HW8

5.1.1ace, 5.1.16ace, 5.2.4ac, 5.2.7acej, 5.2.19

We often write vectors in $\mathbb{R}^n$ as rows.

Exercise 5.1.1 In each case determine whether U is a subspace of $\mathbb{R}^3$. Support your answer.

a. $U = \{(1, s, t) \mid s \text{ and } t \text{ in } \mathbb{R}\}$.

1. 0 vector is in U
2. U is closed under vectoraddition
3. U is closed under scalar multiplication

No matter what s, t are, U will never be (0, 0, 0).

So U is not a subspace of $\mathbb{R}^3$

c. $U = \{(r, s, t) \mid r, s, \text{ and } t \text{ in } \mathbb{R}, -r+3s+2t=0\}$.

(r, s, t) can be 0 vector

$(r_1, s_1, t_1) + (r_2, s_2, t_2)$

$= (r_1 + r_2, s_1 + s_2, t_1 + t_2)$

since $-r + 3s + 2t = 0$,

$-(r_1 + r_2) + 3(s_1 + s_2) + 2(t_1 + t_2)$

$= (-r_1 + 3s_1 + 2t_1) + (-r_2 + 3s_2 + 2t_2)$

$= 0 + 0 = 0$

So V is closed under vector addition.

$c \cdot (r, s, t) = (cr, cs, ct)$

$-cr + 3cs + 2ct$

$-c(-r+3s+2t)=c(0)=0$

So U is closed under scalar multiplication

So U is a subspace of $\mathbb{R}^3$

e. $U = \{(r, 0, s) \mid r^2 + s^2 = 0,\ r \text{ and } s \text{ in } \mathbb{R}\}$.

$(r, 0, s)$ can be 0 vector

$(r_1, 0, s_1) + (r_2, 0, s_2) = (r_1 + r_2, 0, s_1 + s_2)$

$(r_1 + r_2, 0, s_1 + s_2)$

$(r_1 + r_2)^2 + (s_1 + s_2)^2$

$= r_1^2 + 2r_1 r_2 + r_2^2 + s_1^2 + 2s_1 s_2 + s_2^2$

$= r_1^2 + s_1^2 + r_2^2 + s_2^2 + 2r_1 r_2 + 2s_1 s_2$

$= 0 + 0 + 2r_1 r_2 + 2s_1 s_2 \neq 0$

U is NOT closed under vector addition

So U is NOT a subspace of $\mathbb{R}^3$

Exercise 5.1.16 In each case either show that the statement is true or give an example showing that it is false.

a. If $U \neq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ and $\mathbf{x}+\mathbf{y}$ is in U, then $\mathbf{x}$ and $\mathbf{y}$ are both in U.

True, because of the subspace theorem.

Second rule says

'U is closed under vector addition'

and the vectors have to be in U.

In other words

If $x+y$ is in U

x and y must be in U

c. If U is a subspace of $\mathbb{R}^n$ and $\mathbf{x}$ is in U, then $-\mathbf{x}$ is also in U.

True. The third rule says

'U is closed under scalar multiplication.'

So if x is in U

$-1 \cdot x$ must be in U

e. The empty set of vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

False, because the first rule says,

'0 vector is in U.'

but an empty set of vectors has nothing.

Exercise 5.2.4 Find a basis and calculate the dimension of the following subspaces of $\mathbb{R}^4$.

a. $U = \left\{ \begin{bmatrix} a \\ a+b \\ a-b \\ b \end{bmatrix} \middle| \, a \text{ and } b \text{ in } \mathbb{R} \right\}$

$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} a + \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} b$ $\quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

with rank of 2

which means the dimension is $\mathbb{R}^2$

c. $U = \left\{ \begin{bmatrix} a \\ b \\ c+a \\ c \end{bmatrix} \middle| \, a, b, \text{ and } c \text{ in } \mathbb{R} \right\}$

$$\begin{pmatrix}1\\0\\1\\0\end{pmatrix}a+\begin{pmatrix}0\\1\\0\\0\end{pmatrix}b+\begin{pmatrix}0\\0\\1\\1\end{pmatrix}c \qquad \begin{bmatrix}1&0&0\\0&1&0\\1&0&1\\0&0&1\end{bmatrix}\to\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

with rank of 3

which means the dimension is $\mathbb{R}^3$

Exercise 5.2.7 In each case show that the statement is true or give an example showing that it is false.

a. If $\{\mathbf{x}, \mathbf{y}\}$ is independent, then $\{\mathbf{x}, \mathbf{y}, \mathbf{x}+\mathbf{y}\}$ is independent.

True, since

$ax + by = 0$, if and only if

$a = b = 0$.

So $ax + by + c(x+y)$

$= ax + by + cx + cy$

$(a+c)x + (b+c)y = 0$

if and only if

$a = b = c = 0$

c. If {**y**, **z**} is dependent, then {**x**, **y**, **z**} is dependent for any **x**.

$y = kz$

$akz + bz = 0$

$ak = -b$

$a \neq b$

since $a \neq b$,

$ay + bz + cx$

cannot be independent.

Because to be independent

$a = b = c$.

e. If one of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is zero, then $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ is dependent.

True, because even if one of the elements is zero then the coefficient can have infinitely many possibilities which means,

say $x_k = 0$,

then $\{a_1 x_1, a_2 x_2 \cdots a_k x_k\} = 0$

is true even if

$$a_1 = a_2 = \cdots = a_{k-1} \neq a_k$$

which means dependent.

j. **Every non-empty subset of a linearly independent set is again linearly independent.**

True, because a linearly independent set only carries linearly independent subsets.

Exercise 5.2.19 Let U and W denote subspaces of $\mathbb{R}^n$, and assume that $U \subseteq W$. If $\dim U = n-1$, show that either $W = U$ or $W = \mathbb{R}^n$.

Since U is in W and W is in $\mathbb{R}^n$,

$\mathbb{R}^n \geq W \geq U$.

and W can't have dimension

n and $n-1$ at the same time,

So it has to be either $W=U$ or $W=\mathbb{R}^n$