Math 644 - Homework 8 - Due Friday, Nov. 9, 2012

1. (Reflection of traveling waves) For $0 < L < +\infty$, solve the equation:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \text{ on } (0, L) \times (0, +\infty) \\ u = g, u_t = 0 \text{ on } (0, L) \times \{t = 0\} \\ u = 0 \text{ on } \big(\{0\} \times (0, +\infty)\big) \cup \big(\{L\} \times (0, +\infty)\big). \end{cases} \tag{1}$$

   Solve it by converting to a problem on $\mathbb{R}$.

2. (Stokes rule) [Evans, Problem 18 in Chapter 2]

   Assume u solves the initial value problem

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ on } \mathbb{R}^n \times (0, +\infty) \\ u = 0, u_t = h \text{ on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

   Show that $v \equiv u_t$ solves

$$\begin{cases} v_{tt} - \Delta v = 0 \text{ on } \mathbb{R}^n \times (0, +\infty) \\ v = h, v_t = 0 \text{ on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

   This is Stokes rule.

3. (Maxwell's equations) [Evans, Problem 21 in Chapter 2]

4. (A derivation of d'Alembert's formula by using a change of variables) [Evans, Problem 19 in Chapter 2]

   a) Show that the general solution of the PDE $u_{xy} = 0$ is given by:

$$u(x, y) = F(x) + G(y)$$

   for arbitrary differentiable functions F and G.

   b) Using the change of variables $\xi = x + t, \eta = x - t$, show that $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$.

   c) Use a) and b) to rederive d'Alembert's formula.

   d) Under what conditions on the initial data g, h, is the solution u a right-moving wave? A left-moving wave?

5. (Equipartition of energy) [Evans, Problem 24 in Chapter 2]

Let $u \in C^2(\mathbb{R} \times [0, +\infty))$ solve the initial value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 \text{ on } \mathbb{R} \times (0, +\infty) \\ u = g, u_t = h \text{ on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{2}$$

Suppose that $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$ both have compact support. The **kinetic energy** is defined by:

$$k(t) := \frac{1}{2} \int_{-\infty}^{+\infty} u_t^2(x,t)dx$$

and the **potential energy** is defined by:

$$p(t) := \frac{1}{2} \int_{-\infty}^{+\infty} u_x^2(x,t)dx.$$

a) Show that $k(t) + p(t)$ is constant in time by using d' Alembert's formula. Hence, the total energy is conserved in time.

b) Moreover, show that $k(t) = p(t)$ for sufficiently large t. In other words, the total energy gets equally partitioned into the kinetic and potential part over a sufficiently long time.