Math 644 - Homework 11 - Due Friday, Nov. 30, 2012

Final HW Instructions: This homework is mandatory; all problems must be attempted to get credit. You may discuss this homework with Prof. Strain, and only Prof. Strain. You may not consult your peers, the internet, or other text books for help in solving these problems. You can consult your course materials, and any books that we used in class this semester. The challenge is to think up solutions to the problems yourself. Good luck!

This homework should be handed in directly to Janet Burns in the main office by Friday Dec. 7 before 4pm; or alternatively via email to Prof. Strain.

1. Let $S(t)$ denote the linear Schrödinger propagator defined in class. Show that:

$$(S(t))^* = S(-t)$$

where $\cdot^*$ denotes the adjoint with respect to the $L^2(\mathbb{R}^n)$ inner product.

2. For $v \in \mathbb{R}^n_v$, consider the following Transport equation:

$$\begin{cases} \frac{\partial u}{\partial t} + v \cdot \nabla_x u = 0 \text{ in } \mathbb{R}^n_x \times \mathbb{R}^n_v \times (0,\infty) \\ u = f, \text{ on } \mathbb{R}^n_x \times \mathbb{R}^n_v \times \{t = 0\}. \end{cases} \tag{1}$$

Find the exact solution.

Prove the following time decay / blow up estimate for solutions to (1):

$$\left\| \| u(t) \|_{L^p(\mathbb{R}^n_v)} \right\|_{L^q(\mathbb{R}^n_x)} \leq t^{-\left(\frac{n}{p} - \frac{n}{q}\right)} \left\| \| f \|_{L^q(\mathbb{R}^n_v)} \right\|_{L^p(\mathbb{R}^n_x)},$$

whenever $q \geq p \geq 1$. Hint: First prove the case when $p = 1$ and $q = \infty$.

3. Consider the Poisson equation on $\mathbb{R}^3$:

$$-\Delta u = f, \quad \text{on } \mathbb{R}^3_x. \tag{2}$$

Suppose that $f(x)$ is a radial function, e.g. $f(x) = f(|x|)$. Prove that

$$(-\Delta)^{-1} f = \int_0^\infty \frac{f(\rho)}{\max\{|x|, \rho\}} \rho^2 d\rho$$

Hint: Use the fundamental solution to Poisson's equation and polar coordinates.

4. For $\epsilon > 0$, consider the following wave equation on $\mathbb{R}^2_x$:

$$\begin{cases} (\frac{\partial^2}{\partial t^2} - \Delta_x)u^\epsilon = 0 \text{ on } \mathbb{R}^2_x \times (0,4], \\ u^\epsilon = f^\epsilon, \quad u^\epsilon_t = 0 \text{ on } \mathbb{R}^2_x \times \{t=0\}. \end{cases} \tag{3}$$

We define $f^\epsilon(r)$ where $r = |x|$ by

$$f^\epsilon(r) = \begin{cases} \exp\left(-\epsilon \frac{(r-2)^2}{(r-1)(3-r)}\right) \text{ if } 1 < r < 3, \\ 0, \quad \text{elsewhere.} \end{cases}$$

Show that even though $|f^\epsilon(r)| \leq 1$ for all $\epsilon > 0$ we still have

$$\max_{\mathbb{R}^2_x \times [0,4]} |u^\epsilon| \to \infty \quad \text{as} \quad \epsilon \to 0.$$

To do this, use the solution formula for the wave equation in $\mathbb{R}^2_x$.

5. Consider the following problem:

$$\begin{cases} \frac{\partial u}{\partial t} + \Lambda u = 0 \text{ in } \mathbb{R}^n_x \times (0,\infty) \\ u = f, \text{ on } \mathbb{R}^n_x \times \{t=0\}. \end{cases} \tag{4}$$

Here $\Lambda = \sqrt{-\Delta}$ is defined by $\widehat{\Lambda f}(\xi) = 2\pi|\xi|\hat{f}(\xi)$. Use the Fourier transform to show that $u = (G_t * f)(x)$. Find the physical space representation of $G_t(x)$. Furthermore prove the following time decay/ blow up estimate:

$$\|u(t)\|_{L^q(\mathbb{R}^n)} \leq C\ t^{-n\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^n)},$$

whenever $q \geq p \geq 1$ for some $C > 0$.

Good Luck!