Chapter 14 - Review

#10 True/False: If $(2,1)$ is a critical point of f and $f_{xx}(2,1)\,f_{yy}(2,1) < f_{xy}(2,1)^2$ then f has a saddle point at $(2,1)$.

$\boxed{\text{True}}$ $\quad D(2,1) = f_{xx}(2,1)\cdot f_{yy}(2,1) - f_{xy}(2,1)^2 < 0$

So by the Second derivative test f has a saddle point at $(2,1)$.

#16 Find the first partial derivatives of $G(x,y,z) = e^{xy}\cdot \sin\left(\frac{y}{z}\right)$

$$\boxed{G_x(x,y,z) = e^{xy}\cdot x \cdot \sin\left(\tfrac{y}{z}\right)} \quad \boxed{G_y(x,y,z) = e^{xy}\cdot x \sin\left(\tfrac{y}{z}\right) + e^{xy}\cdot \cos\left(\tfrac{y}{z}\right)\cdot \tfrac{1}{z}}$$

$$\boxed{G_z(x,y,z) = e^{xy}\cos\left(\tfrac{y}{z}\right)\cdot\left(-\tfrac{y}{z^2}\right)}$$

#36 If $v = x^2\sin y + ye^{xy}$ where $x = s+2t$ and $y = st$, use the chain rule to find $\partial v/\partial t$ and $\partial v/\partial s$ when $s=0$ and $t=1$.

$$\frac{\partial v}{\partial t} = 2x\frac{\partial x}{\partial t}\sin y + x^2\cos y\frac{\partial y}{\partial t} + y^2e^{xy}\frac{\partial x}{\partial t} + e^{xy}\frac{\partial y}{\partial t} + xye^{xy}\frac{\partial y}{\partial t}$$

$$\left.\frac{\partial v}{\partial t}\right|_{\substack{s=0\\t=1}} = \underline{2(2)(2)\sin(0)} + \underline{(2)^2\cos(0)(0)} + \underline{(0)^2e^{2\cdot 0}(2)} + \underline{e^{2\cdot 0}(0)} + \underline{(2)(0)e^{2\cdot 0}(0)} = \boxed{0}$$

$$\frac{\partial v}{\partial s} = 2x\frac{\partial x}{\partial s}\sin y + x^2\cos y\frac{\partial y}{\partial s} + y^2e^{xy}\frac{\partial x}{\partial s} + e^{xy}\frac{\partial y}{\partial s} + xye^{xy}\frac{\partial y}{\partial s}$$

$$\left.\frac{\partial v}{\partial s}\right|_{\substack{s=0\\t=1}} = \underline{2(2)(1)\sin(0)} + \underline{(2)^2\cos(0)(1)} + \underline{(0)^2e^{2\cdot 0}(1)} + \underline{e^{2\cdot 0}(1)} + \underline{2(0)e^{2\cdot 0}(1)} = \boxed{5}$$

#47 Find the maximum rate of change of $f(x,y) = x^2y + \sqrt{x}$ at the point $(2,1)$. In which direction does it occur?

$$\nabla f(x,y) = \left\langle 2xy + \frac{1}{2\sqrt{x}},\ x^2\right\rangle$$

Maximum rate at $(2,1)$ $= |\nabla f(2,1)| = \left|\left\langle 4 + \frac{1}{2\sqrt{2}}, 4\right\rangle\right| = \boxed{\sqrt{\left(4+\frac{\sqrt{2}}{4}\right)^2 + 4^2}}$

in the direction of $\boxed{\left\langle 4 + \frac{\sqrt{2}}{4}, 4\right\rangle}$.

Chapter 14 - Review

$\boxed{\#56}$ Find the absolute max and min values of f on the set D,

$$f(x,y) = e^{-x^2-y^2}(x^2+2y^2) \qquad D = \{(x,y) \mid x^2+y^2 \le 4\}$$

Critical Points : $\nabla f(x,y) = \vec{0}$, DNE or on the boundary $x^2+y^2=4$

$$\vec{0} = \nabla f(x,y) = \left\langle e^{-x^2-y^2}(-2x)(x^2+2y^2) + e^{-x^2-y^2}(2x),\ e^{-x^2-y^2}(-2y)(x^2+2y^2) + e^{-x^2-y^2}(4y) \right\rangle$$

$$= e^{-x^2-y^2}\left\langle 2x(1-x^2-2y^2),\ 2y(2-x^2-2y^2) \right\rangle$$

Since $e^{-x^2-y^2} > 0$ for all (x,y) we have:

$$2x(1-x^2-2y^2) = 0 \qquad \text{and} \qquad 2y(2-x^2-2y^2) = 0$$

$x = 0$ or $x^2+2y^2 = 1$ when $y=0$ $x = \pm 1$ $\qquad$ $y = 0$ or $x^2+2y^2 = 2$ when $x=0$ $y = \pm 1$

$\boxed{\text{Critical Points}}$ $\boxed{(0,0),\ (0,1),\ (0,-1),\ (1,0),\ (-1,0)}$

Check Critical Points:

$f(0,0) = \boxed{0}$ $\quad f(0,1) = \boxed{\frac{2}{e}}$ ≈ 0.7357 $\quad f(0,-1) = \boxed{\frac{2}{e}}$ ≈ 0.7357 $\quad f(1,0) = \boxed{\frac{1}{e}}$ $\quad f(-1,0) = \boxed{\frac{1}{e}}$

Check boundary: $x^2+y^2 = 4$ $\quad -2 \le x \le 2$ and $-2 \le y \le 2$

$f(x,y) = e^{-4}(4+y^2)$ smallest when $y=0$ $f(x,0) = \boxed{\frac{4}{e^4}}$

Largest when $y = \pm 2$ $f(x,\pm 2) = \boxed{\frac{8}{e^4}} \approx 0.3983$

Therefore the absolute max on D for $f(x,y)$ is at $(0,1)$ and $(0,-1)$ and is $2/e$. Also the absolute min on D for $f(x,y)$ is at $(0,0)$ and is 0.

Chapter 14 - Review

$\boxed{\#60}$ Use Lagrange multipliers to find the max/min values of $f(x,y) = \frac{1}{x} + \frac{1}{y}$ subject to the constraint $\frac{1}{x^2} + \frac{1}{y^2} = 1$.

[Note I will only ask for the set up of these problems]

Let $g(x,y) = x^{-2} + y^{-2}$. Then

$$f_x = \lambda g_x \quad f_y = \lambda g_y \quad \text{and} \quad g(x,y) = 1$$

Set Up: $\left(-\frac{1}{x^2}\right) = \lambda \cdot \left(\frac{-2}{x^3}\right) \quad \left(-\frac{1}{y^2}\right) = \lambda\left(-\frac{2}{y^3}\right) \quad x^{-2} + y^{-2} = 1$

All you need on the test →

$$\boxed{① \ x^3 = 2x^2\lambda \qquad ② \ y^3 = 2y^2\lambda \qquad ③ \ x^{-2} + y^{-2} = 1}$$

Solving: from ① get $x = 0$ or $x = 2\lambda$. Since $x \neq 0$ must have

④ $x = 2\lambda$.

From ③ get ⑤ $\lambda = \frac{1}{2}\frac{y}{\sqrt{y^2-1}}$.

From ② get $y^3 = 2y^2 \cdot \frac{1}{2}\frac{y}{\sqrt{y^2-1}} \Rightarrow \sqrt{y^2-1} = 1$

$\Rightarrow y = \pm\sqrt{2}$

From ⑤ get $\lambda = \pm\frac{\sqrt{2}}{2}$

From ④ get $x = \pm\sqrt{2}$

Two points: $(\sqrt{2}, \sqrt{2})$, and $(-\sqrt{2}, -\sqrt{2})$

$$f(\sqrt{2}, \sqrt{2}) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \qquad f(-\sqrt{2}, -\sqrt{2}) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$$

Thus the max value of f is $\sqrt{2}$ and the min value is $-\sqrt{2}$.