Lecture 21

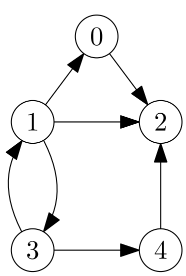
Graph definitions

Graphs are important and general mathematical objects that are widely used in theory and practice. They distill the basic idea of a relationship among a set of objects. Informally we can think of a graph as a collection of dots (the set of objects) with lines connecting them (describing the relationship). The lines can be either directed (arrows) or undirected.

We are interested in the algorithmic aspects of graph theory ("how can we do it efficiently and systematically?"). To talk about this precisely, we must start with precise definitions.

Definition 21.1. A ***digraph*** $G = (V, E)$ is a finite nonempty set V of ***nodes*** together with a (possibly empty) set E of ordered pairs of nodes of G called ***arcs***. Digraph stands for **di**rected **graph**.

Example 21.2. For the digraph shown, write down the sets V and E.

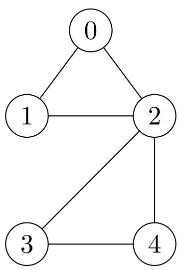


$V = \{0, 1, 2, 3, 4\}$

$E = \{(0,2), (1,2), (1,0), (1,3), (3,1), (3,4), (4,2)\}$

Definition 21.3. A ***graph*** $G = (V, E)$ is a finite nonempty set V of ***vertices*** together with a (possibly empty) set E of unordered pairs of vertices of G called ***edges***. Note that the singular of vertices is ***vertex***.

Example 21.4. For the graph shown, write down the sets V and E.



$V = \{0, 1, 2, 3, 4\}$

$E = \{(0,1), (0,2), (1,2), (2,3), (2,4), (3,4)\}$

Some notes on graphs vs digraphs.

- In order to save writing "(di)graph" too many times, we treat the digraph as the fundamental concept.
- When we say something about digraphs, nodes and arcs, it is understood to also hold for graphs, vertices and edges unless explicitly stated otherwise.
- However, if we talk about graphs, vertices and edges, our statement is not necessarily true for digraphs.
- Some authors use "undirected graph" to mean graph and use the term "graph" to mean what we call a directed graph. We always use digraph and graph.
- E is a set so there are no multiple arcs between a pair of nodes.
- An arc that begins and ends at the same node is called a ***loop***. We make the convention that **loops are not allowed in our digraphs**.
- For a digraph G we may denote the node set $V(G)$ and arc set $E(G)$ for clarity.
- A graph can be viewed as a digraph where every unordered edge $\{u, v\}$ is replaced by two directed arcs (u, v) and (v, u). This works

in most instances and has the advantage of allowing us to consider only digraphs.

Definition 21.5. If $(u, v) \in E$ (that is, if there is an arc going from u to v) we say that v is ***adjacent*** to u, that v is an ***out-neighbour*** of u, and that u is an ***in-neighbour*** of v. In an (undirected) graph G, if $\{u, v\} \in E$, then u is a ***neighbour*** of v and v is a neighbour of u.

Example 21.6. In the digraph in Example 21.2, find all in-neighbours of node 2 and all out-neighbours of node 0.

In-neighbours of node 2 = $\{0, 1, 4\}$

Out-neighbours of node 0 = $\{2\}$

Definition 21.7. The ***order*** of a digraph $G = (V, E)$ is $|V|$, the number of nodes. The ***size*** of G is $|E|$, the number of arcs. We usually use n to denote $|V|$ and m to denote $|E|$.

For a given order n, the size m can be as low as 0 (a digraph consisting of n nodes and no arcs) and as high as $n(n-1)$ (each node can point to each other node; recall that we do not allow loops).

Definition 21.8. If m is toward the low end, the digraph is called ***sparse***, and if m is toward the high end, then the digraph is called ***dense***. These terms are obviously very informal. For our purposes we will call a class of digraphs sparse if m is $O(n)$ and dense if m is $\Omega(n^2)$.

Definition 21.9. A ***walk*** in a digraph G is a sequence of nodes $v_0\, v_1\, \ldots\, v_l$ such that, for each i with $0 \le i < l$, (v_i, v_{i+1}) is an arc in G.

The ***length*** of the walk $v_0\, v_1\, \ldots\, v_l$ is the number l (that is, the number of arcs involved).

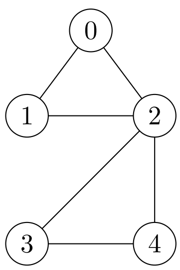
A ***path*** is a walk in which no node is repeated.

A ***cycle*** is a walk in which $v_0 = v_l$ and no other nodes are repeated.

(simple cycle)

In a graph, a walk of the form $u\,v\,u$ – going back and forth along the same edge – is not considered a cycle. A cycle in a graph must be of length at least 3. Note that a walk and a path can have length 0.

Example 21.10. For the graph on the left the following sequences of vertices are classified as being walks, paths, or cycles. Complete the table.



vertex sequence	walk?	path?	cycle?
0 3 2	no	no	no
0 1 2 3 4	yes	yes	no
0 1 2 0	yes	no	yes
0 1 0	yes	no	no
1 2 3 4 2 0	yes	no	no
3	yes	yes	no

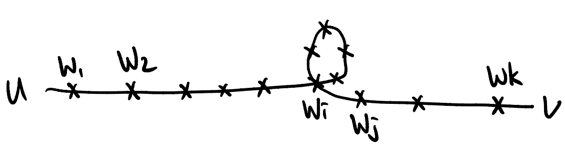
Example 21.11. Show that if there is a walk from u to v, then we can find a path from u to v.

Suppose path from u to v is of form $uw_1 \cdots w_k v$

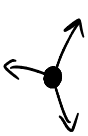
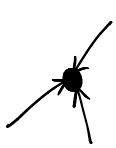
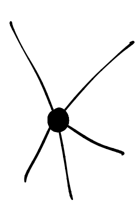
If $w_1 \cdots w_k$ is unique, then done

Else find $w_i = w_j$ for $i < j$, then $uw_1 \cdots w_i w_{j+1} \cdots w_k v$ is a walk

Either this is a path, or repeat process



Definition 21.12. In a graph, the ***degree*** of a vertex v is the number of edges meeting v. In a digraph, the ***outdegree*** of a node v is the number of out-neighbours of v, and the ***indegree*** of v is the number of

 = outdegree  = indegree  = degree of 5

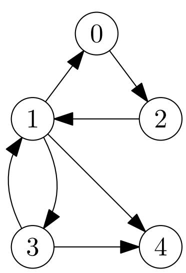
in-neighbours of v.

A node of indegree 0 is called a ***source*** and a node of outdegree 0 is called a ***sink***.

Definition 21.13. The ***distance*** from u to v in G, denoted by $d(u,v)$, is the number of arcs on a shortest path from u to v. If no path from u to v exists, the distance is undefined (or $+\infty$).

For graphs, we have $d(u,v) = d(v,u)$ for all vertices u,v.

Example 21.14. Give the following distances for the digraph on the left.



$d(0,1) =$ 2 $\quad$ $d(0,4) =$ 3

$d(2,3) =$ 2 $\quad$ $d(3,2) =$ 3

$d(2,2) =$ 0 $\quad$ $d(4,0) =$ undefined

Why are the values $d(4,v)$ not defined unless $v = 4$?

because there are no paths of form $4 \cdots v$

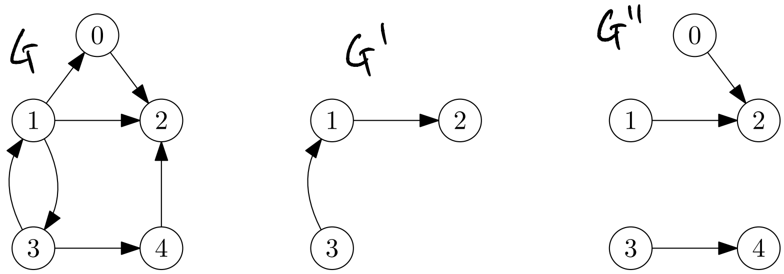
4 is a sink

21.1 Creating new digraphs from old ones

There are several ways to create new digraphs from old ones. One way is to delete nodes and arcs in such a way that the resulting object is still a digraph (no arcs missing endpoints).

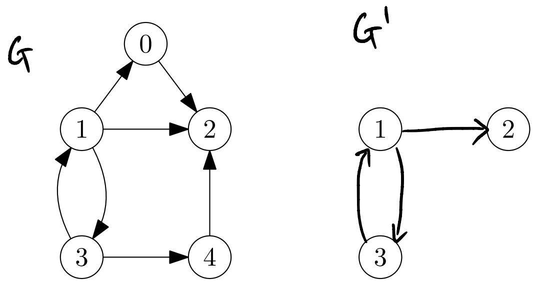
Definition 21.15. A ***subdigraph*** of a digraph $G = (V,E)$ is a digraph $G' = (V',E')$ where $V' \subseteq V$ and $E' \subseteq E$. A ***spanning*** subdigraph is one with $V' = V$; that is, it contains all nodes.

Example 21.16. A digraph (left) with a subdigraph (middle) and a spanning subdigraph (right).



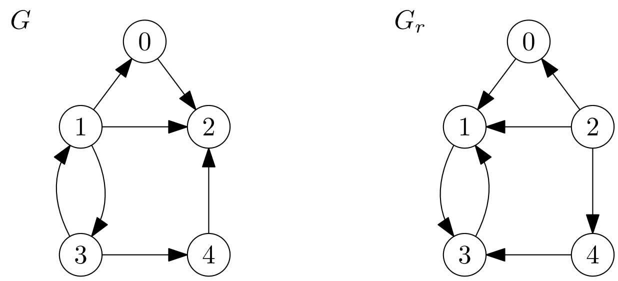
Definition 21.17. The subdigraph ***induced*** by a subset V' of V is the digraph $G' = (V', E')$ where $E' = \{(u, v) \in E \mid u \in V' \text{ and } v \in V'\}$.

Example 21.18. For the digraph shown on the left, draw the subdigraph induced by $\{1, 2, 3\}$ on the right.



Definition 21.19. The ***reverse digraph*** of the digraph $G = (V, E)$, is the digraph $G_r = (V, E')$ where $(u, v) \in E'$ if and only if $(v, u) \in E$.

Example 21.20. Digraph G and its reverse G_r. We simply reverse all the arrows.



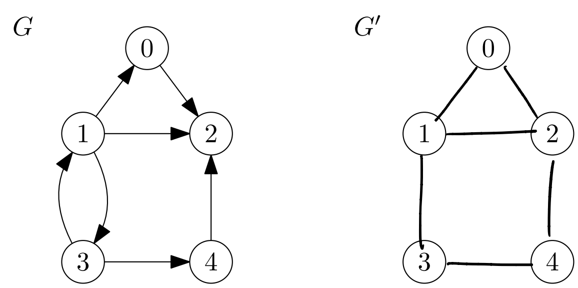
It is sometimes useful to ignore the direction of arcs in a digraph to find the associated 'underlying graph'.

Definition 21.21. The ***underlying graph*** of a digraph $G = (V, E)$ is the graph $G' = (V, E')$ where $E' = \{\{u, v\} \mid (u, v) \in E\}$.

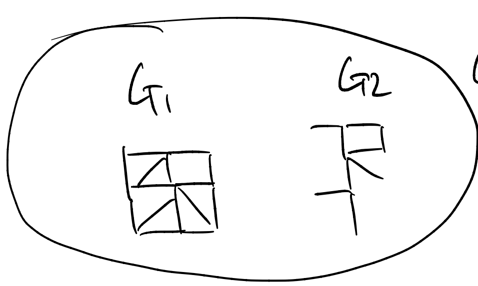
Note that the underlying graph does not have multiple edges even when there are arcs (u, v) and (v, u). In that case, only one edge joins u and v in the underlying graph G'.

$(u,v) \downarrow \{u,v\}$

Example 21.22. Draw the underlying graph G' of the digraph G.



Definition 21.23. We can combine two or more digraphs $G_1, G_2, \ldots G_k$ into a single graph where the vertices of each G_i are completely disjoint from each other and no arc goes between the different G_i. The constructed graph G is called the ***graph union***, where $V(G) = V(G_1) \cup V(G_2) \cup \ldots \cup V(G_k)$ and $E(G) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_k)$.



Graph Union.