

HW6

3.2.7ac,

3.3.1bhi, 3.3.3, 3.3.7, 3.3.17

Exercise 3.2.7 If $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = -2$ calculate:

a. $\det \begin{bmatrix} 2 & -2 & 0 \\ c+1 & -1 & 2a \\ d-2 & 2 & 2b \end{bmatrix}$

$$ad - bc = -2$$

$$\begin{array}{ccccc} 2 & -2 & 0 & 2 & -2 \\ c+1 & -1 & 2a & c+1 & -1 \\ d-2 & 2 & 2b & d-2 & 2 \end{array}$$

$$-4b + (-4a)(d-2) + 0 - (0 + 8a + (-4b)(c+1))$$

$$\cancel{-4b} - 4ad \cancel{+ 8a - 8a} + 4bc \cancel{+ 4b}$$

$$-4(ad - bc)$$

$$-4(-2) = 8$$

c. $\det(3A^{-1})$ where $A = \begin{bmatrix} 3c & a+c \\ 3d & b+d \end{bmatrix}$

$$ad - bc = -2$$

~~$$3bc + 3cd - 3ad = 3cd$$~~

$$-3(ad - bc)$$

$$-3(-2) = 6$$

$$3^2 \cdot \left(\frac{1}{6}\right) = 9 \cdot \frac{1}{6} = \frac{9}{6} = \frac{3}{2}$$


Exercise 3.3.1 In each case find the characteristic polynomial, eigenvalues, eigenvectors, and (if possible) an invertible matrix P such that $P^{-1}AP$ is diagonal.

b. $A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$

$$2-\lambda \quad -4$$

$$-2+\lambda-2\lambda+\lambda^2-4$$

$$-1 \quad -1-\lambda$$

$$\begin{array}{r} \lambda^2 - \lambda - 6 \\ -3 \\ \hline 2 \end{array}$$

$$\lambda = 3, -2$$

$$2-3 \quad -4$$

$$-1 \quad -(-3)$$

$$\Rightarrow \begin{array}{rr} -1 & -4 \\ -1 & -4 \end{array}$$

$$\begin{array}{rr} 1 & 4 \\ 0 & 0 \end{array} \rightarrow$$

$$\begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$2+2 \quad -4$$

$$-1 \quad -1+2$$

$$\Rightarrow \begin{array}{rr} 4 & -4 \\ -1 & 1 \end{array} \rightarrow$$

$$\begin{array}{rr} 1 & -1 \\ 0 & 0 \end{array}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix}}^{-1}$$

$$= \begin{array}{rrr} 4 & 1 & 1 \ 0 \\ -1 & 1 & 0 \ 1 \end{array} \rightarrow \begin{array}{rrr} 1 & \frac{1}{4} & \frac{1}{4} \ 0 \\ 1 & -1 & 0 \ -1 \end{array}$$



$$\begin{array}{rrr} 1 & \frac{1}{4} & \frac{1}{4} \ 0 \\ 0 & -\frac{5}{4} & -\frac{1}{4} \ 1 \end{array} \rightarrow \begin{array}{rr} 1 & 0 \\ 0 & 1 \end{array} \left(\begin{array}{cc} \frac{1}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{4}{5} \end{array} \right)$$

$$\begin{pmatrix} \frac{1}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\frac{2}{5} + \frac{1}{5} = \frac{3}{5} \quad -\frac{4}{5} + \frac{1}{5} = -\frac{3}{5}$$

$$\frac{2}{5} - \frac{4}{5} = -\frac{2}{5} \quad -\frac{4}{5} - \frac{4}{5} = -\frac{8}{5}$$

$$\begin{bmatrix} \frac{3}{5} & -\frac{3}{5} \\ -\frac{2}{5} & -\frac{8}{5} \end{bmatrix} \begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}}$$

h. $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$

$$\begin{vmatrix} 2-\lambda & 1 & 1 & 2-\lambda & 1 \\ 0 & 1-\lambda & 0 & 0 & 1-\lambda \\ 1 & -1 & 2-\lambda & 1 & -1 \end{vmatrix}$$

$$(2-\lambda)(1-\lambda)(2-\lambda) + 0 + 0 - (1-\lambda + 0 + 0) \\ -(1-\lambda)$$

$$-\lambda^3 + 5\lambda^2 - 8\lambda + 4 - 1 + \lambda$$

$$-\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0$$

$$\boxed{\lambda = 1}$$

$$-1 + 5 - 7 + 3 = 8 - 8 = 0$$

$$\begin{array}{r} 1 \left[\begin{array}{rrr} -1 & 5 & -7 \\ & -1 & 4 & -3 \end{array} \right] \\ \hline -1 & 4 & -3 & \boxed{0} \end{array}$$

$$-\lambda^2 + 4\lambda - 3 = 0$$

$$\boxed{\lambda = 1}$$

$$-1 + 4 - 3 = 4 - 4 = 0$$

$$\begin{array}{r} \left| \begin{array}{ccc} -1 & 4 & -3 \\ & -1 & 3 \\ \hline -1 & 3 & 0 \end{array} \right. \\ -\lambda + 3 \end{array}$$

$$(\lambda - 1)(\lambda - 1)(-\lambda + 3) = 0$$

$$(\lambda - 1)^2 (-\lambda + 3) = 0$$

$$\boxed{\lambda = 1, 3}$$

$$\begin{array}{ccccc} 2-1 & | & | & | & | \\ - & 1-1 & 0 & - & - \end{array}$$

$$\begin{matrix} 0 & 1-1 & - \\ 1 & -1 & 2-1 \end{matrix} \rightarrow \begin{matrix} 0 & 0 & 0 \\ 1 & -1 & 1 \end{matrix}$$

$\leftarrow \frac{\text{swap } R_2 \times R_3}{R_2 - R_1}$

$$\begin{matrix} 1 & 1 \\ 0 & -2 & 0 \end{matrix} \rightarrow \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix}$$

$\leftarrow \frac{-\frac{1}{2}R_2}{R_1 - R_3}$

$$\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \quad , \quad \boxed{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

$$\begin{matrix} 2-3 & 1 & 1 \\ 0 & 1-3 & 0 \\ 1 & -1 & 2-3 \end{matrix} \rightarrow \begin{matrix} -1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & -1 & -1 \end{matrix}$$

$\leftarrow \frac{-R_1}{R_3 - R_1}$

$$\begin{array}{ccc} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \xrightarrow{R_1 - R_2} \begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}$$

So

$$\left[\begin{array}{ccc} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \quad \left[\begin{array}{ccc} 1 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{array} \right]$$

$$\begin{array}{cccccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array} \xrightarrow{\substack{-R_3 \\ R_3 - R_1}} \begin{array}{cccccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & 0 & -1 \end{array}$$

Swap R₂ & R₃

$$\begin{array}{c} -\frac{1}{2}R_2 \\ R_1 - R_2 \end{array}$$

$$\begin{array}{c|cc} 1 & -1 & 0 \\ 0 & 0 & 1 \end{array} \begin{array}{c|cc} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{array}$$

$$0 \ 0 \ 0 | 0 \ 1 \ 0$$

$$P^{-1} A P = D$$

$$\begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\frac{1}{2} \cdot 2 + 0 \cdot 0 + -\frac{1}{2} \cdot -1 = \frac{1}{2} \quad \frac{1}{2} \cdot 1 + 0 \cdot 1 + -\frac{1}{2} \cdot -1 = 1 \quad \frac{1}{2} \cdot 1 + 0 \cdot 0 + -\frac{1}{2} \cdot 2 = -\frac{1}{2}$$

$$\frac{1}{2} \cdot 2 + 0 \cdot 0 + \frac{1}{2} \cdot 1 = \frac{3}{2} \quad \frac{1}{2} \cdot 1 + 0 \cdot 1 + \frac{1}{2} \cdot -1 = 0 \quad \frac{1}{2} \cdot 1 + 0 \cdot 0 + \frac{1}{2} \cdot 2 = \frac{3}{2}$$

$$0 \cdot 2 + 1 \cdot 0 + 0 \cdot 1 = 0 \quad 0 \cdot 1 + 1 \cdot 1 + 0 \cdot -1 = 1 \quad 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 2 = 0$$

$$\begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{3}{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}}$$

$$\frac{1}{2} \cdot 1 + 1 \cdot 0 + -\frac{1}{2} \cdot -1 = 1 \quad \frac{1}{2} \cdot -1 + 1 \cdot 0 + -\frac{1}{2} \cdot 1 = -1 \quad \frac{1}{2} \cdot 1 + 1 \cdot 0 + -\frac{1}{2} \cdot 1 = 0$$

$$\frac{3}{2} \cdot 1 + 0 \cdot 0 + \frac{3}{2} \cdot -1 = 0 \quad \frac{3}{2} \cdot -1 + 0 \cdot 0 + \frac{3}{2} \cdot 1 = 0 \quad \frac{3}{2} \cdot 1 + 0 \cdot 0 + \frac{3}{2} \cdot 1 = 3$$

$$0 \cdot 1 + 1 \cdot 0 + 0 \cdot -1 = 0 \quad 0 \cdot -1 + 1 \cdot 0 + 0 \cdot 1 = 0 \quad 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 = 0$$

$$\text{i. } A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}, \lambda \neq \mu$$

$$\begin{bmatrix} \lambda - x & 0 & 0 \\ 0 & \lambda - x & 0 \\ 0 & 0 & M - x \end{bmatrix} \quad \begin{aligned} \det &= (\lambda - x)(\lambda - x)(M - x) + 0 \dots \\ &= (\lambda - x)^2(M - x) \end{aligned}$$

eigen value

$$x = \lambda, M$$

$$\begin{bmatrix} \lambda - \lambda & 0 & 0 \\ 0 & \lambda - \lambda & 0 \\ 0 & 0 & M - \lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M - \lambda \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ (M - \lambda)z \end{bmatrix} = 0$$

eigen vector :

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda - M & 0 & 0 \\ 0 & \lambda - M & 0 \\ 0 & 0 & M - M \end{bmatrix} = \begin{bmatrix} \lambda - M & 0 & 0 \\ 0 & \lambda - M & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{pmatrix} \lambda - M & 0 & 0 \\ 0 & \lambda - M & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

since A is already a diagonal matrix

any matrix can be P such that

$$P^T A P = D.$$

Exercise 3.3.3 Show that A has $\lambda = 0$ as an eigenvalue if and only if A is not invertible.

A with $\lambda=0$ as its eigenvalue means

$$Ax = \lambda x = 0.$$

for a nonzero vector x , $Ax=0$ means,
the columns of A are linearly dependent.

$\therefore A$ is not full rank / NOT invertible.

If A is not invertible, A doesn't have full rank
and A 's columns are linearly dependent.

Then there exists non zero vector x
such that $Ax=0$. ie. $Ax=0x$

which means

$\lambda=0$ is an eigenvalue of A .

Exercise 3.3.7 Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ show that:

- a. $c_A(x) = x^2 - \text{tr } Ax + \det A$, where $\text{tr } A = a + d$ is called the **trace** of A .
- b. The eigenvalues are $\frac{1}{2} \left[(a + d) \pm \sqrt{(a - d)^2 + 4bc} \right]$.

a.

$$\begin{bmatrix} a-x & b \\ c & d-x \end{bmatrix} \quad \det = (a-x)(d-x) - bc$$

$$x^2 - (a+d)x + ad - bc = x^2 - (\text{tr } A)x + \det(A)$$
$$= x^2 - (a+d)x + ad - bc$$

truc

b.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{-(-(a+d)) \pm \sqrt{(-(a+d))^2 - 4(1)(ad - bc)}}{2(1)}$$

$$(-(a+d))^2 = (-a-d)^2 = a^2 + 2ad + d^2$$

$$a^2 + 2ad - 4ad + d^2 = a^2 - 2ad + d^2$$
$$= (a-d)^2$$

$$= \frac{1}{2} \left((a+d) \pm \sqrt{(a-d)^2 + 4bc} \right)$$

true.

Exercise 3.3.17 A square matrix A is called **nilpotent** if $A^n = 0$ for some $n \geq 1$. Find all nilpotent diagonalizable matrices. [Hint: Theorem 3.3.1.]

$P^{-1}AP = 0$, then $A = PDP^{-1}$,

Then $A^n = (PDP^{-1})(PDP^{-1}) \dots (PDP^{-1})$

$$= P D^n P^{-1}$$

If $P D^n P^{-1} = 0$

then $D^n = P^{-1}0P = 0$

which means

$$A^n = \begin{bmatrix} \lambda^n & & & \\ & \lambda^n & & \\ & & \ddots & \\ & & & \lambda^n \end{bmatrix} = 0$$

\therefore All A that has its only eigen value $\lambda = 0$.

