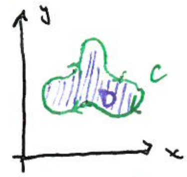
★ Relationship between Line integrals around simple closed curves C and a double integral over the region D bounded by C

Definition - a positive orientation of a simple closed curve C refers to a single counterclockwise traversal of C.

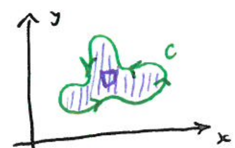
Notation: $\int_C \vec{F}\cdot d\vec{r} = \oint_C \vec{F}\cdot d\vec{r} = \oint_C \vec{F}\cdot d\vec{r}$

Green's Theorem - Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane. D bounded by C, P and Q have continuous partials on D

$$\boxed{\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$



Negative orientation



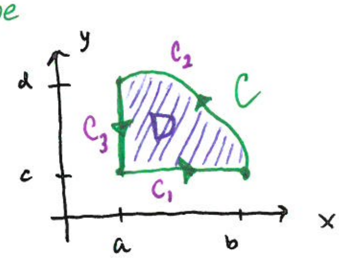
Positive orientation

Notation: boundary of $D = \partial D = C$

Proof: Any curve C piecewise-smooth, simple closed with region D can be broken into rectangles or regions as follows:

$$D = \{(x,y) \mid a \le x \le b,\ c \le y \le f(x)\}$$
$$= \{(x,y) \mid c \le y \le d,\ a \le x \le g(y)\}$$

where f and g are inverse functions of each other



$C = C_1 \cup C_2 \cup C_3$

Show: ① $\int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA$ ② $\int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA$

① $\int_C P\,dx = \int_{C_1} P\,dx + \int_{C_2} P\,dx + \int_{C_3} P\,dx = \int_a^b P(x,c)\,dx + \int_b^a P(x,f(x))\,dx + \int_a^a P(x,y)\,dx$

$= \int_a^b P(x,c)\,dx - \int_a^b P(x,f(x))\,dx + 0 = -\int_a^b P(x,f(x)) - P(x,c)\,dx$

$= -\int_a^b \int_c^{f(x)} \frac{\partial P}{\partial y}(x,y)\,dy\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA$ ✓

② $\int_C Q\,dy = \int_{C_1} Q\,dy + \int_{C_2} Q\,dy + \int_{C_3} Q\,dy = \int_c^c Q(x,y)\,dy + \int_c^d Q(g(y),y)\,dy + \int_d^c Q(a,y)\,dy$

$= 0 + \int_c^d Q(g(y),y)\,dy - \int_c^d Q(a,y)\,dy = \int_c^d Q(g(y),y) - Q(a,y)\,dy$

$= \int_c^d \int_a^{g(y)} \frac{\partial Q}{\partial x}(x,y)\,dx\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA$ ✓

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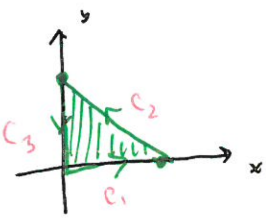
Section 16.4 - Green's Theorem

Ex 1 Evaluate $\int_C x^4 dx + xy\,dy$, where C is the triangular curve from $(0,0)$ to $(1,0)$ to $(0,1)$.



$$\int_C x^4 dx + xy\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_0^1 \int_0^{1-x} (y - 0)\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 (1-x)^2\,dx = -\frac{1}{6}(1-x)^3\Big|_0^1 = \boxed{\frac{1}{6}}$$

Note: $\int_C x^4 dx + xy\,dy = \int_{C_1} x^4 dx + xy\,dy + \int_{C_2} x^4 dx + xy\,dy + \int_{C_3} x^4 dx + xy\,dy$

Not fun ...

Reverse Application of Green's Theorem

A = Area of D is $\iint_D 1\,dA$. Choose P, Q s.t. $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$

Possibilies: $P = 0$, $Q = x$ $\quad$ $P = -y$, $Q = 0$ $\quad$ $P = -\frac{1}{2}y$, $Q = \frac{1}{2}x$

Then $\boxed{A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx}$

Ex 3 Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$x = a\cos t$, $y = b\sin t$, $\quad 0 \le t \le 2\pi$

$$A = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t\,dt) - (b\sin t)(-a\sin t\,dt)$$

$$= \frac{ab}{2}\int_0^{2\pi} dt = \boxed{\pi ab}$$

★ Green's Theorem works for regions with holes:

Cancel since opposite direction

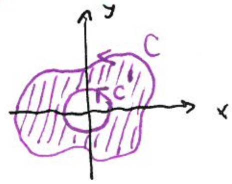
$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \iint_{D_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial x}\,dA + \iint_{D_2} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

$$= \int_{\partial D_1} P\,dx + Q\,dy + \int_{\partial D_2} P\,dx + Q\,dy$$

$$= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy$$

$$= \int_C P\,dx + Q\,dy$$

Example 5 $\vec{F}(x,y) = \frac{\langle -y, x\rangle}{x^2+y^2}$. Show $\int_C \vec{F}\cdot d\vec{r} = 2\pi$ for every positively oriented simple closed path enclosing origin.



$$\int_C P\,dx + Q\,dy + \int_{-C'} P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \iint_D \frac{y^2 - x^2}{(x^2+y^2)^2} - \frac{y^2 - x^2}{(x^2+y^2)^2}\,dA = 0$$

$$\Rightarrow \int_C P\,dx + Q\,dy = \int_{C'} P\,dx + Q\,dy$$

$x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$

$$\int_{C'} P\,dx + Q\,dy = \int_0^{2\pi} \frac{\sin^2 t + \cos^2 t}{1}\,dt = \boxed{2\pi}$$

<u>Proof of 16.3.6</u> (partial converse of Thm 5)

$\vec{F} = \langle P, Q \rangle$ on open simply-connected region D

P, Q have continuous partials with $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D

Then for any simple closed curve C in D with R the region bounded by C we have by Green's Theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P\,dx + Q\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = 0$$

Any closed curve in D can be broken into simple closed curves.

Thus $\int_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve C in D, thus

$\int_C \vec{F} \cdot d\vec{r}$ is path independent and FTC applies showing

$\vec{F}$ is a conservative vector field. ∎