Math 645 - Homework 7 - Due Friday, March 22, 2013

1. Evans PDE (Second Edition) Chapter 5: 17
2. Evans PDE (Second Edition) Chapter 5: 19
3. (Fractional Chain Rule, Tao Dispersive Eq book, exercise A.13). Let $p > 1$ and let $F \in C^1_{loc}(\mathbb{R})$ satisfy $|F(x)| \leq C|x|^p$ and $|\nabla F(x)| \leq C|x|^{p-1}$. Suppose $0 \leq s < 1$ and $1 < q < r < \infty$ obey the scaling condition:

$$\frac{n}{q} = \frac{np}{r} - (p-1)s.$$

Prove the following inequality for all $f \in W^{s,r}(\mathbb{R}^n)$:

$$\|F(f)\|_{W^{s,q}(\mathbb{R}^n)} \leq C\|f\|^p_{W^{s,r}(\mathbb{R}^n)},$$

where $C = C(d, p, q, r, s)$.

If additionally $p > 2$, $F \in C^2_{loc}(\mathbb{R})$ and $|\nabla^2 F(x)| \leq C|x|^{p-2}$. Then prove the following stronger estimate

$$\|F(f) - F(g)\|_{W^{s,q}(\mathbb{R}^n)} \leq C\left(\|f\|_{W^{s,r}(\mathbb{R}^n)} + \|g\|_{W^{s,r}(\mathbb{R}^n)}\right)^{p-1}\|f - g\|^p_{W^{s,r}(\mathbb{R}^n)},,$$

for all $f, g \in W^{s,r}(\mathbb{R}^n)$.

Hint: Use LP theory.