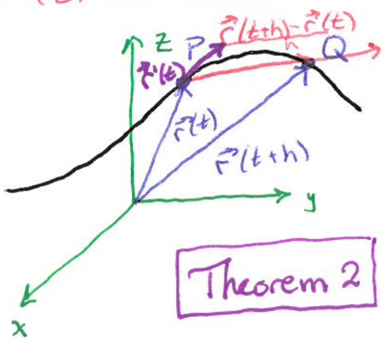
Derivative $\vec{r}\,'$ of a vector function $\vec{r}$ is defined as

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$, provided the limit exists.

$\vec{r}\,'(t)$ is called the tangent vector and



$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$ is the unit tangent vector.

Theorem 2 If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ where f, g, h are differentiable functions then

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t)\rangle$$

Proof: $$\vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \lim_{h\to 0} \frac{1}{h}\langle f(t+h) - f(t), g(t+h) - g(t), h(t+h) - h(t)\rangle$$

$$= \left\langle \lim_{h\to 0} \frac{f(t+h)-f(t)}{h}, \lim_{h\to 0} \frac{g(t+h)-g(t)}{h}, \lim_{h\to 0} \frac{h(t+h)-h(t)}{h} \right\rangle$$

$$= \langle f'(t), g'(t), h'(t)\rangle \quad \blacksquare$$

Example 1 (a) Find the derivative of $\vec{r}(t) = (1+t^3)\vec{i} + te^{-t}\vec{j} + \sin 2t\,\vec{k}$

(b) Unit tangent vector when $t=0$.

(a) $\boxed{\vec{r}\,'(t) = 3t^2\vec{i} + (e^{-t} - te^{-t})\vec{j} + 2\cos 2t\,\vec{k}}$

★ All rules for derivatives of functions in one variable still apply!

(b) $\vec{r}\,'(0) = 0\vec{i} + \vec{j} + 2\vec{k}$ $\quad \vec{T}(0) = \frac{\vec{r}\,'(0)}{|\vec{r}\,'(0)|} = \boxed{\frac{1}{\sqrt{5}}\vec{j} + \frac{2}{\sqrt{5}}\vec{k}}$

Example 3 Find parametric equations for the tangent line to the helix with parametric equations:

$$x = 2\cos t \qquad y = \sin t \qquad z = t$$

at the point $(0, 1, \pi/2)$.

$\vec{r}(t) = \langle 2\cos t, \sin t, t\rangle$ so $\vec{r}\,'(t) = \langle -2\sin t, \cos t, 1\rangle$

at $(0, 1, \pi/2)$ $t = \pi/2$ so the tangent vector is $\vec{r}\,'(\pi/2) = \langle -2, 0, 1\rangle$

point $(0, 1, \pi/2)$ so

$$\boxed{x = 0 - 2t \qquad y = 1 + 0t \qquad z = \frac{\pi}{2} + t}$$

Differentiation Rules: $\vec{u}, \vec{v}$ differentiable vector functions, c a scalar and f a real-valued function.

1. $\frac{d}{dt}(\vec{u}(t) + \vec{v}(t)) = \vec{u}'(t) + \vec{v}'(t)$
2. $\frac{d}{dt}(c\vec{u}(t)) = c\vec{u}'(t)$
3. $\frac{d}{dt}(f(t)\vec{u}(t)) = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$
4. $\frac{d}{dt}(\vec{u}(t) \cdot \vec{v}(t)) = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$
5. $\frac{d}{dt}(\vec{u}(t) \times \vec{v}(t)) = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$
6. $\frac{d}{dt}(\vec{u}(f(t))) = f'(t)\vec{u}'(f(t))$

(Rules 3–5: product Rules; Rule 6: Chain Rule)

Example 4 Show that if $|\vec{r}(t)| = c$ then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$

Show $\vec{r}'(t) \cdot \vec{r}(t) = 0$

Know $\vec{r}(t) \cdot \vec{r}(t) = |\vec{r}(t)|^2 = c^2$

so $0 = \frac{d}{dt}[\vec{r}(t) \cdot \vec{r}(t)] = 2\vec{r}'(t) \cdot \vec{r}(t)$

so $\vec{r}'(t) \cdot \vec{r}(t) = 0$ ∎

Definite integral:

$$\int_a^b \vec{r}(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{r}(t_i)\Delta t = \left\langle \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i^*)\Delta t, \lim_{n\to\infty} \sum_{i=1}^{n} g(t_i^*)\Delta t, \lim_{n\to\infty} \sum_{i=1}^{n} h(t_i^*)\Delta t \right\rangle$$

$$= \boxed{\left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle}$$

FTC: $\int_a^b \vec{r}(t)\,dt = \vec{R}(b) - \vec{R}(a)$ where $\vec{R}'(t) = \vec{r}(t)$.

Example 5 If $\vec{r}(t) = 2\cos t\,\vec{i} + \sin t\,\vec{j} + 2t\,\vec{k}$, find $\int \vec{r}(t)\,dt$ and $\int_0^{\pi/2} \vec{r}(t)\,dt$

$$\int \vec{r}(t)\,dt = \int 2\cos t\,dt\,\vec{i} + \int \sin t\,dt\,\vec{j} + \int 2t\,dt\,\vec{k}$$

$$= \boxed{2\sin t\,\vec{i} - \cos t\,\vec{j} + t^2\vec{k} + \vec{C}}$$

vector constant of integration

$$\int_0^{\pi/2} \vec{r}(t)\,dt = \boxed{2\vec{i} + \vec{j} + \frac{\pi^2}{4}\vec{k}}$$