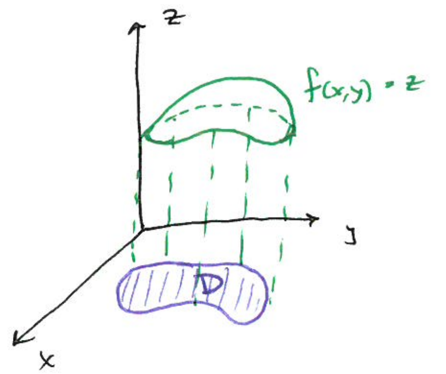
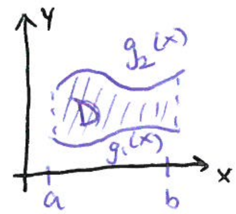
Section 15.3 – Double Integrals over General Regions

★ Want to integrate over regions of a general shape.

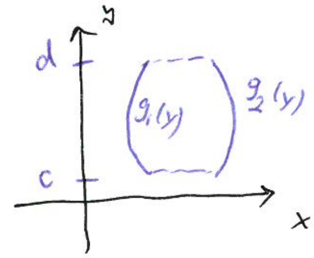


Type I – regions D : (Top and bottom function in xy-plane)

$$D=\{(x,y)\mid a\le x\le b,\ g_1(x)\le y\le g_2(x)\}$$



$$\boxed{\iint_D f(x,y)\,dA=\int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx \quad \text{If } f \text{ is continuous}}$$

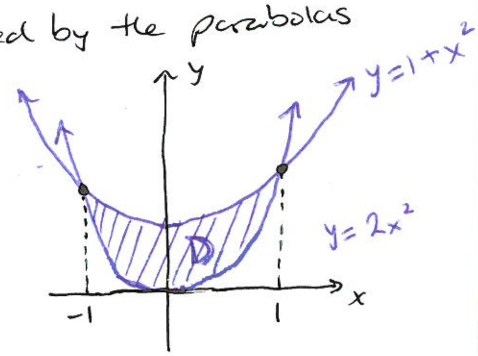


Type II – regions D : (Left and right function in xy-plane)

$$D=\{(x,y)\mid c\le y\le d,\ g_1(y)\le x\le g_2(y)\}$$

$$\boxed{\iint_D f(x,y)\,dA=\int_c^d\int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy \quad \text{if } f \text{ is continuous}}$$

Ex 1 Evaluate $\iint_D (x+2y)\,dA$, where D is the region bounded by the parabolas $y=2x^2$ and $y=1+x^2$.



$$D=\{(x,y)\mid -1\le x\le 1,\ 2x^2\le y\le 1+x^2\}$$

$$\iint_D (x+2y)\,dA=\int_{-1}^{1}\int_{2x^2}^{1+x^2}(x+2y)\,dy\,dx$$

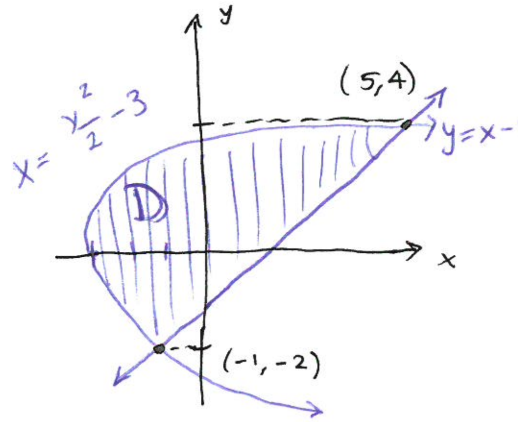
$$=\int_{-1}^{1}(xy+y^2)\Big|_{2x^2}^{1+x^2}dx=\int_{-1}^{1}\left(x+x^3+(1+x^2)^2-2x^3-4x^4\right)dx$$

$$=\int_{-1}^{1}\left(1+2x^2+x-x^3-3x^4\right)dx$$

$$=\left(x+\tfrac{2}{3}x^3+\tfrac{x^2}{2}-\tfrac{x^4}{4}-\tfrac{3}{5}x^5\right)\Big|_{-1}^{1}=\boxed{\frac{32}{15}}$$

Section 15.3 - Double Integrals over General Regions

Ex 3 Evaluate $\iint_D xy\,dA$, where D is the region bounded by the line $y = x-1$ and the parabola $y^2 = 2x+6$.



$$D = \{(x,y) \mid -2 \le y \le 4,\ \frac{y^2}{2} - 3 \le x \le y+1\}$$

$$\iint_D xy\,dA = \int_{-2}^{4}\int_{\frac{y^2}{2}-3}^{y+1} xy\,dx\,dy$$

$$= \int_{-2}^{4} \frac{x^2 y}{2}\Big|_{y^2/2-3}^{y+1} dy = \int_{-2}^{4}\left(\frac{(y+1)^2 y}{2} - \frac{(\frac{y^2}{2}-3)^2 y}{2}\right) dy$$

$$= \frac{1}{2}\int_{-2}^{4} \left(y^3 + 2y^2 + y - \left(\frac{y^4}{4} - 3y^2 + 9\right)y\right) dy$$

$$= \frac{1}{2}\left[\frac{y^4}{4} + \frac{2}{3}y^3 + \frac{y^2}{2} - \frac{y^6}{24} + \frac{3}{3}y^3 - \frac{9}{2}y^2\right]\Big|_{-2}^{4} = \boxed{36}$$

Properties of Double Integrals:

① $\iint_D [f(x,y) + g(x,y)]\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$

② $\iint_D c \cdot f(x,y)\,dA = c \cdot \iint_D f(x,y)\,dA$

③ $f(x,y) \ge g(x,y)$ on D, then $\iint_D f(x,y)\,dA \ge \iint_D g(x,y)\,dA$

④ If $D = D_1 \cup D_2$ and $D_1 \cap D_2$ only on their boundary then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

⑤ $\iint_D 1\,dA = A(D)$

⑥ If $m \le f(x,y) \le M$ for all (x,y) in D then

$$m \cdot A(D) \le \iint_D f(x,y)\,dA \le M \cdot A(D)$$

New properties