Math 645 - Homework 8 - Due Friday, March 29, 2013

1. For $s > 0$, prove the following inequality:

$$\|f\|_{L^2(\mathbb{R}^n)}^{1+2s/n} \leq C\|\Lambda^s f\|_{L^2(\mathbb{R}^n)}\|f\|_{L^1(\mathbb{R}^n)}^{2s/n}$$

   HINT: Use the Fourier Transform.

2. Recall the homogeneous Besov-Lipshitz spaces, with semi-norms given by:

$$\|f\|_{\dot{B}_p^{k,q}} = \left\|2^{kj}\|\Delta_j f\|_{L^p(\mathbb{R}^n)}\right\|_{\ell_j^q}.$$

   Suppose that $m \neq \rho$. Prove the following interpolation estimate:

$$\|f\|_{\dot{B}_p^{k,1}} \lesssim \|f\|_{\dot{B}_r^{m,\infty}}^{1-\theta}\|f\|_{\dot{B}_r^{\rho,\infty}}^{\theta},$$

   where $0 < \theta < 1$ and $1 \leq r \leq p \leq \infty$. We also require by scaling that:

$$k + \frac{n}{r} - \frac{n}{p} = m(1-\theta) + \rho\theta.$$

   Also, what is θ?

   HINT: Use Littlewood-Paley type arguments.

3. Fix $m > \ell \geq k$, and $1 \leq p \leq q \leq r \leq \infty$. Prove that

$$\|g\|_{\dot{B}_q^{\ell,q'}} \leq \|g\|_{\dot{B}_r^{k,r'}}^{\theta}\|g\|_{\dot{B}_p^{m,p'}}^{1-\theta}.$$

   These parameters satisfy the following restrictions

$$\ell = k\theta + m(1-\theta), \quad \frac{1}{q} = \frac{\theta}{r} + \frac{1-\theta}{p}, \quad \frac{1}{q'} = \frac{\theta}{r'} + \frac{1-\theta}{p'}.$$

   Also $1 \leq p' \leq q' \leq r' \leq \infty$, and solving we have $\theta = \frac{m-\ell}{m-k} \in (0,1]$.

   HINT: There exists a short proof. Use Holder's inequality once for integrals, and another time for sums. Plus there is one additional intermediate step.