Splitting integrators for spiking neuronal models

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A neuroscientific mutual exclusion

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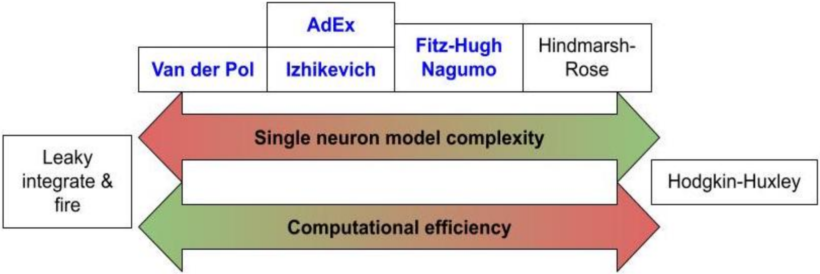
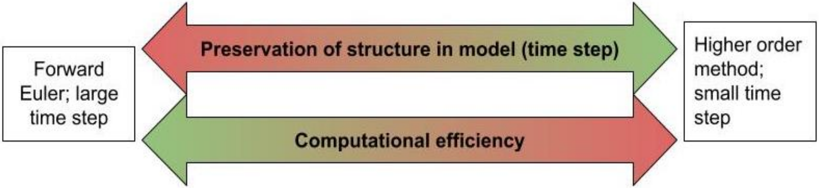


A neuroscientific mutual exclusion



Solution: splitting methods

Splitting methods:

- Splitting methods are a class of numerical methods originally designed for Hamiltonian systems
- They have been shown to preserve structure well for oscillators (leaky)[CRS20, p.2]
- This preservation holds for larger time steps making them more efficient for networks or inference algorithms. [Buc+21, p.28]

How do splitting methods work?

Consider a dynamical system IVP of the form:

$$\frac{dx}{dt} = f(x); \quad x(0) = x_0; \quad t \in [0, T] \tag{1}$$

We split the vector field into two (or more) components

$$\frac{dx}{dt} = f_1(x) + f_2(x); \quad x(0) = x_0; \quad t \in [0, T] \tag{2}$$

We can use the splitting to generate two sub equations that can be solved explicitly (choice of f_i)

How do splitting methods work?

This gives us two problems to solve:

$$\frac{dx_1}{dt} = f_1(x_1); \quad x_1(0) = x_0^{[1]}; \quad t \in [0, T] \tag{3}$$

$$\frac{dx_2}{dt} = f_2(x_2); \quad x_2(0) = x_0^{[2]}; \quad t \in [0, T] \tag{4}$$

Assuming these can be solved explicitly, let $\varphi_t^k(x_0)$ denote the flows of the k-th sub-equation at time t starting from x_0. Now we must recompose these flows to approximate the solution to the original problem (1).

Lie-Trotter & Strang approaches

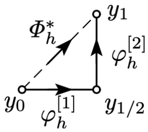
We consider two main approaches to the recomposition of the flows:

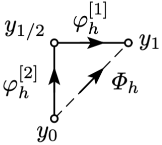
Lie-Trotter:

$$\tilde{x}^{LT}(t_i) = (\varphi_h^{[1]} \circ \varphi_h^{[2]})(\tilde{x}^{LT}(t_{i-1}))$$

$$\Phi_h^* = \varphi_h^{[2]} \circ \varphi_h^{[1]}$$

$$\Phi_h = \varphi_h^{[1]} \circ \varphi_h^{[2]}$$





Lie-Trotter & Strang approaches

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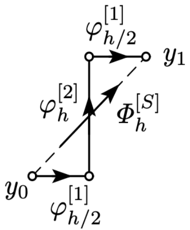
Strang:

$$\tilde{x}^S(t_i) = (\varphi^{[1]}_{h/2} \circ \varphi^{[2]}_{h} \circ \varphi^{[1]}_{h/2})(\tilde{x}^S(t_{i-1}))$$

$$\Phi_h^{[S]} = \varphi^{[1]}_{h/2} \circ \varphi^{[2]}_{h} \circ \varphi^{[1]}_{h/2} \, ,$$



The canonical splitting

The question now becomes, how do we choose an appropriate splitting?

The canonical splitting:

One approach, proposed by Buckwar et al [Buc+21, p.23], is to split into linear and non-linear parts:

$$f_1(x(t)) = Ax(t) \tag{5}$$

$$f_2(x(t)) = N(x(t)) \tag{6}$$

We assume here that the non-linear part can be solved exactly.

The canonical splitting

This gives sub-systems

$$dx^{[1]}(t) = Ax^{[1]}(t)\ dt; \quad x^{[1]}(0) = x_0^{[1]}; \quad t \in [0, T], \tag{7}$$

$$dx^{[2]}(t) = N(x^{[2]}(t))\ dt; \quad x^{[2]}(0) = x_0^{[2]}; \quad t \in [0, T]. \tag{8}$$

Integrating the first system gives us $h-$time flow,

$$\varphi_h^{[1]}(x^{[1]}(t_{i-1})) := x^{[1]}(t_i) = e^{Ah}x^{[1]}(t_{i-1}), \tag{9}$$

where e^{Ah} is an exponential matrix. Assuming N fulfills the assumption, we can write down the second flow,

$$\varphi_h^{[2]}(x^{[2]}(t_{i-1})) := x^{[2]}(t_i) = g(x^{[2]}(t_{i-1}), h), \tag{10}$$

where $g : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ and $g(x_0^{[2]}, t)$ solves the second IVP (8).

The canonical splitting

Applying the Lie-Trotter and Strang compositions, we obtain the following explicit numerical schemes,

$$\tilde{x}^{LT}(t_i) = (\varphi_h^{[1]} \circ \varphi_h^{[2]})(\tilde{x}^{LT}(t_{i-1})) = e^{Ah} g(\tilde{x}^{LT}(t_{i-1}), h), \quad (11)$$

$$\tilde{x}^{S}(t_i) = (\varphi_{h/2}^{[2]} \circ \varphi_h^{[1]} \circ \varphi_{h/2}^{[2]})(\tilde{x}^{S}(t_{i-1})) \quad (12)$$

$$= g(e^{Ah} g(\tilde{x}^{LT}(t_{i-1}), h/2), h/2). \quad (13)$$

Stochastic canonical splitting

Consider a stochastic differential equation (SDE) of additive noise type,

$$dx(t) = f(x(t))\, dt + \Sigma\, dW(t); \quad x(0) = x_0; \quad t \in [0, T], \tag{14}$$

We split the vector field f as before but including the noise term in the linear sub-equation. This gives us two sub-systems,

$$dx^{[1]}(t) = Ax^{[1]}(t)\, dt + \Sigma\, dW(t); \quad x^{[1]}(0) = x_0^{[1]}; \quad t \in [0, T] \tag{15}$$

$$dx^{[2]}(t) = N(x^{[2]}(t))\, dt; \quad x^{[2]}(0) = x_0^{[2]}; \quad t \in [0, T]. \tag{16}$$

Stochastic canonical splitting

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The first sub-equation now has exact solution,

$$x^{[1]}(t) = e^{At}x_0^{[1]} + \int_0^t e^{A(t-s)}\Sigma \, dW(s). \tag{17}$$

The stochastic integral is normally distributed with mean 0 and covariance matrix,

$$C(t) = \int_0^t e^{A(t-s)}\Sigma\Sigma^\top (e^{A(t-s)})^\top \, ds, \tag{18}$$

The first flow is,

$$\varphi_h^{[1]}(x^{[1]}(t_{i-1})) := x^{[1]}(t_i) = e^{Ah}x^{[1]}(t_{i-1}) + \xi_{i-1}, \tag{19}$$

ξ_i are i.i.d $d-$dimensional Gaussian vectors with mean 0 and covariance matrix $C(h)$ [Buc+21].

Stern's approach

Chen et al propose a different splitting for conditionally linear systems [CRS20, p.3].

Conditional linearity:

Conditionally linear system are ODEs of the form

$$dx_i(t) = a_i(x(t))x_i(t) + b_i(x(t))\,dt \quad i = 1, ..., d, \tag{20}$$

Where a_i, b_i are (real-valued) functions depending only on x_j where $j \neq i$.

If x_j with $j \neq i$ are constants, then x_i satisfies a first-order linear ODE with constant coefficients - where the exact solution is known.

Stern's approach

Consider the following vector fields ($i = 1, \ldots, d$),

$$f_j^{[i]}(x) = \begin{cases} a_i(x)x_i + b_i(x) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{21}$$

Then we can write the j-th component of the exact time-h flow in closed form:

$$\varphi_{h,j}^{[i]}(x) = \begin{cases} \exp(ha_i(x))x_i + \frac{\exp(ha_i(x))-1}{ha_i(x)} hb_i(x) & \text{if } i = j \\ x_j & \text{if } i \neq j \end{cases} \tag{22}$$

We can compose these flows with both the Lie-Trotter and Strang approaches.

Generalisation to Ornstein-Uhlenbeck equations

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We extend Stern's approach to the systems of Orstein-Uhlenbeck type.

Ornstein-Uhlenbeck equation

An **Ornstein-Uhlenbeck equation** is an SDE of the form,

$$dx(t) = (ax(t) + b)\, dt + c\, dW(t) \tag{23}$$

for constants $a, b \in \mathbb{R}$, $c > 0$, where $x : \mathbb{R} \to \mathbb{R}$ and W is the standard 1-d Wiener process.

We need to be able to write down the exact solution of an SDE of this form, as we could for the linear ODE with constant coefficients.

Solving Ornstein-Uhlenbeck equations

Following [KP99], we have the exact solution,

$$x(t) = x_0 e^{at} - \frac{b}{a}(1 - e^{at}) + c e^{at} \int_0^t e^{-as}\, dW(s), \tag{24}$$

for $a \neq 0$. This expression is normally distributed with mean $x_0 e^{at} - \frac{b}{a}(1 - e^{at})$. We can calculate the variance,

$$\mathrm{Var}(x(t)) = c^2 e^{2at} \mathbb{E}\left[\left(\int_0^t e^{-au}\, dW(u) \right)^2 \right]. \tag{25}$$

We apply Itô's isometry to get a deterministic integral.

$$\mathrm{Var}(x) = c^2 e^{2at} \mathbb{E}\left[\int_0^t e^{-2au}\, du \right] = c^2 e^{2at} \mathbb{E}\left[\frac{-1}{2a}(e^{-2at} - 1) \right] \tag{26}$$

$$= c^2 e^{2at} \cdot \frac{-1}{2a}(e^{-2at} - 1) = \frac{c^2}{2a}(e^{2at} - 1) \tag{27}$$

Solving Ornstein-Uhlenbeck equations

For the case $a = 0$, the SDE reduces to,

$$dx(t) = b\,dt + c\,dW(t), \tag{28}$$

which means $x(t)$ is normally distributed with mean $x_0 + bt$ and variance $c^2 t$.

Conditionally linear SDEs

A system of SDEs is said to be **conditionally linear** if every component of the system has the form,

$$dx_i(t) = (a_i(x(t))x_i(t) + b_i(x(t)))\,dt + c_i(x(t))\,dW_i(t), \tag{29}$$

where $a_i(x(t)), b_i(x(t)), c_i(x(t))$ depend only on the components $x_j(t)$ for $j \neq i$.

Generalisation to Ornstein-Uhlenbeck equations

We define $f_i(x(t))$ and $f^{(i)}(x(t))$ as before and we denote the exact h-time flow of $f^{(i)}$ by $\varphi_h^{(i)}$. The j-th component is therefore given by,

$$\varphi_{h,j}^{(i)}(x) = \begin{cases} \psi_i & \text{if } i = j \text{ and } a_i(x) \neq 0 \\ \nu_i & \text{if } i = j \text{ and } a_i(x) = 0 \\ x_j & \text{if } i \neq j \end{cases}, \tag{30}$$

where

$$\psi_i \sim N(x_i e^{a_i(x)h} - \frac{b_i(x)}{a_i(x)}(1 - e^{a_i(x)h}), \frac{c_i^2(x)}{2a_i(x)}(e^{2a_i(x)h} - 1)) \tag{31}$$

$$\nu_i \sim N(x_i + b_i(x)h, c_i^2(x)h) \tag{32}$$

What if the model is not conditionally linear?

Some of the models (FHN, IZ, AdEx) are not conditionally linear, so we cannot directly apply Stern's approach. They are, however, of the following form

$$d\begin{pmatrix} v(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} g(v(t)) + ku(t) \\ a(v(t))u(t) + b(v(t)) \end{pmatrix} dt, \tag{33}$$

where $g, a, b : \mathbb{R} \to \mathbb{R}$ are non-linear functions and $g \in C^2(\mathbb{R})$. We can apply a coordinate transform so that the resulting system is conditionally linear and we can apply Stern's approach.

Changing coordinates

León and Samson proposed a coordinate change for the stochastic FHN in order to transform the model into a stochastic damped oscillator [LS18]. We apply a generalised version of this coordinate change. Consider a new coordinate

$$y(t) := g(v(t)) + ku(t). \tag{34}$$

Applying the chain rule, we have that

$$\begin{aligned}
\dot{y} &= \dot{v}g'(v) + \dot{u}k = yg'(v) + (a(v)u + b(v))k && (35)\\
&= yg'(v) + (a(v)\left(\frac{y - g(v)}{k}\right) + b(v))k && (36)\\
&= (g'(v) + a(v))y + (b(v)k - g(v)a(v)). && (37)
\end{aligned}$$

Changing coordinates

The resulting system in (v, y) is

$$d\begin{pmatrix} v \\ y \end{pmatrix} = \begin{pmatrix} y \\ (g'(v) + a(v))y + (b(v)k - g(v)a(v)) \end{pmatrix} dt, \quad (38)$$

which is conditionally linear. Taking a non-conditionally linear stochastic model we have,

$$d\begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} g(v) + ku \\ a(v)u + b(v) \end{pmatrix} dt + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} dW(t). \quad (39)$$

Changing coordinates

We perform the same coordinate change $y(t) = g(v(t)) + ku(t)$, but now proceed using Itô's lemma [Itô44].

$$dy = \left\{ [g'(v) + a(v)]y + [kb(v) - a(v)g(v) + \frac{1}{2}\sigma_1^2 g''(v)] \right\} dt \qquad (40)$$

$$+ g'(v)\sigma_1 \, dW_1 + k\sigma_2 \, dW_2$$

Changing coordinates

In our new coordinate system (v, y), omitting t to simplify notation, the model has the conditionally linear form

$$d \begin{pmatrix} v \\ y \end{pmatrix} = \begin{pmatrix} y \\ (g'(v) + a(v))y + (kb(v) - a(v)g(v) + \frac{1}{2}\sigma_1^2 g''(v)) \end{pmatrix} d \tag{41}$$

$$+ \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sqrt{(g'(v)\sigma_1)^2 + (k\sigma_2)^2} \end{pmatrix} dW(t),$$

where now $W = (W_1, W_3)$ is a 2-dimensional Wiener process and its components are no longer independent.

Hypoellipticity

Hypoellipticity

An SDE with additive noise of the form,

$$dX(t) = f(X(t))\,dt + \Sigma\,dW(t), \tag{42}$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ and $\Sigma \in \mathbb{R}^{d \times d}$, is called **hypoelliptic** if $\Sigma\Sigma^\top$ is not of full rank yet f is such that the transition probability admits a smooth density [Buc+21].

A component of $X(t)$ is called *smooth* if the noise term does not enter directly into its dynamics, otherwise it is *rough*. A common hypoelliptic scenario occurs when one component of the state $X(t)$ is smooth, therefore yielding a degenerate matrix Σ, but a rough component enters into the smooth component.

Hypoellipticity

$k-$step transition probability

The $k-$**step transition probability** of a numerical solution $\tilde{X}(t_i)$ of the SDE above is,

$$\tilde{P}_{t_k}(\mathcal{A}, x) := \mathbb{P}(\tilde{X}(t_k) \in \mathcal{A} | \tilde{X}(0) = x) \tag{43}$$

If $\kappa \in \mathbb{N}$ is the smallest natural such that the transition probability above has smooth density, we say the method is $\kappa-$**step hypoelliptic** [Buc+21].

The 1-step transition probability plays an important role in the field of likelihood-based parameter estimation [Buc+21]. Particular interest lies in the situation where (43) corresponds to a multivariate normal distribution i.e. when $\tilde{X}(t_i)|\tilde{X}(t_{i-1})$ is normally distributed [Buc+21].

Hypoellipticity

- The canonical splitting is 1-step hypoelliptic and therefore generates a well-defined parameter estimator [Buc+21].
- Euler-Maruyama has been proven to be 2-step hypoelliptic i.e. the covariance matrix of $\tilde{X}(t_i)|\tilde{X}(t_{i-1})$ is degenerate [MSH02].
- We prove that under the stochastic Stern approach, $\tilde{X}(t_i)|\tilde{X}(t_{i-1})$ only follows a normal distribution under specific conditions

Distribution under Stern's approach

Consider a 2-d conditionally linear SDE of the form,

$$d\begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} \alpha_1(u)v + \beta_1(u) \\ \alpha_2(v)u + \beta_2(v) \end{pmatrix} dt + \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} dW \tag{44}$$

Given $\tilde{X}(t_{i-1}) = (v_{i-1}, u_{i-1})$, we have that,

$$\tilde{X}(t_i) = \left(\varphi_h^{[1]} \circ \varphi_h^{[2]}\right) \begin{pmatrix} v_{i-1} \\ u_{i-1} \end{pmatrix} = \varphi_h^{[1]} \begin{pmatrix} v_{i-1} \\ \xi \end{pmatrix} = \begin{pmatrix} \eta \\ \xi \end{pmatrix}, \tag{45}$$

Hypoellipticity

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where $\xi \sim N(\mu, \sigma^2)$ with

$$\mu = \begin{cases} u_{i-1} e^{\alpha_2(v_{i-1})h} - \frac{\beta_2(v_{i-1})}{\alpha_2(v_{i-1})}(1 - e^{\alpha_2(v_{i-1})h}) & \text{if } \alpha_2(v_{i-1}) \neq 0 \\ u_{i-1} + \beta_2(v_{i-1})h & \text{if } \alpha_2(v_{i-1}) = 0 \end{cases} \tag{46}$$

$$\sigma^2 = \begin{cases} \frac{\gamma_2^2}{2\alpha_2(v_{i-1})}(e^{2\alpha_2(v_{i-1})h} - 1) & \text{if } \alpha_2(v_{i-1}) \neq 0 \\ \gamma_2^2 h & \text{if } \alpha_2(v_{i-1}) = 0 \end{cases} \tag{47}$$

Hypoellipticity

and $\eta|\xi, v_{i-1} \sim N(\pi(\xi), \kappa^2(\xi))$ where,

$$\pi(\xi) = \begin{cases} v_{i-1}e^{\alpha_1(\xi)h} - \frac{\beta_1(\xi)}{\alpha_1(\xi)}(1 - e^{\alpha_1(\xi)h}) & \text{if } \alpha_1(\xi) \neq 0 \\ v_{i-1} + \beta_1(\xi)h & \text{if } \alpha_1(\xi) = 0 \end{cases} \tag{48}$$

$$\kappa^2(\xi) = \begin{cases} \frac{\gamma_1^2}{2\alpha_1(\xi)}(e^{2\alpha_1(\xi)h} - 1) & \text{if } \alpha_1(\xi) \neq 0 \\ \gamma_1^2 h & \text{if } \alpha_1(\xi) = 0 \end{cases} \tag{49}$$

When is η Gaussian?

Recall that the characteristic function of the normal distribution is given by,

$$\Phi_{X\sim N(\mu,\sigma^2)}(t) = e^{\mathrm{i}t\mu-\frac{1}{2}\sigma^2t^2}. \tag{50}$$

Consider the characteristic function of η,

$$\Phi_\eta(t) = \mathbb{E}[e^{\mathrm{i}t\eta}] = \mathbb{E}[\mathbb{E}[e^{\mathrm{i}t\eta}|\xi]] = \mathbb{E}[e^{\mathrm{i}t\pi(\xi)-\frac{1}{2}\kappa^2(\xi)t^2}] \tag{51}$$

In the case that $\alpha_1(\xi)\neq 0$, $\pi(\xi)$ and $\kappa^2(\xi)$ are non-linear functions of ξ so Φ_η cannot have the form given in (50) thus η does not follow a normal distribution.

When is η Gaussian?

If $\alpha_1(\xi) = 0$, then we have that $\kappa^2(\xi) = \gamma_1^2 h$,

$$\Phi_\eta(t) = e^{-\frac{1}{2}\gamma_1^2 ht^2}\mathbb{E}[e^{it\pi(\xi)}], \tag{52}$$

and therefore, η is normally distributed in the case $\pi(\xi)$ is linear i.e. $\beta_1(\xi)$ is a linear function.

Hypoellipticity

Furthermore consider the hypoelliptic case $\gamma_2 = 0$, where $u(t)$ is now a smooth component. In this case, we have,

$$\tilde{X}(t_i) = \left(\varphi_h^{[1]} \circ \varphi_h^{[2]}\right) \begin{pmatrix} v_{i-1} \\ u_{i-1} \end{pmatrix} = \begin{pmatrix} \eta \\ \mu \end{pmatrix}, \tag{53}$$

where $\eta | \mu \sim N(\pi(\mu), \kappa^2(\mu))$. In other words, the second component has variance 0 even though, in the true solution, noise would propagate through the rough component $v(t)$ into $u(t)$. Thus, composing the flows in this order yields a method that is not 1-step hypoelliptic.

Oscillatory models

Fitz-Hugh Nagumo model

$$d\begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} \frac{1}{\epsilon}(v - v^3 - u) \\ \gamma v - u + \beta \end{pmatrix} dt + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} dW(t) \tag{54}$$

Van der Pol model

$$d\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \epsilon(1 - x_1^2)x_2 - x_1 \end{pmatrix} dt + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} dW(t) \tag{55}$$

Spike-reset models

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Izhikevich model

$$d\begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} av^2 + bv + c + du + fI \\ \alpha(\beta v - u) \end{pmatrix} dt + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} dW(t), \tag{56}$$

with a reset condition,

$$\begin{aligned} &\text{If } v \geqslant 30 \\ &\quad v \to \theta && (57) \\ &\quad u \to u + \delta. && (58) \end{aligned}$$

Spike-reset models

Adaptive exponential integrate and fire model

$$d\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} a\exp(\frac{v}{c}) + dv + fw + gI \\ jv + lw + k \end{pmatrix} dt + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} dW(t) \quad (59)$$

with a reset condition,

$$\text{If } v \geqslant 18$$

$$v \to \theta \quad (60)$$

$$w \to w + \delta. \quad (61)$$

The "stiff" Van der Pol model

If $\epsilon \gg 1$ in the VDP model, it is known as "stiff" [Pol26].

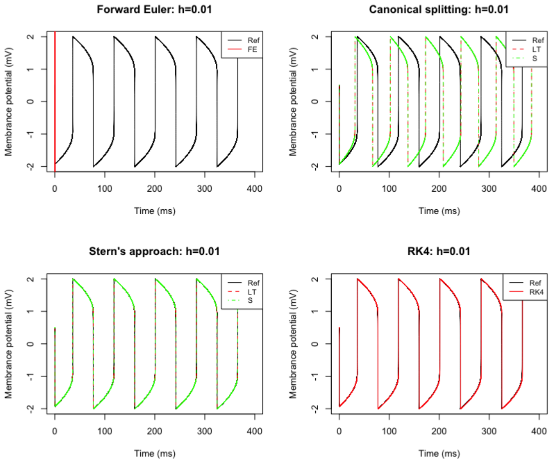

Figure: x_1 component, $\epsilon = 50$, $h = 0.01$, $(x_0, y_0) = (0.5, 0)$

Quadratic loss

We define the **quadratic loss (QL) to** $X(t)$ as,

$$QL(\tilde{X}) := \frac{1}{n}\sqrt{\Sigma_{i=1}^{n}(X(t_i) - \tilde{X}(t_i))^2} \tag{62}$$

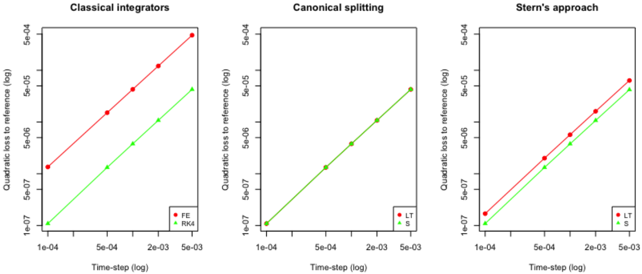

Figure: Stiff VDP: Quadratic loss to reference solution

The Fitz-Hugh Nagumo model: limit cycles

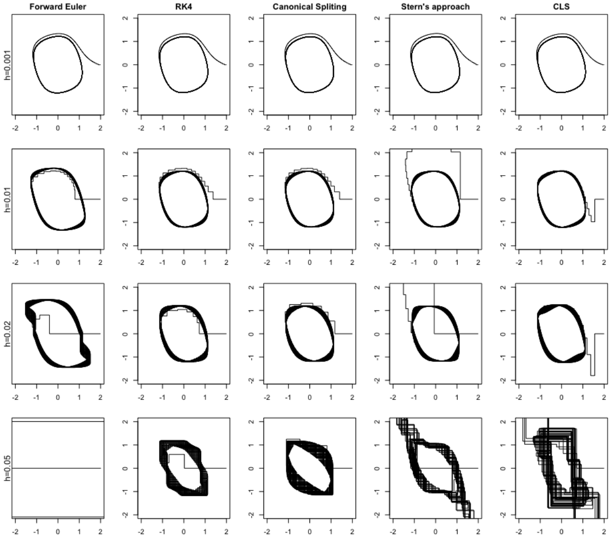

Figure: FHN limit cycles in (v, u).

The Izhikevich model

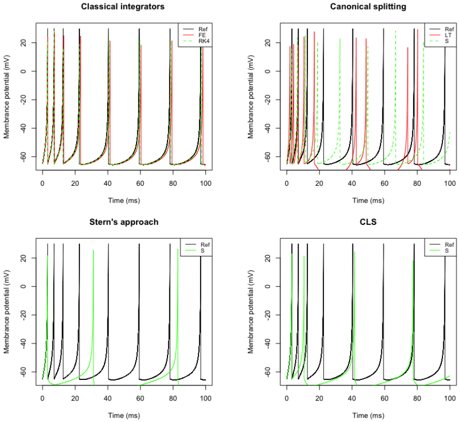

Figure: IZ: $h = 0.05$, $a = 0.04$; $b = 5$; $c = 140$; $d = -1$; $\alpha = 0.02$; $\beta = 0.2$; $I = 10$; $f = 1$; $(v_0, u_0) = (-65, -13)$.

The Izhikevich model, smaller time-step

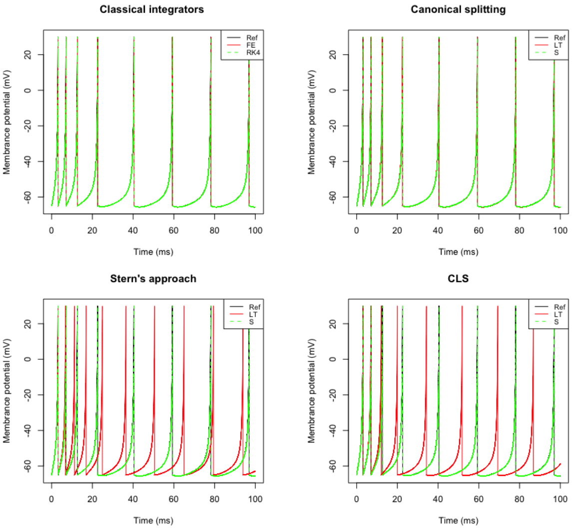

Figure: IZ: $h = 0.0005$

The Izhikevich model, grouping schemes by composition method

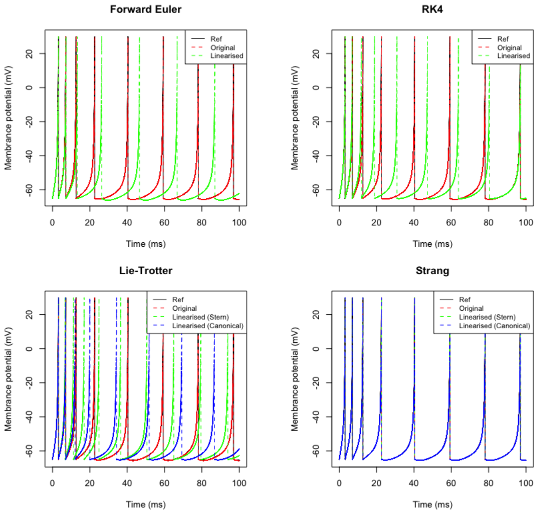

Figure: IZ: $h = 0.0005$ alternative grouping.

The AdEx model

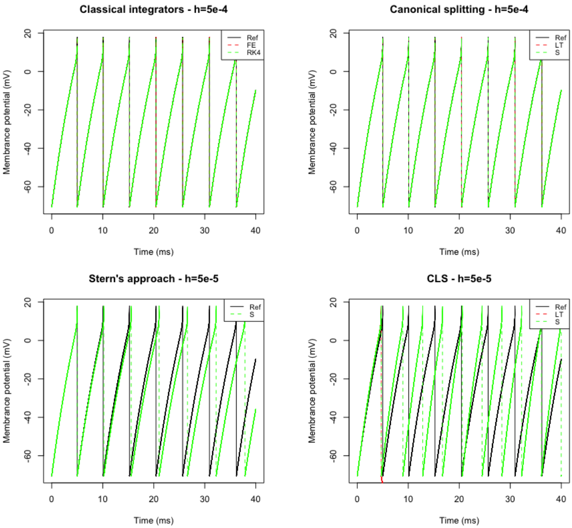

Figure: AdEx $v-$trajectories.

Stochastic models: Non-stiff Van der Pol

Ramón Nartallo-Kaluarachchi

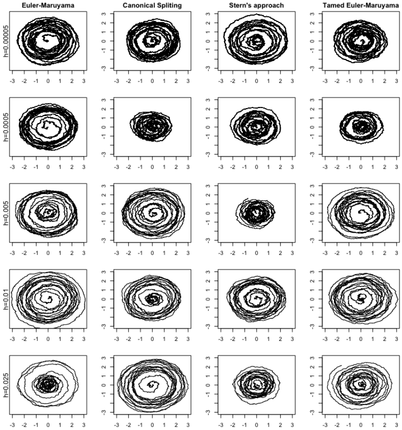

Figure: VDP: phase portraits for the stochastic VDP model in (x, y) space showing limit cycle of true radius 2. The splitting methods are composed with Strang.

Stochastic models: Fitz-Hugh Nagumo

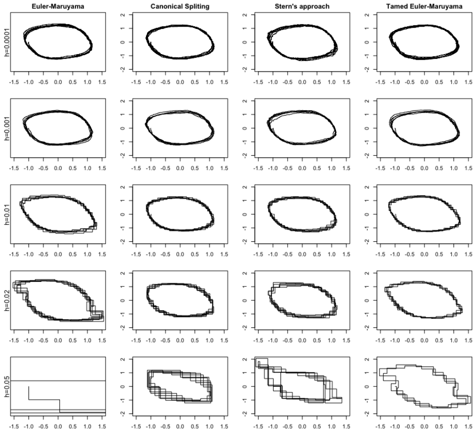

Figure: FHN: phase portraits in (v, u) for various numerical schemes, showing the relative preservation of the limit cycle. The Strang composition is used for the splitting methods.

Absolute Integral Loss and Inter-Spike-Interval

- For a stochastic model, the time between two adjacent spikes varies from path to path.
- We will consider the distribution of the inter-spike interval (ISI) for a given interval. Over 500 iterations, we use the normal kernel density approximation to calculate an approximate distribution of the ISI_j for $j \in \{1, 2, 3, 4, 5\}$.
- We then calculate the **absolute integral loss**, the area between the two curves in absolute value, between this density curve and a reference distribution.
- The reference solution in each case is computed using the mean-square convergent tamed Euler-Maruyama scheme and a small time-step.

Integral loss: Fitz-Hugh Nagumo

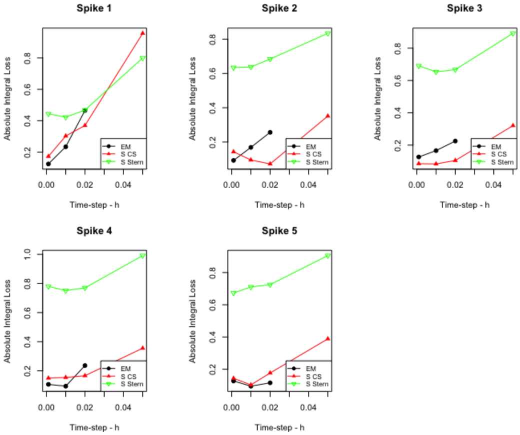

Figure: FHN: Absolute integral loss between reference *ISI* density and the density obtained from the studied methods for the first 5 spikes.

Integral loss: Izhikevich

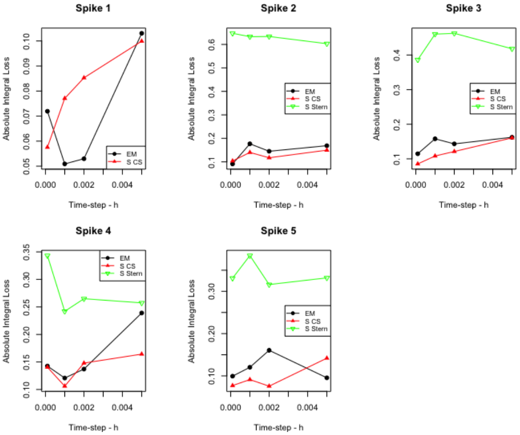

Figure: IZ: Absolute integral loss between reference *ISI* density and the density obtained from the studied methods for the first 5 spikes.

Conclusion - recap

- We extended the work done by Buckwar et al [Buc+21] for the stochastic FHN to a range of stochastic and deterministic models.
- We extended the work done by Chen et al [CRS20] for the deterministic VDP to a range of stochastic and deterministic models, by generalising the approach to SDEs and by changing coordinates for non-conditionally linear models.
- We gave conditions for the resulting method to be normally distributed and showed that in some situations is it not hypoelliptic.
- We performed a range of numerical experiments to compare the preservation of limit cycles, trajectories and ISI distributions.

Conclusion - key results

- The results showed that in several situations the canonical splitting outperformed other methods.
- The Strang composition was the most robust to coordinate changes in the spike-reset models.
- The splitting methods were less effective on spike-reset models because of the spike-reset problem.
- The spike-reset models were more sensitive to the coordinate change.

Conclusion - further work

- The mean-square convergence of the Stern method remains unproved.
- Stewart and Bair propose a method of reducing reset error by numerically approximating the exact spike-time and integrating this information to reset more accurately [SB09].
- This project neglects to study the computational efficiency of the various methods. Preliminary investigations suggest a large disparity between the splitting methods, with Stern's approach being faster than the canonical splitting.

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