A NON-LOCAL INEQUALITY AND GLOBAL EXISTENCE

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Abstract. In this article we prove a collection of new non-linear and non-local integral inequalities. As an example for $u \geq 0$ and $p \in (0, \infty)$ we obtain

$$\int_{\mathbb{R}^3} dx\ u^{p+1}(x) \leq \left(\frac{p+1}{p}\right)^2 \int_{\mathbb{R}^3} dx\ \{(-\triangle)^{-1}u(x)\}|\nabla u^{\frac{p}{2}}(x)|^2.$$

We use these inequalities to deduce global existence of solutions to a non-local heat equation with a quadratic non-linearity for large radial monotonic positive initial conditions. Specifically, we improve [4] to include all $\alpha \in (0, \frac{74}{75})$.

1. Introduction

In this article we study the following non-local quadratically non-linear heat equation for $\alpha > 0$ given as follows:

$$\partial_t u = \left\{(-\triangle)^{-1}u\right\} \triangle u + \alpha u^2, \quad u(0,x) = u_0(x) \geq 0, \tag{1}$$

where as usual

$$(-\triangle)^{-1}u = \left(\frac{1}{4\pi|\cdot|} * u\right)(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} dy\ \frac{u(y)}{|x-y|}.$$

We also define, for simplicity, the kernel of the Laplacian "$(-\triangle)$" as $G(x) = \frac{1}{4\pi|x|}$. Note that then (1) satisfies the following conservation law:

$$\int_{\mathbb{R}^3} dx\ u(t,x) + (1-\alpha)\int_0^t ds \int_{\mathbb{R}^3} dx\ |u(s,x)|^2 = \int_{\mathbb{R}^3} dx\ u_0(x).$$

In this model the variables are $(t,x) \in [0,\infty) \times \mathbb{R}^3$.

Equation (1) was, as far as we know, first introduced in [4] as a model problem for the spatially homogeneous Landau equation from 1936 [5], which takes the form

$$\partial_t f = \mathcal{Q}(f,f).$$

We define $\partial_i = \frac{\partial}{\partial v_i}$, and then we have the Landau collision operator

$$\mathcal{Q}(g,f) \stackrel{\text{def}}{=} \partial_i \int_{\mathbb{R}^3} dv_*\ a^{ij}(v-v_*)\left\{g(v_*)(\partial_j f)(v) - f(v)(\partial_j g)(v_*)\right\}.$$

Here the projection matrix is given by

$$a^{ij}(v) = \frac{L}{8\pi}\frac{1}{|v|}\left(\delta_{ij} - \frac{v_i v_j}{|v|^2}\right), \quad L > 0, \quad v = (v_1, v_2, v_3) \in \mathbb{R}^3. \tag{2}$$

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Throughout this article we use the Einstein convention of implicitly summing over repeated indices so that, for example, $a^{ij}(v)v_i v_j = \sum_{i,j=1}^{3} a^{ij}(v)v_i v_j$. Above furthermore δ_{ij} is the standard Kronecker delta. Then the following equivalent formulation of the Landau equation is well known

$$\partial_t f = \left(a^{ij} * f\right) \partial_i \partial_j f + L f^2, \quad (t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^3. \tag{3}$$

See for example [7, Page 170, Eq. (257)]. We can set $L = 1$ for simplicity. Standard references on the Landau equation include [1–3, 5–7] and the references therein.

It is known that non-negative solutions to (3) preserve the L^1 mass. This grants the point of view that (1) with $\alpha = 1$ may be a good model for solutions to the Landau equation (3). Note that (1) preserves the "Coulomb" singularity in (3), although it removes the projection matrix. Furthermore (1) maintains the quadratic non-linearity in (3). It appears that neither existence of global strong solutions for general large data, nor formation of singularities is known for either (3), or (1) with $\alpha = 1$. More comparisons can be found in [4].

The main result of [4] was to prove the following theorem.

Theorem 1. [4]. *Let $0 \leq \alpha < \frac{2}{3}$. Suppose that $u_0(x)$ is positive, radial, and non-increasing with $u_0 \in L^1(\mathbb{R}^3) \cap L^{2+\delta}(\mathbb{R}^3)$ for some small $\delta > 0$. Additionally suppose that $-\triangle \tilde{u}_0 \in L^2(\mathbb{R}^3)$, where $\tilde{u}_0 \overset{def}{=} \langle x \rangle^{\frac{1}{2}} u_0$ and $\langle x \rangle \overset{def}{=} \sqrt{1+|x|^2}$. Then there exists a non-negative global solution, $u(t,x)$, with*

$$u(t,x) \in C^0([0,\infty), L^1 \cap L^{2+\delta}(\mathbb{R}^3)) \cap C^0(\mathbb{R}_{\geq 0}, H^2(\mathbb{R}^3)),$$
$$\langle x \rangle^{\frac{1}{2}} (-\triangle) u(t,x) \in C^0([0,\infty), L^2(\mathbb{R}^3)).$$

Additionally the solution satisfying all of these conditions is unique. Furthermore this solution decays toward zero at $t = +\infty$, in the following sense:

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{L^q(\mathbb{R}^3)} = 0, \quad \forall q \in (1, (2 \wedge 1/\alpha)].$$

Above we use the notation $(2 \wedge 1/\alpha) \overset{def}{=} \min\{2, 1/\alpha\}$.

The purpose of the present article is to improve the previous Theorem 1 to a substantially larger range of $\alpha \in (0, \frac{74}{75})$ in the following main theorem.

Theorem 2. *Let $0 \leq \alpha < \frac{74}{75}$, and u_0 be as in Theorem 1. Then there exists a global solution in the same spaces as in Theorem 1; this solution further satisfies*

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{L^q(\mathbb{R}^3)} = 0, \quad q \in (1, 75/74].$$

Now our Theorem 2 is in some sense a consequence of the following non-linear and non-local inequality, which as far as we know is completely new:

$$\int_{\mathbb{R}^3} dx \; u^{p+1}(x) \leq \left(\frac{p+1}{p}\right)^2 \int_{\mathbb{R}^3} dx \; \{(-\triangle)^{-1} u(x)\} |\nabla u^{\frac{p}{2}}(x)|^2. \tag{4}$$

This inequality will hold for $p \in (0,\infty)$ and for suitable functions $u(x) \geq 0$.

Furthermore a similar inequality holds in the case of the Landau equation (2) and (3). In this situation we can prove under the same conditions the inequality

$$\int_{\mathbb{R}^3} dv \; f^{p+1} \leq \left(\frac{p+1}{p}\right)^2 \int_{\mathbb{R}^3} dv \; \left(a^{ij} * f\right) \partial_i f^{\frac{p}{2}} \partial_j f^{\frac{p}{2}}. \tag{5}$$

To be precise, the inequality we prove in Theorem 3 below is more general than both (4) and (5). These inequalities may also be interesting on their own.

The rest of this article is organized as follows. In the next Section 1.1 we supply some computations which help in proving the propagation of L^p norms for solutions to (1). After that in Section 1.2 we show that for (5) the constant is sharp when $p = 1$. Then in Section 2 we will state the main non-linear and non-local inequality in Theorem 3, and give its proof. We finish the article with Section 3 where we use the new inequality (4) and arguments from [4] to establish Theorem 2.

1.1. **Computations regarding L^p norms.** Considering the non-local equation (1), for $p \in (0, \infty)$, we use the equation (1) obtain the following

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3} dx\ u^p &= \int_{\mathbb{R}^3} dx\ \partial_t u\ u^{p-1} \\
&= -\int_{\mathbb{R}^3} dx\ \left\{(G*u)\nabla u \cdot \nabla u^{p-1} + ((\nabla G * u)\cdot \nabla u)\, u^{p-1} - \alpha u^{p+1}\right\} \\
&= -\frac{4}{p}\left(\frac{p-1}{p}\right)\int_{\mathbb{R}^3} dx\ (G*u)\left|\nabla u^{\frac{p}{2}}\right|^2 - \frac{1}{p}\int_{\mathbb{R}^3} dx\ u^{p+1} + \alpha\int_{\mathbb{R}^3} dx\ u^{p+1}.
\end{aligned} \tag{6}$$

In other words, in order to propagate an L^p norm of a solution, it would suffice for suitable non-negative functions u to establish the following inequality

$$\left(\alpha - \frac{1}{p}\right)\int_{\mathbb{R}^3} dx\ u^{p+1} \le \frac{4}{p}\left(\frac{p-1}{p}\right)\int_{\mathbb{R}^3} dx\ (G*u)\left|\nabla u^{\frac{p}{2}}\right|^2. \tag{7}$$

In Section 3 we will relate (7) to (4) and prove Theorem 2. In Section 2 we will prove a more general collection of inequalities than (7). Next we look at (5).

1.2. **The constant in the Landau inequality.** In this sub-section we consider inequality (5), and we argue that the constant is sharp when $p = 1$. It is well known that the Landau equation (3) has steady states given by the Maxwellian equilibrium, e.g. [7]; for example $\mu(v) \stackrel{\text{def}}{=} (2\pi)^{-3/2}e^{-|v|^2/2}$. Then from (3):

$$0 = \left(a^{ij} * \mu\right)\partial_i\partial_j\mu + \mu^2.$$

Similar to (6), we multiply this by μ^{p-1} and integrate over $v \in \mathbb{R}^3$ to obtain

$$0 = -\frac{4}{p}\left(\frac{p-1}{p}\right)\int_{\mathbb{R}^3} dv\ \left(a^{ij} * \mu\right)\partial_i\mu^{\frac{p}{2}}\partial_j\mu^{\frac{p}{2}} + \left(\frac{p-1}{p}\right)\int_{\mathbb{R}^3} dv\ \mu^{p+1}.$$

For say $p > 1$ we multiply both sides by $\frac{p}{p-1}$ and take the limit as $p \downarrow 1$ to observe

$$4\int_{\mathbb{R}^3} dv\ \left(a^{ij} * \mu\right)\partial_i\mu^{\frac{1}{2}}\partial_j\mu^{\frac{1}{2}} = \int_{\mathbb{R}^3} dv\ \mu^2.$$

Hence the constant in (5) is sharp when $p = 1$.

2. The non-linear and non-local inequality

In this section we suppose that $\mathbf{b}(v) = (b^{ij}(v))$ is a $n \times n$ matrix for every $v \in \mathbb{R}^n$ where $i, j \in \{1, \dots, n\}$ and say $n \ge 2$. We furthermore suppose

$$\mathbf{b}(v) = \mathbf{b}(-v), \quad b^{ij}(v)\xi_i\xi_j \ge 0, \quad \forall v = (v_1, \dots, v_n),\ \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n. \tag{8}$$

Additionally we suppose that the sum over all the second derivatives in v of the matrix $-\mathbf{b}(v - v_*)$ is a standard delta function at the point $v - v_* = 0$; precisely

$$-\partial_i\partial_j b^{ij}(v - v_*) = \delta_0(v - v_*). \tag{9}$$

Under these basic assumptions, we have the following main inequality.

Theorem 3. *For suitable $g \geq 0$, under* (8) *and* (9) *we have the inequality:*

$$\int_{\mathbb{R}^n} dv\ g^{p+1} \leq \left(\frac{p+1}{p}\right)^2 \int_{\mathbb{R}^n} dv\ (b^{ij} * g)\partial_i g^{\frac{p}{2}} \partial_j g^{\frac{p}{2}}.$$

In this inequality we can allow any $p \in (0, \infty)$.

For simplicity and without loss of generality in the above theorem we can suppose that g is a function in the Schwartz class.

Note that (for example in dimension $n = 3$) inequality (4) follows from Theorem 3 since $b^{ij}(x) = \frac{\delta_{ij}}{4\pi|x|}$ satisfies both (8) and (9). Similarly inequality (5) follows since $b^{ij}(v) = a^{ij}(v)$ from (2) is known to also satisfy (8) and (9).

Proof. In the proof below we can assume without loss of generality that g is strictly positive; then the estimate for $g \geq 0$ will follow by approximation. As a result of (8), we consider the following quadratic form

$$|\partial g(v)|^2_{\mathbf{b}(v-v_*)} \overset{\text{def}}{=} b^{ij}(v - v_*)(\partial_i g)(v)(\partial_j g)(v).$$

Then for $p \in (0, \infty)$ we expand the upper bound in Theorem 3 as

$$\begin{aligned}\int_{\mathbb{R}^n} dv\ (b^{ij} * g)(v)\partial_i g^{\frac{p}{2}} \partial_j g^{\frac{p}{2}} &= \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_*\ b^{ij}(v - v_*)g(v_*)(\partial_i g^{\frac{p}{2}})(v)(\partial_j g^{\frac{p}{2}})(v) \\ &= \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_*\ g(v_*)|\partial g^{\frac{p}{2}}(v)|^2_{\mathbf{b}(v-v_*)}.\end{aligned}$$

By symmetry the above is

$$\begin{aligned}&= \frac{1}{2}\int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_*\ g(v_*)|\partial g^{\frac{p}{2}}(v)|^2_{\mathbf{b}(v-v_*)} + \frac{1}{2}\int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_*\ g(v)|\partial g^{\frac{p}{2}}(v_*)|^2_{\mathbf{b}(v-v_*)} \\ &\geq \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_*\ \sqrt{g(v_*)g(v)}|\partial g^{\frac{p}{2}}(v_*)|_{\mathbf{b}(v-v_*)}|\partial g^{\frac{p}{2}}(v)|_{\mathbf{b}(v-v_*)} \\ &= \left(\frac{p}{p+1}\right)^2 \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_*\ |\partial g^{\frac{p+1}{2}}(v_*)|_{\mathbf{b}(v-v_*)}|\partial g^{\frac{p+1}{2}}(v)|_{\mathbf{b}(v-v_*)}.\end{aligned}$$

Now we use the Cauchy-Schwartz inequality for quadratic forms to obtain

$$\begin{aligned}&\left(\frac{p}{p+1}\right)^2 \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_*\ |\partial g^{\frac{p+1}{2}}(v_*)|_{\mathbf{b}(v-v_*)}|\partial g^{\frac{p+1}{2}}(v)|_{\mathbf{b}(v-v_*)} \\ &\quad \geq \left(\frac{p}{p+1}\right)^2 \int_{\mathbb{R}^n} dv \int_{\mathbb{R}^n} dv_*\ b^{ij}(v - v_*)(\partial_i g^{\frac{p+1}{2}})(v)(\partial_j g^{\frac{p+1}{2}})(v_*) \\ &\quad = \left(\frac{p}{p+1}\right)^2 \int_{\mathbb{R}^n} dv\ g^{p+1}(v).\end{aligned}$$

This last computation used both (8), (9) and standard integration by parts. Collecting the above estimates then yields Theorem 3. □

In the next section we will discuss how this inequality can be used in combination with the techniques from [4] to obtain Theorem 2.

3. The Implications

In this last section, we will explain the proof of Theorem 2. This proof will be derived from the developments in [4] and the inequality (4). In particular Theorem 2 (except for the decay to zero of the L^q norm) follows directly from the arguments in the proof of [4, Theorem 1.3] once we show the monotonicity estimate

$$\|u(t)\|_{L^{\frac{3}{2}+\gamma}} \leq \|u_0\|_{L^{\frac{3}{2}+\gamma}}, \quad 0 \leq t < T, \tag{10}$$

for some $\gamma > 0$ sufficiently small. Here $u(t,x) \geq 0$ is a local in time solution of (1) which is defined on $[0,T) \times \mathbb{R}^3$ and satisfies the properties in Theorem 1 on $[0,T)$.

To establish (10), from (6), (7) and (4) it suffices to check that $\alpha > 0$ satisfies

$$-\frac{4}{p}\left(\frac{p-1}{p}\right) + \left(\alpha - \frac{1}{p}\right)\left(\frac{p+1}{p}\right)^2 \leq 0,$$

for some $p = \frac{3}{2} + \gamma$. This inequality is equivalent to the following

$$\alpha \leq 4\frac{(p-1)}{(p+1)^2} + \frac{1}{p} \stackrel{\text{def}}{=} h(p).$$

Now $h(p)$ is continuous and $h(3/2) = \frac{74}{75}$ so that for any $\alpha \in (0, \frac{74}{75})$ we can find a small $\gamma > 0$ such that (10) holds. The decay of the L^q norms in given in Theorem 2 then follows directly from the arguments in the decay at infinity proof of [4, Theorem 1.1] combined with the monotonicity in (10). **Q. E. D.**

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