Inference Rules

- Codify valid laws of logical inference
- Define mathematical relations inductively. ← this class

Review A binary relation R over set S is a subset of $S \times S$.

Example Take $S = \{0, 1, 2, \dots\}$ the set of nats and $R = \{(0,0), (1,1), (2,2), \dots\} \subseteq S \times S$.
Then $(a, b) \in R$ iff $a = b$.

Note on notation For binary relations, people usually use infix notation. Instead of writing $(a,b) \in R$, usually write $a \, R \, b$.

Examples include "$a \in b$" "$a = b$", "$a \subseteq b$", "$a \to b$" "$a \Downarrow b$".

Two roles : Designer vs Client

Designer ↙ design a system of rules to define some relation

Client ↓ use the system to answer questions about the relation

As a designer, you need to

1) define an expression language

2) propose a notation for the relation

3) define the relation inductively using <u>rules</u>.

A <u>rule</u> looks like

(Many) Hypotheses

rule name

$$\frac{H_1 \qquad H_2 \qquad H_3 \qquad \cdots}{C}\ R$$

(One) Conclusion

both hypo & concl are just statements involving the relation

interpretations

A. Assuming $H_1, H_2, H_3 \ldots$, rule R allows you to prove C.

inventory

materials

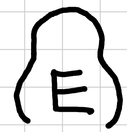
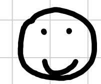
$$\frac{H_1 \quad H_2 \quad H_3}{C}\ \text{Recipe}$$

new material.

B. (Minecraft) If you have materials $H_1, H_2, H_3, \ldots$ in your inventory, then R is a recipe that says you can use those materials to produce material C.

Example 1 (Russian nesting dolls).

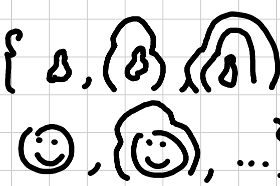
Design:

1. E =  |  | 

"atomic" smallest doll — "cover" add one more layer — "smiley"

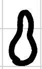
2. " Is $\underline{E}$ a Russian doll ?"

$\underline{\text{Notation}}$: " E ✓ "

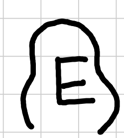
Note that this is a $\underline{\text{unary relation}}$, i.e., elements that satisfy it will form a subset of E = {  , ... }.

3.

$$\frac{}{\text{atomic doll} \ ✓} R_1 \quad \text{"Axiom"}$$



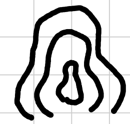
$$\frac{E \ ✓}{\text{cover}(E) \ ✓} R_2$$



Interpretation: " The atomic doll is a Russian doll ".

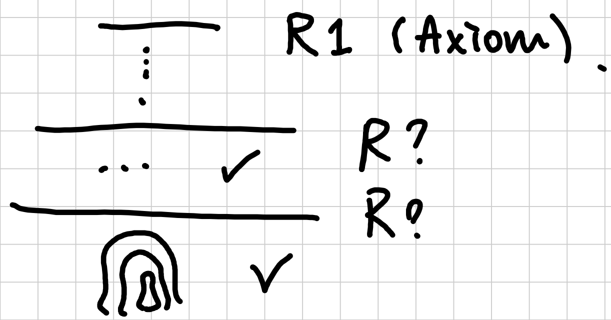
" If the thing has 1 layer, and the inner is a doll, then overall it's a doll ".

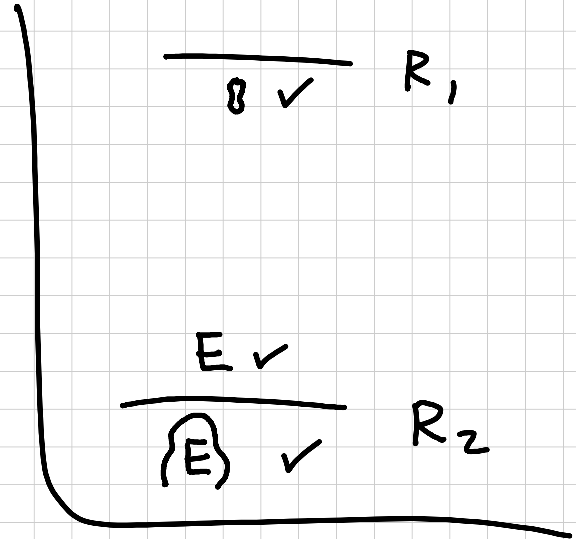
As a <u>client</u>, we can ask questions like
"Is E a Russian doll?" for different choices of E.

<u>Example</u> Let's show that " ✓".

Strategy: <u>work backwards</u>. Start with the ultimate goal, and see which rule applies & make our goal <u>simpler</u>, until we reach "<u>axiom</u>" rules which have no hypothesis, i.e. they're recipes that allow you produce stuff out of thin air.

So our proof / derivation / production plan will look like

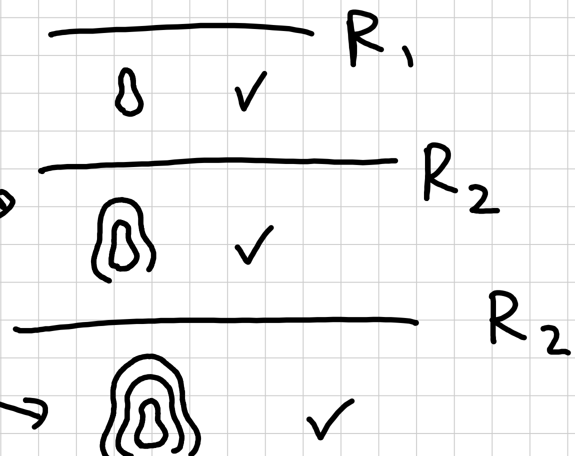




step 4: we're done! since R_1 has no hypothesis.

step3: only R_1 matches

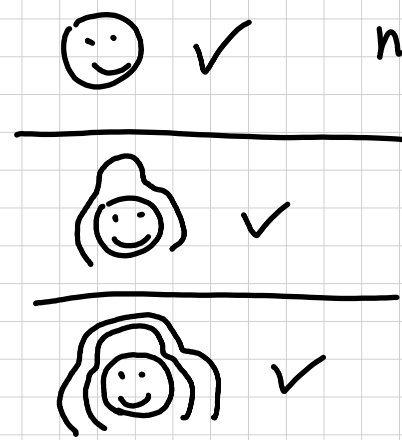
step2: pattern-match again.



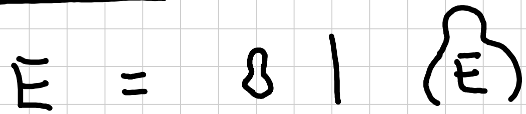
step1. Pattern match this with R_1 & R_2.

Only R_2 matches.

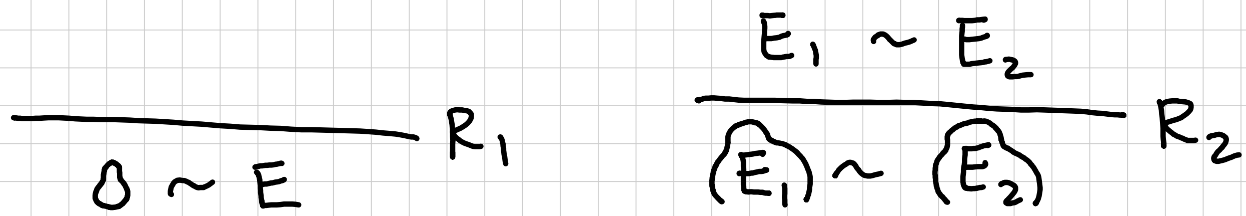
"stuck"! Neither R_1 nor R_2 applies.

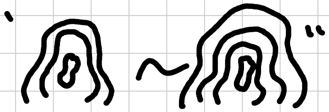
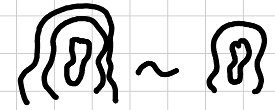


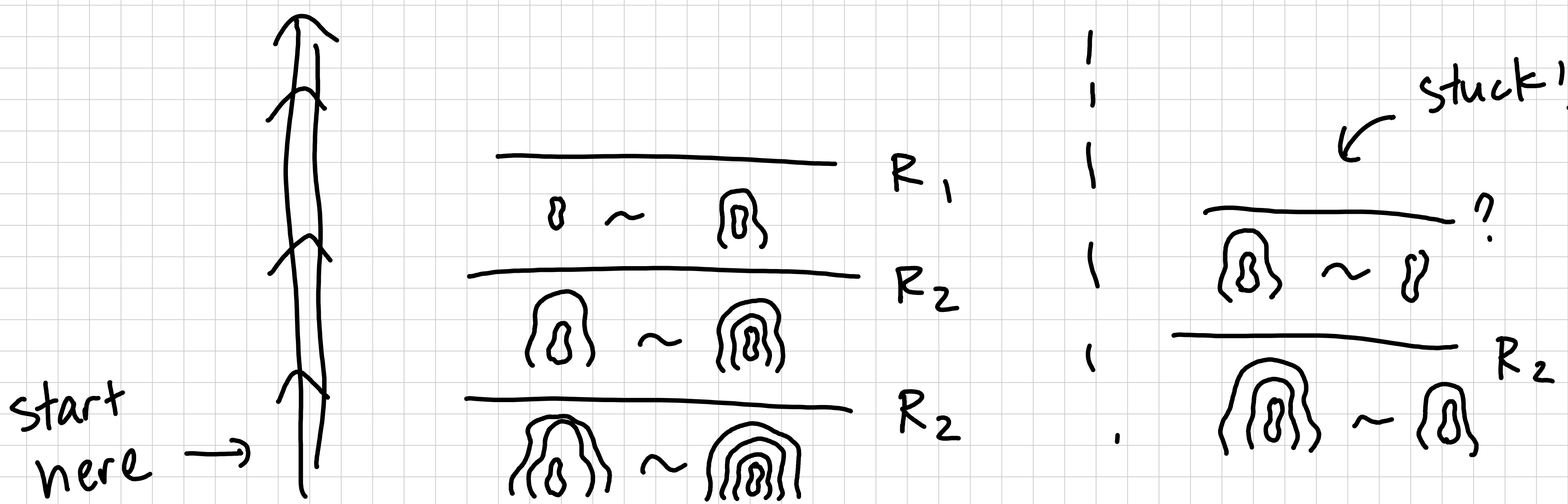
Example 2

1. 

2. "E ~ E"

3. 

Client: let's show "" holds. and "" doesn't.



In hindsight,

$E = 8 \mid \{E\}$ is just $nat = zero \mid succ\ nat$

"$E \sim E$" "$nat \leq nat$"

$$\frac{}{8 \sim E}\ R_1 \qquad \frac{}{0 \leq n}\ R_1$$

$$\frac{E_1 \sim E_2}{\{E_1\} \sim \{E_2\}}\ R_2 \qquad \frac{n \leq m}{n+1 \leq m+1}\ R_2$$

We just showed

$\{\{8\}\} \sim \{\{\{8\}\}\}$ $\qquad 2 \leq 3$

$\{\{8\}\} \not\sim \{8\}$ $\qquad 2 \not\leq 1$

Example 3

Design

$E = 0 \mid S\,E \mid E \otimes E$

"zero" "successor"

Relation " $E \Downarrow E$ "

Rules

$$\frac{}{0 \otimes E \Downarrow E}\, R_1 \qquad \frac{E_1 \otimes E_2 \Downarrow E_3}{(S\,E_1) \otimes E_2 \Downarrow S\,E_3}\, R_2$$

Client question: "For some E, does there exist E_2 such that $E \Downarrow E_2$?"

E.g. does there exist "?" s.t. $\boxed{S\,S\,E \otimes S\,E \Downarrow ?}$?

We're now asking two questions:

1) Can we find E_2?

2) Does $E \Downarrow E_2$?

The strategy is very similar.

1) Start with the end goal.
2) Work backwards " $E \Downarrow \ldots$ " using pattern-match on E to figure out which rules to use. Don't worry about " $\ldots$ " yet.
3) Once we hit an axiom, switch direction, working forward to figure out " $\ldots$ ".

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backwards

Rules: R_2, R_2, R_1

start here ⇒

$$\dfrac{\dfrac{\dfrac{}{O \otimes SO \Downarrow ?}\,R_1}{SO \otimes SO \Downarrow ?}\,R_2}{SSO \otimes SO \Downarrow ?}\,R_2$$

Then start here

$$\dfrac{\dfrac{\dfrac{}{O \otimes SO \Downarrow \boxed{SO}}\,R_1}{SO \otimes SO \Downarrow \boxed{SSO}}\,R_2}{SSO \otimes SO \Downarrow \boxed{SSSO}}\,R_2$$

In hindsight,

$E = 0 \mid S\,E \mid E \otimes E$ is just → $\otimes$ is addition "+"!

"$E \Downarrow E$" → "$E \Downarrow E$" is called (big-step) evaluation relation of operational semantics

$$\frac{}{0 \otimes E \Downarrow E}\ R_1 \quad \leadsto \quad 0 + n = n \quad \forall n$$

$$\frac{E_1 \otimes E_2 \Downarrow E_3}{S\,E_1 \otimes E_2 \Downarrow S\,E_3}\ R_2 \quad \leadsto \quad (1+m)+n = 1+(m+n) \quad \forall m, n.$$