Green's Theorem in Vector Form on $\mathbb{R}^2$:

$$\boxed{\int_C \vec{F}\cdot\vec{n}\,ds} = \int_a^b \vec{F}\cdot\frac{\langle y'(t), -x'(t)\rangle}{\cancel{|\vec{r}'(t)|}}\cdot|\vec{r}\,'(t)|\,dt$$

$$= \int_a^b \langle P, Q\rangle\cdot\langle y'(t), -x'(t)\rangle\,dt$$

$$= \int_C P\,dy - Q\,dx$$

$$= \iint_D \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) dA$$

$$\boxed{= \iint_D \operatorname{div}\vec{F}\,dA}$$

★ Think box, Sphere, ellipsoids, etc.

★ Extension to $\mathbb{R}^3$: The Divergence Theorem

E simple solid region, S the boundary surface of E with positive orientation. $\vec{F}$ has components with continuous paritions on an open region containing E:

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV}$$

Proof: $\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_S \langle P, Q, R\rangle\cdot\vec{n}\,dS$

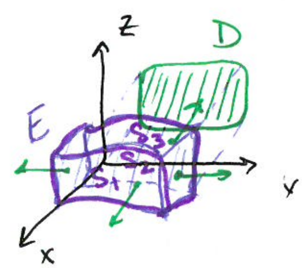
$$= \iint_S \langle P,0,0\rangle\cdot\vec{n}\,dS + \iint_S \langle 0,Q,0\rangle\cdot\vec{n}\,dS + \iint_S \langle 0,0,R\rangle\cdot\vec{n}\,dS$$

$\parallel$ $\parallel$ $\parallel$ ← Show

$$\iiint_E \operatorname{div}\vec{F}\,dV = \iiint_E \frac{\partial P}{\partial x}\,dV + \iiint_E \frac{\partial Q}{\partial y}\,dV + \iiint_E \frac{\partial R}{\partial z}\,dV$$

Enought to Show: $\iint_S \langle P,0,0\rangle\cdot\vec{n}\,dS = \iiint_E \frac{\partial P}{\partial x}\,dV$

Assume: $E = \{(x,y,z) \mid (y,z)\in D \;\; g_1(y,z)\le x\le g_2(y,z)\}$



$$\iiint_E \frac{\partial P}{\partial x}\,dV = \boxed{\iint_D P(g_2, y, z) - P(g_1, y, z)\,dA}$$

0 x-component =0

$$\iint_S P\cdot\vec{i}\cdot\vec{n}\,dS = \iint_{S_1} P\vec{i}\cdot\vec{n}\,dS + \iint_{S_2} P\vec{i}\cdot\vec{n}\,dS + \iint_{S_3} P\vec{i}\cdot\vec{n}\,dS$$

$$= \boxed{\iint_D P(g_2, y, z)\,dA - \iint_D P(g_1, y, z)\,dA}$$

Section 16.9 - The Divergence Theorem

Vector Calc

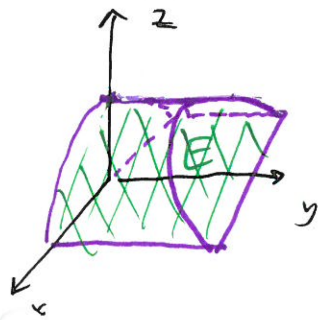
Ex. Find the Flux of the vector field $\vec{F} = \langle x, y, z \rangle$ over the unit sphere.

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV = \iiint_E \frac{d}{dx}(x) + \frac{d}{dy}(y) + \frac{d}{dz}(z)\, dV$$

$$= \iiint_E 3\, dV = 3 \cdot \frac{4}{3}\pi (1)^3 = \boxed{4\pi}$$

Ex 2 Evaluate $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = \langle xy, (y^2 + e^{x z^2}), \sin(xy) \rangle$

and S is the surface of E bounded by $z = 1 - x^2$, $z = 0$, $y = 0$, $y + z = 2$.



$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV$$

$$= \iiint_E y + 2y + 0\, dV$$

$E = \{(x,y,z) \mid -1 \le x \le 1,\ 0 \le z \le 1 - x^2,\ 0 \le y \le 2 - z\}$

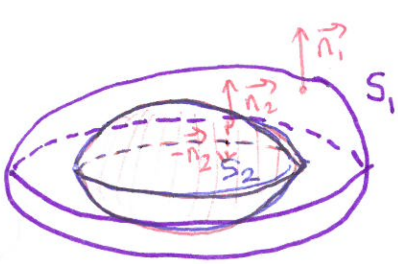
$$= \iiint_E 3y\, dV = \int_{-1}^{1} \int_0^{1-x^2} \int_0^{2-z} 3y\, dy\, dz\, dx$$

$$= \int_{-1}^{1} \int_0^{1-x^2} \frac{3}{2}(2-z)^2\, dz\, dx$$

$$= \int_{-1}^{1} -\frac{1}{2}\left(2 - (1-x^2)\right)^3 + \frac{1}{2}(2)^3\, dx$$

$$= \Big[4x\Big]_{-1}^{1} - \frac{1}{2}\int_{-1}^{1} (x^2+1)^3\, dx = 8 - \frac{1}{2}\int_{-1}^{1} x^6 + 3x^4 + 3x^2 + 1\, dx$$

$$= 8 - \left[\frac{1}{7} + \frac{3}{5} + 1 + 1\right] = \frac{8 \cdot 35 - 5 - 21 - 35 - 35}{35} = \boxed{\frac{184}{35}}$$



Boundary of E is $S = S_1 \cup S_2$, $\vec{n_1}, \vec{n_2}$ the outward normal on S_1, S_2 then the normal to E is $\vec{n} = \begin{cases} \vec{n_1} & \text{on } S_1 \\ -\vec{n_2} & \text{on } S_2 \end{cases}$

$$\iiint_E \operatorname{div} \vec{F}\, dV = \iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_{S_1} \vec{F} \cdot \vec{n_1}\, dS - \iint_{S_2} \vec{F} \cdot \vec{n_2}\, dS$$

$$= \boxed{\iint_{S_1} \vec{F} \cdot d\vec{S} - \iint_{S_2} \vec{F} \cdot d\vec{S}}$$