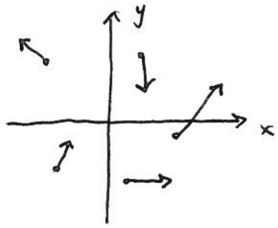
Section 16.1 - Vector Fields

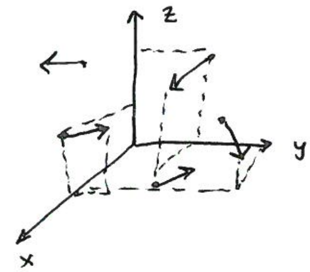
A vector field is a function whose domain is a set of points in $\mathbb{R}^2$ (or $\mathbb{R}^3$) who assigns to each point a vector in V_2 (or in V_3).

See page 1080 for Examples: wind speeds & Currents

★ Show The Wind Map!

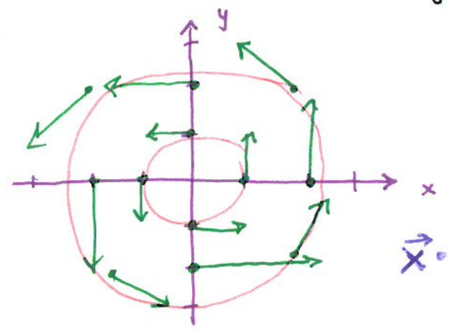
- Best way to picture a vector field is by drawing a vector at each point
- Can't do it for all but we get a reasonable impression for a few representative points.





Example 1 A vector field on $\mathbb{R}^2$ is defined by $\vec{F}(x,y) = -y\vec{i} + x\vec{j}$. Describe $\vec{F}$ by sketching some of the vectors.

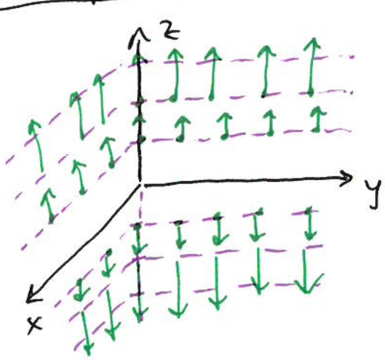
x	y	F(x, y)
0	1	$\langle -1, 0\rangle$
1	0	$\langle 0, 1\rangle$
-1	0	$\langle 0, -1\rangle$
0	-1	$\langle 1, 0\rangle$



Looks like each vector is tangent to a circle at (0, 0)

$$\vec{x} \cdot \vec{F}(x,y) = \langle x, y\rangle \cdot \langle -y, x\rangle$$
$$= -xy + xy = \boxed{0}$$

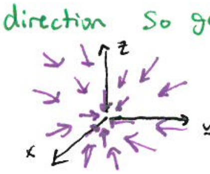
Example 2 Sketch the vector field on $\mathbb{R}^3$ given by $\vec{F}(x,y,z) = z\vec{k}$.



This is a reatively simple vector field in V^3 most are more complicated so we use Computer algebra systems to graph and visualize. Examples see page 1083.

Example 4 Newton's Law of Gravitation states that the magnitude of the gravitational force between two objects with masses m and M is $|\vec{F}| = \frac{mMG}{r^2}$ where r is the distance between the objects and G is the gravitational constant. Assume M is located at the origin. Let $\vec{r} = \langle x, y, z\rangle$ be the position vector for m. Gravitational force exerted on the second object acts towards the origin: $-\frac{\vec{r}}{|\vec{r}|}$ direction so gravitational force on the object is $\vec{F}(\vec{r}) = \frac{-mMG}{|\vec{r}|^3}\vec{r}$

Called the gravitational field.



Gradient fields - recall $\nabla f(x,y,z) = f_x(x,y,z)\vec{i} + f_y(x,y,z)\vec{j} + f_z(x,y,z)\vec{k}$

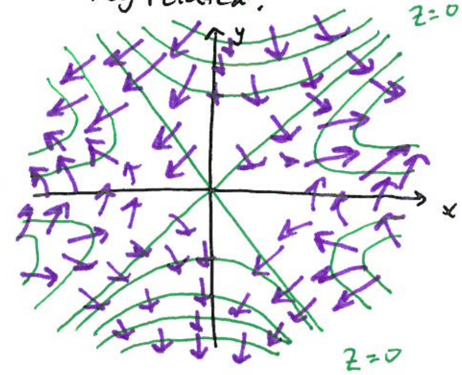
or $\nabla f(x,y) = f_x \vec{i} + f_y \vec{j}$

So the gradient ∇f is really a vector field.

skip

Example 6 Find the gradient vector field of $f(x,y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f. How are they related?

$\nabla f(x,y) = f_x \vec{i} + f_y \vec{j} = \boxed{2xy\,\vec{i} + (x^2 - 3y^2)\,\vec{j}}$



★ Plot Contour with contour plotter

★ Plot vector field with vector field plotter

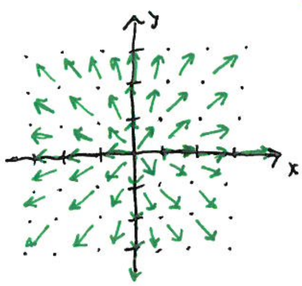
★ Remember that the gradient vectors are pointed in the direction of greatest assent perpendicular to the tangent vector / contour.

Example Find the gradient vector field of f and sketch it.

$$f(x,y) = \sqrt{x^2+y^2}$$

$$\nabla f(x,y) = \tfrac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x\,\vec{i} + \tfrac{1}{2}(x^2+y^2)^{-1/2}\cdot 2y\,\vec{j}$$

$$= \frac{1}{\sqrt{x^2+y^2}}\langle x, y\rangle = \frac{\langle x,y\rangle}{|\langle x,y\rangle|} \quad \leftarrow \text{all unit vectors!}$$

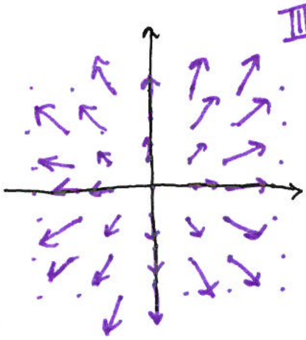


Example #29-32 (a) $f(x,y) = x^2+y^2$ (c) $f(x,y) = x(x+y)$

(b) $f(x,y) = (x+y)^2$ (d) $f(x,y) = \sin\sqrt{x^2+y^2}$

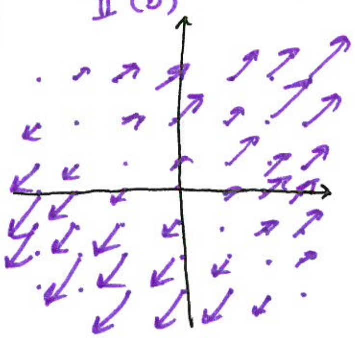
Match to gradient vector field

III (a)



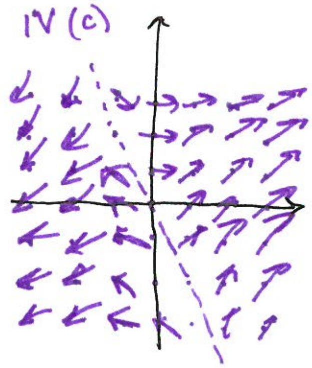
$\langle 2x, 2y\rangle$

II (b)



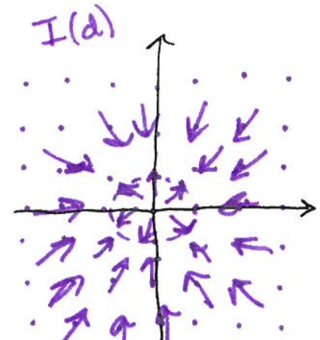
$2(x+y)\langle 1,1\rangle$

IV (c)



$\langle 2x+y, x\rangle$

I (d)



$\frac{\cos\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\langle x, y\rangle$