Section 15.2 - Iterated Integrals

★ Hard to evaluate single integrals directly from definition
Even harder for double integrals → Iterated integrals

Fix x on $R = [a,b] \times [c,d]$,

$$A(x) = \int_c^d f(x,y)\,dy \quad \text{depends on } x$$

then integrate A with respect to x:

$\boxed{\text{Iterated Integral}}$

$$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

★ must work from inside to outside, but each is a single integral
⇒ Can use FTC!!!

$\boxed{\text{Ex. 1}}$ Evaluate

(a) $\displaystyle\int_0^3 \int_1^2 x^2 y\,dy\,dx$

$$= \int_0^3 \left[x^2 \frac{y^2}{2}\right]\Big|_1^2 dx$$

$$= \int_0^3 \left(2x^2 - \frac{x^2}{2}\right) dx$$

$$= \frac{1}{2} x^3 \Big|_0^3$$

$$= \boxed{27/2}$$

(b) $\displaystyle\int_1^2 \int_0^3 x^2 y\,dx\,dy$

$$= \int_1^2 \left[\frac{x^3 y}{3}\right]_0^3 dy$$

$$= \int_1^2 9y\,dy$$

$$= \frac{9y^2}{2}\Big|_1^2$$

$$= \boxed{27/2}$$

$\boxed{\text{Fubini's Theorem}}$ If f is continuous on $R = [a,b] \times [c,d]$ then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

★ more general, f bounded on R, f discontinuous only on a finite # of smooth curves

$\boxed{\text{Ex 3}}$ Evaluate $\displaystyle\iint_R y\sin(xy)\,dA$ where $R = [1,2] \times [0,\pi]$

★ Integrate wrt x first:

$$\int_0^\pi \int_1^2 y\sin(xy)\,dx\,dy = \int_0^\pi -y\frac{\cos(xy)}{y}\Big|_1^2 dy = \int_0^\pi (-\cos(2y) + \cos(y))\,dy$$

$$= -\frac{\sin(2y)}{2} + \sin(y)\Big|_0^\pi = \boxed{0}$$

Reverse order: $\displaystyle\int_1^2 \int_0^\pi y\sin(xy)\,dy\,dx = \int_1^2 \left[-y\frac{\cos(xy)}{x}\Big|_0^\pi + \frac{1}{x}\int_0^\pi \cos(xy)\,dy\right] dx = \boxed{0}$

Integrate by Parts

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Vector Calc

$$\boxed{\iint_R g(x)\cdot h(y)\,dA = \int_a^b\int_c^d g(x)\,h(y)\,dy\,dx = \int_a^b g(x)\,dx \cdot \int_c^d h(x)\,dy}$$

$\boxed{\text{Ex}}$

$$\int_0^3\int_1^2 x^2 y\,dy\,dx = \int_0^3 x^2\,dx \cdot \int_1^2 y\,dy$$

$$= \left.\frac{x^3}{3}\right|_0^3 \cdot \left.\frac{y^2}{2}\right|_1^2$$

$$= 9 \cdot \frac{3}{2} = \boxed{\frac{27}{2}}$$