HW15

3.3.5, 3.5.2, 3.5.5, 3.5.6, 6.6.2

Exercise 3.3.5 Show that the eigenvalues of $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ are $e^{i\theta}$ and $e^{-i\theta}$.
(See Appendix A)

$$\det \begin{bmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix} = 0$$

$$\hookrightarrow (\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$$\hookrightarrow \lambda^2 - 2\cos + 1 = 0$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \cos\theta \pm i\sin\theta$$

Euler's formula $\quad e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

Exercise 3.5.2 Show that the solution to $f' = af$ satisfying $f(x_0) = k$ is $f(x) = ke^{a(x-x_0)}$.

$$f'(x) = \frac{d}{dx}\left(ke^{a(x-x_0)}\right)$$

$$= ake^{a(x-x_0)} = af(x)$$

a)

$$f = g' - (A^{-1}b)'$$

$$= g'$$

$$= Ag$$

$$= A(f + A^{-1}b)$$

$$= Af + b$$

b)

homogeneous equation:

$$g' = Ag$$

Let g_h be its general solution

non-homogeneous equation:

$$f' = Af + b$$

assume $\exists$ a particular solution f_p to this equation

by the principle of superposition,

$$f = g_h + f_p$$

$f_p = -A^{-1} b$, then

$$Af_p + b = -b + b = 0$$

So $f = g_h - A^{-1} b$

if $g' = Ag$,

g is indeed a solution to

$g' = Ag$ because g_h is by definition the general solution to this homogeneous equation.

$\therefore$ every solution f to the nonhomogeneous equation can be written in the form $f = g - A^{-1}b$ with g being a solution to the homogeneous equation $g' = Ag$

Exercise 3.5.6 Denote the second derivative of f by $f'' = (f')'$. Consider the second order differential equation

$$f'' - a_1 f' - a_2 f = 0, \quad a_1 \text{ and } a_2 \text{ real numbers} \tag{3.15}$$

a. If f is a solution to Equation 3.15 let $f_1 = f$ and $f_2 = f' - a_1 f$. Show that
$$\begin{cases} f_1' = a_1 f_1 + f_2 \\ f_2' = a_2 f_1 \end{cases},$$
that is $\begin{bmatrix} f_1' \\ f_2' \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ a_2 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$

b. Conversely, if $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ is a solution to the system in (a), show that f_1 is a solution to Equation 3.15.

let $f_1 = f$, $f_2 = f' - a_1 f$

$$f_1' = f'$$

$$f_2' = f'' - a_1 f'$$

since f is a solution to

$$f'' - a_1 f' - a_2 f = 0,$$

$$f_2' + a_1 f' - a_1 f' - a_2 f = 0$$

$$f_2' = a_2 f$$

$$\begin{pmatrix} f_1' \\ f_2' \end{pmatrix} = \begin{bmatrix} a_1 & 1 \\ a_2 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f \end{bmatrix}$$

b)

$$f_1' = a_1 f_1 + f_2$$

$$f_2' = a_2 f_1$$

$$f_1'' = a_1 f_1' + a_2 f_1$$

$$a_1(-a_1 f_1(t) - f_2(t) + f_1'(t))$$

$$= a_1(-a_1 f_1 - (f_1' - a_1 f_1) + f_1')$$

$- a_1 (- \cancel{y_1(t_1)} + \cancel{t_1} + \cancel{y_1(t_1)} + \cancel{f_1})$

$= a_1 \cdot 0 = 0$

f_1 also satisfies the original second order diff. eq

Exercise 6.6.2 If the characteristic polynomial of $f'' + af' + bf = 0$ has real roots, show that $f = 0$ is the only solution satisfying $f(0) = 0 = f(1)$.

$$f_{(x)} = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

$$f_{(0)} = C_1 e^{r_1 \cdot 0} + C_2 e^{r_2 \cdot 0} = C_1 + C_2 = 0$$

$$f_{(1)} = C_1 e^{r_1 \cdot 1} + C_2 e^{r_2 \cdot 1} = C_1 e^{r_1} + C_2 e^{r_2} = 0$$

So C_1 and C_2 must be 0

then

$f(x) = 0$ is the only solution

satisfying $f_{(0)} = 0$ and $f_{(1)} = 0$