HW6

3.2.7ac,
3.3.1bhi, 3.3.3, 3.3.7, 3.3.17

Exercise 3.2.7 If $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = -2$ calculate:

a. $\det \begin{bmatrix} 2 & -2 & 0 \\ c+1 & -1 & 2a \\ d-2 & 2 & 2b \end{bmatrix}$

$ad - bc = -2$

$$\begin{matrix} 2 & -2 & 0 & 2 & -2 \\ c+1 & -1 & 2a & c+1 & -1 \\ d-2 & 2 & 2b & d-2 & 2 \end{matrix}$$

$-4b + (-4a)(d-2) + 0 - (0 + 8a + (-4b)(c+1))$

$\cancel{-4b} - 4ad \cancel{+8a} \cancel{-8a} + 4bc \cancel{+4b}$

$-4(ad - bc)$

$-4(-2) = 8$

c. $\det(3A^{-1})$ where $A = \begin{bmatrix} 3c & a+c \\ 3d & b+d \end{bmatrix}$

$ad - bc = -2$

$3bc + \cancel{3cd} - 3ad - \cancel{3cd}$

$-3(ad - bc)$

$-3(-2) = 6$

$3^2 \cdot (1/6) = 9 \cdot \frac{1}{6} = \frac{9}{6} = \frac{3}{2}$

Exercise 3.3.1 In each case find the characteristic polynomial, eigenvalues, eigenvectors, and (if possible) an invertible matrix P such that $P^{-1}AP$ is diagonal.

b. $A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$

$$\begin{matrix} 2-\lambda & -4 \\ -1 & -1-\lambda \end{matrix} \qquad -2+\lambda-2\lambda+\lambda^2-4$$

$$\lambda^2-\lambda-6$$

$$\begin{matrix} -3 \\ 2 \end{matrix}$$

$$\boxed{\lambda = 3, -2}$$

$$\begin{matrix} 2-3 & -4 \\ -1 & -1-3 \end{matrix} \Rightarrow \begin{matrix} -1 & -4 \\ -1 & -4 \end{matrix} \to \begin{matrix} 1 & 4 \\ 0 & 0 \end{matrix} \quad \boxed{\begin{bmatrix} 4 \\ -1 \end{bmatrix}}$$

$$\begin{matrix} 2+2 & -4 \\ -1 & -1+2 \end{matrix} \Rightarrow \begin{matrix} 4 & -4 \\ -1 & 1 \end{matrix} \to \begin{matrix} 1 & -1 \\ 0 & 0 \end{matrix} \quad \boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$\boxed{\begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix}}^{-1} = \begin{matrix} 4 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{matrix} \to \begin{matrix} 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 1 & -1 & 0 & -1 \end{matrix}$$

$$\begin{matrix} 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & -\frac{5}{4} & -\frac{1}{4} & 1 \end{matrix} \to \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \begin{bmatrix} \frac{1}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\frac{2}{5}+\frac{1}{5}=\frac{3}{5} \qquad -\frac{4}{5}+\frac{1}{5}=-\frac{3}{5}$$

$$\frac{2}{5}-\frac{4}{5}=-\frac{2}{5} \qquad -\frac{4}{5}-\frac{4}{5}=-\frac{8}{5}$$

$$\begin{bmatrix} \frac{3}{5} & -\frac{3}{5} \\ -\frac{2}{5} & -\frac{8}{5} \end{bmatrix}\begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}}$$

h. $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$

$$\begin{matrix} 2-\lambda & 1 & 1 & 2-\lambda & 1 \\ 0 & 1-\lambda & 0 & 0 & 1-\lambda \\ 1 & -1 & 2-\lambda & 1 & -1 \end{matrix}$$

$$(2-\lambda)(1-\lambda)(2-\lambda)+0+0-(1-\lambda+0+0)$$

$$-(1-\lambda)$$

$$-\lambda^3+5\lambda^2-8\lambda+4 \quad -1+\lambda$$

$$-\lambda^3+5\lambda^2-7\lambda+3=0$$

$$\boxed{\lambda=1}$$

$$-1+5-7+3=8-8=0$$

$$\begin{array}{r|rrrr} 1 & -1 & 5 & -7 & 3 \\ & & -1 & 4 & -3 \\ \hline & -1 & 4 & -3 & \boxed{0} \end{array}$$

$$-1\lambda^2+4\lambda-3=0$$

$\boxed{\lambda = 1}$

$$-1+4-3 = 4-4 = 0$$

$$\begin{array}{c|ccc}
1 & -1 & 4 & -3 \\
 & & -1 & 3 \\
\hline
 & -1 & 3 & \boxed{0}
\end{array}$$

$$-\lambda + 3$$

$$(\lambda-1)(\lambda-1)(-\lambda+3) = 0$$

$$(\lambda-1)^2(-\lambda+3) = 0$$

$$\lambda = 1, 3$$

2 -1 1 1 1 1 1

$$\begin{matrix} 0 & 1-1 & \cdot \\ 1 & -1 & 2-1 \end{matrix} \rightarrow \begin{matrix} 0 & 0 & 0 \\ 1 & -1 & 1 \end{matrix}$$

$\swarrow$ swap R_2 & R_3

$R_2 - R_1$

$$\begin{matrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{matrix} \xrightarrow{-\frac{1}{2}R_2 \quad R_1 - R_3} \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \qquad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{matrix} 2-3 & 1 & 1 \\ 0 & 1-3 & 0 \\ 1 & -1 & 2-3 \end{matrix} \rightarrow \begin{matrix} -1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & -1 & -1 \end{matrix}$$

$\swarrow$ $-R_1$ $R_3 - R_1$

$$\begin{matrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \xrightarrow{R_1 - R_2} \begin{matrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix}$$

$$\boxed{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} \text{ so } \boxed{\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}}$$

NOT invertible! NOP.

$$\left[\begin{matrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{matrix}\right. \xrightarrow[R_3 - R_1]{-R_3} \begin{matrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & 0 & -1 \end{matrix}$$

swap R_2 & R_3

$-\frac{1}{2}R_2$

$R_1 - R_2$

$$\left.\begin{matrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{matrix}\,\right|\,\begin{matrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{matrix}$$

$0 \quad 0 \quad 0 | 0 \quad 1 \quad 0$

$P^{-1}AP = D$

$$\begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

$\frac{1}{2}\cdot 2 + 0\cdot 0 + -\frac{1}{2}\cdot -1 = \frac{1}{2}$ $\quad \frac{1}{2}\cdot 1 + 0\cdot 1 + -\frac{1}{2}\cdot -1 = 1$ $\quad \frac{1}{2}\cdot 1 + 0\cdot 0 + -\frac{1}{2}\cdot 2 = -\frac{1}{2}$

$\frac{1}{2}\cdot 2 + 0\cdot 0 + \frac{1}{2}\cdot 1 = \frac{3}{2}$ $\quad \frac{1}{2}\cdot 1 + 0\cdot 1 + \frac{1}{2}\cdot -1 = 0$ $\quad \frac{1}{2}\cdot 1 + 0\cdot 0 + \frac{1}{2}\cdot 2 = \frac{3}{2}$

$0\cdot 2 + 1\cdot 0 + 0\cdot 1 = 0$ $\quad 0\cdot 1 + 1\cdot 1 + 0\cdot -1 = 1$ $\quad 0\cdot 1 + 1\cdot 0 + 0\cdot 2 = 0$

$$\begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{3}{2} & 0 & \frac{3}{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}}$$

$\frac{1}{2}\cdot 1 + 1\cdot 0 + -\frac{1}{2}\cdot -1 = 1$ $\quad \frac{1}{2}\cdot -1 + 1\cdot 0 + -\frac{1}{2}\cdot 1 = -1$ $\quad \frac{1}{2}\cdot 1 + 1\cdot 0 + -\frac{1}{2}\cdot 1 = 0$

$\frac{3}{2}\cdot 1 + 0\cdot 0 + \frac{3}{2}\cdot -1 = 0$ $\quad \frac{3}{2}\cdot -1 + 0\cdot 0 + \frac{3}{2}\cdot 1 = 0$ $\quad \frac{3}{2}\cdot 1 + 0\cdot 0 + \frac{3}{2}\cdot 1 = 3$

$0\cdot 1 + 1\cdot 0 + 0\cdot -1 = 0$ $\quad 0\cdot -1 + 1\cdot 0 + 0\cdot 1 = 0$ $\quad 0\cdot 1 + 1\cdot 0 + 0\cdot 1 = 0$

i. $A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}, \lambda \neq \mu$

$$\begin{bmatrix} \lambda - x & 0 & 0 \\ 0 & \lambda - x & 0 \\ 0 & 0 & \mu - x \end{bmatrix} \quad \det = (\lambda - x)(\lambda - x)(\mu - x) + 0 \cdots$$

$$= (\lambda - x)^2 (\mu - x)$$

eigenvalue

$x = \lambda, \mu$

$$\begin{bmatrix} \lambda - \lambda & 0 & 0 \\ 0 & \lambda - \lambda & 0 \\ 0 & 0 & \mu - \lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu - \lambda \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ (\mu - \lambda) z \end{bmatrix} = 0$$

eigen vector : $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$

$$\begin{bmatrix} \lambda - \mu & 0 & 0 \\ 0 & \lambda - \mu & 0 \\ 0 & 0 & \mu - \mu \end{bmatrix} = \begin{bmatrix} \lambda - \mu & 0 & 0 \\ 0 & \lambda - \mu & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda-\mu & 0 & 0 \\ 0 & \lambda-\mu & 0 \\ 0 & 0 & 0 \end{bmatrix} \boxed{\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}}$$

since A is already a diagonal matrix

any matrix can be P such that

$P^{-1} A P = D$.

Exercise 3.3.3 Show that A has $\lambda = 0$ as an eigenvalue if and only if A is not invertible.

A with $\lambda=0$ as its eigenvalue means,

$$Ax = \lambda x = 0.$$

for a nonzero vector x, $Ax=0$ means,
the colums of A are linearly dependent.
$\therefore$ A is not full rank / NOT invertible.

If A is not invertible, A doesn't have full rank
and A's columns are linearly dependent.
Then there exists non zero vector x
such that $Ax = 0$. i.e. $Ax = 0x$

which means

$\lambda = 0$ is an eigenvalue of A.

Exercise 3.3.7 Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ show that:

a. $c_A(x) = x^2 - \operatorname{tr} Ax + \det A$, where $\operatorname{tr} A = a + d$ is called the **trace** of A.

b. The eigenvalues are $\frac{1}{2}\left[(a+d) \pm \sqrt{(a-b)^2 + 4bc}\right]$.

a.

$$\begin{bmatrix} a-x & b \\ c & d-x \end{bmatrix}$$

$\det = (a-x)(d-x) - bc$

$$x^2-(a+d)x+ad-bc = x^2-(\mathrm{tr}A)x+\det(A)$$
$$= x^2-(a+d)x+ad-bc$$

true

b. $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$\frac{-(-(a+d)) \pm \sqrt{(-(a+d))^2 - 4(1)(ad-bc)}}{2(1)}$$

$$\boxed{(-(a+d))^2 = (-a-d)^2 = a^2+2ad+d^2}$$

$$\boxed{\begin{aligned} a^2+2ad-4ad+d^2 &= a^2-2ad+d^2 \\ &= (a-d)^2 \end{aligned}}$$

$= \frac{1}{2}\left((a+d) \pm \sqrt{(a-d)^2 + 4bc}\right.$

true.

Exercise 3.3.17 A square matrix A is called **nilpotent** if $A^n = 0$ for some $n \geq 1$. Find all nilpotent diagonalizable matrices. [*Hint*: Theorem 3.3.1.]

$P^{-1}AP = D$, then $A = PDP^{-1}$,

Then $A^n = (PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}}) \cdots (\cancel{P}DP^{-1})$

$= PD^nP^{-1}$

If $PD^nP^{-1} = 0$

then $D^n = P^{-1}0P = 0$

which means

$$A^n = \begin{bmatrix} \lambda^n & \cdots & \\ & \lambda^n & \vdots \\ \vdots & \ddots & \\ \cdots & & \lambda^n \end{bmatrix} = 0$$

$\therefore$ All A that has its only eigenvalue $\lambda = 0$.