Recurrence Formulas

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Semester 2, 2021

Recurrences

Definition

A **recurrence relation** or **difference equation** is an equation that defines an unknown function F recursively. The value $F(n)$ is determined by the values $F(0), F(1), \cdots, F(n-1)$ for sufficiently large n. The smaller values of n are called the **initial conditions**.

Fibonacci example revisited

function SLOWFIB(integer n)
  if $n < 0$ then return 0
  else if $n = 0$ or $n = 1$ then return n
  else return SLOWFIB(n-1) + SLOWFIB(n-2)

If we want to count the number of addition operations it does, we can define a recurrence formula $F(n)$ so that:
Initial condition: $T(0) = T(1) = 0$.
$T(n) = T(n-1) + T(n-2) + 1$, if $n \geqslant 2$.

Question

What is the time complexity of SLOWFIB? e.g. Is $T(n) \in \Theta(n)$ *or* $\Theta(n^2)$*?*

Fibonacci example revisited

Question

What is the time complexity of SLOWFIB? e.g. Is $T(n) \in \Theta(n)$ *or* $\Theta(n^2)$*?*

We can obtain some bounds by simple arguments. e.g. $T(n) \in \Omega(F(n))$ where $F(n)$ is the nth Fibonacci number. Harder to obtain a more precise solution.

Explicit solution

Recurrence formulas are very common in algorithm analysis.
Not very good for analytical purposes.
We'd like to have an **explicit expression** or **closed-form expression** for the function.
Values of explicit formulas are much easier to compute and we can compare it with other functions.
We call the process of deriving the explicit expression 'solving the recurrence formula'.

Linear homogeneous recurrence relations

Definition

A **linear homogeneous recurrence relation** of degree k with constant coefficients is a recurrence relation of the form

$$F(n) = c_1 F(n-1) + c_2 F(n-2) + \cdots + c_k F(n-k)$$

where $c_1, c_2, \cdots, c_k$ are real numbers and $c_k \neq 0$.

Continued

The basic approach for solving linear homogeneous recurrence relations is to look for solutions of the form $F(n) = r^n$ where r is a constant.

$F(n) = r^n$ is a solution of

$F(n) = c_1F(n-1) + c_2F(n-2) + \cdots + c_kF(n-k)$ if and only if

$r^n = c_1r^{n-1} + c_2r^{n-2} + \cdots + c_kr^{n-l}$.

Divide both sides of the equation by r^{n-k} we get

$r^k - c_1r^{k-1} - \cdots - c_{k-1}r - c_k = 0$.

This is called the **characteristic equation** of the recurrence formula.

The roots of the characteristic equation can help us in finding the explicit formula.

We focus on recurrence relations of degree 2 here.

Continued

Theorem

Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then, the sequence generated by the function $F(n)$ is a solution of the recurrence relation $F(n) = c_1 F(n-1) + c_2 F(n-2)$ if and only if $F(n) = \beta_1 r_1^n + \beta_2 r_2^n$ for $n \in \mathbb{N}$ where β_1 and β_2 are constants.

Useful formula: $ax^2 + bx + c = 0$ has solutions $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example

Solve the recurrence relation $F(n) = 3F(n-2) - 2F(n-1)$,
$F(0) = 1$ and $F(1) = -1$.
We have $F(n) + 2F(n-1) - 3F(n-2) = 0$, so the characteristic equation is $r^2 + 2r - 3 = 0$. $(r+3)(r-1) = 0$.
$r_1 = -3$ and $r_2 = 1$.
Solution is $\beta_1(-3)^n + \beta_2(1)^n$.
$F(0) = 1 = \beta_1 + \beta_2$, $F(1) = -1 = \beta_2 - 3\beta_1$.
$\beta_1 = \beta_2 = \frac{1}{2}$.
$F(n) = \frac{1}{2} + \frac{1}{2}(-3)^n$.

Example

Exercise

Solve the recurrence relation $F(n) = 6F(n-1) - 9F(n-2)$, $F(0) = 1$ *and* $F(1) = 6$*?*

Characteristic equation is $r^2 - 6r - 9 = 0$.
$r_1 = r_2 = 3$.
Special case, we have a repeated root.
Solution is of the form $F(n) = \beta_1 r_1^n + \beta_2 n r_1^n$.

SLOWFIB continued

$T(n) = T(n-1) + T(n-2) + 1,\ T(0) = T(1) = 0.$
Non-homogeneous relation.
Need to add a constant to the homogeneous solution.
Characteristic equation of the homogeneous relation is
$r^2 - r - 1 = 0$, so $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$.
Solution is $T(n) = c_1(\frac{1+\sqrt{5}}{2})^n + c_2(\frac{1-\sqrt{5}}{2})^n + c_3$.
c_3 must be a solution to the recurrence equation as well, i.e.
$c_3 = c_3 + c_3 + 1$ so $c_3 = -1$.
The solution is $F(n)$.
Solving with initial conditions gives us $c_1 = \frac{1}{2} + \frac{1}{2\sqrt{5}}$ and
$c_2 = \frac{1}{2} - \frac{1}{2\sqrt{5}}$.
Overall, $T(n) \in \Theta(\phi^n)$ where ϕ is the golden ratio.

Actual exercise

Exercise

Solve the recurrence formula $T(n) = T(n-1) + 2T(n-2)$, $T(0) = 2$ *and* $T(1) = 7$.

Recurrence of high degrees

Example

Suppose we have an algorithm which on input, a list of size n, does the following:
Does a $\Theta(n)$ amount of work. And recursively run on each halves again.
Write down a suitable recurrence formula for the algorithm.

Recurrence of high degrees

Example

Suppose we have an algorithm which on input, a list of size n, does the following:
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Write down a suitable recurrence formula for the algorithm.

$T(n) = 2T(n/2) + n,\ T(1) = 1.$

A linear recurrence, but what we have seen won't work here.

Need something else.

Guess and prove (bad idea in my opinion).

Telescoping.

Telescoping

$T(n) = 2T(n/2) + n,\ T(1) = 1.$
Suppose $n = 2^k$ for some integer k.
$T(2^k) = 2T(2^{k-1}) + 2^k,\ T(2^0) = 1.$
What is $T(2^{k-1}) =?$.
$T(2^{k-1}) = 2T(2^{k-2}) + 2^{k-1}.$
Substitute $T(2^{k-1})$ in the original formula we get
$T(2^k) = 2(2T(2^{k-2}) + 2^{k-1}) + 2^k = 2^2T(2^{k-2}) + 2^k + 2^k.$
Do it again, $T(2^{k-2}) = 2T(2^{k-3}) + 2^{k-2}.$
So
$T(2^k) = 2^2(2T(2^{k-3})+2^{k-2})+2^k+2^k = 2^3T(2^{k-3})+2^k+2^k+2^k.$
..... Expands like a telescopic tube.

Telescoping

We can repeat the process any number of times we'd like.

We can stop and go to the base case when we see the pattern.

$T(2^k) = cT(2^{k-k}) + \cdots.$

$T(2^k) = 2^k T(2^{k-k}) + \cdots.$

$T(2^k) = 2^k T(2^{k-k}) + 2^k + 2^k + \cdots.$

$T(2^k) = 2^k T(0) + k2^k = (k+1)2^k.$

Note that $n = 2^k$ so $k = \lg n$ and we have

$T(n) = (1 + \lg n)n \in \Theta(n \log n).$

Additional notes

What if n is not an exact power of 2?
We could have been more precise in the recurrence definition. e.g. $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + n$.
Unnecessary for asymptotic analysis.
See textbook for formal proof.

Additional notes

There exist very useful parallels between differentiation/integration in calculus and telescoping.

Example: The equation $T(n) - 2T(n-1) = c$, $\frac{T(n)-T(n-1)}{1} = T(n-1) + c$ resembles the differential equation $\frac{dT(x)}{dx} = T(x)$. Solving it with telescoping $T(n) = c(2^n - 1)$ and integration gives us $T(n) = ce^n$. This can sometimes help us in deriving the closed-form solution of complicated recurrences.

Exercise

Exercise

Solve each of the following recurrences:

- *Fibonacci sequence,* $F(n) = F(n-1) + F(n-2)$, $F(0) = 0$, $F(1) = 1$. *Also answer the question what is* $\lim_{n\to\infty} \frac{F(n+1)}{F(n)}$*?*
- $T(n) = T(n/2) + n$, $T(0) = 0$.
- $T(n) = 7T(n/3) + 1$, $T(1) = 1$.