Chapter 16 Review - Vector Calculus

Vector field $\vec{F}$ assigns a vector to each point.
<u>Example</u>: ∇f, the gradient is a vector field. (16.1)

<u>Line integrals</u>:

$$\int_C f(x,y)\,ds = \int_a^b f(r(t)) \cdot |\vec{r}\,'(t)|\,dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(r(t)) \cdot \vec{r}\,'(t)\,dt = \int_C P\,dx + Q\,dy + R\,dz$$

(16.2)

<u>FTC for Line Integrals</u>:

$$\int_C \nabla f \cdot d\vec{r} = f(r(b)) - f(r(a))$$

$\vec{F}$ is <u>conservative</u> if there is f with $\nabla f = \vec{F}$

f is called a <u>potential function</u> for $\vec{F}$.

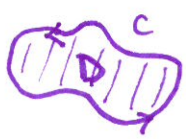
(16.3)

★ Thm: $\vec{F} = \langle P, Q \rangle$ Continuous partials $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} \Rightarrow \vec{F}$ conservative

<u>Green's Thm</u>:

$$\oint_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

(16.4)



$$\text{Curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \qquad \text{div}\,\vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

<u>Thm</u>: $\vec{F}$ on $\mathbb{R}^3$ $\text{Curl}\,\vec{F} = \vec{0}$ then $\vec{F}$ conservative

(16.5)

<u>Parametric Surface</u>: S: $x = x(u,v)$ $y = y(u,v)$ $z = z(u,v)$ $(u,v) \in D$

<u>tangent plane</u>: $\vec{n} = (\vec{r_u} \times \vec{r_v})$ <u>Surface Area</u>: $A(s) = \iint_D |\vec{r_u} \times \vec{r_v}|\,dA$

(16.6)

<u>Surface Integrals</u>:

$$\iint_S f\,ds = \iint_D f(\vec{r}(u,v))\,|\vec{r_u} \times \vec{r_v}|\,dA$$

$$\text{Flux} = \iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\,dS = \iint_D \vec{F} \cdot (\vec{r_u} \times \vec{r_v})\,dA$$

$$= \iint_D -P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R\,dA$$

(16.7)

<u>Stokes</u>: $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{Curl}\,\vec{F} \cdot d\vec{S}$ (16.8) <u>Divergence Thm</u>: $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div}\,\vec{F}\,dV$ (16.9)

Chapter 16 Practice

① find the equation and parametric equations of the tangent plane at $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -2\right)$ to the parametric surface $S: \vec{r}(u,v) = \langle v\cos u, v\sin u, 2v\rangle$ for $-2 < v \le 2$ and $0 \le u \le \pi$.

$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -2\right) \to v = -1 \quad u = \frac{3\pi}{4}$

$\vec{r}_u = \langle -v\sin u, v\cos u, 0\rangle \qquad \vec{r}_v = \langle \cos u, \sin u, 2\rangle$

$\vec{r}_u\left(\frac{3\pi}{4}, -1\right) = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right\rangle \qquad \vec{r}_v = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 2\right\rangle$

$\vec{n} = \vec{r}_u \times \vec{r}_v = \langle \sqrt{2}, -\sqrt{2}, 1\rangle$

$\boxed{x = \frac{\sqrt{2}}{2} + \sqrt{2}t \quad y = -\frac{\sqrt{2}}{2} - \sqrt{2}t \quad z = -2 + t}$

$\boxed{\sqrt{2}\left(x - \frac{\sqrt{2}}{2}\right) - \sqrt{2}\left(y + \frac{\sqrt{2}}{2}\right) + (z+2) = 0}$

② Find a parametrization of

$$S: \quad 3x + x^2 + 2y^2 - z^2 = 3 \quad \text{for } z \ge 0$$

$$\boxed{\begin{array}{lll} z = \sqrt{3x + x^2 + 2y^2 - 3} & x = x & y = y \\ z \ge 0 & & 3x + x^2 + 2y^2 \ge 3 \end{array}}$$

③ $\vec{F} = \langle xye^z, yze^x, xze^y\rangle$

$\text{Div}\,\vec{F} = ye^z + ze^x + xe^y$

$\text{Curl}\,\vec{F} = (xze^y - ye^x)\vec{i} - (ze^y - xye^z)\vec{j} + (yze^x - xe^z)\vec{k}$

$\vec{F}$ Conservative? $\text{Curl}\,\vec{F} \neq 0$ thus $\vec{F}$ is not conservative.

④ $\vec{F} = \langle y\cos z, x\cos z, -xy\sin z\rangle$ find $\int_C \vec{F}\cdot d\vec{r}$ for any curve with initial point $(0,0,0)$ and terminal point $(1,1,0)$.

$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(1,1,0) - f(0,0,0)$ where $f(x,y,z) = xy\cos z$

$= 1 - 0 = \boxed{1}$

Chapter 16 Practice

(5) Set up $\iint_S xy\,dS$ over D, where S is part of the graph of $z^2 = 4x^2 + 4y^2$ between the planes $z = -2$ and $z = -4$ and D is the region for your parameters.

D: $x^2 + y^2 \geq 1$ $\quad x^2 + y^2 \leq 4$

S: $\vec{r}(x,y) = \sqrt{4x^2 + 4y^2}$, $x = x$, $y = y$

$r(r,\theta) = \langle r\cos\theta, r\sin\theta, 2r\rangle$

$\vec{r}_r = \langle \cos\theta, \sin\theta, 2\rangle$

$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$

$\vec{r}_r \times \vec{r}_\theta = \langle 2r\cos\theta, -2r\sin\theta, r\rangle$

$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{4r^2 + r^2} = \sqrt{5}r$

$$\iint_S xy\,dS = \iint_D xy\,|\vec{r}_r \times \vec{r}_\theta|\,dA$$

$$= \boxed{\int_0^{2\pi}\int_1^2 r^2\cos\theta\sin\theta\,(\sqrt{5}r)\,r\,dr\,d\theta}$$

(6) Use Stokes's: $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$ where $\vec{F} = \langle y, -x, z^2\rangle$ and S is part of $z = -x^2 - y^2$ above $z = -4$.

$$\iint_S \operatorname{Curl}\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(2\cos\theta, 2\sin\theta, -4)\cdot\langle -2\sin\theta, 2\cos\theta, 0\rangle\,d\theta$$

$$= \int_0^{2\pi} -4\sin^2\theta - 4\cos^2\theta\,d\theta$$

$$= \boxed{-8\pi}$$

(7) Use Div Thm: $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = \langle zy, 2y, 3z\rangle$ and S is the surface of the solid Right Cone $z^2 = x^2 + y^2$ for $0 \leq z \leq 2$.

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV = \iiint_E 5\,dV$$

$$= 5\cdot\frac{1}{3}\pi(2)^2(2) = \boxed{\frac{40}{3}\pi}$$