Math 644 - Homework 10 - Due Friday, Nov. 30, 2012

1. (Solving the wave equation by using the Fourier transform)

   a) Suppose that $f, g \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz class). Consider the initial value problem:

$$\begin{cases} (\frac{\partial^2}{\partial t^2} - \Delta_x)u = 0 \text{ on } \mathbb{R}^n_x \times \mathbb{R}_t \\ u = f, u_t = g \text{ on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{1}$$

   Find $\widehat{u}(\xi, t)$, where $\widehat{\cdot}$ denotes the Fourier transform in the x variable.

   b) Use the Fourier transform and part a) to show that:

$$\|\nabla u(x,t)\|^2_{L^2(\mathbb{R}^n_x)} + \|\frac{\partial}{\partial t}u(x,t)\|^2_{L^2(\mathbb{R}^n_x)} = \|\nabla f(x)\|^2_{L^2(\mathbb{R}^n_x)} + \|g(x)\|^2_{L^2(\mathbb{R}^n_x)}.$$

   This method gives us an alternative derivation of the conservation of energy for the wave equation.

2. (Localization of functions in the frequency space)

   a) Given $N > 1$, and $\psi \in C^\infty(\mathbb{R}^d)$ which is radial and satisfies:

$$\begin{cases} \psi = 1 \text{ on } \frac{3}{4} \leq |\xi| \leq \frac{5}{4} \\ \psi = 0 \text{ on } |\xi| \leq \frac{1}{2} \text{ and on } |\xi| \geq 2. \end{cases} \tag{2}$$

   We note that ψ defined in this way is a smooth approximation of the characteristic function of the annulus $|\xi| \sim 1$. For the ψ defined in (2), we define an operator P_N on $L^2(\mathbb{R}^d)$ by:

$$(P_N f)\widehat{\ }(\xi) := \psi(\frac{\xi}{N})\widehat{f}(\xi).$$

   a) Explain why it is clear by construction that:

$$\|P_N f\|_{L^2(\mathbb{R}^d)} \leq \|f\|_{L^2(\mathbb{R}^d)}.$$

   b) Express P_N as a convolution operator, i.e. find K_N such that:

$$P_N f = K_N * f.$$

c) Using part b), prove that for $1 \leq p \leq \infty$, one has:

$$\|P_N f\|_{L^p(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)}.$$

(Strictly speaking, we are defining K_N on $L^2 \cap L^p$ and we are extending the definition by density.)

[HINT: Use Young's Inequality.]

d) Show more generally that for all $1 \leq p \leq q \leq \infty$, one has:

$$\|P_N f\|_{L^q(\mathbb{R}^d)} \leq CN^{\frac{d}{p}-\frac{d}{q}}\|f\|_{L^p(\mathbb{R}^d)}.$$

e) Show that the result in d) can be improved to

$$\|P_N f\|_{L^q(\mathbb{R}^d)} \leq CN^{\frac{d}{p}-\frac{d}{q}}\|P_N f\|_{L^p(\mathbb{R}^d)}. \quad \text{(See next page)}.$$

[HINT: Write $K_N f = \tilde{K}_N K_N f$ where $\tilde{K}_N$ is an operator of a similar type as K_N which localizes to a slightly larger region in the frequency space.]

3. (Square problem) [Evans, Problem 23 in Chapter 2]