Exact and Approximated Expressions of the Log-Likelihood Ratio for 16-QAM Signals

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Abstract – This paper discusses the exact and approximate calculation of the bit log-likelihood ratio (LLR) for a 16-QAM signal. In particular, we introduce a new modified *max* function, which allows for an effective implementation of the LLR formula. The results of the paper are applicable to 16-QAM soft-demapping in digital receivers that perform soft decision decoding or iterative decoding.

I. INTRODUCTION

In modern communication receivers employing powerful error correction codes, the channel decoder performance is generally improved by the availability of reliability information on the encoded bits. Besides the performance advantage offered by soft decision decoding, the evaluation of the bit *a posteriori* probability is a central aspect in iterative decoding schemes based on the turbo concept [1]-[3]. Soft-in/soft-out decoding using the symbol-by-symbol maximum *a posteriori* probability (MAP) algorithm [4]-[7] is naturally based on log-likelihood values, and can be seen as a nonlinear filter that improves the quality of the bit log-likelihoods by using the relationship between the bits introduced by the code [5]. Turbo codes have been recently adopted in wireless standards like 3GPP WCDMA. The calculation of the log-likelihood ratio (LLR) on the coded bits is therefore an important function of practical wireless receivers. 3GPP high-speed downlink packet access (HSDPA), the high-speed evolution of the WCDMA standard [8], implements link adaptation by means of adaptive modulation and coding based on QPSK and 16-QAM modulation schemes, with different code rates obtained by varying the amount of puncturing from a basic rate 1/3 turbo code [9].

This paper discusses exact and approximated expressions of the bit LLR for a 16-QAM signal. In particular, we make use of a new modified max function that allows for an effective implementation of the LLR calculation.

The paper is organized as follows. In Section II, we introduce the signal model and the notation used in the paper. Section III provides the derivation of exact formulas for the efficient computation of the bit LLR, while approximate expressions are given in Section IV. Implementation aspects are further discussed in Section V.

II. SIGNAL MODEL

We consider a 16-QAM transmitted symbol sequence $s_k = s_{Ik} + j\, s_{Qk}$, with $s_{Ik}, s_{Qk} \in \{\pm 1, \pm 3\}$. Each modulation symbol is obtained from the information bits b_{ik}, $i = 1, \ldots, 4$, according to the mapping rule $s_k = \mathcal{M}(b_{1k}, b_{2k}, b_{3k}, b_{4k})$. Without loss of generality, we assume $s_{Ik} = \mathcal{M}_I(b_{1k}, b_{3k})$, $s_{Qk} = \mathcal{M}_Q(b_{2k}, b_{4k})$. An example of the above symbol mapping is given in Fig. 1 [10].

Let r_k denote the baseband received signal sample corresponding to symbol s_k,

$$r_k = r_{Ik} + jr_{Qk} = a\sqrt{E_s}\, s_k + n_k \ . \tag{1}$$

In (1), $a \in \mathrm{R}^+$, E_s represents the received signal symbol energy, and $n_k = n_{Ik} + jn_{Qk} = \mathcal{N}(0, N_0)$ is additive white complex noise, with n_{Ik} and n_{Qk} independent Gaussian processes with zero mean and variance $\sigma^2 = N_0 / 2$.

The above model is applicable to the despread signal at the output of the rake receiver or chip-level equalizer in a HSDPA mobile terminal.

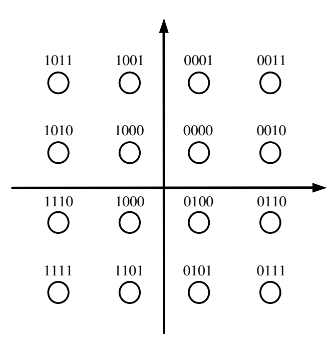

Fig.1. 16-QAM bit to symbol mapping.

III. Log-Likelihood Ratio Calculation

In this section we compute the bit LLR for the 16-QAM signal (1).

Let $\Lambda(b_{ik} \mid r_k)$ indicate the LLR of bit b_{ik} given r_k

$$\Lambda(b_{ik} \mid r_k) = \log \frac{\Pr(b_{ik} = +1 \mid r_k)}{\Pr(b_{ik} = -1 \mid r_k)} \ . \tag{2}$$

Define by $S(b_i = +1) = \{q : s^{(q)} = \mathcal{M}(\ldots, b_i^{(q)} = +1, \ldots)\}$ and $S(b_i = -1) = \{q : s^{(q)} = \mathcal{M}(\ldots, b_i^{(q)} = -1, \ldots)\}$ the subsets of symbol indexes corresponding to $b_i = +1$ and $b_i = -1$, respectively. Then

$$\Pr(b_{ik} = +1 \mid r_k) = \sum_{q \in S(b_{ik} = +1)} \Pr(s_k = s^{(q)} \mid r_k) \tag{3}$$

$$\Pr(b_{ik} = -1 \mid r_k) = \sum_{q \in S(b_{ik} = -1)} \Pr(s_k = s^{(q)} \mid r_k) \tag{4}$$

and in the case of equiprobable symbols

$$\Lambda(b_{ik} \mid r_k) = \log \frac{\sum_{q \in S(b_{ik} = +1)} p(r_k \mid s^{(q)})}{\sum_{q \in S(b_{ik} = -1)} p(r_k \mid s^{(q)})} \ . \tag{5}$$

Fig. 2 and Fig. 3 show the symbol set partitioning for the bits b_{1k} and b_{3k}, respectively, relative to the mapping introduced in Section II.

From the assumption on the noise n_k, we have

$$p(r_k \mid s^{(q)}) = \frac{1}{\pi N_0} \exp\left(- \frac{| r_k - a\sqrt{E_s}\, s^{(q)} |^2}{N_0} \right) \ . \tag{6}$$

Therefore, straightforward calculation gives

$$\Lambda(b_{1k} \mid r_k) = \log \left\{ \left[\exp\left(- \frac{(r_{Ik} - a\sqrt{E_s})^2}{N_0} \right) + \exp\left(- \frac{(r_{Ik} - 3a\sqrt{E_s})^2}{N_0} \right) \right] \Big/ \left[\exp\left(- \frac{(r_{Ik} + a\sqrt{E_s})^2}{N_0} \right) + \exp\left(- \frac{(r_{Ik} + 3a\sqrt{E_s})^2}{N_0} \right) \right] \right\} , \tag{7}$$

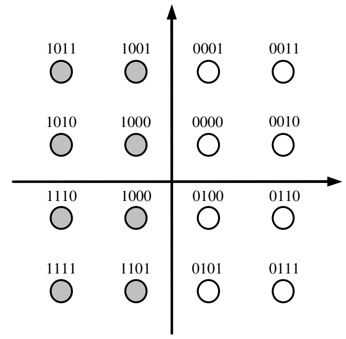

Fig.2. Symbol set partitioning for bit b_{1k}.

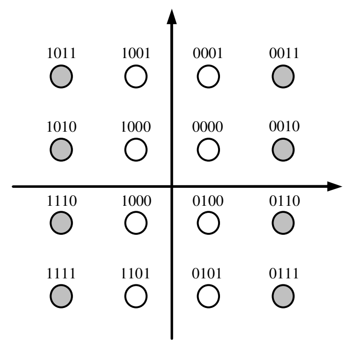

Fig.3. Symbol set partitioning for bit b_{3k}.

$$\Lambda(b_{3k} \mid r_k) = \log \left\{ \left[\exp\left(- \frac{(r_{Ik} - a\sqrt{E_s})^2}{N_0} \right) + \exp\left(- \frac{(r_{Ik} + a\sqrt{E_s})^2}{N_0} \right) \right] \Big/ \left[\exp\left(- \frac{(r_{Ik} - 3a\sqrt{E_s})^2}{N_0} \right) + \exp\left(- \frac{(r_{Ik} + 3a\sqrt{E_s})^2}{N_0} \right) \right] \right\} \ . \tag{8}$$

Note that $\Lambda(b_{1k} \mid r_k)$ and $\Lambda(b_{3k} \mid r_k)$ depend only on the in-phase signal component r_{Ik}. Analogous expressions are obtained for $\Lambda(b_{2k} \mid r_k)$ and $\Lambda(b_{4k} \mid r_k)$, as a function of the quadrature signal component r_{Qk}.

From (7) and (8), we derive

$$\Lambda(b_{1k} \mid r_k) = \log \left\{ \left[\exp\left(\frac{2ar_{Ik}\sqrt{E_s}}{N_0} \right) + \exp\left(\frac{6ar_{Ik}\sqrt{E_s}}{N_0} - \frac{8a^2 E_s}{N_0} \right) \right] \Big/ \left[\exp\left(- \frac{2ar_{Ik}\sqrt{E_s}}{N_0} \right) \right. \right.$$

These are linear approximations of log functions - error rate? LUT space?

$$+\exp\left(-\frac{6ar_{Ik}\sqrt{E_s}}{N_0}-\frac{8a^2E_s}{N_0}\right)\Bigg]\Bigg\} \qquad (9)$$

and

$$\Lambda(b_{3k}\mid r_k)=\log\Bigg\{\Bigg[\exp\left(\frac{2ar_{Ik}\sqrt{E_s}}{N_0}\right)$$

$$+\exp\left(-\frac{2ar_{Ik}\sqrt{E_s}}{N_0}\right)\Bigg]\Bigg/\Bigg[\exp\left(\frac{6ar_{Ik}\sqrt{E_s}}{N_0}-\frac{8a^2E_s}{N_0}\right)$$

$$+\exp\left(-\frac{6ar_{Ik}\sqrt{E_s}}{N_0}-\frac{8a^2E_s}{N_0}\right)\Bigg]\Bigg\}\ . \qquad (10)$$

Equations (9) and (10) can be efficiently computed by observing that [6], [7]

$$\log(e^x+e^y)=\max{}^*(x,y)\ , \qquad (11)$$

with

$$\max{}^*(x,y)=\max(x,y)+\log(1+\exp[-\mid x-y\mid])\ . \qquad (12)$$

Using (11),

$$\Lambda(b_{1k}\mid r_k)=\frac{4ar_{Ik}\sqrt{E_s}}{N_0}+\max{}^*\left(\frac{4ar_{Ik}\sqrt{E_s}}{N_0},\frac{8a^2E_s}{N_0}\right)$$

$$-\max{}^*\left(-\frac{4ar_{Ik}\sqrt{E_s}}{N_0},\frac{8a^2E_s}{N_0}\right)\ , \qquad (13)$$

$$\Lambda(b_{3k}\mid r_k)=\frac{8a^2E_s}{N_0}+\max{}^*\left(\frac{2ar_{Ik}\sqrt{E_s}}{N_0},-\frac{2ar_{Ik}\sqrt{E_s}}{N_0}\right)$$

$$-\max{}^*\left(\frac{6ar_{Ik}\sqrt{E_s}}{N_0},-\frac{6ar_{Ik}\sqrt{E_s}}{N_0}\right)\ . \qquad (14)$$

We observe that the computation of (14) can be further simplified by defining the new modified max function

$$\max{}^\$(x,y)=\max(x,y)+\log(1+\exp[-\mid x-y\mid]$$

$$-\exp[-\frac{\mid x-y\mid}{2}])\ . \qquad (15)$$

Making use of (15), (14) can be rewritten as

$$\Lambda(b_{3k}\mid r_k)=-\max{}^\$\left(\frac{4ar_{Ik}\sqrt{E_s}}{N_0}-\frac{8a^2E_s}{N_0},-\frac{4ar_{Ik}\sqrt{E_s}}{N_0}-\frac{8a^2E_s}{N_0}\right) \qquad (16)$$

The logarithmic correction terms of the modified max functions $\max^*(x,y)$ and $\max^\$(x,y)$ are compared in Fig. 4. A discussion on the implementation of (13) and (16) using the above functions is provided in Section V.

IV. Approximated Expressions

An approximation of $\Lambda(b_{1k}\mid r_k)$ is obtained from (13) by taking $\max^*(x,y)\approx\max(x,y)$ [6], [7]. This gives

$$\Lambda(b_{1k}\mid r_k)\approx\frac{4ar_{Ik}\sqrt{E_s}}{N_0}+\max\left(\frac{4ar_{Ik}\sqrt{E_s}}{N_0},\frac{8a^2E_s}{N_0}\right)$$

$$-\max\left(-\frac{4ar_{Ik}\sqrt{E_s}}{N_0},\frac{8a^2E_s}{N_0}\right)$$

$$=\begin{cases}\dfrac{8ar_{Ik}\sqrt{E_s}}{N_0}+\dfrac{8a^2E_s}{N_0} & r_{Ik}\le-2a\sqrt{E_s}\\[2ex] \dfrac{4ar_{Ik}\sqrt{E_s}}{N_0} & -2a\sqrt{E_s}\le r_{Ik}\le 2a\sqrt{E_s}\\[2ex] \dfrac{8ar_{Ik}\sqrt{E_s}}{N_0}-\dfrac{8a^2E_s}{N_0} & r_{Ik}\ge 2a\sqrt{E_s}\end{cases} \qquad (17)$$

An analogous result is obtained for $\Lambda(b_{2k}\mid r_k)$ substituting r_{Ik} with r_{Qk}. A comparison between the log-likelihood functions (13) and (17) is shown in Fig. 5 for $E_s/N_0=5\,\text{dB}$.

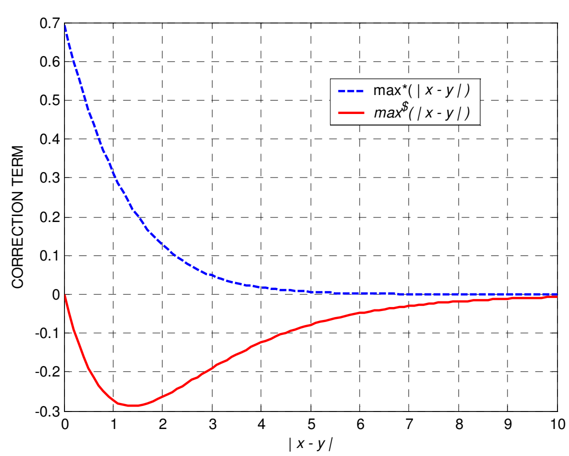

Fig. 4. Correction terms for the functions $\max^*(x, y)$ and $\max^\$(x, y)$.

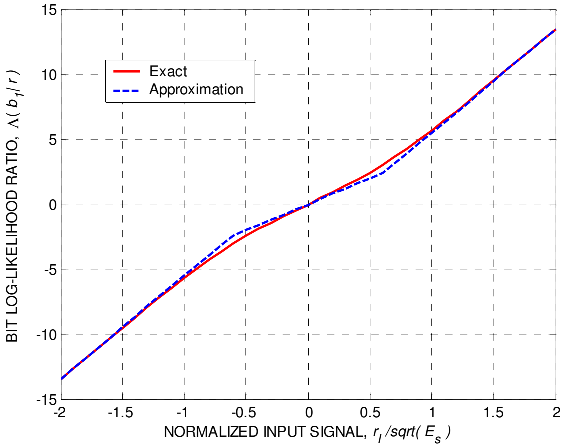

Fig. 5. Bit log-likelihood ratio $\Lambda(b_{1k}|r_k)$. $E_s/N_0 = 5$ dB, $a^2 = 1/10$.

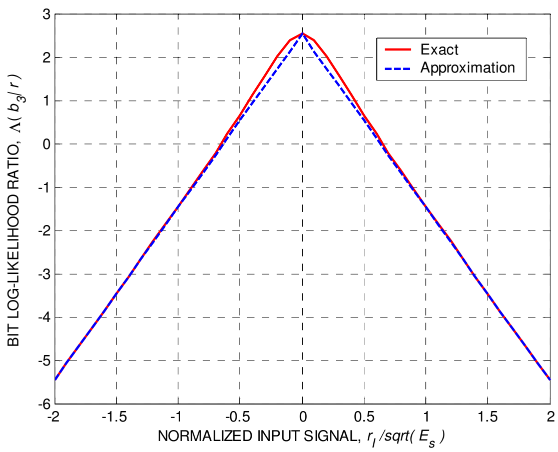

Fig. 6. Bit log-likelihood ratio $\Lambda(b_{3k}|r_k)$. $E_s/N_0 = 5$ dB, $a^2 = 1/10$.

An approximation of $\Lambda(b_{3k}\,|\,r_k)$ can be obtained by ignoring the correction factor of the $\max^{\$}$ function in (16) (or, equivalently, ignoring the correction term of the $\max^*$ functions in (14)),

$$\Lambda(b_{3k}\,|\,r_k) \approx \frac{8a^2E_s}{N_0} + \max\left(\frac{2ar_{Ik}\sqrt{E_s}}{N_0}, -\frac{2ar_{Ik}\sqrt{E_s}}{N_0}\right) - \max\left(\frac{6ar_{Ik}\sqrt{E_s}}{N_0}, -\frac{6ar_{Ik}\sqrt{E_s}}{N_0}\right)$$

$$= \begin{cases} \dfrac{4ar_{Ik}\sqrt{E_s}}{N_0} + \dfrac{8a^2E_s}{N_0} & r_{Ik} \le 0 \\ -\dfrac{4ar_{Ik}\sqrt{E_s}}{N_0} + \dfrac{8a^2E_s}{N_0} & r_{Ik} \ge 0 \end{cases} \tag{18}$$

Again, a similar result is obtained for $\Lambda(b_{4k}\,|\,r_k)$ substituting r_{Ik} with r_{Qk}. Fig. 6 reports the log-likelihood functions (16) and (18) for $E_s/N_0 = 5\,\text{dB}$.

V. IMPLEMENTATION

The calculation of the LLR functions (13) and (16), or (17) and (18), can be efficiently performed by pre-computing the offset $\Delta = 8a^2E_s/N_0$, and by pre-scaling the in-phase and quadrature input signal components r_{Ik} and r_{Qk} by $\eta = 4a\sqrt{E_s}/N_0$. The above quantities may be applied to the LLR calculation of a given block of data, and recomputed at predefined intervals. Letting $z_{Ik} = \eta\, r_{Ik}$ and $z_{Qk} = \eta\, r_{Qk}$, the exact bit LLR calculations for each 16-QAM symbol are summarized as

$$\Lambda(b_{1k}\,|\,r_k) = z_{Ik} + \max{}^*(z_{Ik}, \Delta) - \max{}^*(-z_{Ik}, \Delta) \tag{19}$$

$$\Lambda(b_{2k}\,|\,r_k) = z_{Qk} + \max{}^*(z_{Qk}, \Delta) - \max{}^*(-z_{Qk}, \Delta) \tag{20}$$

$$\Lambda(b_{3k}\,|\,r_k) = -\max{}^{\$}(-z_{Ik} - \Delta, z_{Ik} - \Delta) \tag{21}$$

$$\Lambda(b_{4k}\,|\,r_k) = -\max{}^{\$}(-z_{Qk} - \Delta, z_{Qk} - \Delta) \tag{22}$$

The modified max functions $\max^*(x, y)$ and $\max^{\$}(x, y)$ may be effectively implemented by means of look-up tables.

The implementation of $\Lambda(b_{3k}\,|\,r_k)$ and $\Lambda(b_{4k}\,|\,r_k)$ in (21) and (22) requires a comparator of z with zero, followed by the calculation of the sum $z + \Delta$ or $-z + \Delta$, and a table look-up that outputs the $\max^{\$}$ correction term given the input z. The approximated implementation (18) is obtained by omitting the correction term.

From the definition of $\max^*$, the implementation of $\Lambda(b_{1k}\,|\,r_k)$ and $\Lambda(b_{2k}\,|\,r_k)$ in (19) and (20) requires to calculate the quantities $z - \Delta$ and $z + \Delta$, compare z with the thresholds Δ and $-\Delta$, compute the sum of $z - \Delta$ or $z + \Delta$ and z (if $|z| \ge \Delta$), and finally obtain the $\max^*$ correction terms from table took-up addressed by $z - \Delta$ and $z + \Delta$. The implementation of the approximated LLR formulas is obtained by omitting the operation of table look-up.

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