Math 644 - Homework 6 - Due Friday, Oct. 19, 2012

1. Solve Evans, Problem 17 in Chapter 2. (subsolutions to Heat Eq.)

2. (A method for deducing decay estimates.) Building upon the previous problem... Suppose that $U = (0, 1)$ and $u \in C_1^2(U_T)$ solves:

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0 \text{ for } (x,t) \in (0,1) \times (0,+\infty) \\ u(x,0) = x(1-x) \text{ for } x \in [0,1] \\ u(0,t) = u(1,t) = 0 \text{ for } t > 0. \end{cases} \tag{1}$$

Show that $u \geq 0$.

c) For u as in (1), show that there exist constants $\alpha, \beta > 0$ such that:

$$u(x,t) \leq \alpha x(1-x)e^{-\beta t}.$$

Deduce that $u(x,t) \to 0$ as $t \to +\infty$.

3. (Heat eqn on a periodic domain.) Suppose $g \in C(\mathbb{T}^n) \cap L^\infty(\mathbb{T}^n)$ on the periodic domain $\mathbb{T}^n = [-\pi, \pi]^n$. We suppose that g is periodic in the following sense: $g(x) = g(y)$ if $x, y \in \partial\mathbb{T}^n$ and $x - y = 2\pi j$ for some $j \in \mathbb{Z}^n$.

We aim to solve the Heat equation

$$\begin{cases} u_t - \Delta u = 0 \text{ for } (x,t) \in (-\pi,\pi)^n \times (0,+\infty) \\ u(x,0) = g(x) \text{ for } x \in \mathbb{T}^n. \\ u(x,t) = u(y,t) \text{ for } t > 0,\ x, y \in \partial\mathbb{T}^n, \frac{x-y}{2\pi} \in \mathbb{Z}^n. \end{cases} \tag{2}$$

The last line in (2) assures the boundary condition that u remains periodic.

Show that $u(x,t) = (K_t * g)(x) = \int_{\mathbb{T}^n} K_t(x-y)g(y)dy$ solves (2) where

$$K_t(x) = \sum_{j \in \mathbb{Z}^n} \Phi(x + 2\pi j, t),$$

and $\Phi(x,t)$ is the fundamental solution to the Heat equation on $\mathbb{R}^n$. Is the equation satisfied? does the initial condition hold? Is $u \in C^\infty$? Is u periodic? Justify all calculations rigorously.

(Hint: define $\tilde{g} : \mathbb{R}^n \to \mathbb{R}$ by assuming that $\tilde{g}(x) = g(x)$ for $x \in \mathbb{T}^n$ and $\tilde{g}(x) = \tilde{g}(x + 2\pi j)$ for all $j \in \mathbb{Z}^n$.)

4. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary. Suppose that we can find $F_j \in C^\infty(\Omega) \cap L^\infty(\Omega)$ such that $\sup_{j\in J} \|F_j\|_{L^\infty(\Omega)} \le C < \infty$ and

$$\begin{cases} \Delta F_j = \lambda_j F_j \text{ in } \Omega \\ F_j(x) = 0 \text{ for } x \in \partial\Omega. \end{cases} \tag{3}$$

We further suppose $\lambda_j \le 0$ and $\{F_j\}_{j\in J}$ form an orthonomial basis for $L^2(\Omega)$. (For example on $\mathbb{T}^n$, we have $\{e^{x\cdot j\sqrt{-1}}\}_{j\in\mathbb{Z}^n}$ with $\lambda_j = -|j|^2$.)

Let $f = \sum_{j\in J} a_j F_j$, where $\sum_{j\in J} |a_j| < \infty$. We would like to solve the heat equation:

$$\begin{cases} u_t - \Delta u = 0 \text{ for } (x,t) \in \Omega \times (0,+\infty) \\ u(x,0) = f(x) \text{ for } x \in \Omega. \end{cases} \tag{4}$$

Solve (4) explicitly as an infinite series in terms of the basis $\{F_j\}_{j\in J}$ and prove that the series converges in a strong sense.

5. Building upon the last question, we will solve the inhomogeneous problem

$$\begin{cases} u_t - \Delta u = g(x,t) \text{ for } (x,t) \in \Omega \times (0,+\infty) \\ u(x,0) = 0 \text{ for } x \in \Omega. \end{cases} \tag{5}$$

In (5) we suppose $g(x,t) = \sum_{j\in J} b_j(t) F_j(x)$, where

$$\sum_{j\in J} \int_0^\infty |b_j(t)| dt \le C < \infty.$$

In this problem solve (5) explicitly as an infinite series in terms of the basis $\{F_j\}_{j\in J}$ and prove that the series converges in a strong sense. (The solution end-up as an inhomogenous version of the solution in the previous problem.)