CS 162 Programming languages

Lecture 6: λ-calculus II

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Design a programming language

- Syntax: what do programs look like?
    - Grammar: what programs are we allowed to write?
- Semantics: what do programs mean?
    - Operational semantics: how do programs execute step-by-step?

Syntax: what programs look like

$$
\begin{aligned}
e ::= \; & x \\
\mid \; & \lambda x.\, e \\
\mid \; & e_1 \; e_2
\end{aligned}
$$

\x → e (Haskell)

fun x → e (OCaml)

lambda x. e (λ+)

- Programs are expressions e (also called λ-terms) of one of three kinds:
  - Variable x, y, z
  - Abstraction (i.e. nameless function definition)
    - $\lambda x.\, e$
    - x is the formal parameter, e is the function body
  - Application (i.e. function call)
    - $e_1 \; e_2$
    - e_1 is the function, e_2 is the argument

Semantics: variable scope

The part of a program where a variable is visible

In the expression $\lambda x.\ e$

- x is the newly introduced variable
- e is the scope of x
- any occurrence of x in $\lambda x.\ e$ is bound (by the binder λx)

$\lambda x.\ x$

$\lambda x.\ (\lambda y.\ x)$

x is bounded

$x\ y$

$\lambda y.\ x\ y$

$(\lambda x.\ \lambda y.\ \ y)\ x$

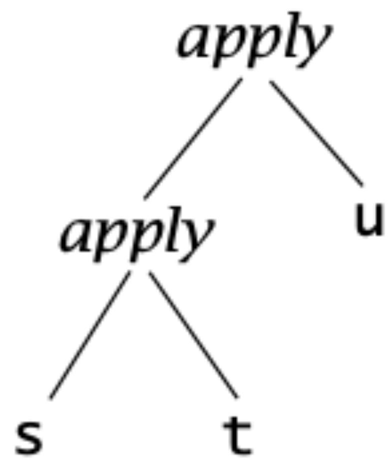
x is free

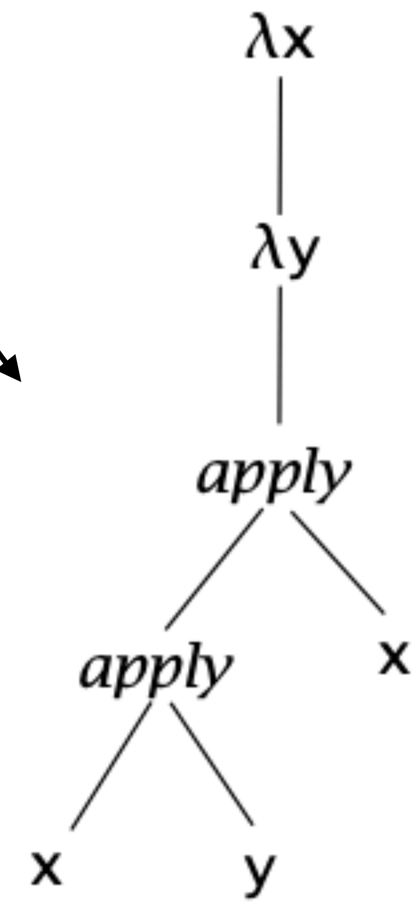
An occurrence of x in e is **free** if it's *not bound* by an enclosing abstraction

Precedence

$$\lambda x\,.\,\lambda y\,.\,x\ y\ x = \lambda x\,.\,(\lambda y\,.\,((x\ y)\ x))$$

$$s\ t\ u = (s\ t)\ u$$





Application associates to the **left**

bodies of abstractions are as far to the **right** as possible

—*Types and programming languages*

Semantics: free variables

An variable x is free in e if there exists a free occurrence of x in e

We use "FV" to represent the set of all free variables in a term:

$FV(x) = x$

$FV(\lambda x.\ e) = FV(e) \setminus x$

$FV(e_1\ e_2) = FV(e_1) \cup FV(e_2)$

$FV(x\ y) = \{x, y\}$

$FV(\lambda y.\ x\ y) = \{x\}$

$FV((\lambda x.\ \lambda y.\ y)\ x) = \{x\}$

If e has no free variables it is said to be closed, or combinators

Semantics: what programs mean

- How do I execute a λ-term?

- "Execute": rewrite step-by-step following simple rules, until no more rules apply

Similar to simplifying (x+1) * (2x -2) using middle-school algebra

$$
\begin{aligned}
e ::= \ & x \\
\mid \ & \lambda x.\, e \\
\mid \ & e_1\ e_2
\end{aligned}
$$

What are the rewrite rules for λ-calculus?

Operational semantics

$$(\lambda x\,.\,t_1)\ t_2 \rightarrow [x \mapsto t_2]t_1$$

β-reduction
(function call)

$[x \mapsto t_2]t_1$ means "t_1 with all ***free occurrences*** of x replaced with t_2"

```
inc1(int x ) {
  return x+1
}

inc1(2);
```

$$(\lambda x\,.\,x + 1)\ 2 \rightarrow [x \mapsto 2]x + 1 = 3$$

$$[x \mapsto y]\lambda x\,.\,x = \lambda x\,.\,y$$

What does free occurrences mean?

Semantics: β-reduction

$$(\lambda x\,.\,t_1)\ t_2 \rightarrow [x \mapsto t_2]t_1$$

β-reduction
(function call)

$[x \mapsto t_2]t_1$ means "t_1 with all ***free occurrences*** of x replaced with t_2"

The core of β-reduction reduces to substitution:

$$[x \mapsto s]x = s$$

$$[x \mapsto s]y = y (x \neq y)$$

$$[x \mapsto s]\lambda y\,.\,t_1 = \lambda y\,.\,[x \mapsto s]t_1 (y \neq x \wedge y \notin FV(s))$$

$$[x \mapsto s]t_1\ t_2 = [x \mapsto s]t_1\ [x \mapsto s]t_2$$

Semantics: α-renaming

$$\lambda x\,.\,e =_\alpha \lambda y\,.\,[x \mapsto y]e$$

- Rename a formal parameter and replace all its occurrences in the body

$$\lambda x\,.\,x =_\alpha \lambda y\,.\,y =_\alpha \lambda z\,.\,z$$

$$[x \mapsto y]\lambda x\,.\,x = \lambda x\,.\,y$$

$$[x \mapsto y]\lambda x\,.\,x =_\alpha [x \mapsto y]\lambda z\,.\,z = \lambda z\,.\,z$$

Call-by-name v.s. Call-by-value

$$(\lambda x\,.\,e_1)\ e_2 =_{\mathsf{name}} [x \mapsto e_2]e_1$$

Call-by-Name: From leftmost/outermost, allowing **no reductions** inside abstractions.

$$(\lambda x\,.\,e_1)\ e_2 =_{\mathsf{value}} [x \mapsto [e_2]]e_1$$

Call-by-Value: only when its right-hand side has already been reduced to a value—a term that **cannot be reduced any further**

Currying: multiple arguments

$$\lambda(x, y)\,.\,e = \lambda x\,.\,\lambda y\,.\,e$$

$$(\lambda(x, y)\,.\,x + y)\ 2\ 3 =$$
$$(\lambda x\,.\,\lambda y\,.\,x + y)\ 2\ 3 = (\lambda y.2 + y)\ 3 = [y \mapsto 3]2 + y = 5$$

Transformation of multi-arguments functions to higher-order functions is called currying (in the honor of Haskell Curry)

What about the others?

- ~~Assignment~~
- ~~Booleans, integers, characters, strings, …~~
- ~~Conditionals~~
- ~~Loops~~
- Functions
- ~~Recursion~~
- ~~References / pointers~~
- ~~Objects and classes~~
- ~~Inheritance~~

λ-calculus:Booleans

- How do we encode Boolean values (`TRUE` and `FALSE`) as functions?
- What do we do with Boolean?
- Make a binary choice
  - if b then e1 else e2

Booleans: API

We need to define three functions

- let TRUE = ???
- let FALSE = ???
- let ITE   = λb x y -> ??? -- if b then x else y

such that

- ITE TRUE apple banana = apple
- ITE FALSE apple banana = banana

Booleans: implementation

Boolean implementation

- let TRUE = λx y. x -- Returns its first argument
- let FALSE = λx y. y -- Returns its second argument
- let ITE = λb x y. b x y -- Applies condition to branches

Why they are correct?

Booleans: examples

eval ite_true:

ITE TRUE e_1 e_2

= (λb x y. b x y) TRUE e_1 e_2 -- expand def ITE

$=_\beta$ (λx y. TRUE x y) e_1 e_2 -- beta-step

$=_\beta$ (λy. TRUE e_1 y) e_2 -- beta-step

$=_\beta$ TRUE e_1 e_2 -- expand def TRUE

= (λx y. x) e_1 e_2 -- beta-step

$=_\beta$ (λy. e_1) e_2 -- beta-step

$=_\beta$ e_1

Other boolean API:

let NOT = λb. ITE b FALSE TRUE

let AND = λb_1 b_2. ITE b_1 b_2 FALSE

let OR = λb_1 b_2. ITE b_1 TRUE b_2

λ-calculus:Numbers

- Church numerals: a number N is encoded as a combinator that calls a function on an argument N times

let ONE = $\lambda f\ \lambda x.\ f\ x$

let TWO = $\lambda f\ \lambda x.\ f\ (f\ x)$

let THREE = $\lambda f\ \lambda x.\ f\ (f\ (f\ x))$

let FOUR = $\lambda f\ \lambda x.\ f\ (f\ (f\ (f\ x)))$

let FIVE = $\lambda f\ \lambda x.\ f\ (f\ (f\ (f\ (f\ x))))$

let SIX = $\lambda f\ \lambda x.\ f\ (f\ (f\ (f\ (f\ (f\ x)))))$

let ZERO = $\lambda f\ \lambda x.\ x$

https://en.wikipedia.org/wiki/Church_encoding

λ-calculus:Numbers API

- Numbers API
  - let INC = (λn λf λx. f (n f x))   -- Call \`f\` on \`x\` one more time than \`n\` does
  - let ADD = λn λm. n INC m. -- Call \`f\` on \`x\` exactly \`n + m\` times

eval inc_zero :
INC ZERO
= (λn λf λx. f (n f x)) ZERO
$=_\beta$ λf λx. f (ZERO f x)
= λf λx. f x
= ONE

eval add_one_zero :
ADD ONE ZERO = ONE

https://en.wikipedia.org/wiki/Church_encoding

TODOs by next lecture

- Install λ+
- Start to work on HW2
- Come to the discussion session if you have questions

let TWO = λf λx. f (f x)

- let INC = (λn λf λx. f (n f x))

INC TWO = ?