Math 644 - Homework 2 - Due Friday, Sept. 21, 2012

1. (Evans, Problem 1 in Chapter 2) Write down an explicit formula for a function u which solves the initial-value problem:

$$\begin{aligned} \partial_t u + b \cdot \nabla u + cu = 0, &\quad \text{in } \mathbb{R}^n \times (0, +\infty), \\ u|_{t=0} = g(x), &\quad \text{on } \mathbb{R}^n_x \times \{t = 0\}. \end{aligned}$$

   Here, $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants.

2. (Evans, Problem 2 in Chapter 2) Prove that Laplace's equation $\Delta u = 0$ is rotationally invariant, i.e. if $\mathcal{O} \in O(n)$ is an orthogonal $n \times n$ matrix, and if $v = u \circ \mathcal{O}$, then $\Delta v = 0$. We recall that this was a useful insight when we were looking for a fundamental solution of the Laplace operator on $\mathbb{R}^n$.

3. (A vanishing theorem) Suppose that $u \in L^2(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ solves $\Delta u = 0$. Show that u is identically equal to zero. (HINT: Use the Mean value property).

4. (Evans, Problem 3 in Chapter 2) (A mean value formula for the Poisson equation on $B(0, r)$.) Modify the proof of the mean-value formulas to show that for $n \geq 3$ and $u \in C^2(B(0, r)) \cap C(\overline{B(0, r)})$ we have

$$u(0) = \fint_{\partial B(0,r)} g(y) dS(y) + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right) f(x) dx,$$

   whenever

$$\begin{cases} -\Delta u = f, \text{ in } B(0, r) \\ u = g, \text{ on } \partial B(0, r). \end{cases}$$

5. (A specific class of harmonic functions) Find all polynomials

$$P(x, y) = \sum_{k=0}^{n} C_k x^k y^{n-k}$$

   (for $x, y \in \mathbb{R}$ and $C_k \in \mathbb{C}$) which are homogeneous of degree n and which are harmonic in $\mathbb{R}^2$ e.g. $\Delta P = 0$.