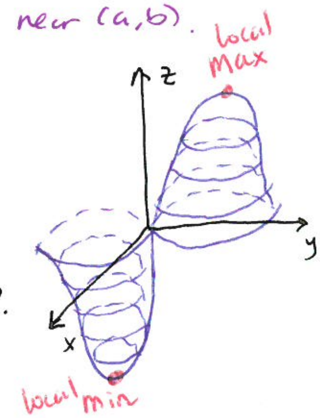
1. (a,b) is a local max of f if $f(x,y) \le f(a,b)$ when (x,y) is near (a,b).
2. (a,b) is a local min of f if $f(x,y) \ge f(a,b)$ when (x,y) is near (a,b).



If ① holds for all (x,y) then (a,b) is an absolute max.
If ② holds for all (x,y) then (a,b) is an absolute min.

Theorem 2 If f has a local max/min at (a,b) and $f_x(a,b)$ and $f_y(a,b)$ exist then $f_x(a,b) = f_y(a,b) = 0$ or $\nabla f(a,b) = \vec{0}$.

proof: Translate to derivatives of curves by fixing x or y.

(a,b) is a critical point of f if $\nabla f(a,b) = \vec{0}$.

↳ Either a local max, Local min or neither.

Second Derivative Test:

Second partial derivatives of f are continuous on a disk containing (a,b),

$$\boxed{D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2$$

If $\nabla f(a,b) = 0$ and if:

(a) $D(a,b) > 0$ with $f_{xx}(a,b) > 0$, then (a,b) is a local min.

(b) $D(a,b) > 0$ with $f_{xx}(a,b) < 0$, then (a,b) is a local max.

(c) If $D(a,b) < 0$, then (a,b) is neither a local max or min, called a Saddle point.

(d) If $D(a,b) = 0$ then nothing can be concluded.

Ex. 3 Find the local max and min values and saddle points of

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

① Critical numbers: $\nabla f = \vec{0}$ $\qquad \vec{0} = \langle 4x^3 - 4y,\ 4y^3 - 4x \rangle$

Critical points: $(0,0), (1,1), (-1,-1)$ $\qquad y = x^3$ or $y = x^{1/3}$

$$0 = x^9 - x = x(x^8 - 1) = x(x^4-1)(x^4+1)$$
$$= x(x-1)(x+1)(x^2+1)(x^4+1$$
$$x = 0, 1, -1$$

$$D(x,y) = f_{xx} f_{yy} - f_{xy}^2$$
$$= 12x^2 \cdot 12y^2 - (-4)^2$$
$$= 12^2 (x^2 y^2) - 16$$
$$= 16(9x^2y^2 - 1)$$

- $D(0,0) = -16$

  By Second Derivative Test $\boxed{(0,0) \text{ is a Saddle point}}$

- $D(1,1) = 16(8) > 0$

  $f_{xx}(1,1) = 12 > 0$

  $\boxed{(1,1) \text{ local min}}$

- $D(-1,-1) = 16(8) > 0$

  $f_{xx}(-1,-1) = 12 > 0$

  $\boxed{(-1,-1) \text{ local min}}$

$\boxed{\text{Ex 5}}$ Find the shortest distance from the point $(1,0,-2)$ to the plane

$$x+2y+z=4$$

minimize f: $d^2=(x-1)^2+(y)^2+(z+2)^2 = (x-1)^2+y^2+(6-x-2y)^2$

① Critical points:

$$\vec{0}=\nabla f = \langle 2(x-1)+2(6-x-2y)(-1),\ 2y+2(6-x-2y)(-2)\rangle$$
$$=\langle 2x-2-12+2x+4y,\ 2y-24+4x+8y\rangle$$
$$=\langle 4x+4y-14,\ 10y+4x-24\rangle$$

$$6y=10$$
$$y=\tfrac{10}{6}\qquad x=\tfrac{11}{6}$$

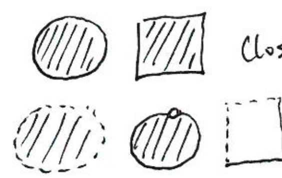
② Second Derivative Test:

$$D(x,y)=f_{xx}f_{yy}-f_{xy}^{\,2} = (4)(10)-(4)^2>0$$

$f_{xx}=4>0$ thus $\boxed{\left(\frac{11}{6},\frac{5}{3}\right)}$ is a local min (must be absolute)

EVT - f continuous on a closed interval $[a,b]$ then f has an absolute max/min.

Translates to a closed set in $\mathbb{R}^2$: i.e. must contain all boundary points



Closed

Not closed

$\boxed{\text{8 EVT for functions of Two Variables}}$

f continuous on a closed bounded set D in $\mathbb{R}^2$, then f attains an absolute max and min value at some points in D. ★ Existence Thm

Steps to find the absolute max/min:

① find all critical points of f in D

② find extreme values of f on the boundary of D

③ the largest function value from ① and ② is the absolute max (similar for abs. min)

$\boxed{\text{Ex. 7}}$ Find the absolute max/min of

$$f(x,y)=x^2-2xy+2y$$

on the rectangle $D=\{(x,y)\mid 0\le x\le 3,\ 0\le y\le 2\}$

① Critical points: $\vec{0}=\nabla f=\langle 2x-2y,\ -2x+2\rangle$

$x=1\quad y=1\quad f(1,1)=1$

② Extreme values on D: 4 edges

$x=0\quad f_{max}=4$ ($y=2$) $\quad \boxed{f_{min}=0}$ ($y=0$)

$x=3\quad f(3,y)=9-6y+2y=9-4y$

$\boxed{f_{max}=9}$ ($y=0$) $\quad f_{min}=1$ ($y=2$)

$y=0\quad \boxed{f_{max}=9}$ ($x=3$) $\quad \boxed{f_{min}=0}$ ($x=0$)

$y=2\quad f(x,2)=x^2-4x+4=(x-2)^2$

$f_{max}=4$ ($x=0$) $\quad \boxed{f_{min}=0}$ ($x=2$)

③ Absolute max points:

$(3,0)$ or $(0,3)$

max value is $\boxed{9}$

Absolute min points:

$(0,0)$, $(2,2)$

min value is $\boxed{0}$