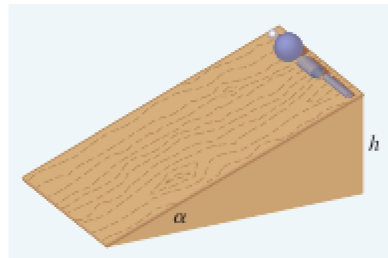
Roller Derby Project

We are going to race 4 objects down a ramp. Suppose you roll a marble (solid ball), a squash ball (hollow ball), streel bar (solid cylinder), and a lead pipe (hollow cylinder) down a ramp at the same time.



1. Make a guess about which object will reach the bottom first.

   Up to the student; no correct solution.

2. To answer definitely we consider a ball and cylinder with mass m, radius r, moment of inertia I (about the axis of rotation). If the highest place on the ramp is h, then what is the potential energy of the object?

   The potential energy of the object is mgh.

3. If the object reaches the bottom with velocity v and angular velocity ω (with $v = r\omega$) then what is the kinetic energy at the bottom of the ramp? (Hint: Remember that kinentic energy consists of translational and rotational energy.)

   The kinetic energy is $\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$.

4. If the energy lost by friction is negligible then conservation of energy gives:

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

5. Show that $v^2 = \dfrac{2gh}{1+I^*}$ where $I^* = \dfrac{I}{mr^2}$.

   Replacing $\omega = \frac{v}{r}$ and solving for v^2 in the conservation of energy equation gives:

$$2mgh = mv^2 + I\frac{v^2}{r^2}$$

$$v^2(m + Ir^2) = 2mgh$$

$$v^2 = \frac{2mgh}{m + \frac{I}{r^2}}$$

$$v^2 = \frac{2gh}{1 + \frac{I}{r^2 m}}$$

$$v^2 = \frac{2gh}{1 + I^*}$$

6. Let $y(t)$ be the vertical distance traveled at time t. Show that $v^2 = \dfrac{2gy(t)}{1+I^*}$ at any time t.

   The potential energy of the object at time t is $mgy(t)$ and it's kinetic energy is still $\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$. Thus by conservation of energy and the work in 5 we have:

$$v^2 = \frac{2gy(t)}{1+I^*}$$

7. Show that y satisfies the differential equation:

$$\frac{dy}{dt} = \sqrt{\frac{2g}{1+I^*}}(\sin\alpha)\sqrt{y}$$

   where α is the angle of inclination of the ramp.

   Solving for $y(t)$ in the equation from 6 gives:

$$y = \frac{v^2(1+I^*)}{2g}$$

   Substituting this in for y in the differential equation gives:

$$\frac{dy}{dt} = \sqrt{\frac{2g}{1+I^*}}\sin(\alpha)\sqrt{\frac{v^2(1+I^*)}{2g}}$$

$$\frac{dy}{dt} = v\sin\alpha$$

   Which is the velocity of the object in the y-direction on a ramp of angle α with linear velocity of v. Thus the differential equation is satisfied.

8. Solve the differential equation above and use it to find the total travel time T.

   Separating the variables gives:

$$y^{-1/2}dy = \sqrt{\frac{2g}{1+I^*}}\sin(\alpha)dt$$

   Then Integrating both sides:

$$2y^{1/2} = \sqrt{\frac{2g}{1+I^*}}\sin(\alpha)t + C$$

   Since $y(0) = 0$ we have that $C = 0$. Also $y(T) = h$ so we have:

$$2h^{1/2} = \sqrt{\frac{2g}{1+I^*}}\sin(\alpha)T$$

   Solving for T gives:

$$T = \frac{2\sqrt{h(1+I^*)}}{\sqrt{2g}\sin\alpha} = \sqrt{\frac{2h(1+I^*)}{g\sin^2\alpha}}$$

9. Since h, g, α are constants what must be true about I^* to minimize the total travel time T and hence win the race?

   T is minimized when I^* is minimized.

10. Find I^* for a solid cylinder.

    By Symmetry, the solid cylinder's moment of inertia is the same regardless of the coordinate system choosen. I will choose it such that the axis of rotation is the z-axis. Assume the density is constant, ρ, heigh is h and let E represent the region of the cylinder. Then

$$\begin{aligned} I^* &= \frac{I_z}{mr^2} \\ &= \frac{1}{mr^2} \int\int\int_E (x^2+y^2)\rho \, dV \\ &= \frac{\rho}{mr^2} \int_0^h \int_0^{2\pi} \int_0^r R^2 R \, dR \, d\theta \, dz \\ &= \frac{2\pi\rho}{mr^2} \frac{r^4}{4} h \\ &= \frac{\pi\rho r^2 h}{2m} \end{aligned}$$

$$\begin{aligned} m &= \int\int\int_E \rho \, dV \\ &= \rho \int\int\int_E DV \\ &= \rho\pi r^2 h \end{aligned}$$

Thus

$$I^* = \frac{\pi\rho r^2 h}{2\rho\pi r^2 h} = \frac{1}{2}$$

11. Find I^* for a hollow cylinder.

    Solution 1: Using a surface integral where S represents the hollow cylinder,

$$\begin{aligned} I^* &= \frac{1}{mr^2} \int\int_S (x^2+y^2)\rho \, dS \\ &= \frac{1}{mr^2} \int\int_S r^2 \rho \, dS \\ &= \frac{r^2 m}{mr^2} \\ &= 1 \end{aligned}$$

$$m = \int\int\int_E \rho \, dV = \rho\pi(r^2-a^2)h$$

Thus

$$I^* = \frac{\pi\rho(r^4-a^4)h}{2\rho\pi r^2(r^2-a^2)h} = \frac{(r^4-a^4)}{2r^2(r^2-a^2)}$$

Solution 2: Consider a partial hollow cylinder E with inner radius a and outer radius r then

$$\begin{aligned} I^* &= \frac{I_z}{mr^2} = \frac{1}{mr^2} \int\int\int_E (x^2+y^2)\rho \, dV \\ &= \frac{\rho}{mr^2} \int_0^h \int_0^{2\pi} \int_a^r R^2 R \, dR \, d\theta \, dz \\ &= \frac{2\pi\rho}{mr^2} \frac{r^4-a^4}{4} h = \frac{\pi\rho(r^4-a^4)h}{2mr^2} \end{aligned}$$

A hollow cylinder has

$$\begin{aligned} I^* &= \lim_{a\to r} \frac{(r^4-a^4)}{2r^2(r^2-a^2)} \\ &= \lim_{a\to r} \frac{-4a^3}{-4r^2 a} \\ &= 1 \end{aligned}$$

12. Find I^* for a solid sphere.

By Symmetry, the solid cylinder's moment of inertia is the same regardless of the coordinate system choosen. I will choose it such that the axis of rotation is the z-axis. Assume the density is constant, ρ, and let E represent the region of the sphere. Then

$$\begin{aligned}
I^* &= \frac{I_z}{mr^2} \\
&= \frac{1}{mr^2}\int\int\int_E (x^2+y^2)\rho\ dV \\
&= \frac{\rho}{mr^2}\int_0^{\pi}\int_0^{2\pi}\int_0^{r} R^2\sin^2\phi R^2 \sin\phi\ dR\ d\theta\ d\phi \\
&= \frac{2\pi\rho}{mr^2}\frac{r^5}{5}\int_0^{\pi}\sin^3\phi\ d\phi \\
&= \frac{2\pi r^3\rho}{5m}\left(2-\frac{2}{3}\right) \\
&= \frac{8\pi r^3\rho}{15m}
\end{aligned}$$

$$\begin{aligned}
m &= \int\int\int_E \rho\ dV \\
&= \rho\int\int\int_E DV \\
&= \rho\frac{4}{3}\pi r^3
\end{aligned}$$

Thus

$$I^* = \frac{8\pi r^3\rho}{15m}\cdot\frac{3}{4\pi r^3\rho} = \frac{2}{5}$$

13. Find I^* for a hollow sphere.

Solution 1: Using a surface integral where S represents the hollow sphere,

$$\begin{aligned}
I^* &= \frac{1}{mr^2}\int\int_S (x^2+y^2)\rho\ dS \\
&= \frac{\rho}{mr^2}\int_0^{\pi}\int_0^{2\pi} r^2\sin^2\phi r^2\sin\phi\ d\theta\ d\phi \\
&= \frac{2\pi\rho r^4}{mr^2}\int_0^{\pi}\sin^3\phi\ d\phi \\
&= \frac{2\pi\rho r^4}{mr^2}\cdot\frac{4}{3} \\
&= \frac{8\pi\rho r^4}{3mr^2} \\
&= \frac{8\pi\rho r^4}{3(4\pi r^2\rho)r^2} \\
&= \frac{2}{3}
\end{aligned}$$

Solution 2: Consider a partial hollow sphere E with inner radius a and outer radius r then

$$\begin{aligned}
I^* &= \frac{I_z}{mr^2} = \frac{1}{mr^2}\int\int\int_E (x^2+y^2)\rho\ dV \\
&= \frac{\rho}{mr^2}\int_0^{\pi}\int_0^{2\pi}\int_a^{r} R^2\sin^2\phi R^2\sin\phi\ dR\ d\theta\ d\phi \\
&= \frac{2\pi\rho}{mr^2}\frac{r^5-a^5}{5}\cdot\frac{4}{3} = \frac{8\pi\rho(r^5-a^5)}{15mr^2}
\end{aligned}$$

$$m = \int\int\int_E \rho\ dV = \rho\frac{4}{3}\pi(r^3-a^3)$$

So $I^* = \dfrac{8\pi\rho(r^5-a^5)}{5mr^2\cdot 4\pi(r^3-a^3)r^2\rho} = \dfrac{2(r^5-a^5)}{5r^2(r^3-a^3)}$

Thus a hollow sphere has

$$I^* = \lim_{a\to r}\frac{2(r^5-a^5)}{5r^2(r^3-a^3)} = \lim_{a\to r}\frac{-10a^4}{-15r^2a^2} = \frac{2}{3}$$

14. Conclude the finishing order of the race.

The marble (solid sphere) will be first, the squash ball (hollow sphere) will be second, the steel pipe (solid cylinder) will be third, and the lead pipe (hollow cylinder) will be last.