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Stability of the Relativistic Maxwellian in a Collisional Plasma

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Abstract: The relativistic Landau-Maxwell system is the most fundamental and complete model for describing the dynamics of a dilute collisional plasma in which particles interact through Coulombic collisions and through their self-consistent electromagnetic field. We construct the first global in time classical solutions. Our solutions are constructed in a periodic box and near the relativistic Maxwellian, the Jüttner solution.

Contents

1. Collisional Plasma

A dilute hot plasma is a collection of fast moving charged particles [7]. Such plasmas appear commonly in such important physical problems as in Nuclear fusion and Tokamaks. Landau, in 1936, introduced the kinetic equation used to model a dilute plasma in which particles interact through binary Coulombic collisions. Landau did not, however, incorporate Einstein's theory of special relativity into his model. When particle velocities are close to the speed of light, denoted by c, relativistic effects become important. The relativistic version of Landau's equation was proposed by Bubker and Beliaev in 1956 [1]. It is widely accepted as the most complete model for describing the dynamics of a dilute collisional fully ionized plasma.

The relativistic Landau-Maxwell system is given by

$$\partial_t F_+ + c\frac{p}{p_0^+}\cdot\nabla_x F_+ + e_+\left(E + \frac{p}{p_0^+}\times B\right)\cdot\nabla_p F_+ = \mathcal{C}(F_+, F_+) + \mathcal{C}(F_+, F_-),$$

$$\partial_t F_- + c\frac{p}{p_0^-}\cdot\nabla_x F_- - e_-\left(E + \frac{p}{p_0^-}\times B\right)\cdot\nabla_p F_- = \mathcal{C}(F_-, F_-) + \mathcal{C}(F_-, F_+),$$

with initial condition $F_\pm(0,x,p) = F_{0,\pm}(x,p)$. Here $F_\pm(t,x,p) \geq 0$ are the spatially periodic number density functions for ions $(+)$ and electrons $(-)$, at time $t \geq 0$, position $x = (x_1, x_2, x_3) \in \mathbb{T}^3 \equiv [-\pi, \pi]^3$ and momentum $p = (p_1, p_2, p_3) \in \mathbb{R}^3$. The constants $\pm e_\pm$ and $m_\pm$ are the magnitude of the particles' charges and rest masses respectively. The energy of a particle is given by $p_0^\pm = \sqrt{(m_\pm c)^2 + |p|^2}$.

The l.h.s. of the relativistic Landau-Maxwell system models the transport of the particle density functions and the r.h.s. models the effect of collisions between particles on the transport. The heuristic derivation of this equation is

total derivative along particle trajectories = rate of change due to collisions,

where the total derivative of $F_\pm$ is given by Newton's laws

$$\dot{x} = \text{the relativistic velocity} = \frac{p}{\sqrt{m_\pm + |p|^2/c}},$$

$$\dot{p} = \text{the Lorentzian force} = \pm e_\pm\left(E + \frac{p}{p_0^\pm}\times B\right).$$

The collision between particles is modelled by the relativistic Landau collision operator $\mathcal{C}$ in (1) and [1, 9] (sometimes called the relativistic Fokker-Plank-Landau collision operator).

To completely describe a dilute plasma, the electromagnetic field $E(t,x)$ and $B(t,x)$ is generated by the plasma, coupled with $F_\pm(t,x,p)$ through the celebrated Maxwell system:

$$\partial_t E - c\nabla_x \times B = -4\pi\mathcal{J} = -4\pi\int_{\mathbb{R}^3}\left\{e_+\frac{p}{p_0^+}F_+ - e_-\frac{p}{p_0^-}F_-\right\}dp,$$

$$\partial_t B + c\nabla_x \times E = 0,$$

with constraints

$$\nabla_x\cdot B = 0, \quad \nabla_x\cdot E = 4\pi\rho = 4\pi\int_{\mathbb{R}^3}\{e_+F_+ - e_-F_-\}\,dp,$$

and initial conditions $E(0,x) = E_0(x)$ and $B(0,x) = B_0(x)$. The charge density and current density due to all particles are denoted ρ and $\mathcal{J}$ respectively.

We define relativistic four vectors as $P_+ = (p_0^+, p) = (p_0^+, p_1, p_2, p_3)$ and $Q_- = (q_0^-, q)$. Let $g_+(p)$, $h_-(p)$ be two number density functions for two types of particles, then the Landau collision operator is defined by

$$\mathcal{C}(g_+, h_-)(p) \equiv \nabla_p\cdot\int_{\mathbb{R}^3}\Phi(P_+, Q_-)\left\{\nabla_p g_+(p)h_-(q) - g_+(p)\nabla_q h_-(q)\right\}dq. \quad (1)$$

The ordering of the $+, -$ in the kernel $\Phi(P_+, Q_-)$ corresponds to the order of the functions in the argument of the collision operator $\mathcal{C}(g_+, h_-)(p)$. The collision kernel is given by the 3×3 non-negative matrix

$$\Phi(P_+, Q_-) \equiv \frac{2\pi}{c} e_+ e_- L_{+,-} \left(\frac{p_0^+}{m_+ c} \frac{q_0^-}{m_- c} \right)^{-1} \Lambda(P_+, Q_-) S(P_+, Q_-),$$

where $L_{+,-}$ is the Couloumb logarithm for $+-$ interactions. The Lorentz inner product with signature $(+ - --)$ is given by

$$P_+ \cdot Q_- = p_0^+ q_0^- - p \cdot q.$$

We distinguish between the standard inner product and the Lorentz inner product of relativistic four-vectors by using capital letters P_+ and Q_- to denote the four-vectors. Then, for the convenience of future analysis, we define

$$\Lambda \equiv \left(\frac{P_+}{m_+ c} \cdot \frac{Q_-}{m_- c} \right)^2 \left\{ \left(\frac{P_+}{m_+ c} \cdot \frac{Q_-}{m_- c} \right)^2 - 1 \right\}^{-3/2},$$

$$\begin{aligned} S \equiv & \left\{ \left(\frac{P_+}{m_+ c} \cdot \frac{Q_-}{m_- c} \right)^2 - 1 \right\} I_3 - \left(\frac{p}{m_+ c} - \frac{q}{m_- c} \right) \otimes \left(\frac{p}{m_+ c} - \frac{q}{m_- c} \right) \\ & + \left\{ \left(\frac{P_+}{m_+ c} \cdot \frac{Q_-}{m_- c} \right) - 1 \right\} \left(\frac{p}{m_+ c} \otimes \frac{q}{m_- c} + \frac{q}{m_- c} \otimes \frac{p}{m_+ c} \right). \end{aligned}$$

This kernel is the relativtistic counterpart of the celebrated classical (non-relativistic) Landau collision operator.

It is well known that the collision kernel Φ is a non-negative matrix satisfying

$$\sum_{i=1}^{3} \Phi^{ij}(P_+, Q_-) \left(\frac{q_i}{q_0^-} - \frac{p_i}{p_0^+} \right) = \sum_{j=1}^{3} \Phi^{ij}(P_+, Q_-) \left(\frac{q_j}{q_0^-} - \frac{p_j}{p_0^+} \right) = 0, \tag{2}$$

and [8, 9]

$$\sum_{i,j} \Phi^{ij}(P_+, Q_-) w_i w_j > 0 \ \text{ if } \ w \neq d \left(\frac{p}{p_0^+} - \frac{q}{q_0^-} \right) \ \forall d \in \mathbb{R}.$$

The same is true for each other sign configuration $((+, +), (-, +), (-, -))$. This property represents the physical assumption that so-called "grazing collisions" dominate, e.g., the change in "momentum of the colliding particles is perpendicular to their relative velocity" [9], p. 170. This is also the key property used to derive the conservation laws and the entropy dissipation below.

It formally follows from (2) that for number density functions $g_+(p), h_-(p)$,

$$\int_{\mathbb{R}^3} \left\{ \begin{pmatrix} 1 \\ p \\ p_0^+ \end{pmatrix} \mathcal{C}(h_+, g_-)(p) + \begin{pmatrix} 1 \\ p \\ p_0^- \end{pmatrix} \mathcal{C}(g_-, h_+)(p) \right\} dp = 0.$$

The same property holds for other sign configurations. By integrating the relativistic Landau-Maxwell system and plugging in this identity, we obtain the conservation of mass, total momentum and total energy for solutions as

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^3\times\mathbb{R}^3} m_+F_+(t) = \frac{d}{dt}\int_{\mathbb{T}^3\times\mathbb{R}^3} m_-F_-(t) = 0,\\
&\frac{d}{dt}\left\{\int_{\mathbb{T}^3\times\mathbb{R}^3} p(m_+F_+(t)+m_-F_-(t)) + \frac{1}{4\pi}\int_{\mathbb{T}^3} E(t)\times B(t)\right\} = 0,\\
&\frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{T}^3\times\mathbb{R}^3} (m_+p_0^+F_+(t)+m_-p_0^-F_-(t)) + \frac{1}{8\pi}\int_{\mathbb{T}^3} |E(t)|^2+|B(t)|^2\right\} = 0.
\end{aligned}
$$

The entropy of the relativistic Landau-Maxwell system is defined as

$$
\mathcal{H}(t) \equiv \int_{\mathbb{T}^3\times\mathbb{R}^3} \{F_+(t,x,p)\ln F_+(t,x,p) + F_-(t,x,p)\ln F_-(t,x,p)\}\,dxdp \geq 0.
$$

Boltzmann's famous H-Theorem for the relativistic Landau-Maxwell system is

$$
\frac{d}{dt}\mathcal{H}(t) \leq 0,
$$

e.g., the entropy of solutions is non-increasing as time passes.

The global relativistic Maxwellian (a.k.a. the Jüttner solution) is given by

$$
J_\pm(p) = \frac{\exp\left(-cp_0^\pm/(k_BT)\right)}{4\pi e_\pm m_\pm^2 ck_BT K_2(m_\pm c^2/(k_BT))},
$$

where $K_2(\cdot)$ is the Bessel function $K_2(z) \equiv \frac{z^2}{3}\int_1^\infty e^{-zt}(t^2-1)^{3/2}dt$, T is the temperature and k_B is Boltzmann's constant. From the Maxwell system and the periodic boundary condition of $E(t,x)$, we see that $\frac{d}{dt}\int_{\mathbb{T}^3} B(t,x)dx \equiv 0$. We thus have a constant $\bar{B}$ such that

$$
\frac{1}{|\mathbb{T}^3|}\int_{\mathbb{T}^3} B(t,x)dx = \bar{B}. \tag{3}
$$

Let $[\cdot,\cdot]$ denote a column vector. We then have the following steady state solution to the relativisitic Landau-Maxwell system

$$
[F_\pm(t,x,p), E(t,x), B(t,x)] = [J_\pm, 0, \bar{B}],
$$

which minimizes the entropy ($\mathcal{H}(t) = 0$).

It is our purpose to study the effects of collisions in a hot plasma and to construct global in time classical solutions for the relativistic Landau-Maxwell system with initial data close to the relativistic Maxwellian (Theorem 1). Our construction implies the asymptotic stability of the relativistic Maxwellian, which is suggested by the H-Theorem.

2. Main Results

We define the standard perturbation $f_\pm(t, x, p)$ to $J_\pm$ as

$$F_\pm \equiv J_\pm + \sqrt{J}_\pm f_\pm.$$

We will plug this perturbation into the Landau-Maxwell system of equations to derive a perturbed Landau-Maxwell system for $f_\pm(t, x, p)$, $E(t, x)$ and $B(t, x)$. The two Landau-Maxwell equations for the perturbation $f = [f_+, f_-]$ take the form

$$\begin{aligned}&\left\{\partial_t + c\frac{p}{p_0^\pm}\cdot\nabla_x \pm e_\pm\left(E + \frac{p}{p_0^\pm}\times B\right)\cdot\nabla_p\right\} f_\pm \mp \frac{e_\pm c}{k_B T}\left\{E\cdot\frac{p}{p_0^\pm}\right\}\sqrt{J}_\pm + L_\pm f\\ &\quad = \pm\frac{e_\pm c}{2k_B T}\left\{E\cdot\frac{p}{p_0^\pm}\right\} f_\pm + \Gamma_\pm(f, f), \qquad (4)\end{aligned}$$

with $f(0, x, p) = f_0(x, p) = [f_{0,+}(x, p), f_{0,-}(x, p)]$. The linear operator $L_\pm f$ defined in (21) and the non-linear operator $\Gamma_\pm(f, f)$ defined in (23) are derived from an expansion of the Landau collision operator (1). The coupled Maxwell system takes the form

$$\begin{aligned}&\partial_t E - c\nabla_x\times B = -4\pi\mathcal{J} = -4\pi\int_{\mathbb{R}^3}\left\{e_+\frac{p}{p_0^+}\sqrt{J}_+ f_+ - e_-\frac{p}{p_0^-}\sqrt{J}_- f_-\right\} dp,\\ &\partial_t B + c\nabla_x\times E = 0, \qquad (5)\end{aligned}$$

with constraints

$$\nabla_x\cdot E = 4\pi\rho = 4\pi\int_{\mathbb{R}^3}\left\{e_+\sqrt{J}_+ f_+ - e_-\sqrt{J}_- f_-\right\} dp, \quad \nabla_x\cdot B = 0, \qquad (6)$$

with $E(0, x) = E_0(x)$, $B(0, x) = B_0(x)$. In computing the charge ρ, we have used the normalization $\int_{\mathbb{R}^3} J_\pm(p)dp = \frac{1}{e_\pm}$.

Notation. For notational simplicity, we shall use $\langle\cdot,\cdot\rangle$ to denote the standard L^2 inner product in $\mathbb{R}^3$ and $(\cdot,\cdot)$ to denote the standard L^2 inner product in $\mathbb{T}^3\times\mathbb{R}^3$. We define the collision frequency as the 3×3 matrix

$$\sigma^{ij}_{\pm,\mp}(p) \equiv \int \Phi^{ij}(P_\pm, Q_\mp)J_\mp(q)dq. \qquad (7)$$

These four weights (corresponding to signatures $(+,+), (+,-), (-,+), (-,-)$) are used to measure the dissipation of the relativistic Landau collision term. Unless otherwise stated $g = [g_+, g_-]$ and $h = [h_+, h_-]$ are functions which map $\{t\geq 0\}\times\mathbb{T}^3\times\mathbb{R}^3\to\mathbb{R}^2$. We define

$$\begin{aligned}\langle g, h\rangle_\sigma \equiv &\int_{\mathbb{R}^3}\left\{\left(\sigma^{ij}_{+,+} + \sigma^{ij}_{+,-}\right)\partial_{p_j} g_+\partial_{p_i} h_+ + \left(\sigma^{ij}_{-,-} + \sigma^{ij}_{-,+}\right)\partial_{p_j} g_-\partial_{p_i} h_-\right\} dp,\\ &+\frac{1}{4}\int_{\mathbb{R}^3}\left(\sigma^{ij}_{+,+} + \sigma^{ij}_{+,-}\right)\frac{p_i}{p_0^+}\frac{p_j}{p_0^+} g_+ h_+ dp \qquad (8)\\ &+\frac{1}{4}\int_{\mathbb{R}^3}\left(\sigma^{ij}_{-,-} + \sigma^{ij}_{-,+}\right)\frac{p_i}{p_0^-}\frac{p_j}{p_0^-} g_- h_- dp,\end{aligned}$$

where in (8) and the rest of the paper we use the Einstein convention of implicitly summing over $i, j \in \{1, 2, 3\}$ (unless otherwise stated). This complicated inner product is motivated by following splitting, which is a crucial element of the energy method used in this paper (Lemma 6 and Lemma 8):

$$\langle Lg, h\rangle = \langle [L_+g, L_-g], h\rangle = \langle g, h\rangle_\sigma + \text{a “compact” term.}$$

We will also use the corresponding L^2 norms

$$|g|_\sigma^2 \equiv \langle g, g\rangle_\sigma, \quad \|g\|_\sigma^2 \equiv (g, g)_\sigma \equiv \int_{\mathbb{T}^3} \langle g, g\rangle_\sigma dx.$$

We use $|\cdot|_2$ to denote the L^2 norm in $\mathbb{R}^3$ and $\|\cdot\|$ to denote the L^2 norm in either $\mathbb{T}^3 \times \mathbb{R}^3$ or $\mathbb{T}^3$ (depending on whether the function depends on both (x, p) or only on x). Let the multi-indices γ and β be $\gamma = [\gamma^0, \gamma^1, \gamma^2, \gamma^3]$, $\beta = [\beta^1, \beta^2, \beta^3]$. We use the following notation for a high order derivative

$$\partial_\beta^\gamma \equiv \partial_t^{\gamma^0} \partial_{x_1}^{\gamma^1} \partial_{x_2}^{\gamma^2} \partial_{x_3}^{\gamma^3} \partial_{p_1}^{\beta^1} \partial_{p_2}^{\beta^2} \partial_{p_3}^{\beta^3}.$$

If each component of β is not greater than that of $\bar{\beta}$'s, we denote by $\beta \le \bar{\beta}$; $\beta < \bar{\beta}$ means $\beta \le \bar{\beta}$, and $|\beta| < |\bar{\beta}|$. We also denote $\binom{\beta}{\bar{\beta}}$ by $C_\beta^{\bar{\beta}}$. Let

$$|||f|||^2(t) \equiv \sum_{|\gamma|+|\beta|\le N} ||\partial_\beta^\gamma f(t)||^2,$$
$$|||f|||_\sigma^2(t) \equiv \sum_{|\gamma|+|\beta|\le N} ||\partial_\beta^\gamma f(t)||_\sigma^2,$$
$$|||[E, B]|||^2(t) \equiv \sum_{|\gamma|\le N} ||[\partial^\gamma E(t), \partial^\gamma B(t)]||^2.$$

It is important to note that our norms include the temporal derivatives. For a function independent of t, we use the same notation but we drop the (t). The above norms and their associated spaces are used throughout the paper for arbitrary functions.

We further define the high order energy norm for a solution $f(t, x, p)$, $E(t, x)$ and $B(t, x)$ to the relativistic Landau-Maxwell system (4) and (5) as

$$\mathcal{E}(t) \equiv \frac{1}{2}|||f|||^2(t) + |||[E, B]|||^2(t) + \int_0^t |||f|||_\sigma^2(s)ds. \tag{9}$$

Given initial datum $[f_0(x, p), E_0(x), B_0(x)]$, we define

$$\mathcal{E}(0) = \frac{1}{2}|||f_0|||^2 + |||[E_0, B_0]|||^2,$$

where the temporal derivatives of $[f_0, E_0, B_0]$ are defined naturally through Eqs. (4) and (5). The high order energy norm is consistent at $t = 0$ for a smooth solution and $\mathcal{E}(t)$ is continuous (Theorem 6).

Assume that initially $[F_0, E_{0,} B_0]$ has the same mass, total momentum and total energy as the steady state $[J_\pm, 0, \bar{B}]$, then we can rewrite the conservation laws in terms of the perturbation $[f, E, B]$:

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} m_+ f_+(t)\sqrt{J}_+ \equiv \int_{\mathbb{T}^3\times\mathbb{R}^3} m_- f_-(t)\sqrt{J}_- \equiv 0, \tag{10}$$

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} p\left\{m_+ f_+(t)\sqrt{J}_+ + m_- f_-(t)\sqrt{J}_-\right\} \equiv -\frac{1}{4\pi}\int_{\mathbb{T}^3} E(t)\times B(t), \tag{11}$$

$$\begin{aligned}&\int_{\mathbb{T}^3\times\mathbb{R}^3} m_+ p_0^+ f_+(t)\sqrt{J}_+ + m_- p_0^- f_-(t)\sqrt{J}_- \\ &\quad\equiv -\frac{1}{8\pi}\int_{\mathbb{T}^3} |E(t)|^2 + |B(t)-\bar{B}|^2.\end{aligned} \tag{12}$$

We have used (3) for the normalized energy conservation (12).

The effect of this restriction is to guarantee that a solution can only converge to the specific relativistic Maxwellian that we perturb away from (if the solution converges to a relativistic Maxwellian). The value of the steady state $\bar{B}$ is also defined by the initial conditions (3).

We are now ready to state our main results:

Theorem 1. *Fix N, the total number of derivatives in (9), with $N \geq 4$. Assume that $[f_0, E_0, B_0]$ satisfies the conservation laws (10), (11), (12) and the constraint (6) initially. Let*

$$F_{0,\pm}(x,p) = J_\pm + \sqrt{J}_\pm f_{0,\pm}(x,p) \geq 0.$$

There exist $C_0 > 0$ and $M > 0$ such that if

$$\mathcal{E}(0) \leq M,$$

then there exists a unique global solution $[f(t,x,p), E(t,x), B(t,x)]$ to the perturbed Landau-Maxwell system (4), (5) with (6). Moreover,

$$F_\pm(t,x,p) = J_\pm + \sqrt{J}_\pm f_\pm(t,x,p) \geq 0$$

solves the relativistic Landau-Maxwell system and

$$\sup_{0\leq s\leq\infty} \mathcal{E}(s) \leq C_0\mathcal{E}(0).$$

Remarks. – These solutions are C^1, and in fact C^k, for N large enough.

– Since $\int_0^\infty |||f|||_\sigma^2(t)dt < +\infty$, $f(t,x,p)$ gains one momentum derivative over its initial data and $|||f|||_\sigma^2(t) \to 0$ in a certain sense.
– Further, Lemma 5 together with Lemma 13 imply that

$$\sum_{|\gamma|\leq N-1}\left\{||\partial^\gamma E(t)|| + ||\partial^\gamma\{B(t)-\bar{B}\}||\right\} \leq C\sum_{|\gamma|\leq N} ||\partial^\gamma f(t)||_\sigma.$$

Therefore, except for the highest order derivatives, the field also converges.
– It is an interesting open question to determine the asymptotic behavior of the highest order derivatives of the electromagnetic field.

Recently, global in time solutions to the related classical Vlasov-Maxwell-Boltzmann equation were constructed by the second author in [6]. The Boltzmann equation is a widely accepted model for binary interactions in a dilute gas, however it fails to hold for a dilute plasma in which grazing collisions dominate.

The following classical Landau collision operator (with normalized constants) was designed to model such a plasma:

$$\mathcal{C}_{cl}(F_-, F_+) \equiv \nabla_v \cdot \left\{ \int_{\mathbf{R}^3} \phi(v - v') \left\{ \nabla_v F_-(v) F_+(v') - F_-(v) \nabla_{v'} F_+(v') \right\} dv' \right\}.$$

The non-negative 3×3 matrix is

$$\phi^{ij}(v) = \left\{ \delta_{ij} - \frac{v_i v_j}{|v|^2} \right\} \frac{1}{|v|}. \tag{13}$$

Unfortunately, because of the crucial hard sphere assumption, the construction in [6] fails to apply to a non-relativistic Coulombic plasma interacting with its electromagnetic field. The key problem is that the classical Landau collision operator, which was studied in detail in [5], offers weak dissipation of the form $\int_{\mathbb{R}^3} (1 + |v|)^{-1} |f|^2 dv$. The global existence argument in Sect. 6 (from [6]) does not work because of this weak dissipation. Further, the unbounded velocity v, which is inconsistent with Einstein's theory of special relativity, in particular makes it impossible to control a nonlinear term like $\{E \cdot v\} f_\pm$ in the classical theory.

On the other hand, in the relativistic case our key observation is that the corresponding nonlinear term $c \left\{ E \cdot p / p_0^\pm \right\} f_\pm$ can be easily controlled by the dissipation because $|cp/p_0^\pm| \leq c$ and the dissipation in the relativistic Landau operator is $\int_{\mathbb{R}^3} |f|^2 dp$ (Lemma 5 and Lemou [8]).

However, it is well-known that the relativity effect can produce severe mathematical difficulties. Even for the related pure relativistic Boltzmann equation, global smooth solutions were only constructed in [3, 4].

The first new difficulty is due to the complexity of the relativistic Landau collision kernel $\Phi(P_+, Q_-)$. Since

$$\frac{P_+}{m_+ c} \cdot \frac{Q_-}{m_- c} - 1 \sim \frac{1}{2c} \left| \frac{p}{m_+} - \frac{q}{m_-} \right|^2 \text{ when } \frac{P_+}{m_+ c} \approx \frac{Q_-}{m_- c},$$

the kernel in (1) has a first order singularity. Hence it can not absorb many derivatives in high order estimates (Lemma 7 and Theorem 4). The same issue exists for the classical Landau kernel $\phi(v - v')$, but the obvious symmetry makes it easy to express v derivatives of ϕ in terms of v' derivatives. It is then possible to integrate by parts and move derivatives off the singular kernel in the estimates of high order derivatives. On the contrary, no apparent symmetry exists beween p and q in the relativistic case. We overcome this severe difficulty with the splitting

$$\partial_{p_j} \Phi^{ij}(P_+, Q_-) = -\frac{q_0^-}{p_0^+} \partial_{q_j} \Phi^{ij}(P_+, Q_-) + \left(\partial_{p_j} + \frac{q_0^-}{p_0^+} \partial_{q_j} \right) \Phi^{ij}(P_+, Q_-),$$

where the operator $\left(\partial_{p_j} + \frac{q_0^-}{p_0^+} \partial_{q_j} \right)$ does not increase the order of the singularity mainly because

$$\left(\partial_{p_j} + \frac{q_0^-}{p_0^+} \partial_{q_j} \right) P_+ \cdot Q_- = 0.$$

This splitting is crucial for performing the integration by parts in all of our estimates (Lemma 2 and Theorem 3). We believe that such an integration by parts technique should shed new light on the study of the relativistic Boltzmann equation.

As in [5, 6], another key point in our construction is to show that the linearized collision operator L is in fact coercive for solutions of small amplitude to the full nonlinear system (4), (5) and (6):

Theorem 2. *Let* $[f(t,x,p), E(t,x), B(t,x)]$ *be a classical solution to (4) and (5) satisfying (6), (10), (11) and (12). There exists* M_0, $\delta_0 = \delta_0(M_0) > 0$ *such that if*

$$\sum_{|\gamma|\le N} \left\{ \frac{1}{2}||\partial^\gamma f(t)||^2 + ||\partial^\gamma E(t)||^2 + ||\partial^\gamma B(t)||^2 \right\} \le M_0, \tag{14}$$

then

$$\sum_{|\gamma|\le N} \big(L\partial^\gamma f(t), \partial^\gamma f(t)\big) \ge \delta_0 \sum_{|\gamma|\le N} ||\partial^\gamma f(t)||_\sigma^2.$$

Theorem 2 is proven through a careful study of the macroscopic equations (98) - (102). These macroscopic equations come from a careful study of solutions f to the perturbed relativistic Landau-Maxwell system (4), (5) with (6) projected onto the null space $\mathcal{N}$ of the linearized collision operator $L = [L_+, L_-]$ defined in (21).

As expected from the H-theorem, L is non-negative and for every fixed (t, x) the null space of L is given by the six dimensional space $(1 \le i \le 3)$

$$\mathcal{N} \equiv \text{span}\{[\sqrt{J_+}, 0],\ [0, \sqrt{J_-}],\ [p_i\sqrt{J_+}, p_i\sqrt{J_-}],\ [p_0^+\sqrt{J_+}, p_0^-\sqrt{J_-}]\}. \tag{15}$$

This is shown in Lemma 1. We define the orthogonal projection from $L^2(\mathbb{R}^3_p)$ onto the null space $\mathcal{N}$ by $\mathbf{P}$. We then decompose $f(t,x,p)$ as

$$f = \mathbf{P}f + \{\mathbf{I} - \mathbf{P}\}f.$$

We call $\mathbf{P}f = [\mathbf{P}_+ f, \mathbf{P}_- f] \in \mathbb{R}^2$ the hydrodynamic part of f and $\{\mathbf{I}-\mathbf{P}\}f = [\{\mathbf{I}-\mathbf{P}\}_+ f, \{\mathbf{I}-\mathbf{P}\}_- f]$ is called the microscopic part. By separating its linear and nonlinear part, and using $L_\pm\{\mathbf{P}f\} = 0$, we can express the hydrodynamic part of f through the microscopic part up to a higher order term $h(f)$:

$$\left\{\partial_t + c\frac{p}{p_0^\pm}\cdot\nabla_x\right\}\mathbf{P}_\pm f \mp \frac{e_\pm c}{k_B T}\left\{E\cdot\frac{p}{p_0^\pm}\right\}\sqrt{J_\pm} = l_\pm(\{\mathbf{I}-\mathbf{P}\}f) + h_\pm(f), \tag{16}$$

where

$$l_\pm(\{\mathbf{I}-\mathbf{P}\}f) \equiv -\left\{\partial_t + \frac{p}{p_0^\pm}\cdot\nabla_x\right\}\{\mathbf{I}-\mathbf{P}\}_\pm f + L_\pm\left\{\{\mathbf{I}-\mathbf{P}\}f\right\}, \tag{17}$$

$$\begin{aligned} h_\pm(f) &\equiv \mp e_\pm\left(E + \frac{p}{p_0^\pm}\times B\right)\cdot\nabla_p f_\pm \\ &\quad \pm\frac{e_\pm c}{2k_B T}\left\{E\cdot\frac{p}{p_0^\pm}\right\} f_\pm + \Gamma_\pm(f,f). \end{aligned} \tag{18}$$

We further expand $\mathbf{P}_{\pm} f$ as a linear combination of the basis in (15),

$$\mathbf{P}_{\pm} f \equiv \left\{ a_{\pm}(t,x) + \sum_{j=1}^{3} b_j(t,x) p_j + c(t,x) p_0^{\pm} \right\} \sqrt{J_{\pm}}. \tag{19}$$

A precise definition of these coefficients will be given in (94). The relativistic system of macroscopic equations (98) - (102) are obtained by plugging (19) into (16).

These macroscopic equations for the coefficients in (19) enable us to show that there exists a constant $C > 0$ such that solutions to (4) which satisfy the smallness constraint (14) (for $M_0 > 0$ small enough) will also satisfy

$$\sum_{|\gamma| \le N} \left\{ ||\partial^{\gamma} a_{\pm}|| + ||\partial^{\gamma} b|| + ||\partial^{\gamma} c|| \right\} \le C(M_0) \sum_{|\gamma| \le N} ||\{\mathbf{I} - \mathbf{P}\} \partial^{\gamma} f(t)||_{\sigma}. \tag{20}$$

This implies Theorem 2 since $\|\mathbf{P} f\|_{\sigma}$ is trivially bounded above by the l.h.s. (Proposition 2) and L is coercive with respect to $\{\mathbf{I} - \mathbf{P}\}\partial^{\gamma} f(t)$ (Lemma 8).

Since our smallness assumption (14) involves no momentum derivatives, in proving (20) the presence of momentum derivatives in the collision operator (1) causes another serious mathematical difficulty. We develop a new estimate (Theorem 5) which involves purely spatial derivatives of the linear term (21) and the nonlinear term (23) to overcome this difficulty.

To the best of the authors' knowledge, until now there were no known solutions for the relativistic Landau-Maxwell system. However in 2000, Lemou [8] studied the linearized relativistic Landau equation with no electromagnetic field. We will use one of his findings (Lemma 5) in the present work.

For the classical Landau equation, the 1990's have seen the first solutions. In 1994, Zhan [10] proved local existence and uniqueness of classical solutions to the Landau-Poisson equation ($B \equiv 0$) with Coulomb potential and a smallness assumption on the initial data at infinity. In the same year, Zhan [11] proved local existence of weak solutions to the Landau-Maxwell equation with Coulomb potential and large initial data.

On the other hand, in the absence of an electromagnetic field we have the following results. In 2000, Desvillettes and Villani [2] proved global existence and uniqueness of classical solutions for the spatially homogeneous Landau equation for hard potentials and a large class of initial data. In 2002, the second author [5] constructed global in time classical solutions near the Maxwellian for a general Landau equation (both hard and soft potentials) in a periodic box based on a nonlinear energy method.

Our paper is organized as follows. In Sect. 3 we establish linear and nonlinear estimates for the relativistic Landau collision operator. In Sect. 4 we construct local in time solutions to the relativistic Landau-Maxwell system. In Sect. 5 we prove Theorem 2. And in Sect. 6 we extend the solutions to $T = \infty$.

Remark 1. It turns out that the presence of the physical constants do not cause essential mathematical difficulties. Therefore, for notational simplicity, after the proof of Lemma 1 we will normalize all constants in the relativistic Landau-Maxwell system (4), (5) with (6) and in all related quantities to be one.

3. The Relativistic Landau Operator

Our main results in this section include the crucial Theorem 3, which allows us to express p derivatives of $\Phi(P, Q)$ in terms of q derivatives of $\Phi(P, Q)$. This is vital for establishing the estimates found at the end of the section (Lemma 7, Theorem 4 and Theorem

5). Other important results include the equivalence of the norm $|\cdot|_\sigma$ with the standard Sobolev space norm for H^1 (Lemma 5) and a weak formulation of compactness for K which is enough to prove coercivity for L away from the null space $\mathcal{N}$ (Lemma 8). We also compute the sum of second order derivatives of the Landau kernel (Lemma 3).

We first introduce some notation. Using (2), we observe that the quadratic collision operator (1) satisfies

$$\mathcal{C}(J_+, J_+) = \mathcal{C}(J_+, J_-) = \mathcal{C}(J_-, J_+) = \mathcal{C}(J_-, J_-) = 0.$$

Therefore, the linearized collision operator Lg is defined by

$$Lg = [L_+ g, L_- g], \quad L_\pm g \equiv -A_\pm g - K_\pm g, \tag{21}$$

where

$$\begin{aligned}
A_+ g &\equiv J_+^{-1/2}\mathcal{C}(\sqrt{J}_+ g_+, J_+) + J_+^{-1/2}\mathcal{C}(\sqrt{J}_+ g_+, J_-),\\
A_- g &\equiv J_-^{-1/2}\mathcal{C}(\sqrt{J}_- g_-, J_-) + J_-^{-1/2}\mathcal{C}(\sqrt{J}_- g_-, J_+),\\
K_+ g &\equiv J_+^{-1/2}\mathcal{C}(J_+, \sqrt{J}_+ g_+) + J_+^{-1/2}\mathcal{C}(J_+, \sqrt{J}_- g_-),\\
K_- g &\equiv J_-^{-1/2}\mathcal{C}(J_-, \sqrt{J}_- g_-) + J_-^{-1/2}\mathcal{C}(J_-, \sqrt{J}_+ g_+).
\end{aligned} \tag{22}$$

And the nonlinear part of the collision operator (1) is defined by

$$\Gamma(g, h) = [\Gamma_+(g, h), \Gamma_-(g, h)],$$

where

$$\begin{aligned}
\Gamma_+(g, h) &\equiv J_+^{-1/2}\mathcal{C}(\sqrt{J}_+ g_+, \sqrt{J}_+ h_+) + J_+^{-1/2}\mathcal{C}(\sqrt{J}_+ g_+, \sqrt{J}_- h_-),\\
\Gamma_-(g, h) &\equiv J_-^{-1/2}\mathcal{C}(\sqrt{J}_- g_-, \sqrt{J}_- h_-) + J_-^{-1/2}\mathcal{C}(\sqrt{J}_- g_-, \sqrt{J}_+ h_+).
\end{aligned} \tag{23}$$

We will next derive the null space (15) of the linear operator in the presence of all the physical constants.

Lemma 1. $\langle Lg, h\rangle = \langle Lh, g\rangle$, $\langle Lg, g\rangle \ge 0$. *And* $Lg = 0$ *if and only if* $g = \mathbf{P}g$.

Proof. From (21) we split $\langle Lg, h\rangle$, with $Lg = [L_+ g, L_- g]$, as

$$\begin{aligned}
&-\int_{\mathbb{R}^3} \frac{h_+}{\sqrt{J_+}}\{\mathcal{C}(\sqrt{J_+} g_+, J_+) + \mathcal{C}(J_+, \sqrt{J_+} g_+)\}dp\\
&-\int_{\mathbb{R}^3} \left\{\frac{h_+}{\sqrt{J_+}}\{\mathcal{C}(\sqrt{J_+} g_+, J_-) + \mathcal{C}(J_+, \sqrt{J_-} g_-)\}\right\} dp\\
&-\int_{\mathbb{R}^3} \left\{\frac{h_-}{\sqrt{J_-}}\{\mathcal{C}(\sqrt{J_-} g_-, J_+) + \mathcal{C}(J_-, \sqrt{J_+} g_+)\}\right\} dp\\
&-\int_{\mathbb{R}^3} \frac{h_-}{\sqrt{J_-}}\{\mathcal{C}(\sqrt{J_-} g_-, J_-) + \mathcal{C}(J_-, \sqrt{J_-} g_-)\}dp.
\end{aligned} \tag{24}$$

We use the fact that $\partial_{q_i} J_-(q) = -\frac{c}{k_B T}\frac{q_i}{q_0^-} J_-(q)$ and $\partial_{p_i} J_+^{1/2}(p) = -\frac{c}{2k_B T}\frac{p_i}{p_0^+} J_+^{1/2}(p)$ as well as the null space of Φ in (2) to show that

$$\begin{aligned}
&\mathcal{C}(J_+^{1/2} g_+, J_-) \\
&= \partial_{p_i} \int_{\mathbb{R}^3} \Phi^{ij}(P_+, Q_-) J_-(q) J_+^{1/2}(p) \left\{ \left(\frac{q_i}{q_0^-} - \frac{p_i}{2p_0^+} \right) g_+(p) + \partial_{p_j} g_+(p) \right\} dq \\
&= \partial_{p_i} \int_{\mathbb{R}^3} \Phi^{ij}(P_+, Q_-) J_-(q) J_+^{1/2}(p) \left\{ \frac{p_i}{2p_0^+} g_+(p) + \partial_{p_j} g_+(p) \right\} dq \\
&= \partial_{p_i} \int_{\mathbb{R}^3} \Phi^{ij}(P_+, Q_-) J_-(q) J_+(p) \partial_{p_j}(J_+^{-1/2} g_+(p)) dq.
\end{aligned}$$

And similarly

$$\begin{aligned}
&\mathcal{C}(J_-, J_+^{1/2} g_+) \\
&= -\partial_{p_i} \int_{\mathbb{R}^3} \Phi^{ij}(P_-, Q_+) J_-(p) J_+^{1/2}(q) \left\{ \left(\frac{p_i}{p_0^-} - \frac{q_i}{2q_0^+} \right) g_+(q) + \partial_{q_j} g_+(q) \right\} dq \\
&= -\partial_{p_i} \int_{\mathbb{R}^3} \Phi^{ij}(P_-, Q_+) J_-(p) J_+^{1/2}(q) \left\{ \frac{q_i}{2q_0^+} g_+(p) + \partial_{q_j} g_+(q) \right\} dq \\
&= -\partial_{p_i} \int_{\mathbb{R}^3} \Phi^{ij}(P_-, Q_+) J_-(p) J_+(q) \partial_{q_j}(J_+^{-1/2} g_+(q)) dq. \qquad (25)
\end{aligned}$$

Similar expressions hold by exchanging the $+$ terms and the $-$ terms in the appropriate places. For the first term in (24), we integrate by parts over p variables on the first line, then relabel the variables switching p and q on the second line and finally adding them up on the last line to obtain

$$\begin{aligned}
&= \iint \Phi^{ij}(P_+, Q_+) J_+(p) J_+(q) \partial_{p_i}(h_+ J_+^{-1/2}(p)) \\
&\quad \times \{\partial_{p_j}(g_+ J_+^{-1/2}(p)) - \partial_{q_j}(g_+ J_+^{-1/2}(q))\} dp dq \\
&= \iint \Phi^{ij}(P_+, Q_+) J_+(p) J_+(q) \partial_{q_i}(h_+ J_+^{-1/2}(q)) \\
&\quad \times \{\partial_{q_j}(g_+ J_+^{-1/2}(q)) - \partial_{p_j}(g_+ J_+^{-1/2}(p))\} dp dq \\
&= \frac{1}{2} \iint \Phi^{ij}(P_+, Q_+) J_+(p) J_+(q) \{\partial_{p_i}(h_+ J_+^{-1/2}(p)) - \partial_{q_i}(h_+ J_+^{-1/2}(q))\} \\
&\quad \times \{\partial_{p_j}(g_+ J_+^{-1/2}(p)) - \partial_{q_j}(g_+ J_+^{-1/2}(q))\} dp dq.
\end{aligned}$$

By (2) the first term in (24) is symmetric and ≥ 0 if $h = g$. The fourth term can be treated similarly (with $+$ replaced by $-$ everywhere. We combine the second and third terms in (24); again we integrate by parts over p variables to compute

$$= \iint \Phi^{ij}(P_+, Q_-) J_+(p) J_-(q) \partial_{p_i}(h_+ J_+^{-1/2}(p)) \times \{\partial_{p_j}(g_+ J_+^{-1/2}(p)) - \partial_{q_j}(g_- J_-^{-1/2}(q))\} dp dq + \iint \Phi^{ij}(P_-, Q_+) J_-(p) J_+(q) \partial_{p_i}(h_- J_-^{-1/2}(p)) \times \{\partial_{p_j}(g_- J_-^{-1/2}(p)) - \partial_{q_j}(g_+ J_+^{-1/2}(q))\} dp dq.$$

We switch the role of p and q in the second term to obtain

$$= \iint \Phi^{ij}(P_+, Q_-) J_+(p) J_-(q) \{\partial_{p_i}(h_+ J_+^{-1/2}(p)) - \partial_{q_i}(h_- J_-^{-1/2}(q))\} \times \{\partial_{p_j}(g_+ J_+^{-1/2}(p)) - \partial_{q_j}(g_- J_-^{-1/2}(q))\} dp dq.$$

Again by (2) this piece of the operator is symmetric and ≥ 0 if $g = h$. We therefore conclude that L is a non-negative symmetric operator.

We will now determine the null space (15) of the linear operator. Assume $Lg = 0$. From $\langle Lg, g \rangle = 0$ we deduce, by (2), that there are scalar functions $\zeta_l(p, q)$ ($l = \pm$) such that

$$\partial_{p_i}(g_l J_l^{-1/2}(p)) - \partial_{q_i}(g_l J_l^{-1/2}(q)) \equiv \zeta_l(p, q) \left(\frac{p_i}{p_0^l} - \frac{q_i}{q_0^l} \right), \quad i \in \{1, 2, 3\}.$$

Setting $q = 0$, $\partial_{p_i}(g_l J_l^{-1/2}(p)) = \zeta_l(p, 0) \frac{p_i}{p_0^l} + b_{li}$. By replacing p by q and subtracting we obtain

$$\begin{aligned} \partial_{p_i}(g_l J_l^{-1/2}(p)) - \partial_{q_i}(g_l J_l^{-1/2}(q)) &= \zeta_l(p, 0) \frac{p_i}{p_0^l} - \zeta_l(q, 0) \frac{q_i}{q_0^l} \\ &= \zeta_l(p, 0) \left(\frac{p_i}{p_0^l} - \frac{q_i}{q_0^l} \right) + (\zeta_l(p, 0) - \zeta_l(q, 0)) \frac{q_i}{q_0^l}. \end{aligned}$$

We deduce, again by (2), that $\zeta_l(p, 0) - \zeta_l(q, 0) = 0$ and therefore that $\zeta_l(p, 0) \equiv c_l$ (a constant). We integrate $\partial_{p_i}(g_l J_l^{-1/2}(p)) = c_l \frac{p_i}{p_0^l} + b_{li}$ to obtain

$$g_l = \{a_l^g + \sum_{i=1}^{3} b_{li}^g p_j + c_l^g p_0^l\} J_l^{1/2}.$$

Here a_l^g, b_{lj}^g and c_l^g are constants with respect to p (but could be functions of t and x). Moreover, we deduce from the middle terms in (24) as well as (2) that

$$\partial_{p_i}(g_+ J_+^{-1/2}(p)) - \partial_{q_i}(g_- J_-^{-1/2}(q)) \equiv \tilde{\zeta}(p, q) \left(\frac{p_i}{p_0^+} - \frac{q_i}{q_0^-} \right).$$

Therefore $b_{+i}^g - b_{-i}^g + c_+^g \frac{p_i}{p_0^+} - c_-^g \frac{q_i}{q_0^-} = \tilde{\zeta}(p, q) \left(\frac{p_i}{p_0} - \frac{q_i}{q_0} \right)$. We conclude

$$\begin{aligned} b_{+i}^g &\equiv b_{-i}^g, \quad i = 1, 2, 3; \\ c_+^g &\equiv c_-^g. \end{aligned}$$

That means $g(t, x, p) \in \mathcal{N}$ as in (15), so that $g = \mathbf{P}g$. Conversely, $L\{\mathbf{P}g\} = 0$ by a direct calculation. $\square$

For notational simplicity, as in Remark 1, we will normalize all the constants to be one. Accordingly, we write $p_0 = \sqrt{1 + |p|^2}$, $P = (p_0, p)$, and the collision kernel $\Phi(P, Q)$ takes the form

$$\Phi(P, Q) \equiv \frac{\Lambda(P, Q)}{p_0 q_0} S(P, Q), \tag{26}$$

where

$$\Lambda \equiv (P \cdot Q)^2 \left\{ (P \cdot Q)^2 - 1 \right\}^{-3/2},$$
$$S \equiv \left\{ (P \cdot Q)^2 - 1 \right\} I_3 - (p - q) \otimes (p - q) + \{(P \cdot Q) - 1\} (p \otimes q + q \otimes p).$$

We normalize the relativistic Maxwellian as

$$J(p) \equiv J_+(p) = J_-(p) = e^{-p_0}.$$

We further normalize the collision freqency

$$\sigma^{ij}_{\pm,\mp}(p) = \sigma^{ij}(p) = \int \Phi^{ij}(P, Q) J(q) dq, \tag{27}$$

and the inner product $\langle \cdot, \cdot \rangle_\sigma$ takes the form

$$\begin{aligned} \langle g, h \rangle_\sigma \ &\equiv 2 \int_{\mathbb{R}^3} \sigma^{ij} \left\{ \partial_{p_j} g_+ \partial_{p_i} h_+ + \partial_{p_j} g_- \partial_{p_i} h_- \right\} dp, \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^3} \sigma^{ij} \frac{p_i}{p_0} \frac{p_j}{p_0} \{ g_+ h_+ dp + g_- h_- \} dp. \end{aligned} \tag{28}$$

The norms are, as before, naturally built from this normalized inner product.

The normalized vector-valued Landau-Maxwell equation for the perturbation f in (4) now takes the form

$$\begin{aligned} &\left\{ \partial_t + \frac{p}{p_0} \cdot \nabla_x + \xi \left(E + \frac{p}{p_0} \times B \right) \cdot \nabla_p \right\} f - \left\{ E \cdot \frac{p}{p_0} \right\} \sqrt{J} \xi_1 + Lf \\ &= \frac{\xi}{2} \left\{ E \cdot \frac{p}{p_0} \right\} f + \Gamma(f, f), \end{aligned} \tag{29}$$

with $f(0, x, v) = f_0(x, v)$, $\xi_1 = [1, -1]$, and the 2×2 matrix ξ is $\mathrm{diag}(1, -1)$. Further, the normalized Maxwell system in (5) and (6) takes the form

$$\partial_t E - \nabla_x \times B \ = -\mathcal{J} = -\int_{\mathbb{R}^3} \frac{p}{p_0} \sqrt{J} (f_+ - f_-) dp, \quad \partial_t B + \nabla_x \times E = 0, \tag{30}$$

$$\nabla_x \cdot E \ = \rho = \int_{\mathbb{R}^3} \sqrt{J} (f_+ - f_-) dp, \quad \nabla_x \cdot B = 0, \tag{31}$$

with $E(0, x) = E_0(x)$, $B(0, x) = B_0(x)$.

We have a basic (but useful) inequality taken from Glassey & Strauss [3].

Proposition 1. *Let $p, q \in \mathbb{R}^3$ with $P = (p_0, p)$ and $Q = (q_0, q)$, then*

$$\frac{|p-q|^2 + |p \times q|^2}{2p_0 q_0} \leq P \cdot Q - 1 \leq \frac{1}{2}|p-q|^2. \tag{32}$$

This inequality will be used many times for estimating high order derivatives of the collision kernel.

Notice that

$$\begin{aligned}\left(\partial_{p_i} + \frac{q_0}{p_0}\partial_{q_i}\right) P \cdot Q &= \left(\partial_{p_i} + \frac{q_0}{p_0}\partial_{q_i}\right)(p_0 q_0 - p \cdot q) \\ &= \frac{p_i}{p_0} q_0 - q_i + \frac{q_0}{p_0}\left(\frac{q_i}{q_0} p_0 - p_i\right) = 0.\end{aligned}$$

This is the key observation which allows us do analysis on the relativistic Landau Operator (Lemma 2). We define the following relativistic differential operator:

$$\Theta_\alpha(p, q) \equiv \left(\partial_{p_3} + \frac{q_0}{p_0}\partial_{q_3}\right)^{\alpha^3} \left(\partial_{p_2} + \frac{q_0}{p_0}\partial_{q_2}\right)^{\alpha^2} \left(\partial_{p_1} + \frac{q_0}{p_0}\partial_{q_1}\right)^{\alpha^1}. \tag{33}$$

Unless otherwise stated, we omit the p, q dependence and write $\Theta_\alpha = \Theta_\alpha(p, q)$. Note that the three terms in Θ_α do not commute (and we choose this order for no special reason).

We will use the following splitting many times in the rest of this section,

$$\mathcal{A} = \{|p-q| + |p \times q| \geq [|p| + 1]/2\}, \quad \mathcal{B} = \{|p-q| + |p \times q| \leq [|p| + 1]/2\}. \tag{34}$$

The set $\mathcal{A}$ is designed to be away from the first order singularity in the collision kernel $\Phi(P, Q)$ (Proposition 1). And the set $\mathcal{B}$ contains a $\Phi(P, Q)$ singularity ((26) and (32)) but we will exploit the fact that we can compare the size of p and q. We now develop crucial estimates for $\Theta_\alpha \Phi(P, Q)$:

Lemma 2. *For any multi-index α, the Lorentz inner product of P and Q is in the null space of Θ_α,*

$$\Theta_\alpha(P \cdot Q) = 0.$$

Further, recalling (26), for p and q on the set $\mathcal{A}$ we have the estimate

$$|\Theta_\alpha(p, q)\Phi(P, Q)| \leq C p_0^{-|\alpha|} q_0^6. \tag{35}$$

And on $\mathcal{B}$,

$$\frac{1}{6} q_0 \leq p_0 \leq 6 q_0. \tag{36}$$

Using this inequality, we have the following estimate on $\mathcal{B}$:

$$|\Theta_\alpha(p, q)\Phi(P, Q)| \leq C q_0^7 p_0^{-|\alpha|} |p-q|^{-1}. \tag{37}$$

Proof. Let e_i $(i = 1, 2, 3)$ be an element of the standard basis in $\mathbb{R}^3$. We have seen that $\Theta_{e_i}(P \cdot Q) = 0$. And the general case follows from a simple induction over $|\alpha|$.

By (26) and (33), we can now write

$$\Theta_\alpha(p,q)\Phi^{ij}(P,Q) = \Lambda(P,Q)\Theta_\alpha(p,q)\left(\frac{S^{ij}(p,q)}{p_0q_0}\right),$$

where

$$\begin{aligned}\Theta_\alpha\left(\frac{S^{ij}(p,q)}{p_0q_0}\right) &= \left\{(P\cdot Q)^2 - 1\right\}\Theta_\alpha\left\{\delta_{ij}/(p_0q_0)\right\} \\ &\quad + (P\cdot Q - 1)\Theta_\alpha\left\{(p_iq_j + p_jq_i)/(p_0q_0)\right\} \\ &\quad - \Theta_\alpha\left\{(p_i - q_i)(p_j - q_j)/(p_0q_0)\right\}.\end{aligned} \tag{38}$$

We will break up this expression and estimate the different pieces.

Using (33), the following estimates are straightforward:

$$\left|\Theta_\alpha\left\{\delta_{ij}/(p_0q_0)\right\}\right| \le Cq_0^{-1}p_0^{-1-|\alpha|}, \tag{39}$$

$$\left|\Theta_\alpha\left\{(p_iq_j + p_jq_i)/(p_0q_0)\right\}\right| \le Cp_0^{-|\alpha|}. \tag{40}$$

On the other hand, we *claim* that

$$\left|\Theta_\alpha\left\{(p_i - q_i)(p_j - q_j)/(p_0q_0)\right\}\right| \le C\frac{|p-q|^2}{p_0q_0}p_0^{-|\alpha|}. \tag{41}$$

This last estimate is not so trivial because only a lower order estimate of $|p - q|$ is expected after applying even a first order derivative like Θ_{e_i}. The key observation is that

$$\left(\partial_{p_i} + \frac{q_0}{p_0}\partial_{q_i}\right)(p_i - q_i)(p_j - q_j) = \left(1 - \frac{q_0}{p_0}\right)(p_j - q_j),$$

and the r.h.s. is again second order. Therefore the operator Θ_α can maintain the order of the cancellation.

Proof of claim. To prove (41), it is sufficient to show that for any multi-index α and any $i, j, k, l \in \{1, 2, 3\}$ there exists a smooth function $G_{kl}^{\alpha,ij}(p,q)$ satisfying

$$\Theta_\alpha\left\{(p_i - q_i)(p_j - q_j)/(p_0q_0)\right\} = \sum_{k,l=1}^{3}(p_k - q_k)(p_l - q_l)G_{kl}^{\alpha,ij}(p,q), \tag{42}$$

as well as the decay

$$\left|\partial_{\nu_1}\partial_{\nu_2}^q G_{kl}^{\alpha,ij}(p,q)\right| \le Cq_0^{-1-|\nu_2|}p_0^{-1-|\alpha|-|\nu_1|}, \tag{43}$$

which holds for any multi-indices ν_1, ν_2. We prove (42) with (43) by a simple induction over $|\alpha|$.

If $|\alpha| = 0$, we define

$$G_{kl}^{0,ij}(p,q) = \frac{\delta_{ki}\delta_{lj}}{q_0p_0}.$$

The decay (43) for $G_{kl}^{0,ij}(p,q)$ is straightforward to check. And (42) holds trivially for $|\alpha| = 0$.

Assume that (42) with (43) holds for $|\alpha| \leq n$. To conclude the proof, let $|\alpha'| = n + 1$ and write $\Theta_{\alpha'} = \Theta_{e_m}\Theta_\alpha$ for some multi-index α with

$$m = \max\{j : (\alpha')^j > 0\}.$$

This specification of m is needed because of our chosen ordering of the three differential operators in (33), which don't commute. Recalling (33),

$$\Theta_{e_m}(p_k - q_k) = \delta_{km}\left(1 - \frac{q_0}{p_0}\right).$$

From the induction assumption and the last display, we have

$$\begin{aligned}
&\Theta_{\alpha'}\left\{(p_i - q_i)(p_j - q_j)/(p_0 q_0)\right\} \\
&\quad = \Theta_{e_m}\sum_{k,l=1}^{3}(p_k - q_k)(p_l - q_l)G_{kl}^{\alpha,ij}(p,q), \\
&\quad = \left(1 - \frac{q_0}{p_0}\right)\sum_{k=1}^{3}(p_k - q_k)\left\{G_{mk}^{\alpha,ij}(p,q) + G_{km}^{\alpha,ij}(p,q)\right\} \\
&\qquad + \sum_{k,l=1}^{3}(p_k - q_k)(p_l - q_l)\Theta_{e_m}G_{kl}^{\alpha,ij}(p,q).
\end{aligned}$$

We compute

$$1 - \frac{q_0}{p_0} = \frac{p_0 - q_0}{p_0} = \frac{p_0^2 - q_0^2}{p_0(p_0 + q_0)} = \frac{(p - q)\cdot(p + q)}{p_0(p_0 + q_0)} = \frac{\sum_l (p_l - q_l)(p_l + q_l)}{p_0(p_0 + q_0)}.$$

We plug this display into the one above it to obtain (42) for α' with the new coefficients

$$G_{kl}^{\alpha',ij}(p,q) \equiv \Theta_{e_m}G_{kl}^{\alpha,ij}(p,q) + \frac{\left\{G_{mk}^{\alpha,ij}(p,q) + G_{km}^{\alpha,ij}(p,q)\right\}(p_l + q_l)}{p_0(p_0 + q_0)}.$$

We check that $G_{kl}^{\alpha',ij}(p,q)$ satisfies (43) using the Leibnitz differentiation formula as well as the induction assumption (43). This establishes the claim (41).

With the estimates (39), (40) and (41) in hand, we return to establishing (35) and (37). We plug the estimates (39), (40) and (41) into $\Theta_\alpha \Phi^{ij}(P,Q)$ from (38) to obtain that

$$\begin{aligned}
\left|\Theta_\alpha \Phi^{ij}(P,Q)\right| &\leq C p_0^{-|\alpha|}(P\cdot Q)^2\left\{(P\cdot Q)^2 - 1\right\}^{-3/2}\frac{(P\cdot Q)^2 - 1}{p_0 q_0} \\
&\quad + C p_0^{-|\alpha|}(P\cdot Q)^2\left\{(P\cdot Q)^2 - 1\right\}^{-3/2}(P\cdot Q - 1) \\
&\quad + C p_0^{-|\alpha|}(P\cdot Q)^2\left\{(P\cdot Q)^2 - 1\right\}^{-3/2}\frac{|p - q|^2}{p_0 q_0}.
\end{aligned} \tag{44}$$

We will use this estimate twice to get (35) and (37).

We first establish (35). On the set $\mathcal{A}$ we have

$$2|p-q|^2+2|p\times q|^2 \geq (|p-q|+|p\times q|)^2 \geq \frac{1}{4}p_0^2+\frac{|p|}{2} \geq \frac{1}{4}p_0^2.$$

From (32) and the last display we have

$$P\cdot Q+1 \geq P\cdot Q-1 \geq \frac{1}{16}\frac{p_0}{q_0}.$$

From the Cauchy-Schwartz inequality we also have

$$0 \leq P\cdot Q-1 \leq P\cdot Q \leq p_0q_0+|p\cdot q| \leq 2p_0q_0.$$

We plug these last two inequalities (one at a time) into (44) to obtain

$$\begin{aligned}
\left|\Theta_\alpha\Phi^{ij}(p,q)\right| &\leq C(P\cdot Q)^2\left\{(P\cdot Q)^2-1\right\}^{-3/2}p_0^{-|\alpha|}\left\{\frac{p_0^2q_0^2+p_0^2q_0^2+p_0^2q_0^2}{p_0q_0}\right\}\\
&\leq C(p_0q_0)^2\left\{(P\cdot Q)^2-1\right\}^{-3/2}p_0^{-|\alpha|}p_0q_0\\
&\leq C(p_0q_0)^3\left\{P\cdot Q-1\right\}^{-3}p_0^{-|\alpha|}\\
&\leq C(p_0q_0)^3\left(\frac{p_0}{q_0}\right)^{-3}p_0^{-|\alpha|}.
\end{aligned}$$

We move on to establishing (36). If $|p|\leq 1$, then $p_0\leq 2\leq 2q_0$. Assume $|p|\geq 1$, using $\mathcal{B}$ we compute

$$q_0 \geq |q| \geq |p|-|p-q| \geq \frac{1}{2}|p|-\frac{1}{2} \geq \frac{1}{4}p_0-\frac{1}{2}.$$

Therefore, $p_0\leq 6q_0$ on $\mathcal{B}$. For the other half of (36),

$$q_0 \leq p_0+|p-q| \leq \frac{3}{2}p_0+\frac{1}{2} \leq 2p_0.$$

We move on to establishing (37). On the set $\mathcal{B}$ we have a first order singularity. Also (32) tells us

$$\frac{|p-q|^2}{2p_0q_0} \leq P\cdot Q-1 \leq \frac{1}{2}|p-q|^2.$$

We plug this into (44) to observe that on $\mathcal{B}$ we have

$$\begin{aligned}
\left|\Theta_\alpha\Phi^{ij}(p,q)\right| &\leq C(P\cdot Q)^2\left\{(P\cdot Q)^2-1\right\}^{-3/2}p_0^{-|\alpha|}\\
&\quad\times\left\{\frac{(P\cdot Q+1)|p-q|^2+|p-q|^2+|p-q|^2q_0p_0}{q_0p_0}\right\}\\
&\leq C(P\cdot Q)^2\left\{(P\cdot Q)^2-1\right\}^{-3/2}p_0^{-|\alpha|}|p-q|^2\\
&\leq C(p_0q_0)^2\left\{(P\cdot Q)^2-1\right\}^{-3/2}p_0^{-|\alpha|}|p-q|^2\\
&\leq C(p_0q_0)^2(p_0q_0)^{3/2}|p-q|^{-1}\left\{P\cdot Q+1\right\}^{-3/2}p_0^{-|\alpha|}\\
&\leq C(p_0q_0)^{7/2}|p-q|^{-1}p_0^{-|\alpha|}.
\end{aligned}$$

We achieve the last inequality because (32) says $P\cdot Q\geq 1$. $\square$

Next, let $\mu(p, q)$ be an arbitrary smooth scalar function which decays rapidly at infinity. We consider the following integral

$$\int_{\mathbb{R}^3} \Phi^{ij}(P, Q) J^{1/2}(q) \mu(p, q) dq.$$

Both the linear term L and the nonlinear term Γ are of this form (Lemma 6). We develop a new integration by parts technique.

Theorem 3. *Given $|\beta| > 0$, we have*

$$\partial_\beta \int_{\mathbb{R}^3} \Phi^{ij}(P, Q) J^{1/2}(q) \mu(p, q) dq$$
$$= \sum_{\beta_1+\beta_2+\beta_3 \leq \beta} \int_{\mathbb{R}^3} \Theta_{\beta_1} \Phi^{ij}(P, Q) J^{1/2}(q) \partial^q_{\beta_2} \partial_{\beta_3} \mu(p, q) \varphi^{\beta}_{\beta_1,\beta_2,\beta_3}(p, q) dq, \quad (45)$$

where $\varphi^{\beta}_{\beta_1,\beta_2,\beta_3}(p, q)$ is a smooth function which satisfies

$$\left| \partial^q_{\nu_1} \partial_{\nu_2} \varphi^{\beta}_{\beta_1,\beta_2,\beta_3}(p, q) \right| \leq C q_0^{|\beta|-|\nu_1|} p_0^{|\beta_1|+|\beta_3|-|\beta|-|\nu_2|}, \quad (46)$$

for all multi-indices ν_1 and ν_2.

Proof. We prove (45) by an induction over the number of derivatives $|\beta|$. Assume $\beta = e_i$ $(i = 1, 2, 3)$. We write

$$\partial_{p_i} = -\frac{q_0}{p_0} \partial_{q_i} + \left(\partial_{p_i} + \frac{q_0}{p_0} \partial_{q_i} \right) = -\frac{q_0}{p_0} \partial_{q_i} + \Theta_{e_i}. \quad (47)$$

Instead of hitting $\Phi^{ij}(P, Q)$ with ∂_{p_i}, we apply the r.h.s. term above and integrate by parts over $-\frac{q_0}{p_0} \partial_{q_i}$ to obtain

$$\partial_{p_i} \int_{\mathbb{R}^3} \Phi^{ij}(P, Q) J^{1/2}(q) \mu(p, q) dq$$
$$= \int_{\mathbb{R}^3} \Phi^{ij}(P, Q) J^{1/2}(q) \partial_{p_i} \mu(p, q) dq$$
$$+ \int_{\mathbb{R}^3} \Phi^{ij}(P, Q) J^{1/2}(q) \frac{q_0}{p_0} \partial_{q_i} \mu(p, q) dq$$
$$+ \int_{\mathbb{R}^3} \Phi^{ij}(P, Q) J^{1/2}(q) \left(\frac{q_i}{q_0 p_0} - \frac{q_i}{2 p_0} \right) \mu(p, q) dq$$
$$+ \int_{\mathbb{R}^3} \Theta_{e_i} \Phi^{ij}(P, Q) J^{1/2}(q) \mu(p, q) dq.$$

We can write the above in the form (45) with the coefficients given by

$$\phi^{e_i}_{0,0,0}(p, q) = \frac{q_i}{q_0 p_0} - \frac{q_i}{2 p_0}, \; \phi^{e_i}_{e_i,0,0}(p, q) = 1, \; \phi^{e_i}_{0,e_i,0}(p, q)$$
$$= \frac{q_0}{p_0}, \; \phi^{e_i}_{0,0,e_i}(p, q) = 1, \quad (48)$$

and define the rest of the coefficients to be zero. Note that these coefficients satisfy the decay (46). This establishes the first step in the induction.

Assume the result holds for all $|\beta| \le n$. Fix an arbitrary β' such that $|\beta'| = n+1$ and write $\partial_{\beta'} = \partial_{p_m}\partial_\beta$ for some multi-index β and

$$m = \max\{j : (\beta')^j > 0\}.$$

This specification of m is needed because of our chosen ordering of the three differential operators in (33), which don't commute.

By the induction assumption

$$\begin{aligned}
&\partial_{\beta'} \int_{\mathbb{R}^3} \Phi^{ij}(P,Q) J^{1/2}(q)\mu(p,q)dq \\
&\quad = \sum_{\bar\beta_1+\bar\beta_2+\bar\beta_3\le\beta} \partial_{p_m} \int_{\mathbb{R}^3} \Theta_{\bar\beta_1}\Phi^{ij}(P,Q)J^{1/2}(q)\partial^q_{\bar\beta_2}\partial_{\bar\beta_3}\mu(p,q)\varphi^\beta_{\bar\beta_1,\bar\beta_2,\bar\beta_3}(p,q)dq,
\end{aligned}$$

We approach applying the last derivative the same as the $|\beta| = 1$ case above. We obtain

$$= \sum \int_{\mathbb{R}^3} \Theta_{\bar\beta_1}\Phi^{ij}(P,Q)J^{1/2}(q)\varphi^\beta_{\bar\beta_1,\bar\beta_2,\bar\beta_3}(p,q)\partial_{p_m}\partial^q_{\bar\beta_2}\partial_{\bar\beta_3}\mu(p,q)dq \tag{49}$$

$$+ \sum \int_{\mathbb{R}^3} \Theta_{\bar\beta_1}\Phi^{ij}(P,Q)J^{1/2}(q)\varphi^\beta_{\bar\beta_1,\bar\beta_2,\bar\beta_3}(p,q)\frac{q_0}{p_0}\partial_{q_m}\partial^q_{\bar\beta_2}\partial_{\bar\beta_3}\mu(p,q)dq \tag{50}$$

$$+ \sum \int_{\mathbb{R}^3} \Theta_{e_m}\Theta_{\bar\beta_1}\Phi^{ij}(P,Q)J^{1/2}(q)\partial^q_{\bar\beta_2}\partial_{\bar\beta_3}\mu(p,q)\varphi^\beta_{\bar\beta_1,\bar\beta_2,\bar\beta_3}(p,q)dq \tag{51}$$

$$\begin{aligned}
&+ \sum \int_{\mathbb{R}^3} \Theta_{\bar\beta_1}\Phi^{ij}(P,Q)J^{1/2}(q)\partial^q_{\bar\beta_2}\partial_{\bar\beta_3}\mu(p,q) \\
&\times \left(\partial_{p_m} + \frac{q_0}{p_0}\partial_{q_m} + \frac{q_m}{p_0 q_0} - \frac{q_m}{2p_0}\right)\varphi^\beta_{\bar\beta_1,\bar\beta_2,\bar\beta_3}(p,q)dq,
\end{aligned} \tag{52}$$

where the unspecified summations above are over $\bar\beta_1 + \bar\beta_2 + \bar\beta_3 \le \beta$. We collect all the terms above with the same order of differentiation to obtain

$$= \sum_{\beta_1+\beta_2+\beta_3\le\beta'} \int_{\mathbb{R}^3} \Theta_{\beta_1}\Phi^{ij}(P,Q)J^{1/2}(q)\partial^q_{\beta_2}\partial_{\beta_3}\mu(p,q)\varphi^{\beta'}_{\beta_1,\beta_2,\beta_3}(p,q)dq,$$

where the functions $\varphi^{\beta'}_{\beta_1,\beta_2,\beta_3}(p,q)$ are defined naturally as the coefficient in front of each term of the form $\Theta_{\beta_1}\Phi^{ij}(P,Q)J^{1/2}(q)\partial^q_{\beta_2}\partial_{\beta_3}\mu(p,q)$ and we recall that $\beta' = \beta + e_m$.

We check (46) by comparing the decay with the order of differentiation in each of the four terms (49-52). For (49), the order of differentiation is

$$\beta_1 = \bar\beta_1,\ \beta_2 = \bar\beta_2,\ \beta_3 = \bar\beta_3 + e_m.$$

And by the induction assumption,

$$\begin{aligned}
\left|\partial^q_{\nu_1}\partial_{\nu_2}\varphi^\beta_{\bar\beta_1,\bar\beta_2,\bar\beta_3}(p,q)\right| &\le C q_0^{|\beta|-|\nu_1|} p_0^{|\bar\beta_1|+|\bar\beta_3|-|\beta|-|\nu_2|}, \\
&\le C q_0^{|\beta+e_m|-|\nu_1|} p_0^{|\beta_1|+|\bar\beta_3+e_m|-|\beta+e_i|-|\nu_2|} \\
&= C q_0^{|\beta'|-|\nu_1|} p_0^{|\beta_1|+|\beta_3|-|\beta'|-|\nu_2|}.
\end{aligned}$$

This establishes (46) for (49).

For (50), the order of differentiation is

$$\beta_1 = \bar{\beta}_1, \ \beta_2 = \bar{\beta}_2 + e_m, \ \beta_3 = \bar{\beta}_3.$$

And by the induction assumption as well as the Leibnitz rule,

$$\begin{aligned}\left|\partial^q_{\nu_1}\partial_{\nu_2}\left(\frac{q_0}{p_0}\varphi^{\beta}_{\bar{\beta}_1,\bar{\beta}_2,\bar{\beta}_3}(p,q)\right)\right| &\le C q_0^{|\beta|+1-|\nu_1|} p_0^{|\bar{\beta}_1|+|\bar{\beta}_3|-|\beta|-1-|\nu_2|}, \\ &= C q_0^{|\beta'|-|\nu_1|} p_0^{|\beta_1|+|\beta_3|-|\beta'|-|\nu_2|}.\end{aligned}$$

This establishes (46) for (50).

For (51), the order of differentiation is

$$\beta_1 = \bar{\beta}_1 + e_m, \ \beta_2 = \bar{\beta}_2, \ \beta_3 = \bar{\beta}_3.$$

And by the induction assumption,

$$\begin{aligned}\left|\partial^q_{\nu_1}\partial_{\nu_2}\varphi^{\beta}_{\bar{\beta}_1,\bar{\beta}_2,\bar{\beta}_3}(p,q)\right| &\le C q_0^{|\beta|-|\nu_1|} p_0^{|\bar{\beta}_1|+|\bar{\beta}_3|-|\beta|-|\nu_2|}, \\ &\le C q_0^{|\beta'|-|\nu_1|} p_0^{|\beta_1|+|\beta_3|-|\beta'|-|\nu_2|}.\end{aligned}$$

This establishes (46) for (51).

For (52), the order of differentiation is

$$\beta_1 = \bar{\beta}_1, \ \beta_2 = \bar{\beta}_2, \ \beta_3 = \bar{\beta}_3.$$

And by the induction assumption as well as the Leibnitz rule,

$$\begin{aligned}&\left|\partial^q_{\nu_1}\partial_{\nu_2}\left\{\left(\partial_{p_m} + \frac{q_0}{p_0}\partial_{q_m} + \frac{q_m}{p_0 q_0} - \frac{q_m}{2p_0}\right)\varphi^{\beta}_{\bar{\beta}_1,\bar{\beta}_2,\bar{\beta}_3}(p,q)\right\}\right| \\ &\le C q_0^{|\beta|+1-|\nu_1|} p_0^{|\bar{\beta}_1|+|\bar{\beta}_3|-|\beta|-1-|\nu_2|}, \\ &= C q_0^{|\beta'|-|\nu_1|} p_0^{|\beta_1|+|\beta_3|-|\beta'|-|\nu_2|}.\end{aligned}$$

This establishes (46) for (52) and therefore for all of the coefficients. □

Next, we compute derivatives of the collision kernel in (26) which will be important for showing that solutions $F_\pm$ to the relativistic Landau-Maxwell system are positive.

Lemma 3. *We compute a sum of first derivatives in q of (26) as*

$$\sum_j \partial_{q_j}\Phi^{ij}(P,Q) = 2\frac{\Lambda(P,Q)}{p_0 q_0}\left(P\cdot Q p_i - q_i\right). \tag{53}$$

This term has a second order singularity at $p = q$. We further compute a sum of (53) over first derivatives in p as

$$\sum_{i,j} \partial_{p_i}\partial_{q_j}\Phi^{ij}(P,Q) = 4\frac{P\cdot Q}{p_0 q_0}\left\{(P\cdot Q)^2 - 1\right\}^{-1/2} \ge 0. \tag{54}$$

This term has a first order singularity.

This result is quite different from the classical theory, it is straightforward to compute the derivative of the classical kernel in (13) as

$$\sum_{i,j} \partial_{v_i} \partial_{v'_j} \phi^{ij}(v - v') = 0.$$

On the contrary, the proof of Lemma 3 is quite technical.

Proof. Throughout this proof, we temporarily suspend our use of the Einstein summation convention. Differentiating (26), we have

$$\begin{aligned}\partial_{q_j} \Phi^{ij}(P, Q) \equiv & \frac{\partial_{q_j} \Lambda(P, Q)}{p_0 q_0} S^{ij}(P, Q) \\ & + \frac{\Lambda(P, Q)}{p_0 q_0} \left(\partial_{q_j} S^{ij}(P, Q) - \frac{q_j}{q_0^2} S^{ij}(P, Q) \right),\end{aligned}$$

and

$$\begin{aligned}\partial_{q_j} \Lambda(P, Q) = & 2(P \cdot Q) \left\{ (P \cdot Q)^2 - 1 \right\}^{-3/2} \left(\frac{q_j}{q_0} p_0 - p_j \right) \\ & -3(P \cdot Q)^3 \left\{ (P \cdot Q)^2 - 1 \right\}^{-5/2} \left(\frac{q_j}{q_0} p_0 - p_j \right).\end{aligned}$$

Since (2) implies $\sum_j S^{ij}(P, Q) \left(\frac{q_j}{q_0} p_0 - p_j \right) = 0$, we conclude

$$\sum_j \frac{\partial_{q_j} \Lambda(P, Q) S^{ij}(P, Q)}{p_0 q_0} = 0.$$

Therefore it remains to evaluate the r.h.s. of

$$\sum_j \partial_{q_j} \Phi^{ij}(P, Q) = \frac{\Lambda(P, Q)}{p_0 q_0} \sum_j \left(\partial_{q_j} S^{ij}(P, Q) - \frac{q_j}{q_0^2} S^{ij}(P, Q) \right). \tag{55}$$

We take a derivative of S^{ij} in (26) as

$$\begin{aligned}\partial_{q_j} S^{ij} &= 2(P \cdot Q) \left(\frac{q_j}{q_0} p_0 - p_j \right) \delta_{ij} + \left(\frac{q_j}{q_0} p_0 - p_j \right) \left(p_i q_j + q_i p_j \right) \\ &\quad + \{P \cdot Q - 1\} \left(p_i + \delta_{ij} p_j \right) + \left(1 + \delta_{ij} \right) (p_i - q_i) \\ &= 2(P \cdot Q) \left(\frac{q_j}{q_0} p_0 - p_j \right) \delta_{ij} + \left(q_j^2 \frac{p_0}{q_0} p_i + p_j q_j \frac{p_0}{q_0} q_i - p_i p_j q_j - q_i p_j^2 \right) \\ &\quad + P \cdot Q \left(1 + \delta_{ij} \right) p_i - \left(1 + \delta_{ij} \right) q_i.\end{aligned}$$

Next, sum this expression over j to obtain

$$\begin{aligned}\sum_j \partial_{q_j} S^{ij} &= 2(P \cdot Q) \left(\frac{q_i}{q_0} p_0 - p_i \right) + \frac{p_0}{q_0} \left(|q|^2 p_i + p \cdot q q_i \right) \\ &\quad - p \cdot q p_i - |p|^2 q_i + 4 P \cdot Q p_i - 4 q_i.\end{aligned}$$

We collect terms which are coefficients of p_i and q_i respectively

$$
\begin{aligned}
\sum_j \partial_{q_j} S^{ij} &= q_i \left\{ 2\,(P\cdot Q)\,\frac{p_0}{q_0} + p\cdot q\,\frac{p_0}{q_0} - |p|^2 - 4 \right\} \\
&\quad + p_i \left\{ -2P\cdot Q + |q|^2 \frac{p_0}{q_0} - p\cdot q + 4P\cdot Q \right\} \\
&= q_i \left(\frac{p_0}{q_0} P\cdot Q - 3 \right) + p_i \left(3P\cdot Q - \frac{p_0}{q_0} \right),
\end{aligned} \tag{56}
$$

where the last line follows from plugging $|p|^2 = p_0^2 - 1 = p_0 q_0 \frac{p_0}{q_0} - 1$ into the first line and plugging

$$
|q|^2 \frac{p_0}{q_0} = q_0^2 \frac{p_0}{q_0} - \frac{p_0}{q_0} = p_0 q_0 - \frac{p_0}{q_0}
$$

into the second line.

Turning to the computation of $-\frac{1}{q_0^2}\sum_j q_j S^{ij}$, we plug (26) into the following:

$$
\begin{aligned}
\sum_j q_j S^{ij}(P,Q) &= \left\{ (P\cdot Q)^2 - 1 \right\} q_i - (p_i - q_i)\, q\cdot(p-q) \\
&\quad + \{(P\cdot Q) - 1\} \left(p\cdot q q_i + |q|^2 p_i \right).
\end{aligned}
$$

We collect terms which are coefficients of p_i and q_i respectively to obtain

$$
\begin{aligned}
&= q_i \left((P\cdot Q)^2 - 1 + q\cdot(p-q) + p\cdot q\,\{P\cdot Q - 1\} \right) \\
&\quad + p_i \left(-q\cdot(p-q) + |q|^2 \{P\cdot Q - 1\} \right) \\
&= q_i \left((P\cdot Q)^2 - 1 - |q|^2 + p\cdot q\,(P\cdot Q) \right) + p_i \left(-p\cdot q + |q|^2 P\cdot Q \right) \\
&= q_i \left(|p|^2 q_0^2 - p_0 q_0\, p\cdot q \right) + p_i \left(|q|^2 P\cdot Q - p\cdot q \right) \\
&= q_i \left(p_0^2 q_0^2 - p_0 q_0\, p\cdot q - q_0^2 \right) + p_i \left(q_0^2 P\cdot Q - P\cdot Q - p\cdot q \right) \\
&= p_0 q_0 q_i \left(P\cdot Q - \frac{q_0}{p_0} \right) + q_0^2 p_i \left(P\cdot Q - \frac{p_0}{q_0} \right).
\end{aligned}
$$

Divide this expression by q_0^2 to conclude

$$
\sum_j \frac{q_j}{q_0^2} S^{ij} = q_i \left(\frac{p_0}{q_0} P\cdot Q - 1 \right) + p_i \left(P\cdot Q - \frac{p_0}{q_0} \right). \tag{57}
$$

This and (56) are very symmetric expressions.

We combine (56) and (57) to obtain

$$
\begin{aligned}
\sum_j \left(\partial_{q_j} S^{ij} - \frac{q_j}{q_0^2} S^{ij} \right) &= q_i \left(\frac{p_0}{q_0} P\cdot Q - 3 \right) + p_i \left(3P\cdot Q - \frac{p_0}{q_0} \right) \\
&\quad - q_i \left(\frac{p_0}{q_0} P\cdot Q - 1 \right) - p_i \left(P\cdot Q - \frac{p_0}{q_0} \right) \\
&= 2\,(P\cdot Q p_i - q_i).
\end{aligned}
$$

We note that this term has a first order cancellation at $p = q$. We plug this last display into (55) to obtain (53).

We differentiate (53) to obtain

$$\sum_i \partial_{p_i} \sum_j \partial_{q_j} \Phi^{ij}(P, Q) = 2\sum_i \frac{\partial_{p_i} \Lambda(P, Q)}{p_0 q_0} (P \cdot Q p_i - q_i) + 2\frac{\Lambda(P, Q)}{p_0 q_0} \sum_i \left(\partial_{p_i} - \frac{p_i}{p_0^2}\right) (P \cdot Q p_i - q_i). \quad (58)$$

And we can write the derivative of Λ as

$$\partial_{p_i} \Lambda(P, Q) = -(P \cdot Q)\left\{(P \cdot Q)^2 - 1\right\}^{-5/2} \left(\frac{p_i}{p_0} q_0 - q_i\right) \left((P \cdot Q)^2 + 2\right).$$

We compute

$$\sum_i \left(\frac{p_i}{p_0} q_0 - q_i\right) ((P \cdot Q) p_i - q_i) = \sum_i \left(\frac{q_0}{p_0}(P \cdot Q) p_i^2 - \frac{q_0}{p_0} p_i q_i - p_i q_i (P \cdot Q) + q_i^2\right)$$
$$= \frac{q_0}{p_0}(P \cdot Q)|p|^2 - \frac{q_0}{p_0} p \cdot q - p \cdot q (P \cdot Q) + |q|^2.$$

We further add and subtract $\frac{q_0}{p_0}(P \cdot Q)$ to obtain

$$= p_0 q_0 (P \cdot Q) - \frac{q_0}{p_0}(P \cdot Q) - \frac{q_0}{p_0} p \cdot q - p \cdot q (P \cdot Q) + |q|^2$$
$$= p_0 q_0 (P \cdot Q) - q_0^2 - p \cdot q (P \cdot Q) + |q|^2$$
$$= p_0 q_0 (P \cdot Q) - p \cdot q (P \cdot Q) - 1$$
$$= (P \cdot Q)^2 - 1.$$

We conclude that

$$\sum_i \frac{\partial_{p_i} \Lambda(P, Q)}{p_0 q_0} (P \cdot Q p_i - q_i) = -\frac{P \cdot Q}{p_0 q_0} \frac{\left((P \cdot Q)^2 + 2\right)}{\left\{(P \cdot Q)^2 - 1\right\}^{3/2}}. \quad (59)$$

This term has a third order singularity. We will find that the second term in (58) also has a third order singularity, but there is second order cancellation between the two terms in (58).

We now evaluate the sum in the second term in (58) as

$$\sum_i \left(\partial_{p_i} - \frac{p_i}{p_0^2}\right) (P \cdot Q p_i - q_i) = \sum_i \left(P \cdot Q + p_i \left(p_i \frac{q_0}{p_0} - q_i\right) - P \cdot Q \frac{p_i^2}{p_0^2} + \frac{p_i q_i}{p_0^2}\right)$$
$$= 3P \cdot Q + |p|^2 \frac{q_0}{p_0} - p \cdot q - P \cdot Q \frac{|p|^2}{p_0^2} + \frac{p \cdot q}{p_0^2}.$$

We add and subtract $\frac{q_0}{p_0}$ as well as $\frac{P \cdot Q}{p_0^2}$ to obtain

$$= 3P \cdot Q - \frac{q_0}{p_0} + p_0 q_0 - p \cdot q - P \cdot Q + \frac{P \cdot Q}{p_0^2} + \frac{p \cdot q}{p_0^2}$$
$$= 3P \cdot Q - \frac{q_0}{p_0} + \frac{p_0 q_0}{p_0^2} = 3P \cdot Q.$$

Therefore, plugging in (26), we obtain

$$\frac{\Lambda(P,Q)}{p_0q_0}\sum_i\left(\partial_{p_i}-\frac{p_i^2}{p_0^2}\right)(P\cdot Qp_i-q_i)=3\frac{(P\cdot Q)^3}{p_0q_0}\left\{(P\cdot Q)^2-1\right\}^{-3/2}.$$

Further plugging this and (59) into (58) we obtain (54). □

In the following lemma, we will use Lemma 3 to obtain a simplified expression for part of the collision operator (1) which will be used to prove the positivity of our solutions $F_\pm$ to the relativistic Landau-Maxwell system.

Lemma 4. *Given a smooth scalar function $G(q)$ which decays rapidly at infinity, we have*

$$-\partial_{p_i}\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)\partial_{q_j}G(q)dq=4\int_{\mathbb{R}^3}\frac{P\cdot Q}{p_0q_0}\left\{(P\cdot Q)^2-1\right\}^{-1/2}G(q)dq \\ +\kappa(p)G(p),$$

where $\kappa(p)=2^{7/2}\pi p_0\int_0^\pi\left(1+|p|^2\sin^2\theta\right)^{-3/2}\sin\theta d\theta.$

Proof. We write out ∂_{p_i} as in (47) to observe

$$\begin{aligned}-\partial_{p_i}\int\Phi^{ij}(P,Q)\partial_{q_j}G(q)dq &=-\int\left\{-\frac{q_0}{p_0}\partial_{q_i}+\Theta_{e_i}\right\}\Phi^{ij}(P,Q)\partial_{q_j}G(q)dq\\ &=-\int\Theta_{e_i}\Phi^{ij}(P,Q)\partial_{q_j}G(q)dq\\ &\quad+\int\frac{q_0}{p_0}\partial_{q_i}\Phi^{ij}(P,Q)\partial_{q_j}G(q)dq.\end{aligned}$$

We split these integrals into $|p-q|\le\epsilon$ and $|p-q|>\epsilon$ for $\epsilon>0$. We note that the integrals over $|p-q|\le\epsilon$ converge to zero as $\epsilon\downarrow 0$. We will eventually send $\epsilon\downarrow 0$, so we focus on the region $|p-q|>\epsilon$. We rewrite

$$\begin{aligned}\frac{q_0}{p_0}\partial_{q_i}\Phi^{ij}(P,Q)\partial_{q_j}G(q) &=\partial_{q_j}\left\{\frac{q_0}{p_0}\partial_{q_i}\Phi^{ij}(P,Q)G(q)\right\}\\ &\quad-\partial_{q_j}\left\{\frac{q_0}{p_0}\partial_{q_i}\Phi^{ij}(P,Q)\right\}G(q).\end{aligned}$$

After an integration by parts, the integrals over $|p-q|>\epsilon$ are

$$\begin{aligned}&=\int_{|p-q|>\epsilon}\partial_{q_j}\left\{\Theta_{e_i}\Phi^{ij}(P,Q)\right\}G(q)dq\\ &\quad-\int_{|p-q|>\epsilon}\partial_{q_j}\left\{\frac{q_0}{p_0}\partial_{q_i}\Phi^{ij}(P,Q)\right\}G(q)dq\\ &\quad+\int_{|p-q|>\epsilon}\partial_{q_j}\left\{\frac{q_0}{p_0}\partial_{q_i}\Phi^{ij}(P,Q)G(q)\right\}dq.\end{aligned}$$

By the definition of Θ_{e_i} in (33), this is

$$=\int_{|p-q|>\epsilon}\partial_{q_j}\partial_{p_i}\Phi^{ij}(P,Q)G(q)dq+\int_{|p-q|>\epsilon}\partial_{q_j}\left\{\frac{q_0}{p_0}\partial_{q_i}\Phi^{ij}(P,Q)G(q)\right\}dq.$$

We plug (54) into the first term above to obtain the first term on the r.h.s. of this lemma as $\epsilon \downarrow 0$. For the second term above, we apply the divergence theorem to obtain

$$\int_{|p-q|>\epsilon} \partial_{q_j} \left\{ \frac{q_0}{p_0} \partial_{q_i} \Phi^{ij}(P,Q) G(q) \right\} dq$$
$$= \int_{|p-q|=\epsilon} \frac{q_0}{p_0} \partial_{q_i} \Phi^{ij}(P,Q) \frac{p_j - q_j}{|p-q|} G(q) dS,$$

where dS is given below. By a Taylor expansion, $P \cdot Q = 1 + O(|p-q|^2)$. Using this and (53) we have

$$\frac{q_0}{p_0} \partial_{q_i} \Phi^{ij}(P,Q) \frac{p_j - q_j}{|p-q|} = 2 \frac{\Lambda(P,Q)}{p_0^2} |p-q| + O(|p-q|^{-1}).$$

And the integral over $|p-q| = \epsilon$ which includes the terms in $O(|p-q|^{-1})$ goes to zero as $\epsilon \downarrow 0$. We focus on the main part

$$2p_0^{-2} \int_{|p-q|=\epsilon} \Lambda(P,Q)|p-q|G(q)dS.$$

We multiply and divide by $p_0 q_0 + p \cdot q + 1$ to observe that

$$P \cdot Q - 1 = \frac{|p-q|^2 + |p \times q|^2}{p_0 q_0 + p \cdot q + 1}.$$

This and (26) imply

$$\Lambda = (P \cdot Q)^2 \left(\frac{p_0 q_0 + p \cdot q + 1}{p_0 q_0 - p \cdot q + 1} \right)^{3/2} \left(|p-q|^2 + |p \times q|^2 \right)^{-3/2}.$$

We change variables as $q \to p - q$ so that the integrand becomes $\Lambda |q| G(p-q)$, and we define $\bar{q}_0 = \sqrt{1 + |p-q|^2}$ so that after the change of variables

$$\Lambda = (p_0 \bar{q}_0 - |p|^2 + p \cdot q)^2 \left(\frac{p_0 \bar{q}_0 + |p|^2 - p \cdot q + 1}{p_0 \bar{q}_0 - |p|^2 + p \cdot q + 1} \right)^{3/2} \left(|q|^2 + |p \times q|^2 \right)^{-3/2}.$$

We choose the angular integration over $|q| = \epsilon$ in such a way that $p \cdot q = |p||q| \cos\theta$ and $dS = \epsilon^2 \sin\theta d\theta d\phi = \epsilon^2 d\omega$ with $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$. Note that as $\epsilon \downarrow 0$ (on $|q| = \epsilon$)

$$\epsilon^3 \Lambda \to 2^{3/2} \left(1 + |p|^2 \sin^2\theta \right)^{-3/2} p_0^3.$$

Hence, as $\epsilon \downarrow 0$,

$$2p_0^{-2} \int_{|q|=\epsilon} \Lambda |q| G(p-q) dS = 2p_0^{-2} \int_{S^2} \epsilon^3 \Lambda G(p - \epsilon\omega) d\omega \to \kappa(p) G(p),$$

with κ defined in the statement of this lemma. $\square$

Lemma 5. *There exists $C > 0$, such that*

$$\frac{1}{C}\left\{\left|\partial_{p_i} g\right|_2^2 + |g|_2^2\right\} \leq |g|_\sigma^2 \leq C\left\{\left|\partial_{p_i} g\right|_2^2 + |g|_2^2\right\}. \tag{60}$$

Further, $\sigma^{ij}(p)$ is a smooth function satisfying

$$\left|\partial_\beta \sigma^{ij}(p)\right| \leq C p_0^{-|\beta|}. \tag{61}$$

Proof. The spectrum of $\sigma^{ij}(p)$, (27), consists of a simple eigenvalue $\lambda_1(p) > 0$ associated with the vector p and a double eigenvalue $\lambda_2(p) > 0$ associated with $p^\perp$; there are constants $c_1, c_2 > 0$ such that, as $|p| \to \infty$, $\lambda_1(p) \to c_1$, $\lambda_2(p) \to c_2$. In Lemou [8] there is a full discussion of these eigenvalues. This is enough to prove (60); see [5] for more details on a similar argument.

We move on to (61). We combine (27) and (45) (with $\mu(p,q) = J^{1/2}(q)$) to obtain

$$\begin{aligned}
\partial_\beta \sigma^{ij}(p) &= \partial_\beta \int_{\mathbb{R}^3} \Phi^{ij}(P,Q) J(q)\,dq \\
&= \sum_{\beta_1+\beta_2\leq\beta} \int_{\mathbb{R}^3} \Theta_{\beta_1}\Phi(P,Q) J^{1/2}(q)\partial_{\beta_2}^q J^{1/2}(q)\varphi^\beta_{\beta_1,\beta_2,0}(p,q)\,dq.
\end{aligned}$$

By (46) then

$$\left|\partial_\beta \sigma^{ij}(p)\right| \leq C \sum_{\beta_1+\beta_2\leq\beta} p_0^{|\beta_1|-|\beta|} \int \left|\Theta_{\beta_1}\Phi(P,Q)\right| J^{1/2}(q)\,dq.$$

Recall (34), we split this integration into the sets $\mathcal{A}$, $\mathcal{B}$. We plug in the estimate (35) to get

$$p_0^{|\beta_1|-|\beta|} \int_{\mathcal{A}} \left|\Theta_{\beta_1}\Phi(P,Q)\right| J^{1/2}(q)\,dq \leq C p_0^{|\beta_1|-|\beta|} p_0^{-|\beta_1|} = C p_0^{-|\beta|}.$$

On $\mathcal{B}$ we have a first order singularity but q is larger than p, in fact we use (36) to get exponential decay in p over this region. With (37) we obtain

$$\int_{\mathcal{B}} \left|\Theta_{\beta_1}\Phi(P,Q)\right| J^{1/2}(q)\,dq \leq C J^{1/16}(p) \int |p-q|^{-1} J^{1/4}(q)\,dq.$$

We can now deduce (61). □

We now write the Landau Operators A, K, Γ in a new form which will be used throughout the rest of the paper.

Lemma 6. *We have the following representations for $A, K, \Gamma \in \mathbb{R}^2$, which are defined in (22) and (23),*

$$\begin{aligned}
Ag &= 2J^{-1/2}\partial_{p_i}\left\{J^{1/2}\sigma^{ij}(\partial_{p_j} g + \frac{p_j}{2p_0} g)\right\} \\
&= 2\partial_{p_i}(\sigma^{ij}\partial_{p_j} g) - \frac{1}{2}\sigma^{ij}\frac{p_i}{p_0}\frac{p_j}{p_0} g + \partial_{p_i}\left\{\sigma^{ij}\frac{p_j}{p_0}\right\} g,
\end{aligned} \tag{62}$$

$$\begin{aligned} Kg &= -J(p)^{-1/2}\partial_{p_i}\left\{J(p)\int_{\mathbb{R}^3}\Phi^{ij}J^{1/2}(q)\partial_{q_j}(g(q)\cdot[1,1])dq\right\}[1,1] \\ &\quad -J(p)^{-1/2}\partial_{p_i}\left\{J(p)\int_{\mathbb{R}^3}\Phi^{ij}J^{1/2}(q)\frac{q_j}{2q_0}(g(q)\cdot[1,1])dq\right\}[1,1], \end{aligned} \tag{63}$$

where $\Phi^{ij}=\Phi^{ij}(P,Q)$. *Further*

$$\Gamma(g,h)=[\Gamma_+(g,h),\Gamma_-(g,h)], \tag{64}$$

where

$$\begin{aligned} \Gamma_\pm(g,h) &= \left(\partial_{p_i}-\frac{p_i}{2p_0}\right)\int\Phi^{ij}J^{1/2}(q)\partial_{p_j}g_\pm(p)\,(h(q)\cdot[1,1])\,dq, \\ &\quad -\left(\partial_{p_i}-\frac{p_i}{2p_0}\right)\int\Phi^{ij}J^{1/2}(q)g_\pm(p)\partial_{q_j}\,(h(q)\cdot[1,1])\,dq. \end{aligned}$$

Proof. For (62) it suffices to consider $2J(p)^{-1/2}\mathcal{C}(J^{1/2}g_\pm,J)$:

$$\begin{aligned} &\equiv 2J(p)^{-1/2}\partial_{p_i}\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)\left\{\partial_{p_j}\left(J^{1/2}g_\pm(p)\right)J(q)-\left(J^{1/2}g_\pm(p)\right)\partial_{q_j}J(q)\right\}dq \\ &= 2J(p)^{-1/2}\partial_{p_i}\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)J(q)J(p)^{1/2}\left\{\partial_{p_j}g_\pm+\left(\frac{q_j}{q_0}-\frac{p_j}{2p_0}\right)g_\pm\right\}dq \\ &= 2J(p)^{-1/2}\partial_{p_i}\left\{\sigma^{ij}(p)J(p)^{1/2}\left(\partial_{p_j}g_\pm+\frac{p_j}{2p_0}g_\pm\right)\right\}. \end{aligned}$$

Above, we have used the null space of Φ in (2). Below, we move some derivatives inside and cancel out one term.

$$\begin{aligned} &= 2\partial_{p_i}\left\{\sigma^{ij}(p)\left(\partial_{p_j}g_\pm+\frac{p_j}{2p_0}g_\pm\right)\right\}-\frac{p_i}{p_0}\left\{\sigma^{ij}(p)\left(\partial_{p_j}g_\pm+\frac{p_j}{2p_0}g_\pm\right)\right\} \\ &= 2\partial_{p_i}\left\{\sigma^{ij}(p)\partial_{p_j}g_\pm\right\}+\partial_{p_i}\left\{\sigma^{ij}(p)\frac{p_j}{p_0}\right\}g_\pm-\frac{1}{2}\sigma^{ij}(p)\frac{p_i}{p_0}\frac{p_j}{p_0}g_\pm. \end{aligned}$$

For K simply plug (25) with normalized constants into (22). For Γ, we use the null condition (2) to compute $J(p)^{-1/2}\mathcal{C}(\sqrt{J}g_+,\sqrt{J}h_-)$

$$\begin{aligned} &= J(p)^{-1/2}\partial_{p_i}\int\Phi^{ij}(P,Q)J^{1/2}(q)J^{1/2}(p)\left\{h_-(q)\partial_{p_j}g_+(p)-\partial_{q_j}h_-(q)g_+(p)\right\}dq \\ &\quad +J(p)^{-1/2}\partial_{p_i}\int\Phi^{ij}(P,Q)J^{1/2}(q)J^{1/2}(p)\left\{\frac{q_j}{2q_0}-\frac{p_j}{2p_0}\right\}h_-(q)g_+(p)dq \\ &= J(p)^{-1/2}\partial_{p_i}\int\Phi^{ij}(P,Q)J^{1/2}(q)J^{1/2}(p)\left\{h_-(q)\partial_{p_j}g_+(p)-\partial_{q_j}h_-(q)g_+(p)\right\}dq \\ &= \left(\partial_{p_i}-\frac{p_i}{2p_0}\right)\int\Phi^{ij}(P,Q)J^{1/2}(q)\left\{h_-(q)\partial_{p_j}g_+(p)-\partial_{q_j}h_-(q)g_+(p)\right\}dq. \end{aligned}$$

Plug four of these type of calculations into (23) to obtain (64). □

We will use these expressions just proven to get the estimates below.

Lemma 7. *Let $|\beta| > 0$. For any small $\eta > 0$, there exists $C_\eta > 0$ such that*

$$-\langle \partial_\beta\{Ag\}, \partial_\beta g\rangle \geq |\partial_\beta g|_\sigma^2 - \eta \sum_{|\alpha|\leq|\beta|} |\partial_\alpha g|_\sigma^2 - C_\eta |g|_2^2, \tag{65}$$

$$|\langle \partial_\beta\{Kg\}, \partial_\beta h\rangle| \leq \left\{ \eta \sum_{|\bar\beta|\leq|\beta|} \left|\partial_{\bar\beta} g\right|_\sigma + C_\eta \, |g|_2 \right\} \left|\partial_\beta h\right|_\sigma . \tag{66}$$

Proof. We will prove (65) for a real valued function g to make the notation less cumbersome, although the result follows trivially for $g = [g_+, g_-]$. We write out the inner product in (65) using (62) to achieve

$$\begin{aligned}
\langle \partial_\beta\{Ag\}, \partial_\beta g\rangle &= \int_{\mathbb{R}^3} \partial_\beta \left(\partial_{p_i} \left\{ \sigma^{ij} \frac{p_i}{p_0} \right\} g \right) \partial_\beta g dp. \\
&\quad - \int_{\mathbb{R}^3} \left\{ \frac{1}{2} \partial_\beta \left(\sigma^{ij} \frac{p_i}{p_0} \frac{p_j}{p_0} g \right) \partial_\beta g + 2\partial_\beta[\sigma^{ij} \partial_{p_j} g] \partial_{p_i} \partial_\beta g \right\} dp \\
&= -|\partial_\beta g|_\sigma^2 + \sum_{\alpha\leq\beta} C_\beta^\alpha \int_{\mathbb{R}^3} \partial_{\beta-\alpha} \partial_{p_i} \left\{ \sigma^{ij} \frac{p_i}{p_0} \right\} \partial_\alpha g \partial_\beta g dp \qquad (67) \\
&\quad - \sum_{\alpha<\beta} C_\beta^\alpha \int_{\mathbb{R}^3} 2\partial_{\beta-\alpha} \sigma^{ij} \partial_\alpha \partial_{p_j} g \partial_{p_i} \partial_\beta g dp \\
&\quad - \sum_{\alpha<\beta} C_\beta^\alpha \int_{\mathbb{R}^3} \frac{1}{2} \partial_{\beta-\alpha} \left(\sigma^{ij} \frac{p_i}{p_0} \frac{p_j}{p_0} \right) \partial_\alpha g \partial_\beta g dp.
\end{aligned}$$

Since $\alpha < \beta$ in the last two terms below, (61) gives us the following estimate:

$$\left| \partial_{\beta-\alpha} \partial_{p_i} \left\{ \sigma^{ij} \frac{p_i}{p_0} \right\} \right| + \left| \partial_{\beta-\alpha} \sigma^{ij} \right| + \left| \partial_{\beta-\alpha} \left(\sigma^{ij} \frac{p_i}{p_0} \frac{p_j}{p_0} \right) \right| \leq C p_0^{-1}.$$

We bound the second and fourth terms in (67) by

$$C \sum_{\alpha\leq\beta} \int_{\mathbb{R}^3} p_0^{-1} \left| \partial_\alpha g \partial_\beta g \right| dp = \int_{|p|\leq m} + \int_{|p|>m} .$$

On the unbounded part we use Cauchy-Schwartz

$$\sum_{\alpha\leq\beta} \int_{|p|>m} p_0^{-1} \left| \partial_\alpha g \partial_\beta g \right| dp \leq \frac{C}{m} |\partial_\beta g|_\sigma \sum_{\alpha\leq\beta} |\partial_\alpha g|_\sigma \leq \frac{C}{m} \sum_{\alpha\leq\beta} |\partial_\alpha g|_\sigma^2.$$

On the compact part we use the compact interpolation of Sobolev-spaces

$$\begin{aligned}
\int_{|p|\leq m} \sum_{\alpha\leq\beta} \left| \partial_\alpha g \partial_\beta g \right| dp &\leq \int_{|p|\leq m} \sum_{\alpha\leq\beta} |\partial_\alpha g|^2 + \left|\partial_\beta g\right|^2 dp \\
&\leq \eta' \sum_{|\alpha|=|\beta|+1} \int_{|p|\leq m} |\partial_\alpha g|^2 dp + C_{\eta'} \int_{|p|\leq m} |g|^2 dp \\
&\leq \eta \sum_{|\alpha|\leq|\beta|} |\partial_\alpha g|_\sigma^2 + C_\eta |g|_2^2.
\end{aligned}$$

For the third term in (67), we split into two cases; first suppose $|\alpha| < |\beta| - 1$ and integrate by parts on ∂_{p_i} to obtain

$$\sum_{|\alpha|<|\beta|-1} \int_{\mathbb{R}^3} 2\left(\partial_{\beta-\alpha}\partial_{p_i}\sigma^{ij}\partial_\alpha\partial_{p_j}g + \partial_{\beta-\alpha}\sigma^{ij}\partial_{p_i}\partial_\alpha\partial_{p_j}g\right)\partial_\beta g dp.$$

We bound this term by

$$\begin{aligned} C\sum_{|\alpha|<|\beta|}\int_{\mathbb{R}^3} p_0^{-1}\left|\partial_\alpha\partial_{p_j}g\partial_\beta g\right|dp &= \int_{|p|\le m} + \int_{|p|>m} \\ &\le \int_{|p|\le m} + \frac{C}{m}|\partial_\beta g|_\sigma \sum_{|\alpha|<|\beta|}|\partial_\alpha g|_\sigma \\ &\le \int_{|p|\le m} + \frac{C}{m}\sum_{|\alpha|\le|\beta|}|\partial_\alpha g|_\sigma^2. \end{aligned}$$

By the compact interpolation of Sobolev spaces

$$\sum_{|\alpha|<|\beta|}\int_{|p|\le m}\left|\partial_\alpha\partial_{p_j}g\partial_\beta g\right|dp \le \eta\sum_{|\alpha|\le|\beta|}|\partial_\alpha g|_\sigma^2 + C_\eta|g|_2^2.$$

Finally, if $|\alpha| = |\beta| - 1$ for the third term in (67), we integrate by parts and use symmetry

$$\begin{aligned} 2\int_{\mathbb{R}^3}\partial_{\beta-\alpha}\sigma^{ij}\partial_\alpha\partial_{p_j}g\partial_{p_i}\partial_\beta g dp &= 2\int_{\mathbb{R}^3}\partial_{\beta-\alpha}\sigma^{ij}\partial_\alpha\partial_{p_j}g\partial_{\beta-\alpha}\partial_\alpha\partial_{p_i}g dp \\ &= -\int_{\mathbb{R}^3}\partial^2_{\beta-\alpha}\sigma^{ij}\partial_\alpha\partial_{p_j}g\partial_\alpha\partial_{p_i}g dp. \end{aligned}$$

Because the order of the derivatives on g is now $= |\beta|$, we can again use the compact interpolation of Sobolev spaces and (61) to get the same bounds as for the last case $|\alpha| < |\beta| - 1$. We obtain

$$-\langle\partial_\beta\{Ag\}, \partial_\beta g\rangle \ge |\partial_\beta g|_\sigma^2 - \left(\eta + \frac{C}{m}\right)\sum_{|\alpha|\le|\beta|}|\partial_\alpha g|_\sigma^2 - C_\eta|g|_2^2.$$

This completes the estimate (65).

We now consider $\langle\partial_\beta\{Kg\}, \partial_\beta h\rangle$ and (66). Recalling (63), we use (45) with $\mu(p,q) = \left\{\partial_{q_j} + \frac{q_j}{2q_0}\right\}(g(q)\cdot[1,1])$, the Leibnitz formula as well as an integration by parts to express $\langle\partial_\beta\{Kg\}, \partial_\beta h\rangle$ as

$$\begin{aligned} &\sum\iint \Theta_{\alpha_1}\Phi^{ij}(P,Q)\sqrt{J(p)J(q)}\partial^q_{\alpha_2}\left\{\partial_{q_j} + \frac{q_j}{2q_0}\right\}(g(q)\cdot[1,1]) \\ &\quad\times\partial_{\beta_1}(h(p)\cdot[1,1])\bar{\varphi}^{\beta,\beta_1}_{\alpha_1,\alpha_2,0}(p,q)dqdp, \end{aligned}$$

where the sum is $\alpha_1 + \alpha_2 \le \beta$, $|\beta| \le |\beta_1| \le |\beta| + 1$. And $\bar{\varphi}^{\beta,\beta_1}_{\alpha_1,\alpha_2,0}(p,q)$ is a collection of the inessential terms; it satisfies the decay estimate (46) independent of the value of β_1. Therefore we can further express $\langle\partial_\beta\{Kg\}, \partial_\beta h\rangle$ as

$$\sum\iint \bar{\mu}_{\beta_1\beta_2}(p,q)J(p)^{1/4}J(q)^{1/4}\partial_{\beta_1}(h(p)\cdot[1,1])\partial^q_{\beta_2}(g(q)\cdot[1,1])dqdp,$$

where the sum is over $|\beta_2| \le |\beta| + 1$, $|\beta| \le |\beta_1| \le |\beta| + 1$. And, using (35) and (37), we see that $\bar{\mu}_{\beta_1\beta_2}(p, q)$ is a collection of L^2 functions. Therefore, as in (80), we split $\langle \partial_\beta\{Kg\}, \partial_\beta h\rangle$ to get

$$\sum \iint \psi_{ij}(p,q) J(p)^{1/4} J(q)^{1/4} \partial_{\beta_1}(h(p)\cdot[1,1])\partial^q_{\beta_2}(g(q)\cdot[1,1])dqdp$$
$$+\sum \iint \left\{\bar{\mu}_{\beta_1\beta_2} - \psi_{ij}\right\} J(p)^{1/4} J(q)^{1/4} \partial_{\beta_1}(h(p)\cdot[1,1])\partial^q_{\beta_2}(g(q)\cdot[1,1])dqdp.$$

In the same fashion as J_2 in (80), for any $m > 0$, we estimate the second term above by

$$\frac{C}{m}\left|\partial_\beta h\right|_\sigma \sum_{|\bar{\beta}|\le|\beta|}\left|\partial_{\bar{\beta}} g\right|_\sigma .$$

Since $\psi_{ij}(p, q)$ is a smooth function with compact support, we integrate by parts over q repeatedly to bound the first term by

$$\sum\left|\iint \partial_{\beta_1}\{\psi_{ij}(p,q)J(q)^{1/4}\}J(p)^{1/4}(g(q)\cdot[1,1])\partial_{\beta_1}(h(p)\cdot[1,1])dqdp\right|$$
$$\le C(m)\sum\iint J(q)^{1/8}J(p)^{1/4}\left|(g(q)\cdot[1,1])\partial_{\beta_1}(h(p)\cdot[1,1])\right|dqdp$$
$$\le C(m)\,|g|_2\sum\left|\partial_{\beta_1}h\right|_2 \le C(m)\,|g|_2\left|\partial_\beta h\right|_\sigma ,$$

where the final sum above is over $|\beta| \le |\beta_1| \le |\beta| + 1$. And we have used (60) to get the last inequality. We conclude our lemma by choosing m large. □

We now estimate the nonlinear term $\Gamma(f, g)$:

Theorem 4. *Let* $|\gamma| + |\beta| \le N$, *then*

$$\left|\langle\partial^\gamma_\beta\Gamma(f,g),\ \partial^\gamma_\beta h\rangle\right| \le C\sum\{|\partial^{\gamma_1}_{\beta_3}f|_2|\partial^{\gamma-\gamma_1}_{\beta_2}g|_\sigma + |\partial^{\gamma_1}_{\beta_3}f|_\sigma|\partial^{\gamma-\gamma_1}_{\beta_2}g|_2\}|\partial^\gamma_\beta h|_\sigma, \quad (68)$$

where the summation is over $\gamma_1 \le \gamma$, $\beta_2 + \beta_3 \le \beta$. *Further*

$$\left|(\partial^\gamma_\beta\Gamma(f,g),\ \partial^\gamma_\beta h)\right| \le C\|\partial^\gamma_\beta h\|_\sigma\left\{|||g|||_\sigma|||f||| + |||f|||_\sigma|||g|||\right\}.$$

Proof. Notice $\partial^\gamma\Gamma(f,g) = \sum_{\gamma_1\le\gamma} C^{\gamma_1}_\gamma\Gamma(\partial^{\gamma_1}f, \partial^{\gamma-\gamma_1}g)$; thus it suffices to only consider the p derivatives. From (64), using (45), we can write $\partial_\beta\Gamma(f,g)$ as

$$\partial_{p_i}\int\Theta_{\beta_1}\Phi^{ij}(P,Q)\sqrt{J(q)}\partial^q_{\beta_2}\partial_{\beta_3}\left\{\partial_{p_j}f_l(p)g_k(q)\right\}\varphi^\beta_{\beta_1,\beta_2,\beta_3}(p,q)dq$$
$$-\partial_{p_i}\int\Theta_{\beta_1}\Phi^{ij}(P,Q)\sqrt{J(q)}\partial^q_{\beta_2}\partial_{\beta_3}\left\{f_l(p)\partial_{q_j}g_k(q)\right\}\varphi^\beta_{\beta_1,\beta_2,\beta_3}(p,q)dq$$
$$+\int\Theta_{\beta_1}\Phi^{ij}(P,Q)\sqrt{J(q)}\partial^q_{\beta_2}\partial_{\beta_3}\left\{f_l(p)\partial_{q_j}g_k(q)\frac{p_i}{2p_0}\right\}\varphi^\beta_{\beta_1,\beta_2,\beta_3}(p,q)dq$$
$$-\int\Theta_{\beta_1}\Phi^{ij}(P,Q)\sqrt{J(q)}\partial^q_{\beta_2}\partial_{\beta_3}\left\{\partial_{p_j}f_l(p)g_k(q)\frac{p_i}{2p_0}\right\}\varphi^\beta_{\beta_1,\beta_2,\beta_3}(p,q)dq.$$

Above, we implicitly sum over $i, j \in \{1, 2, 3\}$, $\beta_1 + \beta_2 + \beta_3 \le \beta$ and $k \in \{+, -\}$. And $l \in \{+, -\}$. Recall $\varphi^{\beta}_{\beta_1,\beta_2,\beta_3}(p, q)$ from Theorem 3. Further (46) implies

$$\left|\varphi^{\beta}_{\beta_1,\beta_2,\beta_3}(p, q)\right| \le C q_0^{|\beta|}.$$

We use the above two displays and integrate by parts over ∂_{p_i} in the first two terms, to bound $\left|\langle \partial_\beta \Gamma(f, g), \partial_\beta h\rangle\right|$ above by

$$C \iint J^{1/4}(q) \left|\Theta_{\beta_1} \Phi^{ij}(P, Q)\partial^q_{\beta_2} g_k(q)\partial_{\beta_3}\partial_{p_j} f_l(p)\partial_{p_i}\partial_\beta h_l(p)\right| dqdp \tag{69}$$

$$+C \iint J^{1/4}(q) \left|\Theta_{\beta_1} \Phi^{ij}(P, Q)\partial^q_{\beta_2}\partial_{q_j} g_k(q)\partial_{\beta_3} f_l(p)\partial_{p_i}\partial_\beta h_l(p)\right| dqdp \tag{70}$$

$$+C \iint J^{1/4}(q) \left|\Theta_{\beta_1} \Phi^{ij}(P, Q)\partial^q_{\beta_2}\partial_{q_j} g_k(q)\partial_{\beta_3} f_l(p)\partial_\beta h_l(p)\right| dqdp$$

$$+C \iint J^{1/4}(q) \left|\Theta_{\beta_1} \Phi^{ij}(P, Q)\partial^q_{\beta_2} g_k(q)\partial_{\beta_3}\partial_{p_j} f_l(p)\partial_\beta h_l(p)\right| dqdp.$$

In the above expressions, we add the summation over $l \in \{+, -\}$. It suffices to estimate (69) and (70); the other two terms are similar. While estimating these terms we will repeatedly use the equivalence of the norms in (60) without mention. We start with (69).

We next split (69) into the two regions, $\mathcal{A}$ and $\mathcal{B}$, defined in (34). We then use (37) and the Cauchy-Schwartz inequality to estimate (69) over $\mathcal{B}$,

$$\begin{aligned}
&\iint_{\mathcal{B}} J^{1/4}(q) \left|\Theta_{\beta_1} \Phi^{ij}(P, Q)\partial^q_{\beta_2} g_k(q)\partial_{\beta_3}\partial_{p_j} f_l(p)\partial_{p_i}\partial_\beta h_l(p)\right| dqdp \\
&\le C \iint_{\mathcal{B}} |p-q|^{-1} J^{1/8}(q) \left|\partial^q_{\beta_2} g_k(q)\partial_{\beta_3}\partial_{p_j} f_l(p)\partial_{p_i}\partial_\beta h_l(p)\right| dqdp \\
&\le C|\partial_\beta h|_\sigma \left(\int_{\mathbb{R}^3} \left|\partial_{\beta_3}\partial_{p_j} f_l(p)\right|^2 \left(\int_{\mathcal{B}} |p-q|^{-1} J^{1/8}(q) \left|\partial^q_{\beta_2} g_k(q)\right| dq\right)^2 dp\right)^{1/2} \\
&\le C|\partial_\beta h|_\sigma |\partial_{\beta_2} g|_2 \left(\int_{\mathbb{R}^3} \left|\partial_{\beta_3}\partial_{p_j} f_l(p)\right|^2 \left(\int_{\mathcal{B}} |p-q|^{-2} J^{1/4}(q) dq\right) dp\right)^{1/2} \\
&\le C|\partial_\beta h|_\sigma |\partial_{\beta_2} g|_2 |\partial_{\beta_3} f|_\sigma,
\end{aligned}$$

where we use (36) to say that $\int_{\mathcal{B}} |p-q|^{-2} J^{1/4}(q)dq \le C$. This completes the estimate of (69) over $\mathcal{B}$.

We estimate (69) over $\mathcal{A}$ using (35)

$$\begin{aligned}
&\iint_{\mathcal{A}} J^{1/4}(q) \left|\Theta_{\beta_1} \Phi^{ij}(P, Q)\partial^q_{\beta_2} g_k(q)\partial_{\beta_3}\partial_{p_j} f_l(p)\partial_{p_i}\partial_\beta h_l(p)\right| dqdp \\
&\le C \iint J^{1/8}(q) \left|\partial^q_{\beta_2} g_k(q)\partial_{\beta_3}\partial_{p_j} f_l(p)\partial_{p_i}\partial_\beta h_l(p)\right| dqdp \\
&\le C|\partial_\beta h|_\sigma |\partial_{\beta_2} g|_2 |\partial_{\beta_3} f|_\sigma.
\end{aligned}$$

This completes the estimate for (69).

Note that the difference between (70) and (69) is that ∂_{q_j} hits g in (70) whereas ∂_{p_j} hit f in (69). This difference means we will need to use the norm $|\cdot|_\sigma$ to estimate g but we are able to use the smaller norm $|\cdot|_2$ to estimate f. This is in contrast to the estimate for (69) where the opposite situation held. The main point is that $|\cdot|_\sigma$ includes first order p derivatives.

Using the same splitting over $\mathcal{A}$ and $\mathcal{B}$ as well as the same type calculation, (70) is bounded by $|\partial_\beta h|_\sigma |\partial_{\beta_2} g|_\sigma |\partial_{\beta_3} f|_2$. This completes the estimate for (70).

The proof of (69) follows from the Sobolev embedding: $H^2(\mathbb{T}^3) \subset L^\infty(\mathbb{T}^3)$. Without loss of generality, assume $|\gamma_1| \leq N/2$. This Sobolev embedding grants us that

$$\left(\sup_x |\partial_{\beta_2}^{\gamma_1} f(x)|_2\right) |\partial_{\beta_3}^{\gamma-\gamma_1} g(x)|_\sigma + \left(\sup_x |\partial_{\beta_2}^{\gamma_1} f(x)|_\sigma\right) |\partial_{\beta_3}^{\gamma-\gamma_1} g(x)|_2$$
$$\leq \left(\sum \|\partial_{\beta_2}^{\bar{\gamma}} f\|\right) |\partial_{\beta_3}^{\gamma-\gamma_1} g(x)|_\sigma + \left(\sum \|\partial_{\beta_2}^{\bar{\gamma}} f\|_\sigma\right) |\partial_{\beta_3}^{\gamma-\gamma_1} g(x)|_2,$$

where summation is over $|\bar{\gamma}| \leq |\gamma_1| + 2 \leq N$ since $N \geq 4$. We conclude (69) by integrating (68) further over $\mathbb{T}^3$. $\square$

We next prove the important estimates which are needed to prove Theorem 2 in Sect. 5.

Theorem 5. *Let $|\gamma| \leq N$. Let $g(x,p)$ be a smooth vector valued $L^2(\mathbb{T}_x^3 \times \mathbb{R}_p^3; \mathbb{R}^2)$ function and $h(p)$ a smooth vector valued $L^2(\mathbb{R}_p^3; \mathbb{R}^2)$ function, we have*

$$\|\langle \partial^\gamma \Gamma(g,g), h\rangle\| \leq C \sum_{|\beta|\leq 2} |\partial_\beta h|_2 \sum_{|\gamma|\leq N} \|\partial^\gamma g\| \sum_{|\gamma|\leq N} \|\partial^\gamma g\|_\sigma. \tag{71}$$

Moreover,

$$\|\langle L\partial^\gamma g, h\rangle\| \leq C\|\partial^\gamma g\| \sum_{|\beta|\leq 2} |\partial_\beta h|_2. \tag{72}$$

Proof. We begin with the linear term. By Lemma 6, (21) and two integrations by parts $\langle L\partial^\gamma g, h\rangle$ is given by

$$\int \left\{-\partial^\gamma g \cdot \partial_{p_j}\left(\partial_{p_i} h(p) 2\sigma^{ij}\right) + \frac{1}{2}\sigma^{ij}\frac{p_i}{p_0}\frac{p_j}{p_0}\partial^\gamma g \cdot h(p)\right\} dp$$
$$-\int \left\{\partial_{p_i}\left\{\sigma^{ij}\frac{p_i}{p_0}\right\}\partial^\gamma g \cdot h(p)\right\} dp$$
$$-\iint \frac{q_j}{2q_0}\Phi^{ij}(P,Q)\sqrt{J(q)}J(p)\partial^\gamma g_l(q)\partial_{p_i}\left\{J(p)^{-1/2}h_k(p)\right\} dq dp,$$
$$+\iint \partial_{q_j}\left\{\Phi^{ij}(P,Q)\sqrt{J(q)}\right\} J(p)\partial^\gamma g_l(q)\partial_{p_i}\left\{J(p)^{-1/2}h_k(p)\right\} dq dp.$$

We implicitly sum over $i, j \in \{1,2,3\}$ and $k, l \in \{+,-\}$. Using Cauchy's inequality, $\|\langle L\partial^\gamma g, h\rangle\|^2$ is bounded by

$$2\int \left(\int \left\{-\partial^\gamma g \cdot \partial_{p_j}\left(\partial_{p_i} h(p) 2\sigma^{ij}\right) + \frac{1}{2}\sigma^{ij}\frac{p_i}{p_0}\frac{p_j}{p_0}\partial^\gamma g \cdot h(p)\right\} dp\right)^2 dx$$
$$2\int \left(\int \left\{\partial_{p_i}\left\{\sigma^{ij}\frac{p_i}{p_0}\right\}\partial^\gamma g \cdot h(p)\right\} dp\right)^2 dx \tag{73}$$
$$+2\int \left(\iint \frac{q_j}{2q_0}\Phi^{ij}(P,Q)\sqrt{J(q)}J(p)\partial^\gamma g_l(q)\partial_{p_i}\left\{J(p)^{-1/2}h_k(p)\right\} dq dp\right)^2 dx$$

plus

$$2\int\left(\iint \partial_{q_j}\left\{\Phi^{ij}(P,Q)\sqrt{J(q)}\right\}J(p)\partial^\gamma g_l(q)\partial_{p_i}\left\{J(p)^{-1/2}h_k(p)\right\}dqdp.\right)^2 dx. \tag{74}$$

We have split (73) and (74) because we will estimate each one separately.

By the Cauchy-Schwartz inequality as well as (61), and that $h(p)$ is not a function of x, the first and second lines of (73) are bounded by

$$C\sum_{|\beta|\le 2}|\partial_\beta h|_2^2\int|\partial^\gamma g(x)|_2^2dx = C\|\partial^\gamma g\|^2\sum_{|\beta|\le 2}|\partial_\beta h|_2^2.$$

This completes (72) for the first and second lines of (73).

By the Cauchy-Schwartz inequality over dp the third line of (73) is bounded by

$$C\sum_{|\beta|\le 1}|\partial_\beta h|_2^2\int\left(\int J^{1/2}(q)\left|\partial^\gamma g_l(q)\right|\left\{\int\Phi^{ij}(P,Q)^2J^{1/2}(p)dp\right\}^{1/2}dq\right)^2dx.$$

Recall again the splitting in (34), apply (35) and (37) (with $\alpha_1=0$) to obtain

$$\begin{aligned}\int\Phi^{ij}(P,Q)^2J^{1/2}(p)dp &= \int_{\mathcal{A}}+\int_{\mathcal{B}}\\ &\le Cq_0^{12}+q_0^{14}\int_{\mathcal{B}}|p-q|^{-2}J^{1/4}(p)dp\\ &\le Cq_0^{14}.\end{aligned}$$

Using the Cauchy-Schwartz inequality again, this implies that the third line of (73) is bounded by $C\|\partial^\gamma g\|^2\sum_{|\beta|\le 1}|\partial_\beta h|_2^2$.

To establish (72) it remains to estimate (74). From (33), we write

$$\partial_{q_j} = -\frac{p_0}{q_0}\partial_{p_j}+\left(\partial_{q_j}+\frac{p_0}{q_0}\partial_{p_j}\right) = -\frac{p_0}{q_0}\partial_{p_j}+\Theta_{e_j}(q,p),$$

where e_j is an element of the standard basis in $\mathbb{R}^3$. For the rest of this proof, we write $\Theta_{e_j}=\Theta_{e_j}(q,p)$ for notational simplicity (although the reader should note that it is the opposite of the shorthand we were using previously). Further,

$$\begin{aligned}\partial_{q_j}\left\{\Phi^{ij}(P,Q)J^{1/2}(q)\right\} &= -\Phi^{ij}(P,Q)\frac{q_j}{2q_0}\sqrt{J(q)}\\ &\quad -\sqrt{J(q)}\frac{p_0}{q_0}\partial_{p_j}\Phi^{ij}(P,Q)+\sqrt{J(q)}\Theta_{e_j}\Phi^{ij}(P,Q).\end{aligned}$$

We plug the above into (74) and integrate by parts for the middle term in (74) to bound (74) by

$$\begin{aligned}&C\int\left(\iint\sqrt{J(q)}\sqrt{J(p)}\left|\Phi^{ij}(P,Q)\partial^\gamma g_l(q)\right|\left|\partial_\beta h_k(p)\right|dqdp\right)^2dx \qquad (75)\\ &+C\int\left(\iint\sqrt{J(q)}\sqrt{J(p)}\left|\Theta_{e_j}\Phi^{ij}(P,Q)\partial^\gamma g_l(q)\right|\left|\partial_\beta h_k(p)\right|dqdp\right)^2dx.\end{aligned}$$

Above, we add the implicit summation over $|\beta| \le 2$. By the same estimates as for (73), the first term in (75) is $\le C\|\partial^\gamma g\|^2 \sum_{|\beta|\le 2} |\partial_\beta h|_2^2$.

To establish (72), it remains to estimate the second term in (75). To this end, we note that $\Theta_{e_j}(q,p)P\cdot Q = 0$. Further, $\Theta_{e_j}(q,p)\Phi^{ij}(P,Q)$ satisfies the estimates

$$
\begin{aligned}
|\Theta_{e_j}(q,p)\Phi(P,Q)| &\le C q_0^6 q_0^{-1} \ \text{ on } \mathcal{A},\\
|\Theta_{e_j}(q,p)\Phi(P,Q)| &\le C q_0^7 q_0^{-1}|p-q|^{-1} \ \text{ on } \mathcal{B},
\end{aligned}
$$

where we recall the sets from (34). This can be shown directly, by repeating the proof of (35) and (37) using the operator $\Theta_{e_j}(q,p)$ in place of the operator $\Theta_{e_j}(p,q)$. Plugging these estimates into the second term of (75), we can show that it is bounded by $C\|\partial^\gamma g\|^2 \sum_{|\beta|\le 2} |\partial_\beta h|_2^2$ using the same estimates as used for (73). This completes the estimate (72).

We turn to the estimate for the non-linear term (71). This estimate employs the same idea as for the linear term (72), i.e. to move the momentum derivatives around so that we can get an upper bound in terms of at least one $\|\cdot\|$ norm.

The inner product $\langle \partial^\gamma \Gamma(g,g), h\rangle$, using (64) and an integration by parts, is equal to

$$
\begin{aligned}
&-C_\gamma^{\gamma_1} \iint \Phi^{ij}\sqrt{J(q)}\partial_{p_j}\partial^{\gamma_1} g_l(p)\partial^{\gamma-\gamma_1} g_k(q)\left(\partial_{p_i} + \frac{p_i}{2p_0}\right) h_l(p)dqdp,\\
&+C_\gamma^{\gamma_1} \iint \Phi^{ij}\sqrt{J(q)}\partial^{\gamma_1} g_l(p)\partial_{q_j}\partial^{\gamma-\gamma_1} g_k(q)\left(\partial_{p_i} + \frac{p_i}{2p_0}\right) h_l(p)dqdp.
\end{aligned}
$$

Above we implicitly sum over $i,j \in \{1,2,3\}$, $\gamma_1 \le \gamma$ and $l,k \in \{+,-\}$.

Assume, without loss of generality, that $|\gamma_1| \le N/2$ (and $N \ge 4$). We then integrate by parts with respect to ∂_{p_j} for the first term above. After this integration by parts, we obtain a term like $\partial_{p_j}\Phi^{ij}$. For this term we write it as $\partial_{p_j}\Phi^{ij} = -\frac{q_0}{p_0}\partial_{q_j}\Phi^{ij} + \Theta_{e_j}(p,q)\Phi^{ij}$, where we used the notation (33). We then integrate by parts again for this term with respect to ∂_{q_j}. The result is bounded above by

$$
\begin{aligned}
&C\iint J^{1/4}(q)\left|\Theta_{e_j}(p,q)\Phi^{ij}\right|\left|\partial^{\gamma_1} g_l(p)\partial^{\gamma-\gamma_1} g_k(q)\right|\left|\partial_\beta h_l(p)\right| dqdp,\\
&+C\iint \sqrt{J(q)}\left|\Phi^{ij}(P,Q)\right|\left|\partial^{\gamma_1} g_l(p)\partial^q_{\beta_1}\partial^{\gamma-\gamma_1} g_k(q)\right|\left|\partial_\beta h_l(p)\right| dqdp,
\end{aligned}
$$

where we sum over everything from the last display as well as over $|\beta_1| \le 1$ and $|\beta| \le 2$. The main point is that we took ∂_{p_j} off of the function on which we want to have an L^2 estimate (the function with less spatial derivatives). We use the same procedure used to estimate (69) to obtain the upper bound (for both terms above)

$$
C\sum_{|\beta|\le 2} |\partial_\beta h|_2 \sum_{|\gamma_1|\le N/2} |\partial^{\gamma_1} g|_2 |\partial^{\gamma-\gamma_1} g|_\sigma.
$$

Therefore $|\langle \partial^\gamma \Gamma(g,g), h\rangle|^2 \le C\sum_{|\beta|\le 2} |\partial_\beta h|_2^2 \sum_{|\gamma_1|\le N/2} |\partial^{\gamma_1} g|_2^2 |\partial^{\gamma-\gamma_1} g|_\sigma^2$. Further integrating over $\mathbb{T}^3$ we get

$$
\|\langle \partial^\gamma \Gamma(g,g), h\rangle\|^2 \le C\sum_{|\beta|\le 2} |\partial_\beta h|_2^2 \sum_{|\gamma_1|\le N/2} \int |\partial^{\gamma_1} g|_2^2 |\partial^{\gamma-\gamma_1} g|_\sigma^2 dx.
$$

We establish (71) by using the Sobolev embedding: $H^2(\mathbb{T}^3) \subset L^\infty(\mathbb{T}^3)$. $\square$

We end this section with a proof that L is coercive away from its null space $\mathcal{N}$.

Lemma 8. *For any* $m > 1$, *there is* $0 < C(m) < \infty$ *such that*

$$|\langle \partial_{p_i} \left\{ \sigma^{ij} \frac{p_j}{p_0} \right\} g, h\rangle| + |\langle Kg, h\rangle| \\ \leq \frac{C}{m} |g|_\sigma \, |h|_\sigma + C(m) \left\{ \int_{|p|\leq C(m)} |g|^2 dp \right\}^{1/2} \left\{ \int_{|p|\leq C(m)} |h|^2 dp \right\}^{1/2}. \quad (76)$$

Moreover, there is $\delta > 0$, *such that*

$$\langle Lg, g\rangle \geq \delta |\, (\mathbf{I} - \mathbf{P})\, g|_\sigma^2. \quad (77)$$

Proof. We first prove (76). We split

$$\int \partial_{p_i} \left\{ \sigma^{ij} \frac{p_j}{p_0} \right\} \{g_+ h_+ + g_- h_-\} \, dp = \int_{\{|p|\leq m\}} + \int_{\{|p|\geq m\}}. \quad (78)$$

By (61)

$$\left| \partial_{p_i} \left\{ \sigma^{ij} \frac{p_j}{p_0} \right\} \right| \leq C p_0^{-1}.$$

So the first integral in (78) is bounded by the right-hand side of (76). From the Cauchy-Schwartz inequality and (60) we obtain

$$\int_{\{|p|\geq m\}} \leq \frac{C}{m} \int |g||h| dp \leq \frac{C}{m} |g|_\sigma \;\; |h|_\sigma \;. \quad (79)$$

Consider the linear operator K in (63). After an integration by parts we can write

$$\langle Kg, h\rangle = \sum \iint \Phi^{ij} J^{1/4}(p) J^{1/4}(q) \Psi_{\alpha_1 \alpha_2}(p, q) \partial_{\alpha_1} g_k(q) \partial_{\alpha_2} h_l(p) dq dp,$$

where $\Phi^{ij} = \Phi^{ij}(P, Q)$ and the sum is over $i, j \in \{1, 2, 3\}$, $|\alpha_1| \leq 1$, $|\alpha_2| \leq 1$ and $k, l \in \{+, -\}$. Also, $\Psi_{\alpha_1 \alpha_2}(p, q)$ is a collection of smooth functions, in which we collect all the inessential terms, that satisfies

$$|\nabla \Psi_{\alpha_1 \alpha_2}(p, q)| + |\Psi_{\alpha_1 \alpha_2}(p, q)| \leq C J^{1/8}(p) J^{1/8}(q).$$

From (26) as well as Proposition 1,

$$\Phi^{ij}(P, Q) J^{1/4}(p) J^{1/4}(q) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3).$$

Therefore, for any given $m > 0$, we can choose a C_c^∞ function $\psi_{ij}(p, q)$ such that

$$||\Phi^{ij} J^{1/4}(p) J^{1/4}(q) - \psi_{ij}||_{L^2(\mathbb{R}^3_p \times \mathbb{R}^3_q)} \leq \tfrac{1}{m},$$
$$\mathrm{supp}\{\psi_{ij}\} \subset \{|p| + |q| \leq C(m)\}, \quad C(m) < \infty.$$

We split

$$\Phi^{ij} J^{1/4}(p) J^{1/4}(q) = \psi_{ij} + \{\Phi^{ij} J^{1/4}(p) J^{1/4}(q) - \psi_{ij}\}$$

and

$$\langle Kg, h\rangle = J_1(g,h) + J_2(g,h), \tag{80}$$

with

$$J_1 = \sum \iint \psi_{ij}(p,q)\Psi_{\alpha_1\alpha_2}(p,q)\partial_{\alpha_1} g_k(q)\partial_{\alpha_2} h_l(p)dqdp,$$
$$J_2 = \sum \iint \{\Phi^{ij} J^{1/4}(p)J^{1/4}(q) - \psi_{ij}\}\Psi_{\alpha_1\alpha_2}(p,q)\partial_{\alpha_1} g_k(q)\partial_{\alpha_2} h_l(p)dqdp.$$

The second term J_2 is bounded in absolute value by

$$\begin{aligned} &||\Phi^{ij} J^{1/4}(p)J^{1/4}(q) - \psi_{ij}||_{L^2(\mathbb{R}^3_p\times\mathbb{R}^3_q)}||\Psi_{\alpha_1\alpha_2}\partial_{\alpha_1} g_k(q)\partial_{\alpha_2} h_l(p)||_{L^2(\mathbb{R}^3_p\times\mathbb{R}^3_q)} \\ &\quad\le \frac{C}{m}\left|J^{1/8}\partial_{\alpha_1} g_k\right|_2 \left|J^{1/8}\partial_{\alpha_2} h_l\right|_2 \le \frac{C}{m}|g|_\sigma\,|h|_\sigma\,, \end{aligned}$$

where we have used the equivalence of the norms (60). For J_1, an integration by parts over p and q yields

$$\begin{aligned} J_1 &= \sum(-1)^{\alpha_1+\alpha_2}\iint \partial_{\alpha_2}\partial^q_{\alpha_1}\left\{\psi_{ij}(p,q)\Psi_{\alpha_1\alpha_2}(p,q)\right\} g_k(q)h_l(p)dqdp \\ &\le C||\psi_{ij}||_{C^2}\left\{\int_{|p|\le C(m)}|g|^2dp\right\}^{1/2}\left\{\int_{|p|\le C(m)}|h|^2dp\right\}^{1/2}. \end{aligned} \tag{81}$$

This concludes (76).

We use the method of contradiction to prove (77). The converse grants us a sequence of normalized functions $g^n(p) = [g^n_+(p), g^n_-(p)]$ such that $|g^n|_\sigma \equiv 1$ and

$$\int_{\mathbb{R}^3} g^n J^{1/2}dp = \int_{\mathbb{R}^3} p_j g^n J^{1/2}dp = \int_{\mathbb{R}^3} g^n p_0 J^{1/2}dp = 0, \tag{82}$$
$$\langle Lg^n, g^n\rangle = -\langle Ag^n, g^n\rangle - \langle Kg^n, g^n\rangle \le 1/n. \tag{83}$$

We denote the weak limit, with respect to the inner product $\langle\cdot,\cdot\rangle_\sigma$, of g^n (up to a subsequence) by g^0. Lower semi-continuity of the weak limit implies $|g^0|_\sigma \le 1$. From (62), (63) and (21) we have

$$\langle Lg^n, g^n\rangle = |g^n|^2_\sigma - \langle\partial_{p_i}\left\{\sigma^{ij}\frac{p_j}{p_0}\right\}g^n, g^n\rangle - \langle Kg^n, g^n\rangle.$$

We *claim* that

$$\lim_{n\to\infty}\langle\partial_{p_i}\left\{\sigma^{ij}\frac{p_j}{p_0}\right\}g^n, g^n\rangle = \langle\partial_{p_i}\left\{\sigma^{ij}\frac{p_j}{p_0}\right\}g^0, g^0\rangle,\ \lim_{n\to\infty}\langle Kg^n, g^n\rangle \to \langle Kg^0, g^0\rangle.$$

For any given $m > 0$, since $\partial_{p_i} g^n$ are bounded in $L^2\{|p| \le m\}$ from $|g^n|_\sigma = 1$ and (60), the Rellich theorem implies

$$\int_{\{|p|\le m\}}\partial_{p_i}\left\{\sigma^{ij}\frac{p_j}{p_0}\right\}(g^n)^2 \to \int_{\{|p|\le m\}}\partial_{p_i}\left\{\sigma^{ij}\frac{p_j}{p_0}\right\}(g^0)^2.$$

On the other hand, by (79) with $g = h = g^n$, the integral over $\{|p| \geq m\}$ is bounded by $O(1/m)$. By first choosing m sufficiently large and then sending $n \to \infty$, we conclude $\langle \partial_{p_i}\{\sigma^{ij} p_j/p_0\}g^n, g^n\rangle \to \langle \partial_{p_i}\{\sigma^{ij} p_j/p_0\}g^0, g^0\rangle$.

We split $\langle Kg^n, g^n\rangle$ into J_1 and J_2 as in (80), then $J_2(g^n, g^n) \leq \frac{C}{m}$. In the same manner as for (81), we obtain

$$\left|J_1(g^n, g^n) - J_1(g^0, g^0)\right| \leq C(m)\left\{\int_{|p|\leq C(m)} |g^n - g^0|^2 dp\right\}^{1/2}.$$

Then the Rellich theorem implies, up to a subsequence, $J_1(g^n, g^n) \to J_1(g^0, g^0)$. Again by first choosing m large and then letting $n \to \infty$, we conclude that $\langle Kg^n, g^n\rangle \to \langle Kg_0, g_0\rangle$.

Letting $n \to \infty$ in (83), we have shown that

$$0 = 1 - \langle \partial_{p_i}\{\sigma^{ij} p_j/p_0\}g^0, g^0\rangle - \langle Kg^0, g^0\rangle.$$

Equivalently

$$0 = \left(1 - |g^0|_\sigma^2\right) + \langle Lg^0, g^0\rangle.$$

Since both terms are non-negative, $|g^0|_\sigma^2 = 1$ and $\langle Lg^0, g^0\rangle = 0$. By Lemma 1, $g^0 = \mathbf{P}g^0$. On the other hand, letting $n \to \infty$ in (82) we deduce that $g^0 = (\mathbf{I} - \mathbf{P})\, g^0$ or $g^0 \equiv 0$; this contradicts $|g^0|_\sigma^2 = 1$. $\square$

4. Local Solutions

We now construct a unique local-in time solution to the relativistic Landau-Maxwell system with normalized constants (29) and (30), with constraint (31).

Theorem 6. *There exist $M_0 > 0$ and $T^* > 0$ such that if $T^* \leq M_0/2$ and*

$$\mathcal{E}(0) \leq M_0/2,$$

then there exists a unique solution $[f(t,x,p), E(t,x), B(t,x)]$ to the relativistic Landau-Maxwell system (29) and (30) with constraint (31) in $[0, T^*) \times \mathbb{T}^3 \times \mathbb{R}^3$ such that

$$\sup_{0\leq t\leq T^*} \mathcal{E}(t) \leq M_0.$$

The high order energy norm $\mathcal{E}(t)$ is continuous over $[0, T^*)$. If

$$F_0(x,p) = J + J^{1/2} f_0 \geq 0,$$

then $F(t,x,p) = J + J^{1/2} f(t,x,p) \geq 0$. Furthermore, the conservation laws (10), (11), (12) hold for all $0 < t < T^*$ if they are valid initially at $t = 0$.

We consider the following iterating sequence ($n \geq 0$) for solving the relativistic Landau-Maxwell system for the perturbation (29) with normalized constants (Remark 1):

$$\begin{aligned}
&\left\{\partial_t + \frac{p}{p_0}\cdot\nabla_x + \xi\left(E^n + \frac{p}{p_0}\times B^n\right)\cdot\nabla_p - A - \frac{\xi}{2}\left(E^n\cdot\frac{p}{p_0}\right)\right\} f^{n+1} \\
&= \xi_1\left\{E^n\cdot\frac{p}{p_0}\right\}\sqrt{J} + Kf^n + \Gamma(f^{n+1}, f^n) \qquad (84)\\
&\quad + \sqrt{J}\left(f^{n+1} - f^n\right)\partial_{p_i}\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)\partial_{q_i}\left(\sqrt{J(q)}f^n(q)\cdot[1,1]\right)dq \\
&\quad + \sqrt{J}\left(f^{n+1} - f^n\right)\partial_{p_i}\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)\partial_{q_i}J(q)dq, \\
&\partial_t E^n - \nabla_x\times B^n = -\mathcal{J}^n = -\int_{\mathbb{R}^3}\frac{p}{p_0}\sqrt{J}\{f^n_+ - f^n_-\}dp, \\
&\partial_t B^n + \nabla_x\times E^n = 0, \\
&\nabla_x\cdot E^n = \rho^n = \int_{\mathbb{R}^3}\sqrt{J}\{f^n_+ - f^n_-\}dp,\ \nabla_x\cdot B^n = 0.
\end{aligned}$$

Above $\xi_1 = [1, -1]$, and the 2×2 matrix ξ is $\mathrm{diag}(1, -1)$. We start the iteration with

$$f^0(t,x,p) = [f^0_+(t,x,p), f^0_-(t,x,p)] \equiv [f_{0,+}(x,p), f_{0,-}(x,p)].$$

Then solve for $[E^0(t,x), B^0(t,x)]$ through the Maxwell system with initial datum $[E_0(x), B_0(x)]$. We then iteratively solve for

$$f^{n+1}(t,x,p) = [f^{n+1}_+(t,x,p), f^{n+1}_-(t,x,p)],\ E^{n+1}(t,x),\ B^{n+1}(t,x)$$

with initial datum $[f_{0,\pm}(x,p), E_0(x), B_0(x)]$.

It is standard from the linear theory to verify that the sequence $[f^n, E^n, B^n]$ is well-defined for all $n \geq 0$. Our goal is to get an uniform in n estimate for the energy $\mathcal{E}_n(t) \equiv \mathcal{E}(f^n, E^n, B^n)(t)$.

Lemma 9. *There exists* $M_0 > 0$ *and* $T^* > 0$ *such that if* $T^* \leq \frac{M_0}{2}$ *and*

$$\mathcal{E}(0) \leq M_0/2$$

then $\sup_{0\leq t\leq T^*}\mathcal{E}_n(t) \leq M_0$ *implies* $\sup_{0\leq t\leq T^*}\mathcal{E}_{n+1}(t) \leq M_0$.

Proof. Assume $|\gamma| + |\beta| \le N$ and take ∂^γ_β derivatives of (84), we obtain:

$$
\begin{aligned}
&\left\{\partial_t + \frac{p}{p_0}\cdot\nabla_x + \xi\left(E^n + \frac{p}{p_0}\times B^n\right)\cdot\nabla_p\right\}\partial^\gamma_\beta f^{n+1}\\
&\quad -\partial_\beta\{A\partial^\gamma f^{n+1}\} - \xi_1\partial^\gamma E^n\cdot\partial_\beta\left\{\frac{p}{p_0}J^{1/2}\right\}\\
&= -\sum_{\beta_1\neq 0} C^{\beta_1}_\beta\partial_{\beta_1}\left(\frac{p}{p_0}\right)\cdot\nabla_x\partial^\gamma_{\beta-\beta_1} f^{n+1} \qquad (85)\\
&\quad +\sum C^{\beta_1}_\beta\frac{\xi}{2}\left\{\partial^{\gamma_1}E^n\cdot\partial_{\beta_1}\left(\frac{p}{p_0}\right)\right\}\partial^{\gamma-\gamma_1}_{\beta-\beta_1} f^{n+1}\\
&\quad -\xi\sum_{\gamma_1\neq 0} C^{\gamma_1}_\gamma\partial^{\gamma_1}E^n\cdot\nabla_p\partial^{\gamma-\gamma_1}_\beta f^{n+1}\\
&\quad +\xi\sum_{(\gamma_1,\beta_1)\neq(0,0)} C^{\gamma_1}_\gamma C^{\beta_1}_\beta\partial_{\beta_1}\left(\frac{p}{p_0}\right)\times\partial^{\gamma_1}B^n\cdot\nabla_p\partial^{\gamma-\gamma_1}_{\beta-\beta_1} f^{n+1}\\
&\quad +\partial_\beta\{K\partial^\gamma f^n\} + \partial^\gamma_\beta\Gamma(f^{n+1}, f^n)\\
&\quad +\sum C^{\beta_1}_\beta C^{\gamma_1}_\gamma\partial_{\beta-\beta_1}\left\{\sqrt{J}\partial^{\gamma-\gamma_1}\left(f^{n+1}-f^n\right)\right\}\\
&\quad \times\partial_{\beta_1}\partial_{p_i}\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)\partial_{q_i}\left(\sqrt{J(q)}\partial^{\gamma_1}f^n(q)\cdot[1,1]\right)dq\\
&\quad +\sum C^{\beta_1}_\beta\partial_{\beta-\beta_1}\left\{\sqrt{J}\partial^\gamma\left(f^{n+1}-f^n\right)\right\}\partial_{\beta_1}\partial_{p_i}\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)\partial_{q_i}J(q)dq.
\end{aligned}
$$

We take the inner product of (85) with $\partial^\gamma_\beta f^{n+1}$ over $\mathbb{T}^3\times\mathbb{R}^3$ and estimate this inner product term by term.

Using (65), the inner product of the first two terms on the l.h.s of (85) are bounded from below by

$$
\frac{1}{2}\frac{d}{dt}||\partial^\gamma_\beta f^{n+1}(t)||^2 + ||\partial^\gamma_\beta f^{n+1}(t)||^2_\sigma - \eta|||f^{n+1}(t)|||^2_\sigma - C_\eta\|\partial^\gamma f^{n+1}(t)\|^2.
$$

For the third term on the l.h.s. of (85) we separate two cases. If $\beta\neq 0$, its inner product is bounded by

$$
\left|\left(\partial^\gamma E^n\cdot\partial_\beta\{p\sqrt{J}/p_0\}\xi_1, \partial^\gamma_\beta f^{n+1}\right)\right| \le C||\partial^\gamma E^n||\,|||f^{n+1}|||. \qquad (86)
$$

If $\beta = 0$, we have a pure temporal and spatial derivative $\partial^\gamma = \partial_t^{\gamma^0}\partial_{x_1}^{\gamma^1}\partial_{x_2}^{\gamma^2}\partial_{x_3}^{\gamma^3}$. We first split this term as

$$
\begin{aligned}
-\partial^\gamma E^n\cdot\left(\frac{p}{p_0}J^{1/2}\right)\xi_1 &\equiv -\partial^\gamma E^{n+1}\cdot\left(\frac{p}{p_0}J^{1/2}\right)\xi_1\\
&\quad -\{\partial^\gamma E^n - \partial^\gamma E^{n+1}\}\cdot\left(\frac{p}{p_0}J^{1/2}\right)\xi_1. \qquad (87)
\end{aligned}
$$

From the Maxwell system (30) and an integration by parts the inner product of the first part is

$$
\begin{aligned}
&-\left(\partial^\gamma E^{n+1}\cdot\{p\sqrt{J}/p_0\}\xi_1, \partial^\gamma f^{n+1}\right)\\
&= -\iint \partial^\gamma E^{n+1}\cdot\left(\frac{p}{p_0}\sqrt{J}\right)\{\partial^\gamma f_+^{n+1} - \partial^\gamma f_-^{n+1}\}dpdx\\
&= -\int \partial^\gamma E^{n+1}\cdot\partial^\gamma \mathcal{J}^{n+1}dx \qquad (88)\\
&= \frac{1}{2}\frac{d}{dt}\left(||\partial^\gamma E^{n+1}(t)||^2 + ||\partial^\gamma B^{n+1}(t)||^2\right).
\end{aligned}
$$

And the inner product of second part in (87) is bounded by

$$C\{|||E^n||| + |||E^{n+1}|||\}|||f^{n+1}|||.$$

We now turn to the r.h.s. of (85). The first inner product is bounded by ($|\beta_1| \geq 1$) $C|||f^{n+1}|||^2$. The second, third and fourth inner products on the r.h.s. of (85) can be bounded by a collection of terms of the same form

$$
\begin{aligned}
&C\sum\int_{\mathbb{T}^3}\{|\partial^{\gamma_1}E^n| + |\partial^{\gamma_1}B^n|\}\left(\int_{\mathbb{R}^3}|\partial^{\gamma-\gamma_1}_{\beta-\beta_1}f^{n+1}\partial^\gamma_\beta f^{n+1}|dp\right)dx \qquad (89)\\
&+C\sum_{(\gamma_1,\beta_1)\neq(0,0)}\int_{\mathbb{T}^3}\{|\partial^{\gamma_1}E^n| + |\partial^{\gamma_1}B^n|\}\left(\int_{\mathbb{R}^3}|\nabla_p\partial^{\gamma-\gamma_1}_{\beta-\beta_1}f^{n+1}\partial^\gamma_\beta f^{n+1}|dp\right)dx,
\end{aligned}
$$

where the sums are over $\gamma_1 \leq \gamma$, and $\beta_1 \leq \beta$. From the Sobolev embedding $H^2(\mathbb{T}^3) \subset L^\infty(\mathbb{T}^3)$ we have

$$\sup_x\left\{\int_{\mathbb{R}^3}|g(x,q)|^2dq\right\} \leq \int_{\mathbb{R}^3}\sup_x|g(x,q)|^2dq \leq C\sum_{|\gamma|\leq 2}||\partial^\gamma g||^2. \qquad (90)$$

We take the L^∞ norm in x of the one of first two factors in (89) depending on whether $|\gamma_1| \leq N/2$ (take the first term) or $|\gamma_1| > N/2$ (take the second term). Since $N \geq 4$, by (90) and (60) we can majorize (89) by

$$C\{|||E^n||| + |||B^n|||\}|||f^{n+1}|||^2 \leq C\{|||E^n||| + |||B^n|||\}|||f^{n+1}|||^2_\sigma. \qquad (91)$$

We take (66), use Cauchy's inequality with η and integrate over $\mathbb{T}^3$ to obtain

$$\left(\partial_\beta[K\partial^\gamma f^n], \partial^\gamma_\beta f^{n+1}\right) \leq \eta|||f^n|||^2_\sigma + \eta\left\|\partial^\gamma_\beta f^{n+1}\right\|^2_\sigma + C_\eta\left\|\partial^\gamma f^n\right\|^2.$$

For the nonlinear term we use Theorem 4 to obtain

$$
\begin{aligned}
(\partial^\gamma_\beta\Gamma(f^n_, f^{n+1}), \partial^\gamma_\beta f^{n+1}) &\leq C|||f^n(t)||||||f^{n+1}(t)|||_\sigma||\partial^\gamma_\beta f^{n+1}(t)||_\sigma\\
&\quad +C|||f^n(t)|||_\sigma|||f^{n+1}(t)|||||\partial^\gamma_\beta f^{n+1}(t)||_\sigma.
\end{aligned}
$$

We turn our attention to the inner product of the second to last term in (85). We integrate by parts over ∂_{p_i} and apply Theorem 3 to the dq integral differentiated by ∂_{β_1}. Then this term is bounded by

$$\int\left|\Theta_{\bar{\beta}_1}\Phi^{ij}(P,Q)\right|J^{1/4}(q)\left|\partial_{\alpha_2}\partial^{\gamma_1}_{\bar{\beta}_2}f^n_k(q)\right|\left|\partial_{\alpha_2}\left(\partial^{\gamma-\gamma_1}_{\bar{\beta}_3}f^m_l(p)\partial^\gamma_\beta f^{n+1}_l(p)\right)\right|dpdqdx,$$

where we sum over $m \in \{n, n+1\}$, $\bar\beta_1 + \bar\beta_1 + \bar\beta_3 \le \beta$, $i, j \in \{1, 2, 3\}$, $k, l \in \{+, -\}$, $|\alpha_1| \le 1$ $|\alpha_2| \le 1$ and $\gamma_1 \le \gamma$. We remark that a few of these sum's are over estimates used to simplify the presentation. This term is always of the form of one of the four terms in (69)-(70) up to the location of one p derivative. Therefore, as in the proof of Theorem 4, this term is bounded above by

$$C\left(|||f^{n+1}|||_\sigma |||f^n|||_\sigma |||f^n||| + |||f^n|||_\sigma^2 |||f^{n+1}|||\right)$$
$$+C\left(|||f^{n+1}|||_\sigma^2 |||f^n||| + |||f^{n+1}|||_\sigma |||f^n|||_\sigma |||f^{n+1}|||\right),$$

where the sum is over $|\gamma_i| + |\bar\beta_i| \le N$, $\bar\beta_1 + \bar\beta_2 \le \beta$ for the inner product of the last term in (85). The null space in (2) implies

$$\int_{\mathbb{R}^3} \Phi^{ij}(P, Q)\partial_{q_i} J(q)dq = \int_{\mathbb{R}^3} \Phi^{ij}(P, Q)\frac{q_i}{q_0} J(q)dq = \sigma^{ij}\frac{p_i}{p_0}.$$

Therefore (61) applies to the derivatives of the dq integral. Therefore, the inner product of the last term in (85) is bounded by

$$\int \left|\partial^\gamma_{\bar\beta} f^m_k(p)\partial^\gamma_\beta f^{n+1}_k(p)\right| dpdx,$$

where we sum over $m \in \{n, n+1\}$, $|\bar\beta| \le |\beta|$ and $k \in \{+, -\}$. This term is bounded above by

$$C\left(|||f^{n+1}|||^2 + |||f^n|||(t)|||f^{n+1}|||\right) \le C|||f^{n+1}|||^2 + C|||f^n|||^2.$$

By collecting all the above estimates, we obtain the following bound for our iteration:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(||\partial^\gamma_\beta f^{n+1}(t)||^2 + ||\partial^\gamma E^{n+1}(t)||^2 + ||\partial^\gamma B^{n+1}(t)||^2\right) + ||\partial^\gamma_\beta f^{n+1}(t)||^2_\sigma \\
&\le \eta|||f^{n+1}(t)|||^2_\sigma + C_\eta\|\partial^\gamma f^{n+1}(t)\|^2 + C\{|||E^n||| + |||E^{n+1}|||\}|||f^{n+1}||| \\
&\quad +C|||f^{n+1}(t)|||^2 + C\{|||E^n||| + |||B^n|||\}|||f^{n+1}|||^2_\sigma \\
&\quad +\eta|||f^n|||^2_\sigma + \eta\left\|\partial^\gamma_\beta f^{n+1}\right\|^2_\sigma + C_\eta\left\|\partial^\gamma f^n\right\|^2 \\
&\quad +C\left\{|||f^{n+1}|||_\sigma|||f^n||| + |||f^n|||_\sigma|||f^{n+1}|||\right\}|||f^{n+1}|||_\sigma \\
&\quad +C\left(|||f^{n+1}|||_\sigma|||f^n|||_\sigma|||f^n||| + |||f^n|||^2_\sigma|||f^{n+1}|||\right) \\
&\quad +C|||f^{n+1}|||^2 + C|||f^n|||^2.
\end{aligned}$$

Summing over $|\gamma| + |\beta| \le N$ and choosing $\eta \le \frac{1}{4}$ we have

$$\begin{aligned}
\mathcal{E}'_{n+1}(t) &\le C\{\mathcal{E}_{n+1}(t) + \mathcal{E}_n(t) + \mathcal{E}_n^{1/2}(t)|||f^{n+1}|||^2_\sigma(t) \\
&\quad +\mathcal{E}_n^{1/2}(t)|||f^n|||_\sigma(t)|||f^{n+1}|||_\sigma(t) + C\mathcal{E}_{n+1}^{1/2}(t)|||f^n|||^2_\sigma(t) \\
&\quad +\frac{1}{4C}|||f^n|||^2_\sigma + |||f^n|||_\sigma(t)\cdot|||f^{n+1}|||(t)\cdot|||f^{n+1}|||_\sigma(t)\}.
\end{aligned} \tag{92}$$

By the induction assumption, we have

$$\frac{1}{2}|||f^n|||^2(t) + |||E^n|||^2(t) + |||B^n|||^2(t) + \int_0^t |||f^n|||_\sigma^2(s)ds$$
$$= \mathcal{E}_n(t) \le \sup_{0\le s\le t} \mathcal{E}_n(s) \le M_0.$$

Therefore,

$$\int_0^t |||f^n|||_\sigma(s)\cdot|||f^{n+1}|||(s)\cdot|||f^{n+1}|||_\sigma(s)ds$$
$$\le \sup_{0\le s\le t}|||f^{n+1}|||(s)\left\{\int_0^t |||f^n|||_\sigma^2(s)\right\}^{1/2}\left\{\int_0^t |||f^{n+1}|||_\sigma^2(s)\right\}^{1/2}$$
$$\le \sqrt{M_0}\sup_{0\le s\le t}\mathcal{E}_{n+1}(s).$$

Upon further integrating (92) over $[0,t]$ we deduce

$$\begin{aligned}\mathcal{E}_{n+1}(t) &\le \mathcal{E}_{n+1}(0) + C\left(t\sup_{0\le s\le t}\mathcal{E}_{n+1}(s) + M_0 t + \sqrt{M_0}\mathcal{E}_{n+1}(t)\right)\\ &\quad + \frac{M_0}{4} + CM_0\sup_{0\le s\le t}\mathcal{E}_{n+1}^{1/2}(s) + C\sqrt{M_0}\sup_{0\le s\le t}\mathcal{E}_{n+1}(s),\end{aligned}$$

and we will use the inequality

$$M_0\sup_{0\le s\le t}\mathcal{E}_{n+1}(s) \le M_0^{3/2} + \sqrt{M_0}\sup_{0\le s\le t}\mathcal{E}_{n+1}(s).$$

From the initial conditions ($n\ge 0$)

$$\begin{aligned}f_0^{n+1} &\equiv f^{n+1}(0,x,p) = f_0(x,p)\\ E_0^{n+1} &\equiv E^{n+1}(0,x) = E_0(x)\\ B_0^{n+1} &\equiv B^{n+1}(0,x) = B_0(x),\end{aligned}$$

we deduce that

$$\partial_\beta^\gamma f_0^{n+1} = \partial_\beta^\gamma f_0,\ \partial^\gamma E_0^{n+1} = \partial^\gamma E_0,\ \partial^\gamma B_0^{n+1} = \partial^\gamma B_0$$

by a simple induction over the number of temporal derivatives, where the temporal derivatives are defined naturally through (84). Hence

$$\mathcal{E}_{n+1}(0) = \mathcal{E}_{n+1}([f_0^{n+1}, E_0^{n+1}, B_0^{n+1}]) \equiv \mathcal{E}([f_0, E_0, B_0]) \le M_0/2.$$

It follows that for $t\le T^*$,

$$\begin{aligned}(1 - CT^* - CM_0^{1/2})\sup_{0\le t\le T^*}\mathcal{E}_{n+1}(t) &\le \mathcal{E}_{n+1}(0) + CM_0T^* + CM_0^{3/2} + \frac{M_0}{4}\\ &\le \frac{3}{4}M_0 + CM_0\left(T^* + \sqrt{M_0}\right).\end{aligned}$$

We therefore conclude Lemma 9 if $T^*\le \frac{M_0}{2}$ and M_0 is small. □

In order to complete the proof of Theorem 6, we take $n \to \infty$, and obtain a solution f from Lemma 9. Now for uniqueness, we assume that there is another solution $[g, E_g, B_g]$, such that $\sup_{0\le s\le T^*} \mathcal{E}(g(s)) \le M_0$ with $f(0,x,p) = g(0,x,p)$, $E_f(0,x) = E_g(0,x)$ and $B_f(0,x) = B_g(0,x)$. The difference $[f-g, E_f - E_g, B_f - B_g]$ satisfies

$$
\begin{aligned}
&\left\{\partial_t + \frac{p}{p_0}\cdot\nabla_x + \xi\left(E_f + \frac{p}{p_0}\times B_f\right)\cdot\nabla_p - A\right\}(f-g) - (E_f - E_g)\cdot\frac{p}{p_0}\sqrt{J}\xi_1 \\
&= -\xi\left\{E_f - E_g + \frac{p}{p_0}\times(B_f - B_g)\right\}\nabla_p g + K(f-g) \qquad (93)\\
&\quad +\xi\left\{E_f\cdot\frac{p}{p_0}\right\}(f-g) + \xi\left\{(E_f - E_g)\cdot\frac{p}{p_0}\right\}g + \Gamma(f-g,f) + \Gamma(g,f-g);\\
&\partial_t(E_f - E_g) - \nabla_x\times(B_f - B_g) = -\int\frac{p}{p_0}\sqrt{J}\{(f-g)\cdot\xi_1\},\\
&\nabla_x\cdot(E_f - E_g) = \int\sqrt{J}\{(f-g)\cdot\xi_1\},\\
&\partial_t(B_f - B_g) + \nabla_x\times(E_f - E_g) = 0, \quad \nabla_x\cdot(B_f - B_g) = 0.
\end{aligned}
$$

By using the Cauchy-Schwarz inequality in the $p-$integration, and applying (90) for $\sup_x \int|\nabla_p g|^2 dp$, we deduce (for $N \ge 4$)

$$
\begin{aligned}
&\left|\left(u\{E_f - E_g + \frac{p}{p_0}\times(B_f - B_g)\}\cdot\nabla_p g, f-g\right)\right|\\
&\le C\left\{\sum_{|\gamma|\le 2}||\partial^\gamma g||_\sigma\right\}\{||E_f - E_g|| + ||B_f - B_g||\}||f-g||_\sigma\\
&\le C\left\{\sum_{|\gamma|\le 2}||\partial^\gamma g||_\sigma^2\right\}\{||E_f - E_g||^2 + ||B_f - B_g||^2\} + \frac{1}{4}||f-g||_\sigma^2\\
&\le C|||\partial^\gamma g|||^2\{||E_f - E_g||^2 + ||B_f - B_g||^2\} + \frac{1}{4}||f-g||_\sigma^2\\
&\le CM_0\{||E_f - E_g||^2 + ||B_f - B_g||^2\} + \frac{1}{4}||f-g||_\sigma^2.
\end{aligned}
$$

Similarly, we use the Sobolev embedding theorem as well as elementary inequalities to estimate the terms below

$$
\begin{aligned}
\left|\left(E_f\cdot\frac{p}{p_0}(f-g), f-g\right)\right| &\le C\sqrt{M_0}||f-g||^2\\
\left|\left(u\{E_f - E_g\}\cdot\frac{p}{p_0}g, f-g\right)\right| &\le C||f-g||_\sigma||E_f - E_g||\sum_{|\gamma|\le 2}||\partial^\gamma g||_\sigma\\
&\le \frac{1}{4}||f-g||_\sigma^2 + CM_0||E_f - E_g||^2.
\end{aligned}
$$

By Theorem 4 as well as (14),

$$
\begin{aligned}
&|(\Gamma(f-g,f)+\Gamma(g,f-g),f-g)| \\
&\quad \le C\{\|f-g\|\|f\|_\sigma+\|f-g\|_\sigma\|f\| \\
&\qquad +\|f-g\|\|g\|_\sigma+\|f-g\|_\sigma\|g\|\}\|f-g\|_\sigma \\
&\quad = C\{\|f\|+\|g\|\}\|f-g\|_\sigma^2+C\{\|f\|_\sigma+\|g\|_\sigma\}\|f-g\|\|f-g\|_\sigma \\
&\quad \le C\sqrt{M_0}\|f-g\|_\sigma^2+C\left\{\|f\|_\sigma^2+\|g\|_\sigma^2\right\}\|f-g\|^2+\frac{1}{4}\|f-g\|_\sigma^2 \\
&\quad \le C\sqrt{M_0}\|f-g\|_\sigma^2+CM_0\|f-g\|^2+\frac{1}{4}\|f-g\|_\sigma^2.
\end{aligned}
$$

From the Maxwell system in (93), we deduce from (88) that

$$
-\left(2(E_f-E_g)\cdot(p\sqrt{J}/p_0)\xi_1,\,f-g\right)=\frac{d}{dt}\{||E_f-E_g||^2+||B_f-B_g||^2\}.
$$

By taking the inner product of (93) with $f-g$, and collecting the above estimates as well as plugging in the K and A estimates from Lemma 8, we have

$$
\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}||f-g||^2+||E_f-E_g||^2+||B_f-B_g||^2\right\}+||f-g||_\sigma^2 \\
&\quad \le C\{M_0+\sqrt{M_0}+1\}\{||f-g||^2+||E_f-E_g||^2+||B_f-B_g||^2\} \\
&\qquad +\left(\frac{C}{m}+C\sqrt{M_0}+\frac{3}{4}\right)||f-g||_\sigma^2.
\end{aligned}
$$

If we choose m and M_0 so that $\frac{C}{m}+C\sqrt{M_0}<\frac{1}{4}$ then the last term on the r.h.s. can be absorbed by $||f-g||_\sigma^2$ from the right. We deduce $f(t)\equiv g(t)$ from the Gronwall inequality.

To show the continuity of $\mathcal{E}(f(t))$ with respect to t, we have from (92) that as $t\to s$,

$$
|\mathcal{E}(t)-\mathcal{E}(s)|\le CM_0(t-s)+C\left(\sup_{s\le\tau\le t}\mathcal{E}^{1/2}(\tau)+1\right)\int_s^t |||f|||_\sigma^2(\tau)d\tau\to 0.
$$

For the positivity of $F=J+J^{1/2}f$, since f^n solves (84), we see that $F^n=J+J^{1/2}f^n$ solves the iterating sequence ($n\ge 0$):

$$
\left\{\partial_t+\frac{p}{p_0}\cdot\nabla_x+\xi\left(E^n+\frac{p}{p_0}\times B^n\right)\cdot\nabla_p\right\}F^{n+1}=\mathcal{C}^{mod}(F^{n+1},F^n)
$$

together with the coupled Maxwell system:

$$
\begin{aligned}
&\partial_t E^n-\nabla_x\times B^n=-\mathcal{J}^n=-\int_{\mathbb{R}^3}\frac{p}{p_0}\{F_+^n-F_-^n\}dp, \\
&\partial_t B^n+\nabla_x\times E^n=0,\qquad \nabla_x\cdot B^n=0, \\
&\nabla_x\cdot E^n=\rho^n=\int_{\mathbb{R}^3}\{F_+^n-F_-^n\}dp.
\end{aligned}
$$

And, as in (84), the first step in the iteration is given through the initial data

$$
\begin{aligned}
F^0(t,x,p)&=[F_+^0(t,x,p),F_-^0(t,x,p)]=[F_{0,+}(x,p),F_{0,-}(x,p)] \\
&=[J+J^{1/2}f_{0,+}(x,p),J+J^{1/2}f_{0,-}(x,p)].
\end{aligned}
$$

Above we have used the modification $\mathcal{C}^{mod} = [\mathcal{C}^{mod}_+, \mathcal{C}^{mod}_-]$, where

$$\begin{aligned}\mathcal{C}^{mod}_\pm(F^{n+1}, F^n) &= \partial_{p_i}\partial_{p_j} F^{n+1}_\pm(p)\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)\left(F^n_+ + F^n_-\right)dq\\ &\quad+\partial_{p_j}F^{n+1}_\pm(p)\int_{\mathbb{R}^3}\partial_{p_i}\Phi^{ij}(P,Q)\left(F^n_+ + F^n_-\right)dq\\ &\quad-\partial_{p_i}F^{n+1}_\pm(p)\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)\partial_{q_j}\left(F^n_+ + F^n_-\right)dq\\ &\quad-F^n_\pm(p)\partial_{p_i}\int_{\mathbb{R}^3}\Phi^{ij}(P,Q)\partial_{q_j}\left(F^n_+ + F^n_-\right)dq.\end{aligned}$$

Since $F^0(t,x,p)\geq 0$ Lemma 4, the elliptic structure of this iteration and the maximum principle imply that $F^{n+1}(t,x,p)\geq 0$ if $F^n(t,x,p)\geq 0$. This implies $F(t,x,p)\geq 0$.

Finally, since $\mathcal{E}(t) < +\infty$, $[f, \partial^2 E, \partial^2 B]$ is bounded and continuous. By $F = J + J^{1/2}f$, it is straightforward to verify that classical mass, total mometum and total energy conservations hold for such solutions constructed. We thus conclude Theorem 6. □

5. Positivity of L

We establish the positivity of the linear operator L for any small amplitude solution $[f(t,x,p), E(t,x), B(t,x)]$ to the full relativistic Landau-Maxwell system (29) and (30). Recall the orthogonal projection $\mathbf{P}f$ with coefficients $a_\pm, b$ and c in (19). For solutions to the nonlinear system, Lemmas 11 and 12 are devoted to basic estimates for the linear and nonlinear parts in the macroscopic equations. We make the crucial observation in Lemma 13 that the electromagnetic field roughly speaking is bounded by $||f||_\sigma(t)$ at any moment t. Then based on Lemma 10, we finally establish Theorem 2 by a careful study of macroscopic equations coupled with the Maxwell system.

We begin with a formal definition of the orthogonal projection $\mathbf{P}$. Define

$$\begin{aligned}\rho_0 &= \int_{\mathbb{R}^3} J(p)dp, \quad \rho_i = \int_{\mathbb{R}^3} p_i^2 J(p)dp \ \ (i=1,2,3),\\ \rho_4 &= \int_{\mathbb{R}^3}|p|^2 J(p)dp, \quad \rho_5 = \int_{\mathbb{R}^3} p_0 J(p)dp.\end{aligned}$$

We can write an orthonormal basis for $\mathcal{N}$ in (15) with normalized constants as

$$\begin{aligned}\epsilon_1^* &= \rho_0^{-1/2}[J^{1/2},0], \ \ \epsilon_2^* = \rho_0^{-1/2}[0, J^{1/2}],\\ \epsilon_{i+2}^* &= (2\rho_i)^{-1/2}[p_i J^{1/2}, p_i J^{1/2}] \ \ (i=1,2,3),\\ \epsilon_6^* &= c_6\left([p_0,p_0] - \frac{\rho_5}{\rho_0}[1,1]\right)J^{1/2},\end{aligned}$$

where $c_6^{-2} = 2(\rho_0+\rho_4) - 2\frac{\rho_5^2}{\rho_0}$. Now consider $\mathbf{P}f$, $f = [f_+, f_-]$, we define the coefficients in (19) so that $\mathbf{P}$ is an orthogonal projection:

$$\begin{aligned}a_+ &\equiv \rho_0^{-1/2}\langle f, \epsilon_1^*\rangle - \frac{\rho_5}{\rho_0}c, \ \ a_- \equiv \rho_0^{-1/2}\langle f, \epsilon_2^*\rangle - \frac{\rho_5}{\rho_0}c,\\ b_j &\equiv (2\rho_j)^{-1/2}\langle f, \epsilon_{j+2}^*\rangle, \quad c \equiv c_6\langle f, \epsilon_6^*\rangle.\end{aligned} \tag{94}$$

Proposition 2. *Let* $\partial^\gamma = \partial_t^{\gamma_0}\partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}\partial_{x_3}^{\gamma_3}$. *There exists* $C > 1$ *such that*

$$\frac{1}{C}||\partial^\gamma \mathbf{P}f||_\sigma^2 \le ||\partial^\gamma a_\pm||^2 + ||\partial^\gamma b||^2 + ||\partial^\gamma c||^2 \le C||\partial^\gamma \mathbf{P}f||^2.$$

For the rest of the section, we concentrate on a solution $[f, E, B]$ to the nonlinear relativistic Landau-Maxwell system.

Lemma 10. *Let* $[f(t,x,p), E(t,x), B(t,x)]$ *be the solution constructed in Theorem 6 to (29) and (30), which satisfies (31), (10), (11) and (12). Then we have*

$$\frac{2}{3}\rho_4\int_{\mathbb{T}^3} b(t,x) = \int_{\mathbb{T}^3} B(t,x)\times E(t,x), \tag{95}$$

$$\left|\int_{\mathbb{T}^3} a_+(t,x)\right| + \left|\int_{\mathbb{T}^3} a_-(t,x)\right| + \left|\int_{\mathbb{T}^3} c(t,x)\right| \le C\left(||E||^2 + ||B-\bar{B}||^2\right), \tag{96}$$

where $a = [a_+, a_-]$, $b = [b_1, b_2, b_3]$, c *are defined in (94).*

Proof. We use the conservation of mass, momentum and energy. For fixed (t,x), notice that (94) implies

$$\int p\{f_+ + f_-\}\sqrt{J}dp = \frac{2}{3}b(t,x)\int |p|^2 J dp.$$

Hence (95) follows from momentum conservation (11) with normalized constants.

On the other hand, for fixed (t,x), (94) implies

$$\int f_\pm\sqrt{J}dp = \rho_0 a_\pm(t,x) + \rho_5 c(t,x),$$

$$\int p_0\{f_+ + f_-\}\sqrt{J}dp = \rho_5\{a_+(t,x) + a_-(t,x)\} + 2(\rho_0+\rho_4)c(t,x).$$

Upon further integration over $\mathbb{T}^3$, we deduce from the mass conservation (10) that $\int_{\mathbb{T}^3} a_+ = \int_{\mathbb{T}^3} a_- = -\frac{\rho_5}{\rho_0}\int_{\mathbb{T}^3} c$. From the reduced energy conservation (12),

$$-\int_{\mathbb{T}^3}\{|E(t)|^2 + |B(t)-\bar{B}|^2\} = 2\left(\rho_0+\rho_4-\frac{\rho_5^2}{\rho_0}\right)\int_{\mathbb{T}^3} c.$$

By the sharp form of Holder's inequality, $(\rho_0+\rho_4)\rho_0 - \rho_5^2 > 0$. □

We now derive the macroscopic equations for $\mathbf{P}f$'s coefficients $a_\pm$, b and c. Recalling Eq. (16) with (17) and (18) with normalized constants in (104) and (105), we further use (19) to expand entries of l.h.s. of (16) as

$$\left\{\partial^0 a_\pm + \frac{p_j}{p_0}\left\{\partial^j a_\pm \mp E_j\right\} + \frac{p_j p_i}{p_0}\partial^i b_j + p_j\left\{\partial^0 b_j + \partial^j c\right\} + p_0\partial^0 c\right\} J^{1/2}(p),$$

where $\partial^0 = \partial_t$ and $\partial^j = \partial_{x_j}$. For fixed (t,x), this is an expansion of the l.h.s. of (16) with respect to the basis of $(1 \le i, j \le 3)$

$$\begin{gathered}[\sqrt{J},0], [0,\sqrt{J}],\ [p_j\sqrt{J}/p_0, 0],\ [0, p_j\sqrt{J}/p_0],\\ [p_j\sqrt{J}, p_j\sqrt{J}],\ [p_jp_i\sqrt{J}/p_0, p_jp_i\sqrt{J}/p_0],\ [p_0\sqrt{J}, p_0\sqrt{J}].\end{gathered} \tag{97}$$

Expanding the r.h.s. of (16) with respect to the same basis (97) and comparing coefficients on both sides, we obtain the important macroscopic equations for $a(t,x) = [a_+(t,x), a_-(t,x)]$, $b_i(t,x)$ and $c(t,x)$:

$$\partial^0 c = l_c + h_c, \tag{98}$$
$$\partial^i c + \partial^0 b_i = l_i + h_i, \tag{99}$$
$$(1-\delta_{ij})\partial^i b_j + \partial^j b_i = l_{ij} + h_{ij}, \tag{100}$$
$$\partial^i a_\pm \mp E_i = l_{ai}^\pm + h_{ai}^\pm, \tag{101}$$
$$\partial^0 a_\pm = l_a^\pm + h_a^\pm. \tag{102}$$

Here $l_c(t,x)$, $l_i(t,x)$, $l_{ij}(t,x)$, $l_{ai}^\pm(t,x)$ and $l_a^\pm(t,x)$ are the corresponding coefficients of such an expansion of the linear term $l(\{\mathbf{I}-\mathbf{P}\}f)$, and $h_c(t,x)$, $h_i(t,x)$, $h_{ij}(t,x)$, $h_{ai}^\pm(t,x)$ and $h_a^\pm(t,x)$ are the corresponding coefficients of the same expansion of the higher order term $h(f)$.

From (19) and (94) we see that

$$\int [p\sqrt{J}/p_0, -p\sqrt{J}/p_0]\cdot \mathbf{P}f\,dp = 0,$$
$$\int [\sqrt{J}, -\sqrt{J}]\cdot f\,dp = \rho_0\{a_+ - a_-\}.$$

We plug this into the coupled maxwell system, (30) and (31), to obtain

$$\partial_t E - \nabla_x \times B = -\mathcal{J} = \int_{\mathbb{R}^3} [p\sqrt{J}/p_0, -p\sqrt{J}/p_0]\cdot \{\mathbf{I}-\mathbf{P}\}f\,dp, \tag{103}$$
$$\partial_t B + \nabla_x \times E = 0,\ \nabla_x\cdot E = \rho_0\{a_+ - a_-\},\ \nabla_x \cdot B = 0.$$

We rewrite the terms (17) and (18) in (16) with normalized constants as

$$l(\{\mathbf{I}-\mathbf{P}\}f) \equiv -\left\{\partial_t + \frac{p}{p_0}\cdot\nabla_x + L\right\}\{\mathbf{I}-\mathbf{P}\}f, \tag{104}$$
$$h(f) \equiv -\xi\left(E + \frac{p}{p_0}\times B\right)\cdot\nabla_p f + \frac{\xi}{2}\left\{E\cdot\frac{p}{p_0}\right\}f + \Gamma(f,f). \tag{105}$$

Next, we estimate these terms.

Lemma 11. *For any* $1 \le i, j \le 3$,

$$\sum_{|\gamma|\le N-1} ||\partial^\gamma l_c|| + ||\partial^\gamma l_i|| + ||\partial^\gamma l_{ij}|| + ||\partial^\gamma l_{ai}^\pm|| + ||\partial^\gamma l_a^\pm|| + ||\partial^\gamma \mathcal{J}||$$
$$\le C\sum_{|\gamma|\le N} \|\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f\|.$$

Proof. Let $\{\epsilon_n(p)\}$ represent the basis in (97). For fixed (t,x), we can use the Gram-Schmidt procedure to argue that the terms $l_c(t,x)$, $l_i(t,x)$, $l_{ij}(t,x)$, $l_{ai}^\pm(t,x)$ and $l_a^\pm(t,x)$ are of the form

$$\sum_{n=1}^{18} \bar{c}_n\langle l(\{\mathbf{I}-\mathbf{P}\}f), \epsilon_n\rangle,$$

where c_n are constants which do not depend on f. Let $|\gamma| \leq N-1$. By (104)

$$\int \partial^\gamma l(\{\mathbf{I}-\mathbf{P}\}f)\cdot\epsilon_n(p)dp = -\int\left\{\partial_t + \frac{p}{p_0}\cdot\nabla_x + L\right\}\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f(p)\cdot\epsilon_n(p)dp.$$

We estimate the first two terms,

$$\begin{aligned}
&\|\int\{\partial_t + \frac{p}{p_0}\cdot\nabla_x\}(\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f)\cdot\epsilon_n dp\|^2\\
&\leq 2\int|\epsilon_n|dp \times \int_{\mathbb{T}^3\times\mathbb{R}^3}|\epsilon_n(p)|(|\{\mathbf{I}-\mathbf{P}\}\partial^0\partial^\gamma f|^2 + |\{\mathbf{I}-\mathbf{P}\}\nabla_x\partial^\gamma f|^2)dpdx\\
&\leq C\left(||\{\mathbf{I}-\mathbf{P}\}\partial^0\partial^\gamma f||^2 + ||\{\mathbf{I}-\mathbf{P}\}\nabla_x\partial^\gamma f||^2\right).
\end{aligned}$$

Similarly, we have

$$||\partial^\gamma\mathcal{J}|| = ||\int_{\mathbb{R}^3}[-p\sqrt{J}/p_0, p\sqrt{J}/p_0]\cdot\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f dp|| \leq C||\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f||.$$

Using (72) we can estimate the last term

$$\|\langle L\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f, \epsilon_n\rangle\| \leq C\|\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f\|.$$

Indeed (72) was designed to estimate this term.

We now estimate coefficients of the higher order term $h(f)$.

Lemma 12. *Let (14) be valid for some* $M_0 > 0$. *Then*

$$\begin{aligned}
&\sum_{|\gamma|\leq N}\{||\partial^\gamma h_c|| + ||\partial^\gamma h_i|| + ||\partial^\gamma h_{ij}|| + ||\partial^\gamma h^\pm_{ai}|| + ||\partial^\gamma h^\pm_a||\}\\
&\qquad\leq C\sqrt{M_0}\sum_{|\gamma|\leq N}||\partial^\gamma f||_\sigma.
\end{aligned}$$

Proof. Let $|\gamma| \leq N$, recall that $\{\epsilon_n(p)\}$ represents the basis in (97). Notice that $\partial^\gamma h_c$, $\partial^\gamma h_i$, $\partial^\gamma h_{ij}$, $\partial^\gamma h^\pm_{ai}$ and $\partial^\gamma h^\pm_a$ are again of the form

$$\sum_{n=1}^{18}\tilde{c}_n\langle\partial^\gamma h(f), \epsilon_n\rangle.$$

It again suffices to estimate $\langle\partial^\gamma h(f), \epsilon_n\rangle$. For the first term of $h(f)$ in (105), we use an integration by parts over the p variables to get

$$\begin{aligned}
&-\int\partial^\gamma\{\xi(E + \frac{p}{p_0}\times B)\cdot\nabla_p f)\}\cdot\epsilon_n(p)dp\\
&= -\sum C^{\gamma_1}_\gamma\int\nabla_p\cdot\{\xi(\partial^{\gamma_1}E + \frac{p}{p_0}\times\partial^{\gamma_1}B)\partial^{\gamma-\gamma_1}f\}\cdot\epsilon_n(p)dp\\
&= \sum C^{\gamma_1}_\gamma\int\xi(\partial^{\gamma_1}E + \frac{p}{p_0}\times\partial^{\gamma_1}B)\partial^{\gamma-\gamma_1}f\cdot\nabla_p\epsilon_n(p)dp\\
&\leq C\sum\{|\partial^{\gamma_1}E| + |\partial^{\gamma_1}B|\}\left\{\int|\partial^{\gamma-\gamma_1}f|^2dp\right\}^{1/2}.
\end{aligned}$$

The last estimate holds because $\nabla_p \epsilon_n(p)$ has exponential decay. Take the square of the above, whose further integration over $\mathbb{T}^3$ is bounded by

$$C\int_{\mathbb{T}^3}\{|\partial^{\gamma_1}E|+|\partial^{\gamma_1}B|\}^2\left\{\int|\partial^{\gamma-\gamma_1}f|^2dp\right\}dx. \tag{106}$$

If $|\gamma_1|\leq N/2$, by $H^2(\mathbb{T}^3)\subset L^\infty(\mathbb{T}^3)$ and the small amplitude assumption (14), we have

$$\sup_x\{|\partial^{\gamma_1}E|+|\partial^{\gamma_1}B|\}\leq C\sum_{|\gamma|\leq N}\{||\partial^\gamma E(t)||+||\partial^\gamma B(t)||\}\leq C\sqrt{M_0}.$$

If $|\gamma_1|\geq N/2$ then $\int_{\mathbb{T}^3}\{|\partial^{\gamma_1}E|+|\partial^{\gamma_1}B|\}^2dx\leq M_0$ and, by (90),

$$\sup_x\left\{\int|\partial^{\gamma-\gamma_1}f|^2dp\right\}\leq C\sum_{|\gamma|\leq N}||\partial^\gamma f(t)||^2.$$

We thus conclude that (106) is bounded by $C\sqrt{M_0}\sum_{|\gamma|\leq N}||\partial^\gamma f||$.

The second term of $h(f)$ in (105) is easily treated by the same argument, for

$$\begin{aligned}&\int\frac{\xi}{2}\partial^\gamma\{(E\cdot\frac{p}{p_0})f\}\cdot\epsilon_n(p)dp\\&=\sum C_\gamma^{\gamma_1}\int\{\frac{\xi}{2}(\partial^{\gamma_1}E\cdot\frac{p}{p_0})\partial^{\gamma-\gamma_1}f\}\cdot\epsilon_n(p)dp\\&\leq C\sum|\partial^{\gamma_1}E|\left\{\int|\partial^{\gamma-\gamma_1}f|^2dp\right\}^{1/2}.\end{aligned}$$

For the third term of $h(f)$ in (105) we apply (71):

$$\|\langle\partial^\gamma\Gamma(f,f),\epsilon_n\rangle\|\leq C\sum_{|\gamma|\leq N}||\partial^\gamma f(t)||\sum_{|\gamma|\leq N}||\partial^\gamma f(t)||_\sigma\leq C\sqrt{M_0}\sum_{|\gamma|\leq N}||\partial^\gamma f||_\sigma.$$

We designed (71) to estimate this term. □

Next we estimate the electromagnetic field $[E(t,x),B(t,x)]$ in terms of $f(t,x,p)$ through the macroscopic equation (101) and the Maxwell system (103).

Lemma 13. *Let $[f(t,x,p),E(t,x),B(t,x)]$ be the solution to (29), (30) and (31) constructed in Theorem 6. Let the small amplitude assumption (14) be valid for some $M_0>0$. Then there is a constant $C>0$ such that*

$$\sum_{|\gamma|\leq N-1}\{||\partial^\gamma E(t)||+||\partial^\gamma\{B(t)-\bar{B}\}||\}\leq C\sum_{|\gamma|\leq N}\left(||\partial^\gamma f(t)||+\sqrt{M_0}||\partial^\gamma f(t)||_\sigma\right).$$

Proof. We first use the plus part of the macroscopic equation (101) to estimate the electric field $E(t,x)$:

$$-\partial^\gamma E_i=\partial^\gamma l_{ai}^+ +\partial^\gamma h_{ai}^+ -\partial^\gamma\partial^i a_+.$$

Proposition 2 says $||\partial^\gamma \partial^i a_+|| \leq C||\mathbf{P}\partial^\gamma \partial^i f||$. Applying Lemmas 11 and 12 to $\partial^\gamma l_{ai}^+$ and $\partial^\gamma h_{ai}^+$ respectively, we deduce that for $|\gamma| \leq N-1$,

$$||\partial^\gamma E|| \leq C \sum_{|\gamma'|\leq N} \left(||\partial^{\gamma'} f(t)|| + \sqrt{M_0}||\partial^{\gamma'} f(t)||_\sigma \right). \tag{107}$$

We next estimate the magnetic field $B(t,x)$. Let $|\gamma| \leq N-2$. Taking ∂^γ to the Maxwell system (103) we obtain

$$\nabla_x \times \partial^\gamma B = \partial^\gamma \mathcal{J} + \partial_t \partial^\gamma E, \ \nabla_x \cdot \partial^\gamma B = 0.$$

Lemma 11, (107) as well as $\int |\nabla \times \partial^\gamma B|^2 + (\nabla \cdot \partial^\gamma B)^2 dx = \int \sum_{i,j} (\partial_{x_i} \partial^\gamma B_j)^2 dx$ imply

$$||\nabla \partial^\gamma B|| \leq C\{||\partial^\gamma \mathcal{J}|| + ||\partial_t \partial^\gamma E||\} \leq C \sum_{|\gamma'|\leq N} \left(||\partial^{\gamma'} f(t)|| + \sqrt{M_0}||\partial^{\gamma'} f(t)||_\sigma \right).$$

By $\partial_t \partial^\gamma B + \nabla \times \partial^\gamma E = 0$, $||\partial_t \partial^\gamma B|| \leq ||\nabla \times \partial^\gamma E||$. Finally, by the Poincaré inequality $||B - \bar{B}|| \leq C||\nabla B||$, we therefore conclude our lemma. □

We now prove the crucial positivity of L for a small solution $[f(t,x,p), E(t,x), B(t,x)]$ to the relativistic Landau-Maxwell system. The conservation laws (10), (11) and (12) play an important role.

Proof of Theorem 2. From (77) we have

$$\left(L\partial^\gamma f, \partial^\gamma f\right) \geq \delta||\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f||_\sigma^2.$$

By Proposition 2, we need only establish (20). The rest of the proof is devoted to establishing

$$\begin{aligned}
&\sum_{|\gamma|\leq N} \{||\partial^\gamma a_\pm|| + ||\partial^\gamma b|| + ||\partial^\gamma c||\} \\
&\qquad \leq C \sum_{|\gamma|\leq N} ||\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f(t)|| + C\sqrt{M_0} \sum_{|\gamma|\leq N} ||\partial^\gamma f(t)||_\sigma.
\end{aligned} \tag{108}$$

This is sufficient to prove the upper bound in (20) because the second term on the r.h.s. can be neglected for M_0 small:

$$\begin{aligned}
\sum_{|\gamma|\leq N} ||\partial^\gamma f(t)||_\sigma &\leq \sum_{|\gamma|\leq N} ||\mathbf{P}\partial^\gamma f(t)||_\sigma + \sum_{|\gamma|\leq N} ||\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f(t)||_\sigma \\
&\leq C \sum_{|\gamma|\leq N} \left(||\partial^\gamma a_\pm|| + ||\partial^\gamma b|| + ||\partial^\gamma c|| \right) \\
&\quad + \sum_{|\gamma|\leq N} ||\{\mathbf{I}-\mathbf{P}\}\partial^\gamma f(t)||_\sigma.
\end{aligned}$$

We will estimate each of the terms $a_\pm$, b and c in (108) one at a time. □

We first estimate $\nabla\partial^\gamma b$. Let $|\gamma| \le N-1$. From (100)

$$\begin{aligned}\Delta\partial^\gamma b_j + \partial^j(\nabla\cdot\partial^\gamma b) &= \sum_i \partial^i\left(\partial^\gamma\partial^i b_j + \partial^\gamma\partial^j b_i\right)\\ &= \sum_i \partial^i\partial^\gamma\left(l_{ij}+h_{ij}\right)(1+\delta_{ij}).\end{aligned}$$

Multiplying with $\partial^\gamma b_j$ and summing over j yields:

$$\begin{aligned}&\int_{\mathbb{T}^3}\left\{\left(\nabla\cdot\partial^\gamma b\right)^2 + \sum_{i,j}\left(\partial^i\partial^\gamma b_j\right)^2\right\}dx\\ &\quad= \sum_{i,j}\int_{\mathbb{T}^3}\left(\partial^\gamma l_{ij}+\partial^\gamma h_{ij}\right)(1+\delta_{ij})\partial^i\partial^\gamma b_j dx.\end{aligned}$$

Therefore

$$\sum_{i,j}||\partial^i\partial^\gamma b_j||^2 \le C\left(\sum_{i,j}||\partial^i\partial^\gamma b_j||\right)\sum\{||\partial^\gamma l_{ij}|| + ||\partial^\gamma h_{ij}||\},$$

which implies, using $\left(\sum_{i,j}||\partial^i\partial^\gamma b_j||\right)^2 \le C\sum_{i,j}||\partial^i\partial^\gamma b_j||^2$, that

$$\sum_{i,j}||\partial^i\partial^\gamma b_j|| \le C\sum\{||\partial^\gamma l_{ij}|| + ||\partial^\gamma h_{ij}||\}. \tag{109}$$

This is bounded by the r.h.s. of (108) by Lemmas 11 and 12. We estimate purely temporal derivatives of $b_i(t,x)$ with $\gamma = [\gamma^0, 0, 0, 0]$ and $0 < \gamma^0 \le N-1$. From (98) and (99), we have

$$\begin{aligned}\partial^0\partial^\gamma b_i &= \partial^\gamma l_i + \partial^\gamma h_i - \partial^i\partial^\gamma c\\ &= \partial^\gamma l_i + \partial^\gamma h_i - \partial^{\gamma'}\partial^0 c\\ &= \partial^\gamma l_i + \partial^\gamma h_i - \partial^{\gamma'} l_c - \partial^{\gamma'} h_c,\end{aligned}$$

where $|\gamma'| = \gamma^0$. Therefore,

$$\|\partial^0\partial^\gamma b_i\| \le C\left(\|\partial^\gamma l_i\| + \|\partial^\gamma h_i\| + \|\partial^{\gamma'} l_c\| + \|\partial^{\gamma'} h_c\|\right).$$

By Lemmas 11 and 12, this is bounded by the r.h.s. of (108). Next, assume $0 \le \gamma^0 \le 1$. We use the Poincaré inequality and (95) to obtain

$$\begin{aligned}||\partial_t^{\gamma^0} b_i|| &\le C\left\{||\nabla\partial_t^{\gamma^0} b_i|| + \left|\partial_t^{\gamma^0}\int b_i(t,x)dx\right|\right\}\\ &= C\left\{||\nabla\partial_t^{\gamma^0} b_i|| + \left|\partial_t^{\gamma^0}\int E\times B dx\right|\right\}.\end{aligned}$$

By (109), it suffices to estimate the last term above. From Lemma 13 and the assumption (14), with $M_0 \le 1$, the last term is bounded by

$$\begin{aligned}
&||\partial_t^{\gamma^0} B|| \cdot ||E|| + ||B|| \cdot ||\partial_t^{\gamma^0} E|| \\
&\le \sqrt{M_0} C \sum_{|\gamma| \le N} \left(||\partial^\gamma f(t)|| + \sqrt{M_0} ||\partial^\gamma f(t)||_\sigma \right) \\
&\le C\sqrt{M_0} \sum_{|\gamma| \le N} ||\partial^\gamma f(t)||_\sigma .
\end{aligned}$$

We thus conclude the case for b.

Now for $c(t, x)$, from (98) and (99),

$$\begin{aligned}
||\partial^0 \partial^\gamma c|| &\le C\{||\partial^\gamma l_c|| + ||\partial^\gamma h_c||\}, \\
||\nabla \partial^\gamma c|| &\le C||\partial^0 \partial^\gamma b_i|| + ||\partial^\gamma l_i|| + ||\partial^\gamma h_i||.
\end{aligned}$$

Thus, for $|\gamma| \le N - 1$, both $||\partial^0 \partial^\gamma c||$ and $||\nabla \partial^\gamma c||$ are bounded by the r.h.s. of (108) by the above argument for b and Lemmas 11 and 12. Next, to estimate $c(t, x)$ itself, from the Poincaré inequality and Lemma 10,

$$\begin{aligned}
||c|| &\le C \left\{ ||\nabla c|| + \left| \int c dx \right| \right\} \\
&\le C\{||\nabla c|| + ||E||^2 + ||B - \bar{B}||^2\}.
\end{aligned}$$

Notice that from (3) and Jensen's inequality $|\bar{B}| \le \|B\|$. Using this, Lemma 13 and (14), with $M_0 \le 1$, imply

$$||E||^2 + ||B - \bar{B}||^2 \le ||E||^2 + C||B - \bar{B}||(||B|| + ||\bar{B}||) \le C\sqrt{M_0} \sum_{|\gamma| \le N} ||\partial^\gamma f(t)||_\sigma .$$

We thus complete the estimate for $c(t, x)$ in (108).

Now we consider $a(t, x) = [a_+(t, x), a_-(t, x)]$. By (102),

$$||\partial_t \partial^\gamma a_\pm|| \le C \left\{ ||\partial^\gamma l_a^\pm|| + ||\partial^\gamma h_a^\pm|| \right\} .$$

We now use Lemma 11 and 12, for $|\gamma| \le N - 1$, to say that $||\partial_t \partial^\gamma a||$ is bounded by the r.h.s. of (108). We now turn to purely spatial derivatives of $a(t, x)$. Let $|\gamma| \le N - 1$ and $\gamma = [0, \gamma_1, \gamma_2, \gamma_3] \ne 0$. By taking ∂^i of (101) and summing over i we get

$$-\Delta \partial^\gamma a_\pm \pm \nabla \cdot \partial^\gamma E = -\sum_i \partial^i \partial^\gamma \{l_{ai}^\pm + h_{ai}^\pm\}. \tag{110}$$

But from the Maxwell system in (103),

$$\nabla \cdot \partial^\gamma E = \rho_0 (\partial^\gamma a_+ - \partial^\gamma a_-).$$

Multiply (110) with $\partial^\gamma a_\pm$ so that the $\pm$ terms are the same and integrate over $\mathbb{T}^3$. By adding the $\pm$ terms together we have

$$\begin{aligned}
&||\nabla \partial^\gamma a_+||^2 + ||\nabla \partial^\gamma a_-||^2 + \rho_0 ||\partial^\gamma a_+ - \partial^\gamma a_-||^2 \\
&\le C\{||\nabla \partial^\gamma a_+|| + ||\nabla \partial^\gamma a_-||\} \sum_\pm ||\partial^\gamma \{l_{bi}^\pm + h_{bi}^\pm\}||.
\end{aligned}$$

Therefore, $||\nabla\partial^\gamma a_+|| + ||\nabla\partial^\gamma a_-|| \le \sum_\pm ||\partial^\gamma\{l_{bi}^\pm + h_{bi}^\pm\}||$. Since γ is purely spatial, this is bounded by the r.h.s of (108) because of Lemmas 11 and 12. Furthermore, by the Poincaré inequality and Lemma 10, a itself is bounded by

$$\begin{aligned}||a_\pm|| &\le C||\nabla a_\pm|| + C\left|\int a_\pm dx\right| \\ &\le C||\nabla a_\pm|| + C\{||E||^2 + ||B - \bar{B}||^2\},\end{aligned}$$

which is bounded by the r.h.s. of (108) by the same argument as for c. We thus complete the estimate for $a(t, x)$ and our theorem follows. □

6. Global Solutions

In this section we establish Theorem 1. We first derive a refined energy estimate for the relativistic Landau-Maxwell system.

Lemma 14. *Let $[f(t, x, p), E(t, x), B(t, x)]$ be the unique solution constructed in Theorem 6 which also satisfies the conservation laws (10), (11) and (12). Let the small amplitude assumption (14) be valid. For any given $0 \le m \le N$, $|\beta| \le m$, there are constants $C_{|\beta|} > 0$, $C_m^* > 0$ and $\delta_m > 0$ such that*

$$\begin{aligned}&\sum_{|\beta|\le m, |\gamma|+|\beta|\le N} \frac{1}{2}\frac{d}{dt} C_{|\beta|}||\partial_\beta^\gamma f(t)||^2 + \frac{1}{2}\frac{d}{dt}|||[E, B]|||^2(t) \\ &+ \sum_{|\beta|\le m, |\gamma|+|\beta|\le N} \delta_m||\partial_\beta^\gamma f(t)||_\sigma^2 \le C_m^*\sqrt{\mathcal{E}(t)}|||f|||_\sigma^2(t).\end{aligned} \tag{111}$$

Proof. We use an induction over m, the order of the $p-$derivatives. For $m = 0$, by taking the pure ∂^γ derivatives of (29), we obtain:

$$\begin{aligned}&\left\{\partial_t + \frac{p}{p_0}\cdot\nabla_x + \xi\left(E + \frac{p}{p_0}\times B\right)\cdot\nabla_p\right\}\partial^\gamma f \\ &\quad - \left\{\partial^\gamma E\cdot\frac{p}{p_0}\right\}\sqrt{J}\xi_1 + L\{\partial^\gamma f\} \\ &= -\sum_{\gamma_1\ne 0} C_\gamma^{\gamma_1}\xi\left(\partial^{\gamma_1}E + \frac{p}{p_0}\times\partial^{\gamma_1}B\right)\cdot\nabla_p\partial^{\gamma-\gamma_1}f \\ &\quad + \sum_{\gamma_1\le\gamma} C_\gamma^{\gamma_1}\left\{\frac{\xi}{2}\left\{\partial^{\gamma_1}E\cdot\frac{p}{p_0}\right\}\partial^{\gamma-\gamma_1}f + \Gamma(\partial^{\gamma_1}f, \partial^{\gamma-\gamma_1}f)\right\}.\end{aligned} \tag{112}$$

Using the same argument as (88),

$$-\langle\partial^\gamma E\cdot\{p\sqrt{J}/p_0\}\xi_1, \partial^\gamma f\rangle = \frac{1}{2}\frac{d}{dt}\left\{||\partial^\gamma E(t)||^2 + ||\partial^\gamma B(t)||^2\right\}.$$

Take the inner product of $\partial^\gamma f$ with (112), sum over $|\gamma| \le N$ and apply Theorem 2 to $L\{\partial^\gamma f\}$ to deduce the following for some constant $C > 0$,

$$\begin{aligned}&\sum_{|\gamma|\le N}\frac{1}{2}\frac{d}{dt}\left(||\partial^\gamma f(t)||^2 + ||\partial^\gamma E(t)||^2 + ||\partial^\gamma B(t)||^2\right) + \delta_0\sum_{|\gamma|\le N}||\partial^\gamma f(t)||_\sigma^2 \\ &\quad\le C\{|||f|||(t) + |||[E, B]|||(t)\}|||f|||_\sigma^2(t) \le C\sqrt{\mathcal{E}(t)}|||f|||_\sigma^2(t).\end{aligned}$$

We have used estimates (89-91) and Theorem 4 to bound the r.h.s. of (112). This concludes the case for $m = 0$ with $C_0 = 1$ and $C_0^* = C$.

Now assume the lemma is valid for m. For $|\beta| = m+1$, taking $\partial_\beta^\gamma (\beta \neq 0)$ of (29), we obtain:

$$\begin{aligned}
&\left\{\partial_t + \frac{p}{p_0}\cdot\nabla_x + \xi\left(E + \frac{p}{p_0}\times B\right)\cdot\nabla_p\right\}\partial_\beta^\gamma f - \partial^\gamma E\cdot\partial_\beta\left\{\frac{p}{p_0}\sqrt{J}\right\}\xi_1 \\
&\quad + \partial_\beta\{L\partial^\gamma f\} + \sum_{\beta_1\neq 0} C_\beta^{\beta_1}\partial_{\beta_1}\left(\frac{p}{p_0}\right)\cdot\nabla_x\partial_{\beta-\beta_1}^\gamma f \qquad (113)\\
&= \sum C_\gamma^{\gamma_1}C_\beta^{\beta_1}\frac{\xi}{2}\left\{\partial^{\gamma_1}E\cdot\partial_{\beta_1}\left(\frac{p}{p_0}\right)\right\}\partial_{\beta-\beta_1}^{\gamma-\gamma_1} f - \sum_{\gamma_1\neq 0} C_\gamma^{\gamma_1}\xi\partial^{\gamma_1}E\cdot\nabla_p\partial_\beta^{\gamma-\gamma_1} f \\
&\quad - \sum_{(\gamma_1,\beta_1)\neq(0,0)} C_\gamma^{\gamma_1}C_\beta^{\beta_1}\xi\partial_{\beta_1}\left(\frac{p}{p_0}\right)\times\partial^{\gamma_1}B\cdot\nabla_p\partial_{\beta-\beta_1}^{\gamma-\gamma_1} f \\
&\quad + \sum C_\gamma^{\gamma_1}\partial_\beta\Gamma(\partial^{\gamma_1} f, \partial^{\gamma-\gamma_1} f).
\end{aligned}$$

We take the inner product of (113) over $\mathbb{T}^3\times\mathbb{R}^3$ with $\partial_\beta^\gamma f$. The first inner product on the left is equal to $\frac{1}{2}\frac{d}{dt}||\partial_\beta^\gamma f(t)||^2$. Now $|\gamma| \leq N-1$ (since $|\beta| = m+1 > 0$), Lemma 13, (60) and $M_0 \leq 1$ imply (after an integration by parts) that the second inner product on the l.h.s. is bounded by

$$\langle\partial^\gamma E\cdot\partial_\beta\{p\sqrt{J}/p_0\}\xi_1, \partial_\beta^\gamma f\rangle \leq C||\partial^\gamma E||\cdot||\partial^\gamma f|| \leq C||\partial^\gamma f||\sum_{|\gamma'|\leq N}||\partial^{\gamma'} f||_\sigma.$$

From Lemma 7 and Cauchy's inequality we deduce that, for any $\eta > 0$, the inner product of the third term on the l.h.s. is bounded from below as

$$\left(\partial_\beta\{L\partial^\gamma f\}, \partial_\beta^\gamma f\right) \geq ||\partial_\beta^\gamma f||_\sigma^2 - \eta\sum_{|\bar\beta|\leq|\beta|}||\partial_{\bar\beta}^\gamma f||_\sigma^2 - C_\eta||\partial^\gamma f||^2.$$

Using Cauchy's inequality again, the inner product of the last term on the l.h.s. of (113) is bounded by

$$\eta||\partial_\beta^\gamma f(t)||^2 + C_\eta\sum_{|\beta_1|\geq 1}||\nabla_x\partial_{\beta-\beta_1}^\gamma f||^2.$$

By the same estimates, (89-91) and Theorem 4, all the inner products in the r.h.s. of (113) are bounded by $C\sqrt{\mathcal{E}(t)}|||f|||_\sigma^2(t)$. Collecting terms and summing over $|\beta| = m+1$ and $|\gamma| + |\beta| \leq N$, we split the highest order p-derivatives from the lower order derivatives to obtain

$$\sum_{|\beta|=m+1,|\gamma|+|\beta|\le N}\left\{\frac{1}{2}\frac{d}{dt}||\partial_\beta^\gamma f(t)||^2+||\partial_\beta^\gamma f(t)||_\sigma^2\right\}$$
$$\le \sum_{|\beta|=m+1,|\gamma|+|\beta|\le N}\left\{\sum_{|\beta|=m+1}2\eta||\partial_\beta^\gamma f(t)||_\sigma^2+C\sqrt{\mathcal{E}(t)}|||f|||_\sigma^2(t)\right\}$$
$$+\sum_{|\beta|=m+1,|\gamma|+|\beta|\le N}(C+2C_\eta)\sum_{|\beta|\le m,|\gamma|+|\beta|\le N}||\partial_\beta^\gamma f(t)||_\sigma^2$$
$$\le Z_{m+1}\left\{\sum_{|\beta|=m+1,|\gamma|+|\beta|\le N}2\eta||\partial_\beta^\gamma f(t)||_\sigma^2+C\sqrt{\mathcal{E}(t)}|||f|||_\sigma^2(t)\right\}$$
$$+Z_{m+1}(C+2C_\eta)\sum_{|\beta|\le m,|\gamma|+|\beta|\le N}||\partial_\beta^\gamma f(t)||_\sigma^2.$$

Here Z_{m+1} denotes the number of all possible (γ,β) such that $|\beta|\le m+1$, $|\gamma|+|\beta|\le N$. By choosing $\eta=\frac{1}{4Z_{m+1}}$, and absorbing the first term on the r.h.s. by the second term on the left, we have, for some constant $C(Z_{m+1})$,

$$\sum_{|\beta|=m+1,|\gamma|+|\beta|\le N}\left\{\frac{1}{2}\frac{d}{dt}||\partial_\beta^\gamma f(t)||^2+\frac{1}{2}||\partial_\beta^\gamma f(t)||_\sigma^2\right\}$$
$$\le C(Z_{m+1})\left\{\sum_{|\beta|\le m,|\gamma|+|\beta|\le N}||\partial_\beta^\gamma f(t)||_\sigma^2+\sqrt{\mathcal{E}(t)}|||f|||_\sigma^2(t)\right\}. \tag{114}$$

We may assume $C(Z_{m+1})\ge 1$. We multiply (114) by $\frac{\delta_m}{2C(Z_{m+1})}$ and add it to (111) for $|\beta|\le m$ to get

$$\sum_{|\beta|=m+1,|\gamma|+|\beta|\le N}\left\{\frac{\delta_m}{4C(Z_{m+1})}\frac{d}{dt}||\partial_\beta^\gamma f(t)||^2+\frac{\delta_m}{4C(Z_{m+1})}||\partial_\beta^\gamma f(t)||_\sigma^2\right\}$$
$$+\sum_{|\beta|\le m,|\gamma|+|\beta|\le N}\frac{1}{2}\frac{d}{dt}\left(C_{|\beta|}||\partial_\beta^\gamma f(t)||^2+||\partial^\gamma E(t)||^2+||\partial^\gamma B(t)||^2\right)$$
$$+\sum_{|\beta|\le m,|\gamma|+|\beta|\le N}\delta_m||\partial_\beta^\gamma f(t)||_\sigma^2$$
$$\le\frac{\delta_m}{2}\sum_{|\beta|\le m,|\gamma|+|\beta|\le N}||\partial_\beta^\gamma f(t)||_\sigma^2+\left\{C_m^*+\frac{\delta_m}{2}\right\}\sqrt{\mathcal{E}(t)}|||f|||_\sigma^2(t).$$

Absorb the first term on the right by the last term on the left. We conclude our lemma by choosing

$$C_{m+1}=\frac{\delta_m}{4C(Z_{m+1})},\quad \delta_{m+1}=\frac{\delta_m}{4C(Z_{m+1})}\le\frac{\delta_m}{2},\quad C_{m+1}^*=C_m^*+\frac{\delta_m}{2},$$

noting that $C(Z_{m+1})>C(Z_m)$ and $\delta_m<\delta_{m-1}$. $\square$

We are ready to construct global in time solutions to the relativistic Landau-Maxwell system (29) and (30).

Proof of Theorem 1. We first fix $M_0 \le 1$ such that both Theorems 2 and 6 are valid. For such an M_0, we let $m = N$ in (111), and define

$$y(t) \equiv \sum_{|\gamma|+|\beta|\le N} C_{|\beta|}||\partial_\beta^\gamma f(t)||^2 + |||[E,B]|||^2(t).$$

We choose a constant $C_1 > 1$ such that for any $t \ge 0$,

$$\frac{1}{C_1}\left\{y(t) + \frac{\delta_N}{2}\int_0^t |||f|||_\sigma^2(s)ds\right\} \le \mathcal{E}(t),$$
$$\mathcal{E}(t) \le C_1\left\{y(t) + \frac{\delta_N}{2}\int_0^t |||f|||_\sigma^2(s)ds\right\}.$$

Recall the constant C_N^* in (111). We define

$$M \equiv \min\left\{\frac{\delta_N^2}{8C_N^{*2}C_1^2}, \frac{M_0}{2C_1^2}\right\},$$

and choose initial data so that $\mathcal{E}(0) \le M < M_0$. From Theorem 6, we may denote $T > 0$ so that

$$T = \sup_t\{t : \mathcal{E}(t) \le 2C_1^2 M\} > 0.$$

Notice that, for $0 \le t \le T$, $\mathcal{E}(t) \le 2C_1^2 M \le M_0$ so that the small amplitude assumption (14) is valid. We now apply Lemma 14 and the definitions of M and T, with $0 \le t \le T$, to get

$$\begin{aligned} &y'(t) + \delta_N|||f|||_\sigma^2(t) \\ &\le C_N^*\sqrt{\mathcal{E}(t)}|||f|||_\sigma^2(t) \le C_N^* C_1\sqrt{2M}|||f|||_\sigma^2(t) \\ &\le \frac{\delta_N}{2}|||f|||_\sigma^2(t). \end{aligned}$$

Therefore, an integration in t over $0 \le t \le s < T$ yields

$$\begin{aligned} \mathcal{E}(s) &\le C_1\left\{y(s) + \frac{\delta_N}{2}\int_0^s |||f|||_\sigma^2(\tau)d\tau\right\} \le C_1 y(0) \\ &\le C_1^2\mathcal{E}(0) \\ &\le C_1^2 M < 2C_1^2 M. \end{aligned} \tag{115}$$

Since $\mathcal{E}(s)$ is continuous in s, this implies $\mathcal{E}(T) \le C_1^2 M$ if $T < \infty$. This implies $T = \infty$. Furthermore, such a global solution satisfies $\mathcal{E}(t) \le C_1^2\mathcal{E}(0)$ for all $t \ge 0$ from (115). $\square$

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