beta\partial_j g\right|\right)\left(w(\ell,\vartheta)[1+|v|]^{(\gamma+2)/2}\left|\partial_\beta g\right|\right) dv.\end{aligned}$$

Consider the second term in parenthesis. We will use the weight to extract extra polynomial decay and look at this as a term with lower order derivatives in the σ norm. Write $\partial_\beta = \partial_{\beta-e_k}\partial_k$ were e_k is an element of the standard basis. Further, from (4.10) with $\tau \leq -1$, write out

$$\begin{aligned}w(\ell,\vartheta) = (1+|v|^2)^{\tau\ell/2}\exp\left(\frac{q}{4}(1+|v|^2)^{\frac{\vartheta}{2}}\right) &= w(\ell-1,\vartheta)(1+|v|^2)^{\tau/2} \\ &\leq Cw(\ell-1,\vartheta)[1+|v|]^{-1}.\end{aligned}$$

Since $|e_k| = 1$ and $\ell = r - l$ with $r \geq |\beta|$, $\ell - 1 \geq |\beta| - 1 - l = |\beta - e_k| - l$. Thus,

$$w(\ell-1,\vartheta)[1+|v|]^{-1} \leq w(|\beta - e_k| - l,\vartheta)[1+|v|]^{-1}.$$

Hence,

$$w(\ell,\vartheta)[1+|v|]^{(\gamma+2)/2} \leq w(|\beta-e_k|-l,\vartheta)[1+|v|]^{\frac{\gamma}{2}}.$$

Then, for any large $m' > 0$, $|\langle \partial_i(w^2)\sigma^{ij}\partial_\beta\partial_j g, \partial_\beta g\rangle|$ is

$$\begin{aligned}&\leq C\int \left(w(\ell,\vartheta)[1+|v|]^{\frac{\gamma}{2}}\left|\partial_\beta\partial_j g\right|\right)\left(w(|\beta-e_k|-l,\vartheta)[1+|v|]^{\frac{\gamma}{2}}\left|\partial_{\beta-e_k}\partial_k g\right|\right) dv \\ &\leq C|\partial_\beta g|_{\boldsymbol{\sigma},\ell,\vartheta}|\partial_{\beta-e_k} g|_{\boldsymbol{\sigma},|\beta-e_k|-l,\vartheta} \\ &\leq \frac{1}{m'}|\partial_\beta g|^2_{\boldsymbol{\sigma},\ell,\vartheta} + C_{m'}\sum_{|\bar\beta|<|\beta|}|\partial_{\bar\beta} g|^2_{\boldsymbol{\sigma},|\bar\beta|-l,\vartheta}.\end{aligned} \tag{4.60}$$

This completes the estimate for the third term in (4.55) when $|\beta_2| = 0$.

Next consider the third term in (4.55) when $|\beta_2| > 0$. Since $|\beta_2| \geq 1$,

$$\left|\partial_i(w^2(\ell,\vartheta))\partial_{\beta_2}\sigma^{ij}\right| \leq Cw^2(\ell,\vartheta)[1+|v|]^{\gamma+2}$$

Notice that the order of $\partial_{\beta-\beta_2}\partial_j$ in this case is $< |\beta|$. Again we exploit the lower order derivative to gain some decay from the weight. Since $\tau \leq -1$, we split

$$\begin{aligned} w(\ell,\vartheta) = w(\ell-1+1,\vartheta) &= w(\ell-1,\vartheta)(1+|v|^2)^{\tau/2} \\ &\leq w(|\beta-\beta_2|-l,\vartheta)[1+|v|]^{-1}. \end{aligned} \tag{4.61}$$

In this last step we have used $\ell = r - l$, $r \geq |\beta|$ so that $r-1 \geq |\beta-\beta_2|$ since $|\beta_2| \geq 1$. Given $m' > 0$, in this case, the third term in (4.55) has the upper bound

$$\begin{aligned} C\int \left|\partial_i(w^2(\ell,\vartheta))\partial_{\beta_2}\sigma^{ij}\right| |\partial_{\beta-\beta_2}\partial_j g\partial_\beta g| &\leq C\int w^2[1+|v|]^{\gamma+2}|\partial_{\beta-\beta_2}\partial_j g\partial_\beta g| \\ &\leq C|\partial_\beta g|_{\boldsymbol{\sigma},\ell,\vartheta}\left\{\int w^2(\ell,\vartheta)[1+|v|]^{\gamma+2}|\partial_{\beta-\beta_2}\partial_j g|^2 dv\right\}^{1/2} \\ &\leq C|\partial_\beta g|_{\boldsymbol{\sigma},\ell,\vartheta}\left\{\int w^2(|\beta-\beta_2|-l,\vartheta)[1+|v|]^{\gamma}|\partial_{\beta-\beta_2}\partial_j g|^2 dv\right\}^{1/2} \\ &\leq C|\partial_\beta g|_{\boldsymbol{\sigma},\ell,\vartheta}\sum_{|\bar\beta|\leq|\beta|-1}\left|\partial_{\bar\beta} g\right|_{\boldsymbol{\sigma},|\bar\beta|-l,\vartheta} \leq \frac{1}{m'}|\partial_\beta g|^2_{\boldsymbol{\sigma},\ell,\vartheta} + C_{m'}\sum_{|\bar\beta|\leq|\beta|-1}\left|\partial_{\bar\beta} g\right|^2_{\boldsymbol{\sigma},|\bar\beta|-l,\vartheta}. \end{aligned} \tag{4.62}$$

This completes the estimate for the third term in (4.55). By combining (4.56), (4.57), (4.60), (4.62) and (4.58) with m and m' chosen large enough we complete this estimate.

We now estimate $\langle w^2\partial_\beta[Kg_1], g_2\rangle$. Recalling (4.45), we have

$$\begin{aligned} w^2\partial_\beta Kg_1 &= -\partial_i[w^2\partial_\beta\{\mu^{1/2}[\phi^{ij} * \{\mu^{1/2}\partial_j g_1 + \frac{v_j}{2}\mu^{1/2}g_1\}]\}] \\ &\quad +\partial_i(w^2)\partial_\beta\{\mu^{1/2}[\phi^{ij} * \{\mu^{1/2}\partial_j g_1 + \frac{v_j}{2}\mu^{1/2}g_1\}]\} \\ &\quad +w^2\partial_\beta\{\frac{v_i}{2}\mu^{1/2}[\phi^{ij} * \{\mu^{1/2}\partial_j g_1 + \frac{v_j}{2}\mu^{1/2}g_1\}]\}. \end{aligned}$$

We take derivatives only on the factor $\{\mu^{1/2}\partial_j g_1 + v_j\mu^{1/2}g_1\}$ in the convolutions above. Upon integrating by parts for the first term, using (4.51) and (4.52) and collecting terms we can express $\langle w^2\partial_\beta[Kg_1], g_2\rangle$ as

$$\sum_{|\beta_1|\leq|\beta|+1,|\beta_2|\leq 1}\int w^2\phi^{ij}(v-v')\mu^{1/2}(v)\mu^{1/2}(v')\bar{\mu}^{\beta_1\beta_2}(v,v')\partial_{\beta_1}g_1(v')\partial_{\beta_2}g_2(v)dv'dv,$$

where $\bar{\mu}^{\beta_1\beta_2}(v,v')$ is a collection of smooth functions which, for any $k-$th order derivatives, satisfies

$$|\nabla^k_{v,v'}\bar{\mu}^{\beta_1\beta_2}(v',v)|\leq C(1+|v|^2)^{|\beta|/2}(1+|v'|^2)^{|\beta|/2}.$$

Using the same argument as in (4.53) for the same $0<q'<1$ as in (4.53) we can rewrite $\langle w^2\partial_\beta[Kg_1], g_2\rangle$ as

$$\sum_{|\beta_1|\leq|\beta|+1,|\beta_2|\leq 1}\int w(v)\phi^{ij}(v-v')\mu^{q'/4}(v)\mu^{1/4}(v')\mu^{\beta_1\beta_2}(v,v')\partial_{\beta_1}g_1(v')\partial_{\beta_2}g_2(v)dv'dv,$$

where $\mu^{\beta_1\beta_2}(v,v')$ is a collection of smooth functions satisfying (for any $k-$th order derivatives)

$$|\nabla^k_{v,v'}\mu^{\beta_1\beta_2}(v',v)|\leq Ce^{-\frac{q'}{16}|v|^2}e^{-\frac{1}{16}|v'|^2}.$$

We split $\langle w^2\partial_\beta[Kg_1], g_2\rangle$ as in (4.54) to get

$$\sum\int w(v)\psi^{ij}(v,v')\mu^{\beta_1\beta_2}(v,v')\partial_{\beta_1}g_1(v')\partial_{\beta_2}g_2(v)dv'dv$$
$$+\sum\int w(v)\{\phi^{ij}(v-v')\mu^{q'/4}(v)\mu^{1/4}(v')-\psi^{ij}\}\mu^{\beta_1\beta_2}(v,v')\partial_{\beta_1}g_1(v')\partial_{\beta_2}g_2(v)dv'dv.$$

Using the estimates as for J_2 in (4.54) and Lemma 4.5 the last term is bounded by

$$\frac{C}{m}\sum_{|\bar{\beta}|\leq|\beta|}\left|\partial_{\bar{\beta}}g_1\right|_{\boldsymbol{\sigma},\ell}|g_2|_{\boldsymbol{\sigma},\ell,\vartheta}.$$

Since ψ^{ij} has compact support, integrating by parts over v' and v, the first term is equal to

$$\sum_{|\beta_1|\leq|\beta|+1,|\beta_2|\leq 1}(-1)^{|\beta_1|+|\beta_2|}\int\partial_{\beta_2}\{w(v)\partial_{\beta_1}[\psi^{ij}(v,v')\bar{\mu}_{\beta_1\beta_2}(v,v')]\}g_1(v')g_2(v)dv'dv.$$

Then, by Cauchy-Schwartz, this term is $\leq C(m)\left|\bar{\chi}_{C(m)}\mu g_1\right|_\ell\left|\bar{\chi}_{C(m)}g_2\right|_\ell$. And our lemma follows by first choosing $m>0$ large. $\square$

Next, from Lemma 4.8 we get a general lower bound for L with high derivatives. We also prove a lower bound for L with no derivatives.

LEMMA 4.9. *Let $0 \leq \vartheta \leq 2$, $q > 0$ and $l \geq 0$ with $|\beta| > 0$ and $\ell = |\beta| - l$. If $\vartheta = 2$ further restrict $0 < q < 1$. Then for $\eta > 0$ small enough there exists $C(\eta) > 0$ such that*

$$\langle w^2 \partial_\beta [Lg], \partial_\beta g \rangle \geq |\partial_\beta g|^2_{\boldsymbol{\sigma},\ell,\vartheta} - \eta \sum_{|\beta_1| = |\beta|} |\partial_{\beta_1} g|^2_{\boldsymbol{\sigma},\ell,\vartheta} - C(\eta) \sum_{|\beta_1| < |\beta|} |\partial_{\beta_1} g|^2_{\boldsymbol{\sigma},|\beta_1| - l,\vartheta},$$

where $w^2 = w^2(\ell,\vartheta)$. If $|\beta| = 0$ we have

$$\langle w^2(\ell,\vartheta)[Lg], g \rangle \geq \delta_q |g|^2_{\boldsymbol{\sigma},\ell,\vartheta} - C(\eta) \left| \bar{\chi}_{C(\eta)} g \right|^2_{\ell},$$

where $\delta_q = 1 - q^2 - \eta > 0$ for $\eta > 0$ small enough.

It turns out that the lower bound for L with no extra v derivatives and an exponential weight needs a new approach. We need to use exact cancellation to make it work in the $\vartheta = 2$ case.

PROOF. By Lemma 4.8, we need only consider the case with $|\beta| = 0$.

First assume $0 \leq \vartheta < 2$. In this case, after an integration by parts, (4.44) gives

$$\langle w^2 Lg, g \rangle = |g|^2_{\boldsymbol{\sigma},\ell,\vartheta} + \langle \partial_i (w^2) \sigma^{ij} \partial_j g, g \rangle - \langle w^2 \partial_i \sigma^i g, g \rangle - \langle w^2 K g, g \rangle.$$

By Lemma 4.7, the last two terms on the r.h.s. satisfy the $|\beta| = 0$ estimate. Thus, we only consider $\langle \partial_i (w^2) \sigma^{ij} \partial_j g, g \rangle$. By (4.51) and (4.59) we can write

$$\partial_i (w^2(v)) \sigma^{ij}(v) = w^2(v) w_1(v) \lambda_1(v) v_j.$$

From (4.52), $|w_1(v)| \leq C(1 + |v|^2)^{\frac{\vartheta}{2} - 1}$, and by Lemma 4.4, $|\lambda_1(v) v_j| \leq C[1 + |v|]^{\gamma+1}$. Thus for any $m' > 0$

$$\begin{aligned}
\left| \langle \partial_i (w^2) \sigma^{ij} \partial_j g, g \rangle \right| &\leq C \int w^2(\ell,\vartheta)[1 + |v|]^{\gamma+1+\frac{\vartheta}{2}-1} |\partial_j g| \, |g| \, dv \\
&= C \int \left(w(\ell,\vartheta)[1 + |v|]^{\frac{\gamma}{2}} |\partial_j g| \right) \left(w(\ell,\vartheta)[1 + |v|]^{\frac{\gamma+2}{2} + \frac{\vartheta}{2} - 1} |g| \right) \\
&\leq C \, |g|_{\boldsymbol{\sigma},\ell,\vartheta} \left| [1 + |v|]^{\frac{\gamma+2}{2} + \frac{\vartheta}{2} - 1} g \right|_{\ell,\vartheta} \\
&\leq \frac{1}{m'} |g|^2_{\boldsymbol{\sigma},\ell,\vartheta} + C(m') \left| [1 + |v|]^{\frac{\gamma+2}{2} + \frac{\vartheta}{2} - 1} g \right|^2_{\ell,\vartheta}.
\end{aligned}$$

For $m > 0$ further split

$$
\begin{aligned}
\left|[1+|v|]^{(\gamma+2)/2+\frac{\vartheta}{2}-1} g\right|_{\ell,\vartheta}^2 &= \int_{|v|\le m} + \int_{|v|>m} \\
&\le \int_{|v|\le m} + C m^{\vartheta-2} \int_{|v|>m} w^2[1+|v|]^{\gamma+2}|g|^2 dv \qquad (4.63)\\
&\le C(m) \int_{|v|\le m} w^2(\ell,0)|g|^2 dv + C m^{\vartheta-2} |g|_{\boldsymbol{\sigma},\ell,\vartheta}^2 . \\
&\le C(m)|\bar{\chi}_m g|_\ell + C m^{\vartheta-2} |g|_{\boldsymbol{\sigma},\ell,\vartheta}^2 .
\end{aligned}
$$

We thus complete the estimate for $0 \le \vartheta < 2$ by choosing m and m' large.

Finally consider the case $\vartheta = 2$ and $0 < q < 1$. We will prove this case in two steps. Split $L = -A - K$. Define $M(v) \equiv \exp\left(\frac{q}{4}(1+|v|^2)\right)$. First we will show that there is $\delta_q > 0$ such that

$$
-\langle w^2(\ell,2)[Ag], g\rangle \ge \delta_q |Mg|_{\boldsymbol{\sigma},\ell,}^2 - C(\delta_q)\left|\bar{\chi}_{C(\delta_q)} g\right|_\ell^2 . \qquad (4.64)
$$

Second we will establish

$$
|Mg|_{\boldsymbol{\sigma},\ell}^2 \ge \delta_q |g|_{\boldsymbol{\sigma},\ell,2}^2 - C(\delta_q)|g\bar{\chi}_{C(\delta_q)}|_\ell^2, \qquad (4.65)
$$

where $\delta_q = 1 - q^2 - \frac{\eta}{2} > 0$ since $\eta > 0$ can be chosen arbitrarily small. This will be enough to establish the case $\vartheta = 2$ because the K part is controlled by Lemma 4.7.

We now establish (4.64). By (4.44), we obtain

$$
\begin{aligned}
-M[Ag] &= -M\partial_i\{\sigma^{ij}\partial_j g\} + M\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2} g - M\partial_i\sigma^i g \\
&= -\partial_i\{M\sigma^{ij}\partial_j g\} + q\sigma^{ij}\frac{v_i}{2} M\partial_j g + M\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2} g - M\partial_i\sigma^i g \\
&= -\partial_i\{\sigma^{ij}\partial_j[Mg]\} + q\partial_i\{\sigma^{ij}\frac{v_j}{2} Mg\} + q\sigma^{ij}\frac{v_i}{2} M\partial_j g \\
&\quad + M\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2} g - M\partial_i\sigma^i g \\
&= -\partial_i\{\sigma^{ij}\partial_j[Mg]\} + q\partial_i\{\sigma^i Mg\} + q\sigma^j\partial_j[Mg] \\
&\quad - q^2 M\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2} g + M\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2} g - M\partial_i\sigma^i g.
\end{aligned}
$$

Notice that after an integration by parts

$$
\langle (1+|v|^2)^{\tau\ell}\{q\partial_i\{\sigma^i Mg\} + q\sigma^j\partial_j[Mg]\}, Mg\rangle = -q\langle \partial_i(1+|v|^2)^{\tau\ell}\sigma^i Mg, Mg\rangle.
$$

Also, by (4.10),

$$-\langle w^2(\ell,2)[Ag],g\rangle = -\langle (1+|v|^2)^{\tau\ell}M[Ag],Mg\rangle.$$

We therefore have

$$\begin{aligned}
-\langle w^2(\ell,2)[Ag],g\rangle &= \int (1+|v|^2)^{\tau\ell}\left\{\sigma^{ij}\partial_j[Mg]\partial_i[Mg] + (1-q^2)\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}[Mg]^2\right\}dv \\
&\quad - \int w^2(\ell,2)\partial_i\sigma^i g^2 dv \\
&\quad + \int \partial_i(1+|v|^2)^{\tau\ell}\left\{\sigma^{ij}\{\partial_j[Mg]\}[Mg] - q\sigma^i[Mg]^2\right\}dv \\
&\geq (1-q^2)|Mg|^2_{\sigma,\ell} - \int w^2(\ell,2)\partial_i\sigma^i g^2 dv \\
&\quad + \int \partial_i(1+|v|^2)^{\tau\ell}\left\{\sigma^{ij}\{\partial_j[Mg]\}[Mg] - q\sigma^i[Mg]^2\right\}dv.
\end{aligned}$$

By Lemma 4.4, $|\partial_i\sigma^i| \leq C[1+|v|]^{\gamma+1}$. Then as in (4.63), for $m>0$, we have

$$\begin{aligned}
\int w^2(\ell,2)\left|\partial_i\sigma^i\right| g^2 dv &= \int_{|v|\leq m} + \int_{|v|>m} \\
&\leq \int_{|v|\leq m} + \frac{C}{m}\int_{|v|>m}(1+|v|^2)^{\tau\ell}[1+|v|]^{\gamma+2}[Mg]^2 dv \\
&\leq \int_{|v|\leq m} + \frac{C}{m}\left|Mg\right|^2_{\boldsymbol{\sigma},\ell} \\
&\leq C(m)\left|\bar{\chi}_m g\right|^2_\ell + \frac{\eta}{4}\left|Mg\right|^2_{\boldsymbol{\sigma},\ell},
\end{aligned} \tag{4.66}$$

where the last line follows from choosing $m>0$ large enough. We integrate by parts on the next term to obtain

$$\int \partial_i(1+|v|^2)^{\tau\ell}\sigma^{ij}\{\partial_j[Mg]\}[Mg]dv = -\frac{1}{2}\int \partial_j\left\{\partial_i(1+|v|^2)^{\tau\ell}\sigma^{ij}\right\}[Mg]^2 dv$$

By Lemma 4.4 and (4.10),

$$\left|\partial_i(1+|v|^2)^{\tau\ell}\sigma^i\right| + \left|\partial_j\left\{\partial_i(1+|v|^2)^{\tau\ell}\sigma^{ij}\right\}\right| \leq C(1+|v|^2)^{\tau\ell}[1+|v|]^{\gamma+1}.$$

Thus the estimate for the final term follows from the same argument as (4.66). This establishes (4.64).

We finally establish (4.65). Notice that

$$|Mg|^2_{\boldsymbol{\sigma},\ell} = \int (1+|v|^2)^{\tau\ell}\left\{\sigma^{ij}\partial_i[Mg]\partial_j[Mg] + \sigma^{ij}v_iv_j[gM]^2\right\}dv.$$

We expand the first term in $|Mg|^2_{\boldsymbol{\sigma},\ell}$ to obtain

$$\begin{aligned}
\int (1+|v|^2)^{\tau\ell}\sigma^{ij}\partial_i[Mg]\partial_j[Mg]dv &= \int (1+|v|^2)^{\tau\ell}M^2\sigma^{ij}\partial_i g\partial_j g dv \\
&\quad + q^2\int (1+|v|^2)^{\tau\ell}M^2\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g^2 dv \\
&\quad + 2q\int (1+|v|^2)^{\tau\ell}M^2\sigma^{ij}\frac{v_i}{2}\{\partial_j g\}g dv \\
&= \int w^2(\ell,2)\left\{\sigma^{ij}\partial_i g\partial_j g + q^2\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g^2\right\}dv \\
&\quad +2q\int (1+|v|^2)^{\tau\ell}M^2\sigma^j\{\partial_j g\}g dv. \qquad (4.67)
\end{aligned}$$

In the last step we used (4.10). We integrate by parts on the last term to obtain

$$\begin{aligned}
2q\int (1+|v|^2)^{\tau\ell}M^2\sigma^j\{\partial_j g\}g dv &= q\int (1+|v|^2)^{\tau\ell}M^2\sigma^j\{\partial_j g^2\}dv \\
&= -q\int (1+|v|^2)^{\tau\ell}M^2\partial_j\sigma^j g^2 dv \\
&\quad - q\int \partial_j(1+|v|^2)^{\tau\ell}M^2\sigma^j g^2 dv \\
&\quad - 2q^2\int (1+|v|^2)^{\tau\ell}M^2\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g^2 dv.
\end{aligned}$$

Since $\partial_j(1+|v|^2)^{\tau\ell} = (1+|v|^2)^{\tau\ell}\left\{2\tau\ell(1+|v|^2)^{-1}v_j\right\}$, we define the error as

$$\bar{w}(v) = q\left\{\partial_j\sigma^j + 2\tau\ell(1+|v|^2)^{-1}\sigma^j v_j\right\}.$$

Then (4.67) is

$$= \int w^2(\ell,2)\left\{\sigma^{ij}\partial_i g\partial_j g - q^2\sigma^{ij}\frac{v_i}{2}\frac{v_j}{2}g^2\right\}dv - \int w^2(\ell,2)\bar{w}g^2 dv.$$

Adding the second term in $|Mg|^2_{\boldsymbol{\sigma},\ell}$ to both sides of the last display yields

$$|Mg|^2_{\boldsymbol{\sigma},\ell} \geq (1-q^2)|g|^2_{\boldsymbol{\sigma},\ell,2} - \int w^2(\ell,2)\bar{w}g^2 dv.$$

By Lemma 4.4 we have

$$|\bar{w}(v)| \leq C[1+|v|]^{\gamma+1}.$$

Thus, using (4.66), for any small $\eta > 0$ we have

$$\int w^2(\ell,2)|\bar{w}|g^2 dv \leq C(m)|g\bar{\chi}_m|^2_\ell + \frac{\eta}{2}|g|^2_{\boldsymbol{\sigma},\ell,2}.$$

This completes the estimate (4.65) and the proof. $\square$

We thus conclude our estimates for the linear terms and finish the section by estimating the nonlinear term.

LEMMA 4.10. *Let $|\alpha|+|\beta|\le N$, $0\le\vartheta\le 2$, $q>0$ and $l\ge 0$ with $\ell=|\beta|-l$. If $\vartheta=2$ restrict $0<q<1$. Then*

$$\langle w^2(\ell,\vartheta)\partial^\alpha_\beta\Gamma[g_1,g_2],\ \partial^\alpha_\beta g_3\rangle \tag{4.68}$$
$$\le C\sum\left\{|\partial^{\alpha_1}_{\bar\beta}g_1|_\ell|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\boldsymbol{\sigma},\ell,\vartheta}+|\partial^{\alpha_1}_{\bar\beta}g_1|_{\boldsymbol{\sigma},\ell}|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|_{\ell,\vartheta}\right\}|\partial^\alpha_\beta g_3|_{\boldsymbol{\sigma},\ell,\vartheta},$$

where summation is over $|\alpha_1|+|\beta_1|\le N$, $\bar\beta\le\beta_1\le\beta$.

Furthermore,

$$\big(w^2(\ell,\vartheta)\partial^\alpha_\beta\Gamma[g_1,g_2],\ \partial^\alpha_\beta g_3\big) \tag{4.69}$$
$$\le C\left\{\mathcal{E}_l^{1/2}(g_1)\mathcal{D}^{1/2}_{l,\vartheta}(g_2)+\mathcal{D}_l^{1/2}(g_1)\mathcal{E}^{1/2}_{l,\vartheta}(g_2)\right\}\|\partial^\alpha_\beta g_3\|_{\boldsymbol{\sigma},\ell,\vartheta}.$$

The proof of Lemma 4.10 is more or less the same as in [38] save a few details. The differences mainly come from taking derivatives of the exponential weight $w(\ell,\vartheta)(v)$ which creates extra polynomial growth.

PROOF. Recall $\Gamma[g_1,g_2]$ in (4.46). By the product rule, we expand

$$\langle w^2\partial^\alpha_\beta\Gamma[g_1,g_2],\partial^\alpha_\beta g_2\rangle=\sum C^{\alpha_1}_\alpha C^{\beta_1}_\beta\times G_{\alpha_1\beta_1},$$

where $G_{\alpha_1\beta_1}$takes the form:

$$-\langle w^2\{\phi^{ij}*\partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1}g_1]\}\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2,\partial_i\partial^\alpha_\beta g_3\rangle \tag{4.70}$$

$$-\langle w^2\{\phi^{ij}*\partial_{\beta_1}[\frac{v_i}{2}\mu^{1/2}\partial^{\alpha_1}g_1]\}\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2,\partial^\alpha_\beta g_3\rangle \tag{4.71}$$

$$+\langle w^2\ \{\phi^{ij}*\partial_{\beta_1}[\mu^{1/2}\partial_j\partial^{\alpha_1}g_1]\}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2,\partial_i\partial^\alpha_\beta g_3\rangle \tag{4.72}$$

$$+\langle w^2\{\phi^{ij}*\partial_{\beta_1}[\frac{v_i}{2}\mu^{1/2}\partial_j\partial^{\alpha_1}g_1]\}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2,\partial^\alpha_\beta g_3\rangle \tag{4.73}$$

$$-\langle\partial_i[w^2]\{\phi^{ij}*\partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1}g_1]\}\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2,\partial^\alpha_\beta g_3\rangle \tag{4.74}$$

$$+\langle\partial_i[w^2]\{\phi^{ij}*\partial_{\beta_1}[\mu^{1/2}\partial_j\partial^{\alpha_1}g_1]\}\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2,\partial^\alpha_\beta g_3\rangle. \tag{4.75}$$

The last two terms appear when we integrate by parts over the v_i variable.

We estimate the last term (4.75) first. Recall from (4.51) and (4.52) that $\partial_i[w^2] = w^2(v)w_1(v)v_i$ where $|w_1(v)| \le C$. By first summing over i and using (4.47) we can rewrite (4.75) as

$$\langle [w^2 w_1]\{\phi^{ij} * v_i \partial_{\beta_1}[\mu^{1/2}\partial_j \partial^{\alpha_1} g_1]\}\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2, \partial^\alpha_\beta g_3\rangle.$$

Since $-1 \le \gamma + 2 < 0$, $\phi^{ij}(v) \in L^2_{loc}(\mathbb{R}^3)$ and $|\partial_{\beta_1}\{\mu^{1/2}\}| \le C\mu^{1/4}$, we deduce by the Cauchy-Schwartz inequality and Lemma 4.5 that

$$\begin{aligned} \{\phi^{ij} * v_i \partial_{\beta_1}[\mu^{1/2}\partial_j \partial^{\alpha_1} g_1]\} &\le [|\phi^{ij}|^2 * \mu^{1/8}]^{1/2}(v) \sum_{\bar\beta \le \beta_1} |\mu^{1/32}\partial_j \partial^{\alpha_1}_{\bar\beta} g_1|_\ell. \\ &\le C[1+|v|]^{\gamma+2} \sum_{\bar\beta \le \beta_1} \left|\partial^{\alpha_1}_{\bar\beta} g_1\right|_{\boldsymbol{\sigma},\ell}, \end{aligned} \tag{4.76}$$

where we have used Lemma 2 in [38] to argue that

$$[|\phi^{ij}|^2 * \mu^{1/8}]^{1/2}(v) \le C[1+|v|]^{\gamma+2}.$$

Using the above, (4.75) is bounded by Lemma 4.5 as

$$\begin{aligned} &C\sum_{\bar\beta \le \beta_1} \left|\partial^{\alpha_1}_{\bar\beta} g_1\right|_{\boldsymbol{\sigma},\ell} \int w^2(\ell,\vartheta)[1+|v|]^{\gamma+2}|\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2 \partial^\alpha_\beta g_3| dv \\ &\le C\sum_{\bar\beta \le \beta_1} |\partial^{\alpha_1}_{\bar\beta} g_1|_{\boldsymbol{\sigma},\ell} |\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2|_{\ell,\vartheta} |[1+|v|]^{\gamma+2}\partial^\alpha_\beta g_3|_{\ell,\vartheta} \\ &\le C\sum_{\bar\beta \le \beta_1} |\partial^{\alpha_1}_{\bar\beta} g_1|_{\boldsymbol{\sigma},\ell} |\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2|_{\ell,\vartheta} |\partial^\alpha_\beta g_3|_{\boldsymbol{\sigma},\ell,\vartheta}. \end{aligned}$$

This completest the estimate for (4.75).

We now estimate (4.70)-(4.74). We decompose their double integration region $[v,v'] \in \mathbb{R}^3 \times \mathbb{R}^3$ into three parts:

$$\{|v| \le 1\}, \quad \{2|v'| \ge |v|,\ |v| \ge 1\} \quad \text{and} \quad \{2|v'| \le |v|,\ |v| \ge 1\}.$$

Case 1. Terms (4.70)-(4.74) *over* $\{|v| \le 1\}$.

For the first part $\{|v|\leq 1\}$, recall $\phi^{ij}(v)=O(|v|^{\gamma+2})\in L^2_{loc}$. As in (4.76), we have

$$|\phi^{ij}*\partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1}g_1]|+|\phi^{ij}*\partial_{\beta_1}[v_i\mu^{1/2}\partial^{\alpha_1}g_1]|+|\phi^{ij}*v_i\partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1}g_1]|$$
$$\leq C[1+|v|]^{\gamma+2}\sum_{\bar\beta\leq\beta}|\partial^{\alpha_1}_{\bar\beta}g_1|_\ell,$$
$$|\phi^{ij}*\partial_{\beta_1}[\mu^{1/2}\partial_j\partial^{\alpha_1}g_1]|+|\phi^{ij}*\partial_{\beta_1}[\{v_i\mu^{1/2}\}\partial_j\partial^{\alpha_1}g_1]|$$
$$\leq C[1+|v|]^{\gamma+2}\sum_{\bar\beta\leq\beta}|\partial^{\alpha_1}_{\bar\beta}g_1|_{\boldsymbol{\sigma},\ell}.$$

Hence their corresponding integrands over the region $\{|v|\leq 1\}$ are bounded by

$$C\sum\left|\partial^{\alpha_1}_{\bar\beta}g_1\right|_\ell[1+|v|]^{\gamma+2}|\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|\left[|\partial_i\partial^\alpha_\beta g_3|+|\partial^\alpha_\beta g_3|\right]$$
$$+C\sum\left|\partial^{\alpha_1}_{\bar\beta}g_1\right|_{\boldsymbol{\sigma},\ell}[1+|v|]^{\gamma+2}|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|\left[|\partial_i\partial^\alpha_\beta g_3|+|\partial^\alpha_\beta g_3|\right],$$

whose $v-$integral over $\{|v|\leq 1\}$ is clearly bounded by right hand side of (4.68). We thus conclude the first part of $\{|v|\leq 1\}$ for (4.70)-(4.74).

Case 2. Terms (4.70)-(4.74) *over* $\{2|v'|\geq|v|,|v|\geq 1\}$.

For the second part $\{2|v'|\geq|v|,|v|\geq 1\}$, we have

$$|\partial_{\beta_1}\mu^{1/2}(v')|+|\partial_{\beta_1}\{v'_j\mu^{1/2}(v')\}|\leq C\mu^{1/8}(v')\mu^{1/32}(v).$$

Thus, by the same type of estimates as in (4.76), using the region, we have

$$|\phi^{ij}*\partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1}g_1]|+|\phi^{ij}*\partial_{\beta_1}[v_i\mu^{1/2}\partial^{\alpha_1}g_1]|+|\phi^{ij}*v_i\partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1}g_1]|$$
$$\leq C[1+|v|]^{\gamma+2}\mu^{1/64}(v)\sum_{\bar\beta\leq\beta}|\partial^{\alpha_1}_{\bar\beta}g_1|_\ell,$$
$$|\phi^{ij}*\partial_{\beta_1}[\mu^{1/2}\partial_j\partial^{\alpha_1}g_1]|+|\phi^{ij}*\partial_{\beta_1}[\{v_i\mu^{1/2}\}\partial_j\partial^{\alpha_1}g_1]|$$
$$\leq C[1+|v|]^{\gamma+2}\mu^{1/64}(v)\sum_{\bar\beta\leq\beta}|\partial^{\alpha_1}_{\bar\beta}g_1|_{\boldsymbol{\sigma},\ell}.$$

And then the $v-$ integrands in (4.70) to (4.74) over this region are bounded by

$$C\sum\left|\partial^{\alpha_1}_{\bar\beta}g_1\right|_\ell w^2(\ell,\vartheta)[1+|v|]^{\gamma+2}\mu^{1/64}(v)|\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|\left[|\partial_i\partial^\alpha_\beta g_3|+|\partial^\alpha_\beta g_3|\right]$$
$$+C\sum\left|\partial^{\alpha_1}_{\bar\beta}g_1\right|_{\boldsymbol{\sigma},\ell}w^2(\ell,\vartheta)[1+|v|]^{\gamma+2}\mu^{1/64}(v)|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2|\left[|\partial_i\partial^\alpha_\beta g_3|+|\partial^\alpha_\beta g_3|\right],$$

By Lemma 4.5, its $v-$integral is bounded by the right hand side of (4.68) because of the fast decaying factor $\mu^{1/64}(v)$. We thus conclude the estimate for the second region $\{2|v'| \geq |v|, |v| \geq 1\}$ for (4.70) to (4.74).

Case 3. Terms (4.70)-(4.74) *over* $\{2|v'| \leq |v|, |v| \geq 1\}$.

We finally consider the third part of $\{2|v'| \leq |v|, |v| \geq 1\}$, for which we shall estimate each term from (4.70) to (4.74). The key is to taylor expand $\phi^{ij}(v-v')$. To estimate (4.70) over the this region we expand $\phi^{ij}(v-v')$ to get

$$\phi^{ij}(v-v') = \phi^{ij}(v) - \sum_k \partial_k \phi^{ij}(v) v'_k + \frac{1}{2}\sum_{k,l} \partial_{kl}\phi^{ij}(\bar{v}) v'_k v'_l. \tag{4.77}$$

where $\bar{v}$ is between v and $v-v'$. We plug (4.77) into the integrand of (4.70). From (4.40), (4.41) and (4.47), we can decompose $\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2$ and $\partial_i \partial^\alpha_\beta g_3$ into their P_v parts as well as $I-P_v$ parts. For the first term in the expansion (4.77) we have

$$\begin{aligned}
&\sum_{ij} \phi^{ij}(v) \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2(v) \partial_i \partial^\alpha_\beta g_3(v) \\
&= \sum_{ij} \phi^{ij}(v) \{[I-P_v]\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2(v)\}\{[I-P_v]\partial_i \partial^\alpha_\beta g_3(v)\}.
\end{aligned}$$

Here we have used (4.47) so that sum of terms with either $P_v \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2$ or $P_v \partial_i \partial^\alpha_\beta g_3$ vanishes. The absolute value of this is bounded by

$$C[1+|v|]^{\gamma+2} |[I-P_v]\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2(v)| \times |[I-P_v]\partial_i \partial^\alpha_\beta g_3(v)|. \tag{4.78}$$

For the second term in the expansion (4.77), by taking a k derivative of

$$\sum_{i,j} \phi^{ij}(v) v_i v_j = 0$$

we have

$$\sum_{i,j} \partial_k \phi^{ij}(v) v_i v_j = -2\sum_j \phi^{kj}(v) v_j = 0.$$

Therefore

$$\sum_{i,j} \partial_k \phi^{ij}(v) P_v \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2 P_v \partial_i \partial^\alpha_\beta g_3 = 0.$$

Splitting $\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2$ and $\partial_i \partial^\alpha_\beta g_3$ into their P_v and $I - P_v$ parts yields

$$\begin{aligned}
&\sum_{i,j} \partial_k \phi^{ij}(v) \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2(v) \partial_i \partial^\alpha_\beta g_3(v) \\
&= \sum_{i,j} \partial_k \phi^{ij}(v) \{[P_v \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2][I - P_v]\partial_i \partial^\alpha_\beta g_3 + [I - P_v]\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2 [P_v \partial_i \partial^\alpha_\beta g_3]\} \\
&\quad + \sum_{i,j} \partial_k \phi^{ij}(v)[I - P_v]\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2 [I - P_v]\partial_i \partial^\alpha_\beta g_3.
\end{aligned}$$

Notice that $|\partial_k \phi^{ij}(v)| \leq C[1 + |v|]^{\gamma+1}$ for $|v| \geq 1$, we therefore majorize the above by

$$\begin{aligned}
&C[1 + |v|]^{\gamma/2} \{|P_v \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2| + |P_v \partial_i \partial^\alpha_\beta g_3|\} \\
&\times [1 + |v|]^{(\gamma+2)/2} \{|[I - P_v]\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2| + |[I - P_v]\partial_i \partial^\alpha_\beta g_3|\} \\
&+ C[1 + |v|]^{\gamma+1} |[I - P_v]\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2||[I - P_v]\partial_i \partial^\alpha_\beta g_3|.
\end{aligned} \tag{4.79}$$

Next, we estimate third term in (4.77). Using the region we have

$$\frac{1}{2}|v| \leq |v| - |v'| \leq |\bar{v}| \leq |v'| + |v| \leq \frac{3}{2}|v|. \tag{4.80}$$

Thus

$$|\partial_{kl} \phi^{ij}(\bar{v})| \leq C[1 + |v|]^\gamma,$$

and

$$\left|\partial_{kl} \phi^{ij}(\bar{v}) \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2(v) \partial_i \partial^\alpha_\beta g_3(v)\right| \leq C[1 + |v|]^\gamma |\partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2 \partial_i \partial^\alpha_\beta g_3|. \tag{4.81}$$

Combining (4.77), (4.78), (4.79) and (4.81) we obtain the estimate

$$\begin{aligned}
&\left|\sum_{i,j} \phi^{ij}(v - v') \partial_i \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2 \partial_i \partial^\alpha_\beta g_3\right| \\
&\leq C[1 + |v'|]^2 \left|\sum_{i,j} \phi^{ij}(v) \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2(v) \partial_i \partial^\alpha_\beta g_3(v)\right| \\
&+ C[1 + |v'|]^2 \left|\sum_{i,j} \partial_k \phi^{ij}(v) \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2(v) \partial_i \partial^\alpha_\beta g_3(v)\right| \\
&+ C[1 + |v'|]^2 \sum_{i,j} \left|\partial_{kl} \phi^{ij}(\bar{v}) \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2(v) \partial_i \partial^\alpha_\beta g_3(v)\right| \\
&\leq C[1 + |v'|]^2 \{\sigma^{ij} \partial_i \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2 \partial_j \partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2\}^{1/2} \{\sigma^{ij} \partial_i \partial^\alpha_\beta g_3 \partial_j \partial^\alpha_\beta g_3\}^{1/2},
\end{aligned}$$

where we have used (4.42) in the last line. The v integrand over $\{2|v'|\leq|v|,\,|v|\geq 1\}$ in (4.70) is thus bounded by

$$w^2\int[1+|v'|]^2\mu^{1/4}(v')|\partial_{\bar\beta}^{\alpha_1}g_1(v')|dv'$$
$$\times\{\sigma^{ij}\partial_i\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\}^{1/2}\{\sigma^{ij}\partial_i\partial_\beta^\alpha g_3\partial_j\partial_\beta^\alpha g_3\}^{1/2}$$
$$\leq C|\partial_{\bar\beta}^{\alpha}g_1|_\ell\{w^2\sigma^{ij}\partial_i\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\}^{1/2}\{w^2\sigma^{ij}\partial_i\partial_\beta^\alpha g_3\partial_j\partial_\beta^\alpha g_3\}^{1/2}.$$

Its further integration over v is bounded by the right hand side of (4.68).

We now consider the second term (4.71). We again expand $\phi^{ij}(v-v')$ as

$$\phi^{ij}(v-v')=\phi^{ij}(v)-\sum_k\partial_k\phi^{ij}(\bar v)v_k', \tag{4.82}$$

with $\bar v$ between v and $v-v'$. Since $\sum_j\phi^{ij}(v)v_j=0$ we obtain as before

$$\sum_j\phi^{ij}(v)\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2(v)\partial_\beta^\alpha g_3(v)=\sum_j\phi^{ij}(v)\{I-P_v\}\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2(v)\times\partial_\beta^\alpha g_3(v)$$
$$\leq C[1+|v|]^{\gamma+2}|\{I-P_v\}\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2(v)||\partial_\beta^\alpha g_3(v)| \tag{4.83}$$
$$\leq C|[1+|v|]^{\frac{\gamma+2}{2}}\{I-P_v\}\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2(v)||[1+|v|]^{\frac{\gamma+2}{2}}\partial_\beta^\alpha g_3(v)|$$
$$\leq C\{\sigma^{ij}\partial_i\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\}^{1/2}\{\sigma^{ij}v_iv_j|\partial_\beta^\alpha g_3|^2\}^{1/2},$$

where we have used (4.42). From (4.80), $|\partial_k\phi^{ij}(\bar v)|\leq C[1+|v|]^{\gamma+1}$. Hence

$$|\partial_k\phi^{ij}(\bar v)\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2(v)\partial_\beta^\alpha g_3(v)| \tag{4.84}$$
$$\leq C[1+|v|]^{\gamma+1}\{|\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2(v)|\}\{|\partial_\beta^\alpha g_3(v)|\}$$
$$\leq C\{[1+|v|]^{\gamma/2}|\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2(v)|\}\{[1+|v|]^{\frac{\gamma+2}{2}}|\partial_\beta^\alpha g_3(v)|\}$$
$$\leq C\{\sigma^{ij}\partial_i\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\}^{1/2}\{\sigma^{ij}v_iv_j|\partial_\beta^\alpha g_3|^2\}^{1/2}.$$

From (4.83) and (4.84), we thus conclude

$$\left|\sum_{ij}\phi^{ij}(v-v')\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2(v)\partial_\beta^\alpha g_3(v)\right|$$
$$\leq C[1+|v'|]\{\sigma^{ij}\partial_i\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\}^{1/2}\{\sigma^{ij}v_iv_j|\partial_\beta^\alpha g_3|^2\}^{1/2}.$$

We thus conclude that the v integrand in (4.71) can be majorized by

$$C\sum\int[1+|v'|]\mu^{1/4}(v')|\partial_{\bar\beta}^{\alpha_1}g_1(v')|dv'$$
$$\times w^2\{\sigma^{ij}\partial_i\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\}^{1/2}\{\sigma^{ij}\partial_i\partial_\beta^\alpha g_3\partial_j\partial_\beta^\alpha g_3\}^{1/2}$$
$$\leq C\sum|\partial_{\bar\beta}^{\alpha_1}g_1|_\ell\{w^2\sigma^{ij}\partial_i\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\partial_j\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\}^{1/2}\{w^2\sigma^{ij}\partial_i\partial_\beta^\alpha g_3\partial_j\partial_\beta^\alpha g_3\}^{1/2}.$$

Further integration over v shows that this bounded by the right hand side of (4.68).

We now consider the third term (4.72) over $\{2|v'|\leq|v|,|v|\geq 1\}$. We use an integration by parts inside the convolution to split (4.72) into two parts

$$\phi^{ij}*\partial_{\beta_1}[\mu^{1/2}\partial_j\partial^{\alpha_1}g_1]=\partial_j\phi^{ij}*\partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1}g_1]-\phi^{ij}*\partial_{\beta_1}[\partial_j\mu^{1/2}\ \partial^{\alpha_1}g_1]. \tag{4.85}$$

Recall expansion (4.82) and decompose

$$\partial_i\partial_\beta^\alpha g_3=P_v\partial_i\partial_\beta^\alpha g_3+[I-P_v]\partial_i\partial_\beta^\alpha g_3.$$

By similar estimates to (4.83) and (4.84), the second part of (4.72) can be estimated as

$$\int_{\{|v|\geq 1,2|v'|\leq|v|\}}|w^2\ \phi^{ij}(v-v')\partial_{\beta_1}[\partial_j\mu^{1/2}(v')\partial^{\alpha_1}g_1(v')]\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2(v)\partial_i\partial_\beta^\alpha g_3(v)|$$
$$=\int_{|v|\geq 1,2|v'|\leq|v|}|w^2[\phi^{ij}(v)-\partial_k\phi^{ij}(\bar v)v_k']\partial_{\beta_1}[\partial_j\mu^{1/2}(v')\partial^{\alpha_1}g_1(v')]\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\partial_i\partial_\beta^\alpha g_3|$$
$$\leq C\left|\partial_{\bar\beta}^{\alpha_1}g_1\right|_\ell\left|[1+|v|]^{\frac{\gamma+2}{2}}\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\right|_{\ell,\vartheta}\left|w^\vartheta[1+|v|]^{\frac{\gamma+2}{2}}\{I-P_v\}\partial_i\partial_\beta^\alpha g_3\right|_{\ell,\vartheta}$$
$$+C\left|\partial_{\bar\beta}^{\alpha_1}g_1\right|_\ell\left|[1+|v|]^{\frac{\gamma+2}{2}}\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\right|_{\ell,\vartheta}\left|[1+|v|]^{\gamma/2}\partial_i\partial_\beta^\alpha g_3\right|_{\ell,\vartheta}.$$

By Lemma 4.5, this is bounded by the right hand side of (4.68).

For the first part of (4.72) by (4.85) notice that our integration region implies

$$|\partial_j\phi^{ij}(v-v')|\leq C[1+|v|]^{\gamma+1}.$$

We thus have

$$\int_{\{|v|\geq 1,2|v'|\leq|v|\}}w^2\ |\partial_j\phi^{ij}(v-v')|\partial_{\beta_1}[\mu^{1/2}(v')\partial^{\alpha_1}g_1(v')]\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2(v)\partial_i\partial_\beta^\alpha g_3(v)$$
$$\leq C\sum_{\bar\beta\leq\beta_1}\left|\partial_{\bar\beta}^{\alpha_1}g_1\right|_\ell\left|[1+|v|]^{\frac{\gamma+2}{2}}\partial_{\beta-\beta_1}^{\alpha-\alpha_1}g_2\right|_{\ell,\vartheta}\left|[1+|v|]^{\gamma/2}\partial_j\partial_\beta^\alpha g_3\right|_{\ell,\vartheta},$$

which is bounded by the right hand side of (4.68) by Lemma 4.5.

Next consider (4.73) over $\{2|v'| \leq |v|,\ |v| \geq 1\}$. We split (4.73) as in (4.85)

$$\phi^{ij} * \partial_{\beta_1}[v_i\mu^{1/2}\partial_j\partial^{\alpha_1}g_1] = \partial_j\phi^{ij} * \partial_{\beta_1}[v_i\mu^{1/2}\partial^{\alpha_1}g_1] - \phi^{ij} * \partial_{\beta_1}[\partial_j\{v_i\mu^{1/2}\}\ \partial^{\alpha_1}g_1].$$

Since $|\phi^{ij}(v-v')| \leq C[1+|v|]^{\gamma+2}$, and $|\partial_j\phi^{ij}(v-v')| \leq C[1+|v|]^{\gamma+1}$, (4.73) is bounded by

$$\begin{aligned}
&\int w^2[1+|v|]^{\gamma+1}|\partial_{\beta_1}[v_i'\mu^{1/2}(v')\partial^{\alpha_1}g_1(v')]\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial^{\alpha}_{\beta}g_3|dv'dv \\
&+\int w^2[1+|v|]^{\gamma+2}|\partial_{\beta_1}[\partial_j\{v_i'\mu^{1/2}(v')\}\partial^{\alpha_1}g_1(v')]\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial^{\alpha}_{\beta}g_3|dv'dv \\
&\leq C\sum_{\bar\beta\leq\beta_1}\left|\partial^{\alpha_1}_{\bar\beta}g_1\right|_{\ell}\left|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\right|_{\boldsymbol{\sigma},\ell,\vartheta}\left|\partial^{\alpha}_{\beta}g_3\right|_{\boldsymbol{\sigma},\ell,\vartheta}.
\end{aligned}$$

We thus conclude the estimate for (4.73).

Finally, consider the term (4.74) over $\{2|v'| \leq |v|,\ |v| \geq 1\}$. First sum over v_i so that (4.74) is given by

$$\langle[w^2w_1]\{\phi^{ij} * v_i\partial_{\beta_1}[\mu^{1/2}\partial^{\alpha_1}g_1]\}\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2, \partial^{\alpha}_{\beta}g_3\rangle$$

We again expand $\phi^{ij}(v-v')$ as in (4.82). By (4.47), we have the estimates (4.83) and (4.84). Plugging (4.83) and (4.84) into (4.82), we thus conclude that the v integrand in (4.74) can be majorized by

$$\begin{aligned}
&C\sum\int[1+|v'|]^2\mu^{1/4}(v')|\partial^{\alpha_1}_{\bar\beta}g_1(v')|dv' \\
&\times w^2\{\sigma^{ij}\partial_i\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\}^{1/2}\{\sigma^{ij}v_iv_j|\partial^{\alpha}_{\beta}g_3|^2\}^{1/2} \\
&\leq C\sum|\partial^{\alpha_1}_{\bar\beta}g_1|_{\ell}\{w^2\sigma^{ij}\partial_i\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\partial_j\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\}^{1/2}\{w^2\sigma^{ij}v_iv_j|\partial^{\alpha}_{\beta}g_3|^2\}^{1/2}.
\end{aligned}$$

By further integrating over v, this bounded by the right hand side of (4.68). And thus, the proof of (4.68) is complete.

The proof of (4.69) now follows from the Sobolev embedding $H^2(\mathbb{T}^3) \subset L^\infty(\mathbb{T}^3)$ and (4.68). Without loss of generality, assume $|\alpha_1| + |\bar{\beta}| \le N/2$ in (4.68). Then

$$\left(\sup_x |\partial^{\alpha_1}_{\bar{\beta}} g_1(x)|_\ell\right) |\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2|_{\boldsymbol{\sigma},\ell,\vartheta} + \left(\sup_x |\partial^{\alpha_1}_{\bar{\beta}} g_1(x)|_{\boldsymbol{\sigma},\ell}\right) |\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2|_{\ell,\vartheta}$$
$$\le \left(\sum \|\partial^{\alpha'}_{\beta'} g_1\|_\ell\right) |\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2|_{\boldsymbol{\sigma},\ell,\vartheta} + \left(\sum \|\partial^{\alpha'}_{\beta'} g_1\|_{\boldsymbol{\sigma},\ell}\right) |\partial^{\alpha-\alpha_1}_{\beta-\beta_1} g_2|_{\ell,\vartheta},$$

where the summation is over $|\alpha'|+|\beta'| \le \frac{N}{2}+2 \le N$. We deduce (4.69) by integrating (4.68) over $\mathbb{T}^3$ and using the this computation. $\square$

4.5. Energy Estimate and Global Existence

In this section we will prove the energy estimate which is a crucial step in constructing global solutions. By now, it is standard to prove local existence of small solutions using the estimates either in Section 4.3 for the Boltzmann case or Section 4.4 for the Landau case:

THEOREM 4.3. *For any sufficiently small $M^* > 0$, $T^* > 0$ with $T^* \le \frac{M^*}{2}$ and*

$$\frac{1}{2} \sum_{|\alpha|+|\beta|\le N} ||\partial^\alpha_\beta f_0||^2_{|\beta|-l,\vartheta} \le \frac{M^*}{2},$$

there is a unique classical solution $f(t,x,v)$ to (4.5) in either the Boltzmann or the Landau case in $[0,T^*) \times \mathbb{T}^3 \times \mathbb{R}^3$ such that

$$\sup_{0\le t\le T^*} \left\{ \frac{1}{2} \sum_{|\alpha|+|\beta|\le N} ||\partial^\alpha_\beta f||^2_{|\beta|-l,\vartheta}(t) + \int_0^t \mathcal{D}_{l,\vartheta}(f)(s)ds \right\} \le M^*,$$

and $\frac{1}{2}\sum_{|\alpha|+|\beta|\le N} ||\partial^\alpha_\beta f||^2_{|\beta|-l,\vartheta}(t) + \int_0^t \mathcal{D}_{l,\vartheta}(f)(s)ds$ is continuous over $[0,T^*)$.

Next, we define some notation. For fixed $N \ge 8$, $0 \le m \le N$ and $\vartheta, q, l \ge 0$, a modified instant energy functional satisfies

$$\frac{1}{C}\mathcal{E}^m_{l,\vartheta}(g)(t) \le \sum_{|\beta|\le m, |\alpha|+|\beta|\le N} ||\partial^\alpha_\beta g(t)||^2_{|\beta|-l,\vartheta} \le C\mathcal{E}^m_{l,\vartheta}(g)(t). \tag{4.86}$$

Similarly the modified dissipation rate is given by

$$\mathcal{D}^m_{l,\vartheta}(g)(t) \equiv \sum_{|\beta|\le m, |\alpha|+|\beta|\le N} ||\partial^\alpha_\beta g(t)||^2_{\boldsymbol{D},|\beta|-l,\vartheta}. \tag{4.87}$$

Note that, $\mathcal{E}^N_{l,\vartheta}(g)(t) = \mathcal{E}_{l,\vartheta}(g)(t)$ and $\mathcal{D}^N_{l,\vartheta}(g)(t) = \mathcal{D}_{l,\vartheta}(g)(t)$. And as before, we will write $\mathcal{E}^m_{l,0}(g)(t) = \mathcal{E}^m_l(g)(t)$ and $\mathcal{D}^m_{l,0}(g)(t) = \mathcal{D}^m_l(g)(t)$. Now we are ready to state a result from equation (4.5) in [60] using this new notation:

LEMMA 4.11. *Let $f(t,x,v)$ be a classical solution to (4.5) satisfying (4.13) in either the Boltzmann or the Landau case. In the Boltzmann case assume $\tau \le \gamma$ but in the Landau case assume $\tau \le -1$ in (4.10). For any $l \ge 0$, there exists M_l, $\delta_l = \delta_l(M_l) > 0$ such that if*

$$\frac{1}{2}\sum_{|\alpha|+|\beta|\le N} ||\partial^\alpha_\beta f||^2_{|\beta|-l}(t) \le M_l, \tag{4.88}$$

then for any $0 \le m \le N$ we have an instant energy functional such that

$$\frac{d}{dt}\mathcal{E}^m_l(f)(t) + \mathcal{D}^m_l(f)(t) \le C\sqrt{\mathcal{E}_l(f)(t)}\mathcal{D}_l(f)(t). \tag{4.89}$$

We will bootstrap this energy estimate without an exponential weight ($\vartheta = 0$) to Corollary 4.1 in the Boltzmann case and Lemma 4.9 on the Landau case to obtain the following general energy estimate.

LEMMA 4.12. *Fix $N \ge 8$, $0 < \vartheta \le 2$, $q > 0$ and $l \ge 0$. If $\vartheta = 2$ let $0 < q < 1$. In the Boltzmann case assume $\tau \le \gamma$ but in the Landau case assume $\tau \le -1$ in (4.10). Let $f(t,x,v)$ be a classical solution to (4.5) satisfying (4.13) and (4.88) in either the Boltzmann or the Landau case. For any given $0 \le m \le N$ there is a modified instant energy functional such that*

$$\frac{d}{dt}\mathcal{E}^m_{l,\vartheta}(f)(t) + \mathcal{D}^m_{l,\vartheta}(f)(t) \le C\mathcal{E}^{1/2}_{l,\vartheta}(f)(t)\mathcal{D}_{l,\vartheta}(f)(t). \tag{4.90}$$

PROOF. We use an induction over m, the order of the $v-$derivatives. For $m = 0$, by taking the pure ∂^α derivatives of (4.5) we obtain

$$\{\partial_t + v\cdot\nabla_x\}\,\partial^\alpha f + L\{\partial^\alpha f\} = \sum_{\alpha_1\le\alpha} C^{\alpha_1}_\alpha \Gamma(\partial^{\alpha_1} f, \partial^{\alpha-\alpha_1} f) \tag{4.91}$$

Multiply $w^2(-l,\vartheta)\partial^\alpha f$ with (4.91), integrate over $\mathbb{T}^3\times\mathbb{R}^3$ and sum over $|\alpha|\le N$ to deduce the following for some constant $C>0$,

$$\sum_{|\alpha|\le N}\left\{\frac{1}{2}\frac{d}{dt}\|\partial^\alpha f(t)\|^2_{-l,\vartheta}+\big(w^2(-l,\vartheta)L\{\partial^\alpha f(t)\},\partial^\alpha f(t)\big)\right\}$$

$$\le C\mathcal{E}^{1/2}_{l,\vartheta}(f)(t)\mathcal{D}_{l,\vartheta}(f)(t). \tag{4.92}$$

We have used Lemma 4.3 in the Boltzmann case and (4.69) in the Landau case to bound the r.h.s. of (4.91). Notice that Lemma 4.2 (in the Boltzmann case) implies

$$\begin{aligned}\big(w^2(-l,\vartheta)L\{\partial^\alpha f(t)\},\partial^\alpha f(t)\big)&=\|\partial^\alpha f(t)\|^2_{\boldsymbol{\nu},-l,\vartheta}-\big(w^2(-l,\vartheta)K\{\partial^\alpha f(t)\},\partial^\alpha f(t)\big)\\&\ge\frac{1}{2}\|\partial^\alpha f(t)\|^2_{\boldsymbol{\nu},-l,\vartheta}-C\|\partial^\alpha f(t)\|^2_{\boldsymbol{\nu},-l},\end{aligned}$$

where $C>0$ is a large constant. In the Landau case, Lemma 4.9 gives

$$\begin{aligned}\big(w^2(-l,\vartheta)L\{\partial^\alpha f(t)\},\partial^\alpha f(t)\big)&\ge\delta_q\|\partial^\alpha f(t)\|^2_{\boldsymbol{\sigma},-l,\vartheta}-C\|\bar{\chi}_C\partial^\alpha f(t)\|^2_{-l}\\&\ge\delta_q\|\partial^\alpha f(t)\|^2_{\boldsymbol{\sigma},-l,\vartheta}-C\|\partial^\alpha f(t)\|^2_{\boldsymbol{\sigma},-l}.\end{aligned}$$

Without loss of generality assume $0<\delta_q<\frac{1}{2}$. Then in either case, plugging these into (4.92) we have

$$\sum_{|\alpha|\le N}\left\{\frac{1}{2}\frac{d}{dt}\|\partial^\alpha f(t)\|^2_{-l,\vartheta}+\frac{1}{2}\|\partial^\alpha f(t)\|^2_{\boldsymbol{D},-l,\vartheta}-C\|\partial^\alpha f(t)\|^2_{\boldsymbol{D},-l}\right\}$$

$$\le C\sqrt{\mathcal{E}_{l,\vartheta}(f)(t)}\mathcal{D}_{l,\vartheta}(f)(t).$$

Add to this inequality to (4.89) with $m=0$, possibly multiplied by a large constant, to obtain (4.90) with $m=0$.

Now assume the Lemma is valid for some fixed $m>0$. For $|\beta|=m+1$, taking ∂^α_β of (4.5) we obtain

$$\{\partial_t+v\cdot\nabla_x\}\,\partial^\alpha_\beta f+\partial_\beta\{L\partial^\alpha f\} \tag{4.93}$$

$$=-\sum_{|\beta_1|=1}C^{\beta_1}_\beta\partial_{\beta_1}v\cdot\nabla_x\partial^\alpha_{\beta-\beta_1}f+\sum C^{\alpha_1}_\alpha\partial_\beta\Gamma(\partial^{\alpha_1}f,\partial^{\alpha-\alpha_1}f).$$

We take the inner product of (4.93) with $w^2(|\beta|-l,\vartheta)\partial^\alpha_\beta f$ over $\mathbb{T}^3\times\mathbb{R}^3$. The first inner product on the left is equal to $\frac{1}{2}\frac{d}{dt}||\partial^\alpha_\beta f(t)||^2_{|\beta|-l,\vartheta}$. From either Corollary 4.1

in the Boltzmann case or Lemma 4.9 in the Landau case, we deduce that the inner product of $\partial_\beta\{L\partial^\alpha f\}$ is bounded from below as

$$\sum_{|\beta|=m+1}\left(w^2\partial_\beta\{L\partial^\alpha f\},\partial_\beta^\alpha f\right)\geq\frac{1}{2}\sum_{|\beta|=m+1}||\partial_\beta^\alpha f||^2_{\boldsymbol{D},|\beta|-l,\vartheta}-C\sum_{|\bar\beta|\leq m}||\partial_{\bar\beta}^\alpha f||^2_{\boldsymbol{D},|\bar\beta|-l,\vartheta}.$$

Since $|\beta_1|=1$, as in [37] the streaming term on the r.h.s. of (4.93) is bounded by

$$\begin{aligned}(w^2(|\beta|-l,\vartheta)\{\partial_{\beta_1}v_j\}\partial_{x_j}\partial^\alpha_{\beta-\beta_1}f,\partial^\alpha_\beta f)&\leq\int w^2(|\beta|-l,\vartheta)|\partial_{x_j}\partial^\alpha_{\beta-\beta_1}f\partial^\alpha_\beta f|dxdv\\&\leq\left\|w(|\beta|+1/2-l,\vartheta)\partial^\alpha_\beta f\right\|\left\|w(1/2+\{|\beta|-1\}-l,\vartheta)\partial_{x_j}\partial^\alpha_{\beta-\beta_1}f\right\|\\&\leq\eta\left\|\partial^\alpha_\beta f\right\|^2_{\boldsymbol{D},|\beta|-l,\vartheta}+C_\eta\left\|\partial_{x_j}\partial^\alpha_{\beta-\beta_1}f\right\|^2_{\boldsymbol{D},|\beta-\beta_1|-l,\vartheta}.\end{aligned}$$

Further, by Lemma 4.3 in the Boltzmann case and Lemma 4.10 in the Landau case, the inner product involving Γ on the r.h.s. of (4.93) is $\leq C\mathcal{E}^{1/2}_{l,\vartheta}(f)(t)\mathcal{D}_{l,\vartheta}(f)(t)$.

Collect terms and sum over $|\beta|=m+1,|\alpha|+|\beta|\leq N$ to obtain

$$\begin{aligned}&\sum_{|\beta|=m+1,|\alpha|+|\beta|\leq N}\left\{\frac{1}{2}\frac{d}{dt}||\partial^\alpha_\beta f(t)||^2_{|\beta|-l,\vartheta}+\left(\frac{1}{2}-W\eta\right)\left\|\partial^\alpha_\beta f\right\|^2_{\boldsymbol{D},|\beta|-l,\vartheta}\right\}\\&\leq\sum_{|\beta|=m+1,|\alpha|+|\beta|\leq N}C\left(\sum_{|\beta_1|=1}\left\|\partial_{x_j}\partial^\alpha_{\beta-\beta_1}f\right\|^2_{\boldsymbol{D},|\beta-\beta_1|-l,\vartheta}+\sum_{|\bar\beta|\leq m}||\partial^\alpha_{\bar\beta}f||^2_{\boldsymbol{D},|\bar\beta|-l,\vartheta}\right)\\&\qquad+CZ_{m+1}\mathcal{E}^{1/2}_{l,\vartheta}(f)(t)\mathcal{D}_{l,\vartheta}(f)(t).\end{aligned}$$

Here $Z_{m+1}=\sum_{|\beta|=m+1,|\alpha|+|\beta|\leq N}1$ and $W=\sum_{|\beta_1|=1}C^{\beta_1}_\beta$. Choosing $\eta>0$ such that $\frac{1}{2}-W\eta=\frac{1}{4}>0$ we get

$$\begin{aligned}&\sum_{|\beta|=m+1,|\alpha|+|\beta|\leq N}\left\{\frac{1}{2}\frac{d}{dt}||\partial^\alpha_\beta f(t)||^2_{|\beta|-l,\vartheta}+\frac{1}{4}\left\|\partial^\alpha_\beta f\right\|^2_{\boldsymbol{D},|\beta|-l,\vartheta}\right\}\\&\leq\tilde C\sum_{|\beta|\leq m,|\alpha|+|\beta|\leq N}||\partial^\alpha_\beta f||^2_{\boldsymbol{D},|\beta|-l,\vartheta}+CZ_{m+1}\sqrt{\mathcal{E}_{l,\vartheta}(f)(t)}\mathcal{D}_{l,\vartheta}(f)(t).\end{aligned}$$

Choose $\bar A_{m+1}$ such that $\bar A_{m+1}-\tilde C\geq 1$. Now multiply (4.90) for $|\beta|\leq m$ by $\bar A_{m+1}$ and add it to the display above to obtain (4.90) for $|\beta|\leq m+1$. We thus conclude the energy estimate. □

With Lemma 4.12, we can prove existence of global in time solutions with an exponential weight using exactly the same argument as in the last Section of [39].

4.6. Proof of Exponential Decay

In this section we prove exponential decay using the differential inequality (4.14) and the uniform bound (4.15) with $\vartheta > 0$. The main difficulty in establishing decay from (4.14) is rooted in the fact that the dissipation $\mathcal{D}_{l,\vartheta}(f)(t)$ is in general weaker than the instant energy $\mathcal{E}_{l,\vartheta}(f)(t)$. As in the work of Caflisch [8], the key point is to split $\mathcal{E}_l(f)(t)$ into a time dependent low velocity part

$$E = \{|v| \leq \rho t^{p'}\},$$

and it's complementary high velocity part $E^c = \{|v| > \rho t^{p'}\}$, where $p' > 0$ and $\rho > 0$ will be chosen at the end of the proof.

First consider the Boltzmann case. Let $\mathcal{E}_l^{low}(f)(t)$ be the instant energy restricted to E. Then from (4.16), for $t > 0$, we have

$$\mathcal{D}_l(f)(t) \geq C\rho^\gamma t^{\gamma p'} \mathcal{E}_l^{low}(f)(t). \tag{4.94}$$

Plugging this into the the differential inequality (4.14) we obtain

$$\frac{d}{dt}\mathcal{E}_l(f)(t) + C\rho^\gamma t^{\gamma p'} \mathcal{E}_l^{low}(f)(t) \leq 0.$$

Letting $\mathcal{E}_l^{high}(f)(t) = \mathcal{E}_l(f)(t) - \mathcal{E}_l^{low}(f)(t)$ we have

$$\frac{d}{dt}\mathcal{E}_l(f)(t) + C\rho^\gamma t^{\gamma p'} \mathcal{E}_l(f)(t) \leq C\rho^\gamma t^{\gamma p'} \mathcal{E}_l^{high}(f)(t).$$

Define $\lambda = C\rho^\gamma / p$ where for now $p = \gamma p' + 1$ and $p' > 0$ is arbitrary. Then

$$\frac{d}{dt}\mathcal{E}_l(f)(t) + \lambda p t^{p-1} \mathcal{E}_l(f)(t) \leq \lambda p t^{p-1} \mathcal{E}_l^{high}(f)(t).$$

Equivalently

$$\frac{d}{dt}\left(e^{\lambda t^p}\mathcal{E}_l(f)(t)\right) \leq \lambda p t^{p-1} e^{\lambda t^p} \mathcal{E}_l^{high}(f)(t).$$

The integrated form is

$$\mathcal{E}_l(f)(t) \leq e^{-\lambda t^p}\mathcal{E}_l(f_0) + \lambda p e^{-\lambda t^p} \int_0^t s^{p-1} e^{\lambda s^p} \mathcal{E}_l^{high}(f)(s) ds.$$

Above $p > 0$ or equivalently $\gamma p' > -1$ is assumed to guarantee the integral on the r.h.s. is finite. Since $\mathcal{E}_l^{high}(f)(s)$ is on $E^c = \{|v| > \rho s^{p'}\}$

$$\mathcal{E}_l^{high}(f)(s) = \mathcal{E}_{l,0}^{high}(f)(s) \le C e^{-\frac{q}{2}\rho s^{\vartheta p'}} \mathcal{E}_{l,\vartheta}^{high}(f)(s).$$

In the last display we have used the region and

$$1 \le \exp\left(\frac{q}{2}(1+|v|^2)^{\frac{\vartheta}{2}}\right) e^{-\frac{q}{2}|v|^\vartheta} \le \exp\left(\frac{q}{2}(1+|v|^2)^{\frac{\vartheta}{2}}\right) e^{-\frac{q}{2}\rho s^{\vartheta p'}}.$$

Hence (4.15) implies

$$\mathcal{E}_l(f)(t) \le e^{-\lambda t^p}\left(\mathcal{E}_{l,0}(f_0) + \lambda p \mathcal{E}_{l,\vartheta}(f_0) \int_0^t s^{p-1} e^{\lambda s^p - \frac{q}{2}\rho s^{\vartheta p'}} ds\right).$$

The biggest exponent p that we can allow with this splitting is $p = \vartheta p'$; since also $p = \gamma p' + 1$ we have $p' = \frac{1}{\vartheta - \gamma}$ so that

$$p = \frac{\gamma}{\vartheta - \gamma} + 1 = \frac{\vartheta}{\vartheta - \gamma}.$$

Further choose $\rho > 0$ large enough so that $\lambda = C\rho^\gamma / p < \frac{q}{2}\rho$ ($\gamma < 0$) and hence

$$\int_0^\infty s^{p-1} e^{\lambda s^p - \frac{q}{2}\rho s^p} ds < +\infty.$$

This completes the proof of decay in the Boltzmann case.

For the proof of decay in the Landau case, instead of (4.94), we use Lemma 4.5 to see that

$$\mathcal{D}_l(f)(t) \ge C\rho^{2+\gamma} t^{(2+\gamma)p'} \mathcal{E}_l^{low}(f)(t).$$

And the rest of the proof is exactly the same. But we find, in this case, that $p = \frac{\vartheta}{\vartheta - (2+\gamma)}$. **Q.E.D.**

CHAPTER 5

On the Relativistic Boltzmann Equation

In this chapter, we will discuss a few basic mathematical properties of the relativistic Boltzmann equation. The relativistic Boltzmann equation is the central equation in relativistic collisional kinetic theory. In the next few paragraphs, we will review most of the mathematical theory of the relativistic Boltzmann equation. Books which discuss relativistic Kinetic theory include [13, 15, 30, 59, 63].

Lichnerowicz and Marrot [49] are said to be the first to write down the full relativistic Boltzmann equation, including collisional effects, in 1940. In 1967, Bichteler [6] showed that the general relativistic Boltzmann equation has a local solution if the initial distribution function decays exponentially with the energy and if the differential cross-section is bounded. Dudyński and Ekiel-Jeżewska [25], in 1988, showed that the linearized equation admits unique solutions in L^2. Afterwards, Dudyński [24] studied the long time and small-mean-free-path limits of these solutions. Then, in 1992, Dudyński and Ekiel-Jeżewska [26] proved global existence of large data Diperna-Lions solutions [23]. In this paper, they assume the relativistic Boltzmann equation is causal (i.e. solutions depend on the inital data only inside the past interior of the light cone), a result which they had previously established [27, 28]. This result has since been extended in [44, 45].

In 1991, Glassey and Strauss [31] studied the collision map that carries the precollisional momentum of a pair of colliding particles into their momentum postcollision. The collisional map becomes very important as soon as one wants to study uniqueness and regularity of the full non-linear equation. In 1993, Glassey and Strauss [32] proved existence and uniqueness of smooth solutions which are initially close to a relativistic Maxwellian and in a periodic box. They also established exponential convergence to Maxwellian. In 1995, they extended these results to the whole space case [33], where the convergence rate is polynomial.

In 1996, Andréasson [1] showed that the gain term is regularizing. This is a generalization of Lions [51, 52] result in the non-relativistic case. In 1997, Wennberg [71] proved the regularity of the gain term for both the relativistic and non-relativistic case in a unified framework. In 2004, Calogero [10] proved existence of local-in-time solutions independent of the speed of light and established a rigorous Newtonian limit. In the same year Andréasson, Calogero and Illner [2] showed that removal of the loss term for the Boltzmann equation (relativistic or not) leads to finite time blow up of a solution.

Our goal in this chapter is to define carefully many aspects of the relativistic Boltzmann equation. We will discuss two different representations for the collsion operator. And then we write down four examples of differential cross sections in the relativistic case. After that we develop three Lorentz transformations which are useful in relativistic collisional kinetic theory. We then rigorously derive the different representations for the collision operator. We then derive the Hilbert-Schmidt form for the linearized collision operator. And we finish this chapter with some remarks on the relativistic Vlasov-Maxwell-Boltzmann system.

Much of this chapter is expository. However most of this material is scattered and/or written in the language of physics. Our motivation is to present all of this material in one place and in one language. The only thing that is possibly original in this chapter is the precise form of two of the Lorentz Transformations. These may be useful in future mathematical investigations.

5.1. The Relativistic Boltzmann Equation

The relativisitic Boltzmann Equation is

$$\partial_t F + \hat{p} \cdot \nabla_x F = \mathcal{C}(F, F), \;\; F(0, x, p) = F_0(x, p). \tag{5.1}$$

Here $F = F(t, x, p)$ is a spatially periodic function of time $t \in [0, \infty)$, space $x \in \Omega \subset \mathbb{R}^3$ and momentum $p \in \mathbb{R}^3$. Let c denote the speed of light. the energy of a relativistic particle with momentum p is defined by $p_0 = \sqrt{c^2 + |p|^2}$ and the normalized velocity of a particle by

$$\hat{p} = \frac{p}{\sqrt{1 + |p|^2/c^2}}.$$

Moreover, the relativistic Boltzmann collision operator is commonly written in the physics literature [7, 15] as

$$\mathcal{C}(F,G) = \frac{c}{2}\frac{1}{p_0}\int_{\mathbb{R}^3}\frac{dq}{q_0}\int_{\mathbb{R}^3}\frac{dq'}{q'_0}\int_{\mathbb{R}^3}\frac{dp'}{p'_0}W(p,q|p',q')[F(p')G(q') - F(p)G(q)]. \tag{5.2}$$

Here the transition rate, $W(p,q|p',q')$, has the form

$$W(p,q|p',q') = s\sigma(g,\theta)\delta^{(4)}(P + Q - P' - Q'), \tag{5.3}$$

where σ is called the differential cross-section. Denote the relativistic four vector, P, by $P = (p_0, p)^t$. $Q = (q_0, q)^t$ is another four-vector. The post-collisional momentum, (p', q'), and energy, (p'_0, q'_0), satisfy:

$$P + Q = P' + Q'. \tag{5.4}$$

Denote the Lorentz inner product for 4-vectors as

$$P \cdot Q = p_0 q_0 - \sum_{i=1}^{3} p_i q_i.$$

With this notation in hand, we write

$$s = s(P,Q) \equiv |P + Q|^2 \equiv (P + Q)\cdot(P + Q) = 2\left(P \cdot Q + c^2\right), \tag{5.5}$$

where s is the normalized total energy in the center-of-momentum system:

$$p + q = 0. \tag{5.6}$$

Also, $2g$ is called the relative momentum, where

$$g = g(P,Q) \equiv |P - Q| \equiv \sqrt{-(P - Q)\cdot(P - Q)} = \sqrt{2\left(P \cdot Q - c^2\right)}. \tag{5.7}$$

Note that

$$s = 2\left(P \cdot Q + c^2\right) = 2\left(P \cdot Q - c^2\right) + 4c^2 = g^2 + 4c^2. \tag{5.8}$$

Now (5.4) and (5.5) together imply that s and therefore g are collision invariants: $s(P,Q) = s(P',Q')$ and $g(P,Q) = g(P',Q')$.

Next, the angle θ–which is the scattering angle in the center-of-momentum system (5.6)–is defined by

$$\cos\theta \equiv -(P-Q)\cdot(P'-Q')/g^2. \tag{5.9}$$

We remark that *in general* the expression in (5.9) is not well-defined because the r.h.s. could be greater than one. But the conservation of momentum and energy (5.4) makes this a good definition [30, p.113].

A smooth solution to the relativistic Boltzmann equation, (5.1), which decays sufficiently rapidly satisfies the conservation of mass, momentum and energy:

$$\frac{d}{dt}\int_{\mathbb{T}^3\times\mathbb{R}^3} F(t)dxdp = \frac{d}{dt}\int_{\mathbb{T}^3\times\mathbb{R}^3} pF(t)dxdp = \frac{d}{dt}\int_{\mathbb{T}^3\times\mathbb{R}^3} p_0F(t)dxdp = 0.$$

Boltzmann's famous H-Theorem for the relativistic Boltzmann equation (5.1) is

$$\frac{d}{dt}\int_{\mathbb{T}^3\times\mathbb{R}^3} F(t)\ln F(t)dxdp \le 0.$$

We write the relativistic Maxwellian or Jüttner solution as

$$J(p) \equiv \frac{\exp(-cp_0)}{4\pi c K_2(c^2)}. \tag{5.10}$$

Since $\mathcal{C}(J,J)\equiv 0$ this is a steady state solution to (5.1). Here K_2 is a bessel function defined by the restriction $\int_{\mathbb{R}^3} J(p)dp = 1$ [48, p.449].

In the literature, there are two ways to carry out the delta function integrations in the collision operator. These are analagous to two well known expressions for the post-collisional velocities in the classical (non-relativistic) Boltzmann theory.

5.1.1. Center-of-Momentum Collision Operator. Following [7, 15] four of the integrations can be carried out in the *center-of-momentum system* to get rid of the delta functions and obtain

$$\mathcal{C}(F,G) = \int_{\mathbb{R}^3\times S^2} v\sigma(g,\theta)[F(p')G(q') - F(p)G(q)]d\omega dq. \tag{5.11}$$

where $v = v(p,q)$ is the Møller velocity given by

$$v = v(p,q) \equiv \frac{c}{2}\sqrt{\left|\frac{p}{p_0} - \frac{q}{q_0}\right|^2 - \frac{1}{c^2}\left|\frac{p}{p_0} \times \frac{q}{q_0}\right|^2} = \frac{c}{4}\frac{g\sqrt{s}}{p_0 q_0}. \tag{5.12}$$

The post collisional momentum in the expression (5.11) can be written:

$$\begin{aligned} p_i' &= \frac{p_i + q_i}{2} + g\left(\omega_i + (\gamma - 1)(p_i + q_i)\frac{(p+q)\cdot\omega}{|p+q|^2}\right), \\ q_i' &= \frac{p_i + q_i}{2} - g\left(\omega_i + (\gamma - 1)(p_i + q_i)\frac{(p+q)\cdot\omega}{|p+q|^2}\right), \end{aligned} \tag{5.13}$$

where $\gamma = (p_0 + q_0)/\sqrt{s}$. The energies are then

$$\begin{aligned} p_0' &= \frac{p_0 + q_0}{2} + \frac{g}{\sqrt{s}}\omega \cdot (p+q), \\ q_0' &= \frac{p_0 + q_0}{2} - \frac{g}{\sqrt{s}}\omega \cdot (p+q). \end{aligned}$$

These clearly satisfy (5.4). Further

$$\cos\theta = k \cdot \omega$$

where k is a unit vector. Then one can write $d\omega = \sin\theta d\theta d\psi$ with $0 \le \theta \le \pi$ and $0 \le \psi \le 2\pi$. We will perform this computation in Section 5.4. In fact, we will show in Section 5.4 that (P', Q') can be defined more generally only up to a (non-unique) Lorentz Transformation.

This expression is the relativistic analogoue of the center-of-mass variables for the classical Boltzmann collision operator. In the non-relativistic case, the center-of-mass variables are

$$\begin{aligned} p' &= \frac{p+q}{2} + \frac{1}{2}|p-q|\omega, \\ q' &\to \frac{p+q}{2} - \frac{1}{2}|p-q|\omega, \end{aligned}$$

And the collision operator with hard-sphere interaction is

$$\mathcal{C}_\infty(F,G) = \frac{1}{2}\int_{\mathbb{R}^3 \times S^2} |p-q|[F(p')G(q') - F(p)G(q)]d\omega dq.$$

Indeed, when $\sigma = 1$, the formal Newtonian limit ($c \uparrow \infty$) of (5.11) with variables (5.13) is this center-of-mass system.

5.1.2. Another expression for the Collision Operator. Glassey and Strauss showed in [32] that another reduction can be performed, without using the center-of-momentum system, to obtain

$$\mathcal{C}(F,G) = \int_{\mathbb{R}^3\times S^2} \frac{s\sigma(g,\theta)}{p_0 q_0} B(p,q,\omega)[F(p')G(q') - F(p)G(q)]d\omega dq, \tag{5.14}$$

where the kernel is

$$B(p,q,\omega) \equiv c\frac{(p_0+q_0)^2 p_0 q_0 \left|\omega\cdot\left(\frac{p}{p_0} - \frac{q}{q_0}\right)\right|}{\left[(p_0+q_0)^2 - (\omega\cdot[p+q])^2\right]^2}. \tag{5.15}$$

In this expression, the post collisional momentum's are given by

$$\begin{aligned} p' &= p + a(p,q,\omega)\omega, \\ q' &= q - a(p,q,\omega)\omega, \end{aligned} \tag{5.16}$$

where

$$a(p,q,\omega) = \frac{2(p_0+q_0)p_0q_0\left\{\omega\cdot\left(\frac{q}{q_0} - \frac{p}{p_0}\right)\right\}}{(p_0+q_0)^2 - \left\{\omega\cdot(p+q)\right\}^2}.$$

And the energies can be expressed as $p_0' = p_0 + N_0$ and $q_0' = q_0 - N_0$ with

$$N_0 \equiv \frac{2\omega\cdot(p+q)\{p_0(\omega\cdot q) - q_0(\omega\cdot p)\}}{(p_0+q_0)^2 - \{\omega\cdot(p+q)\}^2}.$$

These expressions clearly satisfy the collisional conservations (5.4). The angle is then defined by plugging these into (5.9). The Jacobian for the transformation $(p,q) \to (p',q')$ in these variables [31] is

$$\frac{\partial(p',q')}{\partial(p,q)} = -\frac{p_0'q_0'}{p_0q_0}. \tag{5.17}$$

Moreover we have the following quanities which are invariant before and after collisions:

$$\begin{aligned} p_0q_0\left[\omega\cdot\left(\frac{q}{q_0} - \frac{p}{p_0}\right)\right] &= p_0'q_0'\left[\omega\cdot\left(\frac{p'}{p_0'} - \frac{q'}{q_0'}\right)\right], \\ P\cdot Q &= P'\cdot Q'. \end{aligned}$$

Therefore for fixed $\omega \in S^2$ we have

$$B(p,q,\omega) = B(p',q',\omega). \tag{5.18}$$

One can use these to derive further identities.

Now these variables are analogous to the other common non-relativistic Boltzmann variables

$$
\begin{aligned}
p' &= p + \omega \cdot (q - p)\, \omega, \\
q' &= q - \omega \cdot (q - p)\, \omega.
\end{aligned}
$$

With these post-collisional velocities, the non-relativistic collision operator with hard-sphere interaction is

$$
\mathcal{C}_\infty(F, G) = \int_{\mathbb{R}^3 \times S^2} |\omega \cdot (p - q)| \, [F(p')G(q') - F(p)G(q)] d\omega dq.
$$

And when $\sigma = 1$ the formal Newtonian limit of (5.14) with variables (5.16) is this standard hard-sphere collision operator.

5.2. Examples of relativistic Boltzmann cross-sections

The calculation of the differential cross-section is in the relativistic situation utilzes quantum field theory [55]. In the mathematical literature of the relativistic Boltzmann equation it is hard to find precise physically relevant examples of these differential cross-sections. In this section we write down a few examples of differential cross sections. However, I am not a physicist and I am not claiming these are the physically relevant examples.

Above, we have written everything down with the mass m normalized to one, $m = 1$. Here we write down the mass before normalization.

5.2.0.1. *Short Range Interactions.* [29, 55] For short range interactions,

$$
s\sigma \equiv \text{constant}.
$$

This is the relativistic analouge of the hard-sphere cross-section in the Newtonian case. Indeed, as we have already mentioned, the Newtonian limit of the relativistic Boltzmann equation in this case is the hard-sphere Boltzmann equation.

5.2.0.2. *Møller Scattering.* [15, p.350] Møller scattering is used as an approximation of electron-electron scatttering.

$$\sigma(g,\Theta) = r_0^2 \frac{1}{u^2(u^2-1)^2}\left\{\frac{(2u^2-1)^2}{\sin^4\Theta} - \frac{2u^4-u^2-\frac{1}{4}}{\sin^2\Theta} + \frac{1}{4}(u^2-1)^2\right\}.$$

where the magnitude of total four-momentum scaled w.r.t. total mass is

$$u = \frac{\sqrt{s}}{2mc}$$

and $r_0 = \frac{e^2}{4\pi mc^2}$ is the classical electron radius. Photon-photon scattering is often neglected because the size of the cross-section is 'negligible'.

5.2.0.3. *Compton Scattering.* [15, p.351] Compton scattering is an approximation of photon-electron scattering.

$$\sigma(g,\Theta) = \frac{1}{2}r_0^2(1-\xi)\left\{1 + \frac{1}{4}\frac{\xi^2(1-\cos\Theta)^2}{1-\frac{1}{2}\xi(1-\cos\Theta)} + \left(\frac{1-(1-\frac{1}{2}\xi)(1-\cos\Theta)}{1-\frac{1}{2}\xi(1-\cos\Theta)}\right)^2\right\},$$

where

$$\xi = 1 - \frac{m^2c^2}{s}.$$

See [29, p.81] for the Newtonian limit in this case.

5.2.0.4. *(elastic) Neutrino Gas.* [22, p.478] For a neutrino gas, the differential cross section is independent of the scattering a angle Θ.

$$\sigma(g,\Theta) = \frac{G^2}{\pi\hbar^2c^2}g^2,$$

where G is the weak coupling constant and $2\pi\hbar$ is Plank's constant. These are massless particles! See also [15, p.290].

5.2.0.5. *Israel particles.* [43, p.1173] The Isreal particles are the analogue of the "maxwell molecules" cross section in the Newtonian theory.

$$\sigma(g,\Theta) = \frac{m}{2g}\frac{b(\Theta)}{1+(g/mc)^2}.$$

With this cross section, Israel derives eigenfunctions for the Linearized relativistic Boltzmann collision operator. Variants of this cross section are used in [13, 56]. For instance the cross section for "Maxwell Particles" is formed by removing the factor

$1 + (g/mc)^2$. Note that it converges to the maxwell molecules cross section in the Newtonian limit.

5.3. Lorentz Transformations

In this section, we will write down three Lorentz Transformations which can be useful in relativistic Kinetic Theory. Let Λ be a 4×4 matrix denoted by

$$\Lambda = (\Lambda_{ij})_{0 \le i,j \le 3} = \begin{pmatrix} \Lambda^0 \\ \Lambda^1 \\ \Lambda^2 \\ \Lambda^3 \end{pmatrix} = \begin{pmatrix} \Lambda^0_0 & \Lambda^0_1 & \Lambda^0_2 & \Lambda^0_3 \\ \Lambda^1_0 & \Lambda^1_1 & \Lambda^1_2 & \Lambda^1_3 \\ \Lambda^2_0 & \Lambda^2_1 & \Lambda^2_2 & \Lambda^2_3 \\ \Lambda^3_0 & \Lambda^3_1 & \Lambda^3_2 & \Lambda^3_3 \end{pmatrix} = (\Lambda_0 \ \Lambda_1 \ \Lambda_2 \ \Lambda_3)\,.$$

Here Λ^i $(i = 0, 1, 2, 3)$ are used to denote the row's and Λ_i are used to denote the columns's of the matrix. We assume the entries of the matrix are real. For the basics of Lorentz Transformations, we refer to [14, 54, 70].

DEFINITION 5.1. Λ is a Lorentz Transformation if

$$(\Lambda P) \cdot (\Lambda Q) = P \cdot Q$$

Let $D = \mathrm{diag}(1, -1, -1, -1)$. Equivalently, Λ is a Lorentz Transformation if

$$\Lambda^T D \Lambda = D. \tag{5.19}$$

From now on we will use the notation Λ to exclusively denote a Lorentz Transformation. Notice, from (5.19), that any Lorentz transformation satisfies $|\det(\Lambda)| = 1$. Any Lorentz Transformation, Λ, is invertible and (5.19) implies

$$\Lambda^{-1} = D\Lambda^T D.$$

Therefore, $\Lambda D \Lambda^T = D$, e.g. Λ^T is also a Lorentz Transformation. And Λ^{-1} is a Lorentz Transformation too since

$$(\Lambda^{-1})^T D \Lambda^{-1} = (D\Lambda D) D (D\Lambda^T D) = D(\Lambda D \Lambda^T) D = D^3 = D.$$

From (5.19) it suffices to check

$$\Lambda_i \cdot \Lambda_j = D_{ij}, \quad (i, j = 0, 1, 2, 3). \tag{5.20}$$

Or equivalently, since Λ is Lorentz if and only if Λ^T is Lorentz,

$$\Lambda^i \cdot \Lambda^j = D_{ij}, \quad (i, j = 0, 1, 2, 3). \tag{5.21}$$

This is just the matrix multiplication (5.19) in component form. A proper orthochronous Lorentz transformation satisfies $\det(\Lambda) = 1$ and $\Lambda_0^0 \geq 1$.

Moreover, by symmetry, (5.19) is equivalent to the following ten equations:

$$\begin{aligned}
(\Lambda_0^0)^2 - \left\{(\Lambda_0^1)^2 + (\Lambda_0^2)^2 + (\Lambda_0^3)^2\right\} &= 1 \\
(\Lambda_1^0)^2 - \left\{(\Lambda_1^1)^2 + (\Lambda_1^2)^2 + (\Lambda_1^3)^2\right\} &= -1 \\
(\Lambda_2^0)^2 - \left\{(\Lambda_2^1)^2 + (\Lambda_2^2)^2 + (\Lambda_2^3)^2\right\} &= -1 \\
(\Lambda_3^0)^2 - \left\{(\Lambda_3^1)^2 + (\Lambda_3^2)^2 + (\Lambda_3^3)^2\right\} &= -1 \\
\Lambda_0^0\Lambda_1^0 - \left\{\Lambda_0^1\Lambda_1^1 + \Lambda_0^2\Lambda_1^2 + \Lambda_0^3\Lambda_1^3\right\} &= 0 \\
\Lambda_0^0\Lambda_2^0 - \left\{\Lambda_0^1\Lambda_2^1 + \Lambda_0^2\Lambda_2^2 + \Lambda_0^3\Lambda_2^3\right\} &= 0 \\
\Lambda_0^0\Lambda_3^0 - \left\{\Lambda_0^1\Lambda_3^1 + \Lambda_0^2\Lambda_3^2 + \Lambda_0^3\Lambda_3^3\right\} &= 0 \\
\Lambda_1^0\Lambda_2^0 - \left\{\Lambda_1^1\Lambda_2^1 + \Lambda_1^2\Lambda_2^2 + \Lambda_1^3\Lambda_2^3\right\} &= 0 \\
\Lambda_1^0\Lambda_3^0 - \left\{\Lambda_1^1\Lambda_3^1 + \Lambda_1^2\Lambda_3^2 + \Lambda_1^3\Lambda_3^3\right\} &= 0 \\
\Lambda_2^0\Lambda_3^0 - \left\{\Lambda_2^1\Lambda_3^1 + \Lambda_2^2\Lambda_3^2 + \Lambda_2^3\Lambda_3^3\right\} &= 0.
\end{aligned} \tag{5.22}$$

Since Λ has sixteen components and is restricted by ten equations, one says that Λ has six free parameters.

Perhaps the most well known Lorentz transformation is the boost matrix. Given $v = (v^1, v^2, v^3) \in \mathbb{R}^3$, we write the boost matrix as

$$\Lambda_b = \begin{pmatrix} \gamma & -\gamma v^1 & -\gamma v^2 & -\gamma v^3 \\ -\gamma v^1 & 1 + (\gamma - 1)\frac{v^1 v^1}{|v|^2} & (\gamma - 1)\frac{v^2 v^1}{|v|^2} & (\gamma - 1)\frac{v^3 v^1}{|v|^2} \\ -\gamma v^2 & (\gamma - 1)\frac{v^1 v^2}{|v|^2} & 1 + (\gamma - 1)\frac{v^2 v^2}{|v|^2} & (\gamma - 1)\frac{v^3 v^2}{|v|^2} \\ -\gamma v^3 & (\gamma - 1)\frac{v^1 v^3}{|v|^2} & (\gamma - 1)\frac{v^2 v^3}{|v|^2} & 1 + (\gamma - 1)\frac{v^3 v^3}{|v|^2} \end{pmatrix},$$

where $\gamma = (1-|v|^2)^{-1/2}$. Notice that Λ_b has only three free parameters.

In this section, we are exclusively concerned with proper orthochronous Lorentz transformations Λ, depending on P and Q, such that

$$\Lambda(P+Q) = (\sqrt{s}, 0, 0, 0)^t, \tag{5.23}$$

where we recall that s is defined in (5.5).

5.3.1. Lorentz transformations for Relativistic Kinetic Theory. We will give three examples of Lorentz Transformations satisfying (5.23).

5.3.1.1. *Example 1: The Boost Matrix.* Our goal is to choose v such that Λ_b satisfies (5.23). Let

$$v = \frac{p+q}{p_0+q_0}, \quad \gamma = \frac{p_0+q_0}{\sqrt{s}}.$$

Then Λ_b satisfies (5.23) and is given by

$$\Lambda_b = \begin{pmatrix} \frac{p_0+q_0}{\sqrt{s}} & -\frac{p_1+q_1}{\sqrt{s}} & -\frac{p_2+q_2}{\sqrt{s}} & -\frac{p_3+q_3}{\sqrt{s}} \\ -\frac{p_3+q_3}{\sqrt{s}} & 1+(\gamma-1)\frac{v^1v^1}{|v|^2} & (\gamma-1)\frac{v^2v^1}{|v|^2} & (\gamma-1)\frac{v^3v^1}{|v|^2} \\ -\frac{p_3+q_3}{\sqrt{s}} & (\gamma-1)\frac{v^1v^2}{|v|^2} & 1+(\gamma-1)\frac{v^2v^2}{|v|^2} & (\gamma-1)\frac{v^3v^2}{|v|^2} \\ -\frac{p_3+q_3}{\sqrt{s}} & (\gamma-1)\frac{v^1v^3}{|v|^2} & (\gamma-1)\frac{v^2v^3}{|v|^2} & 1+(\gamma-1)\frac{v^3v^3}{|v|^2} \end{pmatrix},$$

where $\frac{v^iv^j}{|v|^2} = \frac{(p_i+q_i)(p_j+q_j)}{|p+q|^2}$. By a direct calculation, this example satisfies (5.23).

5.3.1.2. *Example 2: To reduce the Collision Integrals.* Given $p, q \in \mathbb{R}^3$ with $p+q \neq 0$, we can always choose three orthonormal vectors

$$w^1, \quad w^2, \quad w^3 = \frac{p+q}{|p+q|}.$$

If, for example, $p_1+q_1 \neq 0$ and $p_2+q_2 \neq 0$, then we can explicitly write

$$\begin{aligned} w^1 &= \frac{(-(p_2+q_2)(p_3+q_3), 2(p_1+q_1)(p_3+q_3), -(p_2+q_2)(p_3+q_3))}{|(-(p_2+q_2)(p_3+q_3), 2(p_1+q_1)(p_3+q_3), -(p_2+q_2)(p_3+q_3))|} \\ w^2 &= w^1 \times w^3/|w^1 \times w^3|. \end{aligned}$$

If instead $p_1+q_1 = 0$, then

$$w^1 = (1,0,0), \quad w^2 = w^1 \times w^3/|w^1 \times w^3|.$$

With this notation, we can write down a LT which maps $p+q \to 0$ as

$$\Lambda = \begin{pmatrix} \frac{p_0+q_0}{|P+Q|} & -\frac{p_1+q_1}{|P+Q|} & -\frac{p_2+q_2}{|P+Q|} & -\frac{p_3+q_3}{|P+Q|} \\ 0 & w_1^1 & w_2^1 & w_3^1 \\ 0 & w_1^2 & w_2^2 & w_3^2 \\ \frac{|p+q|}{|P+Q|} & -\frac{p_1+q_1}{|p+q|}\frac{p_0+q_0}{|P+Q|} & -\frac{p_2+q_2}{|p+q|}\frac{p_0+q_0}{|P+Q|} & -\frac{p_3+q_3}{|p+q|}\frac{p_0+q_0}{|P+Q|} \end{pmatrix}. \tag{5.24}$$

This LT clearly satisfies (5.23).

We will derive this LT from the beginning. First, we clearly assume that

$$\Lambda_0^1 = \Lambda_0^2 = 0.$$

Then we are left with the following restrictions for Λ to satisfy (5.23):

$$\begin{aligned} \Lambda_1^1(p_1+q_1) + \Lambda_2^1(p_2+q_2) + \Lambda_3^1(p_3+q_3) &= 0 \\ \Lambda_1^2(p_1+q_1) + \Lambda_2^2(p_2+q_2) + \Lambda_3^2(p_3+q_3) &= 0 \\ \Lambda_0^3(p_0+q_0) + \Lambda_1^3(p_1+q_1) + \Lambda_2^3(p_2+q_2) + \Lambda_3^3(p_3+q_3) &= 0 \end{aligned} \tag{5.25}$$

And the restriction (5.22) reduces to

$$\begin{aligned} \left(\Lambda_0^0\right)^2 - (\Lambda_0^3)^2 &= 1 \\ (\Lambda_1^0)^2 - \left\{(\Lambda_1^1)^2 + (\Lambda_1^2)^2 + (\Lambda_1^3)^2\right\} &= -1 \\ (\Lambda_2^0)^2 - \left\{(\Lambda_2^1)^2 + (\Lambda_2^2)^2 + (\Lambda_2^3)^2\right\} &= -1 \\ (\Lambda_3^0)^2 - \left\{(\Lambda_3^1)^2 + (\Lambda_3^2)^2 + (\Lambda_3^3)^2\right\} &= -1 \end{aligned} \tag{5.26}$$

As well as

$$\begin{aligned} \Lambda_0^0\Lambda_1^0 - \Lambda_0^3\Lambda_1^3 &= 0 \\ \Lambda_0^0\Lambda_2^0 - \Lambda_0^3\Lambda_2^3 &= 0 \\ \Lambda_0^0\Lambda_3^0 - \Lambda_0^3\Lambda_3^3 &= 0 \\ \Lambda_1^0\Lambda_2^0 - \left\{\Lambda_1^1\Lambda_2^1 + \Lambda_1^2\Lambda_2^2 + \Lambda_1^3\Lambda_2^3\right\} &= 0 \\ \Lambda_1^0\Lambda_3^0 - \left\{\Lambda_1^1\Lambda_3^1 + \Lambda_1^2\Lambda_3^2 + \Lambda_1^3\Lambda_3^3\right\} &= 0 \\ \Lambda_2^0\Lambda_3^0 - \left\{\Lambda_2^1\Lambda_3^1 + \Lambda_2^2\Lambda_3^2 + \Lambda_2^3\Lambda_3^3\right\} &= 0. \end{aligned} \tag{5.27}$$

From (5.26) we choose

$$\Lambda_0^0 = \sqrt{1 + (\Lambda_0^3)^2} \tag{5.28}$$

Defining

$$\beta = \frac{\Lambda_0^3}{\sqrt{1 + (\Lambda_0^3)^2}},$$

we find that the first three equations in (5.27) are

$$\Lambda_1^0 = \beta\Lambda_1^3, \quad \Lambda_2^0 = \beta\Lambda_2^3, \quad \Lambda_3^0 = \beta\Lambda_3^3. \tag{5.29}$$

And we plug (5.29) into the bottom three equations in (5.27) to find

$$\begin{aligned} &\Lambda_1^1\Lambda_2^1 + \Lambda_1^2\Lambda_2^2 + \Lambda_1^3\Lambda_2^3(1-\beta^2) = 0 \\ &\Lambda_1^1\Lambda_3^1 + \Lambda_1^2\Lambda_3^2 + \Lambda_1^3\Lambda_3^3(1-\beta^2) = 0 \\ &\Lambda_2^1\Lambda_3^1 + \Lambda_2^2\Lambda_3^2 + \Lambda_2^3\Lambda_3^3(1-\beta^2) = 0. \end{aligned} \tag{5.30}$$

We further plug (5.29) into the last three equations in (5.26) to obtain

$$\begin{aligned} &(\Lambda_1^1)^2 + (\Lambda_1^2)^2 + (\Lambda_1^3)^2(1-\beta^2) = 1 \\ &(\Lambda_2^1)^2 + (\Lambda_2^2)^2 + (\Lambda_2^3)^2(1-\beta^2) = 1 \\ &(\Lambda_3^1)^2 + (\Lambda_3^2)^2 + (\Lambda_3^3)^2(1-\beta^2) = 1 \end{aligned} \tag{5.31}$$

We have thus reduced the system to nine equations and nine unknowns; it remains to simultaneously satisfy (5.25), (5.30) and (5.31).

We first concentrate on satisfying (5.25). In any case, with these vectors define

$$\begin{aligned} &\Lambda^1 = (0, w^1), \\ &\Lambda^2 = (0, w^2), \\ &\Lambda^3 = \left(\Lambda_0^3, -w^3(1-\beta^2)^{-1/2}\right) = \left(\Lambda_0^3, -w^3\sqrt{1 + (\Lambda_0^3)^2}\right). \end{aligned} \tag{5.32}$$

These choices clearly satisfy (5.25) as soon as

$$\Lambda_0^3(p_0 + q_0) = |p + q|\sqrt{1 + (\Lambda_0^3)^2}.$$

Squaring and choosing the positive root by necessity, we have

$$\Lambda_0^3 = \frac{|p+q|}{|P+Q|}, \quad \Lambda_0^0 = \frac{p_0+q_0}{|P+Q|}.$$

It remains to show that these choices satisfy (5.30) and (5.31).

To this end, we define the 3×3 matrix

$$U = \begin{pmatrix} \Lambda_1^1 & \Lambda_2^1 & \Lambda_3^1 \\ \Lambda_1^2 & \Lambda_2^2 & \Lambda_3^2 \\ \Lambda_1^3\sqrt{1-\beta^2} & \Lambda_2^3\sqrt{1-\beta^2} & \Lambda_3^3\sqrt{1-\beta^2} \end{pmatrix}$$

where the coefficients above are defined by (5.32). By definition, $UU^T = I_3$. Therefore U is invertible and

$$U^TU = U^T(UU^T)U = U^T(UU^T)^{-1}U = U^T(U^T)^{-1}U^{-1}U = I_3,$$

which means $U^T = U^{-1}$. This last display also says that the parameters (5.32) satisfy (5.30) and (5.31).

5.3.1.3. *Example 3: Hilbert-Schmidt form for the Linearized Operator.* We now wish to derive a Lorentz Transformation which satisfies (5.23) but also

$$\Lambda(P-Q) = (0,0,0,g). \tag{5.33}$$

In [15, p.277] and Section 5.6 this transformation is used to write down a Hilbert-Schmidt form for the linearized collision operator. But the transformation is not written down explicitly anywhere. All that is given is the conditions (5.23), (5.33) and the first row Λ_0. However, from this information, we can make the following educated guess for Λ satisfying (5.23) and (5.33):

$$\Lambda = \begin{pmatrix} \frac{p_0+q_0}{\sqrt{s}} & -\frac{p_1+q_1}{\sqrt{s}} & -\frac{p_2+q_2}{\sqrt{s}} & -\frac{p_3+q_3}{\sqrt{s}} \\ \Lambda_0^1 & \Lambda_1^1 & \Lambda_2^1 & \Lambda_3^1 \\ 0 & \frac{(p\times q)_1}{|p\times q|} & \frac{(p\times q)_2}{|p\times q|} & \frac{(p\times q)_3}{|p\times q|} \\ \frac{p_0-q_0}{g} & -\frac{p_1-q_1}{g} & -\frac{p_2-q_2}{g} & -\frac{p_3-q_3}{g} \end{pmatrix}.$$

Clearly, $\Lambda^i \cdot \Lambda^j = D_{ij}$ if $i, j \in \{0, 2, 3\}$. By the Lorentz condition $\Lambda_0 \cdot \Lambda_0 = 1$, we choose Λ_0^1 as

$$(\Lambda_0^1)^2 = \left(\frac{p_0 + q_0}{\sqrt{s}}\right)^2 - \left(\frac{p_0 - q_0}{g}\right)^2 - 1. \tag{5.34}$$

By further computations we have

$$\begin{aligned}
sg^2(\Lambda_0^1)^2 &= g^2 (p_0 + q_0)^2 - s (p_0 - q_0)^2 - sg^2 \\
&= 2(P \cdot Q - c^2) (p_0 + q_0)^2 - 2(P \cdot Q + c^2) (p_0 - q_0)^2 - 4((P \cdot Q)^2 - c^4) \\
&= 2(P \cdot Q - c^2) \left(p_0^2 + q_0^2 + 2p_0 q_0\right) - 2(P \cdot Q + c^2) \left(p_0^2 + q_0^2 - 2p_0 q_0\right) \\
&\quad -4((P \cdot Q)^2 - c^4) \\
&= 8(P \cdot Q) p_0 q_0 - 4c^2(p_0^2 + q_0^2) - 4((P \cdot Q)^2 - c^4) \\
&= 8(P \cdot Q) p_0 q_0 - 4c^2(p_0^2 + q_0^2) - 4((p_0 q_0)^2 + (p \cdot q)^2 - 2p_0 q_0 (p \cdot q) - c^4) \\
&= 4p_0^2 q_0^2 - 4(p \cdot q)^2 - 4c^2(p_0^2 + q_0^2) + 4c^4 \\
&= 4 \left(|p|^2 |q|^2 - (p \cdot q)^2\right) = 4|p \times q|^2.
\end{aligned}$$

Choosing the positive root, we conclude

$$\Lambda_0^1 = \frac{2|p \times q|}{\sqrt{s} g} = \frac{|p \times q|}{\sqrt{(P \cdot Q)^2 - c^4}}.$$

The coefficients Λ_i^1 $(i = 1, 2, 3)$ are similarly completely determined by the Lorentz condition $\Lambda_i \cdot \Lambda_i = -1$. However, this turns out to be a long computation. In the end we obtain

$$\Lambda_i^1 = \frac{2\left(p_i \left\{p_0 - q_0 P \cdot Q\right\} + q_i \left\{q_0 - p_0 P \cdot Q\right\}\right)}{\sqrt{s} g |p \times q|} \quad (i = 1, 2, 3).$$

Now, we have a complete description of this lorentz transformation in terms of p, q.

The only way to derive this matrix was by guessing the first, third and fourth rows and then computing $\Lambda_i \cdot \Lambda_i = D_{ii}$. However, it turns out that it is much easier to verify that this is indeed a Lorentz Transformation by checking $\Lambda^i \cdot \Lambda^j = D_{ij}$ $(i, j = 0, 1, 2, 3)$, e.g. the conditions (5.21). We will do this one row at a time.

The First row. Clearly $\Lambda^0 \cdot \Lambda^j = D_{0j}$ for $j = 0, 2, 3$. To check $\Lambda^0 \cdot \Lambda^1 = D_{01} = 0$ we compute

$$
\begin{aligned}
\sum_{i=1}^{3} \Lambda_i^1 p_i &= \frac{2}{\sqrt{s}g|p \times q|} \left(|p|^2 \left\{ c^2 p_0 - q_0 P \cdot Q \right\} + p \cdot q \left\{ c^2 q_0 - p_0 P \cdot Q \right\} \right) \\
&= \frac{2 \left(|p|^2 c^2 p_0 - |p|^2 p_0 q_0^2 + |p|^2 q_0 p \cdot q + p \cdot q c^2 q_0 - p \cdot q p_0^2 q_0 + p_0 (p \cdot q)^2 \right)}{\sqrt{s}g|p \times q|} \\
&= \frac{2 \left(-p_0 |p|^2 |q|^2 + p_0 (p \cdot q)^2 \right)}{\sqrt{s}g|p \times q|} = \frac{2}{\sqrt{s}g|p \times q|} \left(-|p \times q|^2 p_0 \right) \qquad (5.35) \\
&= -\frac{2|p \times q|}{\sqrt{s}g} p_0.
\end{aligned}
$$

And similarly,

$$
\begin{aligned}
\sum_{i} \Lambda_i^1 q_i &= \frac{2}{\sqrt{s}g|p \times q|} \left(p \cdot q \left\{ c^2 p_0 - q_0 P \cdot Q \right\} + |q|^2 \left\{ c^2 q_0 - p_0 P \cdot Q \right\} \right) \\
&= \frac{2 \left(p \cdot q c^2 p_0 - p \cdot q p_0 q_0^2 + q_0 (p \cdot q)^2 + |q|^2 c^2 q_0 - |q|^2 p_0^2 q_0 + |q|^2 p_0 p \cdot q \right)}{\sqrt{s}g|p \times q|} \qquad (5.36) \\
&= \frac{2 \left(-|q|^2 |p|^2 q_0 + q_0 (p \cdot q)^2 \right)}{\sqrt{s}g|p \times q|} = -\frac{2|p \times q|}{\sqrt{s}g} q_0.
\end{aligned}
$$

We thus conclude that $\Lambda^0 \cdot \Lambda^1 = D_{01} = 0$.

The Second row. The calculation we will check now is $\Lambda^1 \cdot \Lambda^1 = D_{11} = -1$. The rest of the conditions are checked elsewhere. We compute

$$
\begin{aligned}
\frac{sg^2 |p \times q|^2}{4} (\Lambda_i^1)^2 &= \left(p_i \left\{ c^2 p_0 - q_0 P \cdot Q \right\} + q_i \left\{ c^2 q_0 - p_0 P \cdot Q \right\} \right)^2 \\
&= \left(p_i^2 \left\{ c^2 p_0 - q_0 P \cdot Q \right\}^2 + q_i^2 \left\{ c^2 q_0 - p_0 P \cdot Q \right\}^2 \right) \\
&\quad + \left(2 p_i q_i \left\{ c^2 p_0 - q_0 P \cdot Q \right\} \left\{ c^2 q_0 - p_0 P \cdot Q \right\} \right).
\end{aligned}
$$

Summing

$$
\begin{aligned}
\frac{sg^2 |p \times q|^2}{4} \sum_{i=1}^{3} (\Lambda_i^1)^2 &= \left(|p|^2 \left\{ c^2 p_0 - q_0 P \cdot Q \right\}^2 + |q|^2 \left\{ c^2 q_0 - p_0 P \cdot Q \right\}^2 \right) \\
&\quad + \left(2 p \cdot q \left\{ c^2 p_0 - q_0 P \cdot Q \right\} \left\{ c^2 q_0 - p_0 P \cdot Q \right\} \right).
\end{aligned}
$$

Notice that the r.h.s. above is

$$
\begin{aligned}
&= \left(|p|^2\left\{c^2 p_0 - q_0 P\cdot Q\right\} + p\cdot q\left\{c^2 q_0 - p_0 P\cdot Q\right\}\right)\left\{c^2 p_0 - q_0 P\cdot Q\right\} \\
&\quad + \left(|q|^2\left\{c^2 q_0 - p_0 P\cdot Q\right\} + p\cdot q\left\{c^2 p_0 - q_0 P\cdot Q\right\}\right)\left\{c^2 q_0 - p_0 P\cdot Q\right\}.
\end{aligned}
$$

Therefore, by (5.35) and (5.36), we conclude that

$$
\begin{aligned}
\sum_{i=1}^{3}(\Lambda_i^1)^2 &= -\frac{4|p\times q|^2}{sg^2|p\times q|^2}\left(p_0\left\{c^2 p_0 - q_0 P\cdot Q\right\} + q_0\left\{c^2 q_0 - p_0 P\cdot Q\right\}\right) \\
&= -\frac{4}{sg^2}\left(p_0\left\{c^2 p_0 - q_0 P\cdot Q\right\} + q_0\left\{c^2 q_0 - p_0 P\cdot Q\right\}\right) \\
&= \frac{1}{sg^2}\left(-4c^2(p_0^2+q_0^2) + 8p_0q_0(P\cdot Q)\right).
\end{aligned}
$$

Adding and subtracting terms, we obtain

$$
\begin{aligned}
&= \frac{1}{sg^2}\left\{-4c^2(p_0^2+q_0^2) + (2(p_0^2+q_0^2) - 2(p_0^2+q_0^2) + 8p_0q_0)P\cdot Q\right\} \\
&= \frac{1}{sg^2}\left\{-4c^2(p_0^2+q_0^2) + 2((p_0+q_0)^2 - 2(p_0-q_0)^2)P\cdot Q\right\}
\end{aligned}
$$

Adding and subtracting terms again, we obtain

$$
\begin{aligned}
&= \frac{1}{sg^2}\left\{-4c^2(p_0^2+q_0^2+p_0q_0-p_0q_0) + 2((p_0+q_0)^2 - 2(p_0-q_0)^2)P\cdot Q\right\} \\
&= \frac{1}{sg^2}\left\{-2c^2((p_0+q_0)^2 + (p_0-q_0)^2) + 2((p_0+q_0)^2 - 2(p_0-q_0)^2)P\cdot Q\right\} \\
&= \frac{2(P\cdot Q - c^2)}{sg^2}(p_0+q_0)^2 - \frac{2(P\cdot Q + c^2)}{sg^2}(p_0-q_0)^2.
\end{aligned}
$$

And by (5.7) this says

$$
\sum_{i=1}^{3}(\Lambda_i^1)^2 = \left(\frac{p_0+q_0}{\sqrt{s}}\right)^2 - \left(\frac{p_0-q_0}{g}\right)^2.
$$

Therefore, by (5.34), $\Lambda^1\cdot\Lambda^1 = D_{11} = -1$.

The Third row. By the definition of the cross product, we clearly have

$$
\Lambda^2\cdot\Lambda^j = D_{2j},
$$

for $j = 0,1,2,3$.

The Fourth row. This is similar to the first row. Notice that $\Lambda^3\cdot\Lambda^j = D_{3j}$ for $j = 0,2,3$. And again by (5.35) and (5.36) we conclude that $\Lambda^3\cdot\Lambda^1 = D_{31}$

We remark that all the components of this Lorentz Transformation may have been previously unknown in their precise form[1].

5.4. Center of Momentum Reduction of the Collision Integrals

In this section, we will reduce the collision integrals in the center of momentum system. Given (5.3) and a integrable function $G : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ we have

$$\begin{aligned} \frac{c}{2}\frac{1}{p_0}\int_{\mathbb{R}^3}\frac{dq}{q_0}\int_{\mathbb{R}^3}\frac{dq'}{q_0'}\int_{\mathbb{R}^3}\frac{dp'}{p_0'} s\delta^{(4)}(P'+Q'-P-Q)G(P,Q,P',Q') \\ = \int_{\mathbb{R}^3} dq \int_{S^2} d\omega v\sigma(g,\theta)G(P,Q,P',Q') \end{aligned}$$

where v is given by (5.12) and the angle θ is given by

$$\cos\theta = \frac{\bar{p}}{|\bar{p}|}\cdot\omega, \tag{5.37}$$

where $\bar{p}$ is defined by $\Lambda(P-Q) = (0,\bar{p})^t$. The post-collisional velocities satisfy

$$P' = \frac{1}{2}\Lambda^{-1}\begin{pmatrix} s \\ g\omega \end{pmatrix}, \; Q' = \frac{1}{2}\Lambda^{-1}\begin{pmatrix} s \\ -g\omega \end{pmatrix}, \tag{5.38}$$

where Λ is any Lorentz Transformation which satisfies (5.23). In particular any of the Lorentz Transformations in the previous section will do.

We will compute, from (5.2), explicitly

$$\begin{aligned} \int_{\mathbb{R}^3}\frac{dq'}{q_0'}\int_{\mathbb{R}^3}\frac{dp'}{p_0'} s\sigma(g,\theta)\delta^{(4)}(P+Q-P'-Q')G(P,Q,P',Q'), \\ = \int_{S^2} d\omega \frac{g\sqrt{s}}{2}\sigma(g,\theta)G(P,Q,P',Q'). \end{aligned} \tag{5.39}$$

This gives the reduced expression by (5.12).

We first claim that the r.h.s. of (5.39) is

$$= \int_{\mathbb{R}^3}\frac{dq'}{q_0'}\int_{\mathbb{R}^3}\frac{dp'}{p_0'} s\sigma(g,\theta)\delta^{(4)}(\Lambda(P+Q)-P'-Q')G(P,Q,\Lambda^{-1}P',\Lambda^{-1}Q').$$

This holds because $\frac{dq'}{q_0'}$ and $\frac{dp'}{p_0'}$ are Lorentz invariant measures and

$$\delta^{(4)}(\Lambda(P+Q-P'-Q')) = \delta^{(4)}(P+Q-P'-Q').$$

[1]W. A. van Leeuwen, personal communication, 2003

But notice that the claim is not true unless the angle θ is redefined as

$$\cos\theta = \frac{[\Lambda(P-Q)]\cdot(P'-Q')}{|P-Q|^2} = \frac{\bar{p}}{|\bar{p}|}\cdot\frac{(p'-q')}{|P-Q|},$$

where we have used $|P-Q| = |\Lambda(P-Q)| = |\bar{p}|$.

Since, now, $p'+q'=0$, the integration over q' can be carried out immediately and we obtain

$$= \frac{1}{2}\int \frac{dp'}{(p'_0)^2} s\sigma(g,\theta)\delta(\frac{1}{2}\sqrt{s}-p'_0)G(P,Q,\Lambda^{-1}P',-\Lambda^{-1}\bar{P}').$$

where now $\bar{P}' = (-p'_0, p')$ and, using $p'+q'=0$, the angle is

$$\cos(\theta) = 2\frac{\bar{p}}{|\bar{p}|}\cdot\frac{p'}{|P-Q|}$$

Next change to polar coordinates as $p' = |p'|\omega$ ($\omega\in S^2$) and

$$dp' = |p'|^2 d|p'|d\omega.$$

We use the following calculation to compute the delta function

$$\begin{aligned}\frac{1}{2}\sqrt{s}-p'_0 &= \frac{1}{2}\sqrt{s}-\sqrt{c^2+|p'|^2} = \frac{\frac{1}{4}s-(c^2+|p'|^2)}{\frac{1}{2}\sqrt{s}+\sqrt{c^2+|p'|^2}}\\ &= \frac{\frac{1}{4}g^2-|p'|^2}{\frac{1}{2}\sqrt{s}+\sqrt{1+|p'|^2}}.\end{aligned}$$

We have just used (5.8). Thus

$$\frac{1}{2}\sqrt{s}-p'_0 = \frac{(\frac{1}{2}g-|p'|)(\frac{1}{2}g+|p'|)}{\frac{1}{2}\sqrt{s}+\sqrt{c^2+|p'|^2}}.$$

Note that if $\frac{1}{2}g = |p'|$ then (5.8) grants $\sqrt{c^2+|p'|^2} = \frac{1}{2}\sqrt{s}$. Therefore,

$$\delta(\frac{1}{2}\sqrt{s}-p'_0) = \frac{\frac{1}{2}\sqrt{s}+\sqrt{c^2+|p'|^2}}{\frac{1}{2}g+|p'|}\delta(\frac{1}{2}g-|p'|) = \frac{\sqrt{s}}{g}\delta(\frac{1}{2}g-|p'|).$$

We plug this in to obtain

$$= \frac{1}{2}\int_0^\infty d|p'|\int_{S^2} d\omega\frac{|p'|^2}{(p'_0)^2}\frac{s^{3/2}}{g}\sigma(g,\theta)\delta(\frac{1}{2}g-|p'|)F(\Lambda^{-1}\bar{\omega})G(\Lambda^{-1}\tilde{\omega}).$$

where

$$\bar{\omega} = (p'_0, |p'|\omega)^T = \frac{1}{2}(\sqrt{s}, g\omega)^t, \quad \tilde{\omega} = (p'_0, -|p'|\omega)^T = \frac{1}{2}(\sqrt{s}, -g\omega)^t.$$

Further, since $p' = |p'|\omega = \frac{1}{2}g\omega$, the angle θ is redefined (from above) as (5.37). We now evaluate the delta function to obtain (5.39).

5.4.1. Post-Collisional velocities. Notice that, using these definition (5.38) for the post-collisional velocities, we recover the identities of conservation for elastic collisions (5.4). This is verified by multiplying both sides of (5.4) by Λ since

$$\Lambda(P+Q) = (\sqrt{s}, 0, 0, 0)^t = \bar{\omega} + \tilde{\omega} = \Lambda(P' + Q').$$

Since Λ^{-1} is also a Lorentz Transformation, we have

$$P' \cdot P' = Q' \cdot Q' = c^2,$$

which is also as it should be...

We further assume

$$\Lambda_0^0 = \frac{p_0 + q_0}{|P+Q|},\ \Lambda_i^0 = -\frac{p_i + q_i}{|P+Q|}\ (i = 1, 2, 3).$$

Note that this assumption is satisfied by all the examples in this note. Define

$$\bar{\Lambda}_i = (\Lambda_i^1, \Lambda_i^2, \Lambda_i^3)\ (i = 1, 2, 3).$$

We can then write

$$\begin{aligned} p_i' &= \frac{p_i + q_i}{2} + g\frac{\omega \cdot \bar{\Lambda}_i}{2}, \\ q_i' &= \frac{p_i + q_i}{2} - g\frac{\omega \cdot \bar{\Lambda}_i}{2}. \end{aligned}$$

If Λ is the boost matrix, we have exactly (5.13).

5.5. Glassey-Strauss Reduction of the Collision Integrals

Glassey & Strauss have derived [32] an alternative form for collision operator without using the center-of-momentum system in (5.14) with kernel (5.15) and post-collisional momentum (5.16). We will write down their argument as follows.

Given an integrable function $G : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, we will show

$$\begin{aligned}\frac{c}{2}\frac{1}{p_0}\int_{\mathbb{R}^3}\frac{dq}{q_0}\int_{\mathbb{R}^3}\frac{dq'}{q_0'}\int_{\mathbb{R}^3}\frac{dp'}{p_0'}\delta^{(4)}(P'+Q'-P-Q)G(P,Q,P',Q')\\ =\frac{1}{p_0}\int_{\mathbb{R}^3}\frac{dq}{q_0}\int_{S^2}d\omega B(p,q,\omega)G(P,Q,P',Q'),\end{aligned}$$

where $B(p,q,\omega)$ is given by (5.15) and (P',Q') on the r.h.s. are given by (5.16).

We focus on

$$I \equiv \int_{\mathbb{R}^3}\frac{dq'}{q_0'}\int_{\mathbb{R}^3}\frac{dp'}{p_0'}\delta^{(4)}(P'+Q'-P-Q)G(p,q,p',q').$$

It is sufficient to establish

$$I = \frac{2}{c}\int_{S^2}d\omega B(p,q,\omega)G(p,q,p',q'),$$

where (p',q') are given by (5.16). We split $I = \frac{1}{2}I + \frac{1}{2}I$.

Letting $q' = p+q-p'$ we can immediately remove three of the delta functions. Next translate $p' \to p+p'$ so that $q' \to q-p'$. And then switch to polar coordinates as $p' = r\omega$, $dp' = r^2 dr d\omega$ where $r \in [0,\infty)$ and $\omega \in S^2$. Then we have

$$\frac{1}{2}I = \frac{1}{2}\int_0^\infty\int_{S^2}\frac{r^2 dr d\omega}{p_0' q_0'}\delta(p_0'+q_0'-p_0-q_0)G(p,q,p',q'),$$

where $p' = p + r\omega$ and $q' = q - r\omega$.

Now consider the other half $\frac{1}{2}I$. Let $p' = p+q-q'$ and remove three of the delta functions. Next translate $q' \to q+q'$ so that $p' \to p-q'$. And switch to polar coordinates as $q' = r\omega$, $dq' = r^2 dr d\omega$ where $r \in [0,\infty)$ and $\omega \in S^2$. Then

$$\frac{1}{2}I = \frac{1}{2}\int_0^\infty\int_{S^2}\frac{r^2 dr d\omega}{p_0' q_0'}\delta(p_0'+q_0'-p_0-q_0)G(p,q,p',q'),$$

where $p' = p - r\omega$ and $q' = q + r\omega$. Further change variables $r \to -r$ so that

$$\frac{1}{2}I = \frac{1}{2}\int_{-\infty}^0\int_{S^2}\frac{r^2 dr d\omega}{p_0' q_0'}\delta(p_0'+q_0'-p_0-q_0)G(p,q,p',q'),$$

where now $p' = p + r\omega$ and $q' = q - r\omega$.

We combine the last two splittings to conclude that

$$I = \frac{1}{2}I + \frac{1}{2}I = \frac{1}{2}\int_{-\infty}^{+\infty}\int_{S^2}\frac{r^2 dr d\omega}{p_0' q_0'}\delta(p_0'+q_0'-p_0-q_0)G(p,q,p',q'),$$

where $p' = p + r\omega$ and $q' = q - r\omega$.

We now focus on the argument of the delta function. For $\lambda_i > 0$ $(i = 1, 2)$, we use the identity $\delta(\lambda_1 - \lambda_2) = 2\lambda_1\delta(\lambda_1^2 - \lambda_2^2)$ to see that

$$\delta(p_0' + q_0' - p_0 - q_0) = 2(p_0 + q_0)\delta((p_0' + q_0')^2 - (p_0 + q_0)^2)$$
$$= 2(p_0 + q_0)\delta(2p_0'q_0' - \{(p_0 + q_0)^2 - (p_0')^2 - (q_0')^2\}).$$

If $\{(p_0 + q_0)^2 - (p_0')^2 - (q_0')^2\} > 0$ then this is

$$= 8(p_0 + q_0)p_0'q_0'\delta(4(p_0')^2(q_0')^2 - \{(p_0 + q_0)^2 - (p_0')^2 - (q_0')^2\}^2).$$

If $\{(p_0 + q_0)^2 - (p_0')^2 - (q_0')^2\} < 0$, then the delta function is zero. Now write the argument of the delta function as

$$p(r) = -(p_0 + q_0)^4 + 2(p_0 + q_0)^2\{(p_0')^2 + (q_0')^2\} - \{(p_0')^2 + (q_0')^2\}^2 + 4(p_0')^2(q_0')^2$$
$$= -(p_0 + q_0)^4 + 2(p_0 + q_0)^2\{(p_0')^2 + (q_0')^2\} - \{(p_0')^2 - (q_0')^2\}^2.$$

Plugging in $p' = p + r\omega$ and $q' = q - r\omega$ we observe that

$$(p_0')^2 - (q_0')^2 = |p + r\omega|^2 - |q - r\omega|^2 = p_0^2 - q_0^2 + 2r\omega \cdot (p + q).$$

This means that $p(r)$ is quadratic in r. Moreover,

$$p(0) = -(p_0 + q_0)^4 + 2(p_0 + q_0)^2\{(p_0)^2 + (q_0)^2\} - \{(p_0)^2 - (q_0)^2\}^2$$
$$= -(p_0 + q_0)^4 + 2(p_0 + q_0)^2\{(p_0)^2 + (q_0)^2\} - (p_0 + q_0)^2(p_0 - q_0)^2$$
$$= -(p_0 + q_0)^4 + (p_0 + q_0)^2\{(p_0)^2 + (q_0)^2\} + 2p_0q_0(p_0 + q_0)^2 = 0.$$

We conclude that $p(r) = 4D_1r^2 - 8D_2r$ for some $D_1, D_2 \in \mathbb{R}$.

We will now determine D_1, D_2. Expanding

$$(p_0')^2 + (q_0')^2 = 2c^2 + |p + r\omega|^2 + |q - r\omega|^2 = p_0^2 + q_0^2 + 2r\omega \cdot (p - q) + 2r^2.$$

We plug the last few calculations into $p(r)$ to write it out in terms of r as

$$p(r) = -(p_0 + q_0)^4 + 2(p_0 + q_0)^2\{p_0^2 + q_0^2 + 2r\omega \cdot (p - q) + 2r^2\}$$
$$- \{p_0^2 - q_0^2 + 2r\omega \cdot (p + q)\}^2$$
$$= -(p_0 + q_0)^4 + 2(p_0 + q_0)^2\{p_0^2 + q_0^2 + 2r\omega \cdot (p - q) + 2r^2\}$$
$$- \{p_0^2 - q_0^2\}^2 - 4r^2\{\omega \cdot (p + q)\}^2 - 4r\{\omega \cdot (p + q)\}\{p_0^2 - q_0^2\}.$$

Rearrainging the terms

$$
\begin{aligned}
p(r) &= 4\{(p_0+q_0)^2 - \{\omega\cdot(p+q)\}^2\}r^2 \\
&\quad + 4\left\{(p_0+q_0)^2\{\omega\cdot(p-q)\} - \{\omega\cdot(p+q)\}\{p_0^2-q_0^2\}\right\} r \\
&\quad - (p_0+q_0)^4 + 2(p_0+q_0)^2\{p_0^2+q_0^2\} - \{p_0^2-q_0^2\}^2 \\
&= 4\{(p_0+q_0)^2 - \{\omega\cdot(p+q)\}^2\}r^2 \\
&\quad + 4\left\{(p_0+q_0)^2\{\omega\cdot(p-q)\} - \{\omega\cdot(p+q)\}\{p_0^2-q_0^2\}\right\} r.
\end{aligned}
$$

In other words,

$$
\begin{aligned}
D_1 &= \{(p_0+q_0)^2 - \{\omega\cdot(p+q)\}^2\}, \\
2D_2 &= -(p_0+q_0)^2\{\omega\cdot(p-q)\} + \{\omega\cdot(p+q)\}\{p_0^2-q_0^2\}.
\end{aligned}
$$

We further calculate D_2 as

$$
\begin{aligned}
2D_2 &= -(p_0^2+q_0^2+2p_0q_0)\{\omega\cdot(p-q)\} + \{\omega\cdot(p+q)\}\{p_0^2-q_0^2\} \\
&= w\cdot p\left\{-2q_0^2 - 2p_0q_0\right\} + \omega\cdot q\left\{2p_0^2 + 2p_0q_0\right\} \\
&= 2\{(p_0+q_0)\omega\cdot(p_0q - q_0p)\} \\
&= 2(p_0+q_0)p_0q_0\left\{\omega\cdot\left(\frac{q}{q_0} - \frac{p}{p_0}\right)\right\}.
\end{aligned}
$$

Thus, $p(r) = 4D_1r^2 - 8D_2r$ with these definitions.

Plug this formulation for $p(r)$ into the full integral to obtain

$$
I = \frac{1}{2}\int_{-\infty}^{+\infty}\int_{S^2} r^2 dr d\omega 8(p_0+q_0)\delta(4D_1r^2 - 8D_2r)G(p,q,p',q').
$$

Equivalently

$$
I = \int_{-\infty}^{+\infty}\int_{S^2} r^2 dr d\omega \frac{(p_0+q_0)}{D_1}\delta(r\{r - 2D_2/D_1\})G(p,q,p',q').
$$

We use the identity $\delta(r\{r-2D_2/D_1\}) = \left|\frac{D_1}{2D_2}\right|\{\delta(r)+\delta(r-2D_2/D_1)\}$. The first delta function drops out because of the r^2 factor. We thus have

$$
I = \int_{-\infty}^{+\infty}\int_{S^2} r^2 dr d\omega \frac{(p_0+q_0)}{2|D_2|}\delta(r - 2D_2/D_1)G(p,q,p',q'),
$$

where we have used $D_1 \geq 2$ [31]; but D_2 can be negative. Evaluating the delta function we obtain the result.

5.6. Hilbert-Schmidt form

We linearize the collision operator around a relativistic Maxwellian, J, as

$$F = J(1+f),$$

where the relativistic Maxwellian is defined in (5.10). Since $\mathcal{C}(J,J) \equiv 0$, the linearized collision operator takes the form

$$Lf \equiv -J^{-1}(p)\left\{\mathcal{C}(Jf,J) + \mathcal{C}(J,Jf)\right\}.$$

Define the collision frequency by

$$\nu(p) \equiv \frac{c}{2}\frac{1}{p_0}\int_{\mathbb{R}^3}\frac{dq}{q_0}\int_{\mathbb{R}^3}\frac{dq'}{q_0'}\int_{\mathbb{R}^3}\frac{dp'}{p_0'}W(p,q|p',q')J(q).$$

Then we can write

$$Lf = \nu(p)f - Kf,$$

where $K = K_2 - K_1$ and

$$\begin{aligned}
K_1 f &\equiv \frac{c}{2}\frac{1}{p_0}\int_{\mathbb{R}^3}\frac{dq}{q_0}\int_{\mathbb{R}^3}\frac{dq'}{q_0'}\int_{\mathbb{R}^3}\frac{dp'}{p_0'}W(p,q|p',q')J(q)f(q),\\
K_2 f &\equiv \frac{c}{2}\frac{1}{p_0}\int_{\mathbb{R}^3}\frac{dq}{q_0}\int_{\mathbb{R}^3}\frac{dq'}{q_0'}\int_{\mathbb{R}^3}\frac{dp'}{p_0'}W(p,q|p',q')J(q)\left\{f(p') + f(q')\right\}.
\end{aligned}$$

We define

$$k_1(p,q) = \frac{c}{2}\frac{J(q)}{p_0 q_0}\int_{\mathbb{R}^3}\frac{dq'}{q_0'}\int_{\mathbb{R}^3}\frac{dp'}{p_0'}W(p,q|p',q'),$$

which can be simplified as in either Section 5.4 or Section 5.5.

Then we have a Hilbert-Schmidt form for K_1. Our goal in this section is to derive a Hilbert-Schmidt form for K_2. From (5.3) we have

$$K_2 f = \frac{c}{2}\frac{1}{p_0}\int_{\mathbb{R}^3}\frac{dq}{q_0}\int_{\mathbb{R}^3}\frac{dq'}{q_0'}\int_{\mathbb{R}^3}\frac{dp'}{p_0'}s\sigma(g,\theta)\delta^{(4)}(P+Q-P'-Q')J(q)\left\{f(p') + f(q')\right\}.$$

We first reduce this to a Hilbert-Schmidt form and second carry out the delta function integrations in the kernel. This calculation is from [15, p.277], see also [22].

In preparation, we write down some invariant quantities. By (5.4) and (5.5)

$$\begin{aligned}
(P-Q)\cdot(P'-Q') &= 2P\cdot P' + 2Q\cdot Q' - (P+Q)\cdot(P'+Q')\\
&= 2P\cdot P' + 2Q\cdot Q' - s.
\end{aligned}$$

Further notice that (5.4) implies

$$(P - P') \cdot (P - P') = (Q' - Q) \cdot (Q' - Q).$$

Expanding this we have

$$2c^2 - P \cdot P' = 2c^2 - Q' \cdot Q.$$

Thus $P \cdot P' = Q \cdot Q'$. Let $\bar{g} = g(P, P')$ from (5.7). We will always use g to exclusively denote $g = g(P, Q)$. By (5.8), we redefine θ from (5.9) as

$$\cos\theta = -(P - Q) \cdot (P' - Q')/g^2 = 1 - 2\left(\frac{\bar{g}}{g}\right)^2.$$

See [30, p.111-113] for lots of similar calculations. We further *claim* that

$$g^2 = \bar{g}^2 + \frac{1}{2}(P + P') \cdot (Q + Q' - P - P'), \tag{5.40}$$

Let $\bar{s} = s(P, P') = \bar{g}^2 + 4c^2$. Then (5.40) is equivalent to

$$\begin{aligned} g^2 &= \frac{1}{2}\bar{g}^2 - 2c^2 + \frac{1}{2}(P + P') \cdot (Q + Q'), \\ &= \frac{1}{2}\bar{g}^2 + g^2 - 2P \cdot Q + \frac{1}{2}(P + P') \cdot (Q + Q'). \end{aligned}$$

We thus prove (5.40) by showing that

$$\frac{1}{2}\bar{g}^2 - 2P \cdot Q + \frac{1}{2}(P + P') \cdot (Q + Q') = 0.$$

Expanding the r.h.s we obtain

$$P \cdot P' - c^2 - 2P \cdot Q + \frac{1}{2}P \cdot Q + \frac{1}{2}P \cdot Q' + \frac{1}{2}P' \cdot Q + \frac{1}{2}P' \cdot Q'.$$

Since $P \cdot Q = P' \cdot Q'$ and $P \cdot P' = Q \cdot Q'$ because of (5.4), we obtain

$$-P \cdot Q + \frac{1}{2}P \cdot P' + \frac{1}{2}Q \cdot Q' - c^2 + \frac{1}{2}P \cdot Q' + \frac{1}{2}P' \cdot Q,$$

which by (5.4) is

$$-P \cdot Q - c^2 + \frac{1}{2}(P + Q) \cdot (P' + Q') = -P \cdot Q - c^2 + \frac{1}{2}s = 0.$$

This establishes the claim (5.40).

Now we establish the Hilbert-Schmidt form. First consider

$$\frac{c}{2}\frac{1}{p_0}\int_{\mathbb{R}^3}\frac{dq}{q_0}\int_{\mathbb{R}^3}\frac{dq'}{q'_0}\int_{\mathbb{R}^3}\frac{dp'}{p'_0}s\sigma(g,\theta)\delta^{(4)}(P+Q-P'-Q')J(q)f(p').$$

Exchanging q with p' the integral above is equal to

$$\frac{c}{2}\frac{1}{p_0}\int\frac{dq}{q_0}f(q)\left\{\int\frac{dq'}{q'_0}\int\frac{dp'}{p'_0}\bar{s}\sigma(\bar{g},\theta)\delta^{(4)}(P+P'-Q-Q')J(p')\right\},$$

where θ is now

$$\cos\theta = 1-2\left(\frac{g}{\bar{g}}\right)^2, \tag{5.41}$$

and from (5.40)

$$\bar{g}^2 = g^2+\frac{1}{2}(P+Q)\cdot(P'+Q'-P-Q), \tag{5.42}$$

and $\bar{s}$ is defined by

$$\bar{s} = \bar{g}^2+4c^2. \tag{5.43}$$

We do a similar calculation for the second term in $K_2 f$, e.g. exchange q with q' and then swap the q' and p' notation. Therefore, we can define

$$k_2(p,q)\equiv\frac{c}{p_0q_0}\int\frac{dq'}{q'_0}\int\frac{dp'}{p'_0}\bar{s}\sigma(\bar{g},\theta)\delta^{(4)}(P+P'-Q-Q')J(p'). \tag{5.44}$$

And we can now write the Hilbert-Schmidt form $K_2 f = \int k_2(p,q)f(q)dq$.

We will carry out the delta function integrations in $k_2(p,q)$ using a center-of-momentum system. One might try, instead of using the center-of-momentum system, to carry out the delta function integrations using a similar procedure to the Appendix in [32], which we did in Section 5.5.

Next, we want to translate (5.44) into an expression involving the total and relative momentum variables, $P'+Q'$ and $P'-Q'$ respectively. Define u by $u(r)=0$ if $r<0$ and $u(r)=1$ if $r\geq 0$. Let $\underline{g}=g(P',Q')$ and $\underline{s}=s(P',Q')$. We *claim* that

$$\int_{\mathbb{R}^3}\frac{dq'}{q'_0}\int_{\mathbb{R}^3}\frac{dp'}{p'_0}G(P,Q,P',Q') = \frac{1}{16}\int_{\mathbb{R}^4\times\mathbb{R}^4}d\mu(P',Q')G(P,Q,P',Q'), \tag{5.45}$$

where we are now integrating over (P',Q') and

$$d\mu(P',Q') = dP'dQ'u(p'_0+q'_0)u(\underline{s}-4c^2)\delta(\underline{s}-\underline{g}^2-4c^2)\delta((P'+Q')\cdot(P'-Q')).$$

To establish the claim, first notice that

$$(P' + Q') \cdot (P' - Q') = P' \cdot P' - Q' \cdot Q'$$
$$= (p_0')^2 - |p'|^2 - (q_0')^2 + |q'|^2 = A_p - A_q,$$

where now p_0' and q_0' are integration variables and we have defined

$$A_p = (p_0')^2 - [|p'|^2 + c^2], \quad A_q = (q_0')^2 - [|q'|^2 + c^2].$$

Integrating first over dP', see that alternatively

$$(P' + Q') \cdot (P' - Q') = (p_0')^2 - [|p'|^2 + c^2 + A_q]$$
$$= \left\{ p_0' - \sqrt{|p'|^2 + c^2 + A_q} \right\} \left\{ p_0' + \sqrt{|p'|^2 + c^2 + A_q} \right\}.$$

Furthermore, by (5.5) and (5.7) we have

$$\underline{s} - \underline{g}^2 - 4c^2 = (P' + Q') \cdot (P' + Q') + (P' - Q') \cdot (P' - Q') - 4c^2$$
$$= 2P' \cdot P' + 2Q' \cdot Q' - 4c^2$$
$$= 2A_p + 2A_q,$$

Thus, similarly

$$\underline{s} - \underline{g}^2 - 4c^2 = 2(p_0')^2 - 2[|p'|^2 + c^2 - A_q]$$
$$= \left\{ p_0' - \sqrt{|p'|^2 + c^2 - A_q} \right\} \left\{ p_0' + \sqrt{|p'|^2 + c^2 - A_q} \right\}.$$

Further note that $p_0' + q_0' \geq 0$ and $\underline{s} - 4c^2 \geq 0$ together imply $p_0' \geq 0$ and $q_0' \geq 0$. With these expressions, by standard delta function calculations we establish (5.45).

We thus conclude that

$$k_2(p, q) \equiv \frac{c}{p_0 q_0} \frac{1}{16} \int_{\mathbb{R}^4 \times \mathbb{R}^4} d\mu(P', Q') \bar{s} \sigma(\bar{g}, \theta) \delta^{(4)}(P + P' - Q - Q') J(p').$$

Now apply the change of variables

$$\bar{P} = P' + Q', \ \bar{Q} = P' - Q'.$$

This transformation has Jacobian $= 16$ and inverse tranformation as

$$P' = \frac{1}{2}\bar{P} + \frac{1}{2}\bar{Q}, \ Q' = \frac{1}{2}\bar{P} - \frac{1}{2}\bar{Q}.$$

With this change of variable the integral becomes

$$k_2(p,q) \equiv \frac{1}{4\pi K_2(c^2)p_0q_0}\int_{\mathbb{R}^4\times\mathbb{R}^4} d\mu(\bar{P},\bar{Q})\bar{s}\sigma(\bar{g},\theta)\delta^{(4)}(P-Q+\bar{Q})e^{-\frac{c}{2}(\bar{p}_0+\bar{q}_0)},$$

where the measure is

$$d\mu(\bar{P},\bar{Q}) = d\bar{P}d\bar{Q}u(\bar{p}_0)u(\bar{P}\cdot\bar{P}-4c^2)\delta(\bar{P}\cdot\bar{P}+\bar{Q}\cdot\bar{Q}-4c^2)\delta(\bar{P}\cdot\bar{Q}).$$

Also $\bar{g}\geq 0$ from (5.42) is now given by

$$\bar{g}^2 = g^2 + \frac{1}{2}(P+Q)\cdot(\bar{P}-P-Q).$$

And θ and $\bar{s}$ can be defined through the new $\bar{g}$ as in (5.41) and (5.43).

We next carry out the delta function argument for $\delta^{(4)}(P-Q+\bar{Q})$ to obtain

$$k_2(p,q) \equiv \frac{1}{4\pi K_2(c^2)p_0q_0}\int_{\mathbb{R}^4} d\mu(\bar{P})\bar{s}\sigma(\bar{g},\theta)e^{-\frac{c}{2}(\bar{p}_0+q_0-p_0)},$$

where the measure is

$$d\mu(\bar{P}) = d\bar{P}u(\bar{p}_0)u(\bar{P}\cdot\bar{P}-4c^2)\delta(\bar{P}\cdot\bar{P}-g^2-4c^2)\delta(\bar{P}\cdot(Q-P)).$$

By (5.8) we have

$$\begin{aligned} u(\bar{p}_0)\delta(\bar{P}\cdot\bar{P}-g^2-4c^2) &= u(\bar{p}_0)\delta(\bar{P}\cdot\bar{P}-s) \\ &= u(\bar{p}_0)\delta((\bar{p}_0)^2-|\bar{p}|^2-s) \\ &= \frac{\delta(\bar{p}_0-\sqrt{|\bar{p}|^2+s})}{2\sqrt{|\bar{p}|^2+s}}. \end{aligned}$$

We then carry out one integration using delta function to get

$$k_2(p,q) \equiv \frac{e^{-\frac{c}{2}(q_0-p_0)}}{8\pi K_2(c^2)p_0q_0}\int_{\mathbb{R}^3}\frac{d\bar{p}}{\bar{p}_0}u(\bar{P}\cdot\bar{P}-4c^2)\delta(\bar{P}\cdot(Q-P))\bar{s}\sigma(\bar{g},\theta)e^{-\frac{c}{2}\bar{p}_0},$$

with $\bar{p}_0\equiv\sqrt{|\bar{p}|^2+s}$. By (5.8), we now have

$$\bar{P}\cdot\bar{P}-4c^2 = s-4c^2 = g^2\geq 0.$$

So always $u(\bar{P}\cdot\bar{P}-4c^2)=1$ and integral reduces to

$$k_2(p,q) \equiv \frac{e^{-\frac{c}{2}(q_0-p_0)}}{8\pi K_2(c^2)p_0q_0}\int_{\mathbb{R}^3}\frac{d\bar{p}}{\bar{p}_0}\delta(\bar{P}\cdot(Q-P))\bar{s}\sigma(\bar{g},\theta)e^{-\frac{c}{2}\bar{P}\cdot\bar{U}},$$

where $\bar{U}=(1,0,0,0)^t$ and $e^{-\frac{c}{2}\bar{p}_0}=e^{-\frac{c}{2}\bar{P}\cdot\bar{U}}$.

We finish our reduction by moving to a center-of-momentum system (5.6). As this form suggests, we choose the Lorentz Transformation Λ from Section 5.3.1.3 which satisfies (5.23) and (5.33). Define $U = \Lambda \bar{U}$, then Definition 5.1 gives

$$\int \frac{d\bar{p}}{\bar{p}_0} \bar{s}\sigma(\bar{g}, \theta) e^{-\frac{c}{2}\bar{p}_0} \delta(\bar{P} \cdot (Q - P)) = \int \frac{d\bar{p}}{\bar{p}_0} \bar{s}_\Lambda \sigma(\bar{g}_\Lambda, \theta_\Lambda) e^{-\frac{c}{2}\bar{P}\cdot U} \delta(\bar{P} \cdot \Lambda(Q - P)).$$

where $\bar{g}_\Lambda, \bar{s}_\Lambda \geq 0$ are now given by

$$\begin{aligned} \bar{g}_\Lambda^2 &= g^2 + \frac{1}{2}\Lambda(P+Q) \cdot \{\bar{P} - \Lambda(P+Q)\} = g^2 + \frac{1}{2}\sqrt{s}\{\bar{p}_0 - \sqrt{s}\} \\ \bar{s}_\Lambda &= 4c^2 + \bar{g}_\Lambda^2 \\ \cos\theta_\Lambda &= 1 - 2\left(\frac{g}{\bar{g}_\Lambda}\right)^2. \end{aligned} \tag{5.46}$$

The equality of the two integrals holds because $d\bar{p}/\bar{p}_0$ is a Lorentz invariant measure.

We now work with the integral on the left hand side above. In this coordinate system

$$\bar{P} \cdot \Lambda(Q - P) = -\bar{p}_3 g.$$

We switch to polar coordinates in the form

$$d\bar{p} = |\bar{p}|^2 d|\bar{p}| \sin\psi d\psi d\varphi, \;\; \bar{p} \equiv |\bar{p}|(\sin\psi\cos\varphi, \sin\psi\sin\varphi, \cos\psi).$$

Then we can write $k_2(p,q)$ as

$$\frac{e^{-\frac{c}{2}(q_0 - p_0)}}{8\pi K_2(c^2) p_0 q_0} \int_0^{2\pi} d\varphi \int_0^\pi \sin\psi d\psi \int_0^\infty \frac{|\bar{p}|^2 d|\bar{p}|}{\bar{p}_0} \bar{s}_\Lambda \sigma(\bar{g}_\Lambda, \theta_\Lambda) e^{-\frac{c}{2}\bar{P}\cdot U} \delta(|\bar{p}| g \cos\psi).$$

From Section 5.3.1.3 we have

$$U = \Lambda \bar{U} = \left(\frac{p_0 + q_0}{\sqrt{s}}, \frac{2|p \times q|}{g\sqrt{s}}, 0, \frac{p_0 - q_0}{g}\right)^t.$$

We evaluate the last delta function at $\psi = \pi/2$ to write $k_2(p,q)$ as

$$\frac{e^{-\frac{c}{2}(q_0 - p_0)}}{8\pi g K_2(c^2) p_0 q_0} \int_0^{2\pi} d\varphi \int_0^\infty \frac{|\bar{p}| d|\bar{p}|}{\bar{p}_0} \bar{s}_\Lambda \sigma(\bar{g}_\Lambda, \theta_\Lambda) e^{-\frac{c}{2}\bar{p}_0 \frac{p_0 + q_0}{\sqrt{s}}} e^{c\frac{|p \times q|}{g\sqrt{s}}|\bar{p}|\cos\varphi}. \tag{5.47}$$

This is already a useful reduced form for $k_2(p,q)$.

However, we now further convert the integration over $d|\bar{p}|$ to an integration over $d\Theta \equiv d\theta_\Lambda$. First notice that (5.46) implies

$$g = \bar{g}_\Lambda \sqrt{\frac{1-\cos\theta_\Lambda}{2}} = \bar{g}_\Lambda \sin\frac{\theta_\Lambda}{2}. \tag{5.48}$$

Now (5.46) implies $g^2\left(\sin^{-2}\frac{\theta_\Lambda}{2} - 1\right) = \frac{1}{2}\sqrt{s}(\bar{p}_0 - \sqrt{s})$. Equivalently,

$$2g^2\cot^2\frac{\theta_\Lambda}{2} = \sqrt{s}(\bar{p}_0 - \sqrt{s}). \tag{5.49}$$

Thus $2\frac{g^2}{\sqrt{s}} d\left(\cot^2\frac{\Theta}{2}\right) = d\sqrt{|\bar{p}|^2 + s}$, or

$$\frac{g^2}{\sqrt{s}}\frac{d\cos\Theta}{\sin^4\frac{\Theta}{2}} = \frac{|\bar{p}|d|\bar{p}|}{\bar{p}_0}.$$

This last deduction follows since

$$\begin{aligned} d\left(\cot^2\frac{\Theta}{2}\right) = d\left(\frac{\cos^2\frac{\Theta}{2}}{\sin^2\frac{\Theta}{2}}\right) &= -\frac{\cos\frac{\Theta}{2}\sin\frac{\Theta}{2}}{\sin^2\frac{\Theta}{2}}d\Theta - \frac{\cos^3\frac{\Theta}{2}}{\sin^3\frac{\Theta}{2}}d\Theta \\ &= -\cos\frac{\Theta}{2}\left(\frac{\sin^2\frac{\Theta}{2} + \cos^2\frac{\Theta}{2}}{\sin^3\frac{\Theta}{2}}\right)d\Theta \\ &= -\frac{1}{2}\frac{\sin\Theta d\Theta}{\sin^4\frac{\Theta}{2}} = \frac{1}{2}\frac{d\cos\Theta}{\sin^4\frac{\Theta}{2}}, \end{aligned}$$

where we used the trigonometric formula $\cos\frac{\Theta}{2}\sin\frac{\Theta}{2} = \frac{1}{2}\sin\Theta$ in the last step. Alternatively, we can solve (5.49) for the change of variable

$$|\bar{p}| = 2gs^{-1/2}\left(4c^2 + \frac{g^2}{\sin^2\frac{\theta}{2}}\right)^{1/2}\cot\frac{\Theta}{2}. \tag{5.50}$$

Here $0 \le |\bar{p}| \le \infty$ and, therefore, $0 \le \Theta \le \pi$ but the Θ goes from π back to 0 as $|\bar{p}|$ goes from 0 to ∞. We plug this change of variable into the integral (5.47) using (5.46) and (5.48) to establish

$$\begin{aligned} k_2(p,q) = \frac{e^{-\frac{c}{2}(q_0-p_0)}}{8\pi K_2(c^2)p_0q_0}\frac{g}{\sqrt{s}}\int_0^{2\pi}d\varphi\int_0^{\pi}\frac{\sin\Theta d\Theta}{\sin^4\frac{\Theta}{2}}\left(4c^2 + \frac{g^2}{\sin^2\frac{\Theta}{2}}\right)\sigma\left(\frac{g}{\sin\frac{\Theta}{2}},\Theta\right) \\ \times\exp\left(-\frac{c}{2}\frac{p_0+q_0}{\sqrt{s}}\bar{p}_0 + c\frac{|p\times q|}{g\sqrt{s}}|\bar{p}|\cos\varphi\right), \end{aligned} \tag{5.51}$$

where $\bar{p}_0 = \sqrt{s + |\bar{p}|^2}$ and $|\bar{p}|$ defined by (5.50).

We can reduce (5.51) even further and evaluate all the integrals if we assume $\sigma = 1$. We write $k_2(p,q)$ from (5.47), and using (5.46), as

$$\frac{e^{\frac{c}{2}(q_0-p_0)}}{16\pi g K_2(c^2)p_0 q_0}\int_0^\infty \frac{ydy}{\sqrt{s+y^2}}\left(s+\sqrt{s^2+sy^2}\right)e^{-\frac{c}{2}\frac{p_0+q_0}{\sqrt{s}}\sqrt{s+y^2}}I_0\left(c\frac{|p\times q|}{g\sqrt{s}}y\right).$$

where $I_0(y) = \frac{1}{2\pi}\int_0^{2\pi} e^{y\cos\varphi}d\varphi$ is a modified Bessel function of index zero. By the change of variable $y \to y\sqrt{s}$ we can write $k_2(p,q)$ as

$$\frac{s^{3/2}e^{\frac{c}{2}(q_0-p_0)}}{16\pi g K_2(c^2)p_0 q_0}\int_0^\infty \frac{ydy}{\sqrt{1+y^2}}\left(1+\sqrt{1+y^2}\right)e^{-\frac{c}{2}(p_0+q_0)\sqrt{1+y^2}}I_0\left(c\frac{|p\times q|}{g}y\right),$$

This is a Laplace Transform and a known integral, which can be calculated exactly via a taylor expansion [46, p.134]. For instance, [32, Lemma 3.5 on p.322] says that for any $R > r \geq 0$ we have

$$\int_0^\infty \frac{e^{-R\sqrt{1+y^2}}yI_0(ry)}{\sqrt{1+y^2}}dy = \frac{e^{-\sqrt{R^2-r^2}}}{\sqrt{R^2-r^2}},$$
$$\int_0^\infty e^{-R\sqrt{1+y^2}}yI_0(ry)dy = \frac{R}{R^2-r^2}\left\{1+\frac{1}{\sqrt{R^2-r^2}}\right\}e^{-\sqrt{R^2-r^2}}.$$

See also [58] or [57, p.383]. Using these formula's we can express the integral as

$$k_2(p,q) = \frac{s^{3/2}e^{\frac{c}{2}(q_0-p_0)}}{16\pi g K_2(c^2)p_0 q_0}H_1(p,q)\exp\left(-H_2(p,q)\right),$$

where $H_2(p,q) = \sqrt{\{c(p_0+q_0)/2\}^2 - (c|p\times q|/g)^2}$ and

$$H_1(p,q) = \left(1+c\frac{p_0+q_0}{2}\left(H_2(p,q)\right)^{-1}+c\frac{p_0+q_0}{2}\left(H_2(p,q)\right)^{-2}\right)\left(H_2(p,q)\right)^{-1}.$$

Further,

$$H_2(p,q) = \frac{c\sqrt{s}}{2g}|p-q| = c|p-q|\sqrt{\frac{g^2+4c^2}{4g^2}}.$$

See Lemma 3.1 [p.316] of [32]. Therefore, $H_2(p,q) \geq \frac{c}{2}|p-q| + c$. This completes our discussion of the Hilbert-Schmidt form for the linearized collision operator.

5.7. Relativistic Vlasov-Maxwell-Boltzmann equation

One might want to prove a stability theorem for the Vlasov-Maxwell-Boltzmann system similar to our result on the relativistic Landau-Maxwell system [62]. The relativistic Boltzmann equation (5.1) with any field terms takes the form

$$\partial_t F + \hat{p} \cdot \nabla_x F + V(x) \cdot \nabla_p F = \mathcal{C}(F, F).$$

Now if one wants to apply Guo's energy method [39] to such a problem, then a key point is to use high order momentum derivatives. But the momentum derivatives cause severe problems.

On the one hand, there doesn't seem to be any analogue in the relativistic case of the non-relativistic translation invarance of the post-collisional velocities. This translation invariance is a key point in taking the high order derivatives without differentiating the post-collisional velocities directly. On the other hand, taking higher and higher momentum derivatives of the post-collisional velocities in the relativistic case makes it harder and harder to get estimates for the non-linear term involving the post-collisional velocities.

More specifically, in the Glassey-Strauss variables (5.14) and (5.16), it was observed by Glassey & Strauss [31] that

$$|\nabla_p q_i'| + |\nabla_p p_i'| \leq C q_0^5 \left(1 + |p \cdot \omega|^{1/2} \mathbf{1}_{\{|p\cdot\omega| > |p\times\omega|^{3/2}\}}\right).$$

And furthermore

$$|\nabla_q p_i'| + |\nabla_q q_i'| \leq C q_0^5 p_0.$$

The q growth is not an issue. But still, by the chain rule, $|\nabla_p^k f(p')|$ for example is only bounded above by the derivatives of the function times a large polynomial weight (in p) which grows larger for larger k. The region where this polynomial growth occurs is quite small, but not small enough to control an estimate of more than one derivative. This is the main problem.

In the center-of-momentum variables (5.11) and (5.13), or more generally (5.38), the problem is essentially the same but it manifests itself in a different way. Instead of generating polyonmial growth with derivatives, here you generate singularities. And larger singularities result from taking more derivatives.

This is the key difficulty which prevents me, at the moment, from proving a stability theorem for the Vlasov-Maxwell-Boltzmann system using Guo's method.

APPENDIX A

Grad's Reduction of the Linear Boltzmann Collision Operator

Recall the Boltzmann collision operator

$$Q(F,G) \equiv \int_{\mathbb{R}^3\times S^2_+} B(\theta)|u-v|^\gamma\{F(u')G(v')-F(u)G(v)\}du d\omega,$$

where θ is defined by

$$\cos\theta = \frac{u-v}{|u-v|}\cdot\omega,$$

and $B(\theta)$ is assumed to satisfy the Grad angular cutoff condition

$$|B(\theta)| \le C\left|\cos\theta\right|.$$

Also

$$S^2_+ = \{\omega \in S^2 : \omega\cdot(u-v)\ge 0\}.$$

Further

$$v' = v-[(v-u)\cdot\omega]\omega, \quad u' = u-[(u-v)\cdot\omega]\omega, \tag{A.1}$$

$$|u|^2+|v|^2 = |u'|^2+|v'|^2, \tag{A.2}$$

Define the normalized Maxwellian by

$$\mu(v) = (2\pi)^{-3/2}e^{-|v|^2/2}.$$

With the standard perturbation

$$F = \mu+\mu^{1/2}f,$$

we define the linear part of the collision operator by

$$Lf = -\mu^{-1/2}\left\{Q(\mu,\mu^{1/2}f)+Q(\mu^{1/2}f,\mu)\right\} = \nu(v)f-Kf.$$

Here $K = K_2 - K_1$ and these are defined as [30, 34]:

$$\begin{aligned}
K_1 g &= \int_{\mathbb{R}^3 \times S^2_+} B(\theta)|u-v|^\gamma \mu^{1/2}(u)\mu^{1/2}(v) g(u)\, du d\omega, \\
K_2 g &= \int_{\mathbb{R}^3 \times S^2_+} B(\theta)|u-v|^\gamma \mu^{1/2}(u)\{\mu^{1/2}(u')g(v') + \mu^{1/2}(v')g(u')\}\, du d\omega.
\end{aligned} \tag{A.3}$$

In this Appendix, we will follow [12, 34] to write the integral operator K in a useful Hilbert-Schmidt form (see also [30, 37]).

We will write

$$Kg = \int_{\mathbb{R}^3} k(v,\xi) g(\xi) d\xi.$$

We notice immediately that

$$K_1 g = \int_{\mathbb{R}^3} k_1(v,\xi) g(\xi) d\xi,$$

where

$$k_1(v,\xi) = \int_{S^2_+} B(\theta)|\xi - v|^\gamma \mu^{1/2}(\xi)\mu^{1/2}(v) d\omega = c_1 |\xi - v|^\gamma \mu^{1/2}(\xi)\mu^{1/2}(v).$$

In the rest of this Appendix we will focus on K_2.

We use (A.2) to rewrite K_2 in (A.3) as

$$K_2 g = \mu^{1/2}(v) \int_{\mathbb{R}^3 \times S^2_+} B(\theta)|u-v|^\gamma \mu(u) \left\{ \frac{g}{\mu^{1/2}}(v') + \frac{g}{\mu^{1/2}}(u') \right\} du d\omega.$$

Rewrite (A.1) as

$$v' = v + [(u-v)\cdot\omega]\omega, \qquad u' = v + (u-v) - [(u-v)\cdot\omega]\omega.$$

Change variables $u - v \to u$ to obtain

$$K_2 g = \mu^{1/2}(v) \int_{\mathbb{R}^3 \times S^2_+} B(\theta)|u|^\gamma \mu(u+v) \left\{ \frac{g}{\mu^{1/2}}(v+u_\parallel) + \frac{g}{\mu^{1/2}}(v+u_\perp) \right\} du d\omega,$$

where now $\cos\theta = u\cdot\omega/|u|$ and we define

$$u_\parallel = \{u\cdot\omega\}\omega, \quad u_\perp = u - u_\parallel = \omega \times (u \times \omega).$$

We have just used the formula

$$w \times (u \times v) = (v\cdot w)u - (u\cdot w)v \quad (\forall u, v, w \in \mathbb{R}^3).$$

Now $K_2 = K_2^{\parallel} + K_2^{\perp}$ where

$$K_2^{\perp} g = \mu^{1/2}(v) \int_{\mathbb{R}^3 \times S_+^2} B(\theta)|u|^\gamma \mu(u+v) \frac{g}{\mu^{1/2}}(v + u_\perp) du d\omega,$$

The first step is to compare the two parts of $K_2 g$.

To this end we write out the angular coordinates precisely as

$$\omega = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \; 0 \le \theta \le \frac{\pi}{2}, \; 0 \le \phi < 2\pi, \quad d\omega = \sin\theta d\theta d\phi,$$

where $u \cdot \omega = |u|\cos\theta$. Then

$$K_2^{\perp} g = \mu^{1/2}(v) \int_{\mathbb{R}^3} du \int_0^{\frac{\pi}{2}} b^*(\theta) d\theta \int_0^{2\pi} d\phi |u|^\gamma \mu(u+v) \frac{g}{\mu^{1/2}}(v + u_\perp),$$

and we have defined $b^*(\theta) = \sin\theta B(\theta)$. For fixed ω, we complete an orthonormal basis as

$$\omega_\perp = \begin{pmatrix} -\cos\theta\cos\phi \\ -\cos\theta\sin\phi \\ \sin\theta \end{pmatrix}, \quad \omega \times \omega_\perp = \begin{pmatrix} \sin\phi \\ -\cos\phi \\ 0 \end{pmatrix}.$$

Now we will use the relation $u \cdot \omega = |u|\cos\theta$ to consider $u \cdot \omega_\perp$ and $u \cdot (\omega \times \omega_\perp)$. Sending $\theta \to \frac{\pi}{2} - \theta$ and $\phi \to \phi \pm \pi$, we have

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta, \; \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta,$$
$$\sin(\phi \pm \pi) = -\sin\phi, \; \cos(\phi \pm \pi) = -\cos\phi.$$

Under this transformation $\omega \to \omega_\perp$, hence in this coordinate system

$$u \cdot \omega_\perp = |u|\sin\theta$$

Furthermore, by linear algebra,

$$u = \{u \cdot \omega\}\omega + \{u \cdot \omega_\perp\}\omega_\perp + \{u \cdot (\omega \times \omega_\perp)\}\omega \times \omega_\perp.$$

By orthogonality,

$$|u|^2 = \{u \cdot \omega\}^2 + \{u \cdot \omega_\perp\}^2 + \{u \cdot (\omega \times \omega_\perp)\}^2.$$

In other words $u \cdot (\omega \times \omega_\perp) = 0$. We conclude

$$u = \{u \cdot \omega\}\omega + \{u \cdot \omega_\perp\}\omega_\perp.$$

In this coordinate system we can alternatively define

$$\omega_\perp = \frac{\omega \times (u \times \omega)}{|\omega \times (u \times \omega)|}.$$

And still $u \cdot \omega_\perp = \sqrt{|u|^2 - (u \cdot \omega)^2} = |u| \sin\theta$. In either case, we observe that

$$u_\perp = \{u \cdot \omega_\perp\}\omega_\perp.$$

Further, by earlier calculations we see that the change of variable $\theta \to \frac{\pi}{2} - \theta$ and $\phi \to \phi \pm \pi$ sends $\omega_\perp \to \omega$. Restricting to $\theta \to \frac{\pi}{2} - \theta$ and $\phi \to \phi - \pi$ for conveinence, we have

$$K_2^\perp g = \int_{\mathbb{R}^3} du \int_0^{\frac{\pi}{2}} b^* \left(\frac{\pi}{2} - \theta\right) d\theta \int_{-\pi}^{+\pi} d\phi |u|^\gamma \mu^{1/2}(v) \mu(u+v) \frac{g}{\mu^{1/2}}(v + u_\parallel).$$

Over the region $[-\pi, 0]$ we further change variables $\phi \to \phi + 2\pi$ and use periodicity to get

$$K_2^\perp g = \int_{\mathbb{R}^3} du \int_0^{\frac{\pi}{2}} b^* \left(\frac{\pi}{2} - \theta\right) d\theta \int_0^{2\pi} d\phi |u|^\gamma \mu^{1/2}(v) \mu(u+v) \frac{g}{\mu^{1/2}}(v + u_\parallel).$$

But notice that by definition

$$K_2^\parallel g = \mu^{1/2}(v) \int_{\mathbb{R}^3} du \int_0^{\frac{\pi}{2}} b^*(\theta) d\theta \int_0^{2\pi} d\phi |u|^\gamma \mu(u+v) \frac{g}{\mu^{1/2}}(v + u_\parallel).$$

We can therefore write

$$K_2 g = \mu^{1/2}(v) \int_{\mathbb{R}^3} du \int_0^{\frac{\pi}{2}} b(\theta) d\theta \int_0^{2\pi} d\phi |u|^\gamma \mu(u+v) \frac{g}{\mu^{1/2}}(v + u_\parallel),$$

where

$$b(\theta) = b^*(\theta) + b^* \left(\frac{\pi}{2} - \theta\right).$$

Further notice that $|b(\theta)| \le C |\cos\theta \sin\theta|$.

This upper bound motivates $|u_\parallel| = |u \cdot \omega| = |u| |\cos\theta|$. Thus,

$$K_2 g = \mu^{1/2}(v) \int_{\mathbb{R}^3} du \int_0^{\frac{\pi}{2}} \frac{b(\theta)}{|\cos\theta|} d\theta \int_0^{2\pi} d\phi \frac{|u_\parallel|}{|u|^{1-\gamma}} \mu(u+v) \frac{g}{\mu^{1/2}}(v + u_\parallel).$$

Note that $(u_\parallel, u_\perp)$ are invariant under the map $\theta \to \pi - \theta$ and $\phi \to \phi \pm \pi$:

$$\sin(\pi - \theta) = \sin\theta,\ \cos(\pi - \theta) = -\cos\theta,$$

$$\sin(\phi \pm \pi) = -\sin\phi,\ \cos(\phi \pm \pi) = -\cos\phi.$$

So that

$$\omega \to \begin{pmatrix} -\sin\theta\cos\phi \\ -\sin\theta\sin\phi \\ -\cos\theta \end{pmatrix} = -\omega.$$

By defining $b(\theta) = b(\pi - \theta)$ for $\theta \in [\pi/2, \pi]$ we can therefore write

$$K_2 g = \mu^{1/2}(v) \int_{\mathbb{R}^3} du \int_0^\pi \frac{2b(\theta)}{|\cos\theta|} d\theta \int_0^{2\pi} d\phi \frac{|u_\parallel|}{|u|^{1-\gamma}} \mu(u+v) \frac{g}{\mu^{1/2}}(v + u_\parallel).$$

Next we will reduce this expression to Hilbert-Schmidt form.

To this end, expand the exponent of the exponentials as

$$\frac{1}{2}|v|^2 + |u+v|^2 - \frac{1}{2}|v+u_\parallel|^2 = \frac{1}{2}|v|^2 + |u_\perp|^2 + 2u_\perp \cdot (v + u_\parallel) + \frac{1}{2}|v + u_\parallel|^2$$

Next define

$$\zeta = v + \frac{1}{2}u_\parallel.$$

Further split $\zeta = \zeta_\parallel + \zeta_\perp$ where

$$\zeta_\parallel = (v\cdot\omega)\omega + \frac{1}{2}u_\parallel,\ \zeta_\perp = v - (v\cdot\omega)\omega.$$

And now, since $u_\perp \cdot \zeta = u_\perp \cdot (v + u_\parallel)$ and $v + u_\parallel = 2\zeta - v$, we further expand the exponent as

$$\begin{aligned}
\frac{1}{2}|v|^2 + |u_\perp|^2 + 2u_\perp\cdot\zeta + \frac{1}{2}|2\zeta - v|^2 &= |v|^2 + |u_\perp|^2 + 2u_\perp\cdot\zeta + 2|\zeta|^2 - 2\zeta\cdot v \\
&= |v|^2 + |u_\perp + \zeta_\perp|^2 + |\zeta|^2 + |\zeta_\parallel|^2 - 2\zeta\cdot v \\
&= |u_\perp + \zeta_\perp|^2 + |\zeta_\parallel|^2 + |v - \zeta|^2 \\
&= |u_\perp + \zeta_\perp|^2 + |\zeta_\parallel|^2 + \frac{1}{4}|u_\parallel|^2.
\end{aligned}$$

Plugging this in, we can write $K_2 g$ as

$$(2\pi)^{-3/2} \int_{\mathbb{R}^3} du \int_0^\pi \frac{2b(\theta)}{|\cos\theta|} d\theta \int_0^{2\pi} d\phi \frac{|u_\parallel|}{|u|^{1-\gamma}} e^{-\frac{1}{2}|u_\perp + \zeta_\perp|^2 - \frac{1}{2}|\zeta_\parallel|^2 - \frac{1}{8}|u_\parallel|^2} g(v + u_\parallel).$$

Equivalently,

$$K_2 g = (2\pi)^{-3/2} \int_{\mathbb{R}^3 \times S^2} du d\omega \frac{2b(\theta)}{\sin\theta |\cos\theta|} \frac{|u_\|\|}{|u|^{1-\gamma}} e^{-\frac{1}{2}|u_\perp + \zeta_\perp|^2 - \frac{1}{2}|\zeta_\||^2 - \frac{1}{8}|u_\||^2} g(v + u_\|).$$

The last step is to rearrange the integration regions.

Let $O = [\omega_\perp^1 \ \omega_\perp^2 \ \omega]$ be a rotation matrix. Change variables $u \to Ou$ to see that

$$u_\| = \{u \cdot \omega\}\omega \to u_3\omega, \quad u_\perp = u - u_\| \to u_1\omega_\perp^1 + u_2\omega_\perp^2.$$

Define

$$\xi = u_3\omega, \ \xi_\perp = u_1\omega_\perp^1 + u_2\omega_\perp^2.$$

Split the $du = du_1 du_2 du_3$ integration into

$$d\xi_\perp = du_2 du_3,$$

$$d|\xi| = du_1.$$

First integrate over $d\xi_\perp$ to obtain

$$K_2 g = \int_{\mathbb{R} \times S^2} k_2^*(v, \xi) g(v + \xi) |\xi| d|\xi| d\omega,$$

where

$$k_2^*(v, \xi) = (2\pi)^{-3/2} e^{-\frac{1}{8}|\xi|^2 - \frac{1}{2}|\zeta_\||^2} \int_{\mathbb{R}^2} \frac{2b(\theta)}{\sin\theta |\cos\theta|} |u|^{\gamma-1} e^{-\frac{1}{2}|\xi_\perp + \zeta_\perp|^2} du_\perp$$

Since we are integrating over the full sphere, symmetry of the transformation $\omega \to -\omega$ allows us to write

$$K_2 g = 2 \int_0^\infty \int_{S^2} k_2^*(v, \xi) g(v + \xi) |\xi| d|\xi| d\omega.$$

Moreover, the standard transformation from polar coordinates is

$$|\xi| d|\xi| d\omega = \frac{|\xi|^2 d|\xi| d\omega}{|\xi|} = \frac{d\xi}{|\xi|}.$$

Here $d\xi$ now represents integration over $\mathbb{R}^3$. We can now write the Hilbert-Schmidt form for K_2 as

$$K_2 g = \int_{\mathbb{R}^3} k_2(v, \xi) g(v + \xi) d\xi$$

where

$$k_2(v,\xi) = 4\frac{e^{-\frac{1}{8}|\xi|^2-\frac{1}{2}|\zeta_{\|}|^2}}{(2\pi)^{3/2}|\xi|}\int_{\mathbb{R}^2}\left(|\xi|^2+|\xi_\perp|^2\right)^{\frac{\gamma-1}{2}}e^{-\frac{1}{2}|\xi_\perp+\zeta_\perp|^2}\frac{b(\theta)}{\sin\theta|\cos\theta|}d\xi_\perp.$$

Here given ξ, the idea is to extend it to an orthonormal basis for $\mathbb{R}^3$ denoted by $\{\xi^1,\xi^2,\xi/|\xi|\}$. Then $d\xi_\perp = d\xi_\perp^1 d\xi_\perp^1$ and in the formulas above

$$\xi_\perp = \xi_\perp^1\xi^2 + \xi_\perp^3\xi^2$$

where above ξ^1,ξ^2 are vectors and $\xi_\perp^1,\xi_\perp^2$ are scalars. Moreover,

$$\zeta = \zeta(v,\xi) = \zeta_{\|} + \zeta_\perp$$

where

$$\zeta_{\|} = \frac{(v\cdot\xi)\xi}{|\xi|^2}+\frac{1}{2}\xi,\ \zeta_\perp = v - \frac{(v\cdot\xi)\xi}{|\xi|^2} = (v\cdot\xi^1)\xi^1+(v\cdot\xi^2)\xi^2.$$

This completes Grad's reduction.

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