Finite volume methods for the advection equation in 1D

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- MATLAB Code & Paper (in Spanish):
  `https://github.com/rnartallo/volumenesefinitos`
- Simulation videos:
  https://www.youtube.com/watch?v=cp0dO8CouTs
  list=PLXhFGfjdT6WK6j7njosild4ldxc15tycl

Introduction

Abstract:

- In this presentation I will study various finite volume schemes for the advection equation in one dimension with an emphasis on the linear case when deriving numeric theory
- I will present numerical results from implementation of these methods
- Finally, I will present a refinable mesh for the sphere for the generalisation of these methods to more dimensions (motivated by applications in earth science)

Advection

What is advection?

- Advection is the movement or transport of a substance, like mass, or a quantity, like heat, in a liquid.
- In closed systems, these quantities are conserved and can be modelled by a conservation law
- The advection equation is given by

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0 \tag{1}$$

where **u** is the *state* vector vector and **f** is the *flux* vector.

Finite volume methods

Philosophy

- To derive a finite volume method, the first step is to discretise space into *volumes* or *cells*
- If we consider a volume V_i, then the idea is to calculate an average value of the solution $U_i(t)$ that we regard as constant in V_i.
- We calculate an average solution by the following:

$$U_i(t) = \frac{1}{|V_i|} \int_{V_i} \mathrm{u}(x, t) \;\; dx \tag{2}$$

.

Finite volume methods

Derivation

We integrate the equation w.r.t x and we apply divergence theorem

$$\int_{V_i} \frac{\partial \mathrm{u}}{\partial t} \; d\mathrm{x} + \int_{V_i} \nabla \cdot \mathrm{f}(\mathrm{u}) \; d\mathrm{x} = 0 \tag{3}$$

$$|V_i| \frac{dU_i}{dt}(t) + \int_{V_i} \nabla \cdot \mathrm{f}(\mathrm{u}) \; d\mathrm{x} = 0 \tag{4}$$

$$|V_i| \frac{dU_i}{dt}(t) + \oint_{S_i} \mathrm{f}(\mathrm{u}) \cdot \mathrm{n} \; dS = 0 \tag{5}$$

$$\frac{dU_i}{dt}(t) + \frac{1}{|V_i|} \oint_{S_i} \mathrm{f}(\mathrm{u}) \cdot \mathrm{n} \; dS = 0 \tag{6}$$

S_i is the boundary of V_i. n is the outward normal of the volume.

Finite volume methods

Derivation

- The idea now is to use to a form of interpolation to reconstruct the flux value on the border S_i
- The properties of our final method depend not only on the mesh, but also on the chosen form of flux approximation

Method in one dimension

Problem in 1D

- The IVP in 1D is the following:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} = 0 & x \in \mathbb{R} \quad t \geqslant 0 \\ u(x,0) = u_0(x) & x \in \mathbb{R} \end{cases} \tag{P}$$

  where $f \in C^1(\mathbb{R}, \mathbb{R})$.
- The volumes are intervals centred in a point x_i.
- Each volume has the form $V_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ con $|V_i| = h_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$.
- The border is made up of two isolated points, $S_i = \{x_{i-1/2}\} \cup \{x_{i+1/2}\}$ $\forall i \in \mathbb{Z}$.

Method in one dimension

- This means the flux integral reduces to,

$$\frac{1}{|V_i|}\oint_{S_i} \mathrm{f}(\mathrm{u}) \cdot \mathrm{n} \;\; dS = \frac{1}{h_i}\left[f(u(x_{i+\frac{1}{2}}, t)) - f(u(x_{i-\frac{1}{2}}, t))\right] \tag{7}$$

$$= \frac{1}{h_i}\left[f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right] \tag{8}$$

where $f_{i\pm\frac{1}{2}} = f(u(x_{i\pm\frac{1}{2}}, t))$.

- The method in 1D is,

$$\frac{du_i}{dt} + \frac{1}{h_i}\left[f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right] = 0 \;\forall i \in \mathbb{Z},\; t > 0. \tag{9}$$

- We must reconstruct the flux on the border to estimate $f_{i\pm\frac{1}{2}}$.

Weak solution

Let $f \in C^1(\mathbb{R}, \mathbb{R})$ and $u_0 \in L^\infty(\mathbb{R})$. A *weak solution* to the previous problem is a function $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+^*, \mathbb{R})$ such that

$$\int_0^\infty \int_{\mathbb{R}} u(x,t)\partial_t\phi(x,t)\,dxdt + \int_0^\infty \int_{\mathbb{R}} f\left(u(x,t)\right)\partial_x\phi(x,t)\,dxdt + \int_{\mathbb{R}} u_0(x)\phi(x,0)\,dx = 0, \;\; \forall \phi \in C_c^1(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}) \quad (10)$$

[Eym+19] proves the existence of a weak solution to the problem but uniqueness only holds in the linear case.

Weak entropy solution

Let $f \in C^1(\mathbb{R})$ and $u_0 \in L^\infty(\mathbb{R})$; a *weak entropy solution* to the problem is a function $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+^*, \mathbb{R})$ such that

$$\int_{\mathbb{R}} \int_{\mathbb{R}_+} \eta\left(u(x,t)\right) \phi_t(x,t)\, dtdx + \int_{\mathbb{R}} \int_{\mathbb{R}_+} \varphi\left(u(x,t)\right) \phi_x(x,t)\, dtdx + \int_{\mathbb{R}} \eta\left(u(x,0)\right) \phi(x,0)\, dx \geqslant 0,$$

$$\forall \phi \in C_c^1(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}_+),\ \forall \eta \in C^1(\mathbb{R}) \text{ convex function}$$

$$\forall \varphi \in C^1(\mathbb{R}) \text{ such that } \varphi' = \eta' f'. \quad (11)$$

Existence & uniqueness

With this definition [Eym+19, p. 124] proves the following theorem

Theorem

Let $f \in C^1(\mathbb{R})$, $u_0 \in L^\infty(\mathbb{R})$, then there is a unique weak entropy solution to the problem.

Mesh in 1D

We use a uniform mesh both in space and time. We define a mesh to be a set $\mathcal{T}_{h,k}$ whose spacial value form the sequence $\left(x_{i+1/2}\right)_{i\in\mathbb{Z}}$ increasing, such that $x_{i+1/2} - x_{i-1/2} = h > 0,\ \forall i \in \mathbb{Z}$.

- The temporal values $t_n = nk,\ \ n \in \mathbb{N}$
- The volumes are $V_i = (x_{i-1/2}, x_{i+1/2}),\ \ \forall i \in \mathbb{Z}$

Explicit scheme in 1D

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- We consider a scheme based on

$$\frac{dU_i}{dt} + \frac{1}{h}\left[f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right] = 0 \tag{12}$$

- We discretise forward in time,

$$\frac{U_i^{n+1} + U_i^n}{k} + \frac{1}{h}\left[f^n_{i+\frac{1}{2}} - f^n_{i-\frac{1}{2}}\right] = 0 \tag{13}$$

- The resulting explicit form is,

$$\begin{cases} U_i^{n+1} = U_i^n - \frac{k}{h}\left[f^n_{i+\frac{1}{2}} - f^n_{i-\frac{1}{2}}\right] & \forall i \in \mathbb{Z},\ \forall n \in \mathbb{N}_0 \\ U_i^0 = \int_{V_i} u_0(x)\, dx & \forall i \in \mathbb{Z} \end{cases} \tag{14}$$

Approximating the flux

- We use F to refer to the numerical approximation, $F^n_{i\pm\frac{1}{2}} \approx f^n_{i\pm\frac{1}{2}} = f(u(x_{i\pm\frac{1}{2}}, t_n))$
- We assume $F^n_{i+\frac{1}{2}}$ can be approximated using only U^n_i, U^n_{i+1}
- We define the *conservation function* $\mathcal{F} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$F^n_{i+\frac{1}{2}} = \mathcal{F}(U^n_i, U^n_{i+1}) \tag{15}$$

- For a 'good' approximation, we impose the consistency condition

$$\mathcal{F}(c, c) = f(c) \quad \forall c \in \mathbb{R} \tag{16}$$

Upwind scheme

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The idea is that

$$F^n_{i\pm\frac{1}{2}} \approx f(u(x_{i\pm\frac{1}{2}}, nk)) \tag{17}$$

but we don't have an approximation of $u(x_{i\pm\frac{1}{2}}, nk)$ because it is on the border of two volumes with different (but constant) approximate solutions. We must pick one.

- Recall that the solution moves with speed $f'(u)$.
- If the speed in $x_{i+\frac{1}{2}}$ is positive the right hand direction (towards V_{i+1}) is called *downwind* and the left hand direction (toward V_i) is called *upwind* and viceversa.

We take the upwind value.

Upwind scheme

Upwind scheme

$$\begin{cases} U_i^{n+1} & = \begin{cases} U_i^n - \frac{k}{h}[f(U_i^n) - f(U_{i-1}^n)] & \text{if } f'(u) \geqslant 0 \\ U_i^n - \frac{k}{h}[f(U_{i+1}^n) - f(U_i^n)] & \text{if } f'(u) < 0 \end{cases}, \\ & \forall n \in \mathbb{N}, \forall i \in \mathbb{Z} \\ U_i^0 & = \frac{1}{h} \int_{V_i} u_0(x), \\ & \forall i \in \mathbb{Z} \end{cases} \tag{18}$$

The linear problem

To study stability and convergence, consider the linear problem

$$\begin{aligned} \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} &= 0 \quad x \in \mathbb{R}, \quad t \in [0, T] \\ u(x, 0) &= u_0(x) \qquad\qquad x \in \mathbb{R} \end{aligned}. \tag{PLin}$$

with $a \in \mathbb{R}$ constant y $u_0 \in L^\infty(\mathbb{R})$.

- We recall that $u(x, t) = u_0(x - at)\ \forall (x, t) \in \mathbb{R} \times \mathbb{R}_+$ is the unique solution
- The max principle holds as u never surpasses the bounds of u_0. We use the max principle as our notion of stability (we discuss this later for a more general family of methods).

Weak-* convergence of Upwind for (PLin)

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Weak-* convergence

Let $(u_n)_{n\in\mathbb{N}} \subset L^\infty(\mathbb{R} \times \mathbb{R}_+^*)$. The sequence $(u_n)_{n\in\mathbb{N}}$ converges to $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+^*)$ weak-* (or converges in the weak-* topology), if

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} (u_n(x,t) - u(x,t))\phi(x,t)\, dx\, dt \to 0 \tag{19}$$

when $n \to \infty$, $\forall \phi \in L^1(\mathbb{R} \times \mathbb{R}_+^*)$

Theorem of convergence

Let $u_0 \in L^\infty(\mathbb{R})$. We take a mesh satisfying $|a| \leqslant (1-\xi)h/k$ with $\xi \in (0,1)$. Let $u_{h,k}$ be the approximate solution using Upwind and let u be the unique weak solution. When $(h,k) \to 0$, $u_{h,k}$ converges weak-* to u.

Weak-* convergence of Upwind for (PLin)

The prove this we need some other results

Corollary of Banach-Alaoglu-Bourbaki Theoreom [Bre10]

Let $(u_n)_{n\in\mathbb{N}} \subset L^\infty(\Omega)$ bounded. Then, there exists a subsequence of $(u_n)_{n\in\mathbb{N}}$ and a function $u \in L^\infty(\Omega)$ such that the subsequence converges to u weak-*.

Weak-* convergence of Upwind for (PLin)

I will give an overview of the proof:

- We use the max principle to say our approximate solution is bounded
- We apply the corollary to find a convergent subsequence
- We multiply by a test function and integrate by parts
- We treat the temporal and spatial derivatives separately and pass to the limit (using a bounded variation estimator)
- We see that the weak-* limit of the subsequence is the weak solution
- We apply uniqueness of the weak solution to say the whole sequence tends weak-* to the weak solution.

Lax-Friedrichs Method

Here is another explicit scheme. Using the following approximations,

$$\frac{\partial u}{\partial t} \approx \frac{u(x,t+k) - \frac{1}{2}(u(x+h,t) + u(x-h,t))}{k} \tag{20}$$

$$\frac{\partial f(u)}{\partial x} \approx \frac{f(u(x+h,t)) - f(u(x-h,t))}{2h} \tag{21}$$

We get the following method,

$$U_i^{n+1} = \frac{1}{2}(U_{i+1}^n + U_{i-1}^n) - \frac{k}{2h}(f(U_{i+1}^n) - f(U_{i-1}^n)) \tag{22}$$

Lax-Friedrichs Method - inverse CFL condition

The LF method poses an interesting condition. Using Taylor, we are approximating the temporal derivative by

$$\frac{\partial u}{\partial t} \approx \frac{u(x, t+k) - u(x, t) - h^2\left(\frac{\partial^2 u}{\partial x^2}(\mu, t)\right)}{k} \tag{23}$$

where $\mu \in [t-h, t+h]$. Therefore, this only converges if

$$\frac{h^2}{k}\left(\frac{\partial^2 u}{\partial x^2}(\mu, t)\right) \xrightarrow[h,k\to 0]{} 0 \tag{24}$$

So that this terms converge to $0, k$ must be chosen such that $k \geqslant h^2$ when $k, h \to 0$ so we take k of the same order as h. This is called *the inverse CFL condition* in [Eym+19, p. 190]

Monotone flux methods

Monotone flux methods are an important subclass (e.g. Upwind, LF) that satisfy the following

Definition

A scheme with conservation function $\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}$ is of *monotone flux* for the advection equation (P) if given $f \in C^1(\mathbb{R})$ y $u_0 \in L^\infty(\mathbb{R} \times \mathbb{R})$ with bounds $U_m \leqslant u_0 \leqslant U_M$, it satisfies

1. $\mathcal{F}$ if locally Lipschitz from $\mathbb{R}^2 \to \mathbb{R}$.
2. $\mathcal{F}(s, s) = f(s) \quad \forall s \in [U_m, U_M]$.
3. $(a, b) \mapsto \mathcal{F}(a, b)$ from $[U_m, U_M]^2$ to $\mathbb{R}$ is non-decreasing with respect to the first variable and non-increasing with respect to the second variable.

Stability of monotone flux methods

Monotone flux methods are stable under a CFL condition that only depends on the Lipschitz constants of the flux

Stability of monotone flux methods

Consider a mesh $\mathcal{T}_{h,k}$ and let $k > 0$. Let $u_{h,k}$ be the approximate solution given by a monotone flux method. Let g_1, g_2 be the Lipschitz constants of $\mathcal{F}$ in $[U_m, U_M]^2$ with respect to its two variables. Under the CFL condition

$$k \leqslant \frac{h}{g_1 + g_2} \tag{25}$$

the approximate solution satisfies

$$U_m \leqslant u_{h,k}(x,t) \leqslant U_M \qquad \forall x \in \mathbb{R}, t \in \mathbb{R}_+ \tag{26}$$

Stability of monotone flux methods

I will only give an idea of the proof:

- We write the scheme as a combination $U_{i-1}^n, U_i^n, U_{i+1}^n$
- Using the three parts of the monotone flux definition we bound the coefficients of $U_{i-1}^n, U_i^n, U_{i+1}^n$
- We prove this is a convex combination and apply induction.

Lax-Wendroff Method

The Lax-Wendroff method has the following form,

$$U_i^{n+1} = U_i^n - \frac{k}{2h}\left[f(U_{i+1}^n) - f(U_{i-1}^n)\right] + \frac{k^2}{2h^2}\left[f'\left(\frac{U_i^n + U_{i+1}^n}{2}\right)\cdot\right.$$

$$\left. - f'\left(\frac{U_i^n + U_{i-1}^n}{2}\right)\cdot\left(f(U_i^n) - f(U_{i-1}^n)\right)\right] \qquad (27)$$

Which simplifies in the linear case to,

$$U_i^{n+1} = U_i^n - \frac{ak}{2h}[U_{i+1}^n - U_{i-1}^n] + \frac{a^2k^2}{2h^2}[U_{i+1}^n - 2U_i^n + U_{i-1}^n] \qquad (28)$$

Lax-Wendroff Method

We can prove *von Neumann stability* for the linear case by considering the solution in the following form:

$$U_i^n = \psi^n e^{\mathrm{i}Kx_i} \tag{29}$$

where $\mathrm{i}^2 = -1$. The CFL condition is $\frac{k}{h} \leqslant \frac{1}{|a|}$.

Lax-Wendroff variants

To avoid calculating the derivative of f, we consider a variant

Richtmyer

$$U_{i+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}(U_{i+1}^{n} + U_{i}^{n}) - \frac{k}{2h}(f(U_{i+1}^{n}) - f(U_{i}^{n})) \tag{30}$$

$$U_{i-\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}(U_{i}^{n} + U_{i-1}^{n}) - \frac{k}{2h}(f(U_{i}^{n}) - f(U_{i-1}^{n})) \tag{31}$$

$$U_{i}^{n+1} = U_{i}^{n} - \frac{k}{h}\left[f(U_{i+\frac{1}{2}}^{n+\frac{1}{2}}) - f(U_{i-\frac{1}{2}}^{n+\frac{1}{2}})\right] \tag{32}$$

Numerical results

Linear case

$$\text{IVP:} \begin{cases} \dfrac{\partial u}{\partial t} + 7\dfrac{\partial u}{\partial x} = 0 & x \in (0,1) \quad t \in (0,1) \\ u(x,0) = \sin(x) & x \in (0,1) \end{cases} \tag{33}$$

$$\text{Exact solution: } u(x,t) = \sin(x - 7t) \tag{34}$$

Order of convergence:

- Upwind: 1.
- Lax-Friedrichs: 1.
- Richtmyer: 2.

Numerical results

Non-viscous Burger's equation

$$\text{IVP:} \begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 0 & x \in (0,1) \quad t \in (0,10) \\ u(x,0) = 2x+1 & x \in (0,1) \end{cases} \tag{35}$$

$$\text{Exact solution: } u(x,t) = \frac{2x+1}{2t+1} \tag{36}$$

Order of convergence:

- Upwind: 1.
- Lax-Friedrichs: 1.
- Richtmyer: 2.

Examples

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Introduction

Existence & uniqueness

Analysis of explicit schemes in 1D

Numerical results and examples

Spherical mesh

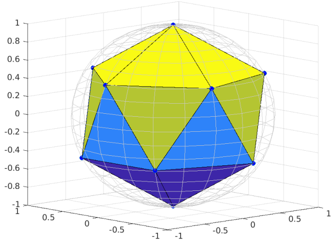
Bibliography

Solutions with shocks

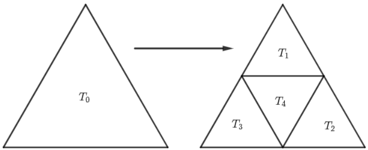
Youtube playlist: `https://www.youtube.com/playlist?list=PLXhFGfjdT6WK6j7njosild4ldxc15tycl`

Spherical mesh

- The algorithm comes from [Gir97].
- We begin with an icosahedron inscribed in a sphere. It has 12 vertices and is made of 20 equilateral triangles
- For each triangle, we refine by dividing it into 4 triangles

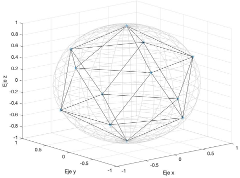


(a) Icosahedron

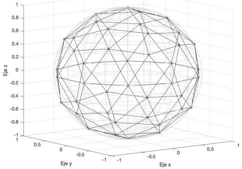


(b) Refining

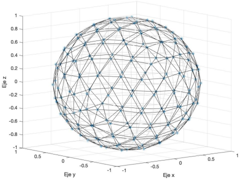
Spherical mesh



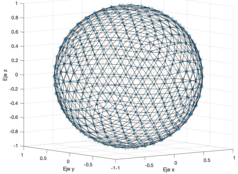
(c) Icosahedron



(d) Iteration 1

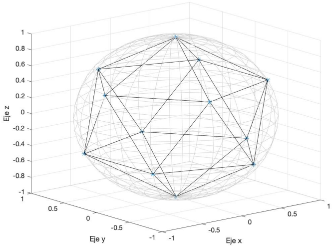


(e) Iteration 2

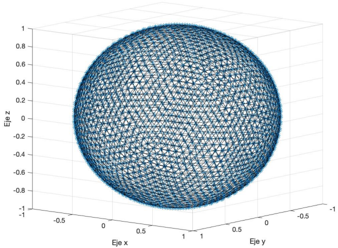


(f) Iteration 3

Spherical mesh



(g) Icosahedron



(h) Iteration 4

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