Exact results on traces of sets

Mingyuan Rong

(University of Science and Technology of China)

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Fudan University

This work is joint with Mingze Li and Jie Ma.

Trace and arrow relation

Definition of trace

For a subset T of V define the **trace** of $\mathcal{F} \subseteq 2^V$ on T by

$$\mathcal{F}_{|T} = \{F \cap T : F \in \mathcal{F}\}.$$

Definition of arrow relation

For integers n, m, a, and b, we say (n, m) **arrows** (a, b) and write

$$(n, m) \to (a, b)$$

if for every family $\mathcal{F} \subseteq 2^V$ with $|\mathcal{F}| \geqslant m$ and $|V| = n$, there is an a-element set $T \subseteq V$ such that $\left|\mathcal{F}_{|T}\right| \geqslant b$.

fixed (a, b) type results

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- Sauer-Shelah lemma (1972): $(n, m) \to (s, 2^s)$ for $m \geqslant 1 + \sum_{i=0}^{s-1} \binom{n}{i}$.
- Frankl (1983): $(n, m) \to (3, 7)$ for $m > \lfloor \frac{n^2}{4} \rfloor + n + 1$.
- Bollobás and Radcliffe (1995): $n \geqslant 4$ and $n \neq 6$ then $(n, m) \to (4, 12)$ for $m > \binom{n}{2} + n + 1$.
- Frankl and Wang (2024): $n \geqslant 25$ then $(n, m) \to (4, 13)$ for $m \geqslant 1 + \Pi_{i=0}^{2} \lfloor \frac{n+3+i}{3} \rfloor$.

Some early results

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if for every family $\mathcal{F} \subseteq 2^V$ with $|\mathcal{F}| \geqslant m$ and $|V| = n$, there is an a-element set $T \subseteq V$ such that $\left|\mathcal{F}_{|T}\right| \geqslant b$.

- Bondy: $(n, m) \to (n-1, m)$ for all $m \leqslant n$.
- Bollobás: $(n, m) \to (n-1, m-1)$ for all $m \leqslant \frac{3}{2}n$.
- Frankl: $(n, m) \to (n-1, m-2)$ for all $m \leqslant 2n$.
  $(n, m) \to (n-1, m-3)$ for all $m \leqslant \frac{7}{3}n$.

Some early results

For arrow relation, it suffices to consider **hereditary families** $\mathcal{F}$:

For any $F' \subseteq F \in \mathcal{F}$, we have $F' \in \mathcal{F}$.

Frankl (1983)

The following are equivalent:

- $(n, m) \to (a, b)$,
- For every **hereditary family** $\mathcal{F} \subseteq 2^{[n]}$ with $|\mathcal{F}| = m$, there exists $T \subseteq [n]$ with $|T| = a$ such that $|\mathcal{F}_{|T}| \geqslant b$.

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Using this, Frankl proved:

- For $s = 2^{d-1} - 1$, $(n, m) \to (n-1, m-s)$ for all $m \leqslant \frac{2^d-1}{d} n$.

Problem

Füredi / Frankl and Watanabe proposed the following problem.

Definition of $m(n, s)$

For n and s, what is the maximum value $\mathbf{m}(\mathbf{n}, \mathbf{s})$ such that for every $m \leqslant m(n, s)$ we have

$$(n, m) \to (n-1, m-s).$$

Frankl and Watanabe proved the following limit exists.

Definition of $m(s)$

$m(s) = \lim_{n \to +\infty} \frac{m(n,s)}{n}$.

Developments

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
m(x)	1	$\frac{3}{2}$	2	$\frac{7}{3}$				$\frac{15}{4}$								$\frac{31}{5}$	

- From previous developments, $m(0) = 1$, $m(1) = \frac{3}{2}$, $m(2) = 2$.
- Frankl (1983) proved $m(2^{d-1} - 1) = \frac{2^d - 1}{d}$ for $d \geqslant 1$. Therefore $m(3) = \frac{7}{3}$, $m(7) = \frac{15}{4}$ and $m(15) = \frac{31}{5}$.

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- Watanabe (1991) proved $m(4) = \frac{17}{6}$, $m(5) = \frac{13}{4}$, $m(6) = \frac{7}{2}$.

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- Frankl and Watanabe (1994) proved $m(9) = \frac{65}{14}$,
  and $m(2^{d-1} - 0) = \frac{2^d - 1}{d} + \frac{1}{2}$ for $d \geqslant 2$,
  and $m(2^{d-1} - 2) = \frac{2^d - 2}{d}$ for $d \geqslant 3$.
  Therefore $m(8) = \frac{17}{4}$, $m(9) = \frac{65}{14}$, $m(14) = 6$, $m(16) = \frac{67}{10}$.

Developments

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
m(x)	1	$\frac{3}{2}$	2	$\frac{7}{3}$	$\frac{17}{6}$	$\frac{13}{4}$	$\frac{7}{2}$	$\frac{15}{4}$	$\frac{17}{4}$	$\frac{65}{14}$	5			$\frac{29}{5}$	6	$\frac{31}{5}$	$\frac{67}{10}$

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- Watanabe (1995) proved $m(10) = 5$ and $m(13) = \frac{29}{5}$.

Developments

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
m(x)	1	$\frac{3}{2}$	2	$\frac{7}{3}$	$\frac{17}{6}$	$\frac{13}{4}$	$\frac{7}{2}$	$\frac{15}{4}$	$\frac{17}{4}$	$\frac{65}{14}$	5	?	?	$\frac{29}{5}$	6	$\frac{31}{5}$	$\frac{67}{10}$

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- Watanabe (1995) proved $m(10) = 5$ and $m(13) = \frac{29}{5}$.
- Frankl and Watanabe (1994) conjectured $m(11) = 5.3$ and $m(12) = 5.6$.

Developments

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
m(x)	1	$\frac{3}{2}$	2	$\frac{7}{3}$	$\frac{17}{6}$	$\frac{13}{4}$	$\frac{7}{2}$	$\frac{15}{4}$	$\frac{17}{4}$	$\frac{65}{14}$	5	?	$\frac{28}{5}$	$\frac{29}{5}$	6	$\frac{31}{5}$	$\frac{67}{10}$

In 2021, Piga and Schülke proved the following theorem.

Theorem (Piga and Schülke 2021)

1. $m(2^{d-1}-3)=\frac{2^d-3}{d}$ *for* $d \geqslant 4$.
2. $m(2^{d-1}-4)=\frac{2^d-4}{d}$ *for* $d \geqslant 5$. $\Longrightarrow m(12)=5.6$.

They also proved the following general bound.

Theorem (Piga and Schülke 2021)

- $m(2^{d-1}-c)=\frac{2^d-c}{d}$ *for all* $1 \leqslant c \leqslant \frac{d}{4}$.

In their article, Piga and Schülke ask the following question.

Question (Piga and Schülke)

For any positive integer d, determining the maximum integer $c_0(d)$ such that for any $c \leqslant c_0(d)$,

$$m(2^{d-1} - c) = \frac{2^d - c}{d}.$$

Our results

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
m(x)	1	$\frac{3}{2}$	2	$\frac{7}{3}$	$\frac{17}{6}$	$\frac{13}{4}$	$\frac{7}{2}$	$\frac{15}{4}$	$\frac{17}{4}$	$\frac{65}{14}$	5	$\frac{53}{10}$	$\frac{28}{5}$	$\frac{29}{5}$	6	$\frac{31}{5}$	$\frac{67}{10}$

Theorem (Li-Ma-R. , 2024+)

$$m(11) = m(2^{5-1} - 5) = \frac{2^5 - 5 - 0.5}{5} = 5.3.$$

- This solves the last open conjecture of Frankl and Watanabe in 1994.

Our results

Theorem (Li-Ma-R., 2024+)

Let $d \geqslant 50$. For $1 \leqslant c \leqslant d$, we have

$$m(2^{d-1} - c) = \mathfrak{B}_c,$$

where

$$\mathfrak{B}_c = \begin{cases} \frac{2^d - c}{d} & \text{for } 1 \leqslant c \leqslant d-1, \\ \frac{2^d - d - \frac{1}{2}}{d} & \text{for } c = d. \end{cases}$$

- This answers the previous question of Piga-Schülke, showing that

$$c_0(d) = d - 1 \text{ for } d \geqslant 50.$$

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- This answers the previous question of Piga-Schülke, showing that

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That is, the maximum c is $d-1$ for which the following holds

$$m(2^{d-1} - c) = \frac{2^d - c}{d}.$$

Notion

- Let n be a positive integers and $\mathcal{F} \subseteq 2^{[n]}$ be a hereditary family.

Notation

- For any $x \in [n]$, we write $\boldsymbol{d_{\mathcal{F}}(x)} = |\{F \in \mathcal{F} : x \in F\}|$ for the **degree** of x in $\mathcal{F}$.
- For any $x \in [n]$, we write $\boldsymbol{\mathcal{F}(x)}$ for the **link** of x in $\mathcal{F}$. That is, $\mathcal{F}(x) = \{F \setminus \{x\} : x \in F \in \mathcal{F}\}$.
- Let $\boldsymbol{\delta(\mathcal{F})} = \min_{x\in[n]} d_{\mathcal{F}}(x)$ be the **minimal degree** of $\mathcal{F}$.

Since for any hereditary family $\mathcal{F} \subseteq 2^V$ and $x \in V$, we have $|\mathcal{F}_{|V\setminus\{x\}}| = |\mathcal{F}| - d_{\mathcal{F}}(x)$. It is easy to get the following corollary.

Corollary

Let n, m and s be positive integers. The following are equivalent.

- $(n, m) \rightarrow (n-1, m-s)$.
- $m(n, s) \geqslant m$.
- *For any hereditary family $\mathcal{F} \subseteq 2^{[n]}$ with $|\mathcal{F}| \leqslant m$, there exists $x \in [n]$ such that $d_{\mathcal{F}}(x) \leqslant s$.*
- *For any hereditary family $\mathcal{F} \subseteq 2^{[n]}$ with $\delta(\mathcal{F}) \geqslant s+1$, we have $|\mathcal{F}| \geqslant m+1$.*

Constructions

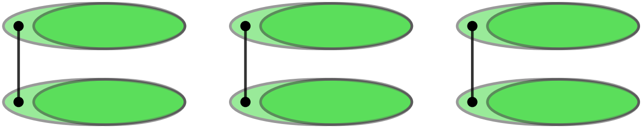
This construction shows $m(n, 2^{d-1} - d) \leqslant \frac{2^d - d - \frac{1}{2}}{d} n$ for $2d \mid n$.

Construction (Piga and Schülke 2021)

Let $U_1, \ldots, U_{2k}$ be a partition of $[n]$ into sets of size d. $x_i \in U_i$.

- $\mathcal{G} = \{S \subseteq V \colon$ there is an i such that $S \subseteq U_i$ and $|S| \leqslant d - 2\}$.
- $\mathcal{H} = \{U_i \setminus \{x_i\} \colon$ for $i \in \{1, 2, \ldots, 2k\}\}$.
- $\mathcal{I} = \{\{x_i, x_{i+1}\} \colon$ for $i \in \{1, 3, 5, \ldots, 2k - 1\}\}$.

Let $\mathcal{F} = \mathcal{G} \cup \mathcal{H} \cup \mathcal{I}$, then it is easy to see that $|\mathcal{F}| = \frac{2^d - d - \frac{1}{2}}{d} n + 1$ and $\delta(\mathcal{F}) \geqslant 2^{d-1} - d + 1$.



Constructions

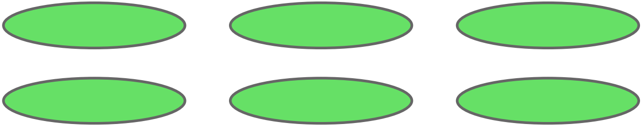
This construction shows $m(n, 2^{d-1} - c) \leqslant \frac{2^d - c}{d} n$ for $d \mid n, c \leqslant d$.

Construction

Let $U_1, \ldots, U_k$ be a partition of V into sets of size d. Let $\mathcal{G}_i \subseteq 2^{U_i}$, $|\mathcal{G}_i| = c - 1$.

- $\mathcal{F} = \{F \subseteq V : F \in 2^{U_i} \setminus \mathcal{G}_i$ for some $i \in [k]\}$.

It is easy to check that $|\mathcal{F}| = \frac{n}{d}(2^d - c) + 1$ and $\delta(\mathcal{F}) \geqslant 2^{d-1} - c + 1$.



Colexicographic order

Colexicographic order

For two finite sets $A, B \subseteq \mathbb{Z}_{>0}$, we say that $A \prec_{col} B$ or A precedes B in the **colexicographic** (**colex** for short) order if $\max(A \triangle B) \in B$.

- Let m be a positive integer, we define $\boldsymbol{\mathcal{R}(m)}$ to be the family containing the first m finite subsets of $\mathbb{Z}_{>0}$ according to the colex order.
- If $m = 2^k + t$, where k and t are non-negative integers with $t < 2^k$, then we have

$$\mathcal{R}(m) = 2^{[k]} \cup \{F \cup \{k+1\} : F \in \mathcal{R}(t)\}.$$

- For example, $\mathcal{R}(7) = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{3\}, \{1,3\}, \{2,3\}\}$.

A result of Katona

The following theorem due to Katona is a generalisation of the well-known Kruskal-Katona theorem.

Theorem (G. Katona 1978)

Let $f : \mathbb{N} \to \mathbb{R}$ be a monotone non-increasing function and let $\mathcal{F}$ be a hereditary family with $|\mathcal{F}| = m$. Then

$$\sum_{F \in \mathcal{F}} f(|F|) \geqslant \sum_{R \in \mathcal{R}(m)} f(|R|).$$

- In the proof, we often choose $\mathcal{F}$ as the link of some vertex and $f(k) = \frac{1}{k+1}$. For convenience, we set

$$\boldsymbol{W(m)} = \sum_{R \in \mathcal{R}(m)} \frac{1}{|R| + 1}$$

Properties on W-function

- Let d be a positive integer. We have

$$W(2^{d-1}) = \sum_{R\subseteq[d-1]} \frac{1}{|R|+1} = \sum_{i=0}^{d-1} \frac{\binom{d-1}{i}}{i+1} = \sum_{i=0}^{d-1} \frac{\binom{d}{i+1}}{d} = \frac{2^d-1}{d}.$$

- For any positive integer $c < 2^{d-2}$, since $A \triangle B = A^c \triangle B^c$ for any sets A and B, we have

$$2^{[d-1]} \setminus \mathcal{R}(2^{d-1}-c) = \{[d-1] \setminus H : H \in \mathcal{R}(c)\}.$$

- Thus we can conclude that

$$W(2^{d-1}-c) = \frac{2^d-1}{d} - \sum_{R\in\mathcal{R}(c)} \frac{1}{d-|R|} \geqslant \frac{2^d-1}{d} - \frac{c}{d-\log c}.$$

Frankl's proof on $m(n, 2^{d-1} - 1)$

- We only need to show that $m(n, 2^{d-1} - 1) \geqslant \frac{2^d - 1}{d} n$, which is equivalent to show that for any hereditary family $\mathcal{F} \subseteq 2^{[n]}$ with $\delta(\mathcal{F}) \geqslant 2^{d-1}$, then $|\mathcal{F}| \geqslant \frac{2^d - 1}{d} n + 1$.
- By Katona's theorem, for any $x \in [n]$,

$$\sum_{H \in \mathcal{F}(x)} \frac{1}{|H| + 1} \geqslant W(2^{d-1}) = \frac{2^d - 1}{d}.$$

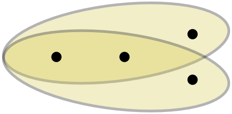
- Then it follows:

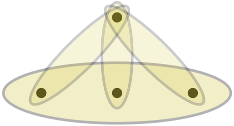
$$|\mathcal{F}| - 1 = |\mathcal{F} \setminus \{\emptyset\}| = \sum_{x \in [n]} \sum_{H \in \mathcal{F}(x)} \frac{1}{|H| + 1} \geqslant \frac{2^d - 1}{d} n \quad \square$$

Proof sketch: our first theorem on $m(11)$

- Let $\mathcal{F} \subseteq 2^{[n]}$ be a hereditary family with $\delta(\mathcal{F}) \geqslant 12$, we want to show that $|\mathcal{F}| \geqslant 5.3n + 1$.

- For the uniform weight $\boldsymbol{\omega(x)} = \sum_{H\in\mathcal{F}(x)} \frac{1}{|H|+1}$, $\omega(x) < 5.3$ if and only if $\mathcal{F}(x)$ is isomorphic with $\mathcal{R}(12)$ or $\binom{[4]}{\leqslant 2} \cup \{\{1,2,3\}\}$





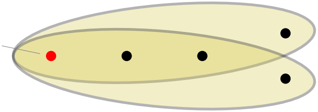
- We use a non-uniform weight $\boldsymbol{u(x)}$, which is slightly different from $\omega(w)$.

- Then we have: $u(x) < 5.3$ if and only if $\mathcal{F}(x) \cong \mathcal{R}(12)$. We call these vertices **mini-weight**.

Proof sketch: our first theorem on $m(11)$

- Note that $u(x) = 5.3 - \frac{2}{15}$ if x is mini-weight.

x is mini-weight



- We define the following "perturbation" $\varepsilon(x)$.

$$\varepsilon(x) = \begin{cases} -\frac{2}{15}, & x \text{ is mini-weight,} \\ \sum_{Q \in \mathcal{Q}(x)} \frac{\frac{1}{15}c(Q)}{4 - c(Q)}, & x \text{ is not mini-weight.} \end{cases}$$

- This ε transforms the "loss" of mini-weight vertices to others, and we can show that $u(x) - \varepsilon(x) \geqslant 5.3$ for any $x \in [n]$.

- Let $\mathcal{F} \subseteq 2^{[n]}$ be a hereditary family with $\delta(\mathcal{F}) \geqslant 2^{d-1}-c+1$ and $c \in [d]$.
- As in Frankl's proof, for any $x \in [n]$ we use the uniform **weight** $\boldsymbol{\omega(x)}$ defined as following

$$\omega(x) = \sum_{x \in F \in \mathcal{F}} \frac{1}{|F|} = \sum_{H \in \mathcal{F}(x)} \frac{1}{|H|+1}.$$

- This weight satisfies the following property.

$$\sum_{x \in [n]} \omega(x) = |\mathcal{F} \setminus \{\emptyset\}| = |\mathcal{F}| - 1.$$

- Since $\delta(\mathcal{F}) \geqslant 2^{d-1} - c + 1 > 2^{d-2}$, we can show that $|N(x)| \geqslant d$ for any $x \in [n]$.

Definition (good/bad vertex)

For any $x \in [n]$,

- we call x **bad** if $|N(x)| = d$;
- we call x **good** if $|N(x)| > d$.

Our second theorem: $m(2^{d-1} - c)$ for $c \leqslant d$

Definition (Pile)

Let $P \subseteq [n]$ and $|P| = d$. We call P a **pile**, if

- For any $y \in P$, we have $P \subseteq N(y)$.
- There exists $z \in P$, such that $P = N(z)$.

- Note that for $d \geqslant 6$, every bad vertex x is in exactly one pile, which is $N(x)$. This is the reason why we have to distinguish between the proofs of two results (they cannot be unified).

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Definition (Different types of piles)

A pile is an **intersecting pile** if it intersects another pile. Otherwise, we call the pile **non-intersecting pile**.

Proof sketch: $m(2^{d-1} - c)$, $1 \leqslant c \leqslant d$

We consider the following partition of piles,

$$\mathcal{P}_1 = \{\text{non-intersecting piles } P\},$$
$$\mathcal{P}_2 = \{\text{intersecting piles } P\}.$$

Then we have the following partition of $[n]$,

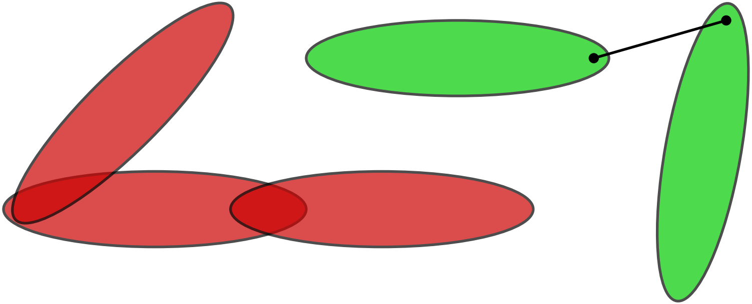
$$[n] = J \cup \left(\bigcup_{P \in \mathcal{P}_1} P \right) \cup \left(\bigcup_{P \in \mathcal{P}_2} P \right).$$

Note that every $x \in J$ is a good vertex.

Proof sketch: $m(2^{d-1} - c)$, $1 \leqslant c \leqslant d$

We will partition the vertex set $[n]$ into three parts and prove the average weight in each part is at least $\mathfrak{B}_c$.

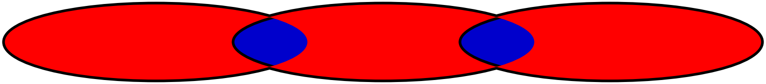
- Vertices that are not in any pile. (shown in white)
- Vertices that are in intersecting piles. (shown in red)
- Vertices that are in non-intersecting piles. (shown in green)



Proof sketch: $m(2^{d-1} - c)$, $1 \leqslant c \leqslant d$

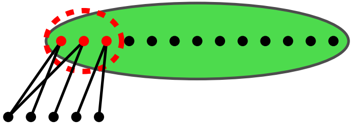
- **Out of piles.** We know that every vertex not contained in any piles is good, hence their weight is at least $\mathfrak{B}_c$.

- **Intersecting piles.** The "gain" of the intersection of piles (shown in blue) is enough to share with others (shown in red).



Proof sketch: $m(2^{d-1} - c)$, $1 \leqslant c \leqslant d$

- **Non-intersecting piles.** Assume the average weight is less than $\mathfrak{B}_c$. By calculating the weights there are at most 7 good vertices. By this the structure is almost fixed and we know the average weight is at least $\mathfrak{B}_c$ (will see soon).



Technical lemmas

Now we give more details for the proof of $m(2^{d-1} - c) = \mathfrak{B}_c$.

Lemma

Given two positive integers n and d with $d \geqslant 50$, $\mathcal{F} \subseteq 2^{[n]}$ is a hereditary family. Let $x \in [n]$ be a vertex with $d_{\mathcal{F}}(x) \geqslant 2^{d-1} - c + 1$, where $c \in [d]$. Then

1. *if $|N(x)| = d$, we have $\omega(x) > \mathfrak{B}_c - \frac{1}{18}$.*
2. *if $|N(x)| > d$, we have $\omega(x) > \mathfrak{B}_c - \frac{1}{18} + \frac{|N(x)|-d}{6} > \mathfrak{B}_c$.*

- By the above lemma, we know that the weight of any vertex is at least $\mathfrak{B}_c - \frac{1}{18}$.
- In particular, the weight of good vertex is at least $\mathfrak{B}_c$.

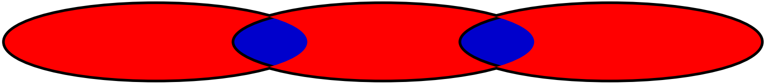
Technical lemmas

For any pile P, let θ_P be the number of vertices in P which only belong to one pile.

Lemma 1 (for intersecting piles)

If the family $\mathcal{P}(x)$ consisting all piles containing x has size at least 2, then we have

$$\omega(x) > \mathfrak{B}_c + \frac{1}{18} \sum_{P \in \mathcal{P}(x)} \frac{\theta_P}{d - \theta_P}$$



Technical lemmas

Let n, d and c be positive integers with $d \geqslant 50$ and $1 \leqslant c \leqslant d$.

Lemma 2 (for non-intersecting piles)

For any hereditary family $\mathcal{F} \subseteq 2^{[n]}$ with $\delta(\mathcal{F}) \geqslant 2^{d-1} - c + 1$. If $P \subseteq [n]$ is a non-intersecting pile, then

$$\sum_{x \in P} \omega(x) \geqslant \mathfrak{B}_c d.$$

This lemma is the most technical part of the proof.

Proof of Theorem

Let $\mathcal{F} \subseteq 2^{[n]}$ be a hereditary family with $\delta(\mathcal{F}) \geqslant 2^{d-1} - c + 1$, where $d \geqslant 50$ and $c \in [d]$. Adapting the uniform weight $\omega(x)$, we only need to show that $\sum_{x \in [n]} \omega(x) \geqslant \mathfrak{B}_c n$. Since we have the partition $[n] = J \cup (\bigcup_{P \in \mathcal{P}_2} P) \cup (\bigcup_{P \in \mathcal{P}_1} P)$, we only need to show that the average weight in each part is at least $\mathfrak{B}_c$.

Proof of Theorem

For any $x \in J$, we know that x is good and $\omega(x) \geqslant \mathfrak{B}_c$.
As for the part $K = \bigcup_{P \in \mathcal{P}_2} P$, we set $K_1 = \{x \in K : |\mathcal{P}(x)| = 1\}$ and $K_2 = K \setminus K_1$.
By Lemma 1, we can get

$$\begin{aligned}\sum_{x \in K_2} \omega(x) &\geqslant \sum_{x \in K_2} \left(\mathfrak{B}_c + \frac{1}{18} \sum_{P \in \mathcal{P}(x)} \frac{\theta_P}{d - \theta_P} \right) \\ &= \mathfrak{B}_c |K_2| + \frac{1}{18} \sum_{P \in \mathcal{P}_2} \theta_P\end{aligned}$$

$$\sum_{x \in K_1} \omega(x) \geqslant \sum_{x \in K_1} (\mathfrak{B}_c - \frac{1}{18}) = \mathfrak{B}_c |K_1| - \frac{1}{18} \sum_{P \in \mathcal{P}_2} \theta_P.$$

Proof of Theorem

Thus we have

$$\sum_{x\in K}\omega(x)=\sum_{x\in K_1}\omega(x)+\sum_{x\in K_2}\omega(x)\geqslant \mathfrak{B}_c(|K_1|+|K_2|)=\mathfrak{B}_c|K|.$$

For any $P\in\mathcal{P}_1$, by Lemma 2, we can get

$$\sum_{x\in P}\omega(x)\geqslant \mathfrak{B}_c d.$$

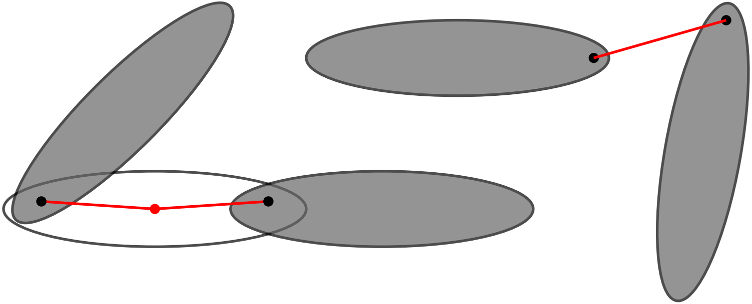
Proof of Theorem

Now combine all results above, we conclude that

$$|\mathcal{F}|-1=\sum_{x\in[n]}\omega(x)=\sum_{x\in J}\omega(x)+\sum_{x\in K}\omega(x)+\sum_{P\in\mathcal{P}_1}\sum_{x\in P}\omega(x)$$
$$\geqslant \mathfrak{B}_c(|J|+|K|+d|\mathcal{P}_1|)=\mathfrak{B}_c n.$$

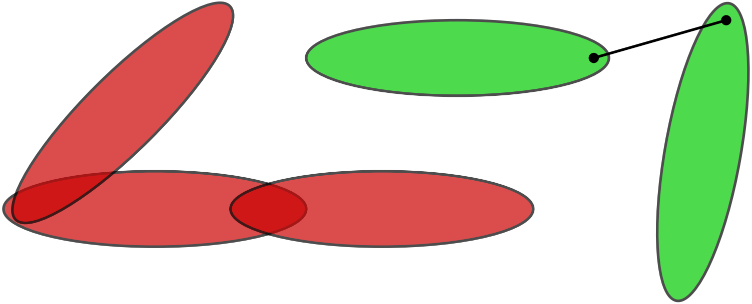
So we conclude that $|\mathcal{F}|\geqslant \mathfrak{B}_c n+1$ and we finish the proof.

Difference between our proof and Piga-Schülke proof



- They choose a maximum collection of disjoint piles and call them **clusters**.
- There could be bad vertices (shown in red) outside of all the clusters. For these bad vertices to have enough weight, they send $\frac{1}{2} - \frac{c-1}{d-c} \geqslant \frac{1}{6}$ weight for every vertex outside of a cluster (shown in red) which have connection with some vertices in the cluster.
- This is why they need $c \leqslant \frac{d}{4}$.

Difference between Our proof and Piga-Schülke proof



- There are no bad vertex outside of piles.
- We have more understanding in the edge structure of a pile, where they just used a simple inequality (works well for $c \leqslant \frac{d}{4}$).
- In order to achieve the range $1 \leqslant c \leqslant d$, we need $d \geqslant 50$ to make certain weight inequalities holds. Weight inequalities though out the proof are far from tight. So there is hope to improve this bound.

Open problems

- The general question: $m(2^{d-1} - c)$ for all possible c?
- What about $m(2^{d-1} + 1)$?

Thank you!

Thanks a lot for your attention!