locally in measure on $\mathbb{R}_t \times \mathbb{R}^d_x$, where (X, P) are the classical particle trajectories induced by the Hamiltonian system

$$\begin{cases} \dot{X} = P, \quad X(0) = x_0, \\ \dot{P} = -\nabla V(X), \quad P(0) = p_0. \end{cases} \tag{6}$$

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Around the Boltzmann equation without angular cut-off

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(joint work with Philip T. Gressman)

In this report, we will describe briefly several recent developments [2, 3, 4, 5] for the Boltzmann equation without the Grad angular cut-off assumption [1]:

$$\frac{\partial F}{\partial t} + v \cdot \nabla_x F = \mathcal{Q}(F, F), \quad F(0, x, v) = F_0(x, v). \tag{1}$$

Here the unknown is $F = F(t, x, v) \geq 0$ with $t \geq 0$. The spatial coordinates we consider are $x \in \Omega$ ($\Omega \in \{\mathbb{T}^n, \mathbb{R}^n\}$), and the velocities are $v \in \mathbb{R}^{\dim}$ with $n \geq 2$. The Boltzmann collision operator, $\mathcal{Q}$, acts only on the velocity variables, v, as

$$\mathcal{Q}(G, F)(v) \overset{\text{def}}{=} \int_{\mathbb{R}^n} dv_* \int_{\mathbb{S}^{n-1}} d\sigma\ B(v - v_*, \sigma) \left[G(v'_*)F(v') - G(v_*)F(v)\right]. \tag{2}$$

Above the velocities of a pair of particles before and after collision are connected by $v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma$ and $v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma$, where $\sigma \in \mathbb{S}^{n-1}$. The collision kernel, $B(v-v_*, \sigma)$, depends upon the deviation angle θ through $\cos\theta = (v-v_*)\cdot\sigma/|v-v_*|$. Furthermore $B(v - v_*, \sigma)$ can be taken to be zero for $\theta > \frac{\pi}{2}$.

We suppose that $B(v - v_*, \sigma) = \Phi(|v - v_*|)\, b(\cos\theta)$ where b and Φ are non-negative. The angular function is not locally integrable; for $c_b > 0$ it satisfies

$$(3)\qquad \frac{c_b}{\theta^{1+2s}} \le \sin^{n-2}\theta\; b(\cos\theta) \le \frac{1}{c_b\theta^{1+2s}}, \quad s \in (0,1), \quad \forall\, \theta \in \left(0, \frac{\pi}{2}\right].$$

Additionally the kinetic factor satisfies for some $C_\Phi > 0$ that

$$(4)\qquad \Phi(|v - v_*|) = C_\Phi |v - v_*|^\gamma, \quad \gamma > -n.$$

We distinguish between the cases $\gamma \ge -2s$, which are called "hard potentials", and the cases $-2s > \gamma > -n$, furthermore called "soft potentials" herein.

These collision kernels are physically motivated since they can be derived from an intermolecular repulsive potential such as $\phi(r) = r^{-(p-1)}$ with $p \in (2,\infty)$ as was shown by Maxwell in 1866. In the physical dimension ($n = 3$), B satisfies the conditions above with $\gamma = (p-5)/(p-1)$ and $s = 1/(p-1)$. The vast majority of previous work requires the Grad angular cut-off assumption [1] from 1963 which usually means either $b(\cos\theta) \in L^\infty(\mathbb{S}^{n-1})$ or $b(\cos\theta) \in L^1(\mathbb{S}^{n-1})$. Neither of these assumptions are satisfied for singular angular factors such as (3).

The Boltzmann H-theorem is a hallmark of statistical physics. Define the H-functional by $H(t) \stackrel{\text{def}}{=} -\int_\Omega dx \int_{\mathbb{R}^{\dim}} dv\ f \log f$. Then the H-theorem predicts that, for solutions of the Boltzmann equation, the entropy is increasing over time:

$$\frac{dH(t)}{dt} = \int_\Omega dx\ D(f,f) \ge 0.$$

This is a demonstration of the second law of thermodynamics. The entropy production functional is $D(g,f) \stackrel{\text{def}}{=} -\int_{\mathbb{R}^{\dim}} dv\ \mathcal{Q}(g,f)\log f$. This functional is zero if and only if it is operating on a Maxwellian equilibrium. The prediction is thus that the Boltzmann equation exhibits irreversible dynamics and should experience convergence to Maxwellian in large time.

We will study the linearization of (1) around the Maxwellian equilibrium

$$(5)\qquad F(t,x,v) = \mu(v) + \sqrt{\mu(v)} f(t,x,v),$$

where the Maxwellian is given by $\mu(v) \stackrel{\text{def}}{=} (2\pi)^{-n/2} e^{-|v|^2/2}$. We linearize the Boltzmann equation (1) around (5). This grants an equation for the perturbation:

$$(6)\qquad \partial_t f + v \cdot \nabla_x f + L(f) = \Gamma(f,f), \quad f(0,x,v) = f_0(x,v),$$

where the bilinear operator, Γ, is given by $\Gamma(g,h) \stackrel{\text{def}}{=} \mu^{-1/2}\mathcal{Q}(\sqrt{\mu}g, \sqrt{\mu}h)$. Then the linearized collision operator, L, is defined as $L(g) \stackrel{\text{def}}{=} -\Gamma(g,\sqrt{\mu}) - \Gamma(\sqrt{\mu}, g)$. The null space of L is: $N(L) \stackrel{\text{def}}{=} \text{span}\left\{\sqrt{\mu},\ v\sqrt{\mu},\ (|v|^2 - n)\sqrt{\mu}\right\}$. Now, for fixed (t,x), we denote the orthogonal projection from $L^2(\mathbb{R}^{\dim}_v)$ into $N(L)$ by $\mathbf{P}$.

In recent works [2, 3, 4], we introduced the norm: $|f|^2_{N^{s,\gamma}} \stackrel{\text{def}}{=} |f|^2_{L^2_{\gamma+2s}} + |f|^2_{\dot{N}^{s,\gamma}}$. Here, for $\ell \in \mathbb{R}$, we use the norm $|f|^p_{L^p_\ell} \stackrel{\text{def}}{=} \int_{\mathbb{R}^{\dim}} dv\ \langle v\rangle^\ell\ |f(v)|^p$ for $p = 1, 2$. The weight is $\langle v\rangle \stackrel{\text{def}}{=} \sqrt{1+|v|^2}$. We also use the "dotted" semi-norm

$$|f|^2_{\dot{N}^{s,\gamma}} \stackrel{\text{def}}{=} \int_{\mathbb{R}^{\dim}} dv \int_{\mathbb{R}^{\dim}} dv' \, (\langle v\rangle \langle v'\rangle)^{\frac{\gamma+2s+1}{2}} \, \frac{(f(v')-f(v))^2}{d(v,v')^{n+2s}} \mathbf{1}_{d(v,v')\le 1}.$$

The fractional differentiation effects are measured using the anisotropic metric:

$$d(v,v') \stackrel{\text{def}}{=} \sqrt{|v-v'|^2 + \frac{1}{4}\left(|v|^2-|v'|^2\right)^2}.$$

Here the quadratic difference $|v|^2-|v'|^2$ is an essential component of the anisotropic fractional differentiation effects induced by the Boltzmann collision operator, which occur on a "lifted" paraboloid. This metric encodes the anisotropic changes in the power of the weight, which are entangled with the fractional differentiation effects.

It is known that $L \ge 0$ and $Lg = 0$ if and only if $g = \mathbf{P}g$. Our anisotropic space then sharply characterizes the Dirichlet form of the linearized collision operator as

$$\frac{1}{C}|\{\mathbf{I}-\mathbf{P}\}g|^2_{N^{s,\gamma}} \le \langle Lg, g\rangle \le C|\{\mathbf{I}-\mathbf{P}\}g|^2_{N^{s,\gamma}},$$

with a constructive constant $C > 0$. Above $\langle\cdot,\cdot\rangle$ is the standard $L^2(\mathbb{R}^{\dim}_v)$ inner product. It follows that a spectral gap exists if and only if $\gamma+2s \ge 0$. Furthermore

Theorem 7 ([4]). *The diffusive behavior of the operator* (2) *in* $L^2(\mathbb{R}^{\dim}_v)$ *is*

$$-\langle \mathcal{Q}(g,f), f\rangle \approx |f|^2_{\dot{N}^{s,\gamma}} + |f|^2_{L^2_\gamma} - l.o.t.$$

Furthermore, the entropy production satisfies the estimate

$$D(g,f) \gtrsim |\sqrt{f}|^2_{\dot{N}^{s,\gamma}} + |f|^2_{L^1_\gamma} - l.o.t.$$

In each statement above $g \ge\ne 0$ is a parameter function. The precise assumptions needed are in [4]. *Also the lower order terms in "l.o.t." are non-differentiating.*

Since this is a three page report, we state our results without complete precision although we illustrate the main conclusions in detail. Otherwise we refer to the precise statements in [2, 3, 4, 5]. In the following we use H^K_ℓ to denote a weighted L^2 Sobolev space with K space-velocity derivatives and ℓ velocity weights.

Theorem 8 ([3], [5]). *Fix $K \ge 2\lfloor \frac{n}{2}+1\rfloor$ and $\ell \ge 0$. Suppose* (3) *and* (4). *Choose initially $f_0(x,v) \in H^K_\ell(\Omega\times\mathbb{R}^n)$ in* (5). *If $\|f_0\|_{H^K_\ell}$ is sufficiently small, then there exists a unique global classical solution to the Boltzmann equation* (1), *in the form* (5). *If $\gamma+2s \ge 0$ then, for some fixed $\lambda > 0$, we have exponential decay as*

$$\|f(t)\|_{H^K_\ell(\mathbb{T}^n\times\mathbb{R}^{\dim})} \lesssim e^{-\lambda t}\|f_0\|_{H^K_\ell(\mathbb{T}^n\times\mathbb{R}^{\dim})}.$$

When $\gamma < -2s$, if $\|f_0\|_{H^K_{\ell+m}}$ is sufficiently small for $\ell, m \ge 0$, then we have

$$\|f(t)\|_{H^K_\ell(\mathbb{T}^n\times\mathbb{R}^n)} \le C_m(1+t)^{-m}\|f_0\|_{H^K_{\ell+m}(\mathbb{T}^n\times\mathbb{R}^n)}.$$

We also have positivity, i.e. $F = \mu + \sqrt{\mu}f \ge 0$ if $F_0 = \mu+\sqrt{\mu}f_0 \ge 0$.

Theorem 8 provides a global existence theorem on Ω, and also shows rapid time decay on the torus $\mathbb{T}^n$ with $n \geq 2$. In the whole space, $\mathbb{R}^n$ with $n \geq 3$, the presence of dispersion slows down the decay rates. For $r \geq 1$, we define the mixed norm $\|g\|_{Z_r} \stackrel{\text{def}}{=} \left(\int_{\mathbb{R}^{\dim}} \left(\int_{\mathbb{R}^{\dim}} |g(x,v)|^r dx\right)^{2/r} dv\right)^{1/2}$. For $\ell \geq 0$, further set $\epsilon_{K,\ell} \stackrel{\text{def}}{=} \|f_0\|_{H^K_\ell} + \|f_0\|^2_{Z_1}$. We then have time decay rates in the Z_r norm:

Theorem 9 ([5])**.** *Let $f(t,x,v)$ be the solution to the Cauchy problem, $\Omega = \mathbb{R}^n$, of the Boltzmann equation from Theorem 8. Suppose $\epsilon_{K,\ell'(n)}$ is sufficiently small, where $\ell'(n) > 0$ is fixed. Then for any $2 \leq r \leq \infty$, we have the uniform estimate:*

$$\|f(t)\|_{Z_r} \lesssim (1+t)^{-\frac{n}{2}+\frac{n}{2r}}. \tag{7}$$

Furthermore $\|\{\mathbf{I}-\mathbf{P}\}f(t)\|_{Z_r} \lesssim (1+t)^{-\frac{n}{2}-\frac{1}{r}+\frac{n}{2r}}$. These hold for any $t \geq 0$.

These time decay rates for the Z_r-norms in (7) are optimal in the sense that they are the same as those for the linearized system, which is studied using Fourier analysis. These rates also coincide with those of the Boltzmann equation for hard-sphere particles, and they are the same in $L^r(\mathbb{R}^{\dim}_x)$ as those for the Heat equation.

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Free path length distributions of a Lorentz gas in a quasi crystal

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The Lorentz gas is a mathematical model for the motion of (point) particles in e.g. a crystal, consisting of spherical, elastic scatterers of radius a with centers at a set of points $\Gamma_\varepsilon \subset \mathbb{R}^n$. A point particle moves in straight lines between the obstacles, on which it is specularly reflected. At least two very different point distributions, Γ_ε have been studied thoroughly: the standard lattice $\mathcal{L} \subset \mathbb{R}^n$, with interstitial distance ε, or a random distribution, where Γ_ε is Poisson distributed with intensity ε^{-n} (almost; one must assume a hard core condition, preventing the obstacles from overlapping).

Here we are interested in the so-called Boltzmann-Grad limit, in which one lets $a \to 0$ and $\varepsilon \to 0$ in such a way that $a\varepsilon^{-n}$ is constant, and the question is: