HW12

7.3.1aeg, 7.3.6ce, 7.3.23, 9.1.15, 9.1.17(for n=4)

Exercise 7.3.1 Verify that each of the following is an isomorphism (Theorem 7.3.3 is useful).

a. $T : \mathbb{R}^3 \to \mathbb{R}^3$; $T(x, y, z) = (x + y, y + z, z + x)$

Theorem 7.3.3 states

If the dimension of V and W is the same, a linear trasformation $T: V \to W$ is an isomorphism if it's either one to one or onto.

1. $R^3 \to R^3$ same dimension

2.

$T(x, y, z) + T(x', y', z')$

$= (x+y, y+z, z+x) + (x'+y', y'+z', z'+x')$

$= (x+y+x'+y', y+z+y'+z', z+y+z'+y'$

$= T(x+x', y+y', z+z')$

closed under add

$c \cdot T(x, y, z) = c \cdot (x+y, y+z, z+x)$

$$= \left(c(x+y),\ c(y+z),\ c(z+x)\right)$$

$$= T(c\cdot x,\ c\cdot y,\ c\cdot z)$$

Closed under scalar mult.

$\forall\ (a,b,c) \in \mathbb{R}^3$, we want to find an (x,y,z)

s.t. $T(x,y,z) = (a,b,c)$

$$\shortparallel$$

$$(x+y,\ y+z,\ z+x)$$

$$\begin{cases} x+y = a \\ y+z = b \\ z+x = c \end{cases}$$

$$z = c - x$$

$$\begin{aligned} y &= b - z \\ &= b - c + x \end{aligned}$$

$$x = a - y$$

$$= a - b + c - x$$

$$2x = a - b + c$$

$$x = \frac{a-b+c}{2}$$

$$y = b - c + x$$

$$= b - c + \frac{a-b+c}{2}$$

$$z = c - \frac{a-b+c}{2}$$

So it is an isomarphism

e. $T : \mathbf{P}_1 \to \mathbb{R}^2;\ T[p(x)] = [p(0),\ p(1)]$

P_1 has basis of $x, 1$

so dim = 2

$\mathbb{R}^2$ dim = 2

$$T(p(x)) + T(p'(x)) = (p(0)+p'(0),\ p(1)+p'(1))$$

$$= T(p(x) + p'(x))$$

$$T(c \cdot p(x)) = (c \cdot p(0),\ c \cdot p(1))$$

$$= c \cdot T(p(x))$$

So it is linear.

then to check onto,

$\forall (a,b) \in \mathbb{R}^2$ we want to find an $cx+d$

such that $T(cx+d) = (a, b)$

$c \cdot (0) + d = a = d$

$c \cdot (1) + d = b = c+d$

So it is an isomorphism

g. $T : \mathbf{M}_{22} \to \mathbb{R}^4$; $T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a+b,\ d,\ c,\ a-b)$

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + T\left(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right) = (a+b, d, c, a-b) + (a'+b', d', c', a'-b')$$

$$= (a+a'+b+b', d+d', c+c', a+a'-b-b')$$

$$= T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right)$$

$$T\left(u \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (u \cdot (a+b), u \cdot d, u \cdot c, u \cdot a-b))$$

$$= u \cdot (a+b, d, c, a-b)$$

$$= c \cdot T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

so T is linear

to check on-to

$\forall (a, b, c, d) \in \mathbb{R}^4$ we need to find

(m, n, u, v) such that

$$T\left(\begin{bmatrix} m & n \\ u & v \end{bmatrix}\right) = (a, b, c, d)$$

$a = m + n$

$b = v$

$c = u$

$d = m - n$

So yes, T is isomorphism

Exercise 7.3.6 Determine whether each of the following transformations T has an inverse and, if so, determine the action of T^{-1}.

c. $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$;

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a-c & b-d \\ 2a-c & 2b-d \end{bmatrix}$$

Matrix representation of T is

$$\begin{array}{c} a \\ b \\ c \\ d \end{array} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right]$$

So

$$T^{-1}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a+c & -b+d \\ -2a+c & -2b+d \end{bmatrix}$$

e. $T : \mathbf{P}_2 \to \mathbb{R}^3; T(a+bx+cx^2) = (a-c,\ 2b,\ a+c)$

$$\begin{array}{c} a \\ b \\ c \end{array} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

So

$$T^{-1}(a+bx+cx^2) = \left(\tfrac{1}{2}c,\ \tfrac{1}{2}b,\ -a+\tfrac{1}{2}c\right)$$

Exercise 7.3.23 Let $T : V \to V$ be a linear transformation such that $T^2 = 0$ is the zero transformation.

a. If $V \neq \{\mathbf{0}\}$, show that T cannot be invertible.

b. If $R : V \to V$ is defined by $R(\mathbf{v}) = \mathbf{v} + T(\mathbf{v})$ for all $\mathbf{v}$ in V, show that R is linear and invertible.

a. assume T is invertible,

the $\exists$ a liner transformation

$S: V \to V$ such that $S \circ T = T \circ S = Id_V$

$$T \circ (S \circ T) = T \circ (T \circ S)$$

$$T \circ Id_V = (T \circ T) \circ S = 0$$

since $V \neq 0$, take $w \in V$ s.t. $w \neq 0$

$$T(Id_V(w)) = T(w) = 0$$

$T(w) = 0 \quad \forall \quad w \neq 0$

and by def. $T(0) = 0$

$\therefore$ T is a zero linear transformation

$\therefore$ T is not invertible

Contradiction

6. proof of linearity:

Let u, V be vectors in V and let c be a scalar

Additivity:

$$R(u+V) = (u+V) + T(u+V)$$
$$= u + T(u) + V + T(V)$$
$$= R(u) + R(V)$$

Homogeneity:

$$R(cV) = cV + T(cV)$$
$$= cV + cT(V)$$
$$= c(V + T(V))$$
$$= cR(V)$$

So R is a linear transformation

Proof of Invertibility:

let $S: V \to V$

$$S(v) = v - T(v)$$

we need to show that

$S(R(v)) = v$ and $R(S(v)) = v$

for all v in V

$$S(R(v)) = S(v + T(v))$$
$$= (v + T(v)) - T(v + T(v))$$
$$= v + T(v) - T(v) - T^2(v)$$
$$= v$$

$$R(S(v)) = R(v - T(v))$$
$$= (v - T(v)) + T(v - T(v))$$
$$= v - T(v) + T(v) - T^2(v)$$

$= V$

So R is both a linear and invertible transformation

Exercise 9.1.15 Let B be an ordered basis of the n-dimensional space V and let $C_B : V \to \mathbb{R}^n$ be the coordinate transformation. If D is the standard basis of $\mathbb{R}^n$, show that $M_{DB}(C_B) = I_n$.

we can write $v \in V$ as

$v = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$,

$C_B(v)$ will take v and send it to

$[a_1, a_2, \cdots, a_n]^T$ in $\mathbb{R}^n$.

$M_{DB}(C_B)$ is the matrix representation, with respect to D of $\mathbb{R}^n$ and B of V.

Since D is set of vectors where each vector has a 1 in a unique position and 0 elsewhere

C_B to b_i in B gets

a column of $M_{DB}(C_B)$

exactly the coordinate vector of b_i in $\mathbb{R}_n$
which is just the i-th standard basis vector

e_i in $\mathbb{R}^n$

$\therefore$ $M_{DB}(C_B)$ has columns that are the standard basis vectors

$e_1, e_2, \dots, e_n$ of $\mathbb{R}$

which is exactly the identity matrix

I_n. Thus, $M_{DB}(C_B) = I_n$

Exercise 9.1.17 Let $T : \mathbf{P}_n \to \mathbf{P}_n$ be defined by $T[p(x)] = p(x) + xp'(x)$, where $p'(x)$ denotes the derivative. Show that T is an isomorphism by finding $M_{BB}(T)$ when $B = \{1,\ x,\ x^2,\ \dots,\ x^n\}$.

For $b_0 = 1$:

$$T[1] = \left(\frac{x}{2}\right)^0 + x \cdot 0' = 1$$

1st column of $M_{BB}(T)$ as

$$[1, 0, 0, \cdots, 0]^T$$

For $b_1 = x$:

$$T[x] = \left(\frac{x}{2}\right)^1 + x \cdot 1'$$

$$= \frac{x}{2} + x \qquad = \frac{3x}{2}$$

2nd column of $M_{BB}(T)$ as

$$\left[0, \frac{3}{2}, 0, 0, \cdots, 0\right]^T$$

For $b_2 = x^2$:

$$T[x^2] = \left(\frac{x}{2}\right)^2 + x \cdot (x^2)'$$

$$= \frac{x^2}{4} + 2x^2 \quad = \frac{9x^2}{4}$$

3rd column of $M_{BB}(T)$ as

$$\left[0, 0, \frac{9}{4}, 0, 0, \cdots, 0\right]^T$$

for $b_i = x^i$ for $i = 3, \cdots, n$

$$T[x^i] = \left(\frac{x}{2}\right)^i + x \cdot (x^i)'$$

$$= \frac{x^i}{2^i} + ix^i$$

The coefficient of x^i in $T[x^i]$ will be the entry in the i-th row and i-th col. of the matrix $M_{BB}(T)$. We'll have to calculate these coefficients for each basis vector up to x^n.

let $n = 3$

$M_{BB}(T)$ for T for P_3 with

basis $B = \{1, x, x^2, x^3\}$

$$M_{BB}(T) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{9}{4} & 0 \\ 0 & 0 & 0 & 25/8 \end{bmatrix}$$

T is invertible, since det is non zero

$\therefore$ T is an isomorphism on P_3,

and by extension on P_n for any n,

since this pattern will continue

for higher powers of x, with the

diagonal entries being $(2^i+1)/2^i$ for x^i.

Each diagonal entry corresponds to the

coefficient of x^i in $T[x^i]$,

which is always non-zero.

Thus, the matrix will always be invertible, confirming that T is an isomorphism for any n.