HW11

6.4.22a, 7.1.1egi, 7.1.11, 7.2.2aei, 7.2.7

Exercise 6.4.22

a. Let $p(x)$ and $q(x)$ lie in $\mathbf{P}_1$ and suppose that $p(1) \neq 0$, $q(2) \neq 0$, and $p(2) = 0 = q(1)$. Show that $\{p(x),\ q(x)\}$ is a basis of $\mathbf{P}_1$. [*Hint*: If $rp(x) + sq(x) = 0$, evaluate at $x = 1$, $x = 2$.]

LI: since $P(1) \neq 0$,

at $x=1$,

$P(1) + q(1) \neq 0$

$\therefore$ $rP(1) + s\,q(1) = 0$ if and only if

$r = s = 0$

and same thing happens at $x=2$

$q(2) \neq 0$

$P(2) + q(2) \neq 0$

$\therefore$ $rP(2) + s\,q(2)$ if and only if

$r = s = 0$

Span: since $P(2) = 0$ & $q(1) = 0$,

$P(x) = a(x-2)$

$q(x) = b(x-1)$

$a \neq 0$ & $b \neq 0$ since $P(1) \neq 0$ & $q(2) \neq 0$

$cx+d$ in P_1 can be expressed by

$rp(x) + sq(x)$

both LI and spans P_1,

So it is a basis.

Exercise 7.1.1 Show that each of the following functions is a linear transformation.

e. $T : \mathbf{M}_{nn} \to \mathbf{M}_{nn}$; $T(A) = A^T + A$

additive:

To show that $T(A+B) = T(A) + T(B)$.

$$T(A+B) = (A+B)^T + (A+B)$$
$$= A^T + B^T + A + B$$

$$T(A) + T(B) = A^T + A + B^T + B$$

$\therefore$ holds additivity.

Scalar mult:

To show that $T(cA) = cT(A)$

$$T(cA) = (cA)^T + (cA)$$
$$= cA^T + cA$$
$$= c(A^T + A)$$

$$cT(A) = c(A^T + A)$$

$\therefore$ holds scalar multiplicity.

T is transformation.

T is a linear

g. $T : \mathbf{P}_n \to \mathbb{R}; T(r_0 + r_1 x + \cdots + r_n x^n) = r_n$

additive:

to show that $T(A+B) = T(A) + T(B)$

$\downarrow$

$$T((a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n)$$
$$= T(a_0+a_1x+\cdots+a_nx^n) + T(b_0+b_1x+\cdots b_nx^n)$$

which is

$$(a_n+b_n) = (a_n) + (b_n)$$

$\therefore$ holds additivity.

Scalar multiplicity:

To show that $T(cA) = c\,T(A)$

$$T(c(a_0+a_1x+\cdots+a_nx^n)) = cT(a_0+a_1x+\cdots+a_nx^n)$$

$\downarrow$

$$c(T(a_0+a_1x+\cdots+a_nx^n)) = cT(a_0+a_1x+\cdots+a_nx^n)$$

$\downarrow$

$c \cdot a_n = c \cdot a_n$

$\therefore$ holds scalar multiplicity.

So T is a linear transformation

i. $T : \mathbf{P}_n \to \mathbf{P}_n; T[p(x)] = p(x+1)$

additivity:

To show that $T(A+B) = T(A) + T(B)$

$$T(A(x) + B(x)) = (A+B)(x+1)$$

$$= A(x+1) + B(x+1)$$

$$T(A(x)) + T(B(x)) = A(x+1) + B(x+1)$$

$\therefore$ holds additivity.

Scalar multiplicity:

to show that $T(cA) = c \cdot T(A)$

$$T(c \cdot A(x)) = c \cdot A(x+1)$$

$$c \cdot T(A(x)) = c \cdot A(x+1)$$

$\therefore$ holds scalar multiplicity.

$\therefore T$ is a linear transformation

Exercise 7.1.11 Let $S : V \to W$ and $T : V \to W$ be linear transformations. Given a in $\mathbb{R}$, define functions $(S+T) : V \to W$ and $(aT) : V \to W$ by $(S+T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v})$ and $(aT)(\mathbf{v}) = aT(\mathbf{v})$ for all $\mathbf{v}$ in V. Show that $S+T$ and aT are linear transformations.

additivity of $(S+T)$:

To show that for $u, v \in V$

$$(S+T)(u+v) = (S+T)(u) + (S+T)(v)$$

$$(S+T)(u+v) = S(u+v) + T(u+v)$$

$$= S(u) + S(v) + T(u) + T(v)$$

$$(S+T)(u) + (S+T(v)) = S(u) + T(u) + S(v) + T(v)$$

$\therefore$ holds additivity.

scalar multiplicity of $(S+T)$:

To show that $(S+T)(c \cdot u) = c \cdot (S+T)(u)$

$$(S+T)(c \cdot u) = S(c \cdot u) + T(c \cdot u)$$

$$= c \cdot S(u) + c \cdot T(u)$$

$$c \cdot (S+T)(u) = c \cdot S(u) + c \cdot T(u)$$

$\therefore$ holds scalar multiplicity.

additivity at (aT):

to show that for $u, v \in V$

$$(aT)(u+v) = (aT)(u) + (aT)(v)$$

$$\downarrow$$

$$(aT)(u) + (aT)(v) = (aT)(u) + (aT)(v)$$

$\therefore$ it holds additivity.

scalar multiplicity for (aT)

To show that $(aT)(c \cdot u) = c \cdot (aT)(u)$

$$= c\left(a \cdot T(u)\right) = a\left(c \cdot T(u)\right) = a\left(T(c \cdot u)\right) = (aT)(c \cdot u)$$

$\therefore$ holds scalar multiplicity.

$\therefore$ $(S+T)$ and (aT) are linear transformation

Exercise 7.2.2 In each case, (i) find a basis of $\ker T$, and (ii) find a basis of $\operatorname{im} T$. You may assume that T is linear.

a. $T : \mathbf{P}_2 \to \mathbb{R}^2$; $T(a+bx+cx^2) = (a, b)$

(i) $a = 0$, $b = 0$ and we have cx^2.

$\therefore$ a basis for $\ker(T)$ is $\{(x^2)\}$

(ii) Since all possibe output is (a, b)

hence $(a, 0)$ and $(0, b)$

$\therefore$ a basis for $\operatorname{im}(T)$ is $\{(1, 0), (0, 1)\}$

e. $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$; $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & b+c \\ c+d & d+a \end{bmatrix}$

(i) $a+b=0$, $b+c=0$, $c+d=0$, $d+a=0$

So $a=-b$, $b=-c$, $c=-d$, $d=-a$

$a=+c$, $a=-d$, $a=+a$

$a=a$, $b=-a$, $c=a$, $d=-a$

So

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ as } \begin{bmatrix} a & -a \\ a & -a \end{bmatrix} = 0$$

$\therefore$ a basis of $\ker T$ is $\left\{ \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \right\}$

(ii)

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = aT\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + bT\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + cT\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + dT\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

is a basis of $\operatorname{im} T$

i. $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$; $T(X) = XA - AX$, where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(i) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$

$$\begin{bmatrix} b & a \\ d & c \end{bmatrix} - \begin{bmatrix} c & d \\ a & b \end{bmatrix} = 0$$

$$\begin{bmatrix} b-c & a-d \\ d-a & c-b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$b-c=0,\ a-d=0,\ d-a=0,\ c-b=0$

$b=c,\ a=d,\ d=a,\ c=b$

$\downarrow$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

$\therefore$ a basis of $\ker T$ is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$

(ii) all passibe out put is $\begin{bmatrix} b-c & a-d \\ d-a & c-b \end{bmatrix}$

from a from b

$\therefore$ a basis for $\operatorname{im} T$ is $\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Exercise 7.2.7 Show that linear independence is preserved by one-to-one transformations and that spanning sets are preserved by onto transformations. More precisely, if $T : V \to W$ is a linear transformation, show that:

a. If T is one-to-one and $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is independent in V, then $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ is independent in W.

b. If T is onto and $V = \operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, then $W = \operatorname{span}\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$.

a. since T is linear transformation,

$$a_1T(v_1) + \cdots + a_nT(v_n)$$

$$= T(a_1v_1 + \cdots + a_nv_n)$$

and since T is one to one

$$0 = T(a_1v_1 + \cdots + a_nv_n)$$

$$0 = (a_1v_1 + \cdots + a_nv_n)$$

then $a_1 = \cdots a_n = 0$

which then $\{T(v_1), \cdots T(v_n)\}$

is independent in W

b.

since T is onto,

for any $w \in W$

there's $v \in V$

such that $T(v) = w$

then

$T(a_1 v_1 + \cdots a_n v_n) = w$

can be rewritten since T is linear Transformation

to

$a_1 T(v_1) + \cdots + a_n T(v_n) = w$

which means

w is span $\{T(v_1), \cdots, T(v_n)\}$