RIGOROUS RESULTS IN FLUID AND KINETIC MODELS

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Dedicated to my parents, Pinki and Janak Patel, for providing me the foundation to pursue my dreams in life.

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ABSTRACT

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In the following, we will consider two different physical systems and their respective PDE models. In the first chapter, we prove time decay of solutions to the Muskat equation, which describes a fluid interface between two incompressible, immiscible fluids with different densities. In [13] and [14], the authors introduce the norms

$$\|f\|_s \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} |\xi|^s |\hat{f}(\xi)| \, d\xi$$

in order to prove global existence of solutions to the Muskat problem. In this paper, for the 3D Muskat problem, given initial data $f_0 \in H^l(\mathbb{R}^2)$ for some $l \geq 3$ such that $\|f_0\|_1 < k_0$ for a constant $k_0 \approx 1/5$, we prove uniform in time bounds of $\|f\|_s(t)$ for $-2 < s < l-1$ and assuming $\|f_0\|_\nu < \infty$ we prove time decay estimates of the form $\|f\|_s(t) \lesssim (1+t)^{-s+\nu}$ for $0 \leq s \leq l-1$ and $-2 \leq \nu < s$. These large time decay rates are the same as the optimal rate for the linear Muskat equation. We prove analogous results in 2D.

In the remaining chapters, we consider sufficient conditions, called continuation criteria, for global existence and uniqueness of classical solutions to the three-

dimensional relativistic Vlasov-Maxwell system. In the compact momentum support setting, we prove that $\|p_0^{\frac{18}{5r}-1+\beta}f\|_{L^\infty_t L^r_x L^1_p} \lesssim 1$ where $1 \leq r \leq 2$ and $\beta > 0$ is arbitrarily small, is a continuation criteria. The previously best known continuation criteria in the compact setting is $\|p_0^{\frac{4}{r}-1+\beta}f\|_{L^\infty_t L^r_x L^1_p} \lesssim 1$, where $1 \leq r < \infty$ and $\beta > 0$ is arbitrarily small, due to Kunze [36]. Our continuation criteria is an improvement in the $1 \leq r \leq 2$ range. We also consider sufficient conditions for a global existence result to the three-dimensional relativistic Vlasov-Maxwell system without compact support in momentum space. In Luk-Strain [38], it was shown that $\|p_0^{\theta}f\|_{L^1_x L^1_p} \lesssim 1$ is a continuation criteria for the relativistic Vlasov-Maxwell system without compact support in momentum space for $\theta > 5$. We improve this result to $\theta > 3$. We also build on another result by Luk-Strain in [37], in which the authors proved the existence of a global classical solution in the compact regime if there exists a fixed two-dimensional plane on which the momentum support of the particle density remains bounded. We prove well-posedness even if the plane varies continuously in time.

Contents

Chapter 1

Muskat Problem

1.1 Introduction

The Muskat problem describes the dynamics between two incompressible immiscible fluids in porous media such that the fluids are of different constant densities. The Muskat problem is an extensively studied well established problem [2–5, 8–14, 16–24, 27, 30–32, 39, 42, 43]. In this paper we consider the interface between the two fluids under the assumption that there is no surface tension and the fluids are of the same constant viscosity. Because the fluids are immiscible, we can assume that we have a sharp interface between the two fluids. Without loss of generality we normalize gravity $g = 1$, permeability $\kappa = 1$ and viscosity $\nu = 1$. Then the 3D Muskat problem is given by

$$\rho_t + \nabla \cdot (u\rho) = 0 \tag{1.1.1}$$

$$u + \nabla P = -(0, 0, \rho) \tag{1.1.2}$$

$$\nabla \cdot u = 0 \tag{1.1.3}$$

where $\rho = \rho(x_1, x_2, x_3, t)$ is the fluid density function, $P = P(x_1, x_2, x_3, t)$ is the pressure, and $u = (u_1(x_1, x_2, x_3, t), u_2(x_1, x_2, x_3, t), u_3(x_1, x_2, x_3, t))$ is the incompressible velocity field. Here $x_i \in \mathbb{R}$ for $i = 1, 2, 3$ and $t \geq 0$. The third equation of this system simply states that the fluids are incompressible. Given the incompressibility condition, the first equation is a conservation of mass equation, as the fluid density is preserved along the characteristic curves given by the velocity field. The second equation is called Darcy's Law, which governs the flow of a fluid through porous medium. When we assume that the two incompressible fluids are of constant density, then the function $\rho(x_1, x_2, x_3, t)$ can be written as

$$\rho(x_1, x_2, x_3, t) = \begin{cases} \rho^1 & (x_1, x_2, x_3) \in \Omega^1(t) = \{x_3 > f(x_1, x_2, t)\} \\ \rho^2 & (x_1, x_2, x_3) \in \Omega^2(t) = \{x_3 < f(x_1, x_2, t)\} \end{cases}$$

where Ω^i for $i = 1, 2$ are the regions in $\mathbb{R}^3$ occupied by the fluids of density ρ^i for $i = 1, 2$ respectively and the equation $x_3 = f(x_1, x_2, t)$ describes the interface between the two fluids. We consider the stable regime (see [20]) in which $\rho_1 < \rho_2$.

The interface function, $f : \mathbb{R}^2_x \times \mathbb{R}^+_t \to \mathbb{R}$ is known to satisfy the equation

$$\frac{\partial f}{\partial t}(x,t) = \frac{\rho^2 - \rho^1}{4\pi} PV \int_{\mathbb{R}^2} \frac{(\nabla f(x,t) - \nabla f(x-y,t)) \cdot y}{[|y|^2 + (f(x,t) - f(x-y,t))^2)]^{\frac{3}{2}}} \, dy \tag{1.1.4}$$

with initial data $f(x,0) = f_0(x)$ for $x = (x_1, x_2) \in \mathbb{R}^2$. Without loss of generality for the results in this paper we can take $\frac{\rho_2 - \rho_1}{2} = 1$. Then, as given in [14], the 3D Muskat interface equation can be written as

$$f_t(x,t) = -\Lambda f - N(f), \tag{1.1.5}$$

where Λ is the square root of the negative Laplacian and

$$N(f)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y}{|y|^2} \cdot \nabla_x \triangle_y f(x) R(\triangle_y f(x)) \, dy, \tag{1.1.6}$$

where

$$R(t) = 1 - \frac{1}{(1+t^2)^{\frac{3}{2}}}$$

and

$$\triangle_y f(x) = \frac{f(x) - f(x-y)}{|y|}.$$

We will use equation (1.1.5) to prove uniform in time norm bounds and large time decay rates for the solution $f(t,x)$ in 3D.

We will also prove uniform in time norm bounds and large time decay rates for the 2D Muskat problem. The 2D Muskat problem is given by the interface equation

$$\frac{\partial f}{\partial t}(x,t) = \frac{\rho^2-\rho^1}{4\pi}\int_{\mathbb{R}} \frac{(\nabla f(x,t)-\nabla f(x-\alpha,t))\alpha}{\alpha^2+(f(x,t)-f(x-\alpha,t))^2}\, d\alpha \tag{1.1.7}$$

with initial data $f(x,0) = f_0(x)$ for $x \in \mathbb{R}$. The density function ρ is given by

$$\rho(x_1,x_2,t) = \begin{cases} \rho^1 & (x_1,x_2) \in \Omega^1(t) = \{x_2 > f(x_1,t)\} \\ \rho^2 & (x_1,x_2) \in \Omega^2(t) = \{x_2 < f(x_1,t)\} \end{cases}.$$

Similarly to above, we can rewrite the 2D interface equation setting $\frac{\rho_2-\rho_1}{2} = 1$, as given by (9) in [13]

$$f_t(x,t) = -\Lambda f - T(f) \tag{1.1.8}$$

where

$$T(f) = \frac{1}{\pi}\int_{\mathbb{R}} \triangle_\alpha \partial_x f(x) \frac{(\triangle_\alpha f(x))^2}{1+(\triangle_\alpha f(x))^2}\, d\alpha \tag{1.1.9}$$

and

$$\triangle_\alpha f(x) = \frac{f(x)-f(x-\alpha)}{\alpha}.$$

The equations (1.1.5) and (1.1.8) will be the relevant formulations of the interface equation that we will use in this paper.

1.1.1 Notation

Typically we have for the dimension that $d \in \{1, 2\}$. Then we consider the following norm introduced in [13]:

$$\|f\|_s \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} |\xi|^s |\hat{f}(\xi)| \, d\xi, \tag{1.1.10}$$

where $\hat{f}$ is the standard Fourier transform of f:

$$\hat{f}(\xi) \stackrel{\text{def}}{=} \mathcal{F}[f](\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx.$$

We will use this norm generally for $s > -d$ and we refer to it as the *s-norm*. To further study the case $s = -d$, then for $s \geq -d$ we define the *Besov-type s-norm*:

$$\|f\|_{s,\infty} \stackrel{\text{def}}{=} \left\| \int_{C_j} |\xi|^s |\hat{f}(\xi)| \, d\xi \right\|_{l_j^\infty} = \sup_{j \in \mathbb{Z}} \int_{C_j} |\xi|^s |\hat{f}(\xi)| \, d\xi, \tag{1.1.11}$$

where $C_j = \{\xi \in \mathbb{R}^d : 2^{j-1} \leq |\xi| < 2^j\}$. Note that we have the inequality

$$\|f\|_{s,\infty} \leq \int_{\mathbb{R}^d} |\xi|^s |\hat{f}(\xi)| \, d\xi = \|f\|_s. \tag{1.1.12}$$

We point out that $\|f\|_{-d/p,\infty} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$ for $p \in [1, 2]$ as is shown in Lemma 5. This and other embeddings are established in Section 1.2.1.

Next, consider the operator $|\nabla|^r$ defined for $r \in \mathbb{R}$ by

$$\widehat{|\nabla|^r f}(\xi) = |\xi|^r \hat{f}(\xi).$$

The Sobolev norms on the homogeneous Sovolev spaces $\dot{W}^{r,p}(\mathbb{R}^d)$ and inhomogeneous Sobolev spaces $W^{r,p}(\mathbb{R}^d)$ for $r \in \mathbb{R}$ and $1 \leq p \leq \infty$ are given by:

$$\|f\|_{\dot{W}^{r,p}} = \||\nabla|^r f\|_{L^p(\mathbb{R}^d)}. \tag{1.1.13}$$

and

$$\|f\|_{W^{r,p}} = \|(1+|\nabla|^2)^{\frac{r}{2}} f\|_{L^p(\mathbb{R}^d)}. \tag{1.1.14}$$

In the special case $p=2$, we write $W^{r,2}(\mathbb{R}^d) = H^r(\mathbb{R}^d)$ and $\dot{W}^{r,2}(\mathbb{R}^d) = \dot{H}^r(\mathbb{R}^d)$. We define the convolution of two functions as usual as

$$(f * g)(x) = \int_{\mathbb{R}^d} f(y)g(x-y)dy.$$

We adopt the following convention for an iterated convolutions of the same function

$$*^n f \stackrel{\text{def}}{=} f * f * \cdots * f$$

where the left-hand side of the above is a convolution of the function f n times. This notation will be useful in some of the estimates.

Finally, we use the notation $f_1 \lesssim f_2$ if there exists a uniform constant $C > 0$, that does not depend upon time, such that $f_1 \leq C f_2$. Also $f_1 \approx f_2$ means that $f_1 \lesssim f_2$ and $f_2 \lesssim f_1$.

1.1.2 Main Results

Given a well-defined fluid interface that exists globally in time, we will study its long-time behavior. We will first use a known well-posedness theory to establish a setting in which to study long-time behavior. In Theorem 3.1, [14] has the following global existence result in 3D:

Theorem 1. *Suppose that $f_0 \in H^l(\mathbb{R}^2)$, for some $l \geq 3$, and $\|f_0\|_1 < k_0$ where $k_0 > 0$ satisfies for some $0 < \delta < 1$ that*

$$\pi \sum_{n \geq 1} (2n+1)^{1+\delta} \frac{(2n+1)!}{(2^n n!\,)^2} k_0^{2n} \leq 1. \tag{1.1.15}$$

Then there exists a unique solution f of (1.1.4) with initial data f_0. Furthermore $f \in C([0,T]; H^l(\mathbb{R}^2))$ for any $T > 0$. Any $0 \leq k_0 \leq \frac{1}{5}$ satisfies (1.1.15) for some $\delta > 0$.

In the proof of Theorem 1, the authors [14] show that $\|f\|_1$ is uniformly bounded in time. They first bound $\mathscr{F}(N(f)) = \widehat{N(f)}$ as follows:

$$\int_{\mathbb{R}^2} |\xi||\mathscr{F}(N(f))| \ d\xi \leq \pi\Big(\frac{1+2\|f\|_1^2}{(1-\|f\|_1^2)^{\frac{5}{2}}} - 1\Big)\|f\|_2.$$

Then using the inequality

$$\frac{d}{dt}\|f\|_1(t) \leq -\int_{\mathbb{R}^2} d\xi \ |\xi|^2|\hat{f}(\xi)| + \int_{\mathbb{R}^2} d\xi \ |\xi||\mathscr{F}(N(f))(\xi)|, \tag{1.1.16}$$

it is shown that

$$\frac{d}{dt}\|f\|_1(t) \leq \Big(\pi\Big(\frac{1+2\|f\|_1^2}{(1-\|f\|_1^2)^{\frac{5}{2}}} - 1\Big) - 1\Big)\|f\|_2.$$

Further since

$$\pi\Big(\frac{1+2k_0^2}{(1-k_0^2)^{\frac{5}{2}}} - 1\Big) - 1 < 0,$$

it is seen for some $C_0 = C_0(\|f_0\|_1) > 0$ that

$$\frac{d}{dt}\|f\|_1(t) \leq -C_0\|f\|_2. \tag{1.1.17}$$

In particular it holds for all $t \geq 0$ that $\|f\|_1(t) \leq \|f_0\|_1 < k_0$.

Related existence results can be shown in 2D [14]:

Theorem 2. [13, 14] *If* $f_0 \in H^l(\mathbb{R})$ *for some* $l \geq 2$ *and* $\|f_0\|_1 < c_0$ *where* c_0 *satisfies*

$$2 \sum_{n \geq 1} (2n+1)^{1+\delta} c_0^{2n} \leq 1 \tag{1.1.18}$$

for some $0 < \delta < \frac{1}{2}$, *then there exists a unique global in time solution* f *of the Muskat problem* (1.1.7) *in 2D with initial data* f_0 *such that* $f \in C([0,T]; H^l(\mathbb{R}))$ *for any* $T > 0$. *Further* (1.1.18) *holds if for example* $0 \leq c_0 \leq 1/3$.

Analogously, in the course of the proof of the 2D existence Theorem 2, it is shown that

$$\frac{d}{dt}\|f\|_1(t) \leq -\beta \|f\|_2(t), \tag{1.1.19}$$

for a constant $\beta > 0$ depending on the c_0 and $\|f_0\|_{H^2(\mathbb{R}^d)}$. These differential inequalities (1.1.17) and (1.1.19) will be very useful for proving the time decay rates.

In this paper, we prove time-decay rates for solutions to the Muskat problem. For simplicity we will state our main theorem so that it holds in either dimension $d \in \{1, 2\}$. We consider a solution to the Muskat problem satisfying all of the assumptions of Theorem 1 (when $d = 2$) or Theorem 2 (when $d = 1$).

Theorem 3. *Suppose* f *is the solution to the Muskat problem either described by Theorem 1 in 3D* (1.1.4) *, or described by Theorem 2 in 2D* (1.1.7). *In this case the initial data satisfies* $f_0 \in H^l(\mathbb{R}^d)$ *for some* $l \geq 1 + d$.

Then, for $-d < s < l-1$, we have the uniform in time estimate

$$\|f\|_s(t) \lesssim 1. \tag{1.1.20}$$

In addition for $0 \leq s < l-1$ we have the uniform time decay estimate

$$\|f\|_s(t) \lesssim (1+t)^{-s+\nu}, \tag{1.1.21}$$

where we allow ν to satisfy $-d \leq \nu < s$.

For (1.1.21)*, when $\nu > -d$ then we require additionally that $\|f_0\|_\nu < \infty$, and when $\nu = -d$ then we alternatively require $\|f_0\|_{-d,\infty} < \infty$. The implicit constants in* (1.1.20) *and* (1.1.21) *depend on $\|f_0\|_s < \infty$ and k_0. In* (1.1.21) *the implicit constant further depends on either $\|f_0\|_\nu$ (when $\nu > -d$) or $\|f_0\|_{-d,\infty}$ (when $\nu = -d$).*

It can be directly seen from the proof that for (1.1.21), when $\nu > -d$ then one only needs to assume $\|f_0\|_{s,\infty} < \infty$ instead of the stronger condition $\|f_0\|_s < \infty$. Also note that it is shown in Proposition 14 that $\|f\|_{-d,\infty}(t) \lesssim 1$ and we more generally have $\|f\|_{s,\infty}(t) \lesssim 1$ for $\nu \geq -d$ from (1.1.12).

The Muskat problem (1.1.4) or (1.1.7) can be linearized around the flat solution, which can be taken as $f(x,t) = 0$, to find the following linearized nonlocal partial differential equation

$$
\begin{aligned}
f_t(x,t) &= -\frac{\rho^2-\rho^1}{2}\Lambda f(x,t), \\
f(\alpha,0) &= f_0(\alpha), \quad \alpha\in\mathbb{R}.
\end{aligned}
\tag{1.1.22}
$$

Here the operator Λ is defined in Fourier variables by $\widehat{\Lambda f}(\xi) = |\xi|\widehat{f}(\xi)$. This linearization shows the parabolic character of the Muskat problem in the stable case which is $\rho^2 > \rho^1$ ([20]).

Notice that the decay rates which we obtain in Theorem 3 are consistent with the optimal large time decay rates for (1.1.22). In particular it can be shown by standard methods that if $g_0(x)$ is a tempered distribution vanishing at infinity and satisfying $\|g_0\|_{\nu,\infty} < \infty$, then one further has

$$
\|g_0\|_{\nu,\infty} \approx \left\| t^{s-\nu} \left\| e^{t\Lambda} g_0 \right\|_s \right\|_{L^\infty_t((0,\infty))}, \quad \text{for any } s\geq \nu.
$$

This equivalence then grants the optimal time decay rate of $t^{-s+\nu}$ for $\left\| e^{t\Lambda} g_0\right\|_s$ that is the same as the non-linear time decay in (1.1.21).

Previously in 2009 in [21] has shown that the Muskat problem satisfies a maximum principle $\|f\|_{L^\infty}(t) \leq \|f_0\|_{L^\infty}$; decay rates are obtained for the periodic case ($x\in\mathbb{T}^d$) as:

$$
\|f\|_{L^\infty(\mathbb{T}^d)}(t) \leq \|f_0\|_{L^\infty(\mathbb{T}^d)} e^{-(\rho_2-\rho_1)C(\|f_0\|_{L^\infty(\mathbb{T}^d)})t},
$$

where the mean zero condition is used. In the whole space case (when the interface is flat at infinity) then again in [21] decay rates are obtained of the form

$$\|f\|_{L^\infty(\mathbb{R}^d)}(t) \le \|f_0\|_{L^\infty(\mathbb{R}^d)}\big(1 + (\rho_2 - \rho_1)C(\|f_0\|_{L^\infty(\mathbb{R}^d)}, \|f_0\|_{L^1(\mathbb{R}^d)})t\big)^{-d}.$$

To prove this time decay in $\mathbb{R}^d$ they suppose that initially either $f_0(x) \ge 0$ or $f_0(x) \le 0$. Notice that by the Hausdorff-Young inequality then (1.1.21) also proves this $L^\infty(\mathbb{R}^d)$ decay rate of t^{-d} under the condition $\|f_0\|_{-d,\infty} < \infty$.

Furthermore [14], it is shown that if $\|\nabla f_0\|_{L^\infty(\mathbb{R}^2)} < 1/3$ then the solution of (1.1.4) with initial data f_0 satisfies the uniform in time bound $\|\nabla f\|_{L^\infty(\mathbb{R}^2)}(t) < 1/3$. Note that (1.1.21) implies in particular when $d = 2$ that

$$\|\nabla f\|_{L^\infty_x} \lesssim \||\xi||\hat{f}|\|_{L^1_\xi} = \|f\|_1 \lesssim (1+t)^{-3}. \tag{1.1.23}$$

However decay estimate (1.1.23) requires $\|f_0\|_1 < k_0$ and $\|f_0\|_{-2,\infty} < \infty$, which is a stronger assumption than $\|\nabla f_0\|_{L^\infty} < 1/3$.

We further obtain the following corollary directly from the Hausdorff-Young inequality; this is explained in the embedding result (1.2.7) below.

Corollary 4. *Suppose f is the solution to the Muskat problem either described by Theorem 1 in 3D* (1.1.4) *, or described by Theorem 2 in 2D* (1.1.7)*. In this case the initial data satisfies $f_0 \in H^l(\mathbb{R}^d)$ for some $l \ge 1 + d$.*

Then, for $-d < s < l-1$, we have the uniform in time estimate

$$\|f\|_{\dot{W}^{s,\infty}}(t) \lesssim 1. \tag{1.1.24}$$

In addition for $0 \leq s < l-1$ we have the uniform time decay estimate

$$\|f\|_{\dot{W}^{s,\infty}}(t) \lesssim (1+t)^{-s+\nu}, \tag{1.1.25}$$

where we allow ν to satisfy $-d \leq \nu < s$.

For (1.1.25)*, when $\nu > -d$ then we require additionally that $\|f_0\|_{\nu} < \infty$, and when $\nu = -d$ then we alternatively require $\|f_0\|_{-d,\infty} < \infty$. The implicit constant in* (1.1.24) *and* (1.1.25) *depend on $\|f_0\|_s < \infty$ and k_0. In* (1.1.25) *the implicit constant further depends on either $\|f_0\|_{\nu}$ (when $\nu > -d$) or $\|f_0\|_{-d,\infty}$ (when $\nu = -d$).*

Thus, under the assumptions of Theorem 3, defining $\nabla^\alpha f \stackrel{\text{def}}{=} \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2} f$ where $\alpha = (\alpha_1, \alpha_2)$ and $|\alpha| = \alpha_1 + \alpha_2$, we know that, up to order $|\alpha| < l-1$, the derivatives $\|\nabla^\alpha f\|_{L^\infty}$ decay in time with the optimal linear decay rate. For comparison to the decay estimates of Corollary 4 in the 2D problem, [16] prove the decay estimate

$$\|f\|_{\dot{W}^{2,\infty}}(t) \leq \frac{\|f_0\|_{\dot{W}^{2,\infty}}}{1 + \frac{\|f_0\|_{\dot{W}^{2,\infty}}}{100B} t}$$

under the assumptions that $f_0 \in L^2(\mathbb{R})$, $\|f_0\|_{\dot{W}^{1,\infty}} \leq B \leq \frac{1}{C_*}$ for a universal large constant C_* and $f_0 \in \dot{W}^{2,\infty}$.

Strategy of proof

We first explain the 3D Muskat problem. Our strategy of this proof is two-fold. We will first prove uniform bounds on $\|f\|_s$ for $-d < s < 2$ and $\|f\|_{s,\infty}$ for $-d \leq s < 2$ including $s = -d$. Then afterwards we use these uniform bounds to prove the large time decay for $0 \leq s < l - 1$.

To this end we prove an embedding lemma, which allows us to bound $\|f\|_s$ for $-1 < s < 2$ as

$$\|f\|_s \lesssim \|f\|_{H^3}.$$

Since our interface solution $f(x,t)$ is uniformly bounded under the H^l Sobolev norm for some $l \geq 3$, we obtain uniform bounds on $\|f\|_s(t)$ for $-1 < s < 2$. Now, we can use (1.1.17) and the general decay Lemma 8 to obtain an initial decay result for $0 \leq s \leq 1$:

$$\|f\|_s \lesssim (1+t)^{-s+\nu}$$

where $-1 < \nu < s$ and the implicit constant depends on $\|f_0\|_\nu$. We then will make use of this decay inequality for $s = 1$ to prove uniform bounds for the range $-2 < s < 1$ as follows.

First, we need an appropriate bound on the time derivative of $\|f\|_s(t)$. To this end, we have the differential inequality

$$\frac{d}{dt}\|f\|_s(t) + C\int_{\mathbb{R}^2} d\xi \ |\xi|^{s+1}|\hat{f}(\xi)| \leq \int_{\mathbb{R}^2} d\xi \ |\xi|^s |\mathscr{F}(N(f))(\xi)|.$$

After several computations we can bound the right hand side of the inequality as

$$\int_{\mathbb{R}^2} d\xi \ |\xi|^s |\mathscr{F}(N(f))(\xi)| \lesssim \|f\|_1,$$

where the implicit constant depends on s, k_0 and $\|f_0\|_{H^3}$. We then use the time decay of $\|f\|_1(t)$ from the previous step as $\|f\|_1(t) \lesssim (1+t)^{-1+\nu}$ for $-1 < \nu < s$, to obtain after integrating in time that $\|f\|_s(t)$ is indeed uniformly bounded in time. We further a uniform bound for the case $s = -1$ by an interpolation argument.

Lastly in the endpoint case $s = -2$ we prove bounds for the norm $\|f\|_{-2,\infty}$. To accomplish this goal we prove uniform bounds on the integral over each annulus C_j.

Once we have these uniform bounds, we use the general decay Lemma 8 to obtain the decay result for $0 \leq s \leq 1$:

$$\|f\|_s(t) \lesssim (1+t)^{-s+\nu}$$

where $-2 \leq \nu < s$ and $\|f_0\|_\nu < \infty$ for $\nu > -2$ and $\|f_0\|_{-2,\infty} < \infty$ for $\nu = -2$.

Finally, to obtain time decay results for $1 < s < l-1$, we utilize the decay of the norm $\|f\|_1(t)$. We control the time derivative of $\|f\|_s(t)$:

$$\frac{d}{dt} \int_{\mathbb{R}^2} |\xi|^s |\hat{f}| \ d\xi \leq -\int_{\mathbb{R}^2} d\xi \ |\xi|^{s+1} |\hat{f}(\xi)| + \int_{\mathbb{R}^2} d\xi \ |\xi|^s |\mathscr{F}(N(f))(\xi)|.$$

Next, for suitably large times we carefully control $\int_{\mathbb{R}^2} d\xi \ |\xi|^s |\mathscr{F}(N(f))(\xi)$ relative to the negative quantity $-\int_{\mathbb{R}^2} d\xi\ |\xi|^{s+1}|\hat{f}(\xi)| = -\|f\|_{s+1}$ by using the previously established time decay rates. This enables us to establish an inequality of the form:

$$\frac{d}{dt}\|f\|_s(t) \leq -\delta\|f\|_{s+1}(t) \tag{1.1.26}$$

given $t \geq T$ for some $T > 0$. We indeed get the existence of such a time $T > 0$ by proving that

$$\int_{\mathbb{R}^2} d\xi\ |\xi|^s|\mathscr{F}(N(f))(\xi) \leq \pi\sum_{n\geq 1} a_n(2n+1)^s\|f\|_1^{2n}\|f\|_{s+1}.$$

Then due to the large time decay of $\|f\|_1(t)$, there exists a time $T > 0$ such that (1.1.26) does indeed hold. By our uniform bound on $\|f\|_{-2,\infty}$ and using the decay Lemma 8, we obtain the large time decay results for $1 < s < l-1$.

1.2 Proof of Theorem

1.2.1 Embedding Theorems

In this section, we prove embeddings for the norms $\|\cdot\|_s$ and $\|\cdot\|_{s,\infty}$. We will later use these embeddings to gain uniform control of $\|f\|_s$ over a certain range of s given by the embedding lemmas. We bound $\|\cdot\|_s$ from above by Sobolev norms because the well-posedness result of [14] is proven in a L^2-Sobolev space. We prove a more general embedding:

Lemma 5. *For $s > -\frac{d}{p}$ and $r > s + d/q$ and $p, q \in [1,2]$ we have the inequality*

$$\|f\|_s \lesssim \|f\|_{L^p(\mathbb{R}^d)}^{1-\theta} \|f\|_{\dot{W}^{r,q}(\mathbb{R}^d)}^{\theta}, \tag{1.2.1}$$

where $\theta = \frac{s+d/p}{r+d\left(\frac{1}{p}-\frac{1}{q}\right)} \in (0,1)$.

For $s = -\frac{d}{p}$ and $p \in [1,2]$ we further have the inequality

$$\|f\|_{s,\infty} \lesssim \|f\|_{L^p(\mathbb{R}^d)}. \tag{1.2.2}$$

In particular for $s = -d$ we take $p = 1$.

Remark 6. *In particular for $s > -\frac{d}{2}$ then (1.2.1) implies that*

$$\|f\|_s \lesssim \|f\|_{H^r(\mathbb{R}^d)} \quad (r > s + d/2). \tag{1.2.3}$$

For exponents $1 \le p \le 2$, $r > s + \frac{d}{p}$ and $s > -\frac{d}{p}$, we also conclude

$$\|f\|_s \lesssim \|f\|_{W^{r,p}(\mathbb{R}^d)}.$$

This follows directly from (1.2.1).

Also notice that generally for $s \in (-d, -d/2]$ in (1.2.1) we require $p \in [1, -d/s)$; in particular this does not include $p = 2$.

Remark 7. *We very briefly introduce the Littlewood-Paley operators, $\triangle_j$ for $j \in \mathbb{Z}$*

is defined on the Fourier side by

$$\widehat{\triangle_j f} = \varphi_j \hat{f} = \varphi(2^{-j}\xi)\hat{f}(\xi),$$

where $\varphi : \mathbb{R}^d \to [0,1]$ *is a standard non-zero test function which is supported inside the annulus* $\tilde{C}_1 = \{3/4 \leq |\xi| \leq 8/3\}$ *which contains the annulus* C_1 *(defined just below* (1.1.11)*). The test function* φ *is then normalized as* $\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1\ \forall \xi \neq 0$.

Proof. We use the Littlewood-Paley operators to obtain the estimate

$$\int_{\mathbb{R}^d} |\xi|^s \varphi_j(\xi)|\hat{f}(\xi)|\ d\xi \approx 2^{js} \int_{\mathbb{R}^d} \varphi_j(\xi)|\hat{f}(\xi)|\ d\xi.$$

We then apply the Bernstein inequality followed by the Hausdorff-Young inequality to $\int_{\mathbb{R}^d} \varphi_j(\xi)|\hat{f}(\xi)|\ d\xi$ to obtain for any $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$ that

$$\int_{\mathbb{R}^d} |\xi|^s \varphi_j(\xi)|\hat{f}(\xi)|\ d\xi \lesssim 2^{js} 2^{j\frac{d}{p}} \|\widehat{\triangle_j f}\|_{L^{p'}(\mathbb{R}^d)} \lesssim 2^{js} 2^{j\frac{d}{p}} \|\triangle_j f\|_{L^p(\mathbb{R}^d)}. \tag{1.2.4}$$

Next we sum (1.2.4) separately over $2^j \leq R$ and $2^j > R$ for some $R > 0$ to be chosen

$$\int_{|\xi|\leq R} |\xi|^s |\hat{f}(\xi)|\ d\xi \lesssim \sum_{2^j \leq R} 2^{js} 2^{j\frac{d}{p}} \|\triangle_j f\|_{L^p(\mathbb{R}^d)} \lesssim R^{s+d/p} \|f\|_{L^p(\mathbb{R}^d)}. \tag{1.2.5}$$

The last inequality holds for $s + d/p > 0$.

Now we sum (1.2.4) over $2^j > R$ and choose a possibly different $p = q$ to obtain

$$
\begin{aligned}
\int_{|\xi|>R} |\xi|^s |\hat{f}(\xi)| \, d\xi &\lesssim \sum_{2^j>R} 2^{js} 2^{j\frac{d}{q}} \|\triangle_j f\|_{L^q(\mathbb{R}^d)} \\
&\lesssim \left(\sum_{2^j>R} 2^{2j\left(-r+s+\frac{d}{q}\right)} \right)^{1/2} \left(\sum_{2^j>R} 2^{2jr} \|\triangle_j f\|^2_{L^q(\mathbb{R}^d)} \right)^{1/2} \\
&\lesssim R^{-r+s+d/q} \left\| \left(\sum_{2^j>R} 2^{2jr} \left|\triangle_j f\right|^2 \right)^{1/2} \right\|_{L^q(\mathbb{R}^d)} \\
&\lesssim R^{-r+s+d/q} \|f\|_{\dot{W}^{r,q}(\mathbb{R}^d)}. \qquad (1.2.6)
\end{aligned}
$$

Here we used the Bernstein inequalities and the Minkowski inequality for norms since $1 \leq q \leq 2$. We further used the Littlewood-Paley characterization of $\dot{W}^{r,q}(\mathbb{R}^d)$. This inequality holds as soon as $r > s + d/q$.

Now to establish (1.2.1), in (1.2.5) and (1.2.6) we further choose

$$R^{r+d\left(\frac{1}{p}-\frac{1}{q}\right)} = \frac{\|f\|_{\dot{W}^{r,q}(\mathbb{R}^d)}}{\|f\|_{L^p(\mathbb{R}^d)}},$$

and then add the two inequalities together.

Lastly we show (1.2.2) by choosing $s = -\frac{d}{p}$ and $p \in [1, 2]$ in (1.2.4). $\square$

We can also obtain lower bounds for the norms $\|f\|_s$ and $\|f\|_{s,\infty}$. In particular

$$\|f\|_{\dot{W}^{s,\infty}(\mathbb{R}^d)} \lesssim \|f\|_s, \qquad (1.2.7)$$

which holds for any $s > -d$. This inequality follows directly from the Hausdorff-Young inequality as $\|f\|_{\dot{W}^{s,\infty}} \leq \||\xi|^s \hat{f}(\xi)\|_{L^1_\xi} = \|f\|_s$.

We also have a lower bound given by the Besov norm:

$$\|f\|_{\dot{B}^s_{\infty,\infty}} \stackrel{\text{def}}{=} \left\| 2^{js} \|\triangle_j f\|_{L^\infty} \right\|_{l^\infty_j(\mathbb{Z})}$$

For this norm, we have the following estimate by the Hausdorff-Young inequality:

$$\|f\|_{\dot{B}^s_{\infty,\infty}} \lesssim \left\| 2^{js} \|\varphi(2^{-j}\xi)\hat{f}(\xi)\|_{L^1} \right\|_{l^\infty_j} \lesssim \|f\|_{s,\infty}. \tag{1.2.8}$$

And this holds for any $s \geq -d$ (including $s = -d$).

We point out here that one can interpolate between the $\|\cdot\|_s$ norms as

$$\|f\|_s \lesssim \|f\|^\theta_{\mu_1,\infty} \|f\|^{1-\theta}_{\mu_2,\infty}, \quad \mu_1 < s < \mu_2, \quad \theta = \frac{\mu_2 - s}{\mu_2 - \mu_1} \tag{1.2.9}$$

This inequality (1.2.9) can be seen in [45, Lemma 4.2]. We will however give a short proof of (1.2.9) for completeness. First notice that (1.2.9) and (1.1.12) imply

$$\|f\|_s \lesssim \|f\|^\theta_{\mu_1} \|f\|^{1-\theta}_{\mu_2}, \quad \mu_1 \leq s \leq \mu_2, \quad \theta = \frac{\mu_2 - s}{\mu_2 - \mu_1}.$$

These inequalities show that if we have uniform control on for example $\|f\|_1$ and $\|f\|_{-2,\infty}$, then we also have uniform bounds on $\|f\|_s$ for $-2 < s \leq 1$.

Now we prove (1.2.9). For $R > 0$ to be chosen later, using (1.2.4) we expand out

$$\|f\|_s \lesssim \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^d} |\xi|^s \varphi_j(\xi)|\hat{f}(\xi)| \, d\xi \lesssim \sum_{j\in\mathbb{Z}} 2^{js} \|\widehat{\triangle_j f}\|_{L^1(\mathbb{R}^d)} \lesssim \sum_{2^j \geq R} + \sum_{2^j < R} .$$

For the first term

$$\sum_{2^j \geq R} \lesssim \|f\|_{\mu_2,\infty} \sum_{2^j \geq R} 2^{j(s-\mu_2)} \lesssim \|f\|_{\mu_2,\infty} R^{s-\mu_2}$$

For the second term

$$\sum_{2^j < R} \lesssim \|f\|_{\mu_1,\infty} \sum_{2^j < R} 2^{j(s-\mu_1)} \lesssim \|f\|_{\mu_2,\infty} R^{s-\mu_1}.$$

Then choose $R = (\|f\|_{\mu_2,\infty}/\|f\|_{\mu_1,\infty})^{1/(\mu_2-\mu_1)}$ to establish (1.2.9).

Having established the relevant norm inequalities, we now move onto the proof of our main result.

1.2.2 Decay Lemma

In this section we now prove the general decay lemma. We will for now continue to work in $\mathbb{R}^d$ for an integer dimension $d \geq 1$. In the next sub-sections we will use the following decay lemma to prove uniform bounds and decay in the $\|\cdot\|_s$ norm. The following lemma proves a general time decay rate for solutions to the given differential inequality.

Lemma 8. *Suppose $g = g(t, x)$ is a smooth function with $g(0, x) = g_0(x)$ and assume that for some $\mu \in \mathbb{R}$, $\|g_0\|_\mu < \infty$ and $\|g(t)\|_{\nu,\infty} \leq C_0$ for some $\nu \geq -d$ satisfying $\nu < \mu$. Let the following differential inequality hold for some $C > 0$:*

$$\frac{d}{dt}\|g\|_\mu \leq -C\|g\|_{\mu+1}.$$

Then we have the uniform in time estimate

$$\|g\|_\mu(t) \lesssim (1+t)^{-\mu+\nu}.$$

Remark 9. *Note that by* (1.1.12) *we have $\|f\|_{\nu,\infty}(t) \leq \|f\|_\nu(t)$. Therefore we can use Lemma 8 if we can bound $\|f\|_\nu(t)$ for $\nu > -d$ uniformly in time.*

Proof. For some $\delta, \kappa > 0$ to be chosen, we initially observe that

$$\begin{aligned}
\|g\|_\kappa &= \int_{\mathbb{R}^d} |\xi|^\kappa |\hat{g}(\xi)| d\xi \\
&\geq \int_{|\xi|>(1+\delta t)^s} |\xi|^\kappa |\hat{g}(\xi)| d\xi \\
&\geq (1+\delta t)^{s\beta} \int_{|\xi|>(1+\delta t)^s} |\xi|^{\kappa-\beta} |\hat{g}(\xi)| d\xi \\
&= (1+\delta t)^{s\beta} \Big(\|g\|_{\kappa-\beta} - \int_{|\xi|\leq(1+\delta t)^s} |\xi|^{\kappa-\beta} |\hat{g}(\xi)| d\xi \Big)
\end{aligned}$$

Using this inequality with $\kappa = \mu + 1$ and $\beta = 1$, we obtain that

$$\frac{d}{dt}\|g\|_\mu + C(1+\delta t)^s\|g\|_\mu \leq -C\|g\|_{\mu+1} + C(1+\delta t)^s\|g\|_\mu \leq C(1+\delta t)^s \int_{|\xi|\leq(1+\delta t)^s} |\xi|^\mu |\hat{g}(\xi)| d\xi.$$

Then, using the sets C_j as in (1.1.11) and defining χ_S to be the characteristic function on a set S, the upper bound in the last inequality can be bounded as follows

$$\begin{aligned}
\int_{|\xi|\leq(1+\delta t)^s} |\xi|^\mu |\hat{g}(\xi)| d\xi &= \sum_{j\in\mathbb{Z}} \int_{C_j} \chi_{\{|\xi|\leq(1+\delta t)^s\}} |\xi|^\mu |\hat{g}| \; d\xi \\
&\approx \sum_{2^j\leq(1+\delta t)^s} \int_{C_j} |\xi|^\mu |\hat{g}| \; d\xi \\
&\lesssim \|g\|_{\nu,\infty} \sum_{2^j\leq(1+\delta t)^s} 2^{j(\mu-\nu)} \\
&\lesssim \|g\|_{\nu,\infty}(1+\delta t)^{s(\mu-\nu)} \sum_{2^j(1+\delta t)^{-s}\leq 1} 2^{j(\mu-\nu)}(1+\delta t)^{-s(\mu-\nu)} \\
&\lesssim \|g\|_{\nu,\infty}(1+\delta t)^{s(\mu-\nu)}
\end{aligned}$$

where the implicit constant in the inequalities do not depend on t. In particular we have used that the following uniform in time estimate holds

$$\sum_{2^j(1+\delta t)^{-s}\leq 1} 2^{j(\mu-\nu)}(1+\delta t)^{-s(\mu-\nu)} \lesssim 1.$$

Combining the above inequalities, we obtain that

$$\frac{d}{dt}\|g\|_\mu + C(1+\delta t)^s\|g\|_\mu \lesssim C_0(1+\delta t)^s(1+\delta t)^{s(\mu-\nu)}. \tag{1.2.10}$$

In the following estimate will use (1.2.10) with $s = -1$, we suppose $a > \mu - \nu > 0$, and we choose $\delta > 0$ such that $a\delta = C$. We then obtain that

$$\begin{aligned} \frac{d}{dt}((1+\delta t)^a \|g\|_\mu) &= (1+\delta t)^a \frac{d}{dt}\|g\|_\mu + a\delta \|g\|_\mu (1+\delta t)^{a-1} \\ &= (1+\delta t)^a \frac{d}{dt}\|g\|_\mu + C\|g\|_\mu (1+\delta t)^{a-1} \\ &\leq (1+\delta t)^a \left(\frac{d}{dt}\|g\|_\mu + C(1+\delta t)^{-1}\|g\|_\mu\right) \\ &\lesssim C_0(1+\delta t)^{a-1-(\mu-\nu)} \end{aligned}$$

Since $a > \mu - \nu$, we integrate in time to obtain that

$$(1+\delta t)^a \|g\|_\mu \lesssim \frac{C_0}{\delta}(1+\delta t)^{a-(\mu-\nu)}.$$

We conclude our proof by dividing both sides of the inequality by $(1+\delta t)^a$. $\square$

Lemma 8 shows that to prove the time decay rates claimed in Theorem 3 then it is sufficient to establish suitable differential inequalities and also to prove uniform in time bounds on the norms $\|\cdot\|_s$. Starting now we will switch our focus to only talking about the 3D case (with $d = 2$) in (1.1.4). Looking at establishing the differential inequality first, from [13, 14] we have the differential inequality (1.1.17) for $\|\cdot\|_1$. Furthermore, from [14], we also know that for $0 < \delta < 1$ and k_0 satisfying (1.1.15) that

$$\|f\|_{1+\delta}(t) + \mu \int_0^t ds\ \|f\|_{2+\delta}(s) \leq \|f_0\|_{1+\delta}$$

where $\mu > 0$ depends on $\|f_0\|_1$. It is also shown in [14] that

$$\|f\|_{H^l(\mathbb{R}^2)}(t) \leq \|f_0\|_{H^l(\mathbb{R}^2)} \exp(CP(k_0)\|f_0\|_{1+\delta}/\mu), \quad l \geq 3, \tag{1.2.11}$$

where $C > 0$ is a constant and $P(k_0)$ is a polynomial in k_0. Furthermore following the exact proof of (1.1.17) in [13, 14] one can directly observe that

$$\frac{d}{dt}\|f\|_s(t) \leq -C\|f\|_{s+1}, \quad 0 \leq s \leq 1. \tag{1.2.12}$$

We will use this differential inequality in the following to prove the time decay rates in Theorem 3. Later in the proof of Proposition 16 we will establish (1.2.12) for $1 \leq s \leq l-1$. First, we use (1.2.11), (1.2.3) and (1.2.12) to obtain uniform bounds on $\|f\|_s(t)$ in the range $-1 < s < 2$ and an initial decay result for $\|f\|_s(t)$ in the range $0 \leq s \leq 1$.

1.2.3 Initial Decay Estimates

In this subsection we will establish uniform in time bounds for $\|f\|_s(t)$ when $-1 < s < 2$ and then we use those to prove an time decay for $\|f\|_s(t)$ when $s \in [0, 1]$.

Lemma 5, in particular (1.2.3), immediately grants the following corollary.

Corollary 10. *Suppose f is the solution to the Muskat problem* (1.1.4) *in 3D described by Theorem 1. Then, for $-1 < s < 2$, we have the uniform in time estimate*

$$\|f\|_s(t) \lesssim 1.$$

Here the implicit constant depends upon $\|f_0\|_{H^3}$.

Proof. By (1.2.11), we know that $\|f\|_{H^3}(t)$ is uniformly bounded in time since from (1.2.1) we have $\|f_0\|_{1+\delta} \lesssim \|f_0\|_{H^3}$. Further directly from (1.2.1) we have that

$$\|f\|_s(t) \lesssim \|f\|_{H^3}(t) \lesssim 1$$

holds uniformly in time for $-1 < s < 2$. □

We now apply the decay Lemma 8 to the special case $\mu = s \in [0, 1]$, then using also Corollary 10 we obtain

Proposition 11. *Suppose f is the solution to the Muskat problem* (1.1.4) *in 3D described by Theorem 1. Then for $s \in [0, 1]$ we have the uniform in time estimate*

$$\|f\|_s \lesssim (1+t)^{-s+\nu}, \tag{1.2.13}$$

for any $-1 < \nu < s$; above the implicit constant depends on $\|f_0\|_\nu$.

Having established some decay of the interface, we will now be able to use the decay for the specific case $s = 1$ to prove uniform bounds for $-2 < s \leq -1$.

1.2.4 Uniform Bounds for $-2 < s \leq -1$

For the 3D Muskat problem (1.1.4), we will use the time decay estimate (1.2.13) to prove uniform in time bounds for $\|f\|_s$ for $-2 < s \leq -1$. First, we establish the following estimate

Proposition 12. *Suppose f is the solution to the Muskat problem* (1.1.4) *in 3D described by Theorem 1 with $\|f_0\|_s < \infty$ for some $-2 < s < -1$. Then*

$$\frac{d}{dt}\|f\|_s(t) \lesssim \|f\|_1, \tag{1.2.14}$$

where the implicit constant depends on s, k_0 and $\|f_0\|_{H^3}$.

Proof. Following the computation of the time derivative of $\|f\|_1(t)$ in the proof of Theorem 3.1 in [13], we can prove that

$$\frac{d}{dt}\|f\|_s(t) + C\int_{\mathbb{R}^2} d\xi\ |\xi|^{s+1}|\hat{f}(\xi)| \leq \int_{\mathbb{R}^2} d\xi\ |\xi|^s|\mathscr{F}(N(f))(\xi)|. \tag{1.2.15}$$

We can bound $|\mathscr{F}(N(f))(\xi)|$ as in [14] to get the bound:

$$\begin{aligned}\int_{\mathbb{R}^2} d\xi\ |\xi|^s|\mathscr{F}(N(f))(\xi)|& \\ &\leq \pi\sum_{n\geq 1} a_n \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2}|\xi|^s|\xi-\xi_1||\hat{f}(\xi-\xi_1)| \\ &\times\prod_{j=1}^{2n-1}|\xi_j-\xi_{j+1}||\hat{f}(\xi_j-\xi_{j+1})||\xi_{2n}||\hat{f}(\xi_{2n})|d\xi d\xi_1\ldots d\xi_{2n} \quad (1.2.16)\end{aligned}$$

where

$$a_n=\frac{(2n+1)!}{(2^n n!\,)^2}.$$

Given a function g, define the corresponding function $\tilde{g}$ by $\tilde{g}(x)=g(-x)$. Then, since $|x-y|=|y-x|$ for any $x,y\in\mathbb{R}^2$, we obtain:

$$\begin{aligned}\int_{\mathbb{R}^2} d\xi\ |\xi|^s|\mathscr{F}(N(f))(\xi)|& \\ &\leq \pi\sum_{n\geq 1} a_n \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\cdots\int_{\mathbb{R}^2}|\xi|^s|\xi_1-\xi||\tilde{\hat{f}}(\xi_1-\xi)| \\ &\times\prod_{j=1}^{2n-1}|\xi_{j+1}-\xi_j||\tilde{\hat{f}}(\xi_{j+1}-\xi_j)||\xi_{2n}||\hat{f}(\xi_{2n})|d\xi d\xi_1\ldots d\xi_{2n}. \quad (1.2.17)\end{aligned}$$

Hence, writing the right hand side in terms of convolutions, we obtain that

$$\int_{\mathbb{R}^2} d\xi\ |\xi|^s|\mathscr{F}(N(f))(\xi)|\leq \pi\sum_{n\geq 1} a_n\int_{\mathbb{R}^2}|\xi_{2n}||\hat{f}(\xi_{2n})|\Big(|\cdot|^s*\Big(*^{2n}\,|\cdot||\tilde{\hat{f}}(\cdot)|\Big)\Big)(\xi_{2n})d\xi_{2n}$$

Applying Holder's inequality:

$$\int_{\mathbb{R}^2} |\xi_{2n}||\hat{f}(\xi_{2n})|\Big(|\cdot|^s *(*^{2n}|\cdot||\tilde{\hat{f}}(\cdot)|)\Big)(\xi_{2n})d\xi_{2n}$$

$$\leq \||\cdot||\hat{f}(\cdot)|\|_{L^1}\||\cdot|^s *(*^{2n}|\cdot||\tilde{\hat{f}}(\cdot)|)\|_{L^\infty} \quad (1.2.18)$$

The first term on the right hand side of (1.2.18) is exactly $\|f\|_1$ which is bounded. The second term can be controlled first by Young's inequality with $\frac{1}{p}+\frac{1}{q}=1$:

$$\||\cdot|^s *(*^{2n}|\cdot||\tilde{\hat{f}}(\cdot)|)\|_{L^\infty} \leq \||\cdot|^s *(*^{2n-1}|\cdot||\tilde{\hat{f}}(\cdot)|)\|_{L^q}\||\cdot||\tilde{\hat{f}}(\cdot)|\|_{L^p} \quad (1.2.19)$$

where we choose $q \in (2,\infty)$ such that $\frac{1}{q}=\frac{-s-1}{2}$. Thus, $p \in (1,2)$, so we use interpolation to obtain for $\frac{\theta}{1}+\frac{1-\theta}{2}=\frac{1}{p}$ that

$$\||\cdot||\tilde{\hat{f}}(\cdot)|\|_{L^p} = \||\cdot||\hat{f}(\cdot)|\|_{L^p} \leq \||\cdot||\hat{f}(\cdot)|\|_{L^1}^{\theta}\||\cdot||\hat{f}(\cdot)|\|_{L^2}^{1-\theta} \leq \|f\|_1^{\theta}\|f\|_{H^3}^{1-\theta}. \quad (1.2.20)$$

We control the other term by the Hardy-Littlewood-Sobolev inequality since $q \in (2,\infty)$:

$$\||\cdot|^s *(*^{2n-1}|\cdot||\tilde{\hat{f}}(\cdot)|)\|_{L^q} \lesssim \|*^{2n-1}|\cdot||\tilde{\hat{f}}(\cdot)|\|_{L^2} \quad (1.2.21)$$

since $-2 < s < -1$ and our choice of q enables the equality $1+\frac{1}{q}=-\frac{s}{2}+\frac{1}{2}$. Finally we use Young's inequality with $1+\frac{1}{2}=1+\frac{1}{2}$ repeatedly to control the $2n-1$ convolutions and get the bound

$$\|*^{2n-1}|\cdot||\tilde{\hat{f}}(\cdot)|\|_{L^2}\leq \||\cdot||\tilde{\hat{f}}(\cdot)|\|_{L^2}\||\cdot||\tilde{\hat{f}}(\cdot)|\|_{L^1}^{2n-2}\leq \|f\|_{H^3}\|f\|_1^{2n-2}, \tag{1.2.22}$$

where we have used the inequality:

$$\||\xi||\hat{f}|\|_{L^2_\xi}\leq \|(1+|\xi|^2)^{\frac{3}{2}}|\hat{f}|\|_{L^2_\xi}= \|f\|_{H^3}$$

Combining the above estimates from (1.2.18), (1.2.19), (4.2.7), (1.2.21) and (1.2.22), we obtain the following bound

$$\int_{\mathbb{R}^2}|\xi_{2n}||\hat{f}(\xi_{2n})|\Big(|\cdot|^s*(*^{2n}|\cdot||\tilde{\hat{f}}(\cdot)|)\Big)(\xi_{2n})d\xi_{2n} \lesssim \|f\|_{H^3}^{2-\theta}\|f\|_1^{2n-1+\theta}.$$

Summing over all n, we get from (1.2.16) that

$$\begin{aligned}\int_{\mathbb{R}^2} d\xi\ |\xi|^s|\mathscr{F}(N(f))(\xi)|&\lesssim \|f\|_{H^3}^{2-\theta}\|f\|_1^{\theta}\sum_{n\geq 1} a_n\|f\|_1^{2n-1}\\ &\lesssim \|f\|_{H^3}^{2-\theta}\|f\|_1^{\theta}\sum_{n\geq 0} a_{n+1}\|f\|_1^{2n+1}. \end{aligned}\tag{1.2.23}$$

By Theorem 3.1 in [14], $\|f\|_1\leq \|f_0\|_1< k_0$. Further, given this bound on $\|f\|_1$, the above series converges. Then by (1.2.11) we also know that $\|f\|_{H^3(\mathbb{R}^2)}(t) \lesssim 1$ uniformly in time. Hence the following uniform bounds hold independently of n

$$
\begin{aligned}
\int_{\mathbb{R}^2} d\xi \ |\xi|^s |\mathscr{F}(N(f))(\xi)| &\lesssim \sum_{n\geq 0} a_{n+1} \|f\|_1^{2n+1} \\
&\lesssim \|f\|_1 \sum_{n\geq 0} a_{n+1} \|f\|_1^{2n} \\
&\lesssim \|f\|_1 \sum_{n\geq 0} a_{n+1} k_0^{2n} \lesssim \\
&\|f\|_1.
\end{aligned}
$$

Here the uniform constant depends on s and k_0. $\square$

Combining Proposition 12 and (1.2.13) we obtain for example

$$\frac{d}{dt}\|f\|_s(t) \lesssim (1+t)^{-1-\epsilon},$$

for a small $\epsilon > 0$. Then we integrate this to obtain

$$\|f\|_s(t) \lesssim \|f_0\|_s + 1 + (1+t)^{-\epsilon}$$

We conclude that $\|f\|_s \lesssim 1$ uniformly in time for $-2 < s < -1$.

In order to obtain the uniform bound for $s=-1$ we observe that

$$\begin{aligned}
\|f\|_{-1} &= \int_{\mathbb{R}^2} |\xi|^{-1}|\hat{f}(\xi)|\ d\xi \qquad (1.2.24)\\
&= \int_{|\xi|\le 1} |\xi|^{-1}|\hat{f}(\xi)|\ d\xi + \int_{|\xi|>1} |\xi|^{-1}|\hat{f}(\xi)|\ d\xi\\
&\le \int_{|\xi|\le 1} |\xi|^{-2+\gamma}|\hat{f}(\xi)|\ d\xi + \int_{|\xi|>1} |\xi||\hat{f}(\xi)|\ d\xi\\
&\le \|f\|_{-2+\gamma} + \|f\|_1 \lesssim 1,
\end{aligned}$$

where $0<\gamma<1$. Hence, we have uniform in time bounds for $\|f\|_s$ for any $-2<s\le -1$. We can now use Lemma 8 to conclude the time decay

$$\|f\|_{\mu}(t) \lesssim (1+t)^{-\mu+\nu}, \qquad (1.2.25)$$

which holds for any $\mu\in[0,1]$ and any $\nu\in(-2,\mu)$ where the implicit constant in particular depends on $\|f\|_{\nu}$ and k_0. We summarize this in the following proposition.

Proposition 13. *Suppose f is the solution to the Muskat problem in 3D described in Theorem 1. Then we have uniformly for $-2<s\le -1$ the following estimate*

$$\|f\|_s(t) \lesssim 1,$$

where the implicit constant depends on k_0 and $\|f_0\|_s < \infty$. And the decay estimate (1.2.25) holds.

This establishes uniform bounds for a larger range of s. We will now prove bounds on the endpoint case.

1.2.5 Endpoint Case

To prove the uniform bounds for the endpoint case $s = -2$, we use the Besov-type norm from (1.1.11) which we recall as

$$\|f\|_{-2,\infty} = \left\| \int_{C_j} |\xi|^{-2} |\hat{f}(\xi)| d\xi \right\|_{l_j^\infty},$$

where we further recall the annulus $C_j = \{2^{j-1} \leq |\xi| < 2^j\}$.

Proposition 14. *Let f be the unique solution to the Muskat problem in 3D from Theorem 1. Then the following estimate holds uniformly in time*

$$\|f\|_{-2,\infty}(t) \lesssim 1, \tag{1.2.26}$$

where the implicit constant depends on $\|f_0\|_{-2,\infty} < \infty$ and k_0.

Proof. To control this endpoint norm, we uniformly bound the integral over C_j for each $j \in \mathbb{Z}$. Analogous to the proof of (1.2.15) from [13, Theorem 3.1], we can use the same exact argument to show that

$$\frac{d}{dt} \int_{C_j} |\xi|^{-2} |\hat{f}(\xi)| d\xi + C \int_{C_j} d\xi \ |\xi|^{-1} |\hat{f}(\xi)| \leq \int_{C_j} d\xi \ |\xi|^{-2} |\mathscr{F}(N(f))(\xi)| \tag{1.2.27}$$

Note that on the annulus C_j the term $|\xi|^{-2}$ is bounded above and below. Next, we control the term on the right hand side using analogous estimates on the integrand as we did for (1.2.15), the difference is that now we control $|\xi|^{-2}$ by the inner radius of the annulus:

$$\int_{C_j} d\xi\ |\xi|^{-2}|\mathscr{F}(N(f))(\xi)|$$
$$\leq 2^{-2j+2}\pi \sum_{n\geq 1} a_n \int_{C_j} d\xi \int_{\mathbb{R}^2} d\xi_1 \cdots \int_{\mathbb{R}^2} d\xi_{2n}\ |\xi-\xi_1||\hat{f}(\xi-\xi_1)|$$
$$\times \prod_{j=1}^{2n-1} |\xi_j-\xi_{j+1}||\hat{f}(\xi_j-\xi_{j+1})||\xi_{2n}||\hat{f}(\xi_{2n})|.$$

Writing this integral in terms of convolutions, we obtain:

$$\int_{C_j} d\xi\ |\xi|^{-2}|\mathscr{F}(N(f))(\xi)| \leq 2^{-2j+2}\pi \sum_{n\geq 1} a_n \int_{C_j} \Big(*^{2n+1}\ |\cdot||\hat{f}(\cdot)|\Big)(\xi)\ d\xi.$$

Next, we obtain:

$$\int_{C_j} d\xi\ |\xi|^{-2}|\mathscr{F}(N(f))(\xi)| \leq 4\pi \sum_{n\geq 1} a_n \| *^{2n+1} |\cdot||\hat{f}(\cdot)|\|_{L^\infty},$$

since the size of the annulus C_j can be bounded above by 2^{2j}. Next by using Young's inequality, first with $1+\frac{1}{\infty}=\frac{1}{2}+\frac{1}{2}$ and then with $1+\frac{1}{2}=1+\frac{1}{2}$, we obtain:

$$\begin{aligned}\int_{C_j} d\xi\ |\xi|^{-2}|\mathscr{F}(N(f))(\xi)| &\le 4\pi\sum_{n\ge 1} a_n \||\cdot||\hat{f}(\cdot)|\|_{L^2}^2 \||\cdot||\hat{f}(\cdot)|\|_{L^1}^{2n-1}\\ &\le 4\pi\|f\|_{H^3}^2\sum_{n\ge 1} a_n\|f\|_1^{2n-1}\\ &\le 4\pi\|f\|_{H^3}^2\sum_{n\ge 0} a_{n+1}\|f\|_1^{2n+1}\end{aligned}$$

Since $\|f\|_1\le k_0$, we obtain that:

$$\int_{C_j} d\xi\ |\xi|^{-2}|\mathscr{F}(N(f))(\xi)|\le 4\pi\|f\|_{H^3}^2\|f\|_1\sum_{n\ge 0} a_{n+1}k_0^{2n}.$$

By the uniform bound on $\|f\|_{H^3}$ and since the series $\sum_{n\ge 0} a_{n+1}k_0^{2n}$ converges, we conclude that we have the following uniform in j estimate

$$\int_{C_j} d\xi\ |\xi|^{-2}|\mathscr{F}(N(f))(\xi)|\lesssim \|f\|_1.$$

Finally, since for example $\|f\|_1\lesssim (1+t)^{-\frac{3}{2}}$ by (1.2.13), we see that

$$\int_{C_j} d\xi\ |\xi|^{-2}|\hat{f}(\xi)|\lesssim (1+t)^{-\frac{3}{2}}$$

for a uniform constant which is independent of j. We then integrate (1.2.27) in time to conclude that we have the uniform in time bound (1.2.26). □

The uniform bound on this endpoint case allows us to prove stronger decay of

$\|f\|_s(t)$ for $0 \leq s \leq 1$ by using Lemma 8. We will also now prove decay for the case $1 < s < l - 1$ using the decay of the norm $\|f\|_1(t)$.

1.2.6 General Decay Estimates

Finally, we obtain the main time decay estimates for the Muskat equation in 3D. Using (1.1.17) and (1.2.26), we can apply Lemma 8 to obtain

Corollary 15. *For the solution f to the Muskat problem in 3D described in Theorem 1, we have the following uniform in time decay estimate:*

$$\|f\|_s(t) \lesssim (1+t)^{-s+\nu}, \tag{1.2.28}$$

where we allow s to satisfy $0 \leq s \leq 1$ and we allow ν to satisfy $-2 \leq \nu < s$.

When $\nu > -2$ then we require additionally that $\|f_0\|_\nu < \infty$, and when $\nu = -2$ then we alternatively require $\|f_0\|_{-2,\infty} < \infty$. The implicit constant in (1.2.28) depends on either $\|f_0\|_\nu$ (when $\nu > -2$) or $\|f_0\|_{-2,\infty}$ (when $\nu = -2$), $\|f_0\|_s$ and k_0.

This corollary is Theorem 3 in 3D for $0 \leq s \leq 1$. To establish Theorem 3 in 3D in the case $s > 1$, we further make the following observation:

Proposition 16. *Suppose s satisfies $1 < s < l - 1$ and $f_0 \in H^l(\mathbb{R}^2)$ for some $l \geq 3$. If $\|f_0\|_1 < k_0$ and $\|f_0\|_s < \infty$, then for the solution f described in Theorem 1, we have* *the decay estimates*

$$\|f\|_s(t) \lesssim (1+t)^{-s+\nu} \tag{1.2.29}$$

where we allow ν to satisfy $-2 \leq \nu < s$.

When $\nu > -2$ then we require additionally that $\|f_0\|_\nu < \infty$, and when $\nu = -2$ then we alternatively require $\|f_0\|_{-2,\infty} < \infty$. The implicit constant in (1.2.29) *depends on either $\|f_0\|_\nu$ (when $\nu > -2$) or $\|f_0\|_{-2,\infty}$ (when $\nu = -2$), $\|f_0\|_s$ and k_0.*

Proof. First, as in (1.2.15), we have the following inequality

$$\frac{d}{dt}\int_{\mathbb{R}^2} |\xi|^s |\hat{f}| \ d\xi \leq -\int_{\mathbb{R}^2} d\xi \ |\xi|^{s+1} |\hat{f}(\xi)| + \int_{\mathbb{R}^2} d\xi \ |\xi|^s |\mathscr{F}(N(f))(\xi)|. \tag{1.2.30}$$

Next, following the arguments of [14] and [13], we directly obtain that

$$\begin{aligned}\int_{\mathbb{R}^2} d\xi \ |\xi|^s |\mathscr{F}(N(f))(\xi)| & \\ \leq \pi \sum_{n\geq} a_n \int_{\mathbb{R}^2} d\xi \int_{\mathbb{R}^2} d\xi_1 \cdots \int_{\mathbb{R}^2} d\xi_{2n} |\xi|^s |\xi - \xi_1| |\hat{f}(\xi - \xi_1)| & \\ \times \prod_{j=1}^{2n-1} |\xi_j - \xi_{j+1}| |\hat{f}(\xi_j - \xi_{j+1})| |\xi_{2n}| |\hat{f}(\xi_{2n})|. &\end{aligned} \tag{1.2.31}$$

We use the inequality for $s > 1$

$$|\xi|^s \leq (2n+1)^{s-1}(|\xi - \xi_1|^s + |\xi_1 - \xi_2|^s + \ldots + |\xi_{2n-1} - \xi_{2n}|^s + |\xi_{2n}|^s). \tag{1.2.32}$$

Applying (1.2.32) and Young's inequality to (1.2.31), it can be shown that

$$\int_{\mathbb{R}^2} d\xi\ |\xi|^s |\mathscr{F}(N(f))(\xi)| \leq \pi \sum_{n\geq 1} a_n (2n+1)^s \|f\|_1^{2n} \int_{\mathbb{R}^2} d\xi\ |\xi|^{s+1} |\hat{f}(\xi)|. \tag{1.2.33}$$

Hence, by (1.2.30) have that

$$\frac{d}{dt}\|f\|_s(t) \leq -\delta(t)\|f\|_{s+1}(t) \tag{1.2.34}$$

where

$$\delta(t) = 1 - \pi \sum_{n\geq 1} a_n (2n+1)^s \|f\|_1(t)^{2n}.$$

By Theorem 3, we know that if $\|f_0\|_1 < k_0$, then (1.2.28) holds.

Thus, there exists some $T > 0$ such that $\|f\|_1(T)$ is small enough such that $\delta(T) > \delta > 0$ for some constant $\delta > 0$. Since $\|f\|_1(t) \leq \|f\|_1(T)$ for $t \geq T$, we know that $\delta(t) > \delta(T) > \delta > 0$. Thus,

$$\frac{d}{dt}\|f\|_s(t) \leq -\delta\|f\|_{s+1}(t) \tag{1.2.35}$$

for all $t \geq T$. Now, consider the interface function f_T defined by $f_T = f(t+T)$ defined for $t \geq T$. Then, f_T satisfies the interface equation (1.1.4) with initial condition $f_T(0) = f(T)$. For the case $\nu > -2$, since $\|f_0\|_\nu < \infty$, we know by Corollary 10 and Proposition 13 that $\|f_T(0)\|_\nu = \|f\|_\nu(T) < \infty$ and $\|f_T\|_\nu \lesssim 1$ uniformly in time. For the case $\nu = 2$, since $\|f_0\|_{-2,\infty} < \infty$, we know by Proposition 14 that $\|f_T(0)\|_{-2,\infty} = \|f\|_{-2,\infty}(T) < \infty$ and $\|f_T\|_{-2,\infty} \lesssim 1$ uniformly in time. Further, by (1.2.35),

$$\frac{d}{dt}\|f_T\|_s(t) \leq -\delta\|f_T\|_{s+1}(t).$$

Hence, we can apply Lemma 8 to f_T to obtain the decay:

$$\|f_T\|_s(t) \leq \gamma(1+t)^{-s+\nu}. \tag{1.2.36}$$

Since $f(t) = f_T(t-T)$, we have the following decay estimate for $t \geq T$,

$$\|f\|_s(t) \leq \gamma(1+t-T)^{-s+\nu} \leq \gamma(1+T)^{s+\nu}(1+t)^{-s+\nu}.$$

Further, for $0 \leq t \leq T$:

$$\|f\|_s(t) \leq \|f\|_{H^l}(t) \leq C_l$$

where $C_l = \|f_0\|_{H^l}\exp(CP(k_0)\|f_0\|_{1+\delta}/\mu)$ is the constant given by (1.2.11). Collecting these last few estimates establishes the result. $\square$

We have now established the decay results for the 3D Muskat problem. Similar results can be summarized for the 2D problem as well.

1.3 A Note on the 2D Problem

In this last section, we will sketch the proof of the large time decay results for the 2D Muskat problem (1.1.7) given in Theorem 3 when $d = 1$. The proof is analogous to the 3D case that was just shown.

To prove the decay, we will first establish the uniform bounds of the relevant norms. Firstly from (1.2.3) we obtain the uniform in time bound

$$\|f\|_s(t) \lesssim \|f\|_{H^2(\mathbb{R})}(t) \lesssim 1, \tag{1.3.1}$$

where in the above inequality we can allow $-\frac{1}{2} < s < \frac{3}{2}$. From the argument in [14] analogous to (1.2.11), it can be shown that for any $0 < \delta < \frac{1}{2}$ we have

$$\|f\|_{H^l(\mathbb{R})}(t) \leq \|f_0\|_{H^l(\mathbb{R})} \exp(CP(c_0)\|f_0\|_{1+\delta}), \quad l \geq 2. \tag{1.3.2}$$

Then the uniform bound of $\|f\|_{H^2(\mathbb{R})}(t) \lesssim 1$ follows from (1.3.2) using the embedding (1.2.3) as in (1.3.1) on the norm $\|f_0\|_{1+\delta}$.

Following the proof of (1.1.19) that is given in [13] it can be directly shown that

$$\frac{d}{dt}\|f\|_s(t) \leq -C\|f\|_{s+1}, \quad 0 \leq s \leq 1. \tag{1.3.3}$$

Now using Lemma 8, (1.3.3) for $\mu = s \in [0,1]$ and (1.3.1) we obtain

$$\|f\|_s \lesssim (1+t)^{-s+\nu}, \tag{1.3.4}$$

for any $-\frac{1}{2} < \nu < s$; here the implicit constant depends on $\|f_0\|_\nu$.

The next step is to obtain uniform in time bounds for $\|f\|_s(t)$ when $-1 < s \leq -\frac{1}{2}$.

Proposition 17. *Suppose f is the solution to the Muskat problem* (1.1.7) *in 2D described by Theorem 2 with $\|f_0\|_s < \infty$ for some $-1 < s < -\frac{1}{2}$. Then*

$$\frac{d}{dt}\|f\|_s(t) \lesssim \|f\|_1, \tag{1.3.5}$$

where the implicit constant depends on s, c_0 and $\|f_0\|_{H^2(\mathbb{R})}$.

Proof. The proof follows similarly to the proof of (1.2.14). The range of s allowed is different due to range of acceptable exponents allowed by the Hardy-Littlewood-Sobolev inequality in one dimension.

Similarly to the proof in the 3D case, we have

$$\frac{d}{dt}\|f\|_s(t) + \int_{\mathbb{R}} d\xi\ |\xi|^{s+1}|\hat{f}(\xi)| \leq \int_{\mathbb{R}} d\xi\ |\xi|^s |\mathscr{F}(N(f))(\xi)|.$$

From the proof of Theorem 3.1 in [13], we obtain the inequality:

$$\int_{\mathbb{R}} d\xi\ |\xi|^s |\mathscr{F}(N(f))(\xi)| \leq 2\sum_{n\geq 1}\int_{\mathbb{R}} |\xi_{2n}||\hat{f}(\xi_{2n})| \Big(|\cdot|^s * \Big(*^{2n}\, |\cdot||\tilde{\hat{f}}(\cdot)| \Big)\Big)(\xi_{2n}) d\xi_{2n}.$$

From here, we can apply the Hardy-Littlewood-Sobolev inequality and Young's inequality to obtain, as in the 3D case, that

$$\begin{aligned}\int_{\mathbb{R}} d\xi\ |\xi|^s |\mathscr{F}(N(f))(\xi)| &\leq 2C_s \||\xi|\hat{f}(\xi)\|_{L^2}^{2-\theta} \|f\|_1^{\theta} \sum_{n\geq 1} \|f\|_1^{2n-1} \\ &= 2C_s \|f\|_{H^2}^{2-\theta} \|f\|_1^{1+\theta} \sum_{n\geq 0} \|f\|_1^{2n},\end{aligned}$$

where we used the fact that $\||\xi|\hat{f}(\xi)\|_{L^2} \leq \|f\|_{H^2(\mathbb{R})}$ by Plancharel's identity. By the uniform bounds on $\|f\|_{H^2(\mathbb{R})}$ and $\|f\|_1$, we obtain the result. $\square$

Then using (1.3.5) combined with (1.3.4), analogous to Proposition 13 we obtain

$$\|f\|_s(t) \lesssim 1, \tag{1.3.6}$$

which now holds uniformly in time for $-1 < s \leq \frac{3}{2}$. The uniform bound when $s = -\frac{1}{2}$ is obtained using the argument from (1.2.24). Further analogous to (1.2.25) using (1.3.3) and Lemma 8 we conclude the time decay

$$\|f\|_s(t) \lesssim (1+t)^{-s+\nu}, \tag{1.3.7}$$

which now holds for any $s \in [0, 1]$ and any $\nu \in (-1, s)$ where the implicit constant in particular depends on $\|f\|_\nu$ and c_0.

Lastly, for the critical case, analogous to Proposition 14 using (1.3.7) we can show

$$\|f\|_{-1,\infty}(t) \lesssim 1,$$

where the implicit constant depends on $\|f_0\|_{-1,\infty} < \infty$ and c_0. This bound enables us to analogously prove the end point decay rate of (1.3.7) with $\nu = -1$. Also the 2D version of Proposition 16 follows similarly. Collecting all of these estimates establishes Theorem 3 in the 2D case. **Q.E.D.**

Chapter 2

Relavistic Vlasov-Maxwell System: An Introduction

Systems of partial differential equations from kinetic theory have been widely studied and are used to model behaviors of gases and plasmas. Some examples of these systems include the Boltzmann equation, the Vlasov-Poisson system, the Vlasov-Maxwell system, Vlasov-Fokker-Planck equation, etc. The relativistic Vlasov-Maxwell system is a system of coupled nonlinear PDE describing the evolution of the particle density and internal electromagnetic fields of a collisionless plasma. It remains a longstanding major open problem to prove global existence of unique classical solutions to the relativistic Vlasov-Maxwell (rVM) system. The existence of a unique local solution on some sufficiently small time interval $[0, T]$ is known. However, by imposing additional assumptions, i.e. *continuation criteria*, one can extend the local solution uniquely and continuously to a larger time interval of existence, $[0, T+\epsilon]$. The goal is to weaken the continuation criteria to the known conserved quantities of the rVM system. In the first half of this thesis, we prove new global wellposedness criteria. We first present the precise statement of the rVM system.

Consider a distribution of charged particles described by a non-negative density function $f : \mathbb{R}_t \times \mathbb{R}^3_x \times \mathbb{R}^3_p \to \mathbb{R}_+$ of time t, space x and momentum p. The Vlasov-Maxwell system describes the evolution of the density function $f(t,x,p)$ under the influence of time-dependent vector fields $E, B : \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{R}^3$. Physically, this system models the behavior of a collisionless plasma:

$$\partial_t f + \hat{p} \cdot \nabla_x f + (E + \hat{p} \times B) \cdot \nabla_p f = 0, \tag{2.0.1}$$

$$\partial_t E = \nabla_x \times B - j, \quad \partial_t B = -\nabla_x \times E, \tag{2.0.2}$$

$$\nabla_x \cdot E = \rho, \quad \nabla_x \cdot B = 0. \tag{2.0.3}$$

Here the charge is

$$\rho(t,x) \stackrel{\text{def}}{=} 4\pi \int_{\mathbb{R}^3} f(t,x,p) dp,$$

and the current is given by

$$j_i(t,x) \stackrel{\text{def}}{=} 4\pi \int_{\mathbb{R}^3} \hat{p}_i f(t,x,p) dp, \quad i = 1, ..., 3$$

with initial data $(f, E, B)|_{t=0}= (f_0, E_0, B_0)$ satisfying the time-independent equations (3). In the above equations, $\hat{p} = \frac{p}{p_0}$ where $p_0 = (1 + |p|^2)^{\frac{1}{2}}$.

2.1 Notation

In this section, we describe the notation that will be utilized in the analysis of the rVM system. For a scalar function, $f = f(t, x, p)$, and real numbers $1 \leq s, r, q \leq \infty$ we define the following norm:

$$\|f\|_{L^s([0,T);L^r_x L^q_p)} \stackrel{\text{def}}{=} \left(\int_0^T \left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |f|^q dp \right)^{\frac{r}{q}} dx \right)^{\frac{s}{r}} dt \right)^{\frac{1}{s}}.$$

Next, we define $K = E + \hat{p} \times B$ and note that $|K|\leq |E|+|B|$ since $|\hat{p}|= 1$. Using the notation from [36], we define:

$$\sigma_{-1}(t, x) \stackrel{\text{def}}{=} \sup_{|\omega|=1} \int_{\mathbb{R}^3} \frac{f(t, x, p)}{p_0(1 + \hat{p} \cdot \omega)} dp.$$

Also, for use in the case where f_0 has compact support in the momentum variable, we define

$$P(t) \stackrel{\text{def}}{=} 2 + \sup\{p \in \mathbb{R}^3 | \; \exists x \in \mathbb{R}^3, s \in [0, t] \text{ such that } f(s, x, p) \neq 0\}. \qquad (2.1.1)$$

The notation $a \lesssim b$ means that there exists some positive inessential constant, C, such that $a \leq Cb$ and $a \approx b$ means that $\frac{1}{C} b \leq a \leq Cb$.

Next, define the integral over the space-time cone $C_{t,x}$ as follows:

$$\int_{C_{t,x}} f d\sigma \stackrel{\text{def}}{=} \int_0^t \int_0^{2\pi} \int_0^{\pi} (t-s)^2 \sin(\theta) f(s, x + (t-s)\omega) d\theta d\phi ds \tag{2.1.2}$$

in which $\omega = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\phi))$.

Finally, for a plane $Q \subset \mathbb{R}^3$ containing the origin we define the projection $\mathbb{P}_Q$ to be the orthogonal projection onto the plane Q.

2.2 Conservation Laws

By the method of characteristics, we obtain that the particle density is conserved over the characteristics described by the system of ordinary differential equations:

$$\frac{dX}{ds}(s; t, x, p) = \hat{V}(s; t, x, p), \tag{2.2.1}$$

$$\frac{dV}{ds}(s; t, x, p) = E(s, X(s; t, x, p)) + \hat{V}(s; t, x, p) \times B(s, X(s; t, x, p)), \tag{2.2.2}$$

together with the conditions

$$X(t; t, x, p) = x, \quad V(t; t, x, p) = p, \tag{2.2.3}$$

where $\hat{V} \stackrel{\text{def}}{=} \frac{V}{\sqrt{1+|V|^2}}$. Further, we also have the conservation laws:

Proposition 18. *Suppose* (f, E, B) *is a solution to the relativistic Vlasov-Maxwell system. Then we have the following conservation laws:*

$$\frac{1}{2}\int_{\{t\}\times\mathbb{R}^3}(|E|^2+|B|^2)dx + 4\pi\int_{\{t\}\times\mathbb{R}^3\times\mathbb{R}^3} p_0 f(t,x,p)dxdp = constant \qquad (2.2.4)$$

and

$$\|f\|_{L^\infty_{x,p}}(t) \leq \|f_0\|_{L^\infty_{x,p}}. \qquad (2.2.5)$$

Note that by interpolation, the conservation laws above imply that $\|f\|_{L^q_{x,p}}(t) \lesssim \|f_0\|_{L^q_{x,p}}$ for $1 \leq q \leq \infty$, where

$$\|g\|_{L^q_{x,p}}(t) \stackrel{\text{def}}{=} \left(\int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_p}|g(x,p)|^q dpdx\right)^{\frac{1}{q}}.$$

Thus, given sufficiently nice initial data, we can assume L^2 bounds on K and L^q bounds on f for $1 \leq q \leq \infty$.

2.3 Global Wellposedness

Luk-Strain [37] stated the following version of the Glassey-Strauss result in [29] in the case where f_0 is compactly supported in momentum space:

Theorem 19. *Consider initial data* (f_0, E_0, B_0) *where* $f_0 \in H^5(\mathbb{R}^3_x \times \mathbb{R}^3_p)$ *is non-negative and has compact support in* (x, p)*, and* $E_0, B_0 \in H^5(\mathbb{R}^3_x)$ *such that (2.0.3)* *holds. Suppose (f, E, B) is the unique classical solution to the relativistic Vlasov-Maxwell system (2.0.1)-(2.0.3) in the time interval $[0, T)$ and there exists a bounded continuous function $P : [0, T) \to \mathbb{R}_+$ such that*

$$f(t, x, p) = 0 \text{ for } |p| \geq P(t) \ \forall x \in \mathbb{R}, t \in [0, T).$$

Then our solution (f, E, B) extends uniquely in C^1 to a larger time interval $[0, T+\epsilon]$ for some $\epsilon > 0$.

Additional assumptions on the Vlasov-Maxwell system, such as the condition that $f(t, x, p) = 0$ for $|p| \geq P(t)$ $\forall x \in \mathbb{R}, t \in [0, T)$ in the above theorem, are known as continuation criteria as they allow us to extend the interval of existence of a solution. A key idea to the proof of the Glassey-Strauss criteria is the decomposition of the field terms E and B. This decomposition allowed Glassey-Strauss to bound the strength of the force, $E + \hat{p} \times B$, on the particles thereby controlling the rate of change in the size of momentum support of the particle density function f.

2.3.1 Decomposition of the Fields E and B

Let $K = E + \hat{p} \times B$. The Glassey-Strauss decomposition is $E = E_0 + E_T + E_S$, where E_0 depends only on the initial data, and E_T and E_S are:

$$E_T = -\int_{|y-x| \leq t} \int_{\mathbb{R}^3} \frac{(\omega + \hat{p})(1 - |\hat{p}|^2)}{(1 + \hat{p} \cdot \omega)^2} f(t - |y - x|, y, p) dp \frac{dy}{|y - x|^2} \tag{2.3.1}$$

$$E_S = -\int_{|y-x|\leq t} \int_{\mathbb{R}^3} \nabla_p \Big(\frac{(\omega+\hat{p})}{1+\hat{p}\cdot\omega} \Big) \cdot K f dp \frac{dy}{|y-x|} \tag{2.3.2}$$

The Glassey-Strauss decomposition for the magnetic field $B = B_0 + B_T + B_S$ is similar. Writing $K_0 = E_0 + \hat{p} \times B_0$, $K_T = E_T + \hat{p} \times B_T$ and $K_S = E_S + \hat{p} \times B_S$, we can write that $|K| \leq |E| + |B|$, $|K_0| \leq |E_0| + |B_0|$, $|K_T| \leq |E_T| + |B_T|$, and $|K_S| \leq |E_S| + |B_S|$, where K_0 depends only on the initial data (f_0, E_0, B_0) of the relativistic Vlasov-Maxwell system. Bounding the other terms as in Propositions 3.1 and 3.2 in [37] we obtain:

$$|K_T| \lesssim \int_{|y|\leq t} \frac{dy}{|y|^2} \int_{\mathbb{R}^3} \frac{f(t-|y|, x+y, p)}{p_0(1+\hat{p}\cdot\omega)} dp$$

and

$$|K_S| \lesssim \int_{|y|\leq t} \frac{dy}{|y|} \int_{\mathbb{R}^3} \frac{(|E|+|B|) f(t-|y|, x+y, p)}{p_0(1+\hat{p}\cdot\omega)} dp$$

Recalling the definition of σ_{-1}, we can bound these expressions by:

$$|K_T| \lesssim \int_{|y|\leq t} \frac{\sigma_{-1}(t-|y|, x+y) dy}{|y|^2} \tag{2.3.3}$$

$$|K_S| \lesssim \int_{|y|\leq t} \frac{((|E|+|B|)\sigma_{-1})(t-|y|, x+y) dy}{|y|} \tag{2.3.4}$$

Note that the right hand side of (2.3.4) is in the form of $\Box^{-1}(|K|\sigma_{-1})$, where $\Box \stackrel{\text{def}}{=} \partial_t^2 - \sum_{i=1}^{3} \partial_{x_i}^2$ and $u = \Box^{-1}F$ satisfies:

$$\Box u = F; \; u|_{t=0} = \partial_t u|_{t=0} = 0 \tag{2.3.5}$$

We will utilize this decomposition in this paper as well to bound the size of the field terms and the momentum support. However, we will also explore the case where the particle density function is not compactly supported, building on the results of Luk-Strain [38] described below.

2.3.2 The Luk-Strain Criterion

Luk-Strain [38] removed the condition of compact support in momentum space and proved continuation criteria in the space $H^D(w_3(p)^2\mathbb{R}_x^3 \times \mathbb{R}_p^3)$, which is the weighted Sobolev space defined by the norm:

$$\|f\|_{H^D(w_3(p)^2\mathbb{R}_x^3 \times \mathbb{R}_p^3)} = \sum_{0 \le k \le D} \|\left(\nabla_{x,p}^k f\right) w_3\|_{L_x^2 L_p^2}$$

where the weight is defined as $w_3(p) = p_0^{\frac{3}{2}} \log(1 + p_0)$. Luk-Strain [38] proved the following in this weighted Sobolev space:

Theorem 20. *Let $(f_0(x,p), E_0(x), B_0(x))$ be a 3D initial data set which satisfies the constraints* (2.0.3) *and such that for some $D \geq 4$, $f_0 \in H^D(w_3(p)^2\mathbb{R}_x^3 \times \mathbb{R}_p^3)$ is non-*

negative and obeys the bounds

$$\sum_{0\leq k\leq D} \| \left(\nabla^k_{x,p} f_0 \right) w_3 \|_{L^2_x L^2_p} < \infty, \tag{2.3.6}$$

$$\| \int_{\mathbb{R}^3} \sup\{ f_0(x+y, p+w) p_0^3 : \ |y| + |w| \leq R \} \, dp \|_{L^\infty_x} \leq C_R, \tag{2.3.7}$$

$$\| \int_{\mathbb{R}^3} \sup\{ |\nabla_{x,p} f_0|(x+y, p+w) p_0^3 : \ |y| + |w| \leq R \} \, dp \|_{L^\infty_x} \leq C_R, \tag{2.3.8}$$

$$\| \int_{\mathbb{R}^3} \sup\{ |\nabla_{x,p} f_0|^2 (x+y, p+w) w_3^2 : \ |y| + |w| \leq R \} \, dp \|_{L^\infty_x} \leq C_R^2, \tag{2.3.9}$$

and

$$\| \int_{\mathbb{R}^3} \sup\{ |\nabla^2_{x,p} f_0|(x+y, p+w) p_0 : \ |y| + |w| \leq R \} dp \|_{L^\infty_x} \leq C_R, \tag{2.3.10}$$

for some different constants $C_R < \infty$ for every $R > 0$; and the initial electromagnetic fields $E_0, B_0 \in H^D(\mathbb{R}^3_x)$ obey the bounds

$$\sum_{0\leq k\leq D} (\| \nabla^k_x E_0 \|_{L^2_x} + \| \nabla^k_x B_0 \|_{L^2_x}) < \infty. \tag{2.3.11}$$

Given this initial data set, there exists a unique local solution (f, E, B) on some time interval $[0, T_{loc}]$ such that $f_0 \in L^\infty([0, T_{loc}]; H^4(w_3(p)^2 \mathbb{R}^3_x \times \mathbb{R}^3_p)$ and the fields

$E, B \in L^\infty([0, T_{loc}]; H^4(\mathbb{R}^3_x))$.

Let (f, E, B) *be the unique solution to* (2.0.1)-(2.0.3) *in* $[0, T_*)$. *Assume that*

$$sup \int_0^{T_*} (|E(s, X(s; t, x, p)|+|B(s, X(s; t, x, p)|)ds < \infty \tag{2.3.12}$$

where the supremum is taken over all $(t, x, p) \in [0, T_*) \times \mathbb{R}^3 \times \mathbb{R}^3$. *Then, there exists* $\epsilon > 0$ *such that the solution extends uniquely beyond* T_* *to an interval* $[0, T_* + \epsilon]$ *such that* $E, B \in L^\infty([0, T_* + \epsilon]; H^D(\mathbb{R}^3_x))$ *and* $f \in L^\infty([0, T_* + \epsilon]; H^D(w_3(p)^2 dp\ dx))$.

We will use the Glassey-Strauss and Luk-Strain continuation criteria to prove new global well-posedness results using moment bounds.

Chapter 3

Continuation Criteria using Moment Bounds

As stated earlier, the goal is to weaken the continuation criteria to the known conserved quantities of the rVM system given by (2.2.4) and (2.2.5). The general interpolation inequality from [38] gives us a range of bounded quantities for the rVM system. We state it here in the three dimensional case with the proof from [38]:

Proposition 21 (General interpolation inequality)**.** *Consider* $g : \mathbb{R}^3_x \times \mathbb{R}^3_p \to \mathbb{R}$. *Suppose that* $1 \leq q < \infty$ *and* $M \geq S > -3$. *Then we have:*

$$\|p_0^S g\|_{L^q_x L^1_p} \lesssim \|p_0^M g\|^{\frac{S+3}{M+3}}_{L_x^{\frac{(S+3)}{M+3}q} L^1_p}.$$

Above the implied constant depends only on $\|g\|_{L^\infty_x L^\infty_p}$.

Proof. We divide the domain of integration in the $|p|$ variable into $|p| \leq R$ and $|p| > R$ for some $R \geq 1$. We can assume without loss of generality that $g \geq 0$. Since $\|g\|_{L^\infty_x L^\infty_p} < \infty$, we have

$$\int_{|p|\leq R} p_0^S g(x,p)dp \lesssim \|g\|_{L^\infty_x L^\infty_p} R^{S+3}.$$

For $|p| > R$, we have

$$\int_{|p|>R} p_0^S g(x,p)dp \leq R^{-(M-S)} \int_{\mathbb{R}^3_p} p_0^M g(x,p)dp.$$

We choose $R = (\int p_0^M g(x,p)dp)^{\frac{1}{M+3}}$ when this quantity is ≥ 1 to obtain

$$\int_{\mathbb{R}^3_p} p_0^S g(x,p)dp \lesssim \left(\int_{\mathbb{R}^3_p} p_0^M g(x,p)dp\right)^{\frac{S+3}{M+3}}.$$

Notice that this inequality is further trivially satisfied when our choice of R satisfies $R \leq 1$. We take the L^q_x of both sides above to achieve the desired inequality. $\square$

Using this interpolation inequality we get the following conserved quantities for the three dimensional rVM system:

Corollary 22. *Suppose (f, E, B) is a solution to the relativistic Vlasov-Maxwell system. Then we have the following conservation law:*

$$\|p_0^{\frac{4}{q}-3} f\|_{L^q_x L^1_p} \lesssim 1 \tag{3.0.1}$$

for $1 \leq q < \infty$.

Proof. By (2.2.4), the quantity $\|p_0 f\|_{L^1_{x,p}} \leq C$ is bounded, where C is some constant depending on the initial data. Using the general interpolation inequality, we obtain

$$\|p_0^S f\|_{L^q_x L^1_p} \lesssim \|p_0 f\|_{L^1_{x,p}} \tag{3.0.2}$$

for S such that $\frac{S+3}{4} q = 1$, i.e. $S = \frac{4}{q} - 3$. □

Due to this range of conserved quantities, it is appropriate to study continuation criteria of the form $\|p_0^\theta f\|_{L^q_x L^1_p} \lesssim 1$, i.e. L^q_x norms of *moments* $\|p_0^\theta f\|_{L^1_p}$. In the following chapter, we outline the prior criteria involving moments of the particle density function f, state our new results on these criteria and explain how the momentum support of f is controlled using moment bounds.

3.1 Previous Results

Most known continuation criteria for the 3D relativistic Vlasov-Maxwell system are in the case where the initial data, f_0, has compact support in the momentum variable, p. These results build off of the Glassey-Strauss criterion of Theorem 19. In the case of initial data without compact momentum support, the prior continuation criteria results using moment bounds are due to J. Luk and R. Strain [38]. Under the additional assumption that

$$\|p_0^N f_0\|_{L_t^\infty([0,T_*);L_x^1 L_p^1)} \lesssim C_N \tag{3.1.1}$$

for a large positive integer $N = N_\theta$ depending on θ, Luk-Strain [38] use Theorem 20 to prove that $\|p_0^\theta f\|_{L_t^\infty([0,T_*);L_x^1 L_p^1)} \lesssim 1$ is a continuation criteria for the relativistic Vlasov-Maxwell system without compact support in momentum space for $\theta > 5$. To do so, Luk-Strain [38] utilized Strichartz estimates on both the K_T and K_S bounds and interpolation inequalities. We note in this paper that we only need the initial data assumption that $\|p_0^N f_0\|_{L_t^\infty([0,T_*);L_x^1 L_p^1)} \lesssim 1$ for some $N > 5$. For comparison to the results in this paper, we state the full result in the noncompact support setting found in [38], which includes a larger range of criteria:

Theorem 23. *Consider initial data (f_0, E_0, B_0) satisfying the constraints (2.0.3) and such that for some $D \geq 4$, $f_0 \in H^D(w_3(p)^2\mathbb{R}_x^3 \times \mathbb{R}_p^3)$ is non-negative and obeys the bounds (2.3.6)-(2.3.10) for some different constants C_R for every $R > 0$; and the initial electromagnetic fields $E_0, B_0 \in H^D(\mathbb{R}_x^3)$ obeys (2.3.11). Additionally, let us assume (3.1.1) for all $N > 0$. Suppose (f, E, B) is the unique solution to (2.0.1)-(2.0.3) in the time interval $[0, T_*]$ and $\|p_0^\theta f\|_{L^\infty t([0,T_*];L_x^q L_p^1)} \lesssim 1$ for θ and q such that $\theta > 2/q$, $2 < q \leq \infty$ or $\theta > \frac{8}{q} - 3$, $1 \leq q \leq 2$. Then we can continuously extend our solution (f, E, B) uniquely to an interval $[0, T_* + \epsilon]$ as in Theorem 20.*

In the compact support setting, there have been several results on continuation criteria using moment bounds. A recent result due to Kunze [36]:

Theorem 24. *Suppose we have initial data $f_0 \in C_0^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and $E_0, B_0 \in C_0^2(\mathbb{R}^3)$ satisfying the constraints (2.0.3). Let (f, E, B) be the unique solution to (2.0.1)-(2.0.3) in the time interval $[0, T_*]$. If*

$$\|p_0^{\frac{4}{r}-1+\beta} f\|_{L^\infty([0,T_*];L_x^r L_p^1)}$$

for some $1 \leq q < \infty$ and $\beta > 0$, then we can continuously extend our solution (f, E, B) uniquely to an interval $[0, T_* + \epsilon]$.

The continuation criteria proven by Kunze improves on the criteria for the range $1 \leq q < \infty$ found in some previous results and is the benchmark for comparison to the results we prove in this paper.

Other continuation criteria using moment bounds $\|p_0 f\|_{L_x^q L_p^1}$ are $\theta = 0$, $q = \infty$ due to Sospedra-Alfonso-Illner [1]; $\theta = 0$, $q = 6$ due to Pallard [41] (note that this criteria implies the Kunze criteria in the case $q = 6$); and $\theta > 4/q$, $6 \leq q \leq \infty$ due to Pallard [40]. As noted in [38], by the general interpolation inequality, the result in [40] also yields criteria $\theta > 22/q - 3$, $1 \leq q < 6$. In the next section, we state our main results which improve on some ranges of the known moment criteria.

3.2 New Criteria

We extend the result of Luk-Strain [38] described above and in Theorem 23 to the case where $\theta > 3$ in Theorem 25 below. Note that we also remove the θ dependence of N in the moment bound, $\|p_0^N f_0\|_{L_t^\infty([0,T_*);L_x^1 L_p^1)} \lesssim C_N$, of the initial data.

Theorem 25. *Consider initial data (f_0, E_0, B_0) satisfying (2.3.6)-(2.3.11) and the additional condition that $\|p_0^{\tilde{N}} f_0\|_{L_t^\infty([0,T_*);L_x^1 L_p^1)} \lesssim C_N$ for some $\tilde{N} > 5$. Let (f, E, B) be the unique solution to $(1)-(3)$ in $[0, T)$ and assume that*

$$\|p_0^\theta f\|_{L_x^1 L_p^1}(t) \leq A(t)$$

for some $\theta > 3$ and some bounded continuous function $A : [0, T) \to \mathbb{R}_+$. Then we can extend our solution (f, E, B) uniquely to an interval $[0, T + \epsilon]$ such that $E, B \in L^\infty([0, T + \epsilon]; H^D(\mathbb{R}_x^3))$ and $f \in L^\infty([0, T + \epsilon]; H^D(w_3(p)^2 dp\ dx))$.

The key to our proof is to gain bounds on $\|p_0^N f\|_{L_t^\infty([0,T);L_x^1 L_p^1)}$ for a power of $N > 5$ since we later prove that the expression in (2.3.12) can be bounded by $\|p_0^N f\|_{L_x^1 L_p^1}$ where $N = 5 + \lambda$ for any $\lambda > 0$. As proven in Proposition 7.3 in Luk-Strain [38], we have the following standard moment estimate for $N > 0$:

$$\|p_0^N f\|_{L_t^\infty([0,T);L_x^1 L_p^1)} \lesssim \|p_0^N f_0\|_{L_x^1 L_p^1} + \|E\|_{L_t^1([0,T);L_x^{N+3})}^{N+3} + \|B\|_{L_t^1([0,T);L_x^{N+3})}^{N+3} \tag{3.2.1}$$

Assume $N > 3$. Our goal now is to bound the terms $\|E\|_{L_t^1([0,T);L_x^{N+3})}^{N+3}$ and

$\|B\|^{N+3}_{L^1_t([0,T);L^{N+3}_x)}$ on the right hand side by $\|p_0^N f\|^{\alpha}_{L^\infty_t([0,T);L^1_xL^1_p)}$ for some $\alpha < 1$. To do so, we employ the Glassey-Strauss decomposition of the field term

$$\tilde{K} \stackrel{\text{def}}{=} (E,B) = (E_0,B_0) + (E_T,B_T) + (E_S,B_S) \tag{3.2.2}$$

where E_0 and B_0 depend only on the initial data of our system. The terms on the right hand side of (3.2.2) have the same bounds as the K_T and K_S bounds in (2.3.3) and (2.3.4) respectively. To bound the K_T term, we utilize estimates for the averaging operator on the sphere and then apply the interpolation inequality used in Luk-Strain [38]. To do so, we define the operator

$$W_\alpha(h(t,x)) \stackrel{\text{def}}{=} \int_0^t s^{2-\alpha} \fint_{\mathbb{S}^2} h(t-s,x+s\omega)d\mu(\omega)ds,$$

where $\fint_X g(x)d\mu(x)$ denotes the average value of the function g over the measure space (X,μ). We can bound K_T with this operator setting $\alpha = 2$:

$$|K_T| \lesssim W_2(\sigma_{-1}).$$

Thus, using a known averaging operator estimate and interpolation inequalities, we obtain the following bound on K_T:

$$\|K_T\|^{N+3}_{L^r_t([0,T];L^{N+3}_x)} \lesssim \|p_0^{\frac{N(1+\gamma)+3+\delta}{2+\gamma}} f\|^{2+\gamma}_{L^\infty_t([0,T);L^1_xL^1_p)} \|p_0^N f\|^{1-\gamma}_{L^\infty_t([0,T);L^1_xL^1_p)}$$

for $1 \leq r \leq \infty$, $\gamma \in (0,1)$, $N > 3$ and $\delta > 0$. To bound the K_S term, we apply Strichartz estimates for the wave equation and utilize the method from Sogge [44] as used in Kunze [36]. This method requires us to use the assumption that

$$\|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^2)} \lesssim 1.$$

We apply wave equation Strichartz estimates on a partition of the interval $[0,T) = \cup_{i=1}^{k-1}[T_i, T_{i+1}]$ such that the quantity $\|\sigma_{-1}\|_{L_t^\infty([T_i,T_{i+1}];L_x^2)}$ is sufficiently small for us to use an iteration scheme to bound K_S over the interval $[0,T]$.

In Luk-Strain [38], the K_T term was bounded by using Hölder's inequality to rewrite the bound (2.3.4) in the form of a solution to the wave equation described by (2.3.5). Then, they used Strichartz estimates for the wave equation. Instead, we use a more direct approach by using averaging operator estimates. This approach also enables us to preserve the singularity in the σ_{-1} denominator, which is useful in reducing the power of p_0 in the bound of K_T. By the bounds on K_T and K_S, we obtain a bound on $\|K\|^{N+3}_{L_t^{N+3}([0,T);L_x^{N+3})}$ for some $\gamma \in (0,1)$:

$$\|K\|^{N+3}_{L_t^1([0,T);L_x^{N+3})} \lesssim 1 + \|p_0^{\frac{N(1+\gamma)+3+\delta}{2+\gamma}} f\|^{2+\gamma}_{L_t^\infty([0,T);L_x^1L_p^1)} \|p_0^N f\|^{1-\gamma}_{L_t^\infty([0,T);L_x^1L_p^1)}$$

where the implicit constant in the above inequality also depends on the quantity $\|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^2)}$.

Thus, assuming that $\|p_0^{\frac{N(1+\gamma)+3+\delta}{2+\gamma}} f\|_{L_t^\infty([0,T);L_x^1L_p^1)} \lesssim 1$, we can insert this estimate into the standard moment estimate above to gain a higher moment bound of the form $\|p_0^N f\|_{L_t^\infty([0,T);L_x^1 L_p^1)} \lesssim 1$. By an iteration of this process, we eventually arrive at the bound $\|p_0^{\hat{N}} f\|_{L_t^\infty([0,T);L_x^1 L_p^1)} \lesssim 1$ for some $\hat{N} > 5$, which proves the result of Theorem 25. Our proof of Theorem 25 relies on this new method of incrementally using lower moment bounds to gain control over slightly higher moment bounds, as compared to directly bounding all arbitrarily large moments by some fixed small moment.

Kunze [36] proves this result in his paper with the assumption of initial compact support in the momentum variable. This method allows him to save an entire power of p_0 and use a Gronwall-type inequality to bound the momentum support at time T. We do not have this extra control given by the momentum support of f and needed a wider range of bounds on the K_T term. Actually, our method for bounding K_T can give us strictly better bounds than those of Kunze [36]. In [36], for $2 \leq r < 6$, Kunze proves the bound:

$$\|K_T\|_{L_t^r([0,T];L_x^r)} \leq C_T \|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^2)}. \tag{3.2.3}$$

In comparison, we prove the bound in (4.1.6) and Proposition 37 which lowers the Lebesgue exponent of the norm on σ_{-1}:

$$\|K_T\|_{L_t^\infty L_x^{mq}} \lesssim \|\sigma_{-1}\|_{L_t^\infty L_x^q}$$

for $1 \leq m \leq 3$, $q > 3 - \frac{3}{m}$ and $\frac{3m-1}{2m} \leq q \leq \infty$. For the purposes of this problem, obtaining lower Lebesgue exponents yields better estimates because by interpolation

$$\|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^q)} \lesssim \|p_0^{2q-1+\nu} f\|_{L^1_{x,p}}$$

for some $\nu > 0$. Thus, lower powers of q yield lower powers of p_0 on the right hand side. This allows us to bound Lebesgue norms of K by lower moments (i.e. lower powers of p_0), which gives us better control on K. We also use this extra control on $\|K_T\|_{L^r_x}$ in the range $2 \leq r \leq 6$ by lower moments to help us in the case of initial data with compact support, in which we prove the following:

Theorem 26. *Consider initial data (f_0, E_0, B_0) satisfying the conditions in Theorem 19 and (f, E, B) is the unique classical solution to (2.0.1)-(2.0.3) in the interval $[0, T)$. Suppose we impose the additional assumption that*

$$\|p_0^{\frac{18}{5r}-1+\beta} f\|_{L_t^\infty L_x^r L_p^1} \lesssim 1 \tag{3.2.4}$$

for some $1 \leq r \leq 2$ and some $\beta > 0$ arbitrarily small. Then, we can continuously extend our solution (f, E, B) to an interval $[0, T + \epsilon]$ in C^1 for some $\epsilon > 0$.

The exponent of p_0 in (3.2.4) is strictly better than the exponent found in the result stated in Theorem 24 since $\frac{18}{5r} - 1 + \beta < \frac{4}{r} - 1 + \beta$. For example, if $r = 1$, our criteria is $\|p_0^{\frac{13}{5}+\beta} f\|_{L_t^\infty L_x^1 L_p^1} \lesssim 1$ which is better than the known criteria of $\|p_0^{3+\beta} f\|_{L_t^\infty L_x^1 L_p^1} \lesssim 1$ due to [36]. Similarly for the $r = 2$ case, our criteria is $\|p_0^{\frac{4}{5}+\beta} f\|_{L_t^\infty L_x^2 L_p^1} \lesssim 1$ which is better than the known criteria of $\|p_0 f\|_{L_t^\infty L_x^2 L_p^1} \lesssim 1$ due to [36].

Outline of Proof

The first step to proving Theorem 26 is to utilize the decomposition

$$|E|+|B| \leq |E_0|+|B_0|+|E_T|+|B_T|+|E_{S,1}|+|B_{S,1}|+|E_{S,2}|+|B_{S,2}| \tag{3.2.5}$$

as in Luk-Strain [37]. The advantage to this decomposition is that it allows us to utilize the conservation of the L^2 norm of $|K_g| = (|E\cdot\omega|^2+|B\cdot\omega|^2+|E-\omega\times B|^2+|B+\omega\times E|^2)^{\frac{1}{2}}$ on the space-time cone and it also reduces the power of $1+\hat{p}\cdot\omega$ in the $K_{S,1}$ term by a power of $\frac{1}{2}$. This decomposition of K_S into the two terms $K_{S,1}$ and $K_{S,2}$ allows us to gain better bounds on the K_S part of the field decomposition. We can bound each element of this decomposition with the operator W_2 or the inverse d-Alembertian $\Box^{-1}$ as follows:

Proposition 27. *We have the following estimates:*

$$|K_T| = |E_T|+|B_T| \lesssim W_2(\sigma_{-1}) \tag{3.2.6}$$

$$|K_{S,1}| = |E_{S,1}|+|B_{S,1}| \lesssim \Box^{-1}(|K|\Phi_{-1}) \tag{3.2.7}$$

$$|K_{S,2}| = |E_{S,2}|+|B_{S,2}| \lesssim (W_2(\sigma_{-1}^2))^{\frac{1}{2}} \tag{3.2.8}$$

where

$$\Phi_{-1}(t,x) \overset{def}{=} \max_{|\omega|=1} \int_{\mathbb{R}^3} \frac{f(t,x,p)dp}{p_0(1+\hat{p}\cdot\omega)^{\frac{1}{2}}}. \tag{3.2.9}$$

Proof. Following the decomposition of [37] we have that

$$(|E_T|+|B_T|)(t,x) \lesssim \int_{C_{t,x}} \int_{\mathbb{R}^3} \frac{f(s,x+(t-s)\omega,p)}{(t-s)^2 p_0(1+\hat{p}\cdot\omega)} dp d\sigma(\omega) \tag{3.2.10}$$

Using the change of variable $t-s \to s$ and writing the integral over the cone $C_{t,x}$ as an integral over spheres of radius s, we obtain:

$$\begin{aligned}(|E_T|+|B_T|)(t,x) &\lesssim \int_{C_{t,x}} \int_{\mathbb{R}^3} \frac{f(t-s,x+s\omega,p)}{s^2 p_0(1+\hat{p}\cdot\omega)} dp d\sigma(\omega) \\ &\leq \int_0^t \fint_{\mathbb{S}^2} \sigma_{-1}(t-s,x+s\omega)d\sigma(\omega) \end{aligned} \tag{3.2.11}$$

which is of the form $W_2(\sigma_{-1})$.

Next, by Proposition 3.4 in [37]:

$$\begin{aligned}(|E_{S,1}|+|B_{S,1}|)(t,x) &\lesssim \int_{C_{t,x}} \int_{\mathbb{R}^3} \frac{|B| f(s,x+(t-s)\omega,p)}{(t-s)p_0(1+\hat{p}\cdot\omega)^{\frac{1}{2}}} dp d\sigma(\omega) \\ &\lesssim \int_{C_{t,x}} \frac{|B|\Phi_{-1}(s,x+(t-s)\omega)}{t-s} d\sigma(\omega)\end{aligned}$$

But since $|B|\leq |K|$, we finally obtain:

$$(|E_{S,1}|+|B_{S,1}|)(t,x) \lesssim \int_{C_{t,x}} \frac{|K|\Phi_{-1}(s,x+(t-s)\omega)}{t-s} d\sigma(\omega) \tag{3.2.12}$$

which is (3.2.8). Recall that this is precisely the representation formula for the inhomogeneous wave equation of the form:

$$\Box u = |K|\Phi_{-1};\ u|_{t=0}= \partial_t u|_{t=0}= 0$$

Finally, our last term has the following bound from Proposition 3.4 in [37]:

$$(|E_{S,2}|+|B_{S,2}|)(t,x) \lesssim \int_{C_{t,x}} \int_{\mathbb{R}^3} \frac{|K_g| f(s,x+(t-s)\omega)}{(t-s)p_0(1+\hat{p}\cdot\omega)} dp d\sigma(\omega) \tag{3.2.13}$$

where $|K_g|^2= |E\cdot\omega|^2+|B\cdot\omega|^2+|E-\omega\times B|^2+|B+\omega\times E|^2$. Recall the conservation law $\|K_g\|_{L^2(C_{t,x})}\lesssim 1$ from Proposition 2.2 in [37] and use Hölder's inequality to obtain:

$$(|E_{S,2}|+|B_{S,2}|)(t,x) \lesssim \left(\int_{C_{t,x}} \left(\int_{\mathbb{R}^3} \frac{f(s,x+(t-s)\omega)}{(t-s)p_0(1+\hat{p}\cdot\omega)} dp \right)^2 d\sigma(\omega) \right)^{\frac{1}{2}} \tag{3.2.14}$$

Finally, using the same change of variables as in (3.2.11), we get (3.2.8). $\Box$

As in the proof of Theorem 25, we can apply averaging operator estimates to the

K_T and $K_{S,2}$ to get the bounds

$$\|K_{S,2}\|_{L_t^\infty L_x^{2mq}} \lesssim \|\sigma_{-1}\|_{L_t^\infty L_x^{2q}}$$

and

$$\|K_T\|_{L_t^\infty L_x^{mq}} \lesssim \|\sigma_{-1}\|_{L_t^\infty L_x^q}$$

where $q > 3 - \frac{3}{m}$ and $\frac{3m-1}{2m} \leq q \leq \infty$ for $1 \leq m \leq 3$. (Note that this is where we will use the improved estimate on $\|K_T\|_{L_x^r}$ in the range $2 \leq r \leq 6$, which also gives us bounds on the $K_{S,2}$ term. Specifically, in this paper, we use the exponent $r = 4 + \delta$ for some $\delta > 0$ appropriately small.) Using these estimates and using Strichartz estimates, we apply similar techniques as in the proof of Theorem 25 to the $K_{S,1}$ term to obtain bounds on K. We are given better control of the $K_{S,1}$ term because in the inequality $|K_{S,1}| \lesssim \Box^{-1}(|K|\Phi_{-1})$, Φ_{-1} has a lower power of singularity in the denominator than σ_{-1}. (We partition our time interval $[0,T]$ under the assumption that $\|\Phi_{-1}\|_{L_t^\infty([0,T];L_x^2)} \lesssim 1$, which is a weaker assumption that $\|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^2)} \lesssim 1$.)

The goal of our bound on K in this proof is not to gain bounds on higher moments, as in the proof of Theorem 25. Instead, we use an idea of Pallard [41] and bound the integral of the electric field over the characteristics by appropriate Lebesgue norms involving f and K:

$$|P(T)| \lesssim 1 + \|\sigma_{-1}\|_{L_t^\infty L_x^{3+}} + \|\sigma_{-1}|K|\ln^{\frac{1}{3}}(1+P(t))\|_{L_t^1 L_x^{\frac{3}{2}}([0,T]\times\mathbb{R}^3)}.$$

Using the bounds on $\|K\|_{L_t^\infty([0,T);L_x^r)}$ for some exponent $r > 4$ appropriately close to 4 and interpolation inequalities, we can then bound these terms by powers of $P(T)$ smaller than 1 to obtain an inequality of the form:

$$P(T) \lesssim 1 + P(T)^\gamma \ln^\lambda(P(t))$$

for some $\gamma \in [0, 1)$ and $\lambda > 0$. From here, we conclude that $P(T) \lesssim 1$.

3.3 Controlling the Momentum Support of f

In the following section, we will demonstrate how the momentum support of the particle density function is bounded by controlling the moments of f and the field K. This argument will be used specifically in the case without compact momentum support. In the case of compact momentum support, we will use a modified argument described later in the paper.

It suffices to bound moments $\|p_0^N f\|_{L_t^\infty([0,T);L_x^1 L_p^1)}$ for sufficiently large $N > 0$ due to the following sharpened estimate from [38] because we can bound the terms on the right hand side of the inequality in Proposition 28 by sufficiently high moments of f (due to field estimates, e.g. (4.2.1), and interpolation inequalities, e.g. (34) found later in this paper).

Proposition 28. *Over any spatial characteristic curve $X(s;t,x,p)$ we have the bound*

$$\sup\int_0^{T_*}(|E(s,X(s;t,x,p)|+|B(s,X(s;t,x,p)|)ds \lesssim 1+\|K\sigma_{-1}\|_{L_t^\infty([0,T];L_x^{2+\lambda}L_p^1)} + \|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^{3+\tilde{\lambda}}L_p^1)} \quad (3.3.1)$$

for any $\lambda, \tilde{\lambda} > 0$ and where the supremum is taken over all $(t,x,p) \in \mathbb{R}_+ \times \mathbb{R}_x^3 \times \mathbb{R}_p^3$.

Proof. We can rewrite the bounds (2.3.3) and (2.3.4) in the form:

$$|K_T|(t,x) \lesssim \int_{C_{t,x}} \frac{\sigma_{-1}(s,y)}{(t-s)^2}d\sigma \quad (3.3.2)$$

$$|K_S|(t,x) \lesssim \int_{C_{t,x}} \frac{(|K|\sigma_{-1})(s,y)}{t-s}d\sigma \quad (3.3.3)$$

where the integral over the cone $C_{t,x}$ is given (2.1.2). Using (3.3.2) and (3.3.3), we can bound our integral over the characteristic $X(s;t,x,p)$ by:

$$sup\int_0^{T_*}(|E(s,X(s;t,x,p)|+|B(s,X(s;t,x,p)|)ds \lesssim 1+\int_0^{T_*}\int_{C_{s,X(s)}}\frac{\sigma_{-1}(\tilde{s},y)}{(s-\tilde{s})^2}d\sigma ds+\int_0^{T_*}\int_{C_{s,X(s)}}\frac{(|K|\sigma_{-1})(\tilde{s},y)}{(s-\tilde{s})}d\sigma ds \quad (3.3.4)$$

where $d\sigma = d\sigma(\tilde{s},y) = (s-\tilde{s})^2\sin(\theta)d\tilde{s}d\phi d\theta$ and $X(s) = X(s;t,x,p)$. The integral terms on the right hand side have the general form:

$$I_i(t,x) \stackrel{\text{def}}{=} \int_0^t \int_{C_{s,X(s)}} \frac{g_i}{(s-\tilde{s})^i} d\sigma ds \quad (i=1,2), \tag{3.3.5}$$

where $g_1 = |K|\sigma_{-1}$ and $g_2 = \sigma_{-1}$. By a change of variables after writing (3.3.5) expanded as (2.1.2), we obtain:

$$\begin{aligned} I_i(t,x) &= \int_0^t \int_{\tilde{s}}^t \int_0^{2\pi} \int_0^{\pi} (s-\tilde{s})^{2-i} \sin(\theta) g_i(s, X(s) + (s-\tilde{s})\omega) d\theta d\phi ds d\tilde{s} \\ &\stackrel{\text{def}}{=} \int_0^t J_i(\tilde{s}, X(\tilde{s})) d\tilde{s}, \end{aligned}$$

where again we have adopted the convention $X(s) = X(s;t,x,p)$.

Following Pallard [40], we define the diffeomorphism $\pi \stackrel{\text{def}}{=} X(s) + (s-\tilde{s})\omega$. This change of variables has Jacobian $J_\pi = (X'(s)\cdot\omega + 1)(s-\tilde{s})^2 \sin(\theta) \neq 0$ on $\theta \in (0,\pi)$ (since $|X'(s)| \leq |\hat{V}(s)| < 1$ and hence $X'(s)\cdot\omega + 1 > 0$). First using Hölder's inequality for Hölder exponents q, q' such that $\frac{1}{q} + \frac{1}{q'} = 1$:

$$\begin{aligned} J_i(\tilde{s}, X(\tilde{s})) \leq & \left(\int_{\tilde{s}}^t \int_0^{2\pi} \int_0^{\pi} \frac{(s-\tilde{s})^{(2-i)q'} \sin^{q'}(\theta)}{J_\pi^{\frac{q'}{q}}} d\theta d\phi ds \right)^{\frac{1}{q'}} \\ & \times \left(\int_{\tilde{s}}^t \int_0^{2\pi} \int_0^{\pi} g_i(s, X(s) + (s-\tilde{s})\omega)^q J_\pi d\theta d\phi ds \right)^{\frac{1}{q}} \end{aligned} \tag{3.3.6}$$

Next, using the change of variables described by the diffeomorphism π in the second integral on the right hand side of (3.3.6):

$$J_i(\tilde{s}, X(\tilde{s})) \leq \left(\int_{\tilde{s}}^{t} \int_0^{2\pi} \int_0^{\pi} \frac{(s-\tilde{s})^{(2-i)q'} \sin^{q'}(\theta)}{J_{\pi}^{\frac{q'}{q}}} d\theta d\phi ds \right)^{\frac{1}{q'}} \|g_i(\tilde{s})\|_{L^q(\mathbb{R}^3)} \quad (3.3.7)$$

Finally, plugging in the expression for J_π into the remaining integral on the right hand side, we see that it is bounded for certain choices of q. To see this, choose coordinates (θ, ϕ) such that $X' \cdot \omega = -|X'||\omega|\cos(\theta) \geq -\cos(\theta)$. Then, using this coordinate system:

$$\int_{\tilde{s}}^{t} \int_0^{2\pi} \int_0^{\pi} \frac{(s-\tilde{s})^{(2-i)q'} \sin^{q'}(\theta)}{J_{\pi}^{\frac{q'}{q}}} d\theta d\phi ds \lesssim \int_{\tilde{s}}^{t} \int_0^{2\pi} \int_0^{\pi} \frac{(s-\tilde{s})^{(2-i)q'-\frac{2q'}{q}} \sin^{q'-\frac{q'}{q}}(\theta)}{(1-\cos(\theta))^{\frac{q'}{q}}} d\theta d\phi ds$$

Now, note that $\frac{1}{1-\cos(\theta)} = \frac{1}{1-\sqrt{1-\sin^2(\theta)}} = \frac{1+\sqrt{1-\sin^2(\theta)}}{\sin^2(\theta)} \lesssim \frac{1}{\sin^2(\theta)}$ since $1+\sqrt{1-\sin^2(\theta)} \leq 2$. Plugging this into the above, we obtain:

$$\int_{\tilde{s}}^{t} \int_0^{2\pi} \int_0^{\pi} \frac{(s-\tilde{s})^{(2-i)q'} \sin^{q'}(\theta)}{J_{\pi}^{\frac{q'}{q}}} d\theta d\phi ds = \int_{\tilde{s}}^{t} \int_0^{2\pi} \int_0^{\pi} (s-\tilde{s})^{(2-i)q'-\frac{2q'}{q}} \sin^{q'-\frac{3q'}{q}}(\theta) d\theta d\phi ds$$

The integral over θ remains bounded when $q' - \frac{3q'}{q} > -1$ and $(2-i)q' - \frac{2q'}{q} > -1$, i.e. when $q > 2$ and $q > \frac{3}{3-i}$ for $i = 1, 2$. $\square$

Hence, it will suffice to bound moments $\|p_0^N f\|_{L_t^\infty([0,T);L_x^1 L_p^1)}$ for sufficiently large

$N > 0$ due to the following sharpened estimate from [38] because we can bound the terms on the right hand side of the inequality in Proposition 28 by sufficiently high moments of f (due to field estimates, e.g. (4.2.1), and interpolation inequalities, e.g. (34) found later in this paper). We will bound the moments using the standard moment estimate recalled from Proposition 7.3 in [38]:

Proposition 29. *Given $N > 0$, we have the uniform estimate*

$$\|p_0^N f\|_{L_t^\infty([0,T);L_x^1 L_p^1)} \lesssim \|p_0^N f_0\|_{L_t^\infty([0,T);L_x^1 L_p^1)} + \|K\|^{N+3}_{L_t^1([0,T);L_x^{N+3})}.$$

In the next chapter, we state the tools we will use to bound the $\|K\|_{L_t^1([0,T);L_x^{N+3})}$ term on the right hand side of the inequality in Proposition 29.

Chapter 4

Estimates on K_T and $K_{S,2}$

In the following chapter, we will begin proving appropriate bounds on the field decomposition terms K_T and $K_{S,2}$. We first derive some estimates for averaging on a sphere.

4.1 Averaging Operator Inequalities

In this section, we prove averaging operator inequalities that will be used to bound field terms, e.g. K_T. We begin by describing the operators we will consider. The average value of a function $g : X \to \mathbb{R}$ defined on a space X with finite measure $\nu(X)$ is denoted by

$$\fint_X g(x)dx = \frac{1}{\nu(X)} \int_X g(x)dx \tag{4.1.1}$$

Define the operator W_α by

$$W_\alpha(h(t,x)) = \int_{|x-y|\leq t} \frac{h(t-|x-y|,y)}{|x-y|^\alpha} dy$$
$$= \int_0^t s^{2-\alpha} \fint_{\mathbb{S}^2} h(t-s, x+s\omega) d\mu(\omega) ds \quad (4.1.2)$$

where $d\mu(\omega)$ is the spherical measure. Then, by (2.3.3) we obtain the following:

Proposition 30. *For the electric and magnetic fields, we have the estimate:*

$$|K_T(t,x)| \lesssim W_2(\sigma_{-1})$$

Thus, we wish to obtain estimates for the operator W_α for $\alpha = 2$. We prove an estimate for general α. Consider:

$$(T_{\alpha,s}h)(t,sx) = s^{2-\alpha} \fint_{\mathbb{S}^2} h(t-s, sx+s\omega) d\mu(\omega) \quad (4.1.3)$$

A Schwartz function is a C^∞ function $f : \mathbb{R}^n \to \mathbb{R}$ such that for any pair of multi-indices, α and β, there exists a finite constant $C_{\alpha,\beta}$ satisfying $\sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)| \leq C_{\alpha,\beta}$. The set of Schwartz functions form a vector space called the Schwartz space, which is dense in the space L^q for $1 \leq q < \infty$. On the Schwartz space, denoted by $\mathscr{S}(\mathbb{R}^n)$, we have known estimates for the averaging operator $Af = \fint_{\mathbb{S}^2} f(x+\omega) d\mu(\omega)$ from (2) in [33]:

Theorem 31. *The estimate*

$$\|Af\|_{L^a} \lesssim \|f\|_{L^q}, \ f \in \mathscr{S}(\mathbb{R}^n) \tag{4.1.4}$$

holds if and only if $(\frac{1}{q}, \frac{1}{a})$ *in the convex hull of* $(0,0)$, $(1,1)$, *and* $(\frac{n}{n+1}, \frac{1}{n+1})$.

For the case $n = 3$, the inequality (4.1.4) holds for $(\frac{1}{q}, \frac{1}{a})$ in the convex hull of $\{(0,0), (1,1), (\frac{3}{4}, \frac{1}{4})\}$. Thus, setting $a = mq$ for $1 \leq m \leq 3$ and for a range of q to be calculated, we have that:

$$\|(T_{\alpha,s}h)(t, sx)\|_{L_x^{mq}} \lesssim s^{2-\alpha}\|h(t-s, sx)\|_{L_x^q} \tag{4.1.5}$$

After a change of variables in the spatial coordinates,

$$\|T_{\alpha,s}h(t)\|_{L_x^{mq}} \lesssim s^{2-\alpha+\frac{3}{mq}-\frac{3}{q}}\|h(t-s)\|_{L_x^q}$$

Applying this estimate to the operator W_α under the additional assumption that

$2-\alpha+\frac{3}{mq}-\frac{3}{q}>-1$,

$$
\begin{aligned}
\|W_\alpha h(t)\|_{L_t^r L_x^{mq}} &\leq \left\| \int_0^t \|T_{\alpha,s}h(t)\|_{L_x^{mq}} ds \right\|_{L_t^r} \\
&\lesssim \left\| \int_0^t s^{2-\alpha+\frac{3}{mq}-\frac{3}{q}} \|h(t-s)\|_{L_x^q} ds \right\|_{L_t^r} \\
&\lesssim \left\| \int_0^t s^{2-\alpha+\frac{3}{mq}-\frac{3}{q}} \|h\|_{L_t^\infty L_x^q} ds \right\|_{L_t^r} \\
&\lesssim \|h\|_{L_t^\infty L_x^q}
\end{aligned}
$$

where $1 \leq r \leq \infty$, $t \in [0,T]$, $x \in \mathbb{R}^3$, and the implicit constant in the upper bound is a continuous function of T, r and α. It remains to check the range of q for which $(\frac{1}{q}, \frac{1}{mq})$ lies in the convex hull described above.

We observe that the line connecting the points $(x,y)=(1,1)$ and $(x,y)=(\frac{3}{4},\frac{1}{4})$ is represented by the equation $y=3x-2$. Thus, the line $y=\frac{1}{m}x$ meets the line $y=3x-2$ when $x=\frac{2m}{3m-1}$.

Summarizing:

Lemma 32. *For $1 \leq r \leq \infty$, $1 \leq m \leq 3$, $2-\alpha+\frac{3}{mq}-\frac{3}{q}>-1$, $\frac{3m-1}{2m} \leq q \leq \infty$,*

$$
\|W_\alpha h(t,x)\|_{L_t^r([0,T];L_x^{mq})} \leq C_{T,\alpha} \|h\|_{L_t^\infty([0,T];L_x^q)} \tag{4.1.6}
$$

for some explicitly computable constant $C_{T,\alpha}$ depending only on T and α.

In the next section, we will use (4.1.6) to control the size of K_T and $K_{S,2}$.

4.2 Bounding K_T and $K_{S,2}$

We can now apply the estimates from the previous section to the K_T term. For the $\alpha = 2$, $m = 3$ case, we need $-\frac{2}{q} > -1$ and $\frac{4}{3} \leq q \leq \infty$. Thus, by Proposition 30 and (4.1.6), we obtain:

Proposition 33. *For $1 \leq r \leq \infty$ and $q > 2$,*

$$\|K_T\|_{L_t^r([0,T];L_x^{3q})} \leq C_T \|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^q)} \tag{4.2.1}$$

for some explicitly computable constant C_T depending only on T.

To get appropriate bounds on K_T, we need to introduce some important inequalities. The statement of the following inequality is analogous to the general interpolation inequality of Proposition 21.

Lemma 34. *Let $1 \leq r, s \leq \infty$ and suppose $\nu > 2\frac{r}{s} - 1$. Then*

$$\|\sigma_{-1}\|_{L_x^r} \lesssim \|p_0^\nu f\|_{L_x^s L_p^1}^{\frac{s}{r}}. \tag{4.2.2}$$

Proof. By Hölder's inequality with $\frac{1}{q} + \frac{1}{q'} = 1$:

$$\int_{\mathbb{R}^3} \frac{f(t,x,p)}{p_0(1+\hat{p}\cdot\omega)} dp \leq \left(\int_{\mathbb{R}^3} \frac{dp}{p_0^{(1+\alpha)q'}(1+\hat{p}\cdot\omega)^{q'}} \right)^{\frac{1}{q'}} \left(\int_{\mathbb{R}^3} p_0^{\alpha q} f(t,x,p)^q dp \right)^{\frac{1}{q}}$$

Call the first term on the right hand side I. In order to bound this term, we use the standard inequality

$$(1+\hat{p}\cdot\omega)^{-1} \lesssim \min\{p_0^2, \theta^{-2}\} \tag{4.2.3}$$

where $\theta = \angle(\frac{p}{|p|}, -\omega) \in [0,\pi]$. Note that for small θ, we can assume $\sin(\theta) \approx \theta$. Assuming $\alpha > 2 - \frac{1}{q}$, we can bound the integral I as follows:

$$\begin{aligned}
I &= \int_{\mathbb{R}^3} \frac{dp}{p_0^{(1+\alpha)q'}(1+\hat{p}\cdot\omega)^{q'}} \\
&\le \int_{\mathbb{R}^3} \frac{dp}{p_0^{(\alpha-1)q'+2}(1+\hat{p}\cdot\omega)} \\
&\lesssim \lim_{P\to\infty} \int_0^P d|p| \int_0^{2\pi} d\phi \left(\int_0^{p_0^{-1}} \frac{p_0^2|p|^2 \sin(\theta) d\theta}{p_0^{(\alpha-1)q'+2}} + \int_{p_0^{-1}}^{\pi} \frac{|p|^2 \sin(\theta) d\theta}{p_0^{(\alpha-1)q'+2}\theta^2} \right) \\
&\lesssim \lim_{P\to\infty} \int_0^P d|p| \frac{1+\log(p_0)}{p_0^{(\alpha-1)q'}} \\
&\lesssim 1
\end{aligned} \tag{4.2.4}$$

since $(\alpha-1)q' > 1$. Taking L^r norm on both sides and using the conservation law $\|f\|_{L^\infty_{x,p}} \lesssim 1$, we obtain:

$$\|\sigma_{-1}\|_{L^r_x} \lesssim \|p_0^{\alpha q} f^q\|^{\frac{1}{q}}_{L_x^{\frac{r}{q}} L_p^1} \lesssim \|p_0^{\alpha q} f\|^{\frac{1}{q}}_{L_x^{\frac{r}{q}} L_p^1} \tag{4.2.5}$$

Hence, setting $q = \frac{r}{s}$, we finally have for $\nu > \frac{2r}{s} - 1$:

$$\|\sigma_{-1}\|_{L^r_x} \lesssim \|p_0^\nu f\|^{\frac{s}{r}}_{L^s_x L^1_p} \tag{4.2.6}$$

This completes the proof of this inequality. $\square$

We also have this interpolation-type inequality from Proposition 10.3 in [38]:

Proposition 35. *Suppose η, ρ, and τ are real numbers such that $0 < q\eta < 1$ and*

$$\tau \geq \frac{\rho - \eta(N + 3 - 3q)}{1 - q\eta}$$

Then,

$$\|fp_0^\rho\|_{L^\infty_t([0,T];L^q_x L^1_p)} \lesssim \|fp_0^\tau\|^{1-q\eta}_{L^\infty_t([0,T];L^q_x L^1_p)} \|fp_0^N\|^\eta_{L^\infty_t([0,T];L^1_x L^1_p)} \tag{4.2.7}$$

Applying Lemma 34 and (4.2.7) to (4.2.1), we obtain for $N > 3$, $1 \leq r \leq \infty$ and some $\delta > 0$:

$$\begin{aligned}
\|K_T\|^{N+3}_{L^r_t([0,T];L^{N+3}_x)} &\lesssim \|\sigma_{-1}\|^{N+3}_{L^\infty_t([0,T];L_x^{\frac{N+3}{3}})} \\
&\lesssim \|p_0^{\frac{2N+3+\delta}{3}} f\|^3_{L^1_x L^1_p} \\
&\lesssim \|p_0^{\frac{2N-3N\eta+3+\delta}{3(1-\eta)}} f\|^{3-3\eta}_{L^1_x L^1_p} \|p_0^N f\|^{3\eta}_{L^1_x L^1_p}
\end{aligned}$$

Setting $\eta = \frac{1-\gamma}{3}$, we obtain the needed estimate for K_T:

Proposition 36. *Given* $1 \leq r \leq \infty$, $\gamma \in (0,1)$, $N > 3$ *and* $\delta > 0$,

$$\|K_T\|^{N+3}_{L^r_t([0,T];L^{N+3}_x)} \lesssim \|p_0^{\frac{N(1+\gamma)+3+\delta}{2+\gamma}} f\|^{2+\gamma}_{L^\infty_t([0,T);L^1_x L^1_p)} \|p_0^N f\|^{1-\gamma}_{L^\infty_t([0,T);L^1_x L^1_p)} \tag{4.2.8}$$

From this point in the paper, we adopt the convention that $\rho+$ denotes some appropriate number $\rho + \epsilon$ where $\epsilon > 0$ is very small, $\epsilon \ll 1$. Note that the size of ϵ may vary depending on the term, but the key point is that ϵ is appropriately small in each of the estimates below. Similarly, we let $\rho-$ denote some appropriate number $\rho - \epsilon$ for $\epsilon \ll 1$ chosen to be appropriately small. Using the full range of estimates on the operator W_α, we can get a more general K_T estimate:

Proposition 37. *Given* $1 \leq m \leq 3$, $\frac{3}{mq} - \frac{3}{q} > -1$ *and* $\frac{3m-1}{2m} \leq q \leq \infty$, *we have the estimate:*

$$\|K_T\|_{L^\infty_t L^{mq}_x} \lesssim \|\sigma_{-1}\|_{L^\infty_t L^q_x} \tag{4.2.9}$$

Proof. By (3.2.6), we can apply (4.1.6) for $\alpha = 2$ to $|K_T|$. $\square$

In particular if $m = 2$, then we need $-\frac{3}{2q} > -1$ (or $q > \frac{3}{2}$) and $q \geq \frac{5}{4}$. Hence, we have for $q > 2$:

$$\|K_T\|_{L^\infty_t L^{4+}_x} \lesssim \|\sigma_{-1}\|_{L^\infty_t L^{2+}_x} \tag{4.2.10}$$

Proposition 38. *Given* $1 \leq m \leq 3$, $\frac{3}{mq} - \frac{3}{q} > -1$ *and* $\frac{3m-1}{2m} \leq q \leq \infty$,

$$\|K_{S,2}\|_{L_t^\infty L_x^{2mq}} \lesssim \|\sigma_{-1}\|_{L_t^\infty L_x^{2q}} \tag{4.2.11}$$

Proof. By (3.2.8):

$$\|K_{S,2}\|_{L_t^\infty L_x^{2mq}} \lesssim \|W_2((\sigma_{-1})^2)^{\frac{1}{2}}\|_{L_t^\infty L_x^{2mq}} = \|W_2((\sigma_{-1})^2)\|^{\frac{1}{2}}_{L_t^\infty L_x^{mq}} \tag{4.2.12}$$

We can apply (4.1.6) now to get:

$$\|K_{S,2}\|_{L_t^\infty L_x^{2mq}} \lesssim \|(\sigma_{-1})^2\|^{\frac{1}{2}}_{L_t^\infty L_x^{q}} = \|\sigma_{-1}\|_{L_t^\infty L_x^{2q}} \tag{4.2.13}$$

$\square$

For reasons that will be clear in Section 9, we use Proposition 37 and Proposition 38 to bound the quantities $\|K_T\|_{L_t^\infty L_x^{4+}}$ and $\|K_{S,2}\|_{L_t^\infty L_x^{4+}}$. Using Proposition 38, we can compute for $|K_{S,2}|$:

$$\|K_{S,2}\|_{L_t^\infty L_x^{4+}} \lesssim \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}} \tag{4.2.14}$$

where we used $m = \frac{5}{3}$ and $q = \frac{6}{5}+$. In particular, setting $q = \frac{6+\epsilon}{5}$ for $\epsilon \ll 1$, we see that m and q satisfy the conditions of Proposition 38. The explicit estimate written in (4.2.14) is

$$\|K_{S,2}\|_{L_t^\infty L_x^{4+\frac{2\epsilon}{3}}} \lesssim \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+\frac{2\epsilon}{5}}}.$$

Similarly, by Proposition 37, we can compute for $|K_T|$:

$$\|K_T\|_{L_t^\infty L_x^{4+}} \lesssim \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}} \tag{4.2.15}$$

where we used $m = \frac{5}{3}$ and $q = \frac{12}{5}+$. Note that this is not the lowest Lebesgue norm exponent that can be chosen for σ_{-1}. However, we do not have a better bound in the $K_{S,2}$ estimate, so a better estimate on the K_T term is not useful.

Chapter 5

Estimates for K_S and $K_{S,1}$

Strichartz estimates are used to study regularity of dispersive equations. These estimates are related to the Fourier restriction problem studied by R. Strichartz in [46]. Since their introduction, they have been widely studied, e.g. Keel-Tao [34], Foschi [25], Taggart [47], etc. The estimates have proven useful in controlling the solutions of linear and nonlinear dispersive partial differential equations. We present the following version of these estimates, as found in (4.8) of [44].

Theorem 39. *(Strichartz Estimates)*

Given $\lambda \in (0,1)$ *and solution* $u : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ *to* $\Box u = F$ *on* $[a,b] \times \mathbb{R}^3$ *with initial data* $u|_{t=a} = u(a)$ *and* $\partial_t u|_{t=a} = \partial_t u(a)$*, there exists a constant* C_λ *such that*

$$\|u\|_{L_t^{\frac{2}{\lambda}}([a,b];L^{\frac{2}{1-\lambda}})} + \|u\|_{L_t^\infty([a,b];\dot{H}^\lambda)} + \|\partial_t u\|_{L_t^\infty([a,b];\dot{H}^\lambda)}$$
$$\leq C_\lambda(\|F\|_{L_t^{\frac{2}{1+\lambda}}([a,b];L^{\frac{2}{2-\lambda}})} + \|u(a)\|_{\dot{H}^\lambda} + \|\partial_t u(a)\|_{\dot{H}^\lambda})$$

We will use the above Strichartz inequality to bound the field terms K_S and $K_{S,2}$ by an iteration argument.

5.1 An Iteration Argument for K_S and $K_{S,1}$

Recall from the Glassey-Strauss decomposition estimates (2.3.4) and (2.3.5) for K_S that

$$|K_S| \lesssim \Box^{-1}(|K|\sigma_{-1}). \tag{5.1.1}$$

Hence, using Strichartz estimates for the wave operator, we can prove the following:

Proposition 40. *Assume $\|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^2)} \lesssim 1$. Given $N > 3$, $\gamma \in (0,1)$ and $\delta > 0$, we obtain the estimate*

$$\|K_S\|^{N+3}_{L_t^1([0,T];L_x^{N+3})} \lesssim 1 + \|p_0^{\frac{N(1+\gamma)+3+\delta}{2+\gamma}} f\|^{2+\gamma}_{L_t^\infty([0,T);L_x^1 L_p^1)} \|p_0^N f\|^{1-\gamma}_{L_t^\infty([0,T);L_x^1 L_p^1)} \tag{5.1.2}$$

Proof. By the above estimate (5.1.1):

$$\|K_S\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} \leq \|\Box^{-1}(|K|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} \tag{5.1.3}$$

Using the decomposition $K = K_0 + K_T + K_S$, we obtain for some time interval

$[a,b] \subset [0,T)$:

$$\|K_S\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} \leq \|\Box^{-1}(|K_0|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} + \\ \|\Box^{-1}(|K_T|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} + \|\Box^{-1}(|K_S|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} \quad (5.1.4)$$

Fix an interval $[a,b] \subset [0,T)$. First notice that

$$\|\Box^{-1}(|K_0|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} \lesssim \|\Box^{-1}(|K_0|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([0,T);L_x^{N+3})}$$

and similarly

$$\|\Box^{-1}(|K_T|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} \lesssim \|\Box^{-1}(|K_T|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([0,T);L_x^{N+3})}$$

Setting $\lambda = \frac{N+1}{N+3}$, we obtain by the Strichartz estimates for the wave operator:

$$\|\Box^{-1}(|K_0|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([0,T);L_x^{N+3})} \lesssim C_{\frac{N+1}{N+3}} \||K_0|\sigma_{-1}\|_{L_t^{\frac{(N+3)}{N+2}}([0,T);L_x^{\frac{2(N+3)}{N+5}})}$$

since we have trivial initial data by (2.3.5). Applying the same argument to the K_T

term and by (5.1.4) we obtain:

$$\|K_S\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} \leq C_{\frac{N+1}{N+3}}\||K_0|\sigma_{-1}\|_{L_t^{\frac{(N+3)}{N+2}}([0,T);L_x^{\frac{2(N+3)}{N+5}})}$$
$$+ C_{\frac{N+1}{N+3}}\||K_T|\sigma_{-1}\|_{L_t^{\frac{(N+3)}{N+2}}([0,T);L_x^{\frac{2(N+3)}{N+5}})}$$
$$+ \|\square^{-1}(|K_S|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} \tag{5.1.5}$$

Applying Hölder's inequality with $\frac{1}{2}+\frac{1}{N+3}=\frac{N+5}{2(N+3)}$:

$$\|K_S\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} \leq C_{\frac{N+1}{N+3}}\|K_0\|_{L_t^{\frac{2(N+3)}{N+1}}([0,T);L_x^{N+3})}\|\sigma_{-1}\|_{L_t^2([0,T);L_x^2)}$$
$$+ C_{\frac{N+1}{N+3}}\|K_T\|_{L_t^{\frac{2(N+3)}{N+1}}([0,T);L_x^{N+3})}\|\sigma_{-1}\|_{L_t^2([0,T);L_x^2)}+\|\square^{-1}(|K_S|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} \tag{5.1.6}$$

Note that we can bound $C_{\frac{N+1}{N+3}}\|K_0\|_{L_t^{\frac{2(N+3)}{N+1}}([0,T);L_x^{N+3})}\|\sigma_{-1}\|_{L_t^2([0,T);L_x^2)}$ by a constant since K_0 depends only on initial data and we have assumed that $\|\sigma_{-1}\|_{L_t^2([0,T);L_x^2)}\lesssim_T \|\sigma_{-1}\|_{L_t^\infty([0,T);L_x^2)}\lesssim 1$. Finally, using the estimate (4.2.8) on K_T from the previous section,

$$\|K_S\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})}$$
$$\leq (data) + C\|p_0^{\frac{N(1+\gamma)+3+\delta}{2+\gamma}}f\|_{L_t^\infty([0,T);L_x^1L_p^1)}^{\frac{2+\gamma}{N+3}}\|p_0^N f\|_{L_t^\infty([0,T);L_x^1L_p^1)}^{\frac{1-\gamma}{N+3}}\|\sigma_{-1}\|_{L_t^2([0,T];L_x^2)}$$
$$+ \|\square^{-1}(|K_S|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} \tag{5.1.7}$$

Similarly, we can now apply Strichartz estimates and Hölder's inequality to the K_S term. Note that we kept the time interval on the K_S term as $[a,b]$. Setting $u = \Box^{-1}(|K_S|\sigma_{-1})$,

$$\|\Box^{-1}(|K_S|\sigma_{-1})\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L_x^{N+3})} \leq C_{\frac{N+1}{N+3}}(\|K_S\|_{L_t^{\frac{2(N+3)}{N+1}}([a,b];L^{N+3})}\|\sigma_{-1}\|_{L_t^2([a,b];L_x^2)} + \|u(a)\|_{\dot{H}^{\frac{N+1}{N+3}}} + \|\partial_t u(a)\|_{\dot{H}^{\frac{N+1}{N+3}}}) \quad (5.1.8)$$

Next, we can choose a partition $0 = T_0 < T_1 < T_2 < \ldots < T_{k-1} < T_k = T$ of $[0,T]$ such that

$$\|\sigma_{-1}\|_{L_t^2([T_i,T_{i+1}];L_x^2)} \leq \frac{1}{2C_{\frac{N+1}{N+3}}} \; for \; i \in \{0,1,\ldots,k-1\}$$

due to the assumption $\|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^2)} \leq \tilde{C}$ for some $\tilde{C}$. (For example, we can choose our partition so that $(T_i - T_{i-1})^{\frac{1}{2}} \leq \frac{1}{2\tilde{C}2C_{\frac{N+1}{N+3}}}$ for $i = 1,2,\ldots,k$.) Using (5.1.7) and (5.1.8):

$$\|K_S\|_{L_t^{\frac{2(N+3)}{N+1}}([T_i,T_{i+1}];L_x^{N+3})} \leq 2(data)_i + 2C\|p_0^{\frac{N(1+\gamma)+3+\delta}{2+\gamma}} f\|_{L_t^\infty([0,T);L_x^1L_p^1)}^{\frac{2+\gamma}{N+3}} \|p_0^N f\|_{L_t^\infty([0,T);L_x^1L_p^1)}^{\frac{1-\gamma}{N+3}} + 2C_{\frac{N+1}{N+3}}(\|u(T_i)\|_{\dot{H}^{\frac{N+1}{N+3}}} + \|\partial_t u(T_i)\|_{\dot{H}^{\frac{N+1}{N+3}}}) \quad (5.1.9)$$

Thus, by Hölder's inequality, our choice of partition, and then (5.1.9):

$$\begin{aligned}
\||K_S|\sigma_{-1}\|_{L_t^{\frac{N+3}{N+2}}([T_i,T_{i+1}];L_x^{\frac{2(N+3)}{N+5}})} &\leq \frac{1}{2C_{\frac{N+1}{N+3}}}\|K_S\|_{L_t^{\frac{2(N+3)}{N+1}}([T_i,T_{i+1}];L_x^{N+3})} \\
&\leq \frac{1}{C_{\frac{N+1}{N+3}}}(data)_i + \frac{1}{C_{\frac{N+1}{N+3}}}C\|p_0^{\frac{N(1+\gamma)+3+\delta}{2+\gamma}}f\|_{L_t^\infty([0,T];L_x^1L_p^1)}^{\frac{2+\gamma}{N+3}}\|p_0^N f\|_{L_t^\infty([0,T];L_x^1L_p^1)}^{\frac{1-\gamma}{N+3}} \\
&\quad + \|u(T_i)\|_{\dot{H}^{\frac{N+1}{N+3}}} + \|\partial_t u(T_i)\|_{\dot{H}^{\frac{N+1}{N+3}}} \qquad (5.1.10)
\end{aligned}$$

Using the Strichartz estimates again for $[T_{i-1}, T_i]$, we obtain:

$$\begin{aligned}
&\|u(T_i)\|_{\dot{H}^{\frac{N+1}{N+3}}} + \|\partial_t u(T_i)\|_{\dot{H}^{\frac{N+1}{N+3}}} \\
&\leq C_{\frac{N+1}{N+3}}\left(\|u(T_{i-1})\|_{\dot{H}^{\frac{N+1}{N+3}}} + \|\partial_t u(T_{i-1})\|_{\dot{H}^{\frac{N+1}{N+3}}} + \||K_S|\sigma_{-1}\|_{L_t^{\frac{N+3}{N+2}}([T_{i-1},T_i];L_x^{\frac{2(N+3)}{N+5}})}\right) \\
&\qquad \leq (data)_i + C\|p_0^{\frac{N(1+\gamma)+3+\delta}{2-\gamma}}f\|_{L_t^\infty([0,T];L_x^1L_p^1)}^{\frac{2+\gamma}{N+3}}\|p_0^N f\|_{L_t^\infty([0,T];L_x^1L_p^1)}^{\frac{1-\gamma}{N+3}} \\
&\qquad\quad + 2C_{\frac{N+1}{N+3}}(\|u(T_{i-1})\|_{\dot{H}^{\frac{N+1}{N+3}}} + \|\partial_t u(T_{i-1})\|_{\dot{H}^{\frac{N+1}{N+3}}})
\end{aligned}$$

Thus, since $u(0) = \partial_t u(0) = 0$, we do an iteration of the above to get the following estimate:

$$\begin{aligned}
&\|u(T_{j+1})\|_{\dot{H}^{\frac{N+1}{N+3}}} + \|\partial_t u(T_{j+1})\|_{\dot{H}^{\frac{N+1}{N+3}}} \\
&\leq \sum_{i=0}^{j}(2C_{\frac{N+1}{N+3}})^{j-i}\left((data)_i + C\|p_0^{\frac{N(1+\gamma)+3+\delta}{2-\gamma}}f\|_{L_t^\infty([0,T];L_x^1L_p^1)}^{\frac{2+\gamma}{N+3}}\|p_0^N f\|_{L_t^\infty([0,T];L_x^1L_p^1)}^{\frac{1-\gamma}{N+3}}\right)
\end{aligned} \qquad (5.1.11)$$

Plugging this estimate into (5.1.9) and using the triangle inequality to sum over the entire partition,

$$\|K_S\|_{L_t^{\frac{2(N+3)}{N+1}}([T_i,T_{i+1}];L_x^{N+3})} \lesssim \sum_{i=0}^{k-1} \|K_S\|_{L_t^{\frac{2(N+3)}{N+1}}([T_i,T_{i+1}];L_x^{N+3})}$$
$$\lesssim 1 + \|p_0^{\frac{N(1+\gamma)+3+\delta}{2+\gamma}} f\|_{L_t^\infty([0,T);L_x^1L_p^1)}^{\frac{2+\gamma}{N+3}} \|p_0^N f\|_{L_t^\infty([0,T);L_x^1L_p^1)}^{\frac{1-\gamma}{N+3}}$$

which implies the estimate (5.1.2) after an application of Hölder's inequality in the time variable. □

Finally, we employ an iteration argument using Strichartz estimates for the inhomogeneous wave equation to gain bounds on $K_{S,1}$. For these estimates, assume that

$$\|\Phi_{-1}\|_{L_t^\infty L_x^2} \lesssim 1 \tag{5.1.12}$$

Proposition 41. *We have the following bound on $K_{S,1}$ assuming (5.1.12):*

$$\|K_{S,1}\|_{L_t^\infty([0,T);L_x^{4+})} \lesssim 1 + \|\sigma_{-1}\|_{L_t^\infty([0,T);L_x^{\frac{12}{5}+})} \tag{5.1.13}$$

Proof. For $\gamma \in (0,1)$, we obtain by (3.2.7) for some interval $[a,b] \subset [0,T)$:

$$\|K_{S,1}\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([a,b]\times\mathbb{R}^3)} \lesssim \|\Box^{-1}(|K|\Phi_{-1})\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([a,b]\times\mathbb{R}^3)} \tag{5.1.14}$$

(Note that we will set $\gamma = \frac{1}{2}+$ later in the proof.) Applying the triangle inequality to the decomposition $|K|\leq|K_0|+|K_T|+|K_{S,1}|+|K_{S,2}|$ and extending the interval $[a,b]$ to $[0,T)$ on certain terms, we obtain:

$$\begin{aligned}
&\|K_{S,1}\|_{L_t^{\frac{2}{\gamma}}L_x^{\frac{2}{1-\gamma}}([a,b]\times\mathbb{R}^3)} \\
&\qquad\leq \|\Box^{-1}(|K_0|\Phi_{-1})\|_{L_t^{\frac{2}{\gamma}}L_x^{\frac{2}{1-\gamma}}([0,T)\times\mathbb{R}^3)} + \|\Box^{-1}(|K_T|\Phi_{-1})\|_{L_t^{\frac{2}{\gamma}}L_x^{\frac{2}{1-\gamma}}([0,T)\times\mathbb{R}^3)} \\
&\quad + \|\Box^{-1}(|K_{S,2}|\Phi_{-1})\|_{L_t^{\frac{2}{\gamma}}L_x^{\frac{2}{1-\gamma}}([0,T)\times\mathbb{R}^3)} + \|\Box^{-1}(|K_{S,1}|\Phi_{-1})\|_{L_t^{\frac{2}{\gamma}}L_x^{\frac{2}{1-\gamma}}([a,b]\times\mathbb{R}^3)} \qquad (5.1.15)
\end{aligned}$$

Note that we now replace the $\lesssim$ symbol with some explicit constant $\tilde{C}$. From here, since $\Box^{-1}$ is the solution operator to the inhomogeneous wave equation on the interval $[0,T)$ with zero initial data as expressed in (2.3.5), we know from Theorem 39 that

$$\|\Box^{-1}(|K_T|\Phi_{-1})\|_{L_t^{\frac{2}{\gamma}}L_x^{\frac{2}{1-\gamma}}([0,T)\times\mathbb{R}^3)} \leq C_\gamma \||K_T|\Phi_{-1}\|_{L_t^{\frac{2}{1+\gamma}}L_x^{\frac{2}{2-\gamma}}([0,T)\times\mathbb{R}^3)} \qquad (5.1.16)$$

and similarly for $|K_0|$ and $|K_{S,2}|$. Next, since $\|\Phi_{-1}\|_{L_t^2L_x^2([0,T)\times\mathbb{R}^3)}\lesssim 1$ by (5.1.12) and $\frac{2}{1-\gamma}=4+$ by the assumption that $\gamma=\frac{1}{2}+$, we can apply Hölder's inequality and (4.2.15) to (5.1.16) to get:

$$\begin{aligned}
\||K_T|\Phi_{-1}\|_{L_t^{\frac{2}{1+\gamma}}L_x^{\frac{2}{2-\gamma}}([0,T)\times\mathbb{R}^3)} &\leq \||K_T|\|_{L_t^{\frac{2}{\gamma}}L_x^{\frac{2}{1-\gamma}}([0,T)\times\mathbb{R}^3)} \|\Phi_{-1}\|_{L_t^2L_x^2([0,T)\times\mathbb{R}^3)} \\
&\lesssim \||K_T|\|_{L_t^{\frac{2}{\gamma}}L_x^{4+}([0,T)\times\mathbb{R}^3)} \\
&\lesssim \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}}
\end{aligned}$$

We obtain the same bound for the $|K_{S,2}|$ term. The $|K_0|$ term can be bounded by a constant since $|K_0|$ depends only on the initial data of the system. Summarizing, there exists a constant C:

$$\|K_{S,1}\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([a,b]\times\mathbb{R}^3)} \leq CC_\gamma(1 + \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}}) + C\|\Box^{-1}(|K_{S,1}|\Phi_{-1})\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([a,b]\times\mathbb{R}^3)} \tag{5.1.17}$$

Now, let us set $u = \Box^{-1}(|K_{S,1}|\Phi_{-1})$ for convenience of notation. Then, by Stichartz estimates and Hölder's inequality, we have the following fact:

$$\begin{aligned} \|u\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([a,b]\times\mathbb{R}^3)} \leq C_\gamma \Big(& \|u(a)\|_{\dot{H}_x^\gamma(\mathbb{R}^3)} + \|\partial_t u(a)\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^3)} \\ & + \||K_{S,1}|\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([0,T)\times\mathbb{R}^3)} \|\Phi_{-1}\|_{L_t^2 L_x^2([0,T)\times\mathbb{R}^3)} \Big). \end{aligned} \tag{5.1.18}$$

Next, due to (5.1.12), we can choose a partition $0 = T_0 < T_1 < T_2 < \ldots < T_N = T$ of the interval $[0, T]$ such that:

$$\|\Phi_{-1}\|_{L_t^2 L_x^2([T_j,T_{j+1}]\times\mathbb{R}^3)} \leq \frac{1}{2CC_\gamma} \tag{5.1.19}$$

for $j = 0, 1, \ldots, N-1$.

Hence, by (5.1.18) and (5.1.17), we obtain:

$$\|K_{S,1}\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([T_j,T_{j+1}]\times\mathbb{R}^3)} \leq CC_\gamma(1 + \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}} + \|u(T_j)\|_{\dot{H}_x^\gamma(\mathbb{R}^3)} + \|\partial_t u(T_j)\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^3)}) + \frac{1}{2}\|K_{S,1}\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([T_j,T_{j+1}]\times\mathbb{R}^3)}. \quad (5.1.20)$$

This implies that for any $j = 0, 1, \ldots N-1$, we have the inequality:

$$\|K_{S,1}\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([T_j,T_{j+1}]\times\mathbb{R}^3)} \leq 2CC_\gamma(1 + \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}} + \|u(T_j)\|_{\dot{H}_x^\gamma(\mathbb{R}^3)} + \|\partial_t u(T_j)\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^3)}). \quad (5.1.21)$$

Using Strichartz estimates again, we obtain the bound:

$$\|u(T_j)\|_{\dot{H}_x^\gamma(\mathbb{R}^3)} + \|\partial_t u(T_j)\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^3)} \leq C_\gamma\Big(\|u(T_{j-1})\|_{\dot{H}_x^\gamma(\mathbb{R}^3)} + \|\partial_t u(T_{j-1})\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^3)} + \||K_{S,1}|\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([T_{j-1},T_j)\times\mathbb{R}^3)} \|\Phi_{-1}\|_{L_t^2 L_x^2([T_{j-1},T_j)\times\mathbb{R}^3)}\Big). \quad (5.1.22)$$

We now apply Hölder's inequality and the bound (5.1.19) to (5.1.22) to get that:

$$\|u(T_j)\|_{\dot{H}_x^\gamma(\mathbb{R}^3)} + \|\partial_t u(T_j)\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^3)} \leq C_\gamma\Big(\|u(T_{j-1})\|_{\dot{H}_x^\gamma(\mathbb{R}^3)} + \|\partial_t u(T_{j-1})\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^3)} + \frac{1}{2CC_\gamma}\||K_{S,1}|\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([T_{j-1},T_j)\times\mathbb{R}^3)}\Big). \quad (5.1.23)$$

Next, by the estimate (5.1.21), we obtain that:

$$\begin{aligned}&\|u(T_j)\|_{\dot{H}^{\gamma}_x(\mathbb{R}^3)}+\|\partial_t u(T_j)\|_{\dot{H}^{\gamma-1}_x(\mathbb{R}^3)}\\&\quad\le C_\gamma\Big(\|u(T_{j-1})\|_{\dot{H}^{\gamma}_x(\mathbb{R}^3)}+\|\partial_t u(T_{j-1})\|_{\dot{H}^{\gamma-1}_x(\mathbb{R}^3)}\Big)+\frac{1}{2C}\Big(2CC_\gamma(1+\|\sigma_{-1}\|_{L^\infty_t L^{\frac{12}{5}+}_x}\\&\qquad\qquad+\|u(T_{j-1})\|_{\dot{H}^{\gamma}_x(\mathbb{R}^3)}+\|\partial_t u(T_{j-1})\|_{\dot{H}^{\gamma-1}_x(\mathbb{R}^3)})\Big).\end{aligned}\tag{5.1.24}$$

Finally, it follows that:

$$\begin{aligned}&\|u(T_j)\|_{\dot{H}^{\gamma}_x(\mathbb{R}^3)}+\|\partial_t u(T_j)\|_{\dot{H}^{\gamma-1}_x(\mathbb{R}^3)}\\&\quad\le 2C_\gamma\Big(\|u(T_{j-1})\|_{\dot{H}^{\gamma}_x(\mathbb{R}^3)}+\|\partial_t u(T_{j-1})\|_{\dot{H}^{\gamma-1}_x(\mathbb{R}^3)}\Big)+C_\gamma(1+\|\sigma_{-1}\|_{L^\infty_t L^{\frac{12}{5}+}_x}).\end{aligned}\tag{5.1.25}$$

Notice that $u(0)=\partial_t u(0)=0$. Thus, performing an iteration of the above estimate (5.1.25), we obtain for any $k\in\{0,1,\dots,N-1\}$:

$$\|u(T_k)\|_{\dot{H}^{\gamma}_x(\mathbb{R}^3)}+\|\partial_t u(T_k)\|_{\dot{H}^{\gamma-1}_x(\mathbb{R}^3)}\le\sum_{j=0}^{k-1}(2C_\gamma)^{k-1-j}(1+\|\sigma_{-1}\|_{L^\infty_t L^{\frac{12}{5}+}_x}).$$

Hence by (5.1.21), it follows that:

$$\|K_{S,1}\|_{L^{\frac{2}{\gamma}}_t L^{\frac{2}{1-\gamma}}_x([T_k,T_{k+1}]\times\mathbb{R}^3)}\le 2CC_\gamma\Big(1+\|\sigma_{-1}\|_{L^\infty_t L^{\frac{12}{5}+}_x}+\sum_{j=0}^{k-1}(2C_\gamma)^{k-1-j}(1+\|\sigma_{-1}\|_{L^\infty_t L^{\frac{12}{5}+}_x})\Big)\tag{5.1.26}$$

Using the triangle inequality and summing (5.1.26) over $k=0,1,\dots,N-1$ and noting that N is some finite positive integer depending on $\|\Phi_{-1}\|_{L_t^\infty L_x^2}$, we get

$$
\begin{aligned}
&\|K_{S,1}\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([0,T)\times\mathbb{R}^3)} \\
&\qquad \le \sum_{k=0}^{N-1} 2CC_\gamma\Big(1+\|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}}+\sum_{j=0}^{k-1}(2C_\gamma)^{k-1-j}(1+\|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}})\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \lesssim 1+\|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}}.
\end{aligned}
$$

Since $\frac{2}{1-\gamma}=4+$, we obtain the desired estimate (5.1.13). $\square$

Chapter 6

Proof of New Moment Criteria

In this chapter, we will prove the stated continuation criteria using moment bounds in both the noncompact momentum support and compact momentum support settings.

6.1 Noncompact Support

We begin by proving the noncompact support criteria. Recall that by Theorem 20, we simply need to control the right hand side of the inequality in Propisition 28. To this end, we first use Hölder's inequality and Lemma 34 to obtain for any $\lambda', \tilde{\lambda}, \hat{\lambda} > 0$:

$$\begin{aligned}
&\|K\sigma_{-1}\|_{L_t^\infty([0,T];L_x^{2+\lambda}L_p^1)} + \|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^{3+\tilde{\lambda}}L_p^1)} \\
&\qquad \lesssim \|K\|_{L_t^\infty([0,T];L_x^{3+\lambda'})} \|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^3L_p^1)} + \|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^{3+\tilde{\lambda}}L_p^1)} \\
&\qquad \lesssim \|K\|_{L_t^\infty([0,T];L_x^{3+\lambda'})} \|p_0^{5+2\hat{\lambda}} f\|_{L_t^\infty([0,T];L_x^1L_p^1)}^{\frac{1}{3+\hat{\lambda}}} + \|p_0^{5+2\tilde{\lambda}} f\|_{L_t^\infty([0,T];L_x^1L_p^1)}^{\frac{1}{3+\tilde{\lambda}}}
\end{aligned} \tag{6.1.1}$$

By interpolation and the conservation law (2.2.4), there exists some $\theta \in (0,1)$ such that

$$\|K\|_{L^\infty_t([0,T];L^{3+\lambda'}_x)} \lesssim \|K\|^{\theta}_{L^\infty_t([0,T];L^{6+\lambda'}_x)} \|K\|^{1-\theta}_{L^\infty_t([0,T];L^2_x)} \lesssim \|K\|^{\theta}_{L^\infty_t([0,T];L^{6+\lambda'}_x)}.$$

By the estimates (4.2.8) and (5.1.2), we can further bound this by:

$$\|K\|_{L^\infty_t([0,T];L^{6+\lambda'}_x)} \lesssim 1 + \|p_0^{\frac{(3+\lambda')(1+\gamma)+3+\delta}{2+\gamma}} f\|^{2+\gamma}_{L^\infty_t([0,T);L^1_x L^1_p)} \|p_0^{3+\lambda'} f\|^{1-\gamma}_{L^\infty_t([0,T);L^1_x L^1_p)} \quad (6.1.2)$$

for any $\delta > 0$ and $\gamma \in (0,1)$. Choosing $\delta < \lambda'$, we obtain that

$$\frac{(3+\lambda')(1+\gamma)+3+\delta}{2+\gamma} < 3+\lambda',$$

and hence by (6.1.2):

$$\|K\|_{L^\infty_t([0,T];L^{6+\lambda'}_x)} \lesssim 1 + \|p_0^{3+\lambda'} f\|^{3}_{L^\infty_t([0,T);L^1_x L^1_p)}.$$

Putting these bounds together, we obtain:

$$\begin{aligned} \|K\sigma_{-1}\|_{L^\infty_t([0,T];L^{2+\lambda}_x L^1_p)} + \|\sigma_{-1}\|_{L^\infty_t([0,T];L^{3+\tilde{\lambda}}_x L^1_p)} & \\ \lesssim (1 + \|p_0^{3+\lambda'} f\|^3_{L^\infty_t([0,T);L^1_x L^1_p)}) \|p_0^{5+2\hat{\lambda}} f\|^{\frac{1}{3+\hat{\lambda}}}_{L^\infty_t([0,T];L^1_x L^1_p)} & \\ + \|p_0^{5+2\tilde{\lambda}} f\|^{\frac{1}{3+\tilde{\lambda}}}_{L^\infty_t([0,T];L^1_x L^1_p)} & \quad (6.1.3) \end{aligned}$$

Thus, in order to satisfy the known continuation criteria stated in Theorem 20, we simply need to bound $\|p_0^{5+\lambda} f\|_{L_t^\infty([0,T];L_x^1 L_p^1)} \lesssim 1$ for some $\lambda > 0$. To this end, we can use the estimates on K to prove that:

Proposition 42. *Consider initial data f_0 such that $\|p_0^N f_0\|_{L_x^1 L_p^1} \lesssim 1$ and suppose we have the bound $\|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^2)} + \|p_0^M f\|_{L_t^\infty([0,T];L_x^1 L_p^1)} \lesssim 1$ where $M > \frac{N+3}{2}$ for some $N > 3$. Then*

$$\|p_0^N f\|_{L_t^\infty([0,T];L_x^1 L_p^1)} \lesssim 1$$

and

$$\|K\|_{L_t^\infty([0,T];L_x^{N+3})} \lesssim 1$$

.

Proof. By the estimates on K_T and K_S given by (4.2.8) and (5.1.2) respectively and Proposition 29, we obtain for some $\gamma \in (0,1)$:

$$\|p_0^N f\|_{L_t^\infty([0,T];L_x^1 L_p^1)} \lesssim 1 + \|p_0^{\frac{N(1+\gamma)+3+\delta}{2+\gamma}} f\|_{L_t^\infty([0,T);L_x^1 L_p^1)}^{2+\gamma} \|p_0^N f\|_{L_t^\infty([0,T);L_x^1 L_p^1)}^{1-\gamma} \tag{6.1.4}$$

Choose appropriate $0 < \gamma < 1$ and $\delta > 0$ such that $\frac{N(1+\gamma)+3+\delta}{2+\gamma} = M$ and let the implicit constant in (6.1.4) be denoted by $C > 0$. (Suppose $M = \frac{N+3+\epsilon}{2}$. Then set $\delta = \epsilon + \left(\frac{N+3+\epsilon}{2} - N\right)\gamma$. For $\gamma \in (0,1)$ sufficiently small, $\delta > 0$.) Thus, since

$\|p_0^M f\|_{L_t^\infty([0,T];L_x^1L_p^1)} \leq B$ for some constant $B > 0$ and by Young's inequality:

$$\begin{aligned}\|p_0^N f\|_{L_t^\infty([0,T];L_x^1L_p^1)} &\leq C + CB^\beta \|p_0^N f\|^{1-\gamma}_{L_t^\infty([0,T);L_x^1L_p^1)} \\ &\leq C + \gamma C^{\frac{1}{\gamma}} B^{\frac{2+\gamma}{\gamma}} + (1-\gamma)\|p_0^N f\|_{L_t^\infty([0,T];L_x^1L_p^1)} \quad (6.1.5)\end{aligned}$$

Thus for some constant $\tilde{C}$,

$$\|p_0^N f\|_{L_t^\infty([0,T];L_x^1L_p^1)} \leq \frac{1}{\gamma}(C + \gamma \tilde{C} B^{\frac{2+\gamma}{\gamma}}) \lesssim 1 \tag{6.1.6}$$

Finally, plugging (6.1.6) into (4.2.8) and (5.1.2), we obtain that

$$\|K\|_{L_t^\infty([0,T];L_x^{N+3})} \lesssim 1.$$

□

Theorem 43. *Suppose $\|p_0^{\tilde{N}} f_0\|_{L_x^1L_p^1} \lesssim 1$ for some $\tilde{N} > 5$. Let $M > 3$. Then $\|p_0^M f\|_{L_t^\infty([0,T];L_x^1L_p^1)} \lesssim 1$ is a continuation criteria for the Vlasov-Maxwell system without compact support.*

Proof. First, if $M > 5$, then by the comment under (6.1.3), we are done. (Note that this is also a known continuation criteria found in [38].) If $3 < M < 5$, note that by Lemma 34

$$\|\sigma_{-1}\|_{L_t^\infty([0,T];L_x^2)} + \|p_0^M f\|_{L_t^\infty([0,T];L_x^1L_p^1)} \lesssim \|p_0^M f\|_{L_t^\infty([0,T];L_x^1L_p^1)} \lesssim 1.$$

Suppose $M > 3$ and $\|p_0^M f\|_{L_t^\infty([0,T];L_x^1L_p^1)} \lesssim 1$. Since $\|p_0^{\tilde{N}} f_0\|_{L_x^1L_p^1} \lesssim 1$ for some $\tilde{N} > 5$, it follows that $\|p_0^N f_0\|_{L_x^1L_p^1} \lesssim 1$ for all $N < 5$. Then, by Proposition 42, we obtain that $\|p_0^N f\|_{L_t^\infty([0,T];L_x^1L_p^1)} \lesssim 1$ for $N = 2M - 3 - \delta$ for $\delta > 0$ as long as $3 < 2M - 3 - \delta < 5$. Note that if $M > 3$, then $2M - 3 = M + M - 3 > M$. Hence setting $\delta = \frac{M-3}{2}$, we obtain that $N = 2M - 3 - \delta = 2M - 3 - \frac{M-3}{2} = M + \frac{M-3}{2} > M > 3$.

Let $M = M_0$ and suppose, as above, that

$$\|p_0^M f\|_{L_t^\infty([0,T];L_x^1L_p^1)} = \|p_0^{M_0} f\|_{L_t^\infty([0,T];L_x^1L_p^1)} \lesssim 1.$$

Then, if $M_1 = M_0 + \frac{M_0-3}{2} < 5$, we know by the above that

$$\|p_0^{M_1} f\|_{L_t^\infty([0,T];L_x^1L_p^1)} \lesssim 1.$$

Define the sequence M_i in this manner: let $M_{i+1} = M_i + \frac{M_i-3}{2}$. Notice that since $M_0 > 3$, by the earlier argument, we obtain that $M_1 > 3$. By induction, we obtain that $M_k > 3$ for all $k \in \mathbb{N}$.

Since $M = M_0 > 3$, there exists an $\epsilon > 0$ such that $M_0 = 3 + \epsilon$. We now claim that $M_k > M_0 + \frac{k\epsilon}{2}$. Indeed, this is true in the case of M_0. Suppose it holds for $k = n$. Then, since $M_n - 3 > M_0 - 3 + \frac{n\epsilon}{2} = \frac{\epsilon}{2} + \frac{n\epsilon}{4} > \frac{\epsilon}{2}$, it follows that $M_{n+1} = M_n + \frac{M_n-3}{2} > M_0 + \frac{n\epsilon}{2} + \frac{\epsilon}{2} = M_0 + \frac{(n+1)\epsilon}{2}$.

Thus, as n tends to infinity, we know that M_n tends to infinity. Thus, there exists some $m \in \mathbb{N}$ such that $3 < M_m < 5$ but $M_{m+1} > 5$. Under our assumption that $\|p_0^{M_0} f\|_{L_t^\infty([0,T];L_x^1 L_p^1)} \lesssim 1$, we can iterate the argument above to obtain that $\|p_0^{M_n} f\|_{L_t^\infty([0,T];L_x^1 L_p^1)} \lesssim 1$ for all positive integers $n \leq m$. Finally, choose some $\delta > 0$ such that $5 < 2M_m - 3 - \delta < \tilde{N}$. (This is certainly possible since choosing $\delta = \frac{M_i - 3}{2}$, we obtain by our choice of m that $M_{m+1} = 2M_m - 3 - \delta > 5$. On the other hand, if $M_{m+1} > \tilde{N} > 5$, we simply choose a large delta such that $2M - 3 - \delta$ is still greater than 5 but is less than $\tilde{N}$.) Let us set $\tilde{M} = 2M_m - 3 - \delta$. Since $\tilde{M} < \tilde{N}$, we know that $\|p_0^{\tilde{M}} f_0\|_{L_x^1 L_p^1} \lesssim 1$. By Proposition 42, we obtain that $\|p_0^{\tilde{M}}\|_{L_x^1 L_p^1} \lesssim 1$. Since $\tilde{M} > 5$, by the comment under (6.1.3), we are done. $\square$

6.2 Compact Support

In this section, we first recall the decomposition method in [41] and then apply the above estimates to gain a bound on the size of the momentum support of f, which we will denote by:

$$P(T) \stackrel{\text{def}}{=} 1 + \sup\{p \in \mathbb{R}^3 | \exists (t,x) \in [0,T) \times \mathbb{R}^3 \text{such that} f(t,x,p) \neq 0\} \tag{6.2.1}$$

By the method of characteristics:

$$\frac{dV}{ds}(s;t,x,p') = E(s, X(s;t,x,p')) + \hat{V}(s;t,x,p') \times B(s, X(s;t,x,p')) \tag{6.2.2}$$

Taking the Euclidean inner product with $\hat{V}(s;t,x,p')$ on both sides and then integrating in time, we obtain:

$$\sqrt{1+|V(s;t,x,p')|^2} = \sqrt{1+|V(0;t,x,p')|^2} + \int_0^T E(s,X(s;t,x,p'))\cdot\hat{V}(s;t,x,p')ds$$

First, for $i=1,2,3$ and $K_j = E_j + (\hat{p}\times B)_j$, we can decompose the electric field:

$$\begin{aligned}E_i(t,x) &= E_i^{(0)}(x) + \int_{\mathbb{R}^3}\Big(\frac{(1-|\hat{p}|^2)(x_i-t\hat{p})}{(t-\hat{p}\cdot x)^2}\Big)Y \star_{t,x} (f\chi_{t\geq 0})dp \\ &\quad - \sum_{j=1}^{3}\int_{\mathbb{R}^3}\Big(\frac{[t(t-\hat{p}\cdot x)(\hat{p}_i\hat{p}_j - e_i) + (x_i - t\hat{p}_i)(x_j - (\hat{p}\cdot x)\hat{p}_j)]}{p_0(t-\hat{p}\cdot x)^2}\Big) \star_{t,x} (K_j f\chi_{t\geq 0})dp \\ &\stackrel{\text{def}}{=} E_i^{(0)}(x) + F_i(t,x) + G_i(t,x) \qquad (6.2.3)\end{aligned}$$

where e_i is the unit vector with all entries equal to 0 except for the ith entry which is equal to 1. Also, the double convolution $\star_{t,x}$ is a binary operation defined by:

$$f_1 \star_{t,x} f_2 = \int_{\mathbb{R}\times\mathbb{R}^3} f_1(t-s,x-y)f_2(s,y)\ ds\ dy \qquad (6.2.4)$$

and

$$Y \stackrel{\text{def}}{=} (4\pi t)^{-1}\delta_{|x|=t} \qquad (6.2.5)$$

Following the scheme of [41], we can decompose the characteristic integral of the electric field into:

$$\int_0^T E(s, X(s;t,x,p')) \cdot \hat{V}(s;t,x,p')ds = I_0 + I_F + I_G \tag{6.2.6}$$

where I_0 depends only on the initial data term $E^{(0)}$ and

$$I_F \stackrel{\text{def}}{=} \int_0^T F(s, X(s;t,x,p')) \cdot \hat{V}(s;t,x,p')ds \tag{6.2.7}$$

and

$$I_G \stackrel{\text{def}}{=} \int_0^T G(s, X(s;t,x,p')) \cdot \hat{V}(s;t,x,p')ds \tag{6.2.8}$$

Pallard then bounds I_G by:

$$\begin{aligned}&|I_G| \\ &\lesssim \int_0^T \int_s^T \int_{|y|=t-s} \int_{\mathbb{R}^3_p} \frac{(f|K|)(s, X(t)-y, p)}{p_0(1-\hat{p}\cdot\omega)} \left(\sqrt{1-\hat{V}(t)\cdot\omega}\right) dp \frac{d\sigma(y)dt}{4\pi|t-s|} ds\end{aligned} \tag{6.2.9}$$

From here, Pallard [41] bounds the integral

$$\int_{\mathbb{R}^3_p} \frac{(f|K|)(s, X(t)-y, p)}{p_0(1-\hat{p}\cdot\omega)} dp$$

using the term $m(t,x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} p_0 f(t,x,p)dp$. Instead, we preserve the singularity in the denominator:

$$|I_G| \lesssim \int_0^T \int_s^T \int_{|y|=t-s} (\sigma_{-1}|K|)(s, X(t)-y)\Big(\sqrt{1-\hat{V}(t)\cdot\omega}\Big)\frac{d\sigma(y)dt}{4\pi|t-s|}ds \quad (6.2.10)$$

Split the integral into $I_G \lesssim I_G^{'} + I_G^{''}$ as follows:

$$\begin{aligned}|I_G| \lesssim &\int_0^T \int_s^{s+\epsilon(s)} \int_{|y|=t-s} (\sigma_{-1}|K|)(s, X(t)-y)\Big(\sqrt{1-\hat{V}(t)\cdot\omega}\Big)\frac{d\sigma(y)dt}{4\pi|t-s|}ds \\ &+ \int_0^T \int_{s+\epsilon(s)}^T \int_{|y|=t-s} (\sigma_{-1}|K|)(s, X(t)-y)\Big(\sqrt{1-\hat{V}(t)\cdot\omega}\Big)\frac{d\sigma(y)dt}{4\pi|t-s|}ds \quad (6.2.11)\end{aligned}$$

for

$$\epsilon(s) = \frac{T-s}{1+P(s)^8} \quad (6.2.12)$$

Note that the power of $P(s)$ in (6.2.12) is useful for bounding $I_G^{'}$ as in [41]. First, let us bound $I_G^{''}$. By computing using Hölder's inequality as in [41]:

$$\begin{aligned}|I_G^{''}| \lesssim &\int_0^T \Big|\int_{s+\epsilon(s)}^T \int_{|y|=t-s} (\sigma_{-1}|K|)^{\frac{3}{2}}(s, X(t)-y)(1-\hat{V}(t)\cdot\omega)d\sigma(y)dt\Big|^{\frac{2}{3}} \\ &\times \Big(\int_{s+\epsilon(s)}^T \int_{|y|=t-s} ((1-\hat{V}(t)\cdot\omega)^{-\frac{1}{6}})^3 d\sigma dt\Big)^{\frac{1}{3}} ds \\ \lesssim &\int_0^T \Big|\int_{s+\epsilon(s)}^T \int_{|y|=t-s} (\sigma_{-1}|K|)^{\frac{3}{2}}(s, X(t)-y)(1-\hat{V}(t)\cdot\omega)d\sigma(y)dt\Big|^{\frac{2}{3}} \ln^{\frac{1}{3}}\Big(\frac{T-s}{\epsilon(s)}\Big)ds \\ &\quad (6.2.13)\end{aligned}$$

Setting $\omega = \omega(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$:

$$\int_{s+\epsilon(s)}^{T} \int_{|y|=t-s} (\sigma_{-1}|K|)^{\frac{3}{2}}(s, X(t)-y)(1-\hat{V}(t)\cdot\omega)d\sigma(y)dt$$
$$= \int_{s+\epsilon(s)}^{T} \int_{|y|=t-s} (\sigma_{-1}|K|)^{\frac{3}{2}}(s, X(t)-(t-s)\omega(\theta,\phi))$$
$$(1-\hat{V}(t)\cdot\omega(\theta,\phi))(t-s)^2 \sin\theta d\theta d\phi dt \quad (6.2.14)$$

Consider the change of variables $\Psi : (s_1, s_2) \times (0,\pi) \times (0, 2\pi) \to \Psi((s_1, s_2) \times (0,\pi) \times (0, 2\pi))$ mapping

$$(t, \theta, \phi) \mapsto X(t) - (t-s)\omega(\theta,\phi) \stackrel{\text{def}}{=} z$$

The Jacobian of this map is $J = (\hat{V}(t)\cdot\omega - 1)(t-s)^2\sin\theta$. Applying this change of variables to (6.2.14) and inserting our choice of $\epsilon(s)$, we obtain:

$$|I_G''| \lesssim \int_0^T \Big| \int_{\Psi((s_1,s_2)\times(0,\pi)\times(0,2\pi))} (\sigma_{-1}|K|)^{\frac{3}{2}}(s,z)dzdt \Big| \ln^{\frac{1}{3}}\Big(1+P(s)\Big)ds \quad (6.2.15)$$

Following [41] precisely, we also know that $I_G' \lesssim 1$. (This is done through first applying Hölder's inequality to isolate the first term and then using conservation law $\|K\|_{L_t^\infty L_x^2} \lesssim 1$. Finally, by the definition of $\epsilon(s)$, the leftover integral is bounded.) Thus, we arrive at the estimate:

$$|I_G| \lesssim 1 + \|\sigma_{-1}|K|\ln^{\frac{1}{3}}(1+P(t))\|_{L_t^1 L_x^{\frac{3}{2}}([0,T]\times\mathbb{R}^3)} \quad (6.2.16)$$

Next, we recognize that F is equivalent to our E_T term as expressed in (2.3.1). Thus, using the proof of Proposition 28:

$$|I_F| \lesssim \|\sigma_{-1}\|_{L_t^\infty L_x^{3+}} \tag{6.2.17}$$

In conclusion:

Proposition 44. *By (6.2.16) and (6.2.17), we have the following bound for $P(T)$:*

$$|P(T)| \lesssim 1 + \|\sigma_{-1}\|_{L_t^\infty L_x^{3+}} + \|\sigma_{-1}|K|\ln^{\frac{1}{3}}(1+P(t))\|_{L_t^1 L_x^{\frac{3}{2}}([0,T]\times\mathbb{R}^3)} \tag{6.2.18}$$

We conclude by applying the estimates given by (4.2.14), (4.2.15) and (5.1.13) on $|K|$ under the assumption that $\|\Phi_{-1}\|_{L_t^\infty L_x^2} \lesssim 1$:

$$\begin{aligned}
\|\sigma_{-1}|K|\ln^{\frac{1}{3}}(1+P(t))\|_{L_t^1 L_x^{\frac{3}{2}}([0,T]\times\mathbb{R}^3)} &\leq \ln^{\frac{1}{3}}(1+P(T))\||K|\|_{L_t^1 L_x^{4+}} \|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}-}} \\
&\lesssim \ln^{\frac{1}{3}}(1+P(T))(1+\|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}})\|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}-}}
\end{aligned}$$

Notice that our choice of Hölder exponents used in the first line above allow for the Lebesgue norm exponents on both terms involving σ_{-1} to be approximately equivalent to $\frac{12}{5}$. This choice of Hölder exponents simplifies our computation. Other choices yield similar results. We can now use Lemma 34 to bound each term in (6.2.18) for some

$\beta > 0$ arbitrarily small:

$$\|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}-}} \lesssim \|p_0^{\frac{24}{5r}-1} f\|_{L_t^\infty L_x^r L_p^1}^{\frac{5r}{12}+} \tag{6.2.19}$$

$$\|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}} \lesssim \|p_0^{\frac{24}{5r}-1+\beta} f\|_{L_t^\infty L_x^r L_p^1}^{\frac{5r}{12}-} \tag{6.2.20}$$

$$\|\sigma_{-1}\|_{L_t^\infty L_x^{3+}} \lesssim \|p_0^{\frac{6}{r}-1+\beta} f\|_{L_t^\infty L_x^r L_p^1}^{\frac{r}{3}-} \tag{6.2.21}$$

We can extract $\frac{12}{10r} - \delta$ power of p_0 for some $\delta > 0$ arbitrarily small from each of (6.2.19) and (6.2.20) and $\frac{3}{r} - \delta$ power of p_0 from (6.2.21). Thus:

$$\|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}-}} \lesssim \|p_0^{\frac{18}{5r}-1} f\|_{L_t^\infty L_x^r L_p^1}^{\frac{5r}{12}+} P(T)^{\frac{1}{2}-} \tag{6.2.22}$$

$$\|\sigma_{-1}\|_{L_t^\infty L_x^{\frac{12}{5}+}} \lesssim \|p_0^{\frac{18}{5r}-1+\beta} f\|_{L_t^\infty L_x^r L_p^1}^{\frac{5r}{12}} P(T)^{\frac{1}{2}-} \tag{6.2.23}$$

$$\|\sigma_{-1}\|_{L_t^\infty L_x^{3+}} \lesssim \|p_0^{\frac{3}{r}-1+\beta} f\|_{L_t^\infty L_x^r L_p^1}^{\frac{r}{3}-} P(T)^{1-} \tag{6.2.24}$$

where $P(T)^{1-}$ indicates a power of $P(T)$ smaller than 1 by an arbitrarily small amount. Assume that $\|p_0^{\frac{18}{5r}-1+\beta} f\|_{L_t^\infty L_x^r L_p^1} \lesssim 1$. Hence

$$\|p_0^{\frac{3}{r}-1+\beta} f\|_{L_t^\infty L_x^r L_p^1} \lesssim \|p_0^{\frac{18}{5r}-1+\beta} f\|_{L_t^\infty L_x^r L_p^1} \lesssim 1.$$

Plugging these into (6.2.18), we obtain the bound:

$$P(T) \lesssim 1 + \ln^{\frac{1}{3}}(1 + P(T))P(T)^{1-} \tag{6.2.25}$$

which implies that $P(T) \lesssim 1$ since $P(T) > 1$. Finally the last term we need to take care of is the assumption that $\|\Phi_{-1}\|_{L_t^\infty L_x^2} \lesssim 1$. By employing similar proof to Lemma 34, we see that:

Proposition 45. *Given $r \in [1, 2]$, we have the estimate:*

$$\|\Phi_{-1}\|_{L_x^2} \lesssim \|p_0^\alpha f\|_{L_x^r L_p^1}^{\frac{r}{2}} \tag{6.2.26}$$

where $\alpha > \frac{2}{r} - 1$.

Proof. Fix some $\omega \in \mathbb{S}^2$ and let $r = \frac{2}{q}$. Then $\frac{q'}{2} \geq 1$ (since $\frac{1}{q} + \frac{1}{q'} = 1$ and $q \geq 2$) and

$\frac{1}{1+\hat{p}\cdot\omega} \lesssim p_0^2$ implies:

$$\int_{\mathbb{R}^3} \frac{f(t,x,p)}{p_0(1+\hat{p}\cdot\omega)^{\frac{1}{2}}} dp \lesssim \Big(\int_{\mathbb{R}^3} \frac{1}{p_0^{(\beta+1)q'}(1+\hat{p}\cdot\omega)^{\frac{q'}{2}}} dp\Big)^{\frac{1}{q'}} \Big(\int_{\mathbb{R}^3} p_0^{\beta q} f(t,x,p) dp\Big)^{\frac{1}{q}} \|f\|_{L^\infty_{t,x,p}}^{\frac{q-1}{q}} \tag{6.2.27}$$

$$\lesssim \Big(\int_{\mathbb{R}^3} \frac{1}{p_0^{\beta q'+2}(1+\hat{p}\cdot\omega)} dp\Big)^{\frac{1}{q'}} \Big(\int_{\mathbb{R}^3} p_0^{\beta q} f(t,x,p) dp\Big)^{\frac{1}{q}} \tag{6.2.28}$$

By (4.2.4) in the proof of Lemma 34, we know that the first integral on the right hand side is bounded by a constant when $\beta q' > 1$, i.e. $\beta q > q-1$. Taking the L^2 norm of this inequality:

$$\left\| \int_{\mathbb{R}^3} \frac{f(t,x,p)}{p_0(1+\hat{p}\cdot\omega)^{\frac{1}{2}}} dp \right\|_{L^2_x} \lesssim \left\| \Big(\int_{\mathbb{R}^3} p_0^{\beta q} f(t,x,p) dp\Big)^{\frac{1}{q}} \right\|_{L^2_x} = \left\| \Big(\int_{\mathbb{R}^3} p_0^{\beta q} f(t,x,p) dp\Big) \right\|_{L^{\frac{2}{q}}_x L^1_p}^{\frac{1}{q}} \tag{6.2.29}$$

Finally, setting $\alpha = \beta q$, we obtain that

$$\alpha > q - 1 = \frac{2}{r} - 1.$$

Taking the maximum over all $\omega \in \mathbb{S}^2$ retains the same upper bound. Hence, this completes the proof. $\square$

In particular, notice that for $1 \le r \le 2$:

$$\|\Phi_{-1}\|_{L^2_x} \lesssim \|p_0^{\frac{2}{r}-1+\beta} f\|_{L^r_x L^1_p} \lesssim \|p_0^{\frac{18}{5r}-1+\beta} f\|_{L^\infty_t L^r_x L^1_p} \lesssim 1 \tag{6.2.30}$$

Thus, if $\|p_0^{\frac{18}{5r}-1+\beta}f\|_{L_t^\infty L_x^r L_p^1} \lesssim 1$, then $\|\Phi_{-1}\|_{L_x^2} \lesssim 1$. Hence, all of the terms in (6.2.18), which implicitly included the assumption $\|\Phi_{-1}\|_{L_x^2} \lesssim 1$, are bounded. Thus, indeed we do know that $P(T) \lesssim 1$. Thus, we can extend our local solution on the time interval $[0, T)$ to a larger time interval $[0, T + \epsilon]$. This concludes the proof of

$$\|p_0^{\frac{18}{5r}-1+\beta}f\|_{L_t^\infty L_x^r L_p^1} \lesssim 1 \tag{6.2.31}$$

as a continuation criteria for $1 \le r \le 2$.

Chapter 7

Another Continuation Criteria

7.1 Luk-Strain Plane Support Result

In [37], Luk-Strain prove that compact momentum support on a fixed two-dimensional plane is sufficient for global wellposedness of the three-dimensional relativistic Vlasov-Maxwell system.

Theorem 46. *Consider initial data* (f_0, E_0, B_0) *where* $f_0 \in H^5(\mathbb{R}^3_x \times \mathbb{R}^3_p)$ *is non-negative and has compact support in* (x, p)*, and* $E_0, B_0 \in H^5(\mathbb{R}^3_x)$ *such that* (3) *holds. Suppose* (f, E, B) *is the unique classical solution to the relativistic Vlasov-Maxwell system* $(1) - (3)$ *in the time interval* $[0, T)$*. Assume that there exists a plane* $Q \subset \mathbb{R}^3$ *with* $0 \in Q$ *and a bounded continuous function* $\kappa : [0, T_+) \to \mathbb{R}^3$ *such that*

$$f(t, x, p) = 0 \;\; for \;\; |\mathbb{P}_Q p| \geq \kappa(t), \;\; \forall x \in \mathbb{R}^3.$$

Then there exists an $\epsilon > 0$ such that the solution extends uniquely in C^1 to a larger time interval $[0, T + \epsilon]$.

In this chapter, we will allow the two-dimensional plane of momentum support to vary continuously in time. Our new result is described in the next section.

7.2 Modification of Luk-Strain Theorem

Our final result on the relativistic Vlasov-Maxwell system improves the continuation criteria due to Luk-Strain in [37]. First, consider a family of planes $\{Q(t)\}_{t\in[0,T]}$. At $t = 0$, we choose a normal vector $n_3(0)$ orthogonal to the plane $Q(0)$ at the origin.

Definition 47. *A family of planes $\{Q(t)\}_{t\in[0,T]}$ containing the origin is considered to be* ***uniformly continuous family of planes*** *in the following sense: There exists a partition $[T_i, T_{i+1})$ of $[0, T)$ such that locally in a small time interval, for say $s \in [T_i, T_{i+1})$, we can let $n_3(s)$ be the normal to $Q(s)$ at the origin that is on the same half of $\mathbb{R}^3$ as $n_3(T_i)$, meaning $\angle(n_3(s), n_3(T_i)) < \angle(n_3(s), -n_3(T_i))$, where $\angle(v, w) \stackrel{def}{=} \cos^{-1}(\frac{v\cdot w}{|v||w|})$. Then, the map $n_3 : [0, T) \to \mathbb{S}^2$ is uniformly continuous.*

Using this definition, we prove the following:

Theorem 48. *Suppose we have initial data $f_0(x, p) \in H^5(\mathbb{R}^3 \times \mathbb{R}^3)$ with compact support in (x, p), $E_0, B_0 \in H^5(\mathbb{R}^3)$. Let (f, E, B) be the unique classical solution in $L^\infty_t([0, T); H^5_{x,p}) \times L^\infty_t([0, T); H^5_x) \times L^\infty_t([0, T); H^5_x)$ to the Vlasov-Maxwell system in*

$[0,T)$. Let $\{Q(t)\}$ be a uniformly continuous family of planes containing the origin such that there exists a bounded, continuous function $\kappa : [0,T) \to \mathbb{R}_+$ such that

$$f(t,x,p) = 0 \textit{ for } |\mathbb{P}_{Q(t)}p| \geq \kappa(t) \ \forall x \in \mathbb{R}$$

Then there exists $\epsilon > 0$ such that our solution can be extended continuously in time in H^5 to $[0, T+\epsilon]$.

A more general theorem can be proven. Theorem 48 will be a special case of this theorem. First, we need to define a time dependent coordinate system on $\mathbb{R}^3$ which will depend on the plane $Q(t)$. Let $\{n_1(t), n_2(t), n_3(t)\}$ be unit vectors such that $\{n_1(t), n_2(t)\}$ span $Q(t)$ and $n_3(t)$ is the unit normal to $Q(t)$ as defined earlier.

Fix a time $t \in [0,T)$. By uniform continuity of $n_3(t)$, there exists a partition of $[0,t) = \cup_{i=0}^{n_t}[T_i, T_{i+1})$ (the number of intervals in the partition n_t depends on t and $T_{n_t+1} = t$) such that for $s \in [T_i, T_{i+1})$, we have:

$$\angle(n_3(s), n_3(T_i)) < \angle(-n_3(s), n_3(T_i)) \tag{7.2.1}$$

and

$$\angle(n_3(s), n_3(T_i)) < \frac{P(t)^{-1}}{4} \tag{7.2.2}$$

We will use this precise partition for the proof of Theorem 49 in this paper.

Theorem 49. *Suppose we have initial data $f_0(x,p) \in H^5(\mathbb{R}^3 \times \mathbb{R}^3)$ with compact support in (x,p), $E_0, B_0 \in H^5(\mathbb{R}^3)$. Let (f,E,B) be the unique classical solution in $L_t^\infty([0,T); H^5_{x,p}) \times L_t^\infty([0,T); H^5_x) \times L_t^\infty([0,T); H^5_x)$ to the Vlasov-Maxwell system in $[0,T)$. Let $\{Q(t)\}$ be a uniformly continuous family of planes. Suppose for each $t \in [0,T)$, there exists a measurable, positive function $\kappa : [0,T) \times [0,2\pi] \to \mathbb{R}_+$ such that $\kappa(t,\gamma) > 1$,*

$$sup\{|\mathbb{P}_{Q(t)}p|\colon \frac{p \cdot n_2(t)}{p \cdot n_1(t)} = tan(\gamma), f(t,x,p) \neq 0 \text{ for some } x \in \mathbb{R}^3\} < \kappa(t,\gamma)$$

and

$$\int_0^T \Big(A(t)^2 + (\int_0^t A(s)^8 ds)^{\frac{1}{2}}\Big) dt < +\infty \text{ where } A(t) = \|\kappa(t,\cdot)\|_{L^4_\gamma}$$

Then there exists $\epsilon > 0$ such that our solution can be extended continuously in time to $[0, T+\epsilon]$.

Note that γ depends on $p \in \mathbb{R}^3$, so we actually have $\tan(\gamma) = \tan(\gamma(p)) = \frac{p \cdot n_2(t)}{p \cdot n_1(t)}$.

We modify methods used in [37] to prove Theorem 49. We wish to show that the quantity

$$P(t) = 2 + \sup\{|p|\colon f(s,x,p) \neq 0 \text{ for some } 0 \leq s \leq t \text{ and } x \in \mathbb{R}^3\}$$

is bounded on $[0,T)$. By the method of characteristics (see [37]), we have the bound

$$P(t) \lesssim 1 + \sup_{(t,x,p)\in\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3} \int_0^t |E(s;X(s;t,x,p))|+|B(s;X(s;t,x,p))|ds.$$

We wish to bound the momentum support quantity $P(t)$. To do so, we first find appropriate estimates on E and B. We again use the decomposition:

$$4\pi E(x,t) = (E)_0 + E_{S,1} + E_{S,2} + E_T$$

$$4\pi B(x,t) = (B)_0 + B_{S,1} + E_{S,2} + B_T$$

where $(E)_0$ and $(B)_0$ depend only on the initial data. We have the following estimates from Proposition 3.1 and Proposition 3.4 in [37]:

$$|E_T(t,x)|+|B_T(t,x)| \lesssim \int_{C_{t,x}} \int_{\mathbb{R}^3} \frac{f(s,x+(t-s)\omega,p)}{(t-s)^2 p_0^2 (1+\hat{p}\cdot\omega)^{\frac{3}{2}}} dp\ d\omega \quad (7.2.3)$$

$$|E_{S,1}(t,x)|+|B_{S,1}(t,x)| \lesssim \int_{C_{t,x}} \int_{\mathbb{R}^3} |B| \frac{f(s,x+(t-s)\omega,p)}{(t-s) p_0 (1+\hat{p}\cdot\omega)^{\frac{1}{2}}} dp\ d\omega \quad (7.2.4)$$

$$|E_{S,2}(t,x)|+|B_{S,2}(t,x)| \lesssim \int_{C_{t,x}} \int_{\mathbb{R}^3} \frac{(|E\cdot\omega|+|B\cdot\omega|+|B+\omega\times E|)f(s,x+(t-s)\omega,p)}{(t-s) p_0 (1+\hat{p}\cdot\omega)} dp\ d\omega \quad (7.2.5)$$

Next, we prove analogous bounds on the momentum integral $\int_{\mathbb{R}^3} \frac{f(s,x+(t-s)\omega,p)}{(t-s)^2 p_0^2 (1+\hat{p}\cdot\omega)^{\frac{3}{2}}} dp$ as those found in [37]. Partitioning the time interval $[0,T)$ into subintervals $[T_i, T_{i+1}]$

small enough as described in (7.2.1) and (7.2.2). Applying these two conditions to bound (7.2.3), (7.2.4) and (7.2.5) on each subinterval $[T_i, T_{i+1}]$ in an analogous method to [37]. (These conditions allow us to approximate the integrals in each time subinterval by the integral at the endpoints, as the variation in the momentum support plane is very small on each subinterval. This observation is the key to proving Theorem 49.) We then sum over the partition to prove analogous bounds on the field terms to those found in [37]. From here, we conclude that $P(T) \lesssim 1$ by the bootstrap argument of [37].

7.3 Proof

In this section, we prove Theorem 49. First, we state the following bounds analogous to [37]. The inequality (7.3.3) is proven analogously to Proposition 4.3 in [37], where we replace the fixed unit vector e_3 with the time-dependent unit vector $n_3(t)$. This change does not affect the proof because our inequality is pointwise in time. Before stating our main propositions, we define the following notation for vectors $v, w \in \mathbb{R}^3$:

$$\angle(v, \pm w) \overset{\text{def}}{=} \min\{\angle(v, w), \angle(v, -w)\},$$

which will be used throughout this section.

Proposition 50. *For any $p \in \mathbb{R}^3$ and $\omega \in \mathbb{S}^2$:*

$$(1+\hat{p}\cdot\omega)^{-1} \lesssim min\{p_0^2, (\angle(\frac{p}{|p|}, -\omega))^{-2}\} \tag{7.3.1}$$

Further, if $\gamma = \tan^{-1}(\frac{p\cdot n_2(t)}{p\cdot n_1(t)})$ and $p \in supp\{f\}$, then

$$|p| \lesssim \frac{\kappa(t,\gamma(p))}{\angle(\frac{p}{|p|}, \pm n_3(t))} \tag{7.3.2}$$

Combining (7.3.1) and (7.3.2), we obtain the following estimate for $p \in supp\{f\}$:

$$(1+\hat{p}\cdot\omega)^{-1} \lesssim min\{\left(\frac{\kappa(t,\gamma(p))}{\angle(\frac{p}{|p|}, \pm n_3(t))}\right)^2, (\angle(\frac{p}{|p|}, -\omega))^{-2}\} \tag{7.3.3}$$

Define $\omega^{(i)} = (\sin(\theta^{(i)})\cos(\phi^{(i)}), \sin(\theta^{(i)})\sin(\phi^{(i)}), \cos(\theta^{(i)}))$ where $\omega^{(i)}$ is the transformation of ω under a rotation matrix that takes e_i to $n_i(T_i)$. Thus, we have that $\theta^{(i)} = \angle(n_3(T_i), \omega^{(i)})$. By similar arguments to Proposition 4.4 in [37], we obtain:

Proposition 51. *We have the uniform estimate*

$$\int_{\mathbb{R}^3} \frac{f(s, x+r\omega^{(i)}, p)}{p_0(1+\hat{p}\cdot\omega^{(i)})} dp \lesssim \min\{P(s)^2\log(P(s)), \frac{A(s)^4\log(P(s))}{(\angle(n_3(s), \pm\omega^{(i)}))^2}\} \tag{7.3.4}$$

for $s \in [T_i, T_{i+1})$.

Proof. We follow the proof of Proposition 4.4 in [37], emphasizing the steps in which we deviate from their proof. First, pick spherical coordinates $\theta_{(i)}, \phi_{(i)}$ such that $-\omega^{(i)}$

lies on the half-axis $\theta_{(i)} = 0$. Then, by (7.3.1), we obtain the estimate

$$(1 + p \cdot \omega^{(i)})^{-1} \lesssim \min\{p_0^2, (\theta_{(i)})^{-2}\}. \tag{7.3.5}$$

By the definition of $P(s)$, the particle density $f(s, x + r\omega^{(i)}, p) = 0$ for $|p| > P(s)$. Thus, the conservation law $\|f\|_{L^\infty_{x,p}} \lesssim 1$ and the inequality (7.3.5) imply that

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{f(s, x + r\omega^{(i)}, p)}{p_0(1 + \hat{p} \cdot \omega^{(i)})} dp &\lesssim \int_{|p| \leq P(s)} \frac{1}{p_0(1 + \hat{p} \cdot \omega^{(i)})} dp \\
&\lesssim \int_0^{P(s)} \int_0^{\pi} \int_0^{2\pi} \frac{1}{p_0(1 + \hat{p} \cdot \omega^{(i)})} d|p| \; d\theta_{(i)} \; d\phi_{(i)} \\
&\lesssim \int_0^{P(s)} \int_0^{P(s)^{-1}} p_0^2 d|p| \; d\theta_{(i)} + \int_0^{P(s)} \int_{P(s)^{-1}}^{\pi} (\theta_{(i)})^{-2} d|p| \; d\theta_{(i)} \\
&\lesssim P(s)^2 \log(P(s)),
\end{aligned}$$

which proves the first part of our proposition. We now move on to prove the second bound we need. Let $\beta_i = \angle(n_3(s), \pm\omega^{(i)})$. We partition the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$ of β_i as in [37]:

$$I_i = \{\angle(n_3(s), \omega^{(i)}) \leq \frac{\beta_i}{2}\} \cup I_i = \{\angle(n_3(s), -\omega^{(i)}) \leq \frac{\beta_i}{2}\}$$

$$II_i = \{\angle(n_3(s), \pm\omega^{(i)}) \geq \frac{\beta_i}{2}\} \cap I_i = \{\angle(n_3(s), \pm\omega^{(i)}) \geq \frac{\beta_i}{2}\}.$$

By the definition of β_i and the triangle inequality:

$$\angle(\frac{p}{|p|}, \omega^{(i)}) \geq |\angle(n_3(s), \omega^{(i)}) - \angle(\frac{p}{|p|}, n_3(s))|$$

if $\angle(n_3(s), \omega^{(i)}) \leq \frac{\beta}{2}$. Similarly, if $\angle(n_3(s), -\omega^{(i)}) \leq \frac{\beta}{2}$, then

$$\angle(\frac{p}{|p|}, \omega^{(i)}) \geq |\angle(n_3(s), \omega^{(i)}) - \angle(\frac{p}{|p|}, n_3(s))|.$$

We can do the same estimate for $\angle(\frac{p}{|p|}, -\omega^{(i)})$, and hence,

$$\angle(n_3(s), \pm\omega^{(i)}) \leq \frac{\beta}{2}.$$

By (7.3.3), we now know that

$$(1 + p \cdot \omega^{(i)})^{-1} \lesssim \beta^{-2}.$$

Using this estimate for region I_i and defining the domains D_i and $\tilde{D}_i$ as

$$D_i = \{p \in \mathbb{R}^3 \mid \exists\, x \in \mathbb{R}^3 \text{ such that } f(s, x, p) \neq 0\}$$

and

$$D_i = \{(p_1, p_2) \in \mathbb{R}^2 \mid \exists\, x \in \mathbb{R}^3, p_3 \in \mathbb{R} \text{ such that } f(s, x, p_1, p_2, p_3) \neq 0\},$$

we obtain the following estimate on region I_i:

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{f(s,x+r\omega^{(i)},p)}{(1+p\cdot\omega^{(i)})^{-1}}dp &\lesssim \beta^{-2}\int_{D_i}\frac{1}{p_0}dp\\ &\lesssim \beta^{-2}\int_{\tilde{D}_i}\int_{-P(s)}^{P(s)}\frac{1}{\sqrt{1+p_3}}dp_3\ dp_1\ dp_2\\ &\lesssim \beta^{-2}\log(P(s))\int_{\tilde{D}_i}dp_1\ dp_2\\ &\lesssim \beta^{-2}\log(P(s))\int_0^{2\pi}\int_0^{\kappa(s,\gamma)} ududγ\\ &\lesssim \beta^{-2}\log(P(s))\|\kappa(s,\gamma)\|^2_{L^4_\gamma}\end{aligned}$$

(In the above, we used polar coordinate to compute the integral over $\tilde{D}$ and Hölder's inequality in γ in the last step.) Thus, we have obtained the bound in region I_i:

$$\int_{I_i}\frac{f(s,x+r\omega^{(i)},p)}{p_0(1+p\cdot\omega^{(i)})}dp \lesssim \beta^{-2}\log(P(s))A(s)^2 \lesssim \frac{\log(P(s))A(s)^4}{(\angle(n_3(s),\pm\omega^{(i)}))^2}.$$

For region II_i, pick a system of polar coordinates (θ_s,ϕ_s) such that $p\cdot n_3(s) = |p|\cos(\theta_s)$, i.e. $\theta_s = \angle(p,n_3(s))$. Hence, by definition of β, we have that $\frac{\beta}{2}\le\theta_s\le\frac{\pi}{2}-\frac{\beta}{2}$ and by definition of $\gamma=\gamma(p)$, we also have that $\phi_s=\gamma(p)$. By (7.3.2), we have that

$$|p|\lesssim \kappa(t,\phi_s)(\theta_s^{-1}+(\pi-\theta_s)^{-1}).$$

Using (7.3.3), we obtain

$$\int_{II_i} \frac{f(s, x + r\omega^{(i)}, p)}{p_0(1 + p \cdot \omega^{(i)})} dp$$
$$\lesssim \int_0^{2\pi} d\phi_s \int_{\frac{\beta}{2}}^{\frac{\pi}{2} - \frac{\beta}{2}} \sin(\theta_s)\ d\theta_s \int_0^{C\kappa(t,\phi_s)(\theta_s^{-1} + (\pi - \theta_s)^{-1})} |p| \kappa(t, \phi_s)^2 (\theta_s^{-2} + (\pi - \theta_s)^{-2}) d|p|$$
$$\lesssim \beta^{-2} A(t)^4 \lesssim \frac{\log(P(s)) A(s)^4}{(\angle(n_3(s), \pm\omega^{(i)}))^2}$$

Summing the integrals over the domains I_i and I_{ii}, we obtain the second bound we wanted. This completes our proof. □

In the above, $\angle(n_3(s), \pm\omega^{(i)}) \stackrel{\text{def}}{=} \min\{\angle(n_3(s), \omega^{(i)}), \angle(n_3(s), -\omega^{(i)})\}$. Notice that the above inequality is pointwise in time. Thus, the proof Proposition 51 differs from the proof of Proposition 4.4 in [37] only in that we replace the unit vector $e_3 = (0, 0, 1)$ with $n_3(s)$ and ω with $\omega^{(i)}$. We now give modified arguments for momentum support on planes changing uniformly continuously in time.

Proposition 52. *For $t \in [0, T)$:*

$$|E_T(t, x)| + |B_T(t, x)| \lesssim \log(P(t)) + (\log(P(t)))^2 \int_0^t A(s)^4 ds \tag{7.3.6}$$

Proof. Using the bound (2.3.1) and partitioning the time interval

$$[0, t] = \cup_0^{n_t} ([T_i, T_{i+1}] \cap [0, t])$$

as given by the conditions (7.2.1) and (7.2.2):

$$
\begin{aligned}
|E_T(t,x)|+|B_T(t,x)| &\lesssim \int_{C_{t,x}} \int_{\mathbb{R}^3} \frac{f(s, x+(t-s)\omega, p)}{(t-s)^2 p_0^2 (1+\hat{p}\cdot\omega)^{\frac{3}{2}}} dp\ d\omega \\
&= \int_0^t \int_0^{2\pi} \int_0^{\pi} \int_{\mathbb{R}^3} \frac{f(s, x+(t-s)\omega, p)}{p_0^2 (1+\hat{p}\cdot\omega)^{\frac{3}{2}}} dp \sin(\theta)\ d\theta\ d\phi\ ds \\
&= \sum_0^{n_t} \int_{T_i}^{T_{i+1}} \int_0^{2\pi} \int_0^{\pi} \int_{\mathbb{R}^3} \frac{f(s, x+(t-s)\omega^{(i)}, p)}{p_0^2 (1+\hat{p}\cdot\omega^{(i)})^{\frac{3}{2}}} dp\ \sin(\theta^{(i)}) d\theta^{(i)}\ d\phi^{(i)}\ ds
\end{aligned}
$$

We can divide the integral over $d\theta^{(i)}$ into three regions:

$$
\int_0^{\pi} \int_{\mathbb{R}^3} \frac{f(s, x+(t-s)\omega^{(i)}, p)}{p_0^2 (1+\hat{p}\cdot\omega^{(i)})^{\frac{3}{2}}} dp\ d\theta^{(i)} = A_i + B_i + C_i
$$

where A_i is the integral over $[0, P(t)^{-1}]$, B_i is the integral over $[P(t)^{-1}, \pi - P(t)^{-1}]$, and C_i is the integral over $[\pi - P(t)^{-1}, \pi]$. We estimate each of these integrals using Proposition 51.

$$
\begin{aligned}
B_i = \int_{P(t)^{-1}}^{\pi - P(t)^{-1}} \int_{\mathbb{R}^3} \frac{f(s, x+(t-s)\omega^{(i)}, p)}{p_0^2 (1+\hat{p}\cdot\omega^{(i)})^{\frac{3}{2}}} dp\ \sin(\theta^{(i)}) d\theta^{(i)} \\
\lesssim \int_{P(t)^{-1}}^{\pi - P(t)^{-1}} \frac{A(s)^4 \log(P(s))}{(\angle(n_3(s), \pm\omega^{(i)}))^2}\ \sin(\theta^{(i)}) d\theta^{(i)} \\
\lesssim \int_{P(t)^{-1}}^{\pi - P(t)^{-1}} \frac{A(s)^4 \log(P(s))}{(\theta^{(i)})^2} \sin(\theta^{(i)}) d\theta^{(i)} \\
+ \int_{P(t)^{-1}}^{\pi - P(t)^{-1}} \frac{A(s)^4 \log(P(s))}{(\pi - \theta^{(i)})^2} \sin(\pi - \theta^{(i)}) d\theta^{(i)}
\end{aligned}
$$

where in the third line we used the fact that $\sin(\theta^{(i)}) = \sin(\pi - \theta^{(i)})$ and we also used the following triangle inequality argument for angles:

$$\angle(n_3(s), \pm\omega^{(i)}) \geq |\angle(n_3(T_i), \pm\omega^{(i)}) - \angle(n_3(s), n_3(T_i))|$$

In the time interval $[T_i, T_{i+1}]$, we have that $\angle(n_3(s), n_3(T_i)) < \frac{P(t)^{-1}}{4}$. Further, we are integrating over the interval $\theta^{(i)} = \angle(n_3(T_i), \omega^{(i)}) \in [P(t)^{-1}, \pi - P(t)^{-1}]$ and $\pi - \theta^{(i)} = \angle(n_3(T_i), -\omega^{(i)}) \in [P(t)^{-1}, \pi - P(t)^{-1}]$. Thus,

$$|\angle(n_3(T_i), \omega^{(i)}) - \angle(n_3(s), n_3(T_i))| \approx \theta^{(i)}$$

and

$$|\angle(n_3(T_i), -\omega^{(i)}) - \angle(n_3(s), n_3(T_i))| \approx \pi - \theta^{(i)}$$

Evaluating the integral, we obtain:

$$B_i \lesssim A(s)^4 \log(P(t))^2$$

and

$$\sum_0^{n_t} \int_{T_i}^{T_{i+1}} B_i \; ds \lesssim \log(P(t))^2 \int_0^t A(s)^4 ds$$

Next, evaluating A_i and C_i using the estimate

$$\int \frac{f(s, x + r\omega, p)}{p_0(1 + \hat{p} \cdot \omega)} dp \lesssim P(s)^2 \log(P(s))$$

we obtain that

$$A_i \lesssim \int_0^{P(t)^{-1}} P(s)^2 \log(P(s)) \sin(\theta^{(i)}) d\theta^{(i)} \tag{7.3.7}$$

$$\lesssim \log(P(t)) \tag{7.3.8}$$

and similarly for $C_i \lesssim \log(P(t))$. Summing over $i = 1, \ldots, n_t$, we obtain our result. □

Next, we bound the $E_{S,1} + B_{S,1}$ term. To do so, we apply the argument directly from [37]:

Proposition 53. *For $s \in [0, T)$:*

$$||\int_{\mathbb{R}^3} f(s, x, p)\ dp||_{L^\infty_x} \lesssim A(s)^2 P(s) \tag{7.3.9}$$

Proof. Consider coordinates on $\mathbb{R}^3$ such that $Q(s)$ is lies in the $(p_1, p_2, 0)$ plane. By the support of f and since f is a bounded function,

$$||\int_{\mathbb{R}^3} f\ dp||_{L^\infty} \lesssim \int_{-P(s)}^{P(s)} dp_3 \int_0^{2\pi} d\gamma \int_0^{\kappa(s,\gamma)} r dr \lesssim A(s)^2 P(s)$$

□

Since we still have the same bound (7.3.9) as in [37], the proof of Proposition 5.3 in [37] follows exactly:

Proposition 54. *For $t \in [0, T)$:*

$$\int_0^t |E_{S,1}| + |B_{S,1}| ds \lesssim \sqrt{\log P(t)} \int_0^t A(s)^2 P(s) ds \qquad (7.3.10)$$

Finally, we have:

Proposition 55. *For $t \in [0, T)$:*

$$|E_{S,2}| + |B_{S,2}| \lesssim P(t) \log P(t) + P(t) \log P(t) \Big(\int_0^t A(s)^8 ds \Big)^{\frac{1}{2}} \qquad (7.3.11)$$

Proof. Applying Hölder's inequality to (7.2.4):

$$\begin{aligned} &|E_{S,2}| + |B_{S,2}| \\ &\lesssim ||K_g||_{L^2(C_{t,x})} \Big(\int_0^t \int_0^{2\pi} \int_0^{\pi} \Big(\int_{\mathbb{R}^3} \frac{f(s, x + (t-s)\omega, p)}{p_0(1 + \hat{p} \cdot \omega)} dp \Big)^2 \sin\theta d\theta d\phi ds \Big)^{\frac{1}{2}} \qquad (7.3.12) \end{aligned}$$

The $||K_g||_{L^2(C_{t,x})}$ term is uniformly bounded so we just have to get an estimate on the second term on the right. We apply the same decomposition as in the proof of Proposition 52. First, we split the integral over θ into three intervals and apply (51) to the momentum integral to obtain the inequality:

$$\begin{aligned} &\int_0^t \int_0^{2\pi} \int_0^{\pi} \Big(\int_{\mathbb{R}^3} \frac{f(s, x + (t-s)\omega, p)}{p_0(1 + \hat{p} \cdot \omega)} dp \Big)^2 \sin\theta d\theta d\phi ds \\ &\qquad \lesssim \sum_{i=0}^{n_t} A_i + B_i + C_i \qquad (7.3.13) \end{aligned}$$

where

$$A_i = \int_{T_i}^{T_{i+1}} \int_0^{2\pi} \int_0^{P(t)^{-1}} P(s)^4 \log(P(s))^2 \sin\theta d\theta d\phi ds$$

$$B_i = \int_{T_i}^{T_{i+1}} \int_0^{2\pi} \int_{P(t)^{-1}}^{\pi - P(t)^{-1}} \frac{A(s)^8 \log(P(s))^2}{(\angle(n_3(s), \pm\omega^{(i)})^4)} \sin\theta d\theta d\phi ds$$

$$C_i = \int_{T_i}^{T_{i+1}} \int_0^{2\pi} \int_{\pi - P(t)^{-1}}^{\pi} P(s)^4 \log(P(s))^2 \sin(\pi - \theta) d\theta d\phi ds$$

Now, we apply the same methods to bound A_i, B_i and C_i as in Proposition 52 to obtain that:

$$\sum_{i=0}^{n_t} A_i + B_i + C_i \lesssim P(t)^2 \log(P(t))^2 + P(t)^2 \log(P(t))^2 \int_0^t A(s)^8 ds \tag{7.3.14}$$

Plugging (7.3.14) into (7.3.12), we obtain our result. $\square$

Notice that we have proven the same bounds on the fields E and B as found in [37]. Thus, we can borrow the same proof from Proposition 6.1 in [37] to obtain that $P(T) \lesssim 1$. Hence, by Theorem 19, we can extend our solution (f, E, B) to a larger time interval $[0, T + \epsilon]$.

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