1. Suppose $a, b \in \mathbb{N}$ are even. Prove using the direct proof method that $a + b$ is even.

> Suppose $a = 2c$ and $b = 2d$ for some numbers c, d (this is using the definition of 'even number'). Then we note that $2c + 2d = 2(c + d)$, which implies that $a + b$ must be even.

2. Prove that for every $n \in \mathbb{N}$ such that $n \geq 1$, $\lceil lg(n + 1)\rceil = 1 + \lfloor lg(n)\rfloor$

> Case 1: $n+1 = 2^k$ for some k (ie $n+1$ is a power of 2). Clearly $\lg(n+1) = k$, hence $\lceil \lg(n+1)\rceil = k$. Further, we have it that $k - 1 \leq lg(n) < k$, so it must be the case that $\lfloor \lg(n)\rfloor = k - 1$ and we get $\lfloor \lg(n)\rfloor + 1 = k$.
>
> Case 2: $n + 1 \neq 2^k$, for all k (ie $n + 1$ is not a power of 2). Clearly there exists some k s.t. $2^{k-1} < n + 1 < 2^k$, then $k - 1 < \lg(n + 1) < k$, implying that $\lceil \lg(n + 1)\rceil = k$. For the other side, we note that $k - 1 \leq n < k$, hence $\lfloor \lg(n)\rfloor = k - 1$ and thus we we have $\lfloor \lg(n)\rfloor + 1 = k$.

3. Prove by contradiction that $\sqrt{2} \notin \mathbb{Q}$.

> Suppose for a contradiction that $\sqrt{2} \in \mathbb{Q}$. By definition, this means that $\sqrt{2} = \frac{a}{b}$ for some integers a and b with $\frac{a}{b}$ being in simplest form (ie you can't divide through by any integer to simplify the fraction further). This last part implies that both a and b can't be even. Taking $\sqrt{2} = \frac{a}{b}$ we square both sides to get $2 = \frac{a^2}{b^2}$ which implies that $2b^2 = a^2$, so a^2 is even, which implies that a must be even. Suppose $a = 2k$ for some integer k. Substituting this back into the original equation, we get $2 = \frac{(2k)^2}{b^2} = \frac{4k^2}{b^2}$, and with some algebraic manipulation we get $2b^2 = 4k^2$, and then $b^2 = 2k^2$. But this implies that b^2 (and consequently b) must be even, and this incurs a contradiction.

4. Prove by induction that for all $n \in \mathbb{N}$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

> Base case: $n = 1$ (Note: depending on whether or not $0 \in \mathbb{N}$, and the specifics of the statement to be proven, the base case may begin with $n = 0$, or even for instance $n = 4$ if "$n \geq 4$" is part of the premise. Here we have to begin with $n = 1$ due to us only considering the sum of numbers starting at 1, but be mindful of the base case in induction proofs). Note that $\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}$. Now suppose the statement is true for some n. We will now establish that it is true for $n + 1$. Namely, given $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, we need to prove $\sum_{i=1}^{(n+1)} i = \frac{(n+1)((n+1)+1)}{2}$. We observe the following:
> $\sum_{i=1}^{(n+1)} i = (n + 1) + \sum_{i=1}^{(n)} i$
> $= n + 1 + \frac{n(n+1)}{2}$
> $= \frac{2(n+1)}{2} + \frac{n(n+1)}{2}$
> $= \frac{n(n+1)+2(n+1)}{2}$
> $= \frac{(n+2)(n+1)}{2}$
> $\frac{(n+1)((n+1)+1)}{2}$

5. Given the below fragment of pseudo-code, how many arithmetic operations occur?

```
var = 0
for (int i = 1; i ≤ n; i = i + 1)
    var = var + 1
    for (int j = 1; j ≤ n; j = j + 1)
        var = var + 1
    end for
end for
```

We first observe that the outer loop iterates n times. Each time it loops, the statement $var = var + 1$ is executed once. We further observe that the inner loop runs for n steps, and similarly each time its ran, the statement $var = var + 1$ is executed. Hence we count the total number of arithemtic operations to be $n(1 + n)$.

6. Given the below fragment of pseudo-code, how many arithmetic operations occur?

```
var = 0
for (int j = 1; j ≤ n; j = j * 3)
    var = var + 5 + j
    if (j < floor(n/2))
        var *= 5
    end if
end for
```

First we note that the outer loop iterates $\lfloor log_3 n \rfloor + 1$. Every time the outer loop iterates, the line $var = var+5+j$ is executed, which is two arithmetic operations. However we also have a conditional, which is triggered $\lceil log_3(\lfloor n/2 \rfloor) \rceil$ many times, and each time the line $var* = 5$ is executed, which is one arithmetic operation. Putting this all together, we get $2\lfloor log_3 n \rfloor + \lceil log_3(\lfloor n/2 \rfloor) \rceil$.