

$$1. \quad \nexists \Gamma, T. \Gamma \vdash e : T$$

$$2. \quad \nexists T. \vdash e : T$$

Let $\Gamma = x : \text{Int}$. Then $\Gamma \vdash x : \text{Int}$
 or $\Gamma = x : \underset{\substack{\uparrow \\ \text{arbitrary}}}{T}$. Then $\Gamma \vdash x : T$.

$$3. \quad \vdash e : \text{Int}$$

$$4. \quad \vdash e : \text{Bool}$$

$$5. \quad \vdash e : \text{List}[\text{Int}]$$

$$6. \quad \nexists \Gamma, T. \Gamma \vdash e : T$$

$$7. \quad \vdash e : T$$

$$8. \quad \nexists \Gamma, T. \Gamma \vdash e : T$$

$$9. \quad \vdash e : \text{Int}$$

$$10. \quad \vdash e : \text{Bool}$$

$$11. \quad \nexists \Gamma, T. \vdash e : T$$

$$12. \quad \vdash e : \text{List}[\text{List}[\text{Int}]]$$

$$13. \quad \vdash e : \text{Int}$$

$$14. \quad \nexists \Gamma, T. \vdash e : T$$

15. $\nexists \Gamma, T. \Gamma \vdash e : T$

16. $\vdash e : \text{Int}$

17. $\vdash e : \text{Int}$

18. $\vdash e : \text{Int}$

19. $\vdash e : \text{Int} \rightarrow \text{Bool}$

20: $\vdash e : \text{Int}$

$$\begin{array}{c}
 \frac{\frac{}{\vdash 3:\text{Int}} \text{T-Int} \quad \frac{}{\vdash 4:\text{Int}} \text{T-Int}}{\vdash 3 > 4:\text{Bool}} \\
 \frac{\vdash 3 > 4:\text{Bool} \quad \frac{}{\vdash 5:\text{Int}} \text{T-Int} \quad \frac{\frac{\frac{}{\vdash 7:\text{Int}} \text{T-Int} \quad \frac{\frac{\frac{}{\vdash 10:\text{Int}} \text{T-Int}}{\vdash 10 * 3:\text{Int}} \text{T-Arith}}{\vdash 7 + 10 * 3:\text{Int}} \text{T-Arith}}{\vdash \text{if } 3 > 4 \text{ then } 5 \text{ else } 7 + 10 * 3:\text{Int}} \text{T-If}}{\vdash (\text{if } 3 > 4 \text{ then } 5 \text{ else } 7 + 10 * 3) = 10:\text{Int}} \text{T-Rel}
 \end{array}$$

Diagram illustrating a type derivation for the expression $(\text{if } 3 > 4 \text{ then } 5 \text{ else } 7 + 10 * 3) = 10$.

The derivation uses the following rules and assumptions:

- T-Int**: $\vdash 3:\text{Int}$, $\vdash 4:\text{Int}$, $\vdash 5:\text{Int}$, $\vdash 10:\text{Int}$, $\vdash 7:\text{Int}$
- T-Rel**: $\vdash (\text{if } 3 > 4 \text{ then } 5 \text{ else } 7 + 10 * 3) = 10:\text{Int}$
- T-If**: $\vdash \text{if } 3 > 4 \text{ then } 5 \text{ else } 7 + 10 * 3:\text{Int}$
- T-Arith**: $\vdash 7 + 10 * 3:\text{Int}$
- T-Arith**: $\vdash 10 * 3:\text{Int}$

Arrows indicate the flow of information from the sub-derivations to the final result.

$$\frac{\frac{}{T-Int} \vdash 1: Int \quad \frac{}{T-Nil} \vdash Nil[Int]: List[Int]}{\vdash 1: Nil[Int]: List[Int]} T-Cons$$

$$\frac{}{T-Int} \vdash 0: Int$$

$$\frac{(\text{hd}: Int, -: List[Int]) (\text{hd}) = Int}{\text{hd}: Int, -: List[Int] \vdash \text{hd}: Int} T-Var$$

$$\frac{}{T-Match} \vdash \text{match } 1::Nil[Int] \text{ with } Nil \rightarrow 0 \mid \text{hd}::_ \rightarrow \text{hd} \text{ end} : Int$$

Let ① = $\text{recur} : \text{Int} \rightarrow \text{Int} \rightarrow \text{Bool}$

Let ② = $n : \text{Int}, \text{recur} : \text{Int} \rightarrow \text{Int} \rightarrow \text{Bool}$

$$\begin{array}{c}
 \text{②}(n) = \text{Int} \quad \text{T-var} \\
 \hline
 \text{②}(\text{recur}) = \text{Int} \rightarrow \text{Int} \rightarrow \text{Bool} \quad \text{②} \vdash n : \text{Int} \quad \text{②} \vdash 1 : \text{Int} \quad \text{T-Int} \\
 \hline
 \text{②} \vdash \text{recur} : \text{Int} \rightarrow \text{Int} \rightarrow \text{Bool} \quad \text{②} \vdash n-1 : \text{Int} \quad \text{T-Aritn} \\
 \hline
 \text{②} \vdash \text{recur}(n-1) : \text{Int} \rightarrow \text{Bool} \quad \text{T-App} \\
 \hline
 \text{②} \vdash \text{recur}(n-1) : \text{Int} \rightarrow \text{Bool} \quad \text{T-Lambda} \\
 \hline
 \text{①} \vdash \lambda n : \text{Int}. \text{recur}(n-1) : \text{Int} \rightarrow \text{Int} \rightarrow \text{Bool} \\
 \hline
 \vdash (\text{fix } \dots) : \text{Int} \rightarrow \text{Int} \rightarrow \text{Bool} \quad \text{T-Fix} \\
 \hline
 \vdash 10 : \text{Int} \quad \text{T-Int} \\
 \hline
 \vdash \text{fix } \text{recur} : \text{Int} \rightarrow \text{Int} \rightarrow \text{Bool} \text{ is } \lambda n : \text{Int}. \text{recur}(n-1)) 10 : \text{Int} \rightarrow \text{Bool} \quad \text{T-App}
 \end{array}$$