HW9

5.4.3ace, 5.4.10, A.11ac, 3.3.5, 5.5.7

Exercise 5.4.3

a. Can a 3×4 matrix have independent columns? Independent rows? Explain.

A 3×4 matrix can have at most 3 linearly independent rows or columns.

So it can't have independent colums, since it can have at most 3 independent, but it has 4 columns 1 column must be dependent

It can have independent rows, since it has 3 rows all 3 rows could be independent

c. If A is an $m \times n$ matrix and rank $A = m$, show that $m \leq n$.

The rank of a matrix A is same as the number of independent rows or columns.

which means A has m independent rows or columns, which implies that m must be less or equal to n, since we define the number of independent rows or columns with the smaller number between the number of rows and columns.

e. Can the null space of a 3×6 matrix have dimension 2? Explain.

maximum rank that a 3×6 matrix can have is 3. The rank - nullity theorem says

rank A + nullity A = dimension of A

which

$3 + 2 \neq 6$

can't have null space with dimension of 2

Exercise 5.4.10 Let A be an $n \times n$ matrix.

a. Show that $A^2 = 0$ if and only if $\operatorname{col} A \subseteq \operatorname{null} A$.

b. Conclude that if $A^2 = 0$, then $\operatorname{rank} A \leq \frac{n}{2}$.

c. Find a matrix A for which $\operatorname{col} A = \operatorname{null} A$.

a.

Assume $A^2 = 0$

let b any vector in $col(A)$

Then $Ax = b$

Consider Ab

$$Ab = A(Ax) = A^2x = 0 \cdot x = 0$$

which means b is in $null(A)$

Therefore $col(A) \subseteq null(A)$

Assume $col(A) \subseteq null(A)$

Consider Ax for any x

Ax is in $col(A)$, since it's a linear combination of the columns A.

Since $col(A) \subseteq null(A)$,

$A(Ax) = 0$

therefore

$A^2 x = 0$

and

$A^2 = 0$

b.

we know that $A^2 \iff col(A) \subseteq null(A)$,

Rank - Nullity theorem says

$dim(null(A)) \leq$ most $n - rank\ A$

which $dim(null(A)) = dim(col(A)) = rank(A)$

therefore

$rank(A) \leq n - rank(A)$

$\Downarrow$

$rank(A) + rank(A) \leq n$

$$\Downarrow$$

$$2 \cdot \text{rank}(A) \le n$$

$$\Downarrow$$

$$\text{rank}(A) \le n/2$$

C.

A would be a matrix that its column vectors ar also in its null space.

which mean A is jus a 0 matrix

So for 2×2 matrix A would be

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{Col}(A) = \text{null}(A) = \{0\}$$

Exercise A.11 Find the roots of each of the following complex quadratic equations.

a. $x^2 + 2x + (1 + i) = 0$

c. $x^2 - (2 - i)x + (3 - i) = 0$

quadratic formula states

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

a.

$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(1+i)}}{2(1)}$$

$$= \frac{-2}{2} \pm \frac{\sqrt{4-4(1+i)}}{2}$$

$$= -1 \pm \frac{\sqrt{4(1-(1+i))}}{2}$$

$$= -1 \pm \frac{2\sqrt{1-1-i}}{2}$$

$$= -1 \pm \sqrt{-i} = -1 \pm \sqrt{-1} \cdot \sqrt{i}$$

$$= -1 \pm i\sqrt{i}$$

c.

$$x = \frac{-(-(2-i)) \pm \sqrt{(-(2-i))^2 - 4(1)(3-i)}}{2(1)}$$

$$= \frac{2-i \pm \sqrt{(-2+i)^2 - 4(3-i)}}{2}$$

$$= \frac{2-i \pm \sqrt{4-4i-1-12+4i}}{2}$$

$$= \frac{2-i \pm \sqrt{-9}}{2}$$

$$= \frac{2-i \pm 3i}{2}$$

$$= \frac{2-i+3i}{2} \quad \alpha \quad \frac{2-i-3i}{2}$$

$$= \frac{2+2i}{2} \quad \alpha \quad \frac{2-4i}{2}$$

$$= 1+i \quad \alpha \quad 1-2i$$

Exercise 3.3.5 Show that the eigenvalues of $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ are $e^{i\theta}$ and $e^{-i\theta}$.
(See Appendix A)

$$\begin{bmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix}$$

$$\det(A - \lambda i) = (\cos\theta - \lambda)(\cos\theta - \lambda) + (\sin\theta)(\sin\theta)$$

$$= \cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta$$

$$= \cos^2\theta + \sin^2\theta + \lambda^2 - 2\lambda\cos\theta$$

$$= 1 + \lambda^2 - 2\lambda\cos\theta$$

$$= \lambda^2 - 2\cos\theta\,\lambda + 1$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2\cos\theta) \pm \sqrt{(-2\cos\theta)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \frac{2\cos\theta \pm \sqrt{4(\cos^2\theta - 1)}}{}$$

$$2$$

$$= \frac{2\cos\theta \pm 2\sqrt{-\sin^2\theta}}{2}$$

$$= \frac{2\cos\theta \pm 2i\sin\theta}{2}$$

$$= \cos\theta \pm i\sin\theta$$

Euler's formula says

$$e^{i\theta} = \cos\theta + i\sin\theta$$

therefore

$$\lambda = e^{i\theta},\ e^{-i\theta}$$

Exercise 5.5.7 Let λ be an eigenvalue of A with corresponding eigenvector $\mathbf{x}$. If $B = P^{-1}AP$ is similar to A, show that $P^{-1}\mathbf{x}$ is an eigenvector of B corresponding to λ.

Let $B \cdot P^{-1}x = \lambda P^{-1}x$

since $B = P^{-1}AP$,

$$B \cdot P^{-1}x = P^{-1}AP \cdot P^{-1}x$$
$$= P^{-1}Ax$$

since $Ax = \lambda x$,

$$P^{-1} \cdot Ax = P^{-1}\lambda x$$
$$= \lambda \cdot P^{-1}x$$

therefore

$$BP^{-1}x = \lambda \cdot P^{-1}x$$