

Bounds in Two-Sample Mendelian Randomization With Summary Statistics

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1 Introduction

The gold standard to estimate the causal effect of a treatment or an exposure on an outcome is a randomized trial where the treatment assignment is randomized. However, in many epidemiological studies, randomized experiments are not feasible. For example, a study estimating the negative effects of smoking on depression (Wootton et al. 2019) would not be feasible with a randomized trial due to ethical concerns. In such settings, epidemiologists rely on different types of observational studies, which introduces potential biases from unmeasured confounders. In recent years, there has been an increase in using instrumental variable (IV) in the form of Mendelian randomization (MR) (Davey Smith and Ebrahim 2003; Lawlor et al. 2008). Briefly, IV is a variable that is (A1) associated with the exposure, (A2) is independent from unmeasured confounders affecting the exposure and the outcome, and (A3) affects the outcome only through its effect on the exposure; see 2.1 for details. MR uses genetic variants, usually single nucleotide polymorphisms and encoded as 0, 1, or 2, as instruments due to an idea that genotypes are randomly assigned when passed on from parents to offspring by meiosis and thus, possibly satisfying (A2). For additional discussions on the plausibility of assumptions (A1)-(A3) in MR, see Davey Smith and Ebrahim (2003), Lawlor et al. (2008), Didelez and Sheehan (2007), Bowden et al. (2017), Hartwig et al. (2017).

Data from MR studies often consist of published summary statistics from two independent genome wide association studies (GWAS), often referred to as the two-sample setting (Burgess et al. 2013, 2015; Davies et al. 2018). Typically, the first GWAS provides information about the exposure and instrument and the second GWAS provides information about the outcome and instrument. With the summary statistics from two studies, investigators often use parametric methods to arrive at estimates and tests for the exposure effect. Examples of such estimators and test are the IVW estimator (Burgess et al. 2013), MR-Egger regression (Bowden et al. 2016b), weighted median (Bowden et al. 2016a) and modes (Hartwig et al. 2017), and MRRAPs (Zhao et al. 2020), to name a few; see Burgess and Thompson (2015) and Burgess et al. (2017) for recent reviews.

An alternative approach to study the exposure effect without parametric assumptions is through nonparametric IV bounds (Balke and Pearl 1997; Cheng and Small 2006; Manski 1990; Richardson and Robins 2014; Robins 1989). Briefly, nonparametric IV bounds only use a minimum set of amount of assumptions, usually (A1)-(A3), to provide a range of plausible values for the exposure effect. They are typically used when the outcome, the exposure, and the instrument are all binary and are simultaneously observed; we refer to this setting as the one-sample setting to contrast it from the two-sample setting in MR. The most well-known are the Balke-Pearl bounds (Balke and Pearl 1997) for the average treatment effect under slight variants of assumptions (A1)-(A3). Also, the conditions underlying these bounds lead to a set of instrumental inequalities to falsify the IV assumptions. Since then, Cheng and Small (2006) and Richardson and Robins (2014) extended the Balke-Pearl bounds to allow for a non-binary instrument. Ramsahai (2012) derived bounds for the two-sample setting. Palmer et al. (2018) provides software to compute IV bounds for two-sample MR studies using only summary statistics. For a recent overview, see Swanson et al. (2018).

Due to their nonparametric nature, IV bounds have been attractive alternatives in non-MR, one-sample settings to analyze treatment effects, especially in settings where some parametric assumptions are suspect or difficult to justify. More generally, if IV bounds using fewer assumptions arrive at similar conclusions about the treatment effect as those based on parametric approaches, the case for the treatment effect becomes stronger. But, there is little work on understanding about the behavior of IV bounds in MR settings where we have summary statistics from two samples. For example,

1. What kind of genetic instruments are needed in two-sample MR studies to provide useful conclusions about the exposure effect, say the bound does not contain the null effect?
2. Can combining multiple genetic instruments from GWAS lead to shorter and tighter bounds on the exposure effect?
3. How do the bounds change if many instruments have weak association with the exposure, which is typically in MR studies where genetic variants only explain a small amount of variation in the exposure?

In one sample setups where individual-level data are available, the Balke-Pearl bounds are known to be wide, often containing the null effect; Balke and Pearl (1997) showed that the width of the bound decrease linearly with the magnitude of the instrument's association to the exposure. However, it is not clear if the same principle holds for two-sample MR studies. The goal of the paper is to characterize the behavior of these bounds, specifically addressing what can be learned from MR studies that choose to use nonparametric IV bounds to analyze the exposure effect.

The paper is divided as follows. Section 2 introduces the counter factual framework and definitions used throughout the paper along with the expressions for the two-sample bounds. Section 3 investigates the behavior of the two-sample bounds in settings that mimic that or most MR analyses. In Section 4, we introduce a method that allows us to illustrate the cost of going from one- to two-sample data and get a sense of the potential of non-parametric bounds in any given setting. Section 5 presents two examples of using these two-sample bounds in MR analyses. Finally, we present our conclusions and a few practical considerations in Section 6.

2 Setup

2.1 Review: Notation and Definitions

In the following, let X and Y be binary exposure and outcome, respectively, Z be a categorical instrumental variable taking values in $\{0, 1, \text{ and } 2\}$, and U an unmeasured confounder for the effect of X on Y . No assumptions about the structure of U are made. Let $Y^{z,x}$ be the potential outcome (Rubin 1974; Splawa-Neyman et al. 1990) had the subject received exposure value $X = x$ and instrument value $Z = z$. Throughout the paper, we assume the stable unit treatment value assumption (SUTVA) (Cox 1958; Rubin 1980), formalized as $Y = \sum_{x,z} I[Z = z, X = x]Y^{x,z}$ and $I[\cdot]$ is the indicator function.

We make the following set of assumptions about the instrument, the exposure, the outcome, and the unmeasured confounder that are typical in MR studies; see Didelez and Sheehan (2007) and Wang and Tchetgen (2018) for details.

- (A1) (*Relevance*): $Z \not\perp X$
- (A2) (*Independent instrument*): $Z \perp U$
- (A3) (*Exclusion restriction*): $Y^{z,x} = Y^{z',x} = Y^x$ for all x, z, z'
- (A4) (*Conditional ignorability of X, Z given U*): $Y^{z,x} \perp Z, X | U$

Briefly, assumption (A1) can be assessed by finding SNPs that have been consistently associated with the exposure through multiple GWAS (Marigorta et al. 2018). Assumption (A2) is usually checked based on scientific theory surrounding how the genetic instrument was inherited from the parents to the offspring. Assumption (A3) states that there is no direct effect of the instrument Z on the outcome Y other than that through the exposure X and like assumption (A2), is assessed by scientific theory. Both assumptions (A2) and (A3) can be violated if the SNP is (i) in linkage disequilibrium with an unmeasured SNP that affects the exposure and outcome, (ii) pleiotropic and has multiple functions beyond affecting the exposure, or (iii) under population stratification, to name a few. For a more in-depth discussion of (A1)-(A3) in MR studies, see Lawlor et al. (2008). Finally (A4) states that if U is observed, then it is sufficient to unconfound the relationship between X and Y .

We make a few additional remarks about assumptions (A1)-(A4). First, most MR studies only make assumptions (A1)-(A3) along with some modeling assumptions (Burgess and Thompson 2015). Second, the role of assumption (A4) is to show the role that an unmeasured confounder U plays in identification of causal effects; Richardson and Robins (2014) showed that one can remove (A4) and strengthen (A2) with $Z \perp U, Y^{z,x}$ without consequence on the IV bounds. Third, under SUTVA and assumptions (A3)-(A4), we have $Y \perp Z|X, U$, which is another common way to express the exclusion restriction assumption in MR studies (Didelez and Sheehan 2007). Fourth, for simplicity, we do not assume the existence of a potential

treatment X^z ; the existence of X^z does not change the IV bounds (Richardson and Robins 2014; Swanson et al. 2018), and its primary purpose is to define a “causal” instrument (Hernán and Robins 2006).

We conclude by introducing two assumptions and defining instrument strength; the assumptions are not necessarily to construct bounds, but will help us explain the behavior of the IV bounds. First, we state the assumptions restricting the direction of the instrument’s effect on the exposure and the outcome.

$$(A5) \text{ (Monotonicity between } Z \text{ and } X\text{)} P(X = 1|Z = z, U) \leq P(X = 1|Z = z + 1, U) \text{ for } z = 0, 1$$

$$(A6) \text{ (Monotonicity between } Z \text{ and } Y\text{)} P(Y = 1|Z = z, U) \leq P(Y = 1|Z = z + 1, U) \text{ for } z = 0, 1$$

A variant of assumption (A5) is common in the IV literature to study noncompliance (Angrist et al. 1996; Baiochi et al. 2014). Assumption (A6) is an extension of assumption (A5) to the outcome variable. Assumption (A5) or (A6) is plausible in MR if the direction of the effects of the genetic instrument on the exposure or the outcome are well-established from scientific theory and replication of findings from many observational studies.

Second, we define instrument strength as the maximum possible contrast between the exposure when instruments take on different values

$$ST = \max_{z_1 \neq z_2} |P(X = 1|Z = z_1) - P(X = 1|Z = z_2)| \quad (1)$$

The formula for ST reduces to the definition of instrument strength used in Balke and Pearl (1997) when the instrument is binary. More importantly, in a binary IV setting, ST was used to characterize the width of the IV bounds. However, we remark that (1) differs from other definitions of instrument strength based on a parametric model between the exposure and the outcome, say the concentration parameter; see Stock et al. (2002) for an overview.

2.2 Review: Study Designs and Target Estimand

For the purposes of studying IV bounds, we can divide IV studies into two designs, the two-sample design and the one-sample design. The two-sample design has two separate data sources, one providing information of (X, Z) and one providing information of (Y, Z) , and is the most popular design in MR studies. The one-sample design has a single data source providing information on all observed variables (X, Y, Z) and is more common in traditional IV studies involving non-genetic instruments. As mentioned in Section 1, the behavior of bounds under a one-sample design has been well-studied (Balke and Pearl 1997; Richardson and Robins 2014; Swanson et al. 2018); we remark that these bounds can also be used when individual-level data are not available, but population summary statistics in the form of $P(Y = y, X = x|Z = z)$ for y, x, z are known.

However, not much is known about the behavior of bounds under a two-sample design. Specifically, an MR study often uses a two-sample design only with summary statistics from GWAS. If both the outcome and the exposure are binary as is the case for case-control study, these summary statistics are computed by running a logistic regression between the exposure X and the outcome Y for each genetic instrument Z and extracting the estimated slope coefficients associated with Z ; it’s also common for the logistic regression to adjust for age, sex, and principal components. To focus our paper on studying behavior of bounds not due to sampling errors, we will assume that we have population-level quantities $P(Y = 1|Z = z)$ from one data source and $P(X = 1|Z = z)$ from another data source for different values of z .

Given the quantities $P(Y = 1|Z = z)$ and $P(X = 1|Z = z)$ for each $z = 0, 1, 2$ from a two-sample design, the goal is to study the average treatment effect (ATE)

$$\text{ATE} = E[Y^1 - Y^0] = \int P(Y = 1 | X = 1, U = u)P(U = u)du - \int P(Y = 1 | X = 0, U = u)P(U = u)du.$$

Here, the second equality follows from SUTVA and assumptions (A3) and (A4). Since U is not observed, additional assumptions are needed to point-identify the ATE. In particular, even with the remaining assumptions (A1), (A2), and (A5), the ATE cannot be point-identified; see Robins (1989), Manski (1990), and Balke and Pearl (1995). In two-sample designs, Ramsahai (2012) showed that under assumptions (A1)-(A4), the bounds for the ATE are

$$\max \left\{ \begin{array}{ll} \max_{z_1 \neq z_2} & P(Y = 1|Z = z_1) - 2 \cdot P(Y = 1|Z = z_2) - 2 \cdot P(X = 1|Z = z_2) \\ \max_{z_1 \neq z_2} & P(Y = 1|Z = z_1) + P(X = 1|Z = z_1) - P(Y = 1|Z = z_2) - P(X = 1|Z = z_2) - 1 \\ \max_{z_1 \neq z_2} & 2 \cdot P(Y = 1|Z = z_1) + 2 \cdot P(X = 1|Z = z_1) - P(Y = 1|Z = z_2) - 3 \\ \max_z & -P(Y = 1|Z = z) - P(X = 1|Z = z) \\ \max_z & P(Y = 1|Z = z) + P(X = 1|Z = z) - 2 \end{array} \right\} \leq ATE \leq \quad (2)$$

$$\min \left\{ \begin{array}{ll} \min_{z_1 \neq z_2} & P(Y = 1|Z = z_1) - 2 \cdot P(Y = 1|Z = z_2) + 2 \cdot P(X = 1|Z = z_2) + 1 \\ \min_{z_1 \neq z_2} & P(Y = 1|Z = z_1) + 2 \cdot P(Y = 1|Z = z_2) - 2 \cdot P(X = 1|Z = z_2) + 1 \\ \min_{z_1 \neq z_2} & P(Y = 1|Z = z_1) - P(X = 1|Z = z_1) + P(X = 1|Z = z_2) - P(Y = 1|Z = z_2) + 1 \\ \min_z & P(X = 1|Z = z) - P(Y = 1|Z = z) + 1 \\ \min_z & P(Y = 1|Z = z) - P(X = 1|Z = z) + 1 \end{array} \right\}$$

Additionally, the data from two-sample designs can be used to check the validity of the IV assumptions

$$\min \left\{ \begin{array}{ll} \min_{z_1 \neq z_2} & P(Y = 1|Z = z_1) - P(X = 1|Z = z_1) - P(Y = 1|Z = z_2) - P(X = 1|Z = z_2) + 2 \\ \min_{z_1 \neq z_2} & P(Y = 1|Z = z_1) + P(X = 1|Z = z_1) - P(Y = 1|Z = z_2) + P(X = 1|Z = z_2) \\ \min_z & P(X = 1|Z = z) \\ \min_z & P(Y = 1|Z = z) \\ \min_z & 1 - P(X = 1|Z = z) \\ \min_z & 1 - P(Y = 1|Z = z) \end{array} \right\} \geq 0 \quad (3)$$

The inequalities in equation (5) are extensions of the “IV inequalities” of Balke and Pearl (1997) used to check the validity of the IV assumptions. Versions of these inequalities have been used in MR studies (Diemer et al. 2020) to check whether the genetic variants satisfy the IV assumptions. In the Appendix, we provide some details on deriving equations (2) and (5) as well as numerically computing the bound using Polymake (Assarf et al. 2017). We also discuss a minor, but important issue concerning ordering of the bounds in order to obtain “proper bounds”, i.e. bounds where the lower bound is less than or equal to the upper bound. We believe this issue is pertinent among investigators who are using a linear-program based software to compute these bounds (Palmer et al. 2011), or who are computing lower and upper bounds separately (Richardson and Robins 2014).

The rest of the sections are devoted to studying the behavior of the bound in (2) under a variety of settings.

3 Properties of Bounds from Summary-Level Data

3.1 Bounds from Two Sample Data

We begin our investigation into the behavior of bounds in equation (2) when there is a single instrument. We are interested in whether we can gain any insights into the direction and magnitude of the ATE by examining the length of the bounds; wide bounds typically provide less information about the magnitude or the sign of the ATE compared to narrower bounds.

First, Theorem 3.1 shows the width of the ATE bound in equation (2) under a near-ideal MR study where all the assumptions (A1)-(A6) hold. That is, in addition to having some evidence in support of assumptions (A1)-(A4) that are needed to obtain the bound in equation (2), the investigator knows that the genetic instrument has a monotonic effect on the exposure and the outcome. The extra assumptions (A5)-(A6) simplify the bound formula in equation (2) and allow us to characterize its width by instrument strength ST.

Theorem 3.1. *Under assumptions (A1)-(A6), the bounds for the ATE in (2) become*

$$\begin{aligned} \max & \left\{ \begin{array}{l} -P(Y = 0|Z = 2) - P(Y = 1|Z = 0) + P(X = 0|Z = 0) - P(X = 0|Z = 2) \\ P(Y = 0|Z = 0) - 2 \cdot P(Y = 0|Z = 2) - P(X = 0|Z = 2) \\ -P(Y = 0|Z = 2) - 2 \cdot P(Y = 1|Z = 0) + P(X = 0|Z = 0) \end{array} \right\} \\ & \leq ATE \leq \\ \min & \left\{ \begin{array}{l} 1 + P(Y = 0|Z = 0) - P(X = 0|Z = 0) \\ 1 + P(Y = 0|Z = 0) - P(Y = 0|Z = 2) - P(X = 0|Z = 0) + P(X = 0|Z = 2) \\ 1 - P(Y = 0|Z = 2) + P(X = 0|Z = 2) \end{array} \right\} \end{aligned}$$

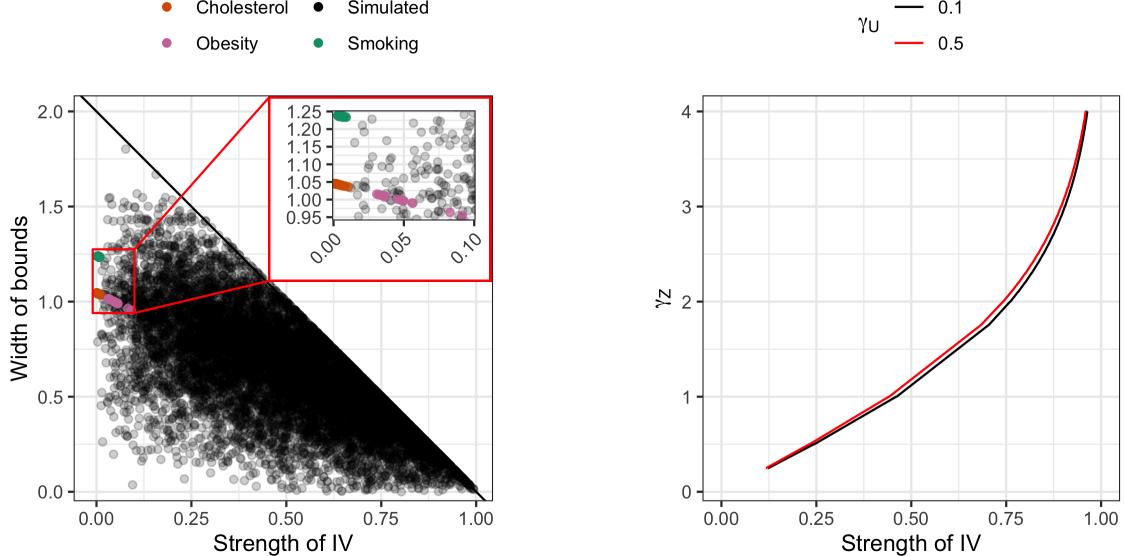
and a sharp upper bound on the width of the bounds is $2 - 2 \cdot ST$, i.e. there exists a data generating process satisfying (A1)-(A6) and has width equal to $2 - 2 \cdot ST$.

The proof is presented in Appendix A. The bounds under the near-ideal MR setting is up to twice as large as the Balke-Pearl bounds with a binary IV in single-sample designs where the width is $1 - ST$. An instrument with strength $ST = 0.6$ would lead to a smaller bound with width 0.4 under a binary IV, single-sample design setting compared to a length of up to 0.8 in the near-ideal MR study. The potential doubling of the bound is a “cost” of using two-samples instead of one-sample. In particular, two-sample designs do not provide any information about the joint distribution of $P(Y, X|Z)$, which can tighten the bounds; see Section 4 where we exploit this phenomena to obtain more informative bounds in MR studies.

Based on Theorem 3.1, the width of the IV bounds in two-sample settings is only guaranteed to be less than 1 when the instrument strength ST is greater than 0.5; we remark that a bound with length greater than 1 provides no information about the existence of the exposure effect since the bound will always cover zero. However, instruments with strength less than 0.5 could still generate a bound with width less than 1 (see Figure 1 for examples).

To illustrate our theorem, we randomly generate 10,000 sets of values of $P(X = 1|Z = z)$ and $P(Y = 1|Z = z)$ that satisfy the IV inequalities, and calculate the corresponding bounds from equation (2). This simulation mimics a scenario where there is a uniform/flat prior over the possible summary statistics that can arise from two-sample MR studies satisfying assumptions (A1)-(A4) and illustrates the large variety of bounds two-sample MR can result in.

Figure 1a shows the widths of the same 10,000 bounds plotted against the strength of the instruments. The black line is the upper bound for the width of the bounds in Theorem 3.1. We see that the width of the bounds often exceed 1 as the instrument strength decreases. Table 1 makes this more precise by showing the proportion of bounds presented in Figure 1 with widths greater than 1, 0.75, and 0.5, stratified by instrument strength. The table reveals that while it is possible to observe bounds with width less than 1 for IVs with strength less than 0.05, 47% of the bounds lead to widths greater than 1 and about 46.4% of bounds from IVs with strength between 0.05 and 0.1 have width greater than 1. Also, only 62% of bounds with strength greater than 0.5 have widths less than 0.5.



(a) Black line has intercept 2 and slope -2. Color indicates real data exposure (see Section 5 for details.)

(b) Coefficients from logistic regression related to the strength of the instrument.

Figure 1: Illustration of the relationship between instrument strength, and width of bounds obtained from two-sample design and coefficients from logistic regression model.

Table 1: Proportion of bounds from distributions where width is greater than 1, 0.75, and 0.5 stratified by strength of the instrument Z on the exposure X .

| Strength | Proportion of bounds with width greater than... | | |
|-------------|-------------------------------------------------|-----------|-----------|
| | 1 | 0.75 | 0.5 |
| [0, 0.05] | 0.4698795 | 0.7831325 | 0.8915663 |
| (0.05, 0.1] | 0.4640884 | 0.7237569 | 0.8950276 |
| (0.1, 0.25] | 0.3261704 | 0.7045280 | 0.9232540 |
| (0.25, 0.5] | 0.1360505 | 0.4974168 | 0.8384041 |
| (0.5, 1] | 0.0000000 | 0.0755708 | 0.3796727 |

To better understand the implications of instrument strength on the width of bounds, Figure 1b characterizes the relationship between instrument strength ST and a popular summary statistic measuring instrument strength reported in MR methods (Burgess 2014; King et al. 2020; Lawlor et al. 2008; Millard et al. 2019; Verma et al. 2018). Specifically, suppose we assume that $P(Z = 0) = P(Z = 2) = 0.25$ and $P(Z = 1) = 0.5$, and a value of an unmeasured confounder U from the standard normal. We assume the exposure X follows $P(X = 1|Z = z, U = u) = \text{logit}(\gamma_0 + \gamma_Z \cdot z + \gamma_U \cdot u)$ where γ_Z corresponds to the regression estimate one would obtain from GWAS studying the relationship between the genetic variant and the exposure. For simplicity, we set $\gamma_0 = -\gamma_Z$; this corresponds to the scenario where difference in the probability $P(X = 1|Z = z, U)$ between z s is large. We then vary γ_Z from 0 to 4 and set γ_U to be either 0.1 or 0.5. For each combination of γ_Z and γ_U , we compute the corresponding ST through monte carlo integration involving 10,000,000 samples.

update x-axis with γ_Z . Also, can we draw a curve without the dots and extend this graph to 0 to 1? Also, rearrange x-y axis to be consistent with figure above? I'm also wondering if we should replace Fig1a with this one? RMT: we can do all of this, but I feel like drawing the curve without the dots would be a bit

misleading, no? Also, curve = connect the dots, or curve = smooth curve (loess)?

From Figure 1b, we see that instrument strength ST of 0.5 corresponds to a regression coefficient γ_Z of 1.16 if $\gamma_U = 0.5$ and 1.1 if $\gamma_U = 0.1$. Such coefficients are rarely encountered in GWAS summary statistics meaning that we have little hope of guaranteeing narrow bounds from MR analyses.

Next, among bounds that have width less than 1, we study whether they can tell investigators about the direction of the exposure effect. More specifically, for an anticipated effect size, we ask what kind of genetic instrument in terms of instrument strength are needed in order for the IV bounds to ~~not only contain this effect size, but also~~ be able to detect the direction of the effect, that is for the bounds to exclude 0 **RMT: bounds always contain the effect size, no?**. We remark that this question is akin to computing the power of bounds where instrument strength roughly stands for sample size; a major difference, though, is that we are using population-level estimates of the probability distributions.

Formally, we again use the exposure model from the previous paragraph and suppose an outcome model $P(Y = 1|X = x, U = u) = \text{logit}(\beta_0 + \beta_X \cdot x + \beta_U \cdot u)$. For simplicity, we set $\beta_U = \gamma_U$ from the exposure model and $\beta_0 = -\beta_X/2$. We then vary β_X to be between 0.25 and 2 and for each β_X , we find the smallest γ_Z needed to produce an IV bound that contains the exposure effect, but does not contain 0. Figure 2 show the results. Similar to the story about instrument strength and width of bounds, we see that to detect even moderate effect sizes of 0.4, the corresponding γ_Z must be around 1.75, a tall order for most GWAS summary statistics.

Overall, in the context of two-sample MR studies where most genetic instruments are weak, the chances that the IV bounds are informative is unlikely. The bounds will often have width greater than 1 for most genetic instruments and detecting even moderate effect sizes require uncharacteristically strong genetic variants.

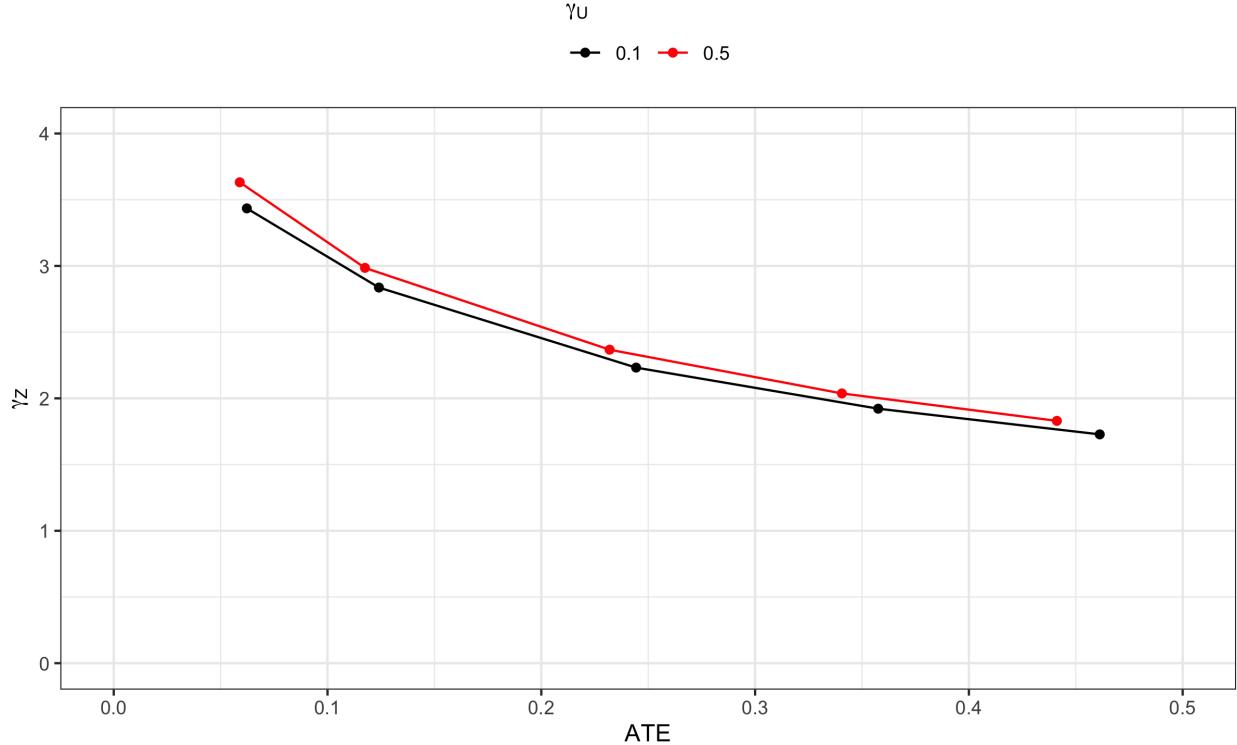


Figure 2: For different values of β_Z, β_U , and γ_X , 10,000,000 observations were simulated from the model described in Section 3.1. This graph shows the estimated size of β_Z needed to detect direction for different values of the average treatment effect based on a loess extrapolation. See Figure 9 in Appendix D.

3.2 Bounds From Two Sample Data With Multiple IVs

Prior section revealed that bounds from two-sample MR studies require a strong instrument to obtain informative bounds. However, it did not address whether the bound can become more informative if multiple valid instruments are available. In this section, we the simplest and most naive approach to aggregate bounds across multiple instruments by taking intersections of separate IV bounds. This may be superior to another alternative where we expand the levels of Z from 0, 1, 2 to accommodate multiple instruments (Swanson 2017), but has the benefit of being applicable to the data that can be easily obtained from two-sample MR. We show that with this simple aggregation strategy, bounds generally do not become more informative as the number of instruments increase.

Formally, consider the following outcome model when there are multiple instruments; this model has been used in MR studies by Burgess (2014) and Burgess and Thompson (2012) so that every instrument estimates the same exposure effect.

$$\text{logit}(P(X = 1|Z_1 = z_1, \dots, Z_p = z_p, U = u)) = \gamma_0 + \sum_i \gamma_i z_i + \gamma_U u$$

$$\text{logit}(P(Y = 1|X = x, U = u)) = \beta_0 + \beta_X x + \beta_U u,$$

where $\text{logit}(a) = \frac{1}{1+\exp(-a)}$, $y \in \{0, 1\}$, $x \in \{0, 1\}$, $z_i \in \{0, 1, 2\}$, $\beta_i, \gamma_j \in \mathbb{R}$, and U is an unmeasured confounder.

We allow either $p = 10$ or $p = 50$ instruments. For each of the p instruments, we set $P(Z_j = 0) = P(Z_j = 2) = 0.25$ and $P(Z_j = 1) = 0.5$, and draw U from a standard normal distribution. We also set $\beta_U = \gamma_U$, which is equal to either 0.1 or 0.5, and set β_X to be either 0.25, 0.5, 1, 1.5, or 2. We then consider four scenarios for setting γ_j :

1. *Many weak instruments:* γ_j are spread out evenly on the interval 0 to 0.2.
2. *Many strong instruments:* γ_j are spread out evenly on the interval 1 to 4. This is the magnitude of γ s that detected the direction of the ATE using bounds in the previous section
3. *Many very weak instruments, one medium strength instrument:* γ_j , $j = 1, 2, \dots, p-1$, are evenly spread out on the interval 0 to 0.01, and $\gamma_p = 0.2$.
4. *Many medium strong instruments, one strong instrument:* γ_j , $j = 1, 2, \dots, p-1$, are evenly spread out on the interval 1 to 1.2, and $\gamma_p = 4$.

The first scenario mimics typical magnitudes of coefficients we see in MR studies, where most genetic variants have a weak effect on the exposure. The scenario is also an example of a genetic architecture where many genetic traits contribute to complex traits (Loh et al. 2015; Nj et al. 2017; Shi et al. 2016). The second scenario is an extension of the first where we increase the magnitude of the genetic variants' effects on the exposure. We don't expect to observe this practice, but these are the magnitudes that our results in Section 3.1 suggests for an investigator to obtain informative bounds from a single instrument. The third scenario represents a genetic architecture where only few genetic variants have strong effects on the exposure while others have weak effects (Yang et al. 2010), while the fourth scenario extends the third in the same way that the second extends the first.

For each scenario, we use Monte Carlo integration with 1 million re-samples to obtain $P(X = 1|Z_j = z_j)$ and $P(Y = 1|Z_j = z_j)$ and obtain IV bounds for each instrument. Figure 3 summarizes the results. We see that in scenarios 1 and 2, every bound is non-informative, with widths close to or exceeding 1. Also, the bounds are nested within each other. Thus, if we were to aggregate the bounds by taking intersections, the width of the intersection bounds will still be close to or exceed 1. In addition, the increase in magnitude of the γ_j coefficient did not improve the bounds. The perhaps more plausible Scenario 3 (many genetic instruments with small effects on the exposure, while one instrument has a somewhat strong effect) show

similar results. From Figure 3c it is clear that on the scale that is often observed in MR studies, two-sample nonparametric bounds are generally non-informative. Furthermore, the bounds are again nested leaving us with the conclusion that the intersection of bounds from multiple instruments will give no more information than the strongest of the instruments itself.

Where we saw previously that a single instrument with coefficient $\gamma_Z = 4$ was sufficient to detect the direction of the average treatment effect even for small effect sizes (see Figure 2), when such an instrument is observed in a setting where many instruments are present, the resulting bounds are non-informative. This is due to a dilution effect from including multiple instruments; see Table 2. In practice, this means that for similar coefficients, the strength of the instrument as measured by ST is much smaller when more instruments are included. This effect is more dominant when the coefficients are larger. The simulation results from Scenario 4 illustrates this point. Figure 3d shows that in a setting where we would be able to detect the direction of the ATE from an instrument with $\gamma_j = 4$ if only 10 instruments are contributing to the exposure, that same coefficient would not be enough if 50 instruments were included. In other words, this dilution effect indicates that nonparametric bounds based on two-sample MR data are very unlikely to be informative if it is suspected that there are many valid genetic instruments.

Table 2: The largest coefficient allowed in each of the four scenarios results in instruments with very different values of ST when more instruments are included. The effect is more prominent when the support of the coefficients is on the larger end.

| Scenario | β_U | γ_j | Strength | |
|----------|-----------|------------|----------|--------|
| | | | p = 10 | p = 50 |
| 1 | 0.1 | 0.2 | 0.098 | 0.092 |
| 1 | 0.5 | 0.2 | 0.094 | 0.087 |
| 2 | 0.1 | 4.0 | 0.547 | 0.242 |
| 2 | 0.5 | 4.0 | 0.546 | 0.241 |
| 3 | 0.1 | 0.2 | 0.102 | 0.094 |
| 3 | 0.5 | 0.2 | 0.096 | 0.094 |
| 4 | 0.1 | 4.0 | 0.826 | 0.514 |
| 4 | 0.5 | 4.0 | 0.821 | 0.512 |

Our results above also have dire implications when some instruments turn out to be invalid. If, as suggested by Swanson (2017), we take the union of IV bounds so that the union bound is guaranteed to cover the true ATE so long as there is at least one valid instrument, the union bound will likely be non-informative because there was at least one IV bound in our scenario that was non-informative. More broadly, without making some assumptions about the nature of the invalid IVs, it would generally be infeasible to obtain useful information from using bound-based analysis.

Overall, combining our investigation into the behavior of bounds from Section 3.1, our conclusion about using nonparametric IV bounds in two-sample MR studies is grim. They generally require very strong instruments and/or effect sizes, which are rare in MR studies, and multiple instruments are no better than having a single, strong instrument. Without having strong instruments in nature, this is generally difficult, if not impossible, to address with any statistical methodology. Also, as illustrated in our theory in Section 3.1, another primarily reason for the non-informative nature of the IV bounds is because of the two-sample setup. While this is also generally difficult to resolve in many studies, in the next section, we discuss how to obtain a plausible range of the joint distribution of the outcome and the exposure given the instrument $P(Y, X|Z)$ given two sample MR data $P(Y|Z)$ and $P(X|Z)$ in order to create more informative bounds from two-sample MR studies.

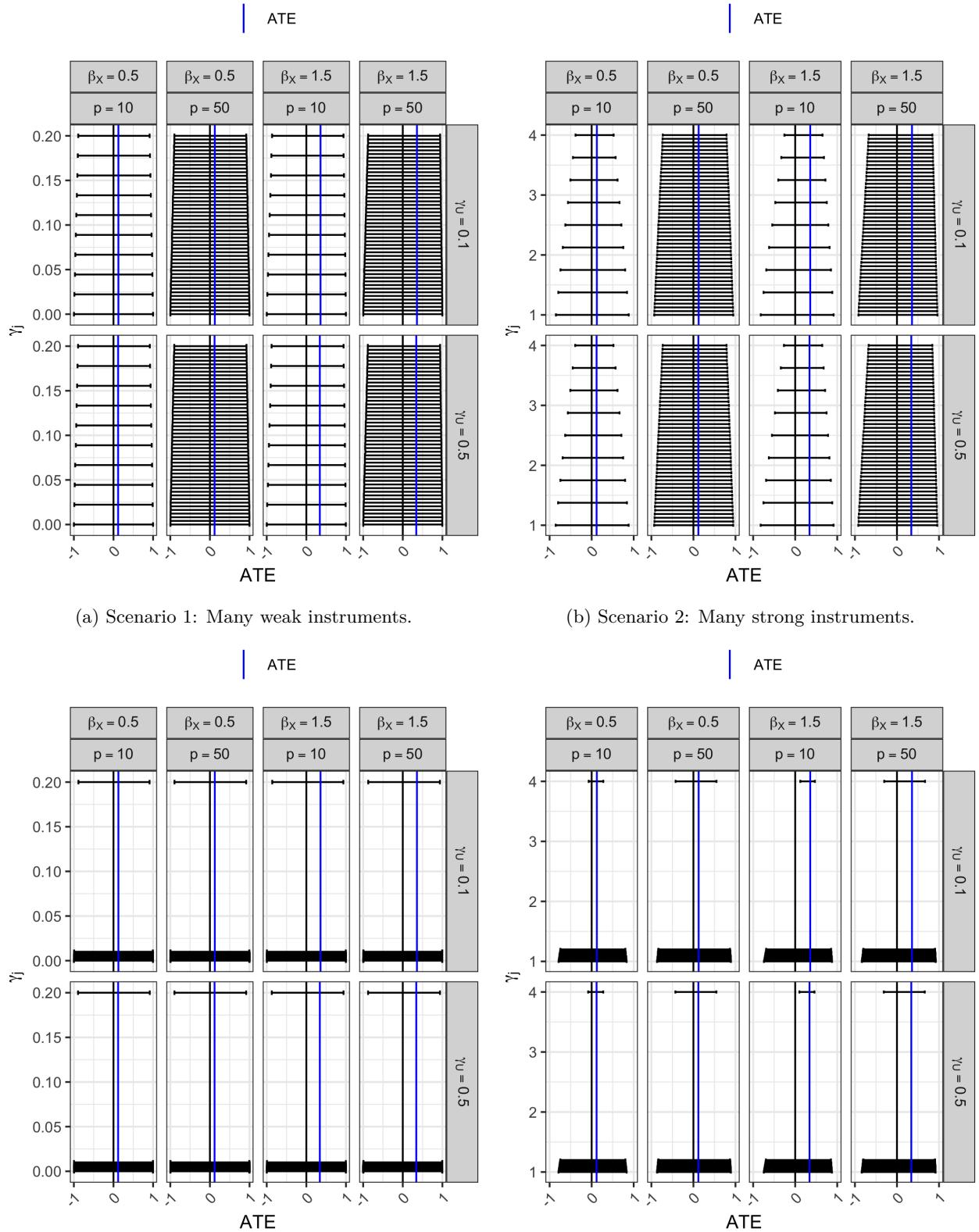


Figure 3: Two-sample bounds for the 10 or 50 instruments from each of the four scenarios. Similar patterns were observed for other values of β_X ; see Appendix D for details.

4 What can you do with summary-level data for bounds? A Quasi-Bayesian Path to More Information

Our method to creating more informative bounds from two-sample MR rests on creating a plausible range of the joint distribution of the outcome and the exposure given the instrument Z , $P(X = x, Y = y|Z = z)$. The plausible range of the joint distribution is informed by quantities available from two-sample MR studies, specifically $P(X = x|Z = z)$ and $P(Y = y|Z = z)$, and a uniform prior on unknown quantities subject to IV assumptions. We then compute an IV bound for each of the plausible sets of $P(X = x, Y = y|Z = z)$ by Balke and Pearl (1997) and Richardson and Robins (2014). In short, the approach addresses the question “had we observed one-sample data that satisfies the constraints of the two-sample data we currently have, could we have detected the presence of an exposure effect?”

To formalize our method, we start by writing the joint conditional distribution $P(X = x, Y = y|Z = z)$ as a function of the marginal conditional distributions $P(X = x|Z = z)$ and $P(Y = y|Z = z)$ and the conditional covariance of the exposure X and Y given $Z = z$ $\text{Cov}(X, Y|Z = z)$ for each z

$$P(X = x, Y = y|Z = z) = P(X = x|Z = z)P(Y = y|Z = z) + (2 \cdot I[x = y] - 1)\text{Cov}(X, Y|Z = z). \quad (4)$$

Because $\text{Cov}(X, Y|Z = z)$ is impossible to estimate from two-sample data, we instead propose to put a prior on this quantity. This prior must not only produce a proper probability distribution of $(X, Y|Z)$, but also satisfy the verifiable constraints (5) from the IV assumptions. Specifically, by the definition of a proper probability distribution, $\text{Cov}(X, Y|Z = z)$ must satisfy

$$\begin{aligned} \max_z & \left\{ \begin{array}{l} -P(X = 1|Z = z)P(Y = 1|Z = z) \\ -P(X = 0|Z = z)P(Y = 0|Z = z) \\ P(X = 1|Z = z)P(Y = 0|Z = z) - 1 \\ P(X = 0|Z = z)P(Y = 1|Z = z) - 1 \end{array} \right\} \\ & \leq \text{Cov}(X, Y|Z = z) \leq \\ & \min_z \left\{ \begin{array}{l} 1 - P(X = 1|Z = z)P(Y = 1|Z = z) \\ 1 - P(X = 0|Z = z)P(Y = 0|Z = z) \\ P(X = 1|Z = z)P(Y = 0|Z = z) \\ P(X = 0|Z = z)P(Y = 1|Z = z) \end{array} \right\} \end{aligned}$$

Additionally, by the IV inequality constraints, for any pair of $(z_1, z_2) \in \{0, 1, 2\} \times \{0, 1, 2\}$, the values of $\text{Cov}(X, Y|Z = z_1)$ and $\text{Cov}(X, Y|Z = z_2)$ must satisfy

$$\begin{aligned} \max & \left\{ \begin{array}{l} -P(X = 0|Z = z_1)P(Y = 0|Z = z_1) - P(X = 0|Z = z_2)P(Y = 1|Z = z_2) \\ P(X = 1|Z = z_1)P(Y = 0|Z = z_1) + P(X = 1|Z = z_2)P(Y = 1|Z = z_2) - 1 \\ P(X = 0|Z = z_2)P(Y = 0|Z = z_2) + P(X = 0|Z = z_1)P(Y = 1|Z = z_1) - 1 \\ -P(X = 1|Z = z_2)P(Y = 0|Z = z_2) - P(X = 1|Z = z_1)P(Y = 1|Z = z_1) \end{array} \right\} \\ & \leq \text{Cov}(X, Y|Z = z_1) - \text{Cov}(X, Y|Z = z_2) \leq \\ \min & \left\{ \begin{array}{l} 1 - P(X = 0|Z = z_1)P(Y = 0|Z = z_1) - P(X = 0|Z = z_2)P(Y = 1|Z = z_2) \\ P(X = 1|Z = z_1)P(Y = 0|Z = z_1) + P(X = 1|Z = z_2)P(Y = 1|Z = z_2) \\ P(X = 0|Z = z_2)P(Y = 0|Z = z_2) + P(X = 0|Z = z_1)P(Y = 1|Z = z_1) \\ 1 - P(X = 1|Z = z_2)P(Y = 0|Z = z_2) - P(X = 1|Z = z_1)P(Y = 1|Z = z_1) \end{array} \right\} \end{aligned}$$

Then, we sequentially sample values of $\text{Cov}(X, Y|Z = 0)$, $\text{Cov}(X, Y|Z = 1)$, $\text{Cov}(X, Y|Z = 2)$, such that the above inequalities plus the existing constraints in (5) are satisfied. Then, among samples of $\text{Cov}(X, Y|Z =$

$0)$, $\text{Cov}(X, Y|Z = 1)$, $\text{Cov}(X, Y|Z = 2)$ that satisfy the constraints, we calculate the joint distribution of $P(X = x, Y = y|Z = z)$ using (4), leading us to a plausible set of the joint distribution $P(X = x, Y = y|Z = z)$.

For each plausible set of the joint distribution of $P(X = x, Y = y|Z = z)$, we use the IV bounds by Balke and Pearl (1997) and Richardson and Robins (2014) from one-sample IV studies to obtain a bound for the ATE. If a large number of the IV bounds do not cover zero, then there is some evidence for a non-zero exposure effect and the only reason we are not able to detect this effect is due to the limitations of the two-sample design. However, if a large number of these bounds do cover zero, there is less evidence for a non-zero causal effect or that utilizing bound-based approaches to obtain some information about the ATE may be a hopeless exercise.

We can also extend our method to handle multiple IVs by simply repeating the above method for each proposed instrument and taking intersections of the one-sample IV bounds. This builds on one assumption that the above sampling are done independently for each instrument; in other words, the assumption implicitly assumes that the covariances of X and Y given Z_1 are independent of the covariances of X and Y given Z_2 . For additional details, see Appendix E.

Finally, we remark that the proposed method above can be thought of as using an empirically bayesian framework for partially identified sets. Specifically, our procedure generates a posterior distribution of IV bounds given the marginalized probabilities from two-sample data (i.e. the likelihood) and a uniform, flat prior on the unknown quantities $\text{Cov}(X, Y|Z = z)$. The constraints that we impose on $\text{Cov}(X, Y|Z = z)$ are almost empirically Bayesian in that they are informed by data from two-sample MR.

4.1 Single Instrument

We illustrate our proposed method from the previous section by considering nine hypothetical MR studies, each using one instrument. An illustration of the sampling of intersection bounds is presented in Appendix F.

Table 3 presents nine different sets of values of the marginal distributions $P(Y|Z)$ and $P(X|Z)$ and Figure 4 shows the resulting one-sample IV bounds from our method.

Table 3: Values of $P(X = 1|Z = z)$ and $P(Y = 1|Z = z)$ used to illustrate our quasi-bayesian approach. These are presented with $\{P(X = 1|Z = 0), P(X = 1|Z = 1), P(X = 1|Z = 2)\}$ on the first row, and $\{P(Y = 1|Z = 0), P(Y = 1|Z = 1), P(Y = 1|Z = 2)\}$ on the second row.

| | Column 1 | Column 2 | Column 3 |
|-------|---------------------------|---------------------------|---------------------------|
| Row A | $\{0.125, 0.399, 0.080\}$ | $\{0.244, 0.275, 0.185\}$ | $\{0.603, 0.469, 0.310\}$ |
| | $\{0.699, 0.840, 0.742\}$ | $\{0.238, 0.089, 0.146\}$ | $\{0.638, 0.346, 0.719\}$ |
| Row B | $\{0.886, 0.968, 0.874\}$ | $\{0.139, 0.441, 0.334\}$ | $\{0.901, 0.909, 0.935\}$ |
| | $\{0.805, 0.822, 0.951\}$ | $\{0.179, 0.359, 0.559\}$ | $\{0.821, 0.810, 0.905\}$ |
| Row C | $\{0.175, 0.079, 0.365\}$ | $\{0.493, 0.911, 0.085\}$ | $\{0.434, 0.045, 0.733\}$ |
| | $\{0.599, 0.358, 0.087\}$ | $\{0.360, 0.480, 0.441\}$ | $\{0.747, 0.370, 0.169\}$ |

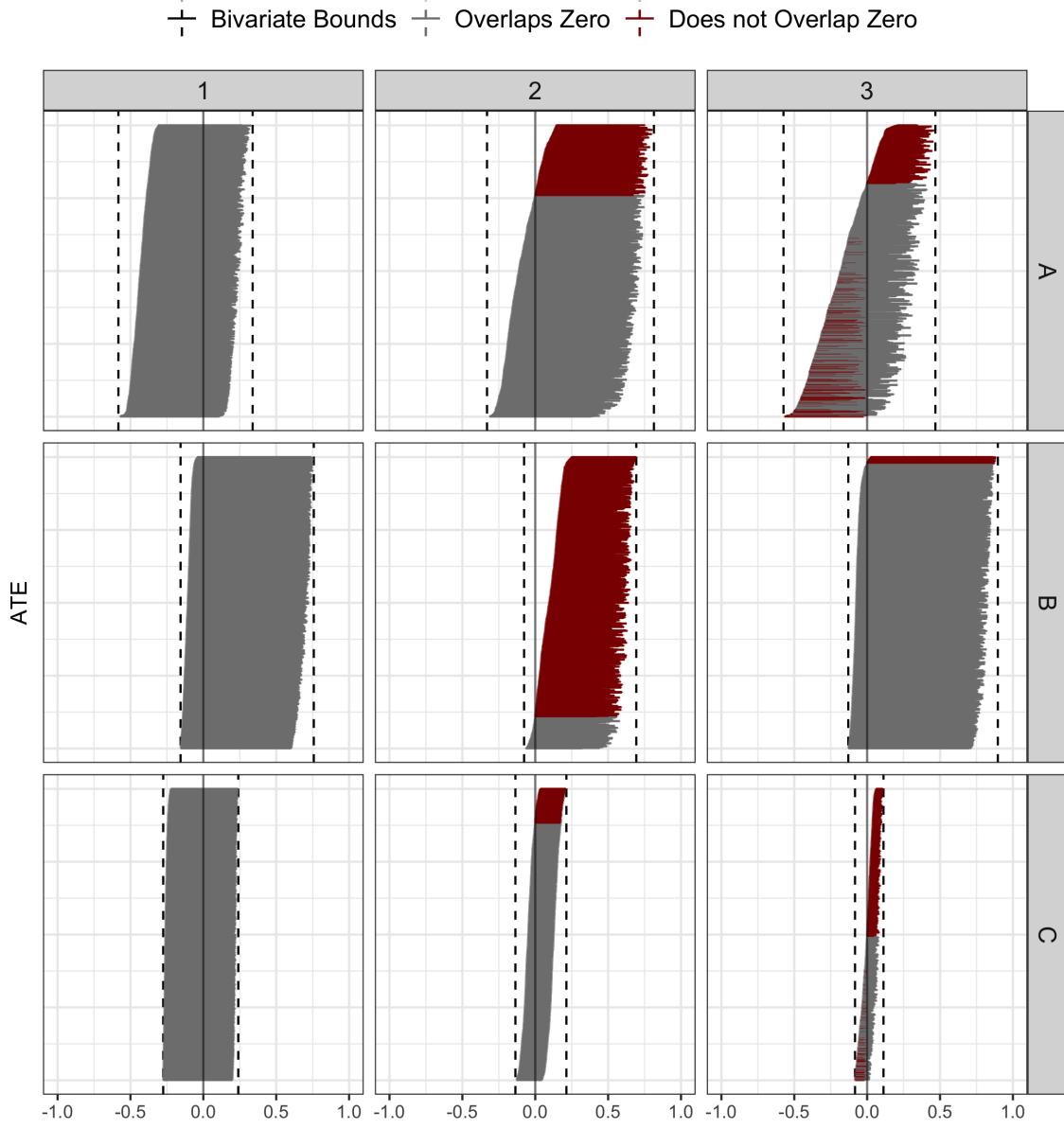


Figure 4: One-sample and two-sample bounds. Even similar bivariate distributions can result in very different insights.

Row A of Figure 4 shows three scenarios where the two-sample bounds are all more or less centered around zero with similar widths. However, the conclusions are rather different. Column 1 shows no one-sample bounds would allow us to determine the presence of a non-zero causal effect. Column 2 indicates that about % of the one-sample IV bounds does not contain 0 while for column 3 that number is approximately %. However, while the direction of the effect is always the same for column 2 (positive), it varies for column 3.

Row B illustrates three scenarios where the two-sample bounds are centered well above zero and have large widths. Here, we see one case where we have no hope of determining direction from the one-sample bounds (column 1), one case where we are most likely to be able to determine the direction of the ATE to be positive from the one-sample bounds (column 2), and one case where we are rather unlikely to be able to determine the direction of the ATE from the one-sample bounds (column 3).

Row C is similar to row A in that all the two-sample bounds are centered around 0, but the width of the two-sample bounds are narrow. The three columns indicate similar conclusions as seen in row a. This shows that even with rather narrow two-sample bounds centered around 0, the one-sample bounds may still be have to reveal some information about presence as well as the direction of the exposure effect.

Despite showing promise about studying the ATE, some caution should be exercised when interpreting the proportion of one-sample bounds not containing 0 from our method. In particular, a scenario like the one resulting in the bounds presented in row B, column 2 only provides information about the one-sample bounds if our prior on $\text{Cov}(X, Y|Z)$ is correctly specified. Under this prior, it tells us that it is much more likely that the ATE is positive. If the prior is mis-specified whereby most one-sample bounds cover negative values of the ATE, a negative value of the ATE is possible. But, even in this case, if the ATE is in fact negative, our method does rule out the possibility of one-sample bounds being able to ascertain this because all one-sample bounds covering a negative ATE also covers 0.

5 Data Analysis

We present two example analyses to demonstrate our findings about the behavior of bounds and our proposed method to obtain more informative bounds. Our first analysis studies the effect of smoking on lung cancer and our second analysis studies the effect of self-reported high cholesterol on incidence of heart attack. The causal effects underlying both analyses are well-established and serve as positive controls. The effect of smoking on lung cancer is known to be strong and positive. The exact nature of the causal connection between high cholesterol and coronary heart disease is still being discussed (Holmes et al. 2015; Richardson et al. 2020), but some meta-analyses of randomized clinical trials of the effect of cholesterol-lowering medication suggest a causal relationship (Cholesterol Treatment Trialists' (CTT) Collaborators 2012; “Efficacy and safety of cholesterol-lowering treatment” 2005). In both cases, we explore the nonparametric bounds obtained from two-sample designs and assess what conclusions are attainable based on bound-based approaches.

The data to study both effects was obtained from the UK Biobank data curated at the IEU GWAS database, which is available in R through the `TwoSampleMR` package (Hemani et al. 2018). Specifically, data on smoking was obtained from data entry with ID `ukb-d-20116_0`, data on lung cancer from entry with ID `ukb-d-40001_C349`, data on cholesterol from entry with ID `ukb-a-108`, and data on heart attack from entry with ID `ukb-a-434`. We followed the defaults of the R package where linkage disequilibrium based clumping ($r^2 \geq 0.001$ within a 10,000 kb window using $p < 5 \times 10^{-8}$ as the level of significance) were performed such that only independent instruments with significant associations are returned. The data was harmonized to make sure that the effects of the SNPs on exposure and outcome were measured with the same allele as reference. Afterwards, we obtain the estimated coefficients from previous GWAS experiments corresponding to the effects of the SNPs on the exposure, and the outcome from a logistic model. Since estimates of the intercept are included in these reported results, but marginal proportions of the outcome, exposure, and allele frequencies are known, we find the intercepts by solving $P(X = 1) = \sum_{z=0}^2 \text{logit}(\beta_0 + \hat{\beta}_1 \cdot z) \cdot P(Z_j = z)$ and $P(Y = 1) = \sum_{z=0}^2 \text{logit}(\gamma_0 + \hat{\gamma}_1 \cdot z) \cdot P(Z_j = z)$ for β_0 and γ_0 , respectively. Overall, the `TwoSampleMR` package along with our estimates of the intercept allowed us to calculate $P(Y = 1|Z_j = z)$ and $P(X = 1|Z_j = z)$ for every j and $z = 0, 1, 2$; see [link to vignette showing analysis on pkgdown page] for the code.

5.1 Effect of Smoking on Chance of Lung Cancer

Our MR analysis of smoking’s effect on lung cancer is based on bounds 84 genetic variants as instruments; detailed information on the 84 instruments can be found in the Appendix F. On average, the instrument strength is around 0.0042, with the strongest instrument having $ST = 0.0091$; this is much smaller than the $ST = 0.5$ needed to guarantee narrow bounds. As such, the two-sample bounds in Figure 5 are rather wide; all of them have width greater than 1 and they convey no truly useful information about smoking’s effect on lung cancer. Additionally, even after applying our method to get more informative bounds, we find that we are unable to determine the direction of the ATE; see Figure 6. In the Appendix, we also show that aggregating bounds through intersections (Figure 19) show similar results.

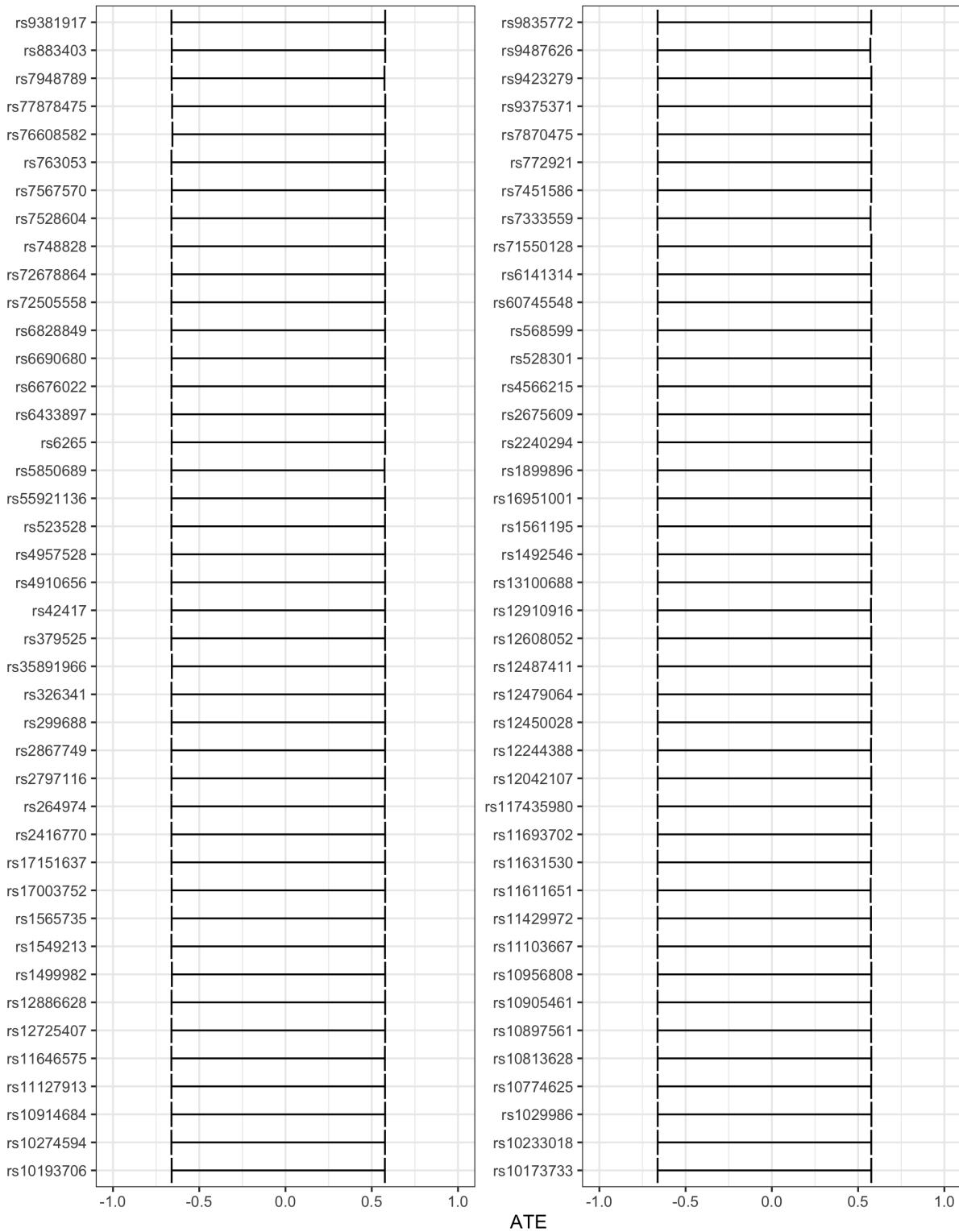


Figure 5: Nonparametric bounds on the average treatment effect of smoking on lung cancer.

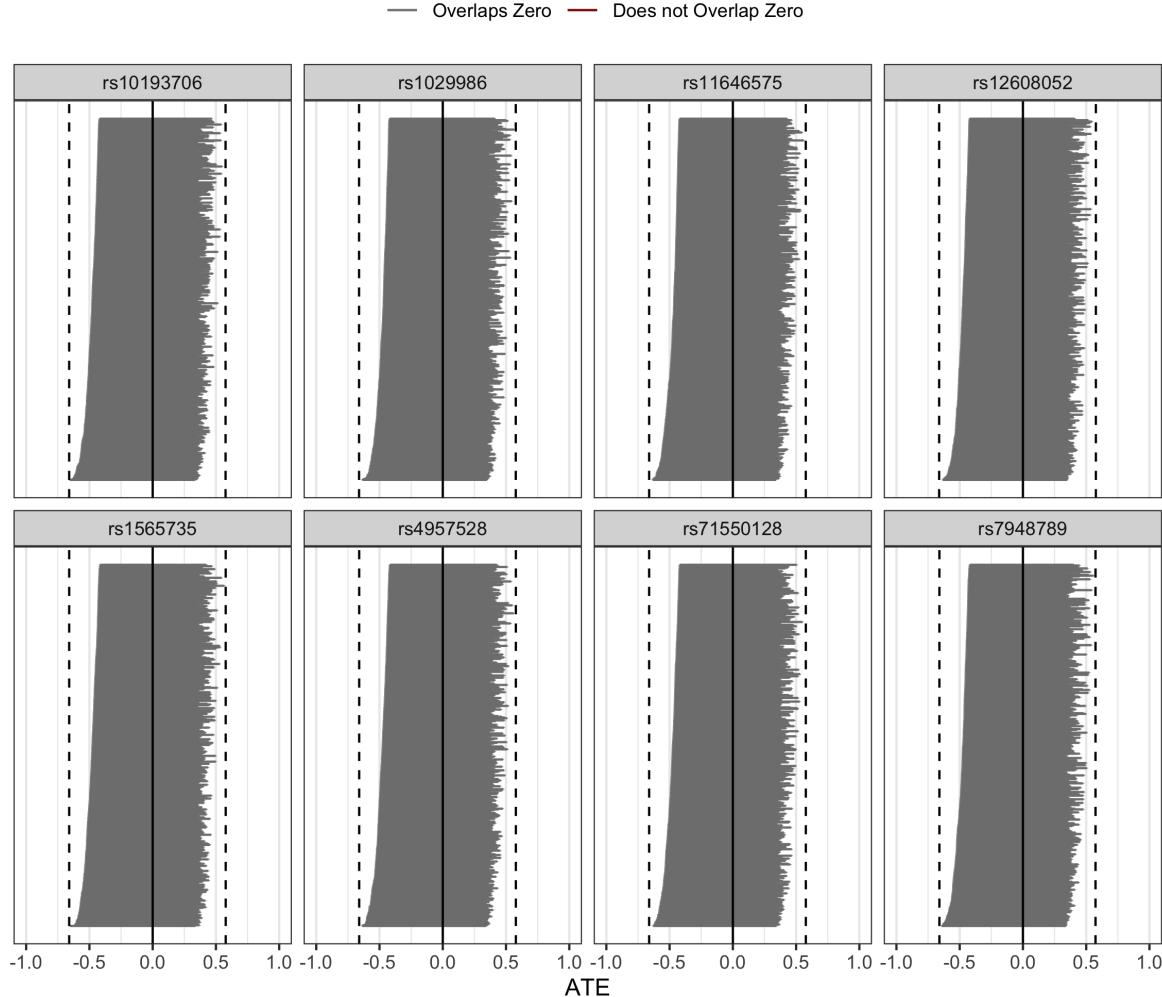


Figure 6: 500 sets of bounds of the average treatment effect of smoking on lung cancer for six of the 84 SNPs. Each bound is based on a set of values for the trivariate distribution randomly sampled. Bounds are color coded to show if they overlap 0 (grey) or do not (red). All bounds overlap 0.

The result is a cause for concern since it is well established that smoking has a strong causal effect on the chances of developing lung cancer Cornfield et al. (1959). The fact that we are unable to say anything about the ATE in this case does not leave much hope in terms of future discoveries based on nonparametric IV bounds from two-sample MR studies. Even more concerning is the fact that our methodology reveals that had we obtained one-sample MR data, we would still be unsuccessful in determining the direction of the effect based on a bound-based analysis of the ATE. In short, while nonparametric bounds allow us to make little assumptions about the data and as such, is robust to some common modeling assumptions in MR, they are often too conservative and are not suited for MR studies with many weak instruments.

5.2 Effect of High Cholesterol on Chance of Heart Attack

Our MR analysis of cholesterol's effect on heart attack is based on 54 genetic variants as instruments; detailed information on the 54 instruments can be found in the Appendix F. On average, the instrument strength is around 0.0005, with the strongest instrument having ST = 0.0022; again, much smaller than the ST = 0.5 needed to guarantee narrow bounds. As was the case in the previous section, all of the two-sample bounds

in Figure 7 have width close to 1, and provide no useful information about the causal effect of interest. However, the two-sample bounds here are all centered close to 0.35, which does give us a very large lower bound of the effect. That is, if high cholesterol has a negative effect on heart attack (i.e. lowers the risk as opposed to increases the risk), that effect is not very large.

Maybe a bit surprising, our proposed method suggests that one-sample bounds would be only minimally more informative. Figure 8 shows such potential bounds for eight of the SNPs; see Appendix for the full figure, where we also see that aggregating bounds through intersections once again provides no further information (Figure 25). Given the lopsided nature of the bivariate bounds, it seems reasonable to have thought that

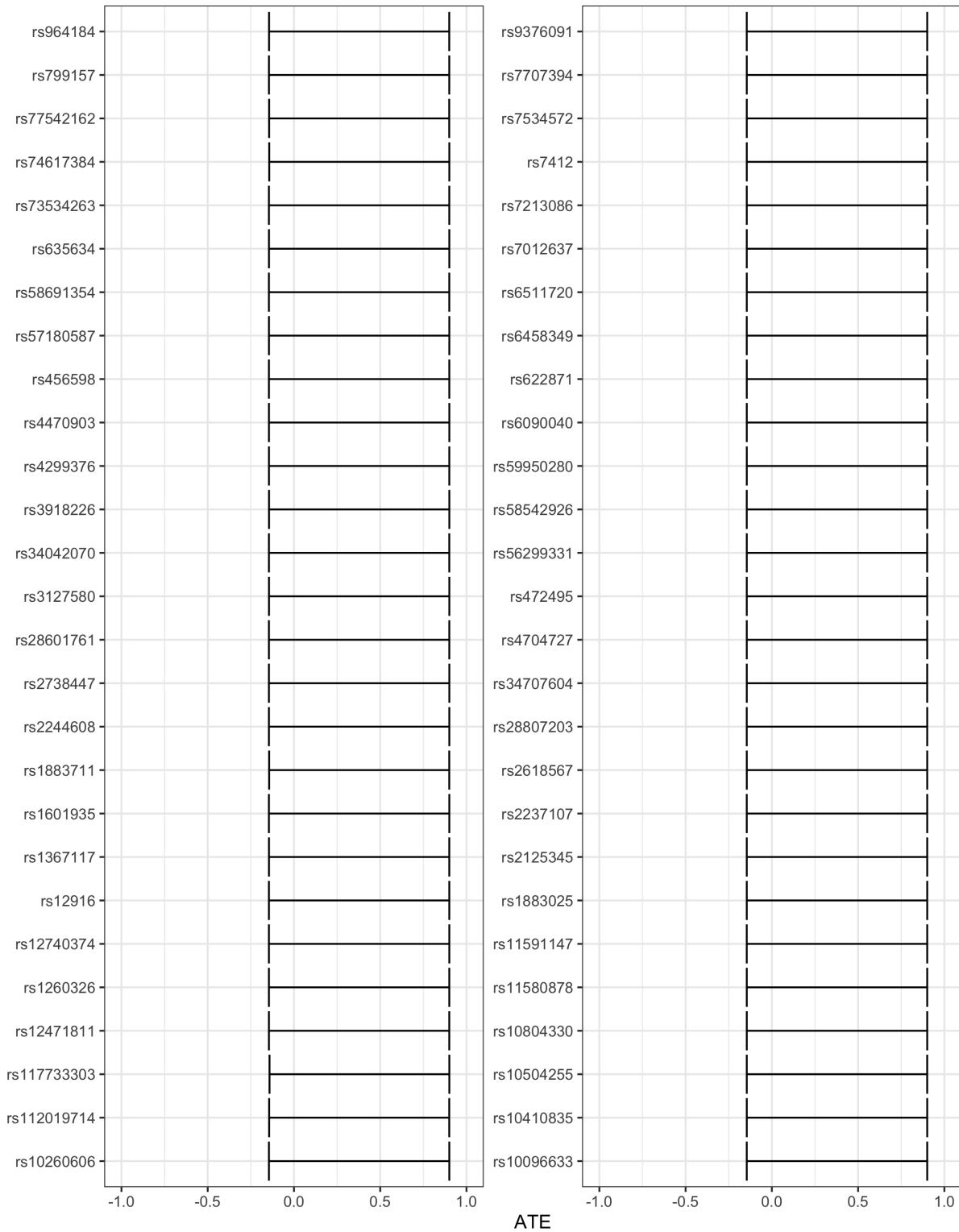


Figure 7: Nonparametric bounds on the average treatment effect of high cholesterol on heart attack.

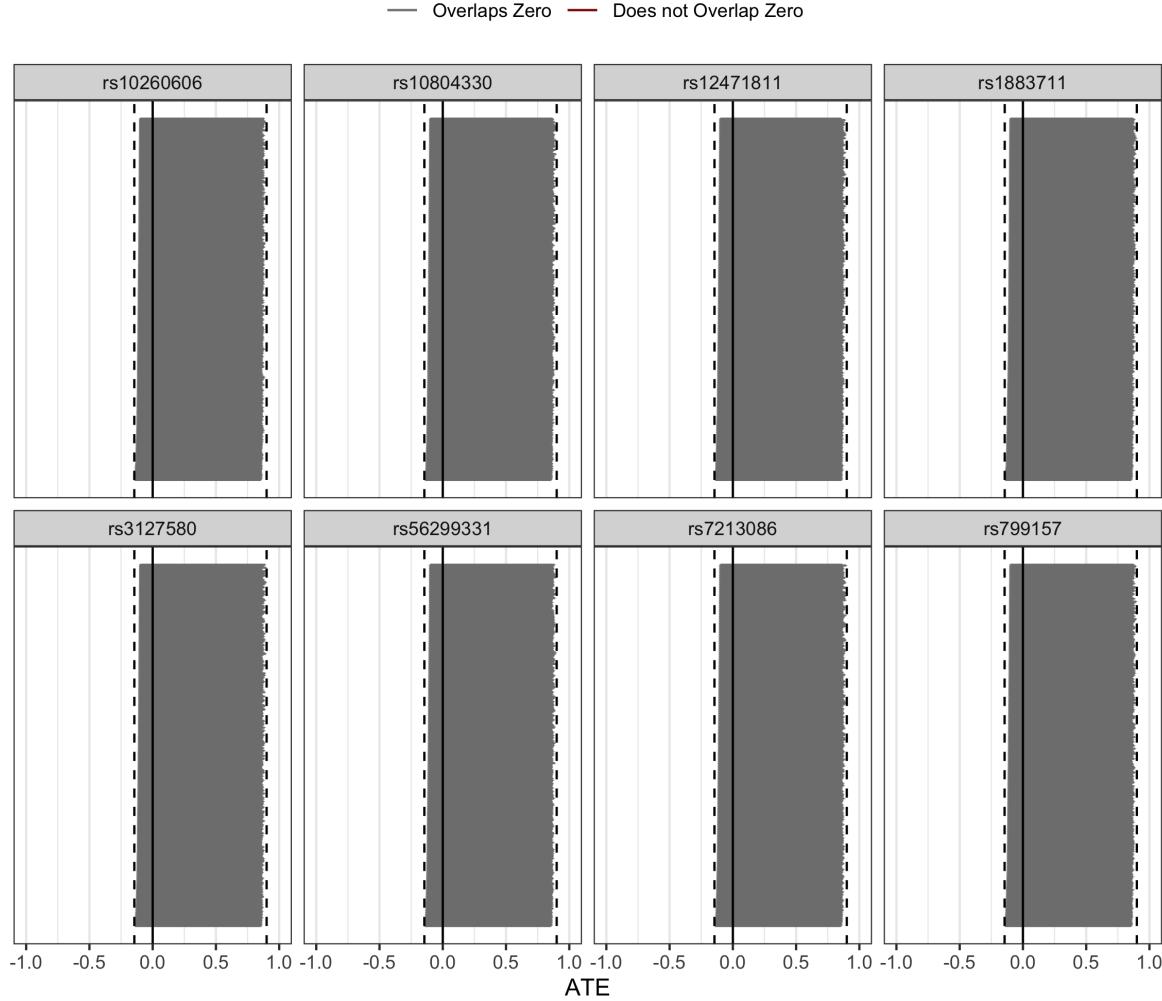


Figure 8: 500 sets of bounds of the average treatment effect of high cholesterol on heart attack for eight of the 54 SNPs. Each bound is based on a set of values for the trivariate distribution randomly sampled. Bounds are color coded to show if they overlap 0 (grey) or do not (red). All bounds overlap 0.

6 Conclusion and Practical Considerations

Nonparametric bounds are without a doubt an attractive concept. With a minimal set of assumptions they let us obtain bounds on the average treatment effect. However, as we have seen here, in typical MR studies with two-sample summary data and many weak instruments, bounds may be too uninformative to make meaningful conclusions about the ATE. Specifically, nonparametric bounds in usual one-sample settings data come with very nice guarantees, such as the width always being less than 1. But, in Mendelian randomization analyses with two-sample data, we lose the strong guarantees on the maximum width of the bounds and strong assumptions about the strength of the IV are often required to make sure that the width is less than 1. Even aggregating information from many instruments through simple intersections will only be as good as using a single strong instrument.

To address the limitations that the two-sample design has in terms of producing informative bounds, we outline an approach to generate a plausible range of one-sample bounds that are in agreement with the two-sample data at hand. This gives us the opportunity to assess the range of conclusions that can be

drawn from bound-based approaches had we had one-sample data. We applied our method to a few different settings of two-sample data and showed the range conclusions about the ATE that can be drawn from it. This exercise also highlighted a significant loss of information in two-sample designs compared to one-sample designs.

need a better way to tie these two paragraphs to rest

To assess the usefulness of two-sample nonparametric bounds in Mendelian randomization analyses, we considered two examples. In the first example, we aimed at finding bounds on the effect of smoking on the chances of developing lung cancer. It has been well established that there is a rather strong causal effect of smoking on the chances of developing lung cancer. Unfortunately, all instruments available were very weak with the strongest instrument having a strength of less than 0.01. This result in bounds that provide very little information. Our approach suggests that even one-sample bounds would not provide much extra information, and in particular would also fail to recover the direction of the effect.

In our second example, we explored the effect of high cholesterol on the chances of heart attack. Unfortunately, the instruments were here even weaker than in our first example, and once again the nonparametric bounds were unable to determine the direction of the effect. Our approach indicates that one-sample bounds would bring only marginal improvement. This example illustrates a scenario where the loss in information by going to two-sample rather than one-sample data is minimal, in that the difference between the two-sample bounds and the possible one-sample bounds is almost non-existing.

Using nonparametric bounds in two-sample MR studies seem a promising idea since many MR analysis rely on a host of potentially unjustifiable modeling assumptions. But, as we have seen above, the nonparametric nature and the two-sample design can make these bounds often meaningless in practice. Nevertheless, one potential use case of nonparametric bounds in two-sample MR studies could be when one has prior knowledge about the direction of the effect, but wish to get a better sense of the magnitude. By knowing the sign of the effect a priori, nonparametric bounds can provide an upper limit on this magnitude. This is especially useful in cases where the exposure is known to cause harm or benefit, for example in our smoking lung cancer example where the direction of the effect of smoking on lung cancer is well known and an upper bound on this effect would tell investigators about the maximum possible effect that smoking could have on increasing the propensity of lung cancer.

A Proof of Theorem 3.1

First of all, we note that the bounds found using the approach previously described when we impose both of the mentioned monotonicity assumptions are as follows:

$$\begin{aligned} \max & \left\{ \begin{array}{l} -P(Y = 0|Z = 2) - P(Y = 1|Z = 0) + P(X = 0|Z = 0) - P(X = 0|Z = 2) \\ P(Y = 0|Z = 0) - 2 \cdot P(Y = 0|Z = 2) - P(X = 0|Z = 2) \\ -P(Y = 0|Z = 2) - 2 \cdot P(Y = 1|Z = 0) + P(X = 0|Z = 0) \end{array} \right\} (L1) \\ & \leq ATE \leq \\ \min & \left\{ \begin{array}{l} 1 + P(Y = 0|Z = 0) - P(X = 0|Z = 0) \\ 1 + P(Y = 0|Z = 0) - P(Y = 0|Z = 2) - P(X = 0|Z = 0) + P(X = 0|Z = 2) \\ 1 - P(Y = 0|Z = 2) + P(X = 0|Z = 2) \end{array} \right\} (U1) \\ & (U2) \\ & (U3) \end{aligned}$$

This gives us a total of nine different expressions for the width of the bounds. Since we assume monotonicity of the effect of Z on X , the strength simplifies to $ST = P(X = 1|Z = 2) - P(X = 1|Z = 0)$.

Width = U1 - L1

If the upper bound is $U1$, $U1 \leq U2$, which implies $P(Y = 0|Z = 2) - P(X = 0|Z = 2) \leq 0$. Therefore,

$$\begin{aligned} U1 - L1 &= 1 + P(Y = 0|Z = 0) - P(X = 0|Z = 0) + P(Y = 0|Z = 2) + \\ &\quad P(Y = 1|Z = 0) - P(X = 0|Z = 0) + P(X = 0|Z = 2) \\ &= 2 - ST + P(Y = 0|Z = 2) - P(X = 0|Z = 0) \\ &= 2 - 2 \cdot ST + P(Y = 0|Z = 2) - P(X = 0|Z = 2) \leq 2 - 2 \cdot ST. \end{aligned}$$

Width = U2 - L1

$$\begin{aligned} U2 - L1 &= 1 + P(Y = 0|Z = 0) - P(Y = 0|Z = 2) - P(X = 0|Z = 0) + P(X = 0|Z = 2) \\ &\quad + P(Y = 0|Z = 2) + P(Y = 1|Z = 0) - P(X = 0|Z = 0) + P(X = 0|Z = 2) \\ &= 2 - 2 \cdot ST \end{aligned}$$

Width = U3 - L1

Since the upper bound is $U3$, $U3 \leq U2$, which implies $P(X = 0|Z = 0) - P(Y = 0|Z = 0) \leq 0$. Therefore,

$$\begin{aligned} U3 - L1 &= 1 - P(Y = 0|Z = 2) + P(X = 0|Z = 2) + P(Y = 0|Z = 2) + \\ &\quad P(Y = 1|Z = 0) - P(X = 0|Z = 0) + P(X = 0|Z = 2) \\ &= 1 + P(Y = 1|Z = 0) - ST + P(X = 0|Z = 2) \\ &= 2 - 2 \cdot ST + P(X = 0|Z = 0) - P(Y = 0|Z = 0) \leq 2 - 2 \cdot ST. \end{aligned}$$

Width = U1 - L2

Since the upper bound is $U1$, $P(Y = 0|Z = 2) \leq P(X = 0|Z = 2)$. Since the lower bound is $L2$, $L2 \geq L1$, which gives us $1 - P(X = 0|Z = 0) \geq P(Y = 0|Z = 2)$. Therefore,

$$\begin{aligned} U1 - L2 &= 1 + P(Y = 0|Z = 0) - P(X = 0|Z = 0) - P(Y = 0|Z = 0) + 2 \cdot P(Y = 0|Z = 2) + P(X = 0|Z = 2) \\ &= 1 - ST + 2P(Y = 0|Z = 2) \\ &\leq 2 - ST - P(X = 0|Z = 0) + P(X = 0|Z = 2) = 2 - 2 \cdot ST. \end{aligned}$$

Width = U2 - L2

Since the lower bound is $L2$, $1 - P(X = 0|Z = 0) \geq P(Y = 0|Z = 2)$. So,

$$\begin{aligned} U2 - L2 &= 1 - P(X = 0|Z = 0) + P(X = 0|Z = 2) + P(Y = 0|Z = 2) + P(X = 0|Z = 2) \\ &= 1 - ST + P(Y = 0|Z = 2) + P(X = 0|Z = 2) \\ &\leq 2 - 2 \cdot ST. \end{aligned}$$

Width = U3 - L2

Since the lower bound is $L2$, $1 - P(X = 0|Z = 0) \geq P(Y = 0|Z = 2)$. Since the upper bound is $U3$, $P(X = 0|Z = 0) \leq P(Y = 0|Z = 0)$. Therefore,

$$\begin{aligned} U3 - L2 &= 1 - P(Y = 0|Z = 2) + P(X = 0|Z = 2) - P(Y = 0|Z = 0) + 2 \cdot P(Y = 0|Z = 2) + P(X = 0|Z = 2) \\ &= 1 + 2 \cdot P(X = 0|Z = 2) + P(Y = 0|Z = 2) - P(Y = 0|Z = 0) \\ &= 1 - 2 \cdot ST + 2P(X = 0|Z = 0) + P(Y = 0|Z = 2) - P(Y = 0|Z = 0) \\ &\leq 2 - 2 \cdot ST \end{aligned}$$

Width = U1 - L3

Since the upper bound is $U1$, $P(Y = 0|Z = 2) \leq P(X = 0|Z = 2)$. Since the lower bound is $L3$, $L3 \geq L1$, which implies $P(Y = 1|Z = 0) \leq P(X = 0|Z = 2)$. So,

$$\begin{aligned} U1 - L3 &= 2 - P(X = 0|Z = 0) + P(Y = 0|Z = 2) + P(Y = 1|Z = 0) - P(X = 0|Z = 0) \\ &= 2 - 2 \cdot ST - 2 \cdot P(X = 0|Z = 2) + P(Y = 0|Z = 2) + P(Y = 1|Z = 0) \\ &\leq 2 - 2 \cdot ST \end{aligned}$$

Width = U2 - L3

Since the lower bound is $L3$, $P(Y = 1|Z = 0) \leq P(X = 0|Z = 2)$

$$\begin{aligned} U2 - L3 &= 2 - 2 \cdot P(X = 0|Z = 0) + P(X = 0|Z = 2) + P(Y = 1|Z = 0) \\ &= 2 - ST + P(Y = 1|Z = 0) - P(X = 0|Z = 0) \\ &= 2 - 2 \cdot ST + P(Y = 1|Z = 0) - P(X = 0|Z = 2) \leq 2 - 2 \cdot ST \end{aligned}$$

Width = U3 - L3

Since the lower bound is $L3$, $P(Y = 1|Z = 0) \leq P(X = 0|Z = 2)$. Since the upper bound is $U3$, $1 - P(X = 0|Z = 0) \geq P(Y = 1|Z = 0)$. Therefore,

$$\begin{aligned} U3 - L3 &= 1 + P(X = 0|Z = 2) + 2 \cdot P(Y = 1|Z = 0) - P(X = 0|Z = 0) \\ &\leq 1 - ST + P(X = 0|Z = 2) + 1 - P(X = 0|Z = 0) \\ &= 2 - 2 \cdot ST. \end{aligned}$$

B Bounds on Average Treatment Effect

We briefly review the method presented by Ramsahai (2012) to bound the average treatment effect using two-sample summary data. Let $\vec{\tau}^* = (P(Y = 1|X = 0, U), P(Y = 1|X = 1, U), P(X = 1|Z = 0, U), \dots, P(X = 1|Z = k - 1, U)) \in [0, 1]^{2+k}$ and $\vec{v}^* = (P(Y = 0|Z = 0, U), \dots, P(Y = 1|Z = k - 1, U), P(X = 0|Z = 0, U), \dots, P(X = 1|Z = k - 1, U), \alpha^*)$ where

$$\alpha^* = P(Y = 1|X = 1, U) - P(Y = 1|X = 0, U).$$

Since $U \perp Z$, $E_U[P(X = x|Z = z, U)] = P(X = x|Z = z)$ and $E_U[P(Y = y|Z = z, U)] = P(Y = y|Z = z)$. Let $\vec{v} = E_U[\vec{v}^*] = (P(Y = 0|Z = 0), \dots, P(Y = 1|Z = k - 1), P(X = 0|Z = 0), \dots, P(X = 1|Z = k - 1), \alpha)$, where

$$\begin{aligned}\alpha &= E_U[P(Y = 1|X = 1, U) - P(Y = 1|X = 0, U)] \\ &= E[Y^1] - E[Y^0] = \text{ATE}.\end{aligned}$$

Note that while $\vec{\tau}^*$ and \vec{v}^* are both entirely unobservable, \vec{v} consists of k observable values, and one unobservable value, the ATE.

By the exclusion restriction, we have

$$P(X = x, Y = y|Z = z, U) = P(Y = 1|X = x, U)P(X = x|Z = z, U),$$

which means we can define a mapping $f : [0, 1]^{2+k} \mapsto \mathcal{V}$ such that $f(\vec{\tau}^*) = \vec{v}^*$ as

$$f(y_0, y_1, x_0, x_1, \dots, x_{k-1}) = \begin{pmatrix} (1 - y_0) \cdot (1 - x_0) + (1 - y_1) \cdot x_0 \\ y_0 \cdot (1 - x_0) + y_1 \cdot x_0 \\ \vdots \\ (1 - y_0) \cdot (1 - x_{k-1}) + (1 - y_1) \cdot x_{k-1} \\ y_0 \cdot (1 - x_{k-1}) + y_1 \cdot x_{k-1} \end{pmatrix}$$

We define $\mathcal{V} = f([0, 1]^{2+k})$.

Since $\vec{v} = E_U[\vec{v}^*]$, \vec{v} must be a convex combination of \vec{v}^* . Let \mathcal{H} be the convex hull of \mathcal{V} . Then \vec{v} will be in \mathcal{H} .

Now, let $\hat{\mathcal{T}}$ be the set of extreme vertices of $[0, 1]^{2+k}$, $\hat{\mathcal{V}} = f(\hat{\mathcal{T}})$, and $\hat{\mathcal{H}}$ be the convex hull of $\hat{\mathcal{V}}$. By Theorem 1 in Appendix B of Ramsahai (2012), $\mathcal{H} = \hat{\mathcal{H}}$. This means that $\vec{v} \in \hat{\mathcal{H}}$. Utilizing a program such as Polymake, we can describe \mathcal{H} with a set of inequalities, which give us constraints that \vec{v} must satisfy.

This means that we can obtain inequalities that the components of \vec{v} must satisfy by describing the extreme vertices of $[0, 1]^{2+k}$, map them to \mathcal{V} using the relatively simple function f , and then use polymake to find inequalities that characterize the convex hull of $f([0, 1])^{2+k}$. This gives us a set of inequalities involving the components of \vec{v} . Some of these will be verifiable, as they will not include the only unobservable quantity α . Others will not be verifiable, but will allow us to obtain bounds on the unobservable quantity α using the observable entries of \vec{v} .

Following the approach from Ramsahai (2012) as outlined above, we obtain bounds on the average treatment effect from the quantities $P(X = 1|Z = z)$ and $P(Y = 1|Z = z)$, $z = 0, 1, 2$. To do so, we first write down the most extreme values of each of $P(Y = 1|X = x, U)$ and $P(X = x|Z = z, U)$ for all $x = 0, 1$, $z = 0, 1, 2$. Since these are probabilities, the extreme values are 0 and 1.

Table 4: Most extreme values of $P(Y = 1|X = x, U)$ and $P(X = 1|Z = z, U)$. Here, PY1XxU = $P(Y = 1|X = x, U)$ and PX1ZzU = $P(X = 1|Z = z, U)$.

| PY1X0U | PY1X1U | PY1Z0U | PX1Z1U | PX1Z2U |
|--------|--------|--------|--------|--------|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 1 | 1 | 0 |
| 0 | 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 | 0 |
| 0 | 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 |
| 1 | 1 | 0 | 1 | 0 |
| 1 | 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 1 | 0 |
| 1 | 1 | 1 | 1 | 1 |

By applying the function f , as presented in (B), to each row, we get the most extreme vertices of $P(X = x|Z = z, U)$ and $P(Y = y|Z = z, U)$ for all $x = 0, 1$, $y = 0, 1$ and $z = 0, 1, 2$.

Table 5: Most extreme values of $P(Y = y|Z = z)$ and $P(X = x|Z = z)$. Here, PYyZz = $P(Y = y|Z = z)$, PXxZz = $P(X = x|Z = z)$, and $\alpha = P(Y = 1|X = 1, U) - P(Y = 1|X = 0, U)$.

| PY0Z0 | PY0Z1 | PY0Z2 | PY1Z0 | PY1Z1 | PY1Z2 | PX0Z0 | PX0Z1 | PX0Z2 | PX1Z0 | PX1Z1 | PX1Z2 | α |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|
| 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | -1 |
| 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | -1 |
| 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 |

| PY0Z0 | PY0Z1 | PY0Z2 | PY1Z0 | PY1Z1 | PY1Z2 | PX0Z0 | PX0Z1 | PX0Z2 | PX1Z0 | PX1Z1 | PX1Z2 | α |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------|
| 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 |
| 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | -1 |
| 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | -1 |
| 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | -1 |
| 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | -1 |
| 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 |
| 0 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | -1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | -1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 |
| 0 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | -1 |
| 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | -1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |

Theorem 1 of Ramsahai (2012) tells us that the values of $P(X = 1|Z = z)$, $P(Y = 1|Z = z)$, $z = 0, 1, 2$ must lie in the convex hull of the vertices given by the rows in Table 5. This means that the vector of these values must be a convex combination of the rows in said table. Using this with the fact that they must sum to 1 is what enables us to use polymake to find inequalities that the values of $P(X = 1|Z = z)$, $P(Y = 1|Z = z)$, and α must satisfy. In this particular case, these are as presented below. This table should be read as rows of coefficients for which it holds that $\sum_{z=0}^2 c_{X1Zz} \cdot P(X = 1|Z = z) + \sum_{z=0}^2 c_{Y0Zz} \cdot P(Y = 0|Z = z) + c_{Y1Z0} \cdot P(Y = 1|Z = 0) + c_\alpha \alpha \geq 0$.

Table 6: Results from polymake. Columns with all zeroes have been removed.

| PY0Z0 | PY0Z1 | PY0Z2 | PY1Z0 | PX1Z0 | PX1Z1 | PX1Z2 | c_α |
|-------|-------|-------|-------|-------|-------|-------|------------|
| 2 | 0 | -1 | 0 | 2 | 0 | 0 | -1 |
| 1 | 0 | -1 | 1 | 0 | 0 | 0 | 0 |
| 1 | -1 | 0 | 1 | 0 | 0 | 0 | 0 |
| 1 | -1 | 0 | 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | -1 | 0 | 1 | 0 | 1 | 0 |
| 2 | 0 | -1 | 1 | 1 | 0 | -1 | -1 |
| 2 | -1 | 0 | 1 | 1 | -1 | 0 | -1 |
| 2 | 0 | -2 | 1 | 0 | 0 | 2 | 1 |
| 2 | -1 | 0 | 1 | -1 | 1 | 0 | 1 |
| 4 | 0 | -2 | 3 | 0 | 0 | -2 | -1 |
| 2 | -2 | 0 | 1 | 0 | 2 | 0 | 1 |
| 4 | -1 | 0 | 2 | -2 | 0 | 0 | 1 |
| 4 | 0 | -1 | 2 | -2 | 0 | 0 | 1 |
| 2 | 0 | -1 | 1 | -1 | 0 | 1 | 1 |
| 1 | 0 | -1 | 1 | 0 | 0 | 1 | 1 |

| PY0Z0 | PY0Z1 | PY0Z2 | PY1Z0 | PX1Z0 | PX1Z1 | PX1Z2 | c_α |
|-------|-------|-------|-------|-------|-------|-------|------------|
| 3 | -1 | 0 | 2 | -1 | -1 | 0 | 0 |
| 2 | -1 | 0 | 0 | 2 | 0 | 0 | -1 |
| 4 | -2 | 0 | 3 | 0 | -2 | 0 | -1 |
| 3 | 0 | -1 | 2 | -1 | 0 | -1 | 0 |
| 1 | -1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 1 | -1 | 1 | 1 | 0 | 1 | -1 | 1 |
| 1 | 0 | 0 | 1 | 0 | -1 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 | 0 | -1 | 0 |
| 1 | 0 | 1 | 1 | 0 | 0 | -1 | 1 |
| 2 | -1 | 2 | 2 | 0 | 0 | -2 | 1 |
| 1 | 1 | 0 | 1 | 0 | -1 | 0 | 1 |
| 0 | 1 | 0 | 1 | 1 | -1 | 0 | 1 |
| 0 | 0 | 1 | 1 | 1 | 0 | -1 | 1 |
| 2 | 2 | -1 | 2 | 0 | -2 | 0 | 1 |
| 2 | 1 | -1 | 2 | 0 | -1 | -1 | 0 |
| 2 | -1 | 1 | 2 | 0 | -1 | -1 | 0 |
| 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| 1 | 1 | -1 | 1 | 0 | -1 | 1 | 1 |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 2 | 0 | 0 | 1 | -1 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 | -1 | 0 | 1 | -1 |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| 1 | -1 | 1 | 1 | 0 | -1 | 1 | -1 |
| -1 | 2 | 0 | 0 | 0 | 2 | 0 | -1 |
| 2 | 0 | -1 | 2 | 0 | 0 | -1 | -1 |
| 1 | 0 | 1 | 3 | -2 | 0 | 0 | -1 |
| 1 | 1 | 0 | 2 | -1 | -1 | 0 | 0 |
| 0 | 1 | -1 | 0 | 0 | 1 | 1 | 0 |
| 0 | 1 | 0 | 1 | -1 | 1 | 0 | -1 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| -1 | 0 | 1 | 1 | 2 | 0 | 0 | 1 |
| 3 | -2 | 1 | 3 | 0 | -2 | 0 | -1 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| 0 | -1 | 1 | 0 | 0 | 1 | 1 | 0 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 3 | -2 | 0 | 0 | -1 |
| 1 | 0 | 0 | 1 | -1 | 0 | 0 | 0 |
| 0 | 2 | -1 | 0 | 0 | 2 | 0 | -1 |
| 1 | 0 | 2 | 2 | 0 | 0 | -2 | 1 |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| 1 | -2 | 1 | 1 | 0 | 2 | 0 | 1 |
| 2 | -1 | 0 | 2 | 0 | -1 | 0 | -1 |
| 1 | 1 | -1 | 1 | 0 | 1 | -1 | -1 |
| -1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 | -1 |
| -1 | 0 | 2 | 0 | 0 | 0 | 2 | -1 |
| 1 | 2 | 0 | 2 | 0 | -2 | 0 | 1 |
| 1 | 1 | -2 | 1 | 0 | 0 | 2 | 1 |
| -1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 1 | 0 | -1 |
| 0 | 0 | 1 | 0 | 0 | 0 | 1 | -1 |
| 1 | 0 | 0 | 2 | -1 | 0 | 0 | -1 |

| PY0Z0 | PY0Z1 | PY0Z2 | PY1Z0 | PX1Z0 | PX1Z1 | PX1Z2 | c_α |
|-------|-------|-------|-------|-------|-------|-------|------------|
| -1 | 1 | 0 | 1 | 2 | 0 | 0 | 1 |
| 3 | 1 | -2 | 3 | 0 | 0 | -2 | -1 |
| 0 | -1 | 2 | 0 | 0 | 0 | 2 | -1 |
| 1 | 0 | 1 | 2 | -1 | 0 | -1 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

The matrix presented in the table above simplifies to the following set of bounds on the average treatment effect. These are obtained by considering the rows above where $c_\alpha \neq 0$.

$$\max \left\{ \begin{array}{ll} \max_{i \neq j} & P(Y = 1|Z = i) - 2 \cdot P(Y = 1|Z = j) - 2 \cdot P(X = 1|Z = j) \\ \max_{i \neq j} & P(Y = 1|Z = i) + P(X = 1|Z = i) - P(Y = 1|Z = j) - P(X = 1|Z = j) - 1 \\ \max_{i \neq j} & 2 \cdot P(Y = 1|Z = i) + 2 \cdot P(X = 1|Z = i) - P(Y = 1|Z = j) - 3 \\ \max_i & -P(Y = 1|Z = i) - P(X = 1|Z = i) \\ \max_i & P(Y = 1|Z = i) + P(X = 1|Z = i) - 2 \end{array} \right\}$$

$$\leq \alpha \leq$$

$$\min \left\{ \begin{array}{ll} \min_{i \neq j} & P(Y = 1|Z = i) - 2 \cdot P(Y = 1|Z = j) + 2 \cdot P(X = 1|Z = j) + 1 \\ \min_{i \neq j} & P(Y = 1|Z = i) + 2 \cdot P(Y = 1|Z = j) - 2 \cdot P(X = 1|Z = j) + 1 \\ \min_{i \neq j} & P(Y = 1|Z = i) - P(X = 1|Z = i) + P(X = 1|Z = j) - P(Y = 1|Z = j) + 1 \\ \min_i & P(X = 1|Z = i) - P(Y = 1|Z = i) + 1 \\ \min_i & P(Y = 1|Z = i) - P(X = 1|Z = i) + 1 \end{array} \right\}$$

Furthermore, we obtain the following checkable constraints from the rows where $\alpha = 0$:

$$\min \left\{ \begin{array}{ll} \min_{i \neq j} & P(Y = 1|Z = i) - P(X = 1|Z = i) - P(Y = 1|Z = j) - P(X = 1|Z = j) + 2 \\ \min_{i \neq j} & P(Y = 1|Z = i) + P(X = 1|Z = i) - P(Y = 1|Z = j) + P(X = 1|Z = j) \\ \min_i & P(X = 1|Z = i) \\ \min_i & P(Y = 1|Z = i) \\ \min_i & 1 - P(X = 1|Z = i) \\ \min_i & 1 - P(Y = 1|Z = i) \end{array} \right\} \geq 0 \quad (5)$$

We notice that the constraints from the law of probability are recovered (the last four expressions above) along with 12 non-trivial constraints.

These bounds involve 24 different expressions on both the lower and upper end, making an algebraic exploration of the width very challenging. However, by imposing the two monotonicity assumptions (??) and (??), the bounds reduce to just three on the lower end and three on the upper end. This is done by removing rows in the matrix of extreme vertices where the monotonicity assumptions are violated before using Polymake to get the inequalities. The resulting bounds are presented below.

$$\max \left\{ \begin{array}{l} -P(Y = 0|Z = 2) - P(Y = 1|Z = 0) + P(X = 0|Z = 0) - P(X = 0|Z = 2) \\ P(Y = 0|Z = 0) - 2 \cdot P(Y = 0|Z = 2) - P(X = 0|Z = 2) \\ -P(Y = 0|Z = 2) - 2 \cdot P(Y = 1|Z = 0) + P(X = 0|Z = 0) \end{array} \right\}$$

$$\leq ATE \leq$$

$$\min \left\{ \begin{array}{l} 1 + P(Y = 0|Z = 0) - P(X = 0|Z = 0) \\ 1 + P(Y = 0|Z = 0) - P(Y = 0|Z = 2) - P(X = 0|Z = 0) + P(X = 0|Z = 2) \\ 1 - P(Y = 0|Z = 2) + P(X = 0|Z = 2) \end{array} \right\}$$

C Exploration of Scenarios Where Bounds are Flipped

Of 10,123 randomly generated sets of values for $P(X = 1|Z = z), P(Y = 1|Z = z)$, $z = 0, 1, 2, 123$ resulted in bounds where the upper limit is smaller than the lower limit without violating any of the verifiable constraints presented in (5). Table 7 gives the values of the marginal conditional distributions with the strength of the IV, the corresponding bounds, and the width. It is notable that the IVs are rather strong in all cases where we see the bounds flip, but the bounds themselves and the widths vary quite a bit.

We first attributed this to the transition from trivariate to bivariate bounds, but later realized similar scenarios arise when dealing with trivariate bounds from four category IVs. Of 100,000 randomly generated sets of values for $P(X = x, Y = y|Z = z)$, $x = 0, 1$, $y = 0, 1$, $z = 0, 1, 2, 3, 37$ result in bounds where the upper limit is smaller than the lower limit without any violation of the verifiable constraints. It is also worth noting that in a similar number of trivariate distributions randomly generated with a trichotomous instrument, we did not see any cases of flipped bounds without a violation of one or more of the verifiable constraints. Table 8 show the bounds from these trivariate distributions with the strengths of the IVs, and the width. Again, it is interesting to see the large span of widths and strengths present.

We have been unable to unearth a reason for why we see this phenomenon. One possible explanation is that the distributions that result in flipped bounds violate some uncheckable assumption.

Table 7: Marginal conditional probabilities resulting in bounds where the upper bound is smaller than the lower bound.

| P(X=1 Z=0) | P(X=1 Z=1) | P(X=1 Z=2) | P(Y=1 Z=0) | P(Y=1 Z=1) | P(Y=1 Z=2) | Strength | Lower Bound | Upper Bound | Width |
|------------|------------|------------|------------|------------|------------|-----------|-------------|-------------|------------|
| 0.2309955 | 0.3669268 | 0.9387298 | 0.8850137 | 0.3013143 | 0.9801302 | 0.7077343 | 0.5364056 | -0.0067221 | -0.5431277 |
| 0.9404491 | 0.4742722 | 0.1448868 | 0.0262469 | 0.5741507 | 0.1155472 | 0.7955623 | 0.0532826 | -0.4025552 | -0.4558377 |
| 0.8243777 | 0.0826950 | 0.6396267 | 0.0984834 | 0.0536095 | 0.6267494 | 0.7416826 | 0.3541403 | -0.0785379 | -0.4326782 |
| 0.6253430 | 0.7940521 | 0.0769966 | 0.7125237 | 0.1332569 | 0.0937761 | 0.7170556 | 0.3709784 | -0.0341142 | -0.4050925 |
| 0.4687418 | 0.9885571 | 0.0147455 | 0.4269904 | 0.0952051 | 0.1145516 | 0.9738116 | 0.1683963 | -0.2136943 | -0.3820906 |
| 0.2384690 | 0.9589127 | 0.4551064 | 0.9411639 | 0.8220534 | 0.2995920 | 0.7204437 | 0.2623402 | -0.1057977 | -0.3681380 |
| 0.1201855 | 0.5087544 | 0.6903413 | 0.1553146 | 0.7813318 | 0.0153936 | 0.5701558 | 0.2303316 | -0.1312272 | -0.3615588 |
| 0.0558596 | 0.8249922 | 0.5150187 | 0.1693588 | 0.0317164 | 0.6019942 | 0.7691326 | 0.1515574 | -0.1885458 | -0.3401031 |
| 0.0601930 | 0.7105220 | 0.7764157 | 0.0349669 | 0.6138605 | 0.1288649 | 0.7162227 | 0.4235408 | 0.0910378 | -0.3325030 |
| 0.9689451 | 0.3369273 | 0.0921191 | 0.9728974 | 0.3379845 | 0.6435396 | 0.8768260 | 0.5457005 | 0.2351435 | -0.3105570 |
| 0.0272617 | 0.9602504 | 0.7090107 | 0.9941238 | 0.7603751 | 0.5393045 | 0.9329888 | -0.0980534 | -0.3944198 | -0.2963664 |
| 0.8593575 | 0.5455747 | 0.0954651 | 0.7493743 | 0.2343858 | 0.8692962 | 0.7638924 | -0.0169223 | -0.3132765 | -0.2963542 |
| 0.0051370 | 0.7930864 | 0.6854693 | 0.0171757 | 0.5039197 | 0.0258429 | 0.7879494 | 0.4592943 | 0.1768274 | -0.2824669 |
| 0.8095621 | 0.0899196 | 0.7315497 | 0.1398438 | 0.0112235 | 0.5721541 | 0.7196425 | 0.3698677 | 0.0884094 | -0.2814583 |
| 0.0312864 | 0.5136612 | 0.7187288 | 0.1782691 | 0.7144743 | 0.0839332 | 0.6874423 | 0.2953632 | 0.0159345 | -0.2794287 |
| 0.2841081 | 0.4642261 | 0.9303618 | 0.9272837 | 0.3015191 | 0.8563395 | 0.6462537 | 0.2718836 | 0.0151680 | -0.2567156 |
| 0.7020589 | 0.0426525 | 0.7537495 | 0.8146495 | 0.9551254 | 0.3030152 | 0.7110970 | -0.2695984 | -0.5219304 | -0.2523321 |
| 0.7299439 | 0.7079992 | 0.0126445 | 0.4179246 | 0.9411138 | 0.9059591 | 0.7172993 | -0.1196986 | -0.3687044 | -0.2490059 |
| 0.8553215 | 0.16111814 | 0.3987327 | 0.0868026 | 0.0650961 | 0.5766878 | 0.6941401 | 0.1241329 | -0.1137256 | -0.2378585 |
| 0.7503627 | 0.8262444 | 0.0255938 | 0.9023691 | 0.4826617 | 0.9697816 | 0.8006505 | -0.1771982 | -0.4057139 | -0.2285157 |
| 0.7516532 | 0.1293625 | 0.6636683 | 0.2319998 | 0.0773707 | 0.8011377 | 0.6222907 | 0.3876713 | 0.1595554 | -0.2281159 |
| 0.1892072 | 0.6542341 | 0.6029697 | 0.9717090 | 0.8941221 | 0.2186525 | 0.4650268 | -0.1219402 | -0.3463509 | -0.2244107 |
| 0.9351863 | 0.1648035 | 0.3655840 | 0.1803887 | 0.1576169 | 0.6793117 | 0.7703828 | 0.0344709 | -0.1889068 | -0.2233777 |
| 0.8913881 | 0.2924893 | 0.1391987 | 0.0678851 | 0.5562612 | 0.1311623 | 0.7521894 | 0.0155394 | -0.2032671 | -0.2188065 |
| 0.2004629 | 0.8817321 | 0.4467427 | 0.2410824 | 0.0446975 | 0.7057212 | 0.6812692 | -0.1773694 | -0.3797903 | -0.2024209 |
| 0.2713706 | 0.9177118 | 0.2155938 | 0.0584116 | 0.0235335 | 0.5341155 | 0.7021180 | -0.1254488 | -0.3224721 | -0.1970232 |
| 0.1716186 | 0.9793879 | 0.4387238 | 0.0758875 | 0.0913810 | 0.4572813 | 0.8077692 | -0.0377310 | -0.2332949 | -0.1955639 |
| 0.0346134 | 0.8601421 | 0.5243412 | 0.7170224 | 0.9940138 | 0.4402146 | 0.8255286 | 0.2680971 | 0.0753966 | -0.1927005 |
| 0.0517557 | 0.9490455 | 0.4763609 | 0.2257054 | 0.0428283 | 0.4666474 | 0.8972898 | -0.0882749 | -0.2790819 | -0.1908070 |
| 0.2097271 | 0.7849572 | 0.5591844 | 0.9851851 | 0.7694310 | 0.2353843 | 0.5752301 | -0.1266079 | -0.3155315 | -0.1889237 |
| 0.8533233 | 0.5437889 | 0.3202183 | 0.0278734 | 0.0138157 | 0.8263378 | 0.5331050 | -0.2888714 | -0.4772378 | -0.1883664 |
| 0.0781475 | 0.4316186 | 0.9562902 | 0.6056942 | 0.2534086 | 0.8616394 | 0.8781427 | 0.3824505 | 0.1983152 | -0.1841354 |
| 0.7343532 | 0.7111032 | 0.0863323 | 0.4004145 | 0.9342732 | 0.9323079 | 0.6480209 | -0.1096618 | -0.2915366 | -0.1818748 |
| 0.4855778 | 0.2600183 | 0.9736867 | 0.3390356 | 0.9283873 | 0.7874292 | 0.7136685 | 0.1831962 | 0.0022975 | -0.1808987 |
| 0.6368154 | 0.0572293 | 0.8159708 | 0.5109590 | 0.0158577 | 0.1663634 | 0.7587416 | 0.3647850 | 0.1898262 | -0.1749588 |

Table 7: Marginal conditional probabilities resulting in bounds where the upper bound is smaller than the lower bound. (*continued*)

| P(X=1 Z=0) | P(X=1 Z=1) | P(X=1 Z=2) | P(Y=1 Z=0) | P(Y=1 Z=1) | P(Y=1 Z=2) | Strength | Lower Bound | Upper Bound | Width | |
|------------|------------|------------|------------|------------|------------|-----------|-------------|-------------|------------|------------|
| 0.8824330 | 0.1367268 | 0.3081087 | 0.0653359 | 0.1951474 | 0.6000460 | 0.7457061 | -0.0637026 | -0.2342401 | -0.1705375 | |
| 0.8090247 | 0.3226145 | 0.5675011 | 0.9402684 | 0.9741885 | 0.3180210 | 0.4864103 | 0.1805653 | 0.0148730 | -0.1656923 | |
| 0.4510693 | 0.0872080 | 0.9033969 | 0.5323388 | 0.1710303 | 0.0969452 | 0.8161888 | 0.0158620 | -0.1452420 | -0.1611040 | |
| 0.1518352 | 0.6975145 | 0.6509167 | 0.0629987 | 0.8097783 | 0.1657477 | 0.5456793 | 0.3801104 | 0.2198838 | -0.1602266 | |
| 0.0653620 | 0.3813488 | 0.9612892 | 0.9275631 | 0.4953530 | 0.7515764 | 0.8959272 | -0.0696219 | -0.2290492 | -0.1594273 | |
| 0.2032074 | 0.7755576 | 0.4991361 | 0.7865987 | 0.9554554 | 0.2348516 | 0.5723502 | 0.2271745 | 0.0680689 | -0.1591056 | |
| 0.0233274 | 0.6660489 | 0.8176706 | 0.8429973 | 0.2798561 | 0.7213751 | 0.7943432 | -0.2017648 | -0.3594838 | -0.1577189 | |
| 0.9294752 | 0.2110150 | 0.4387583 | 0.1560685 | 0.0882931 | 0.6040925 | 0.7184602 | 0.0054762 | -0.1509059 | -0.1563822 | |
| 0.1670113 | 0.6894123 | 0.4795673 | 0.0041910 | 0.8002859 | 0.0345400 | 0.5224010 | 0.4578813 | 0.3096595 | -0.1482218 | |
| 0.3785346 | 0.9143229 | 0.1322393 | 0.3764540 | 0.9927913 | 0.6755701 | 0.7820836 | 0.4377743 | 0.2897923 | -0.1479819 | |
| 0.1776605 | 0.3763786 | 0.8762187 | 0.2525663 | 0.7852824 | 0.1601145 | 0.6985582 | -0.0751713 | -0.2174909 | -0.1423196 | |
| 0.7676593 | 0.0086728 | 0.5238627 | 0.3109642 | 0.8841540 | 0.9821670 | 0.7589865 | -0.2989048 | -0.4399984 | -0.1410937 | |
| 0.8834087 | 0.2154675 | 0.5237259 | 0.9402145 | 0.9094435 | 0.4479360 | 0.6679412 | 0.1993104 | 0.0599839 | -0.1393265 | |
| 0.2128945 | 0.6634662 | 0.7020688 | 0.9859116 | 0.2297734 | 0.8227277 | 0.4891743 | -0.1801804 | -0.3162608 | -0.1360804 | |
| 0.8197957 | 0.4539939 | 0.2933378 | 0.1292782 | 0.6944266 | 0.0241216 | 0.5264579 | 0.0595077 | -0.0754615 | -0.1349692 | |
| 31 | 0.8932091 | 0.2573860 | 0.3789772 | 0.8683447 | 0.8850420 | 0.3218777 | 0.6358231 | 0.2012298 | 0.0665657 | -0.1346641 |
| | 0.3852521 | 0.7681010 | 0.1679198 | 0.6200211 | 0.0286245 | 0.1269667 | 0.6001813 | 0.0302481 | -0.0989742 | -0.1292223 |
| | 0.4450183 | 0.3448027 | 0.9580487 | 0.0334938 | 0.6223715 | 0.0373602 | 0.6132460 | -0.3346527 | -0.4637484 | -0.1290957 |
| | 0.9626206 | 0.3323393 | 0.3615993 | 0.8971357 | 0.8947940 | 0.3577061 | 0.6302814 | 0.3618066 | 0.2327966 | -0.1290100 |
| | 0.9579589 | 0.2856719 | 0.2557011 | 0.0294142 | 0.0312341 | 0.4495460 | 0.7022578 | -0.1842660 | -0.3066353 | -0.1223693 |
| 0.2722892 | 0.1030317 | 0.9532750 | 0.3335194 | 0.0179986 | 0.1046059 | 0.8502432 | 0.0914587 | -0.0308574 | -0.1223161 | |
| 0.2075435 | 0.6267518 | 0.9907035 | 0.0610969 | 0.8711902 | 0.5325762 | 0.7831600 | 0.3339092 | 0.2125552 | -0.1213540 | |
| 0.1309917 | 0.9511009 | 0.6110001 | 0.0092469 | 0.1382892 | 0.3862037 | 0.8201092 | 0.1057264 | -0.0118269 | -0.1175533 | |
| 0.9469203 | 0.4771290 | 0.2975224 | 0.8483259 | 0.2756656 | 0.8366797 | 0.6493979 | 0.3148269 | 0.1973510 | -0.1174758 | |
| 0.9141838 | 0.3947449 | 0.2582693 | 0.1776121 | 0.6284717 | 0.0485084 | 0.6559145 | 0.0149163 | -0.1016151 | -0.1165314 | |
| 0.2539480 | 0.3283935 | 0.9257231 | 0.5855638 | 0.1211694 | 0.0074839 | 0.6717752 | -0.3135619 | -0.4220422 | -0.1084803 | |
| 0.7554315 | 0.0394385 | 0.8166883 | 0.9193390 | 0.1504442 | 0.4920783 | 0.7772497 | 0.5395735 | 0.4314412 | -0.1081323 | |
| 0.5322302 | 0.8442719 | 0.1311744 | 0.7227207 | 0.1174348 | 0.2652317 | 0.7130975 | -0.0700917 | -0.1763950 | -0.1063033 | |
| 0.1022484 | 0.7850567 | 0.3114329 | 0.9983873 | 0.9750404 | 0.6040354 | 0.6828082 | -0.0838413 | -0.1882423 | -0.1044009 | |
| 0.8859779 | 0.1854690 | 0.2675919 | 0.9352886 | 0.8113619 | 0.3954484 | 0.7005089 | 0.2470847 | 0.1436625 | -0.1034222 | |
| 0.8858413 | 0.0577413 | 0.7457014 | 0.9231434 | 0.9814877 | 0.6837953 | 0.8281000 | -0.0658260 | -0.1636975 | -0.0978715 | |
| 0.5688937 | 0.0533840 | 0.9092544 | 0.4161218 | 0.0847550 | 0.1385937 | 0.8558704 | 0.1398438 | 0.0425567 | -0.0972870 | |
| 0.0111502 | 0.5785773 | 0.7360408 | 0.9491940 | 0.9715842 | 0.4417906 | 0.7248905 | -0.3414676 | -0.4342969 | -0.0928294 | |
| 0.8016434 | 0.0919814 | 0.6269118 | 0.0598012 | 0.0080604 | 0.4024806 | 0.7096620 | 0.2023970 | 0.1138349 | -0.0885621 | |
| 0.5613155 | 0.3343263 | 0.9641096 | 0.1739435 | 0.9413168 | 0.6466249 | 0.6297833 | 0.0475254 | -0.0400375 | -0.0875629 | |

Table 7: Marginal conditional probabilities resulting in bounds where the upper bound is smaller than the lower bound. (continued)

| P(X=1 Z=0) | P(X=1 Z=1) | P(X=1 Z=2) | P(Y=1 Z=0) | P(Y=1 Z=1) | P(Y=1 Z=2) | Strength | Lower Bound | Upper Bound | Width | |
|------------|------------|------------|------------|------------|------------|-----------|-------------|-------------|------------|------------|
| 0.9421035 | 0.7800406 | 0.0170238 | 0.6536674 | 0.8584000 | 0.0860958 | 0.9250797 | 0.6521608 | 0.5647278 | -0.0874330 | |
| 0.4856718 | 0.1412137 | 0.8327200 | 0.2353279 | 0.7698770 | 0.8171080 | 0.6915064 | 0.0643282 | -0.0219988 | -0.0863269 | |
| 0.7587967 | 0.2217142 | 0.4642144 | 0.1261614 | 0.0095185 | 0.6397095 | 0.5370825 | 0.1772441 | 0.0950201 | -0.0822241 | |
| 0.8476325 | 0.0321449 | 0.5761561 | 0.7137147 | 0.9222930 | 0.4156565 | 0.8154876 | -0.2929622 | -0.3646398 | -0.0716776 | |
| 0.8443266 | 0.0231323 | 0.6135112 | 0.5114541 | 0.9662261 | 0.9901356 | 0.8211943 | -0.3041605 | -0.3747334 | -0.0705729 | |
| 0.7090756 | 0.0306938 | 0.8591612 | 0.8275547 | 0.1987801 | 0.4221209 | 0.8284674 | 0.3686070 | 0.2983647 | -0.0702424 | |
| 0.5210445 | 0.6877412 | 0.1936365 | 0.2077578 | 0.8583608 | 0.8895555 | 0.4941047 | -0.1155538 | -0.1840802 | -0.0685264 | |
| 0.7325333 | 0.0360979 | 0.7452189 | 0.9243027 | 0.1841382 | 0.4150783 | 0.7091209 | 0.4838304 | 0.4154162 | -0.0684143 | |
| 0.3112649 | 0.5408216 | 0.7700621 | 0.0719339 | 0.8911155 | 0.9844600 | 0.4587973 | 0.4371103 | 0.3713461 | -0.0657642 | |
| 0.6839198 | 0.0601158 | 0.7429099 | 0.3546209 | 0.0832522 | 0.8458772 | 0.6827941 | 0.5591411 | 0.4955250 | -0.0636161 | |
| 0.4925476 | 0.1475428 | 0.6432137 | 0.1357593 | 0.7295215 | 0.9418075 | 0.4956709 | 0.0342830 | -0.0281982 | -0.0624812 | |
| 0.0567614 | 0.4716677 | 0.8412115 | 0.9781020 | 0.6182925 | 0.8866750 | 0.7844501 | -0.1625195 | -0.2243887 | -0.0618691 | |
| 0.1902110 | 0.3836209 | 0.9071890 | 0.8456573 | 0.3088491 | 0.0296753 | 0.7169780 | -0.5392827 | -0.6006846 | -0.0614020 | |
| 0.3772296 | 0.8822068 | 0.2883994 | 0.2173902 | 0.9350335 | 0.7191264 | 0.5938073 | 0.4170904 | 0.3559363 | -0.0611541 | |
| 0.5973862 | 0.8450983 | 0.2624347 | 0.1392309 | 0.6156584 | 0.9712264 | 0.5826636 | -0.2177176 | -0.2783525 | -0.0606348 | |
| 28 | 0.6339672 | 0.0297922 | 0.8123455 | 0.7376053 | 0.9506195 | 0.2630108 | 0.7825533 | -0.5198657 | -0.5786439 | -0.0587783 |
| | 0.0823461 | 0.5840173 | 0.6679903 | 0.9677474 | 0.8284869 | 0.2712011 | 0.5856442 | -0.4461926 | -0.4996015 | -0.0534089 |
| | 0.6535119 | 0.8883952 | 0.1073055 | 0.2820041 | 0.7154519 | 0.8117950 | 0.7810897 | -0.0743099 | -0.1269749 | -0.0526651 |
| | 0.7404535 | 0.1312750 | 0.4474163 | 0.1314948 | 0.9068344 | 0.9347602 | 0.6091785 | -0.3671417 | -0.4196239 | -0.0524822 |
| | 0.0820021 | 0.8994346 | 0.3178099 | 0.4734612 | 0.1446546 | 0.8253918 | 0.8174325 | -0.2855348 | -0.3349518 | -0.0494170 |
| | 0.0143154 | 0.1408971 | 0.9883829 | 0.5259441 | 0.4011591 | 0.9257180 | 0.9740675 | 0.4270428 | 0.3779018 | -0.0491410 |
| | 0.5142074 | 0.8446779 | 0.0753746 | 0.5067568 | 0.0715657 | 0.1808748 | 0.7693032 | -0.0057421 | -0.0529810 | -0.0472389 |
| | 0.1391137 | 0.4452852 | 0.7319911 | 0.0201224 | 0.4730480 | 0.0227584 | 0.5928773 | 0.1545757 | 0.1084867 | -0.0460890 |
| | 0.7671998 | 0.0911903 | 0.9424491 | 0.7190755 | 0.0257481 | 0.5228183 | 0.8512587 | 0.4851985 | 0.4416630 | -0.0435356 |
| | 0.2249334 | 0.9771968 | 0.6502243 | 0.9434316 | 0.7995282 | 0.4743734 | 0.7522634 | 0.0790767 | 0.0373769 | -0.0416998 |
| 0.9124694 | 0.5503730 | 0.0400667 | 0.7951134 | 0.6099932 | 0.9632078 | 0.8724027 | -0.1948275 | -0.2362891 | -0.0414616 | |
| 0.1645046 | 0.8060324 | 0.5635964 | 0.9246119 | 0.7605022 | 0.3061245 | 0.6415279 | -0.1730552 | -0.2140902 | -0.0410350 | |
| 0.7079565 | 0.5723802 | 0.2806847 | 0.8839699 | 0.2430289 | 0.9515723 | 0.4272719 | -0.0591760 | -0.0987463 | -0.0395703 | |
| 0.2097282 | 0.9124687 | 0.2747676 | 0.2570863 | 0.1285457 | 0.7024909 | 0.7027405 | -0.2311382 | -0.2703369 | -0.0391987 | |
| 0.9736240 | 0.0208031 | 0.3737885 | 0.9045140 | 0.4334044 | 0.2716260 | 0.9528209 | 0.4846500 | 0.4464234 | -0.0382266 | |
| 0.1845828 | 0.1851770 | 0.8937890 | 0.8433725 | 0.4857333 | 0.9516657 | 0.7092062 | 0.2051761 | 0.1681541 | -0.0370221 | |
| 0.1904095 | 0.9898458 | 0.0778574 | 0.3241436 | 0.0396418 | 0.5826816 | 0.9119883 | -0.4464247 | -0.4830894 | -0.0366648 | |
| 0.3058563 | 0.8758829 | 0.3221585 | 0.8338573 | 0.0715108 | 0.2981029 | 0.5700266 | -0.4066656 | -0.4426015 | -0.0359359 | |
| 0.5517228 | 0.8850872 | 0.1379439 | 0.7797196 | 0.3208303 | 0.1888349 | 0.7471432 | 0.1261619 | 0.0917667 | -0.0343952 | |
| 0.0614376 | 0.2965834 | 0.9979328 | 0.0027831 | 0.1401460 | 0.0597136 | 0.9364952 | 0.0117046 | -0.0165844 | -0.0282890 | |

Table 7: Marginal conditional probabilities resulting in bounds where the upper bound is smaller than the lower bound. (*continued*)

| P(X=1 Z=0) | P(X=1 Z=1) | P(X=1 Z=2) | P(Y=1 Z=0) | P(Y=1 Z=1) | P(Y=1 Z=2) | Strength | Lower Bound | Upper Bound | Width | |
|------------|------------|------------|------------|------------|------------|-----------|-------------|-------------|------------|------------|
| 0.8779495 | 0.4096741 | 0.2304406 | 0.7998226 | 0.4274697 | 0.9938156 | 0.6475089 | -0.0719255 | -0.0992804 | -0.0273549 | |
| 0.6979215 | 0.7737010 | 0.0234315 | 0.9852010 | 0.4651610 | 0.8182570 | 0.7502694 | -0.0989160 | -0.1244899 | -0.0255739 | |
| 0.6623782 | 0.7107869 | 0.1608789 | 0.9024376 | 0.2805005 | 0.8890312 | 0.5499081 | -0.1508689 | -0.1758042 | -0.0249354 | |
| 0.4107040 | 0.6300393 | 0.0755462 | 0.7135503 | 0.0247311 | 0.2318819 | 0.5544931 | 0.0986941 | 0.0758333 | -0.0228608 | |
| 0.2389620 | 0.9996788 | 0.3607017 | 0.1224239 | 0.2775328 | 0.6499732 | 0.7607167 | -0.0727986 | -0.0942652 | -0.0214665 | |
| 0.2466505 | 0.3150522 | 0.9973913 | 0.7941729 | 0.4943148 | 0.9589104 | 0.7507408 | 0.4182885 | 0.3992699 | -0.0190186 | |
| 0.1047963 | 0.5872602 | 0.6265764 | 0.1702907 | 0.0689137 | 0.7661262 | 0.5217801 | 0.2159521 | 0.1971807 | -0.0187714 | |
| 0.6454304 | 0.5477765 | 0.0021959 | 0.8270074 | 0.1628806 | 0.2007895 | 0.6432345 | 0.4210367 | 0.4032008 | -0.0178359 | |
| 0.0147348 | 0.9403617 | 0.7719393 | 0.1339251 | 0.5201033 | 0.7372833 | 0.9256270 | 0.4399636 | 0.4221999 | -0.0177637 | |
| 0.6149141 | 0.1287129 | 0.8052456 | 0.3774013 | 0.9281094 | 0.7809966 | 0.6765327 | -0.2049168 | -0.2213916 | -0.0164747 | |
| 0.6318831 | 0.8417779 | 0.1046526 | 0.1803197 | 0.6822984 | 0.0227946 | 0.7371254 | 0.4274041 | 0.4145748 | -0.0128292 | |
| 0.4658334 | 0.1177519 | 0.8202813 | 0.3008471 | 0.8740505 | 0.7295855 | 0.7025294 | -0.2011135 | -0.2117500 | -0.0106365 | |
| 0.4692894 | 0.9793264 | 0.2505315 | 0.6858286 | 0.3586177 | 0.0507586 | 0.7287948 | 0.0832484 | 0.0727541 | -0.0104943 | |
| 0.9053262 | 0.4920161 | 0.2908324 | 0.8237065 | 0.8801458 | 0.1128271 | 0.6144939 | 0.3452384 | 0.3365678 | -0.0086706 | |
| 0.8400507 | 0.6066834 | 0.0207922 | 0.8392446 | 0.3014262 | 0.1199182 | 0.8192585 | 0.5578239 | 0.5502410 | -0.0075829 | |
| 88 | 0.2986999 | 0.3574011 | 0.7508847 | 0.7003727 | 0.1246649 | 0.9739429 | 0.4521849 | 0.3249903 | 0.3213192 | -0.0036711 |
| | 0.0463115 | 0.4417234 | 0.7452841 | 0.1110238 | 0.4748895 | 0.0612693 | 0.6989726 | 0.1602189 | 0.1570808 | -0.0031381 |
| | 0.8543023 | 0.0104242 | 0.1896705 | 0.9925313 | 0.2311163 | 0.0674310 | 0.8438782 | 0.6262363 | 0.6260467 | -0.0001896 |

Table 8: Lower and Upper limits of bounds where the upper limit is less than the lower limit for trivariate distributions with four category instruments.

| Lower | Upper | Strength | Width |
|------------|------------|-----------|------------|
| 0.1796920 | 0.0395535 | 0.0853119 | -0.1401385 |
| -0.0038326 | -0.1264492 | 0.1539099 | -0.1226166 |
| -0.0169573 | -0.1304422 | 0.2235469 | -0.1134849 |
| -0.0620851 | -0.1743916 | 0.0805434 | -0.1123066 |
| 0.0996764 | -0.0065497 | 0.2112420 | -0.1062260 |
| -0.0348047 | -0.1393748 | 0.1884223 | -0.1045701 |
| -0.0097177 | -0.1102060 | 0.0874967 | -0.1004882 |
| -0.0470850 | -0.1435686 | 0.1458296 | -0.0964835 |
| -0.1052398 | -0.1993785 | 0.2667633 | -0.0941387 |
| 0.1097975 | 0.0268471 | 0.1774704 | -0.0829504 |
| 0.1884781 | 0.1110487 | 0.3297432 | -0.0774293 |
| 0.0174359 | -0.0580424 | 0.2058740 | -0.0754784 |
| -0.0530855 | -0.1187770 | 0.2521754 | -0.0656915 |
| 0.0534080 | -0.0107149 | 0.1509847 | -0.0641230 |
| -0.0660707 | -0.1258819 | 0.2831483 | -0.0598112 |
| 0.3495840 | 0.2945716 | 0.3633999 | -0.0550124 |
| 0.1665198 | 0.1136389 | 0.2131245 | -0.0528809 |
| -0.0356540 | -0.0879713 | 0.2476628 | -0.0523173 |
| 0.1089847 | 0.0575836 | 0.1941017 | -0.0514012 |
| 0.0086756 | -0.0338341 | 0.2340061 | -0.0425097 |
| 0.1335166 | 0.0930974 | 0.4555966 | -0.0404192 |
| 0.1163970 | 0.0761754 | 0.1573917 | -0.0402216 |
| -0.1249197 | -0.1611461 | 0.1712798 | -0.0362264 |
| -0.1252239 | -0.1581375 | 0.1035529 | -0.0329136 |
| -0.2954311 | -0.3273509 | 0.3077593 | -0.0319199 |
| 0.0274287 | -0.0007244 | 0.0813449 | -0.0281530 |
| -0.1317444 | -0.1586467 | 0.3469784 | -0.0269023 |
| 0.1050533 | 0.0818064 | 0.2388595 | -0.0232469 |
| -0.1980031 | -0.2156885 | 0.2205149 | -0.0176854 |
| 0.0408272 | 0.0265662 | 0.1314643 | -0.0142609 |
| 0.1255375 | 0.1131666 | 0.0426523 | -0.0123709 |
| -0.1421790 | -0.1523644 | 0.1409053 | -0.0101854 |
| -0.0997312 | -0.1083943 | 0.3816466 | -0.0086630 |
| -0.0304169 | -0.0353880 | 0.1323408 | -0.0049711 |
| 0.0094786 | 0.0046709 | 0.2838685 | -0.0048077 |
| -0.0217285 | -0.0245811 | 0.3531008 | -0.0028526 |
| -0.0563955 | -0.0583218 | 0.4092683 | -0.0019263 |

D Complete Results from Simulations Described in Sections 3.1 and 3.2

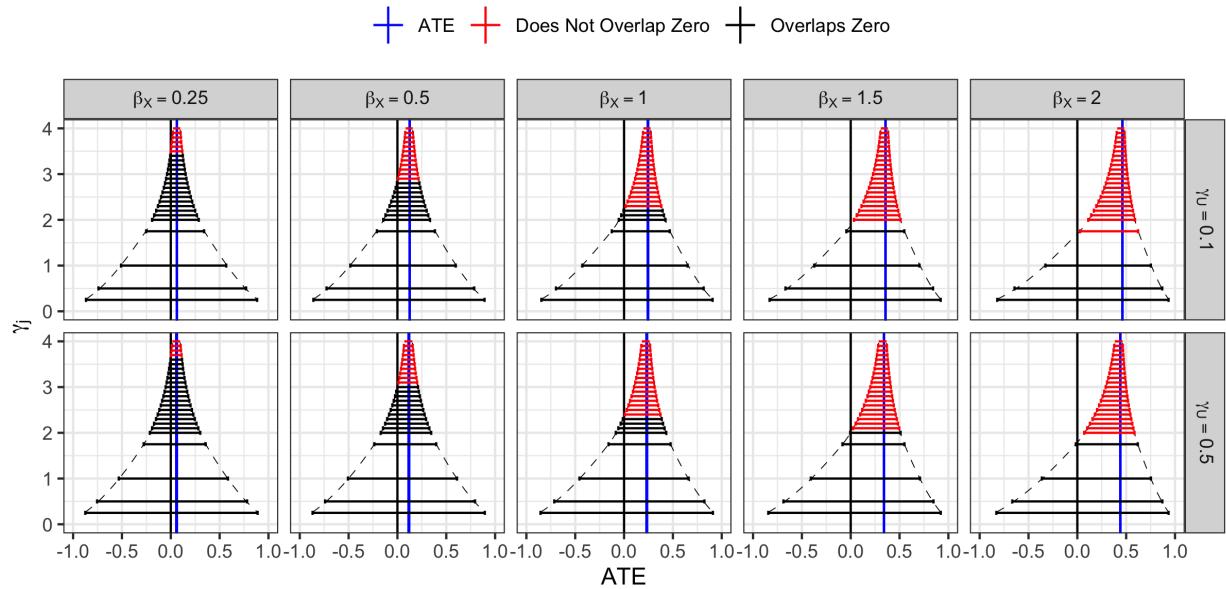


Figure 9: Bounds based on simulations as described in Section 3.1. Upper and lower bounds are connected a curve based on a loess extrapolation. This curve is used to find the smallest coefficients needed to detect direction as plotted on Figure 2.

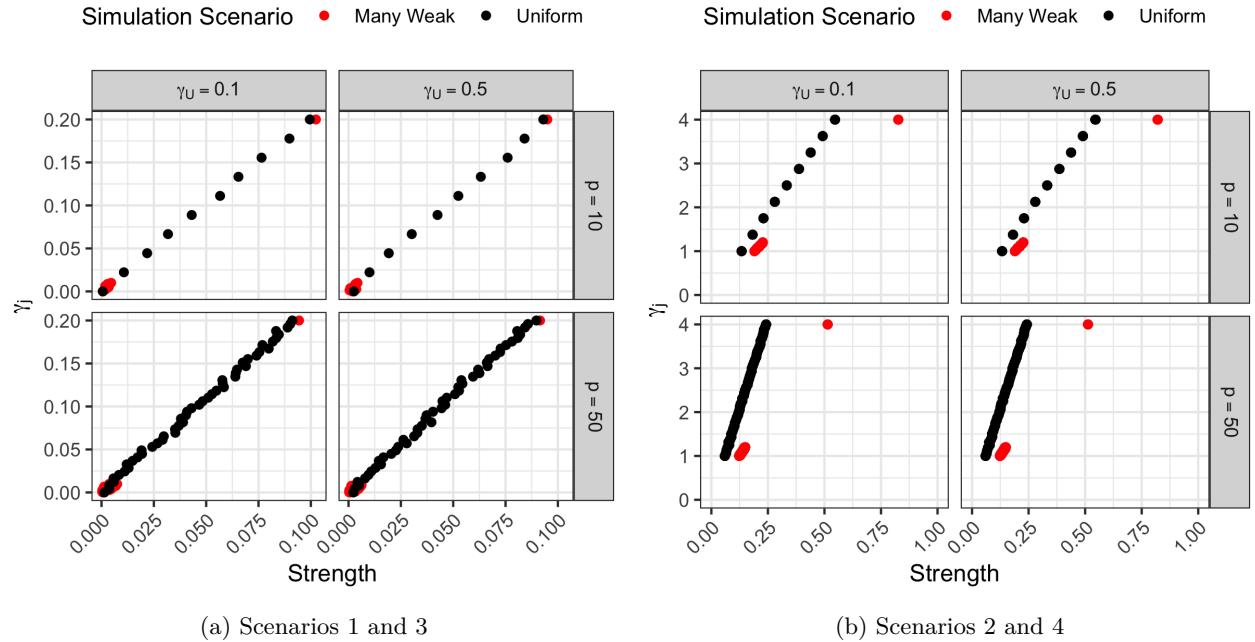


Figure 10: Figure showing the dilution effect described in Section 3.2 in each of the four scenarios. When p is larger, similar sized coefficients lead to lower strength. The effect is smaller when we are in a scenario where one coefficient is relatively much larger than the rest, rather than when the coefficients are evenly spread out.

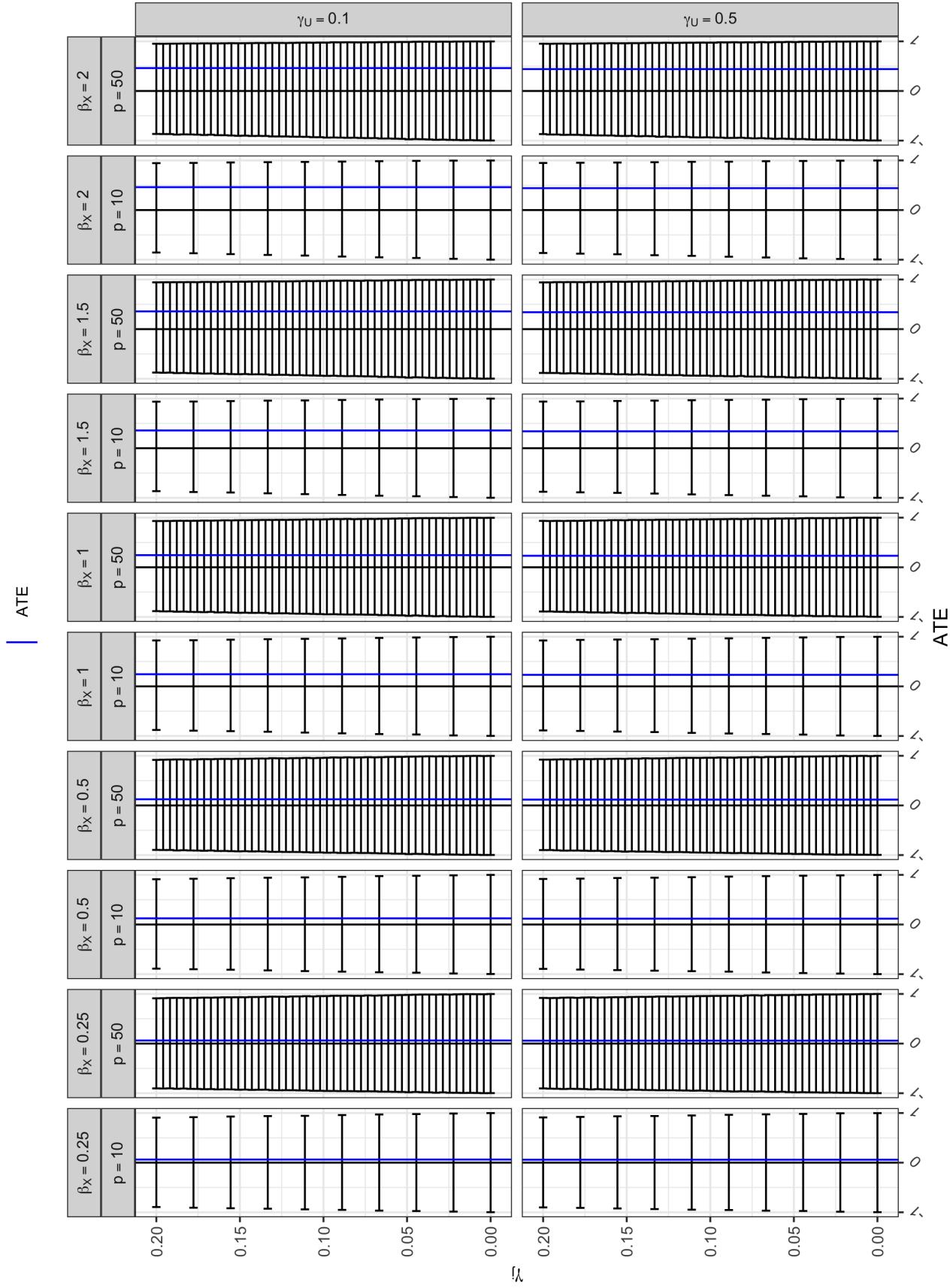


Figure 11: Bounds based on Monte Carlo integration with 1,000,000 resamples in scenario 1.

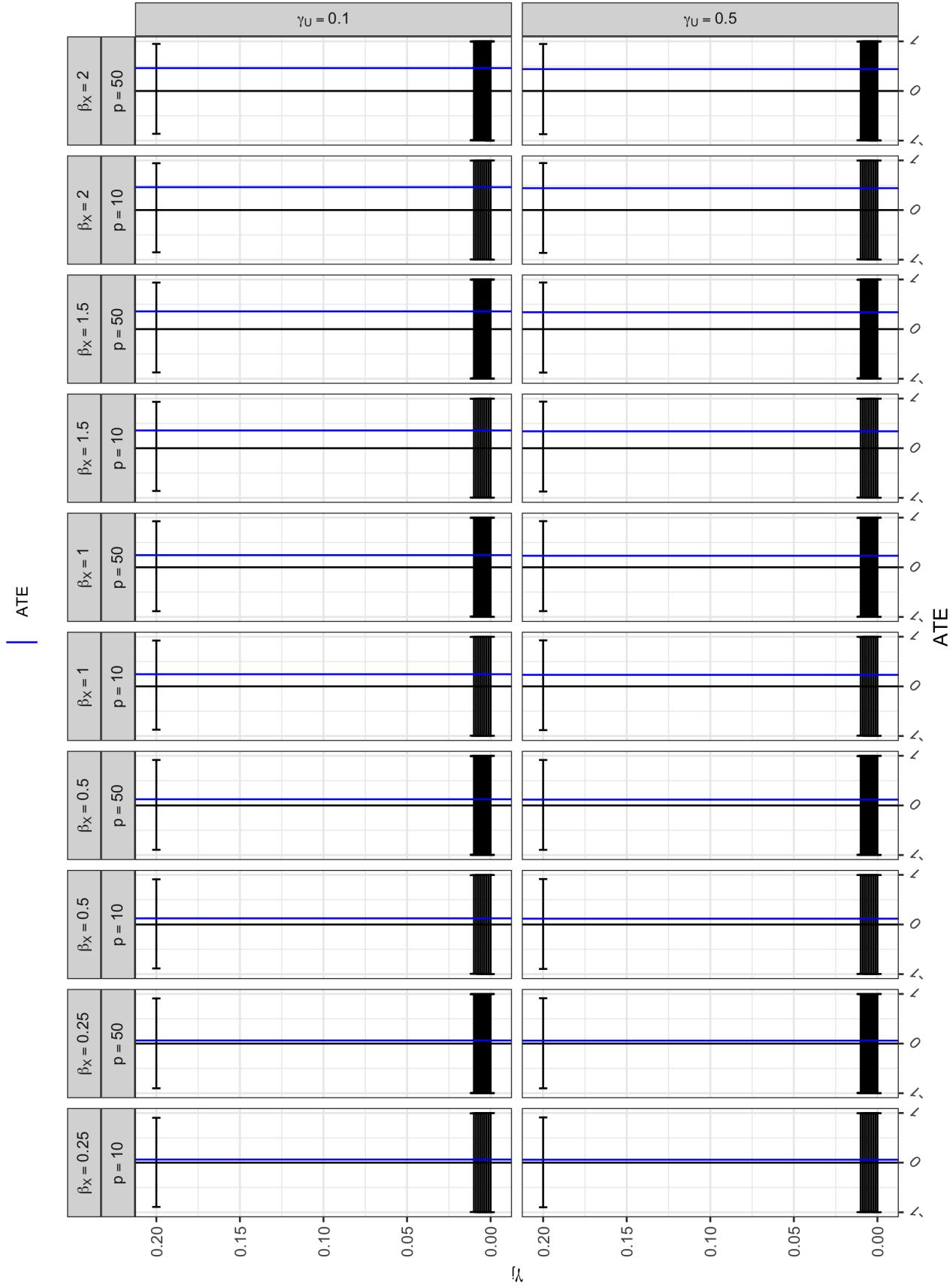


Figure 12: Bounds based on Monte Carlo integration with 1,000,000 resamples in scenario 3.

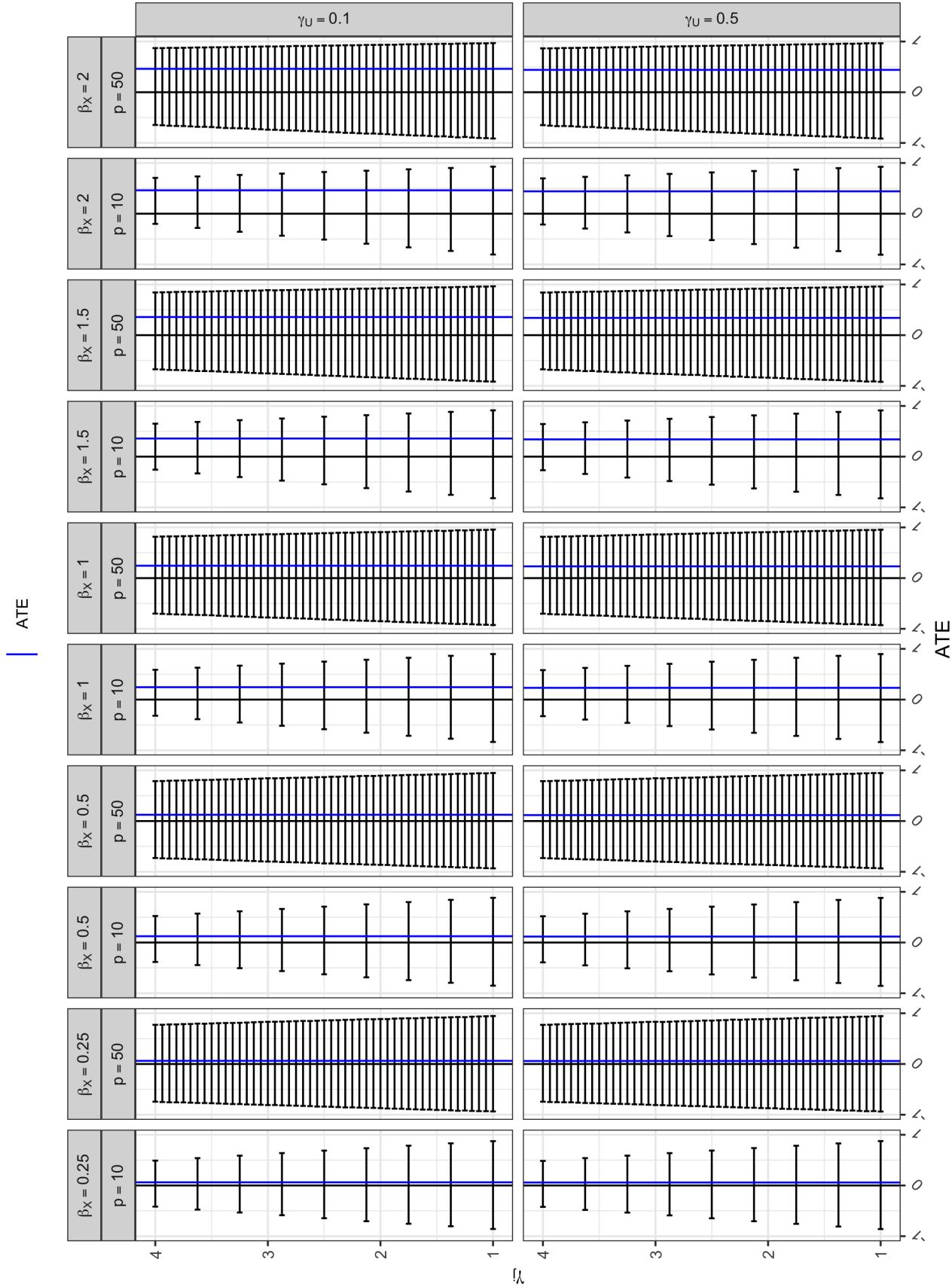


Figure 13: Bounds based on Monte Carlo integration with 1,000,000 resamples in scenario 2.

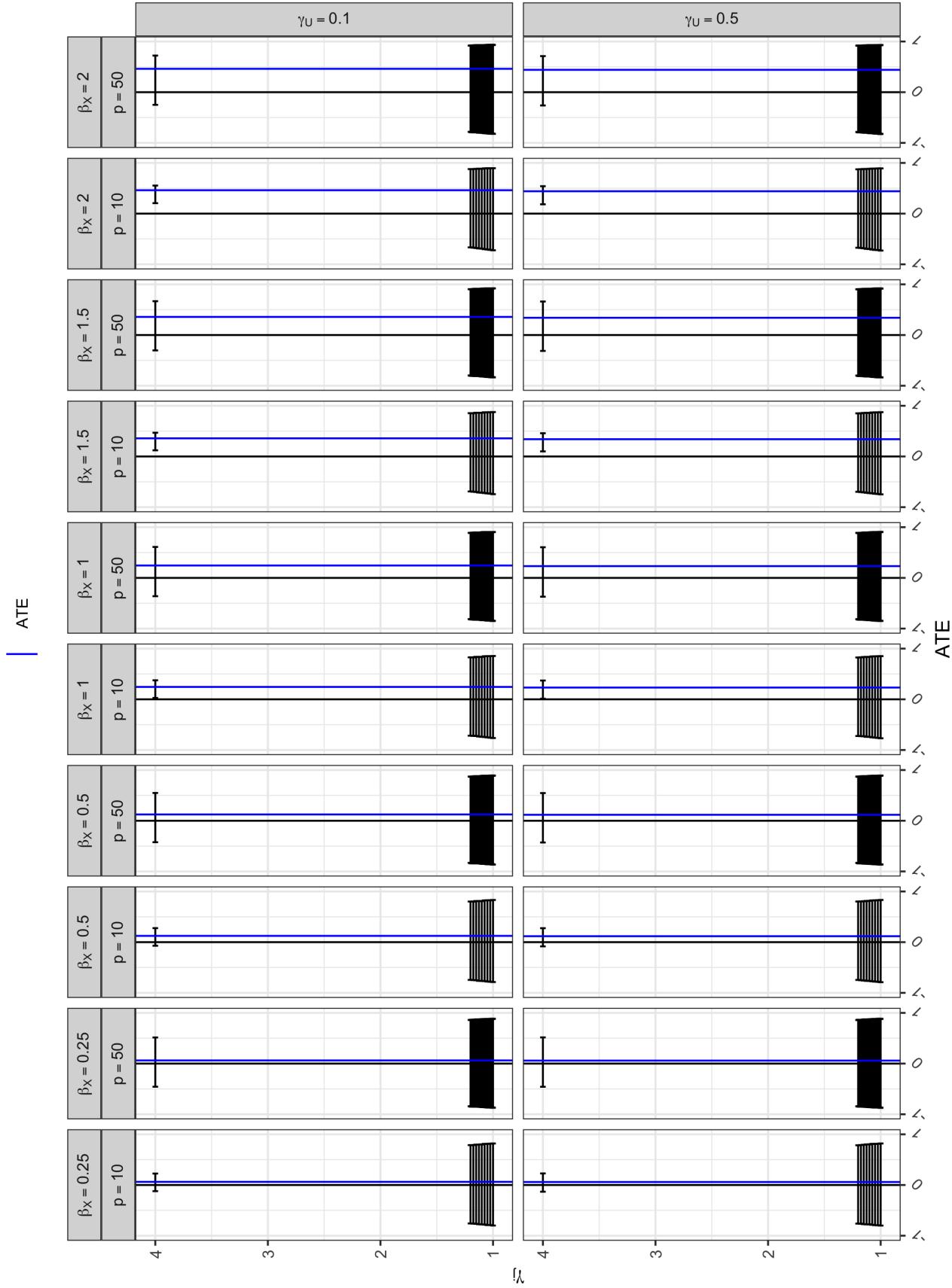


Figure 14: Bounds based on Monte Carlo integration with 1,000,000 resamples in scenario 4.

E Sampling of Intersection Bounds From Two Instruments

To extend our method for sampling plausible joint distributions of $P(X = x, Y = y|Z = z)$ to the scenario where we have multiple instruments available, we simply repeat the one instrument sampling for each instrument. This is equivalent to assuming that the covariances of X and Y given Z_1 are independent of the covariances of X and Y given Z_2 . Once we have obtained bounds for each instrument, we take the intersection to get the intersection bounds.

Specifically, say we get bounds $(LB_{1i}, UB_{1i}), i = 1, 2, \dots, m$ by sampling m trivariate distributions based on the information we have on (X, Z_1) and (Y, Z_1) , and bounds $(LB_{2i}, UB_{2i}), i = 1, 2, \dots, m$ by sampling m trivariate distributions based on the information we have on (X, Z_2) and (Y, Z_2) . We then create the intersection bounds as $(\max_{z \in 1,2} LB_{zi}, \min_{z \in 1,2} UB_{zi}), i = 1, 2, \dots, m$. This, under the assumption that $\text{Cov}(X, Y|Z_1 = z)$ and $\text{Cov}(X, Y|Z_2 = z)$ are independent of each other, gives us a sample from the posterior distribution of intersection bounds. We can use this to assess the potential usefulness of aggregating information from two sets of trivariate data, (X, Y, Z_1) and (X, Y, Z_2) , using intersection bounds.

F Additional Summary Statistics and Figures for Analyses Presented in Section 5

F.1 Effect of Smoking on Lung Cancer

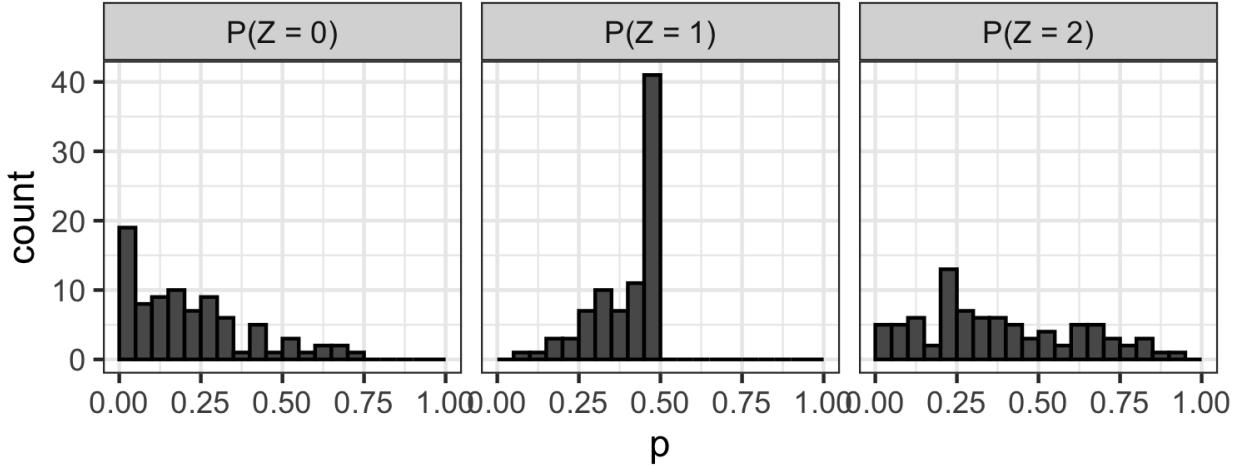


Figure 15: Histograms of the marginal distribution of instruments, $P(Z = z), z = 0, 1, 2$, estimated after preprocessing for analysis in Section 5.1.

Table 9: Table of the marginal distribution of instruments, $P(Z = z)$, $z = 0, 1, 2$, estimated after preprocessing for analysis in Section 5.1

| SNP | $P(Z = 2)$ | $P(Z = 1)$ | $P(Z = 0)$ | SNP | $P(Z = 2)$ | $P(Z = 1)$ | $P(Z = 0)$ |
|-------------|------------|------------|------------|------------|------------|------------|------------|
| rs10173733 | 0.3562119 | 0.4812460 | 0.1625421 | rs2797116 | 0.5370791 | 0.3915554 | 0.0713655 |
| rs10193706 | 0.2254196 | 0.4987283 | 0.2758521 | rs2867749 | 0.4639468 | 0.4343792 | 0.1016740 |
| rs10233018 | 0.2458307 | 0.4999649 | 0.2542044 | rs299688 | 0.0806544 | 0.4066855 | 0.5126601 |
| rs10274594 | 0.2540510 | 0.4999674 | 0.2459816 | rs326341 | 0.2745833 | 0.4988473 | 0.2265693 |
| rs1029986 | 0.1723980 | 0.4856208 | 0.3419813 | rs35891966 | 0.8609698 | 0.1338295 | 0.0052006 |
| rs10774625 | 0.2457332 | 0.4999633 | 0.2543035 | rs379525 | 0.2690001 | 0.4993042 | 0.2316957 |
| rs10813628 | 0.2349574 | 0.4995333 | 0.2655093 | rs42417 | 0.0959979 | 0.4276747 | 0.4763274 |
| rs10897561 | 0.4140371 | 0.4588401 | 0.1271228 | rs4566215 | 0.2184561 | 0.4978736 | 0.2836703 |
| rs10905461 | 0.0654474 | 0.3807590 | 0.5537936 | rs4910656 | 0.4334112 | 0.4498570 | 0.1167317 |
| rs10914684 | 0.4570550 | 0.4380069 | 0.1049382 | rs4957528 | 0.0432505 | 0.3294341 | 0.6273153 |
| rs10956808 | 0.3337643 | 0.4879181 | 0.1783175 | rs523528 | 0.1717181 | 0.4853414 | 0.3429405 |
| rs11103667 | 0.6528207 | 0.3103050 | 0.0368743 | rs528301 | 0.2006916 | 0.4945891 | 0.3047192 |
| rs11127913 | 0.3717426 | 0.4759287 | 0.1523286 | rs55921136 | 0.6351822 | 0.3236020 | 0.0412158 |
| rs11429972 | 0.1128192 | 0.4461330 | 0.4410478 | rs568599 | 0.2090011 | 0.4963306 | 0.2946684 |
| rs11611651 | 0.8323808 | 0.1599365 | 0.0076827 | rs5850689 | 0.1341980 | 0.4642649 | 0.4015371 |
| rs11631530 | 0.7779345 | 0.2081429 | 0.0139226 | rs60745548 | 0.0747101 | 0.3972427 | 0.5280472 |
| rs11646575 | 0.3149600 | 0.4925059 | 0.1925340 | rs6141314 | 0.5735637 | 0.3675524 | 0.0588839 |
| rs11693702 | 0.2849095 | 0.4977193 | 0.2173712 | rs6265 | 0.6582586 | 0.3061456 | 0.0355959 |
| rs117435980 | 0.6998026 | 0.2734789 | 0.0267185 | rs6433897 | 0.0693372 | 0.3879647 | 0.5426982 |
| rs12042107 | 0.2025948 | 0.4950210 | 0.3023842 | rs6676022 | 0.7713790 | 0.2138057 | 0.0148153 |
| rs12244388 | 0.4404143 | 0.4464457 | 0.1131399 | rs6690680 | 0.7094689 | 0.2656618 | 0.0248694 |
| rs12450028 | 0.4293549 | 0.4517938 | 0.1188513 | rs6828849 | 0.3395694 | 0.4863129 | 0.1741177 |
| rs12479064 | 0.6268375 | 0.3297864 | 0.0433761 | rs71550128 | 0.2008017 | 0.4946147 | 0.3045837 |
| rs12487411 | 0.2788384 | 0.4984262 | 0.2227354 | rs72505558 | 0.3617072 | 0.4794276 | 0.1588652 |
| rs12608052 | 0.2306302 | 0.4992191 | 0.2701507 | rs72678864 | 0.6825787 | 0.2872090 | 0.0302123 |
| rs12725407 | 0.6546886 | 0.3088794 | 0.0364320 | rs7333559 | 0.0439935 | 0.3315056 | 0.6245008 |
| rs12886628 | 0.1124522 | 0.4457734 | 0.4417744 | rs7451586 | 0.3541182 | 0.4819202 | 0.1639616 |
| rs12910916 | 0.6206505 | 0.3343265 | 0.0450230 | rs748828 | 0.5139770 | 0.4058898 | 0.0801332 |
| rs13100688 | 0.3932914 | 0.4676762 | 0.1390324 | rs7528604 | 0.3213716 | 0.4910497 | 0.1875787 |
| rs1492546 | 0.2022894 | 0.4949531 | 0.3027575 | rs7567570 | 0.0299625 | 0.2862686 | 0.6837689 |
| rs1499982 | 0.0221071 | 0.2531548 | 0.7247382 | rs763053 | 0.6013164 | 0.3482591 | 0.0504245 |
| rs1549213 | 0.1285982 | 0.4600154 | 0.4113864 | rs76608582 | 0.9070039 | 0.0907272 | 0.0022689 |
| rs1561195 | 0.2279701 | 0.4989841 | 0.2730458 | rs772921 | 0.4315416 | 0.4507533 | 0.1177051 |
| rs1565735 | 0.6376078 | 0.3217914 | 0.0406009 | rs77878475 | 0.8356836 | 0.1569474 | 0.0073690 |
| rs16951001 | 0.3380123 | 0.4867519 | 0.1752358 | rs7870475 | 0.2763346 | 0.4986816 | 0.2249839 |
| rs17003752 | 0.7420669 | 0.2387323 | 0.0192008 | rs7948789 | 0.3767706 | 0.4740916 | 0.1491378 |
| rs17151637 | 0.5166809 | 0.4042486 | 0.0790705 | rs883403 | 0.7156415 | 0.2606289 | 0.0237296 |
| rs1899896 | 0.4934387 | 0.4180265 | 0.0885349 | rs9375371 | 0.5345687 | 0.3931467 | 0.0722846 |
| rs2240294 | 0.3093641 | 0.4936820 | 0.1969539 | rs9381917 | 0.8063218 | 0.1832649 | 0.0104133 |
| rs2416770 | 0.2199058 | 0.4980707 | 0.2820235 | rs9423279 | 0.1179428 | 0.4509704 | 0.4310869 |
| rs264974 | 0.2640248 | 0.4996173 | 0.2363579 | rs9487626 | 0.0332246 | 0.2981030 | 0.6686724 |
| rs2675609 | 0.1387352 | 0.4674731 | 0.3937917 | rs9835772 | 0.5737177 | 0.3674477 | 0.0588346 |

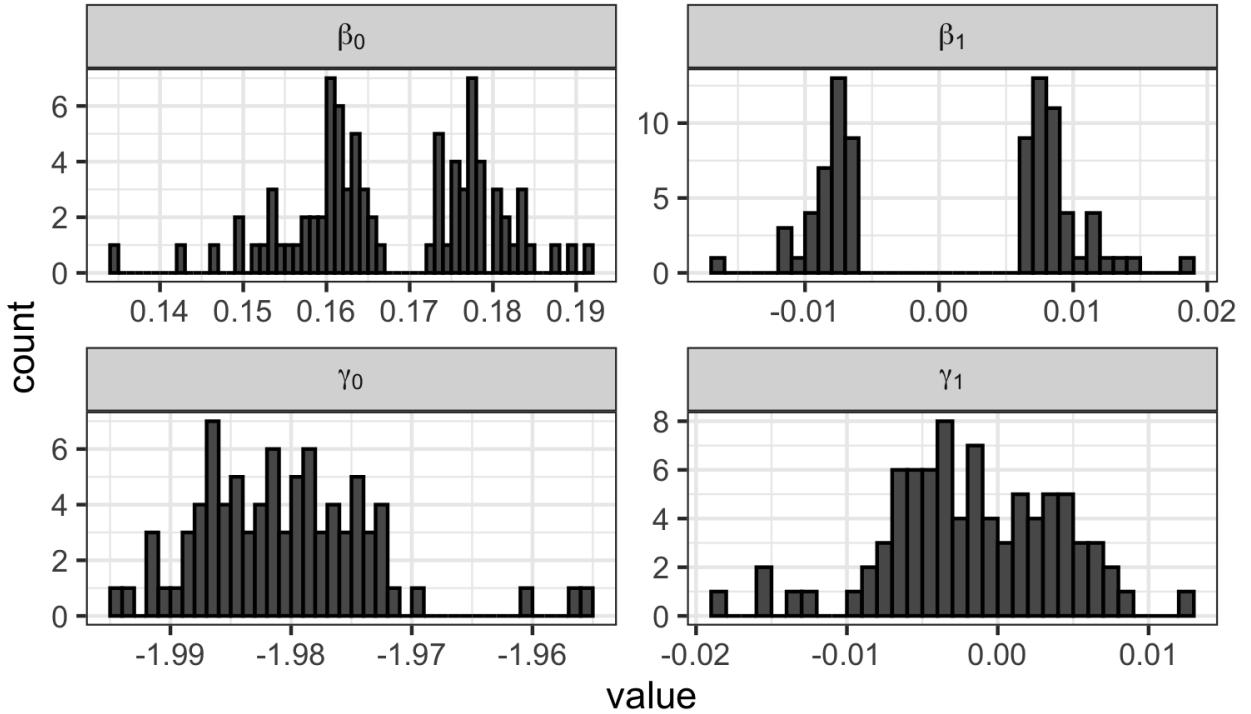


Figure 16: Histograms of the coefficients from GWAS results of logistic regression of the SNPs on smoking status and lung cancer status. Intercepts (β_0 and γ_0) are inferred, while slopes (β_1 and γ_1) are as reported.

Table 10: Coefficients from GWAS results of logistic regression of the SNPs on smoking status and lung cancer status. Intercepts (β_0 and γ_0) are inferred, while slopes (β_1 and γ_1) are as reported.

| SNP | β_1 | β_0 | γ_1 | γ_0 |
|-------------|------------|-----------|------------|------------|
| rs10173733 | -0.0065148 | 0.1773766 | 0.0033363 | -1.987122 |
| rs10193706 | -0.0117667 | 0.1807753 | -0.0015310 | -1.981684 |
| rs10233018 | -0.0076551 | 0.1771914 | 0.0050495 | -1.988150 |
| rs10274594 | 0.0078326 | 0.1617046 | -0.0015364 | -1.981589 |
| rs1029986 | -0.0070208 | 0.1754303 | 0.0035498 | -1.986088 |
| rs10774625 | 0.0074868 | 0.1621777 | -0.0084158 | -1.974806 |
| rs10813628 | -0.0068761 | 0.1762662 | 0.0051706 | -1.988156 |
| rs10897561 | -0.0066917 | 0.1782117 | 0.0066835 | -1.991747 |
| rs10905461 | 0.0072731 | 0.1658787 | -0.0058844 | -1.980131 |
| rs10914684 | 0.0077356 | 0.1591408 | -0.0026047 | -1.979616 |
| rs10956808 | 0.0076247 | 0.1607905 | -0.0063546 | -1.975802 |
| rs11103667 | -0.0086047 | 0.1835048 | 0.0063118 | -1.993343 |
| rs11127913 | 0.0081801 | 0.1596256 | -0.0033969 | -1.978997 |
| rs11429972 | 0.0083148 | 0.1640148 | -0.0096129 | -1.976695 |
| rs11611651 | -0.0119868 | 0.1914724 | 0.0013059 | -1.985521 |
| rs11631530 | -0.0099863 | 0.1872160 | -0.0047887 | -1.974691 |
| rs11646575 | -0.0082446 | 0.1788545 | 0.0012319 | -1.984521 |
| rs11693702 | -0.0080254 | 0.1781679 | 0.0046224 | -1.988077 |
| rs117435980 | -0.0092037 | 0.1849986 | -0.0054804 | -1.973970 |
| rs12042107 | 0.0071759 | 0.1631404 | -0.0020557 | -1.981288 |

Table 10: Coefficients from GWAS results of logistic regression of the SNPs on smoking status and lung cancer status. Intercepts (β_0 and γ_0) are inferred, while slopes (β_1 and γ_1) are as reported. *(continued)*

| SNP | β_1 | β_0 | γ_1 | γ_0 |
|------------|------------|-----------|------------|------------|
| rs12244388 | -0.0104344 | 0.1834505 | 0.0019355 | -1.985707 |
| rs12450028 | -0.0070626 | 0.1788556 | -0.0024536 | -1.979923 |
| rs12479064 | -0.0080362 | 0.1823251 | -0.0088600 | -1.969116 |
| rs12487411 | 0.0075048 | 0.1616745 | -0.0077980 | -1.974913 |
| rs12608052 | 0.0067542 | 0.1631129 | -0.0048100 | -1.978521 |
| rs12725407 | 0.0081386 | 0.1564297 | -0.0067998 | -1.972138 |
| rs12886628 | -0.0071010 | 0.1743626 | -0.0018595 | -1.981891 |
| rs12910916 | -0.0090138 | 0.1838027 | 0.0026458 | -1.987308 |
| rs13100688 | 0.0072663 | 0.1604864 | -0.0055464 | -1.976186 |
| rs1492546 | -0.0068801 | 0.1757890 | 0.0040638 | -1.986797 |
| rs1499982 | -0.0114648 | 0.1730098 | 0.0024892 | -1.983878 |
| rs1549213 | 0.0085270 | 0.1634849 | 0.0056335 | -1.987184 |
| rs1561195 | -0.0078947 | 0.1771393 | 0.0072232 | -1.990046 |
| rs1565735 | 0.0115901 | 0.1510915 | -0.0072487 | -1.971566 |
| rs16951001 | -0.0066035 | 0.1772784 | 0.0070226 | -1.991313 |
| rs17003752 | 0.0098606 | 0.1526117 | -0.0055424 | -1.973591 |
| rs17151637 | 0.0075112 | 0.1588020 | -0.0027771 | -1.979146 |
| rs1899896 | -0.0079928 | 0.1808293 | 0.0047935 | -1.989876 |
| rs2240294 | 0.0069566 | 0.1618616 | -0.0078381 | -1.974429 |
| rs2416770 | -0.0064888 | 0.1756858 | -0.0035668 | -1.979794 |
| rs264974 | 0.0093111 | 0.1600323 | -0.0047198 | -1.978291 |
| rs2675609 | 0.0081586 | 0.1635228 | -0.0069708 | -1.977953 |
| rs2797116 | 0.0079136 | 0.1580011 | -0.0039635 | -1.977330 |
| rs2867749 | 0.0069446 | 0.1601396 | -0.0032894 | -1.978658 |
| rs299688 | -0.0072721 | 0.1737306 | -0.0019058 | -1.982055 |
| rs326341 | 0.0065809 | 0.1627032 | 0.0031753 | -1.986468 |
| rs35891966 | 0.0147752 | 0.1421811 | -0.0122161 | -1.960473 |
| rs379525 | -0.0064906 | 0.1763327 | -0.0018594 | -1.981209 |
| rs42417 | -0.0070331 | 0.1739582 | 0.0003829 | -1.983375 |
| rs4566215 | 0.0066219 | 0.1634100 | -0.0035546 | -1.979817 |
| rs4910656 | 0.0068438 | 0.1605890 | -0.0006962 | -1.982221 |
| rs4957528 | -0.0084750 | 0.1731252 | 0.0036288 | -1.984649 |
| rs523528 | 0.0080708 | 0.1629116 | 0.0029251 | -1.985564 |
| rs528301 | -0.0086008 | 0.1773068 | 0.0124616 | -1.994333 |
| rs55921136 | 0.0085950 | 0.1559000 | -0.0069653 | -1.972040 |
| rs568599 | -0.0067027 | 0.1757286 | 0.0043346 | -1.987105 |
| rs5850689 | 0.0119733 | 0.1608296 | -0.0038879 | -1.980291 |
| rs60745548 | 0.0071946 | 0.1656670 | 0.0062353 | -1.986552 |
| rs6141314 | -0.0080616 | 0.1818108 | 0.0010534 | -1.984733 |
| rs6265 | 0.0101598 | 0.1531146 | -0.0043806 | -1.976031 |
| rs6433897 | -0.0072353 | 0.1734104 | -0.0011588 | -1.982527 |
| rs6676022 | 0.0115926 | 0.1492373 | -0.0153059 | -1.956268 |
| rs6690680 | 0.0088409 | 0.1547067 | -0.0050219 | -1.974679 |
| rs6828849 | 0.0067122 | 0.1617773 | 0.0008050 | -1.984076 |
| rs71550128 | -0.0073950 | 0.1762278 | 0.0034139 | -1.986200 |

Table 10: Coefficients from GWAS results of logistic regression of the SNPs on smoking status and lung cancer status. Intercepts (β_0 and γ_0) are inferred, while slopes (β_1 and γ_1) are as reported. *(continued)*

| SNP | β_1 | β_0 | γ_1 | γ_0 |
|------------|------------|-----------|------------|------------|
| rs72505558 | 0.0067437 | 0.1614885 | -0.0009876 | -1.981950 |
| rs72678864 | 0.0097538 | 0.1534836 | -0.0034394 | -1.977455 |
| rs7333559 | 0.0080523 | 0.1662222 | -0.0183846 | -1.975467 |
| rs7451586 | -0.0066732 | 0.1775422 | 0.0027432 | -1.986404 |
| rs748828 | 0.0086213 | 0.1572389 | -0.0047229 | -1.976368 |
| rs7528604 | 0.0068658 | 0.1618157 | -0.0001820 | -1.982931 |
| rs7567570 | -0.0091324 | 0.1727617 | -0.0002451 | -1.983053 |
| rs763053 | 0.0080618 | 0.1570972 | -0.0069210 | -1.972409 |
| rs76608582 | 0.0182891 | 0.1347646 | -0.0048192 | -1.973958 |
| rs772921 | 0.0072725 | 0.1600453 | -0.0054837 | -1.975937 |
| rs77878475 | 0.0125950 | 0.1465726 | 0.0010985 | -1.985146 |
| rs7870475 | -0.0071900 | 0.1771594 | 0.0082598 | -1.991835 |
| rs7948789 | -0.0161713 | 0.1894568 | 0.0009336 | -1.984284 |
| rs883403 | 0.0094240 | 0.1536556 | -0.0014726 | -1.980646 |
| rs9375371 | -0.0073963 | 0.1804155 | -0.0069852 | -1.972929 |
| rs9381917 | 0.0112569 | 0.1493838 | -0.0155636 | -1.955201 |
| rs9423279 | 0.0076695 | 0.1643324 | 0.0046716 | -1.986350 |
| rs9487626 | 0.0131029 | 0.1648247 | -0.0136868 | -1.978168 |
| rs9835772 | -0.0078024 | 0.1814198 | -0.0031275 | -1.978401 |

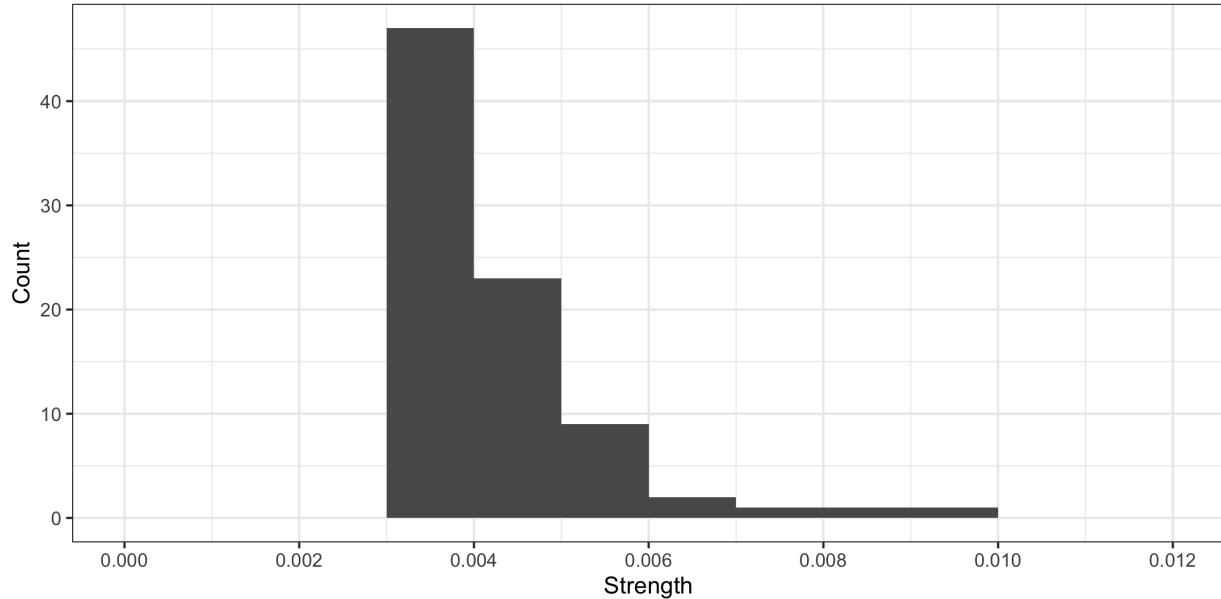


Figure 17: Histogram of strengths of IVs on the exposure. Here, SNPs are IVs, and smoking status (ever/never) is exposure. We see that all IVs are very weak, with the largest value just below 0.01.

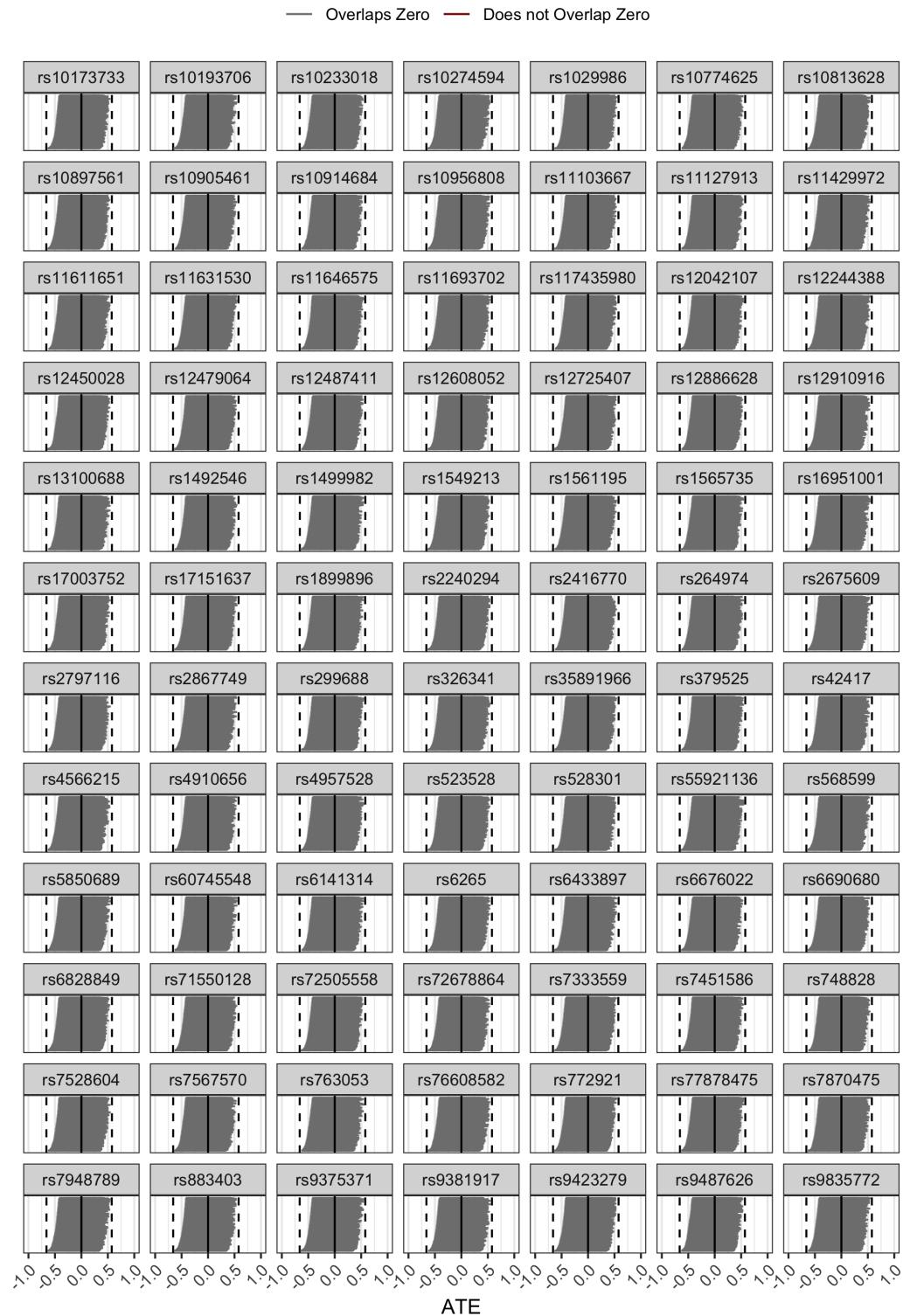


Figure 18: 500 sets of bounds of the average treatment effect of smoking on lung cancer for each of the 84 SNPs. Each bound is based on a set of values for the trivariate distribution randomly sampled. Bounds are color coded to show if they overlap 0 (grey) or do not (red). All bounds overlap 0.

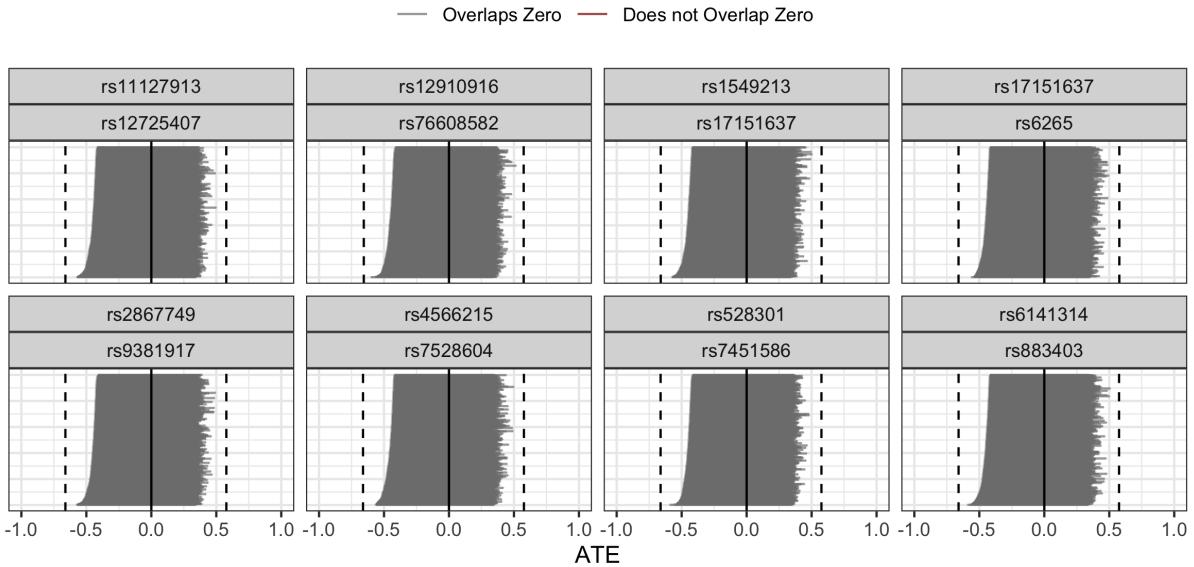


Figure 19: Intersection bounds of the average treatment effect of smoking on lung cancer based on randomly sampled trivariate distributions from pairs of SNPs. These 8 pairs were randomly chosen from all possible pairs.

F.2 Effect of High Cholesterol on Heart Attack

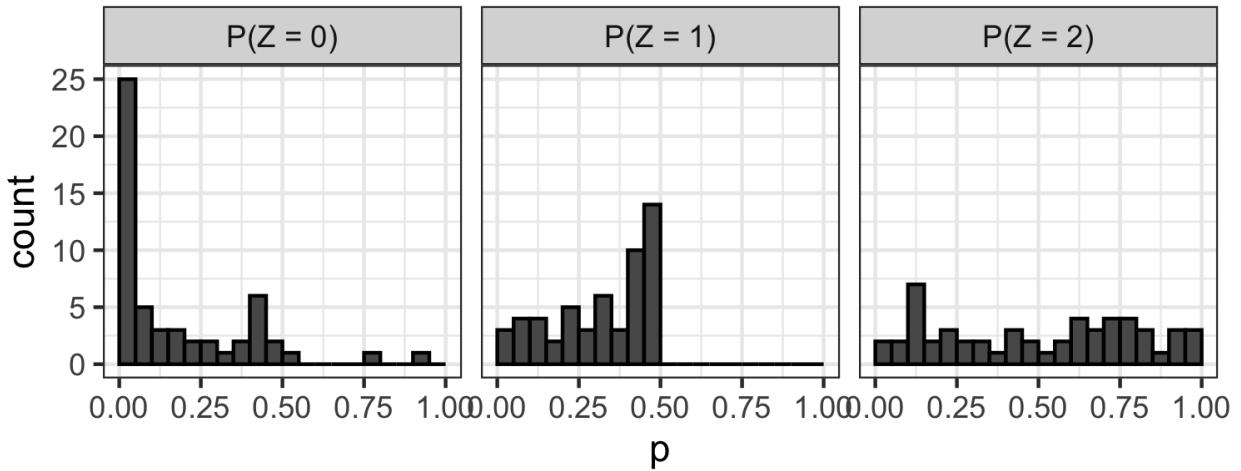


Figure 20: Histograms of the marginal distribution of instruments, $P(Z = z)$, $z = 0, 1, 2$, estimated after preprocessing for analysis in Section 5.2

Table 11: Table of the marginal distribution of instruments, $P(Z = z), z = 0, 1, 2$, estimated after pre-processing for analysis in Section 5.2

| SNP | P(Z = 2) | P(Z = 1) | P(Z = 0) | SNP | P(Z = 2) | P(Z = 1) | P(Z = 0) |
|-------------|-----------|-----------|-----------|------------|-----------|-----------|-----------|
| rs10096633 | 0.7682873 | 0.2164654 | 0.0152473 | rs3918226 | 0.8434773 | 0.1498658 | 0.0066569 |
| rs10260606 | 0.6689457 | 0.2978906 | 0.0331637 | rs4299376 | 0.1044835 | 0.4375111 | 0.4580055 |
| rs10410835 | 0.2261041 | 0.4987999 | 0.2750961 | rs4470903 | 0.6122421 | 0.3404338 | 0.0473241 |
| rs10504255 | 0.1141345 | 0.4474070 | 0.4384585 | rs456598 | 0.7353800 | 0.2443260 | 0.0202940 |
| rs10804330 | 0.3246447 | 0.4902626 | 0.1850927 | rs4704727 | 0.1153479 | 0.4485623 | 0.4360899 |
| rs112019714 | 0.9445278 | 0.0546808 | 0.0007914 | rs472495 | 0.1219232 | 0.4545036 | 0.4235732 |
| rs11580878 | 0.2532012 | 0.4999796 | 0.2468192 | rs56299331 | 0.6368870 | 0.3223300 | 0.0407830 |
| rs11591147 | 0.9653935 | 0.0343018 | 0.0003047 | rs57180587 | 0.7289642 | 0.2496596 | 0.0213762 |
| rs117733303 | 0.9629825 | 0.0366685 | 0.0003491 | rs58542926 | 0.8541959 | 0.1400626 | 0.0057415 |
| rs12471811 | 0.7974669 | 0.1910863 | 0.0114469 | rs58691354 | 0.7129641 | 0.2628159 | 0.0242201 |
| rs1260326 | 0.1542518 | 0.4769944 | 0.3687538 | rs59950280 | 0.4469685 | 0.4431771 | 0.1098545 |
| rs12740374 | 0.6060342 | 0.3448956 | 0.0490702 | rs6090040 | 0.2300488 | 0.4991705 | 0.2707808 |
| rs12916 | 0.3593703 | 0.4802094 | 0.1604203 | rs622871 | 0.0988228 | 0.4310763 | 0.4701008 |
| rs1367117 | 0.4370916 | 0.4480749 | 0.1148336 | rs635634 | 0.6627002 | 0.3027276 | 0.0345722 |
| rs1601935 | 0.1186871 | 0.4516457 | 0.4296671 | rs6458349 | 0.0768498 | 0.4007364 | 0.5224138 |
| rs1883025 | 0.5579089 | 0.3780482 | 0.0640429 | rs6511720 | 0.7764852 | 0.2093975 | 0.0141172 |
| rs1883711 | 0.9385769 | 0.0604497 | 0.0009733 | rs7012637 | 0.2755284 | 0.4987592 | 0.2257124 |
| rs2125345 | 0.4990744 | 0.4147551 | 0.0861704 | rs7213086 | 0.2001050 | 0.4944520 | 0.3054430 |
| rs2237107 | 0.6333104 | 0.3249953 | 0.0416944 | rs73534263 | 0.7971401 | 0.1913739 | 0.0114861 |
| rs2244608 | 0.4686429 | 0.4318641 | 0.0994929 | rs7412 | 0.8445834 | 0.1488576 | 0.0065590 |
| rs2618567 | 0.1161249 | 0.4492923 | 0.4345829 | rs74617384 | 0.8447171 | 0.1487357 | 0.0065473 |
| rs2738447 | 0.1661712 | 0.4829396 | 0.3508892 | rs7534572 | 0.1255675 | 0.4575751 | 0.4168575 |
| rs28601761 | 0.3342690 | 0.4877820 | 0.1779490 | rs7707394 | 0.4169078 | 0.4575523 | 0.1255398 |
| rs28807203 | 0.9046336 | 0.0929773 | 0.0023890 | rs77542162 | 0.9546715 | 0.0448029 | 0.0005257 |
| rs3127580 | 0.7081492 | 0.2667336 | 0.0251172 | rs799157 | 0.0018869 | 0.0831041 | 0.9150089 |
| rs34042070 | 0.6625016 | 0.3028808 | 0.0346176 | rs9376091 | 0.5451282 | 0.3863995 | 0.0684722 |
| rs34707604 | 0.5518930 | 0.3820040 | 0.0661030 | rs964184 | 0.0174433 | 0.2292594 | 0.7532973 |

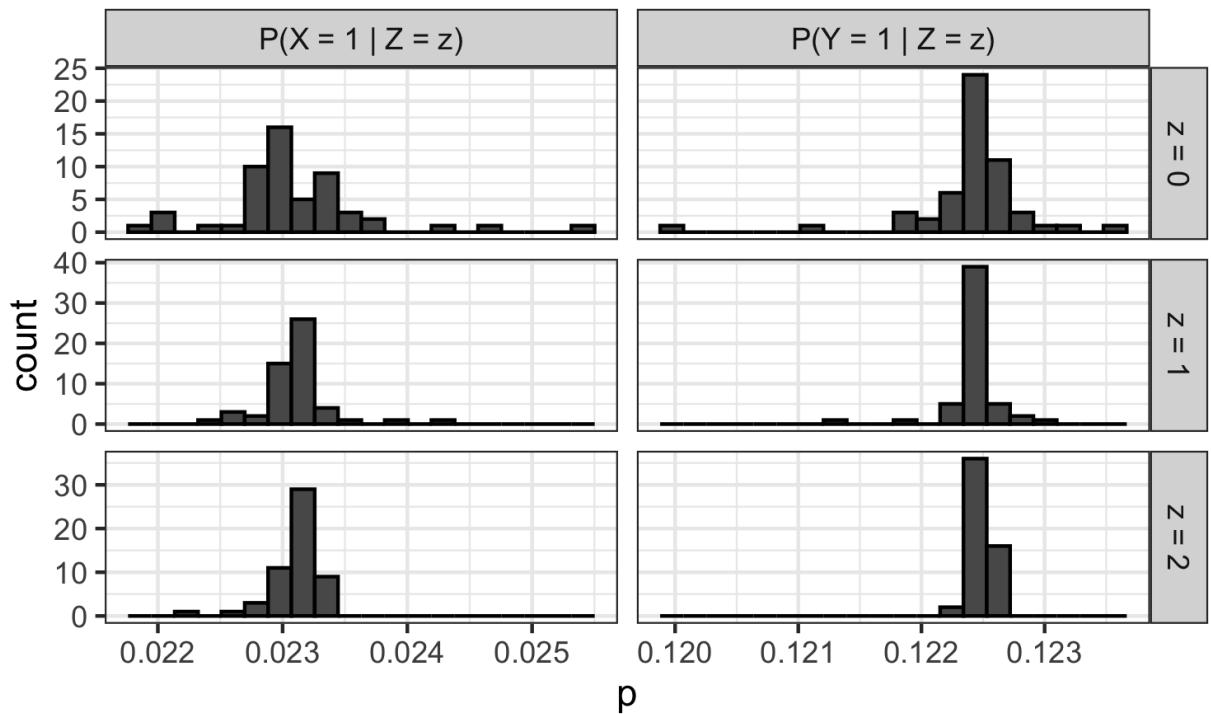


Figure 21: Histograms of the marginal conditional probabilities $P(X = 1 | Z = z), z = 0, 1, 2$ and $P(Y = 1 | Z = z), z = 0, 1, 2$.

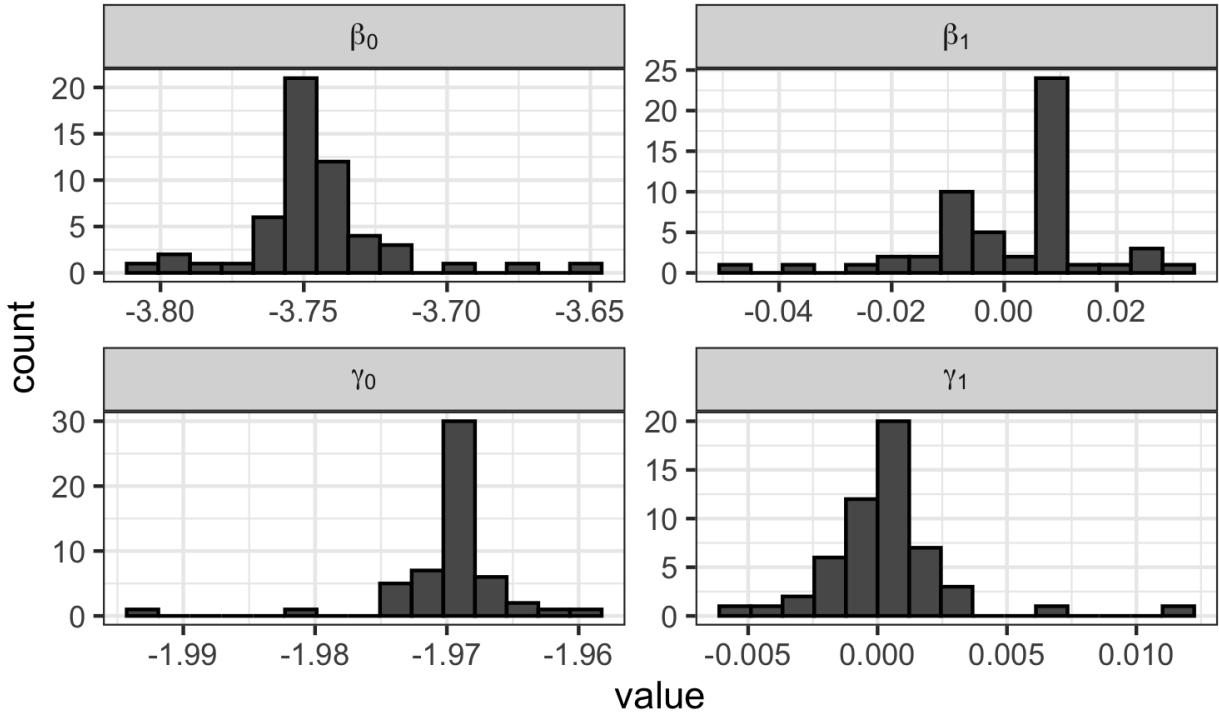


Figure 22: Histograms of the coefficients from GWAS results of logistic regression of the SNPs on high cholesterol and heart attack, respectively. Intercepts (β_0 and γ_0) are inferred, while slopes (β_1 and γ_1) are as reported.

Table 12: Coefficients from GWAS results of logistic regression of the SNPs on high cholesterol and heart attack status. Intercepts (β_0 and γ_0) are inferred, while slopes (β_1 and γ_1) are as reported.

| SNP | β_1 | β_0 | γ_1 | γ_0 |
|-------------|------------|-----------|------------|------------|
| rs10096633 | -0.0089830 | -3.727152 | -0.0012995 | -1.966860 |
| rs10260606 | 0.0076950 | -3.755485 | 0.0007029 | -1.970288 |
| rs10410835 | 0.0071078 | -3.749661 | 0.0007948 | -1.969894 |
| rs10504255 | -0.0056764 | -3.739063 | -0.0000742 | -1.969088 |
| rs10804330 | -0.0050169 | -3.737181 | -0.0012539 | -1.967709 |
| rs112019714 | 0.0251675 | -3.791824 | 0.0025525 | -1.974100 |
| rs11580878 | -0.0051399 | -3.737725 | -0.0006621 | -1.968472 |
| rs11591147 | -0.0476105 | -3.649365 | -0.0054389 | -1.958449 |
| rs117733303 | 0.0311528 | -3.804047 | 0.0116909 | -1.992088 |
| rs12471811 | 0.0084776 | -3.758037 | 0.0000048 | -1.969147 |
| rs1260326 | -0.0102312 | -3.734879 | -0.0003941 | -1.968828 |
| rs12740374 | -0.0183231 | -3.714419 | -0.0025251 | -1.965207 |
| rs12916 | 0.0104793 | -3.755479 | 0.0006700 | -1.969941 |
| rs1367117 | 0.0155585 | -3.763513 | 0.0011495 | -1.970658 |
| rs1601935 | -0.0061378 | -3.738671 | -0.0007014 | -1.968655 |
| rs1883025 | -0.0069826 | -3.732469 | -0.0013153 | -1.967173 |
| rs1883711 | 0.0241076 | -3.789616 | 0.0026734 | -1.974319 |
| rs2125345 | -0.0056374 | -3.734933 | -0.0009408 | -1.967809 |
| rs2237107 | -0.0070166 | -3.731732 | -0.0007194 | -1.967993 |

Table 12: Coefficients from GWAS results of logistic regression of the SNPs on high cholesterol and heart attack status. Intercepts (β_0 and γ_0) are inferred, while slopes (β_1 and γ_1) are as reported. *(continued)*

| SNP | β_1 | β_0 | γ_1 | γ_0 |
|------------|------------|-----------|------------|------------|
| rs2244608 | 0.0070205 | -3.752512 | 0.0010406 | -1.970563 |
| rs2618567 | -0.0047485 | -3.739660 | -0.0007455 | -1.968630 |
| rs2738447 | 0.0081671 | -3.749563 | 0.0016947 | -1.970520 |
| rs28601761 | -0.0140739 | -3.726664 | -0.0011169 | -1.967847 |
| rs28807203 | -0.0106943 | -3.722554 | -0.0002164 | -1.968726 |
| rs3127580 | 0.0076693 | -3.755804 | 0.0022978 | -1.973006 |
| rs34042070 | 0.0094413 | -3.758272 | 0.0002698 | -1.969577 |
| rs34707604 | 0.0058521 | -3.751591 | 0.0002016 | -1.969438 |
| rs3918226 | 0.0081783 | -3.757916 | 0.0028105 | -1.974301 |
| rs4299376 | -0.0111342 | -3.735719 | -0.0012431 | -1.968335 |
| rs4470903 | 0.0067035 | -3.753387 | 0.0014579 | -1.971420 |
| rs456598 | 0.0065720 | -3.754166 | 0.0005768 | -1.970127 |
| rs4704727 | 0.0074887 | -3.747988 | 0.0007432 | -1.969643 |
| rs472495 | 0.0064154 | -3.747379 | 0.0004743 | -1.969469 |
| rs56299331 | 0.0057258 | -3.752033 | 0.0001068 | -1.969308 |
| rs57180587 | 0.0081592 | -3.756830 | 0.0013685 | -1.971475 |
| rs58542926 | -0.0146353 | -3.715853 | -0.0013536 | -1.966636 |
| rs58691354 | 0.0074756 | -3.755521 | 0.0000196 | -1.969171 |
| rs59950280 | 0.0058286 | -3.750690 | 0.0004805 | -1.969780 |
| rs6090040 | -0.0055812 | -3.737545 | -0.0007168 | -1.968450 |
| rs622871 | 0.0065093 | -3.746991 | 0.0013161 | -1.969966 |
| rs635634 | 0.0098788 | -3.758987 | 0.0014151 | -1.971442 |
| rs6458349 | 0.0056558 | -3.746031 | 0.0007529 | -1.969556 |
| rs6511720 | -0.0261322 | -3.696906 | -0.0030216 | -1.963813 |
| rs7012637 | 0.0047984 | -3.747932 | 0.0002456 | -1.969396 |
| rs7213086 | 0.0047773 | -3.747169 | 0.0007846 | -1.969840 |
| rs73534263 | 0.0071810 | -3.755717 | 0.0000767 | -1.969275 |
| rs7412 | -0.0374088 | -3.674234 | -0.0038000 | -1.962153 |
| rs74617384 | 0.0190473 | -3.777927 | 0.0069894 | -1.981990 |
| rs7534572 | 0.0081187 | -3.748658 | 0.0005830 | -1.969551 |
| rs7707394 | 0.0061511 | -3.750841 | 0.0000817 | -1.969243 |
| rs77542162 | 0.0253674 | -3.792474 | 0.0020548 | -1.973154 |
| rs799157 | -0.0108031 | -3.741956 | -0.0003979 | -1.969103 |
| rs9376091 | -0.0053004 | -3.735070 | -0.0005561 | -1.968317 |
| rs964184 | -0.0215630 | -3.737246 | -0.0013629 | -1.968778 |

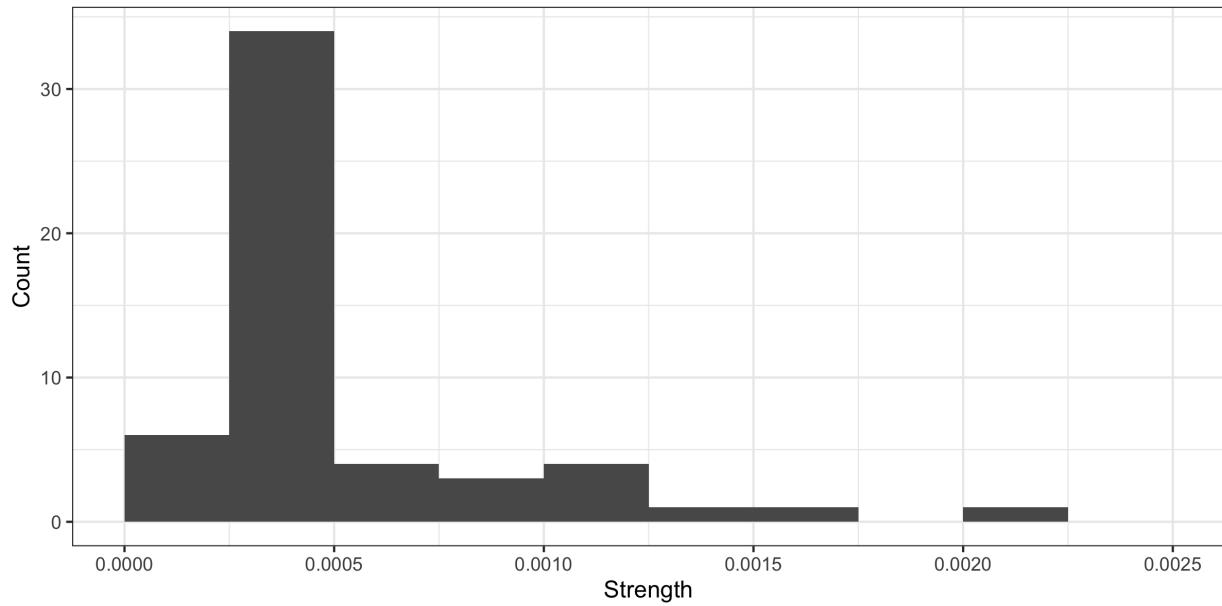


Figure 23: Histogram of strengths of IVs on the exposure. Here, SNPs are IVs, and high cholesterol is the exposure. We see that all IVs are very weak, with the largest value below 0.00225.

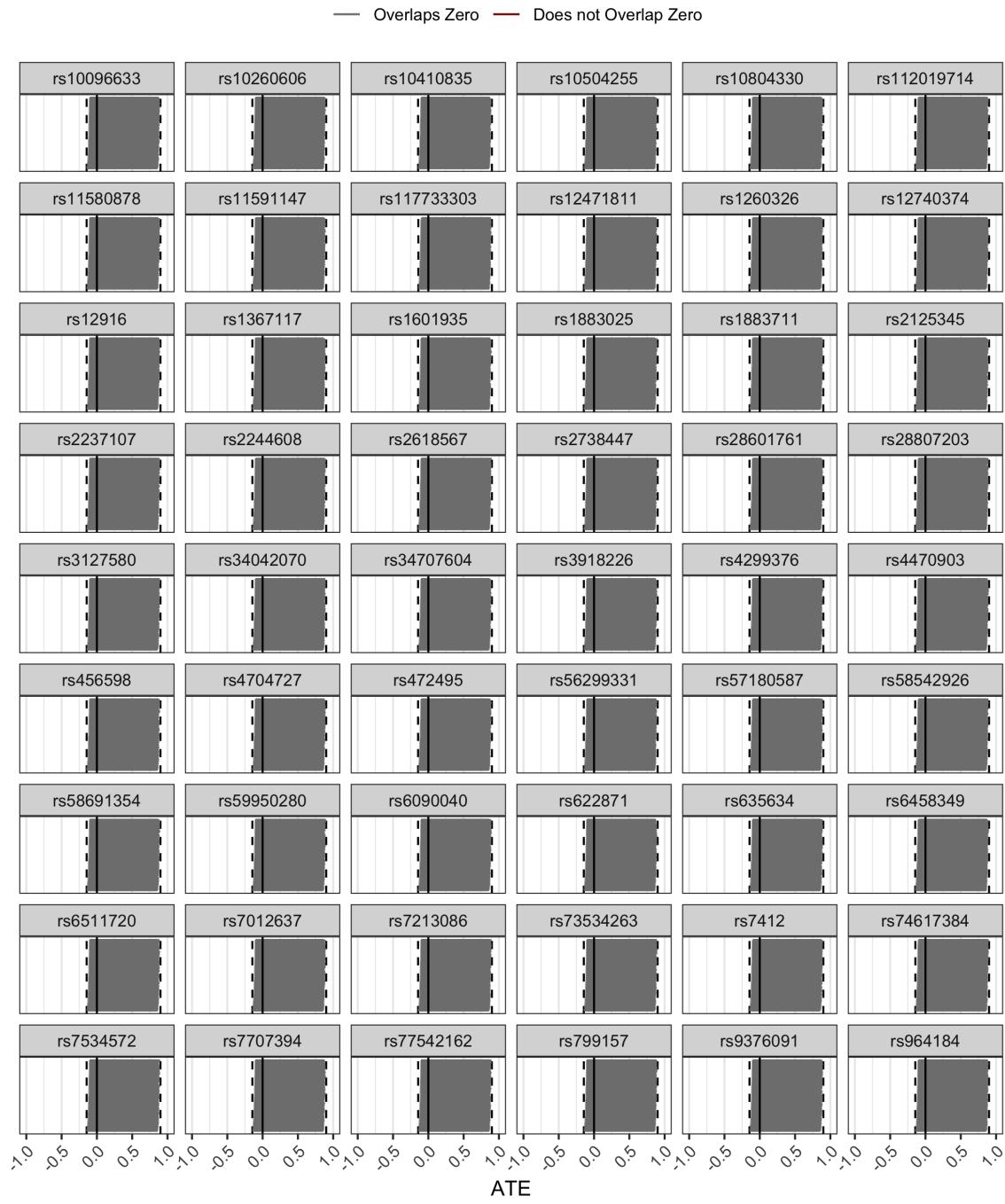


Figure 24: 500 sets of bounds of the average treatment effect of high cholesterol on heart attack for each of the 54 SNPs. Each bound is based on a set of values for the trivariate distribution randomly sampled. Bounds are color coded to show if they overlap 0 (grey) or do not (red). All bounds overlap 0.

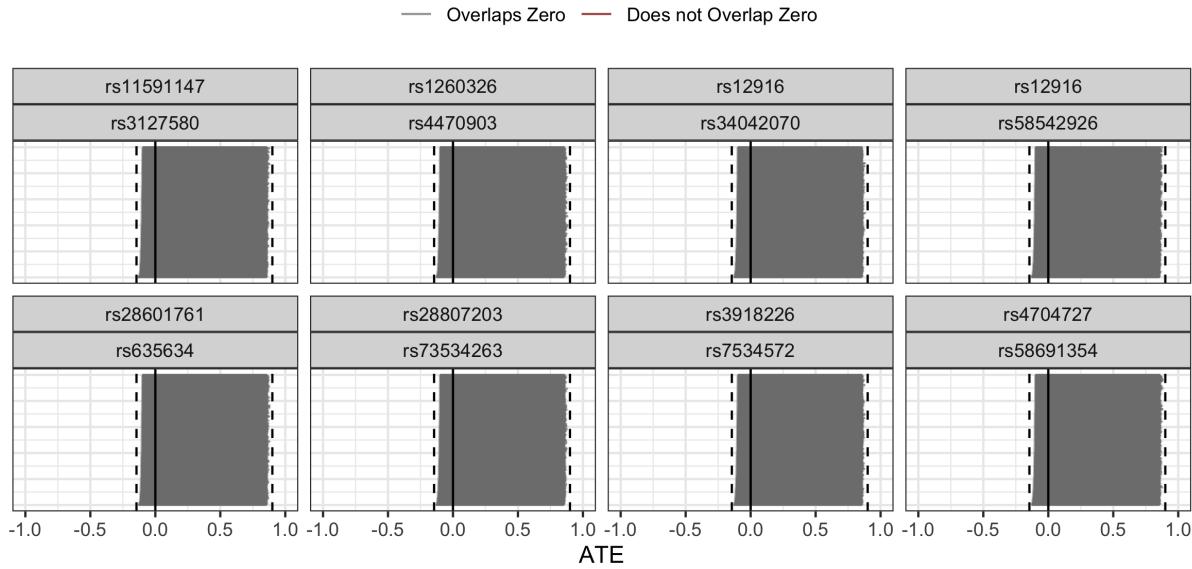


Figure 25: Intersection bounds of the average treatment effect of high cholesterol on heart attack based on randomly sampled trivariate distributions from pairs of SNPs. These 8 pairs were randomly chosen from all possible pairs.

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