Math 644 - Homework 9 - Due Friday, Nov. 16, 2012

1. (A nonlinear problem related to the wave equation)

   Consider the nonlinear PDE:

$$\begin{cases} u_{tt} - \Delta u + u_t^2 - |\nabla u|^2 = 0 \text{ on } \mathbb{R}^3 \times (0, +\infty) \\ u = 0, u_t = g \text{ on } \mathbb{R}^3 \times \{t = 0\}. \end{cases} \tag{1}$$

   Let us suppose that $g \in C^\infty(\mathbb{R}^3)$ and that g has compact support.

   a) Show that $v := e^u$ solves:

$$\begin{cases} v_{tt} - \Delta v = 0 \text{ on } \mathbb{R}^3 \times (0, +\infty) \\ v = 1, v_t = g \text{ on } \mathbb{R}^3 \times \{t = 0\}. \end{cases} \tag{2}$$

   b) Use Kirchoff's formula to solve for v.

   We want to show that for g "sufficiently small", v is positive, and hence $u = \ln(v)$ is well-defined and is a global smooth solution of (1).

   c) Observe that this reduces to showing that: $t \fint_{\partial B(x,t)} g(y) dS(y)$ is uniformly small in t, x.

   d) By fixing $x \in \mathbb{R}^3$ and rescaling, note that one has to consider

$$t \fint_{\partial B(0,1)} g(x + t\xi) dS(\xi).$$

   e) Use the fundamental theorem of Calculus in t to write

$$g(x + t\xi) = -\int_t^{+\infty} \frac{\partial}{\partial s}[g(x + s\xi)] ds.$$

   One is then reduced to estimating an integral in s and ξ. Estimate this integral by using polar coordinates centered at x. Deduce that:

$$\left| t \fint_{\partial B(0,1)} g(x + t\xi) dS(\xi) \right| \leq \frac{C}{t} \|\nabla g\|_{L^1}$$

   This bound is good for large t.

   f) How should we argue for small t?

2. (A decay estimate for the wave equation)

   Let u solve:
$$\begin{cases} u_{tt} - \Delta u = 0 \text{ on } \mathbb{R}^3 \times (0, +\infty) \\ u = g, u_t = h \text{ on } \mathbb{R}^3 \times \{t = 0\}. \end{cases} \tag{3}$$
   Suppose that $g, h \in C^\infty(\mathbb{R}^3)$ have compact support.

   Show that there exists a constant $C > 0$ such that:
$$|u(x,t)| \leq \frac{C}{t},$$
   for all $x \in \mathbb{R}^3$ and $t > 0$.

   (HINT: Use Kirchoff's formula. The bound is not immediately obvious from the formula. One can argue as in the previous problem by using the fundamental theorem of Calculus and polar coordinates centered at a fixed $x \in \mathbb{R}^3$.)

3. (An interpolation inequality)

   Suppose that $1 \leq p, q \leq \infty$ and suppose that $\theta \in [0,1]$ is given. Define r by:
$$\frac{1}{r} := \frac{\theta}{p} + \frac{1-\theta}{q}$$
   Show that:
$$\|f\|_{L^r} \leq \|f\|_{L^p}^{\theta} \|f\|_{L^q}^{1-\theta}.$$

4. (Attenuation function) [Evans, Problem 20 in Chapter 2]

5. (telegraph equations) [Evans, Problem 22 in Chapter 2]