

# Champaign-Urbana Data Science User Group: A Beginner's Guide to Sample Size Calculations

Ryan Muraglia

Oath Champaign

August 4, 2017

# Problem Statement

To improve user engagement (as measured by daily time spent in app in seconds per user), you and your team just finished developing a new news article recommendation algorithm for your app.

Before rolling out the change to all of your users, you've been tasked with "proving" that the new algorithm does indeed improve user engagement.

Your manager says that you'll be able to submit a request to try out the new algorithm on some of the userbase, but to be careful with how much you request, since your request is more likely to get denied as you ask for more and more test users.

# Ring a bell?

A/B testing, hypothesis testing, statistical power, confidence intervals,  $\chi^2$  test vs  $Z$ -test vs student's  $t$ -test vs ANOVA, paired vs unpaired, one-tailed vs two-tailed, type I and type II errors...

# Searching for statistical significance

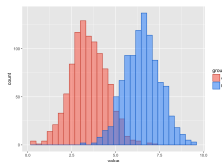
## Null and alternative hypotheses

- $H_0$ : The control and experimental groups are the same
- $H_1$ : The control and experimental groups are different

# Searching for statistical significance

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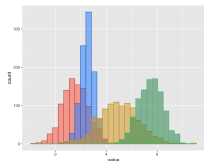
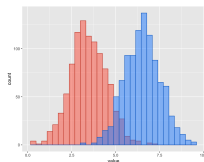
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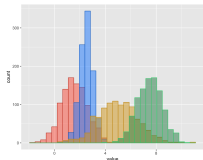
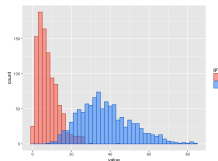
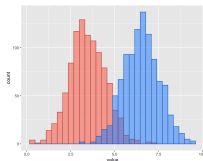
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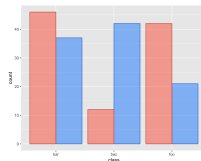
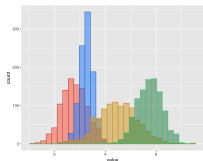
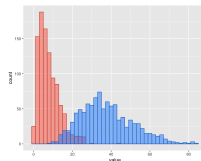
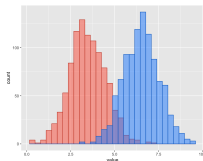
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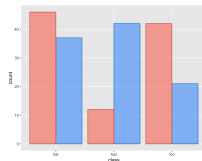
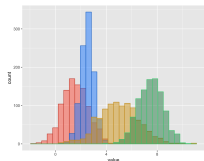
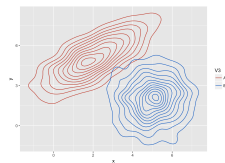
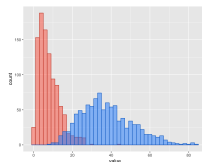
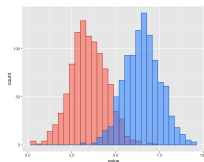




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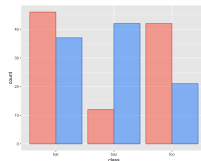
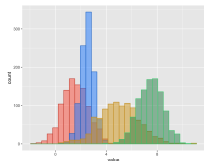
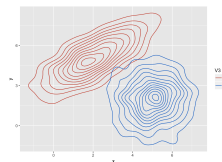
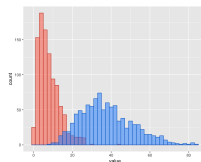
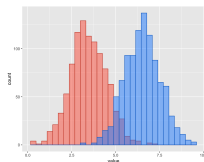
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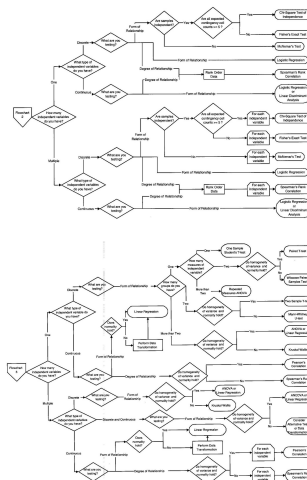
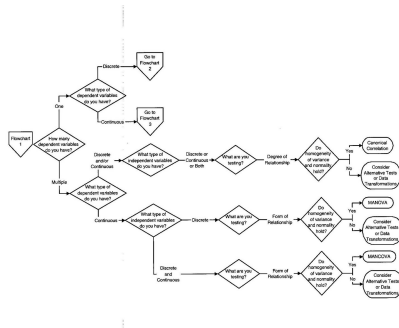
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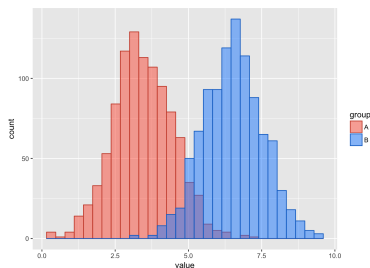
And more...

# The meaning of “different” & choice of statistical test

It depends on what  
kind of data you have



# A/B testing example



## Hypotheses

- $H_0$ : The new algorithm has no impact on average user engagement
- $H_1$ : The new algorithm increases average user engagement

# Choice of statistical test

In this case, we will use an unpaired, two-sample, one-tailed t-test.

## Why this test?

**unpaired:** the samples in each group do not correspond to one another (they aren't before and after measurements)

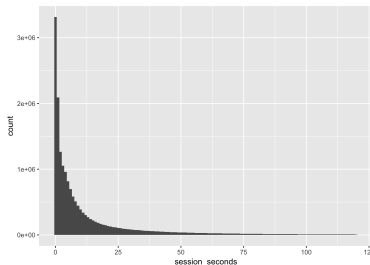
**two-sample:** we are comparing two population means, not one population mean to a known constant

**one-tailed:** we are looking for a directional change (increase), not just any deviation

**t-test:** appropriate for comparison of means of normally distributed data when variances are unknown

Note: when dealing with large sample sizes, the t and Z test are essentially equivalent, since degrees of freedom for t-test gets so large that the t-distribution approaches the standard normal.

# But my data isn't normally distributed!

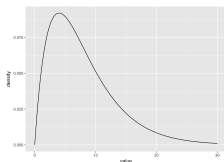


## Central limit theorem

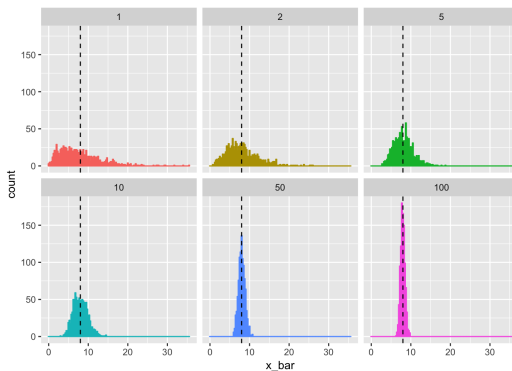
The average of a large number of iid samples will be normally distributed, as  $\mathcal{N}(\mu, \sigma^2/n)$ , where  $\mu$  and  $\sigma^2$  are the mean and variance of the underlying distribution, and  $n$  is the number of samples we are averaging over. This is true, regardless of the underlying distribution.

# CLT in action

Say your underlying distribution is skewed like this...



Watch as the CLT works its magic and makes the distribution of the sampled mean increasingly normal and narrow as the number of samples increases:



# Bringing it all together

What we already have:

- Normally distributed data
- A statistical test, and its associated test statistic:

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - \Delta}{\sqrt{s_1^2/n_1 + s_2^2/n_2}}$$

What we're still missing:

- Error rates
- Expected effect size
- Required sample size



# Back to hypothesis testing

## Type I error

Falsely rejecting  $H_0$  when  $H_0$  is true.

$\alpha$  denotes the probability of committing a type I error.

## Type II error

Falsely accepting  $H_0$  when  $H_1$  is true.

$\beta$  denotes the probability of committing a type II error.

As part of your experiment design, you need to choose error rates you're willing to live with.

Common values are  $\alpha = 0.05$  and  $\beta = 0.1$ , which correspond to critical test statistic values of 1.65 and -1.28.

Look these critical values up in a Z-table – remember that we're in a large sample size regime where the t-distribution is basically normal, so we don't need to mess around with the degrees of freedom in a t-table.

# Expected effect size

## Effect size (Cohen's $d$ )

$$d = \frac{\mu_1 - \mu_2}{\sigma}$$

Effect sizes help to contextualize the severity of a difference in means by scaling it by the standard deviation.

For your experiment, you will have to have some idea of the effect size you are expecting to see – a pilot experiment is a good way to get a feel for this.

Alternatively, you can calculate required sample sizes for a range of effect sizes.

# Two conditions to satisfy

Recall that we have a test statistic:

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - \Delta}{\sqrt{s_1^2/n_1 + s_2^2/n_2}}$$

Let's assume that  $s_1 = s_2$  and  $n_1 = n_2$  and simplify:

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - \Delta}{\sqrt{2s^2/n}}$$

---

## Type I error

We require the computed t-statistic to be  $\leq$  the critical test statistic,  $t_\alpha$ . For the null hypothesis,  $\Delta = 0$ .

$$\frac{(\bar{x}_1 - \bar{x}_2)}{\sqrt{2s^2/n}} \leq t_\alpha$$

## Type II error

We require the computed t-statistic to be  $\geq$  the critical test statistic,  $t_\beta$ . For the alternative hypothesis,  $\Delta = \mu_1 - \mu_2$

$$\frac{(\bar{x}_1 - \bar{x}_2) - \Delta}{\sqrt{2s^2/n}} \geq t_\beta$$

# A threshold value to satisfy both conditions

This gives us a little system of equations we can solve for  $n$ :

$$\frac{(\bar{x}_1 - \bar{x}_2)}{\sqrt{2s^2/n}} \leq t_\alpha$$

$$\bar{x}_1 - \bar{x}_2 \leq t_\alpha \sqrt{2s^2/n}$$

Note: by construction,  $\Delta$  is negative here, which is why we flip the direction of the inequality in the penultimate line.

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$$\frac{(\bar{x}_1 - \bar{x}_2) - \Delta}{\sqrt{2s^2/n}} \geq t_\beta$$

$$t_\alpha - \frac{\Delta}{\sqrt{2s^2/n}} \geq t_\beta$$

$$t_\alpha - t_\beta \geq \frac{\Delta\sqrt{n}}{\sqrt{2s^2}}$$

$$\frac{(t_\alpha - t_\beta)\sqrt{2s^2}}{\Delta} \leq \sqrt{n}$$

$$2 * \left( \frac{(t_\alpha - t_\beta) * s}{\Delta} \right)^2 \leq n$$

Note: by construction,  $\Delta$  is negative here, which is why we flip the direction of the inequality in the penultimate line.

# The formula for 2-sample t-test

## Minimum required sample size per group for 2-sample t-test

$$n = 2 * \left( \frac{(t_{\alpha} - t_{\beta}) * s}{\Delta} \right)^2$$
$$n = 2 * \left( \frac{(t_{\alpha} - t_{\beta})}{d} \right)^2$$

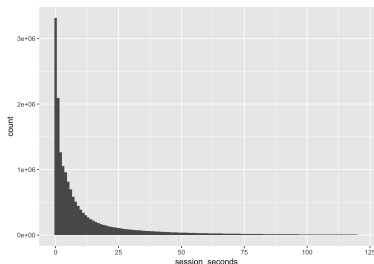
Note that the following all cause  $n$  to increase:

- Decrease in effect size,  $d$  ( $\Delta$  decreasing, or  $s$  increasing)
- Decrease in leniency for type I error ( $t_{\alpha}$  increasing)
- Decrease in leniency for type II error ( $t_{\beta}$  decreasing)

Note that the CLT plays a two-fold role: it both normalizes our data and decreases the variance ( $s$ ) as the sample size increases.

# Our data and test parameters

The distribution of time spent for 18.4 million users looks like:



Super skewed, super high standard deviation:

$$\bar{x} = 27.848$$

$$s = 168.73$$

We will be somewhat relaxed about our type I and type II error requirements:

- $\alpha = 0.1 \rightarrow t_{\alpha} = 1.28$

- $\beta = 0.2 \rightarrow t_{\beta} = -0.84$

We aren't sure what sort of effect size to expect, so we'll compute required sample sizes for a range and see what we think we can get away with.

## Formula reminder

$$n = 2 * \left( \frac{(t_{\alpha} - t_{\beta}) * s}{\Delta} \right)^2$$

The following parameters are invariant:  $t_{\alpha} = 1.28$ ,  $t_{\beta} = 0.84$ ,  $\bar{x} = 27.848$  and  $s = 168.73$ .

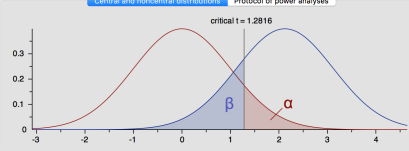
| % ↗ sess. time | $\Delta$ | N (each group) | % of users to request |
|----------------|----------|----------------|-----------------------|
| 0.1            | -0.028   | 329,988,124    | 3,595                 |
| 0.5            | -0.14    | 13,199,525     | 143.8                 |
| 1              | -0.28    | 3,299,881      | 36                    |
| 2              | -0.56    | 824,970        | 9                     |
| 5              | -1.39    | 131,995        | 1.4                   |
| 10             | -2.78    | 32,999         | 0.36                  |



# Using G\*Power

G\*Power 3.1

Central and noncentral distributions | Protocol of power analyses



Test family: t tests | Statistical test: Means: Difference between two independent means (two groups)

Type of power analysis: A priori: Compute required sample size - given  $\alpha$ , power, and effect size

Input parameters

Tail(s): One

Determine

Effect size d: 0.008249867

$\alpha$  err prob: 0.1

Power (1- $\beta$  err prob): 0.8

Allocation ratio N2/N1: 1

Output parameters

Noncentrality parameter  $\delta$ : 2.1231831

Critical t: 1.2815548

Df: 264934

Sample size group 1: 132468

Sample size group 2: 132468

Total sample size: 264936

Actual power: 0.8000019

X-Y plot for a range of values | Calculate

☐ n1 ≠ n2

Mean group 1: 0

Mean group 2: 1

SD  $\sigma$  within each group: 0.5

☒ n1 = n2

Mean group 1: 27.848

Mean group 2: 29.24

SD  $\sigma$  group 1: 168.73

SD  $\sigma$  group 2: 168.73

Calculate | Effect size d: 0.008249867

Calculate and transfer to main window

Close effect size drawer

## In R

```
[R]> mu1 <- 27.848
[R]> mu2 <- mu1 * 1.05
[R]> sigma <- 168.73
[R]>
[R]> alpha <- 0.1
[R]> beta <- 0.2
[R]>
[R]> # in base R:
[R]> power.t.test(delta = (mu2 - mu1),
+               sd = sigma,
+               sig.level=alpha,
+               power=(1-beta),
+               type='two.sample',
+               alternative='one.sided')
```

Two-sample t test power calculation

```
      n = 132391
delta = 1.3924
      sd = 168.73
sig.level = 0.1
      power = 0.8
alternative = one.sided
```

NOTE: n is number in *each* group

```
[R]> # or with the pwr package:
[R]> library(pwr)
[R]> pwr.t.test(d = (mu1-mu2)/sigma,
+             sig.level=alpha,
+             power=(1-beta),
+             type='two.sample',
+             alternative='less')
```

Two-sample t test power calculation

```
      n = 132391
      d = -0.008252237
sig.level = 0.1
      power = 0.8
alternative = less
```

NOTE: n is number in *each* group

# Useful links

## Recommended tools:

- Standalone software: <http://www.gpower.hhu.de/en.html>
- pwr package for R: <https://cran.r-project.org/web/packages/pwr/index.html>

## References for learning:

- <https://onlinecourses.science.psu.edu/stat414/node/306>  
(check the preceding lessons too)
- [http://www.ysumathstat.org/faculty/chang/class/s5817/L/L5817\\_1\\_2\\_PowerSampleSize\\_n.pdf](http://www.ysumathstat.org/faculty/chang/class/s5817/L/L5817_1_2_PowerSampleSize_n.pdf)
- [https://en.wikipedia.org/wiki/Statistical\\_hypothesis\\_testing#Common\\_test\\_statistics](https://en.wikipedia.org/wiki/Statistical_hypothesis_testing#Common_test_statistics)
- [https://github.com/rmuraglia/dsug/blob/master/201708\\_sample\\_size/dsug\\_sample\\_size\\_201708.pdf](https://github.com/rmuraglia/dsug/blob/master/201708_sample_size/dsug_sample_size_201708.pdf)