Math 644 - Homework 4 - Due Friday, Oct. 5, 2012

1. (Evans, Problem 9 from Chapter 2.) (An example of non-smoothness at the boundary of a harmonic function.) Let u be the solution of

$$\begin{cases} \Delta u = 0, \text{ in } \mathbb{R}^n_+ \\ u = g, \text{ on } \partial\mathbb{R}^n_+ \end{cases} \tag{1}$$

given by Poisson's formula for half-space. Suppose that $g \in C(\partial\mathbb{R}^n_+) \cap L^\infty(\partial\mathbb{R}^n_+)$ is non-negative and that $g(x) = |x|$ for $|x| \le 1$. Calculate the limit:

$$\lim_{\lambda \to 0} \frac{u(\lambda e_n) - u(0)}{\lambda}.$$

Deduce that u is not smooth up to $\partial\mathbb{R}^n_+$.

2. (a) Prove Evans, Problem 7 of Chapter 2. (Explicit form of Harnack's inequality.)

(b) Use part (a) to deduce that there exists a constant $C_n > 0$ depending only upon the dimension n such that, whenever u is a non-negative harmonic function on $B(a, 2R) \subset \mathbb{R}^n$ for $R > 0$ then one further has

$$\max_{B(a,R)} u \le C_n \min_{B(a,R)} u.$$

3. (A condition for convergence of a sum of non-negative harmonic functions.) Suppose that $U \subset \mathbb{R}^n$ is open and connected and suppose that (u_n) is a sequence of non-negative harmonic functions on U such that the series $\sum_n u_n$ converges uniformly at some $x_0 \in U$.

(a) Show that the series $\sum_n u_n$ converges uniformly on any compact subset $K \subset U$.

[HINT: Argue as in the proof of Harnacks inequality to cover K by a chain of finitely many balls with non-empty intersection "starting from x_0". Part b) of the previous exercise is useful too.]

(b) Deduce that $\sum_n u_n$ is also harmonic in U.

4. Solve Evans, Problem 10 from Chapter 2. (Reflection principle.)

5. (Removable singularities for the Laplace equation on $\mathbb{R}^2$.) Suppose that $U \subset \mathbb{R}^2$ is open and bounded and suppose that for some $x_0 \in U$, we are given a function $u \in C^2(U \backslash \{x_0\})$ which satisfies for some $0 < M < \infty$ that:

$$\Delta u(x) = 0, \quad |u(x)| \leq M, \quad \forall x \in U \backslash \{x_0\}.$$

Show that there exists $\tilde{u} \in C^2(U)$ which is harmonic on U and agrees with u on $U \backslash \{x_0\}$.

[HINT: It is good to first reduce to the case $x_0 = 0$ and $U = B(0, R)$. By Poissons formula for $B(0, R)$, we can solve:

$$\begin{cases} \Delta v = 0, \text{ in } B(0, R) \\ v = u, \text{ on } \partial B(0, R) \end{cases} \tag{2}$$

We want to show that $w := u - v = 0$ on $B(0, R) \backslash \{0\}$. It is good to consider the function

$$h(x) := 2M \frac{log(|x|/R)}{log(r/R)}$$

Why can we say that $|w(x)| \leq |h(x)|$ for $0 < r < |x| < R$? What happens when $r \to 0$?]