Physical Applications - Computing mass, electric charge, center of mass, moment of inertia

Consider a thin plate (also called a Lamina) with variable density occupying a region D in the xy-plane:

Given density function $\rho(x,y)$ for a Lamina:

- Find total mass of lamina
- Moments of Lamina
- Center of mass of Lamina
- Moments of inertia of Lamina

Total mass of Lamina with density function $\rho(x,y)$ over D:

density $\rho(x,y) = \dfrac{\text{Units of mass}}{\text{Unit Area}} \quad \dfrac{\Delta m(x,y)}{\Delta A}$ over small rectangle $m(x,y) \approx \rho(x,y)\Delta A$

So increasing rectangles $\to \infty$

$$\boxed{m = \iint_D \rho(x,y)\,dA}$$

Moments of Lamina with density function $\rho(x,y)$ over D:

- Moment of a particle about an axes is defined as the product of its mass and its directed distance from the axis.

- about x-axis over small rectangle moment $\approx \rho(x,y)\cdot\Delta A\cdot y$

★ Tendency to rotate about x-axis

$$\boxed{\text{Moment of lamina about x-axis } M_x = \iint_D y\,\rho(x,y)\,dA}$$

- about y-axis over small rectangle moment $\approx \rho(x,y)\,\Delta A\cdot x$

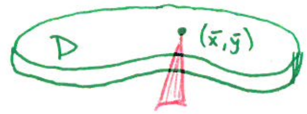
★ Tendency to rotate about y-axis

$$\boxed{\text{Moment of lamina about y-axis } M_y = \iint_D x\,\rho(x,y)\,dA}$$

Center of mass of Lamina with $\rho(x,y)$ over D:

- Point on plate where it balances horizontally $(\bar{x},\bar{y})$

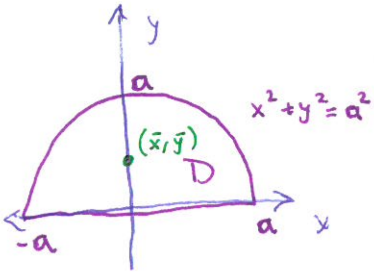
So that $m\bar{x} = M_y$ and $m\bar{y} = M_x$



$$\boxed{\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\,\rho(x,y)\,dA \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\,\rho(x,y)\,dA}$$

Section 15.5 - Applications of Double Integrals

Ex 3 The density at any point on a semicircular lamina is proportional to the distance from the center of the circle. Find the center of mass of the lamina.



$\rho(x,y) = K\sqrt{(x-0)^2+(y-0)^2}$ where K is some constant

Convert to polar coords: $r=\sqrt{x^2+y^2}$ $\quad 0\le r\le a \quad 0\le\theta\le\pi$

$$m = \int_0^{\pi}\int_0^{a} K\cdot r\cdot r\,dr\,d\theta = K\int_0^{\pi}d\theta\int_0^{a} r^2\,dr = \frac{K\pi a^3}{3}$$

$$M_y = \int_0^{\pi}\int_0^{a}(r\cos\theta)\cdot(Kr)\cdot r\,dr\,d\theta = K\underbrace{\int_0^{\pi}\cancel{\cos\theta}\,d\theta}_{0}\int_0^{a} r^3\,dr = 0$$

$$M_x = \int_0^{\pi}\int_0^{a}(r\sin\theta)(Kr)r\,dr\,d\theta = K\int_0^{\pi}\sin\theta\,d\theta\int_0^{a} r^3\,dr$$

$$= K(2)\frac{a^4}{4}$$

$$\boxed{(\bar{x},\bar{y}) = \left(0, \frac{M_x}{m}\right) = \left(0, \frac{3a}{2\pi}\right)}$$

Moments of Inertia (second moment) of Lamina, $\rho(x,y)$ over D:

- about x-axis for a particle of mass m — my^2

  Moment of Inertia about x-axis $\quad \boxed{I_x = \iint_D y^2\rho(x,y)\,dA}$

- about y-axis for a particle of mass m — mx^2

  Moment of Inertia about y-axis $\quad \boxed{I_y = \iint_D x^2\rho(x,y)\,dA}$

- about the origin — $m(x^2+y^2)$ $\quad \boxed{I_0 = \iint_D (x^2+y^2)\rho(x,y)\,dA = I_x + I_y}$

Ex 4 Find the moments of inertia I_x, I_y, I_0 of a homogeneous disk D with density $\rho(x,y)=\rho$, center the origin, radius a.

$D = \{(r,\theta) \mid 0\le r\le a,\ 0\le\theta\le 2\pi\}$

$$I_0 = \int_0^{2\pi}\int_0^{a}(x^2+y^2)\rho\cdot r\,dr\,d\theta = \int_0^{2\pi}\int_0^{a} r^3\rho\,dr\,d\theta = \boxed{\frac{2\pi\rho a^4}{4}}$$

Note: $I_0 = I_x + I_y$ and by symmetry of D $I_x = I_y$

Thus $I_x = I_y = \frac{1}{2}I_0 = \boxed{\frac{\pi\rho a^4}{4}}$