Recall: $y = f(x)$ and $x = g(t)$ where f and g are differentiable functions

$$\boxed{\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}}$$

Chain Rule (Case 1): $z = f(x,y)$ differentiable of x and y, $x = g(t)$ and $y = h(t)$ both differentiable functions of t. Then

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}$$

Def 14.4.7 $\quad \frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}$

$$\frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + 0 \cdot \frac{dx}{dt} + 0 \cdot \frac{dy}{dt} \quad ■$$

Ex 2 The pressure P (in Kilopascals), volume V (in Liters), and temp T (in kelvins) of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the temp is 300K and increasing at a rate of 0.1 K/s and the volume is 100L and increasing at a rate of 0.2 L/s.

$$P = 8.31\frac{T}{V} \qquad T = 300\text{K} \quad \frac{dT}{dt} = 0.1 \text{ K/s}$$

$$V = 100\text{L} \quad \frac{dV}{dt} = 0.2 \text{ L/s}$$

$$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt} \qquad \left.\frac{dP}{dt}\right|_{\substack{T=300\\V=100}} = \frac{8.31}{100}(0.1) - \frac{8.31}{(100)^2}(300)(0.2) = -0.04155$$

$$= \frac{8.31}{V}(0.1) + \left(-\frac{8.31\,T}{V^2}\right)(0.2)$$

$\boxed{\text{Thus, the pressure is decreasing at a rate of } 0.04155 \text{ KPa/s}.}$

Chain Rule (Case 2) $z = f(x,y)$ differentiable function of x and y, $x = g(s,t)$ and $y = h(s,t)$ differentiable functions of s and t then

$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}}$$

☆ Extends to the general case of a function in n-variables

Ex 5 If $u = x^4y + y^2z^3$ where $x = rse^t$, $y = rs^2e^{-t}$ and $z = r^2s\sin t$ then find $\frac{\partial u}{\partial s}$ when $r=2, s=1, t=0$.

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s} = (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2\sin t)$$

$x(2,1,0) = 2 \quad y(2,1,0) = 2 \quad z(2,1,0) = 0$

$$\left.\frac{\partial u}{\partial s}\right|_{\substack{r=2\\s=1\\t=0}} = (4(2)^3\cdot 2)(2) + (16+0)(4) + (3\cdot 4\cdot \cancel{0})(4\cdot 0)$$

$$= 64\cdot 2 + 16\cdot 4 = \boxed{192}$$

Implicit Differentiation : $F(x,y)=0$ defines y implicitly as a differentiable function of x, $y=f(x)$ with $F(x,f(x))=0$

If F is differentiable then by chain rule get

$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}=0$$

If $\partial F/\partial y \neq 0$ then we get

$$\boxed{\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}=\frac{-F_x}{F_y}}$$

Implicit Function Theorem

Ex 8 Find y' if $x^3+y^3=6xy$.

$$F(x,y)=x^3+y^3-6xy=0$$

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{(3x^2-6y)}{(3y^2-6x)}=\boxed{-\frac{(x^2-2y)}{y^2-2x}}$$

Now suppose $z=f(x,y)$ is implicitly defined with $F(x,y,z)=0 \Rightarrow F(x,y,f(x,y))=0$ then if F and f are differentiable

$$\frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial x}+\frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial x}+\frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x}=0$$

$$\left[\ \frac{\partial x}{\partial x}=1 \quad \frac{\partial y}{\partial x}=0 \quad \frac{\partial x}{\partial y}=0 \quad \frac{\partial y}{\partial y}=1\right.$$

So $\boxed{\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}}$ and Similarly $\boxed{\dfrac{\partial z}{\partial y}=-\dfrac{F_y}{F_z}}$