PS2-1 Intersection of a line and a plane

$$\mathbf{p}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},\ \mathbf{p}_2 = \begin{pmatrix} 8 \\ 3 \\ 5 \end{pmatrix},\ \mathbf{p}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ \mathbf{p}_4 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix},\ \mathbf{p}_5 = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

Sol>

Parametric form of line equation

$$\mathbf{p}(t) = \mathbf{p}_1 + t(\mathbf{p}_2 - \mathbf{p}_1)$$

Vector form of plane equation

$$\mathbf{p}(t) \cdot ((\mathbf{p}_4 - \mathbf{p}_3) \times (\mathbf{p}_5 - \mathbf{p}_3)) = \mathbf{p}_3 \cdot ((\mathbf{p}_4 - \mathbf{p}_3) \times (\mathbf{p}_5 - \mathbf{p}_3))$$

So, plug the line equation into the plane equation

$$(\mathbf{p}_1 + t(\mathbf{p}_2 - \mathbf{p}_1)) \cdot ((\mathbf{p}_4 - \mathbf{p}_3) \times (\mathbf{p}_5 - \mathbf{p}_3)) = \mathbf{p}_3 \cdot ((\mathbf{p}_4 - \mathbf{p}_3) \times (\mathbf{p}_5 - \mathbf{p}_3))$$

$$t = \frac{\mathbf{p}_3 \cdot ((\mathbf{p}_4 - \mathbf{p}_3) \times (\mathbf{p}_5 - \mathbf{p}_3)) - \mathbf{p}_1 \cdot ((\mathbf{p}_4 - \mathbf{p}_3) \times (\mathbf{p}_5 - \mathbf{p}_3))}{(\mathbf{p}_2 - \mathbf{p}_1) \cdot ((\mathbf{p}_4 - \mathbf{p}_3) \times (\mathbf{p}_5 - \mathbf{p}_3))}$$

Plug this into the line equation

$$\mathbf{p}(t) = \mathbf{p}_1 + \frac{\mathbf{p}_3 \cdot ((\mathbf{p}_4 - \mathbf{p}_3) \times (\mathbf{p}_5 - \mathbf{p}_3)) - \mathbf{p}_1 \cdot ((\mathbf{p}_4 - \mathbf{p}_3) \times (\mathbf{p}_5 - \mathbf{p}_3))}{(\mathbf{p}_2 - \mathbf{p}_1) \cdot ((\mathbf{p}_4 - \mathbf{p}_3) \times (\mathbf{p}_5 - \mathbf{p}_3))} (\mathbf{p}_2 - \mathbf{p}_1)$$

The intersection point is

$$\mathbf{p} = \begin{bmatrix} \frac{5}{4} \\ \frac{57}{28} \\ \frac{43}{14} \end{bmatrix}$$

PS2-2 Minimum distance from a point to a line

$$\mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{n}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{n}_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Plane1: $\mathbf{p} \cdot \mathbf{n}_1 = 1$

Plane2: $\mathbf{p} \cdot \mathbf{n}_2 = 2$

Sol>

The two planes share the point $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ or (3,0,-2)

So, the parametric form of intersection line equation is

$$\mathbf{p}(t) = \mathbf{p}_2 + t(\mathbf{n}_1 \times \mathbf{n}_2)$$

Here, $(\mathbf{n}_1 \times \mathbf{n}_2)$ =(-3,1,2)

For the minimum distance, the intersection line should be perpendicular to the line from $\mathbf{p}_1$ to $\mathbf{p}(t)$

$$(\mathbf{n}_1 \times \mathbf{n}_2) \cdot (\mathbf{p}(t) - \mathbf{p}_1) = 0$$

Plug the line equation into this constraint

$$(\mathbf{n}_1 \times \mathbf{n}_2) \cdot (\mathbf{p}_2 - \mathbf{p}_1 + t(\mathbf{n}_1 \times \mathbf{n}_2)) = 0$$

$$((\mathbf{n}_1 \times \mathbf{n}_2) \cdot (\mathbf{p}_2 - \mathbf{p}_1) + t(\mathbf{n}_1 \times \mathbf{n}_2) \cdot (\mathbf{n}_1 \times \mathbf{n}_2)) = 0$$

$$t(\mathbf{n}_1 \times \mathbf{n}_2) \cdot (\mathbf{n}_1 \times \mathbf{n}_2) = (\mathbf{n}_1 \times \mathbf{n}_2) \cdot (\mathbf{p}_1 - \mathbf{p}_2)$$

$$t = \frac{(\mathbf{n}_1 \times \mathbf{n}_2) \cdot (\mathbf{p}_1 - \mathbf{p}_2)}{(\mathbf{n}_1 \times \mathbf{n}_2) \cdot (\mathbf{n}_1 \times \mathbf{n}_2)} = \frac{1}{14}$$

So,

$$\mathbf{p}(t) = \mathbf{p}_2 + \frac{(\mathbf{n}_1 \times \mathbf{n}_2) \cdot (\mathbf{p}_1 - \mathbf{p}_2)}{(\mathbf{n}_1 \times \mathbf{n}_2) \cdot (\mathbf{n}_1 \times \mathbf{n}_2)}(\mathbf{n}_1 \times \mathbf{n}_2) = \begin{bmatrix} \frac{-3}{14} \\ \frac{15}{14} \\ \frac{2}{14} \end{bmatrix}$$

$$d = \|\mathbf{p}(t) - \mathbf{p}_1\| = \left\| \mathbf{p}_2 + \frac{(\mathbf{n}_1 \times \mathbf{n}_2) \cdot (\mathbf{p}_1 - \mathbf{p}_2)}{(\mathbf{n}_1 \times \mathbf{n}_2) \cdot (\mathbf{n}_1 \times \mathbf{n}_2)}(\mathbf{n}_1 \times \mathbf{n}_2) - \mathbf{p}_1 \right\|$$

$$d = \sqrt{\frac{27}{14}} = \frac{3\sqrt{42}}{14}$$

PS2-3 Minimum distance between two lines

Sol>

Line 1 : $\mathbf{q}_1(t) = \mathbf{p}_1 + t\mathbf{v}_1$

Line 2 : $\mathbf{q}_2(s) = \mathbf{p}_2 + s\mathbf{v}_2$

If the lines are not parallel, the vector of the line which is perpendicular to line 1 as well as line 2 is $\mathbf{v}_1 \times \mathbf{v}_2$

So, the unit vector of the perpendicular line is

$$\frac{\mathbf{v}_1 \times \mathbf{v}_2}{\|\mathbf{v}_1 \times \mathbf{v}_2\|}$$

The distance between the lines are projection of the line $(\mathbf{p}_1 - \mathbf{p}_2)$ onto this unit vector.

$$d = \left| (\mathbf{p}_1 - \mathbf{p}_2) \cdot \frac{\mathbf{v}_1 \times \mathbf{v}_2}{\|\mathbf{v}_1 \times \mathbf{v}_2\|} \right|$$

If the lines are parallel, let's imagine the triangle composed of three vertex $\mathbf{p}_1$, $\mathbf{p}_1 + \mathbf{v}_1$, $\mathbf{p}_2$

The area of this triangle is $\frac{1}{2}\|\mathbf{v}_1 \times (\mathbf{p}_2 - \mathbf{p}_1)\| = \frac{1}{2}\|\mathbf{v}_1\| d$

So, $d = \dfrac{\|\mathbf{v}_1 \times (\mathbf{p}_2 - \mathbf{p}_1)\|}{\|\mathbf{v}_1\|}$

PS2-4 Tangent distance from a point to a cylinder

Sol>

Imagine the projection of the line onto the plane which is defined by the line $p(t)$ and q_0. The shortest line of q_0 and tangent point is projected onto the shortest line from q_0 to $p(t)$. So, if we find the shortest distance between q_0 and line $p(t)$, namely d', we can calculate d from the Pythagorean theorem.

To calculate the shortest distance d', let's imagine the triangle of point $\mathbf{p}_0$, $\mathbf{p}_0 + \mathbf{v}$ and $\mathbf{q}_0$

The area of this triangle is

$$\frac{1}{2}\left\|(\mathbf{p}_0 - \mathbf{q}_0) \times (\mathbf{p}_0 + \mathbf{v} - \mathbf{q}_0)\right\|$$

And if we consider the edge which connects point $\mathbf{p}_0 - \mathbf{q}_0$ and $\mathbf{p}_0 + \mathbf{v} - \mathbf{q}_0$ as a base, the area should be the same with

$$\frac{1}{2}\|\mathbf{v}\| d'$$

These two areas are same. So,

$$\frac{1}{2}\left\|(\mathbf{p}_0 - \mathbf{p}_0) \times (\mathbf{p}_0 + \mathbf{v} - \mathbf{q}_0)\right\| = \frac{1}{2}\|\mathbf{v}\| d'$$

$$d' = \frac{\left\|(\mathbf{p}_0 - \mathbf{q}_0) \times (\mathbf{p}_0 + \mathbf{v} - \mathbf{q}_0)\right\|}{\|\mathbf{v}\|}$$

As a result, the shortest distance d is

$$d = \sqrt{d'^2 - r^2} = \sqrt{\left(\frac{\left\|(\mathbf{p}_0 - \mathbf{q}_0) \times (\mathbf{p}_0 + \mathbf{v} - \mathbf{q}_0)\right\|}{\|\mathbf{v}\|}\right)^2 - r^2}$$

PS2-5 Solving a vector equation

Sol>

Plane : $\mathbf{p}(u,v)=\mathbf{p}+u\mathbf{a}+v\mathbf{b}$

Line : $\mathbf{p}(t)=\mathbf{q}+t\mathbf{v}$

Intersection point is

$$\mathbf{p}+u\mathbf{a}+v\mathbf{b}=\mathbf{q}+t\mathbf{v}$$
$$\mathbf{p}-\mathbf{q}=t\mathbf{v}-u\mathbf{a}-v\mathbf{b}$$

Unknown variables in this equation are t,u,v

Take dot product $(\mathbf{a}\times\mathbf{b})\cdot$ on both side to cancel out the terms of u and v.

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{p}-\mathbf{q})=t(\mathbf{a}\times\mathbf{b})\cdot\mathbf{v}-u(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}-v(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}$$

Here, $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=0$. So,

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{p}-\mathbf{q})=t(\mathbf{a}\times\mathbf{b})\cdot\mathbf{v}$$
$$t=\frac{(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{p}-\mathbf{q})}{(\mathbf{a}\times\mathbf{b})\cdot\mathbf{v}}$$

Plug this into the line equation $\mathbf{p}(t)=\mathbf{q}+t\mathbf{v}$.

As a final result,

$$\mathbf{p}(t)=\mathbf{q}+\frac{(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{p}-\mathbf{q})}{(\mathbf{a}\times\mathbf{b})\cdot\mathbf{v}}\mathbf{v}$$

P2-6 intersection of three planes

Sol>

What we need to show to verify the point $\mathbf{p}_c = \dfrac{d_1(\mathbf{n}_2 \times \mathbf{n}_3) + d_2(\mathbf{n}_3 \times \mathbf{n}_1) + d_3(\mathbf{n}_1 \times \mathbf{n}_2)}{\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)}$ is the intersection of three plane satisfies following three constraints.

$$\text{i) } \mathbf{n}_1 \cdot \mathbf{p}_c = d_1$$
$$\text{ii) } \mathbf{n}_2 \cdot \mathbf{p}_c = d_2$$
$$\text{iii) } \mathbf{n}_3 \cdot \mathbf{p}_c = d_3$$

i) $\mathbf{n}_1 \cdot \mathbf{p}_c$

$$= \mathbf{n}_1 \cdot \frac{d_1(\mathbf{n}_2 \times \mathbf{n}_3) + d_2(\mathbf{n}_3 \times \mathbf{n}_1) + d_3(\mathbf{n}_1 \times \mathbf{n}_2)}{\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)}$$
$$= \frac{d_1\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3) + d_2\mathbf{n}_1 \cdot (\mathbf{n}_3 \times \mathbf{n}_1) + d_3\mathbf{n}_1 \cdot (\mathbf{n}_1 \times \mathbf{n}_2)}{\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)}$$
$$= \frac{d_1\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3) + 0 + 0}{\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)}$$
$$= d_1$$

As a result,

$$\mathbf{n}_1 \cdot \mathbf{p}_c = d_1$$

In the same way,

ii) $\mathbf{n}_2 \cdot \mathbf{p}_c$

$$= \mathbf{n}_2 \cdot \frac{d_1(\mathbf{n}_2 \times \mathbf{n}_3) + d_2(\mathbf{n}_3 \times \mathbf{n}_1) + d_3(\mathbf{n}_1 \times \mathbf{n}_2)}{\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)}$$
$$= \frac{d_1\mathbf{n}_2 \cdot (\mathbf{n}_2 \times \mathbf{n}_3) + d_2\mathbf{n}_2 \cdot (\mathbf{n}_3 \times \mathbf{n}_1) + d_3\mathbf{n}_2 \cdot (\mathbf{n}_1 \times \mathbf{n}_2)}{\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)}$$
$$= \frac{0 + d_2\mathbf{n}_2 \cdot (\mathbf{n}_3 \times \mathbf{n}_1) + 0}{\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)}$$

From $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$

$$= \frac{d_2\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)}{\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)}$$
$$= d_2$$

iii) $\mathbf{n}_3 \cdot \mathbf{p}_c$

$$= \mathbf{n}_3 \cdot \frac{d_1(\mathbf{n}_2 \times \mathbf{n}_3) + d_2(\mathbf{n}_3 \times \mathbf{n}_1) + d_3(\mathbf{n}_1 \times \mathbf{n}_2)}{\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)}$$
$$= \frac{d_1\mathbf{n}_3 \cdot (\mathbf{n}_2 \times \mathbf{n}_3) + d_2\mathbf{n}_3 \cdot (\mathbf{n}_3 \times \mathbf{n}_1) + d_3\mathbf{n}_3 \cdot (\mathbf{n}_1 \times \mathbf{n}_2)}{\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)}$$

$$= \frac{0 + 0 + d_3 \mathbf{n}_3 \cdot (\mathbf{n}_1 \times \mathbf{n}_2)}{\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)}$$

$$= \frac{d_3 \mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)}{\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)}$$

$$= d_3$$

So, all equalities are satisfied.

As a result, $\mathbf{p}_c$ is the intersection of three planes.

PS2-7 Volume of a polyhedron

Sol>

Let's denote vertex $\begin{bmatrix}1\\0\\2\end{bmatrix}=\mathbf{a}$, $\begin{bmatrix}1\\4\\2\end{bmatrix}=\mathbf{b}$, $\begin{bmatrix}0\\3\\3\end{bmatrix}=\mathbf{c}$, $\begin{bmatrix}0\\1\\4\end{bmatrix}=\mathbf{d}$, $\begin{bmatrix}0\\0\\0\end{bmatrix}=\mathbf{e}$.

Total volume is a sum of the lower tetrahedron's volume and the higher tetrahedron's volume.

So,

$$\frac{1}{6}(\mathbf{d}-\mathbf{b})\cdot\{(\mathbf{c}-\mathbf{b})\times(\mathbf{a}-\mathbf{b})\}+\frac{1}{6}(\mathbf{e}-\mathbf{b})\cdot\{(\mathbf{c}-\mathbf{b})\times(\mathbf{a}-\mathbf{b})\}$$

$$=\frac{1}{6}\begin{bmatrix}-1\\-3\\2\end{bmatrix}\cdot\left\{\begin{bmatrix}-1\\-1\\1\end{bmatrix}\times\begin{bmatrix}0\\-4\\0\end{bmatrix}\right\}+\frac{1}{6}\begin{bmatrix}-1\\-4\\-2\end{bmatrix}\cdot\left\{\begin{bmatrix}-1\\-1\\1\end{bmatrix}\times\begin{bmatrix}0\\-4\\0\end{bmatrix}\right\}$$

$$=\frac{8}{3}$$