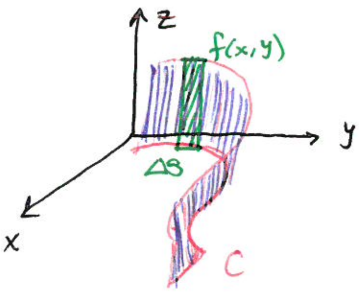
Section 16.2 - Line Integrals

Recall the arc length formula: $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\int_a^b |r'(t)|\,dt = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\,dt$$

Now rather than integrating over an interval $[a,b]$ we integrate over a curve C - Called line integrals



Area of one rectangle =

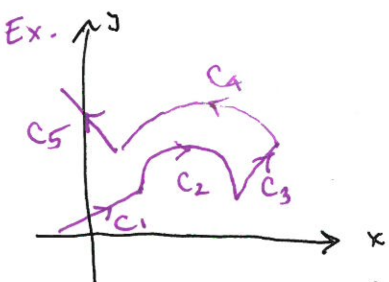
$$f(x(t), y(t))\,\Delta S = f(x(t), y(t))\sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2}\,dt$$

- f defined on a smooth curve C, C given by $x = x(t)$, $y = y(t)$ with $a \le t \le b$ then the line Integral of f on C is

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2}\,dt}$$

If C is piecewise-smooth - union of a finite number of smooth curves

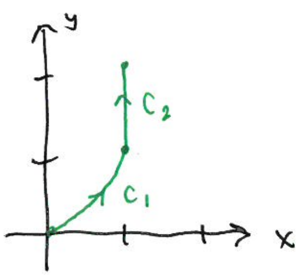
Ex.



$$C = C_1 \cup C_2 \cup C_3 \cup \cdots \cup C_n \text{ then}$$

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \cdots + \int_{C_n} f(x,y)\,ds$$

Example 2 Evaluate $\int_C 2x\,ds$ where C consists of the arc C_1 of a parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line C_2 from $(1,1)$ to $(1,2)$.



$x = t$, $y = t^2$ $\quad C_1: 0 \le t \le 1$

$x = 1$ $\quad y = t$ $\quad C_2: 1 \le t \le 2$

$$\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds$$

$$= \int_0^1 2t\sqrt{1 + (2t)^2}\,dt + \int_1^2 2(1)\sqrt{0^2 + 1^2}\,dt$$

$$= 2\int_0^1 t\sqrt{1 + 4t^2}\,dt + 2\int_1^2 dt$$

$$= 2\left[(1+4t^2)^{3/2} \cdot \frac{2}{3} \cdot \frac{1}{8}\right]\Big|_0^1 + 2(2-1)$$

$$= \boxed{\frac{1}{6}\left[5\sqrt{5} - 1\right] + 2}$$

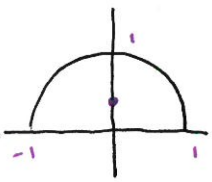
Applications of Line Integrals:

$f = \rho(x,y)$ a density function of a thin wire shaped like a curve C

mass of wire: $m = \int_C \rho(x,y)\,ds$

Center of mass: $(\bar{x}, \bar{y})$ $\quad \bar{x} = \frac{1}{m}\int_C x\,\rho(x,y)\,ds \quad \bar{y} = \frac{1}{m}\int_C y\,\rho(x,y)\,ds$

Example 3 A wire takes the shape of the semicircle $x^2+y^2=1$, $y \geq 0$ and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from $y=1$.



Parametrization of C: $x = \cos t$, $y = \sin t$ $\quad 0 \leq t \leq \pi$

$ds = \sqrt{\sin^2 t + \cos^2 t}\,dt = dt$

$\rho(x,y) = K(1-y)$

$$m = \int_C \rho(x,y)\,ds = \int_0^\pi K(1-y)\,dt = K\int_0^\pi 1 - \sin t\,dt$$

$$= K\left[t + \cos t\right]_0^\pi = \boxed{K(\pi - 2)}$$

$$\bar{y} = \frac{1}{m}\int_C y\,\rho(x,y)\,ds = \frac{K}{m}\int_0^\pi \sin t - \sin^2 t\,dt$$

$$= \frac{1}{\pi-2}\left[-\cos t - \frac{t}{2} + \frac{\sin 2t}{4}\right]_0^\pi = \frac{1 - \pi/2 + 1}{\pi - 2} = \boxed{\frac{4-\pi}{2(\pi-2)}} \approx 0.38$$

By Symmetry $\bar{x} = 0$ $\quad$ Center of mass: $\boxed{\left(0, \frac{4-\pi}{2(\pi-2)}\right)}$

Other Line Integrals: replace Δs with Δx or Δy

Line integral of f along C with respect to x and y:

$$\int_C f(x,y)\,dx \qquad \int_{-C} f\,dx = -\int_C f\,dx \qquad \int_{-C} f\,dy = -\int_C f\,dy \qquad \int_C f(x,y)\,dy$$

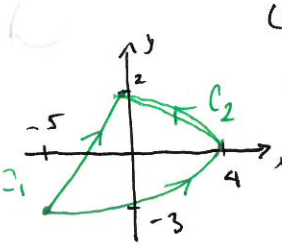
Notation:

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy$$

Line integral of f along C with respect to arc length: $\int_C f(x,y)\,ds$ $\quad \int_{-C} f\,ds = \int_C f\,ds$

Example 4 Evaluate $\int_C y^2\,dx + x\,dy$, where (a) $C = C_1$ line segment from $(-5,3)$ to $(0,2)$

(b) $C = C_2$ is the arc of $x = 4-y^2$ from $(-5,-3)$ to $(0,2)$.



(a) $\vec{r}(t) = \vec{r_0} + \vec{v}t = \langle -5,-3\rangle + t\langle 5,5\rangle$ $\quad x = 5t-5 \quad y = 5t-3 \quad 0 \leq t \leq 1$

$dx = 5\,dt \quad dy = 5\,dt$

$$\int_C y^2\,dx + x\,dy = \int_0^1 (5t-3)^2 \cdot 5\,dt + (5t-5)\,5\,dt = \frac{(5t-3)^3}{3}\Big|_0^1 + \frac{(5t-5)^2}{2}\Big|_0^1 = \frac{8}{3} + \frac{27}{3} + 0 - \frac{25}{2} = \boxed{-\frac{5}{6}}$$

(b) $x = 4-t^2 \quad y = t \quad -3 \leq t \leq 2 \quad dx = -2t\,dt \quad dy = dt$

$$\int_C y^2\,dx + x\,dy = \int_{-3}^2 t^2(-2t)\,dt + (4-t^2)\cdot dt = -\frac{2}{4}t^4 + 4t - \frac{1}{3}t^3\Big|_{-3}^2 = \boxed{40\tfrac{5}{6}}$$

Line integrals are dependent on the Path C!