Math 644 - Homework 7 - Due Friday, Nov. 2, 2012

1. Suppose that $u_1(y,t)$, ..., $u_n(y,t)$ are solutions of the one-dimensional heat equation: $\partial_t u - \partial_y^2 u = 0$. Show that $v(x,t) = \Pi_{j=1}^n u_j(x_j,t)$ solves the n-dimensional heat equation. (What special properties of the heat equation make this work?)

2. Use the L^∞ estimates on derivatives of solutions to the heat equation to prove that the mapping $x \to u(x,t)$ is generally analytic when $u(x,t)$ solves the heat equation. Formulate the appropriate conditions in order to make your statement rigorous and then prove it.

3. Let $U_T = U \times (0,T)$ where $U \subset \mathbb{R}^n$ is a bounded open set. Suppose that $u \in C_1^2(U_T) \cap C(\overline{U_T})$ satisfies the following equation:
$$\partial_t u = D\Delta u + \mathbf{b}(x,t) \cdot \nabla u + c(x,t)u, \quad \text{in } U_T.$$
Here $D > 0$ and $\mathbf{b}(x,t), c(x,t) \in C(\overline{U_T})$. Show that if $u \geq 0$ (or $u \leq 0$) on ∂U_T then $u \geq 0$ (or $u \leq 0$) on U_T.

   (Hint: Assume first that $c(x,t) \leq a < 0$. Then reduce to this case by setting $u = ve^{kt}$ for a suitably large $k > 0$.)

4. Solve the following initial-Dirichlet problem in $B_1(0) = \{x \in \mathbb{R}^3 : |x| < 1\}$:
$$\begin{cases} u_t(x,t) - \Delta u(x,t) = 0 \text{ for } (x,t) \in B_1(0) \times (0,+\infty) \\ u(x,0) = 0 \text{ for } x \in B_1(0) \\ u(\sigma,t) = 1 \text{ for } \sigma \in \partial B_1(0),\ t > 0. \end{cases}$$
Then compute the limit $\lim_{t\to\infty} u$. Discuss the possibility of generalizing your solution to this problem to the case of $\mathbb{R}^n$.

   (Hint: Observe the solution is radial: $u = u(r,t)$ with $r = |x|$. And then let $v = ru$ for the case of $\mathbb{R}^3$.)

5. Solve the following initial-Neumann problem in $B_1(0) = \{x \in \mathbb{R}^3 : |x| < 1\}$:
$$\begin{cases} u_t(x,t) - \Delta u(x,t) = 0 \text{ for } (x,t) \in B_1(0) \times (0,+\infty) \\ u(x,0) = |x| \text{ for } x \in B_1(0) \\ \frac{\partial u}{\partial \nu}(\sigma,t) = 1 \text{ for } \sigma \in \partial B_1(0),\ t > 0. \end{cases}$$
Discuss the possibility of generalizing your solution to the case of $\mathbb{R}^n$.