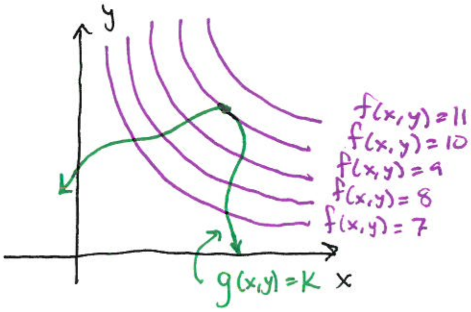
★ Lagrange's Method for maximizing/minimizing $f(x,y,z)$ subject to a Constrain of the form $g(x,y,z) = K$



Find the extreme value of f on the curve $g(x,y) = K$

Happens when the curves touch each other

$\Rightarrow$ Common tangent line

$\Rightarrow$ Parallel gradient vectors

$\boxed{\nabla f = \lambda \cdot \nabla g}$ for some scalar λ

The number λ is called a Lagrange multiplier

Method of Lagrange Multipliers:

Find max/min values of $f(x,y,z)$ subject to $g(x,y,z)=K$ (assuming these values exist and $\nabla g \neq \vec{0}$ on $g(x,y,z)=K$)

1. Find all values of x, y, z, λ such that

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

and $g(x,y,z) = K$

2. Evaluate f on all points (x,y,z) from 1 to find largest/smallest value of f.

$\boxed{\text{Ex. 2}}$ Find the extreme values of $f(x,y) = x^2+2y^2$ on the circle $x^2+y^2=1$

1. $f_x = \lambda g_x$ $\quad f_y = \lambda g_y$ $\quad g(x,y)=1$ where $g(x,y) = x^2+y^2$

$2x = \lambda(2x)$ $\quad 4y = \lambda 2y$ $\quad x^2+y^2=1$

$x=0,\ y=\pm 1$ $\quad y=0\ \ x=\pm 1$

$\lambda = 1\ \ y=0$ $\quad \lambda=2\ \ x=0$

Points: $(0,1)$, $(0,-1)$
$(1,0)$, $(-1,0)$

$\underbrace{f(0,1)=2 \quad f(0,-1)=2}_{\text{maxs}}$ $\quad \underbrace{f(1,0)=1 \quad f(-1,0)=1}_{\text{mins}}$

Ex 4 Find the points on the sphere $x^2+y^2+z^2=4$ that are closest to and furthest from the point $(3,1,-1)$.

$$d^2 = (x-3)^2 + (y-1)^2 + (z+1)^2 \qquad g(x,y,z) = x^2+y^2+z^2 = 4 \quad ⑤$$

① $(2)(x-3) = \lambda(2x) \qquad 2(y-1) = \lambda 2y \qquad 2(z+1) = \lambda 2z$

② $x - 3 = \lambda x \qquad$ ③ $y - 1 = \lambda y \qquad$ ④ $z + 1 = \lambda z$

$$x = \frac{3}{1-\lambda} \qquad y = \frac{1}{1-\lambda} \qquad z = \frac{-1}{1-\lambda}$$

Using ⑤:

$$\frac{9}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} = 4 \implies \lambda = 1 \pm \frac{\sqrt{11}}{2}$$

$$11 = 4(1-\lambda)^2$$

Points: $\underbrace{\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{-2}{\sqrt{11}}\right)}_{\text{closest point, } d^2 \text{ smallest}}$ and $\underbrace{\left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)}_{\text{farthest point, } d^2 \text{ largest}}$

Two Constraints:

$f(x,y,z)$ subject to $g(x,y,z) = K$ and $h(x,y,z) = c$

$\nabla f \perp C$ at P so is ∇g and ∇h

$\Rightarrow \nabla f$ is in the plane determined by ∇g and ∇h

$$\boxed{\nabla f = \lambda \nabla g + \mu \nabla h}$$

Ex. 5 maximize $f(x,y,z) = x + 2x + 3z$ on the curve of intersection of $x - y + z = 1$ and $x^2+y^2=1$

① $1 = \lambda + 2x\mu \ \rightarrow\ -1 = x\mu$

② $2 = -\lambda + 2y\mu \ \rightarrow\ 5/2 = y\mu$

③ $3 = \lambda + 0\mu \ \rightarrow\ \lambda = 3$

④ $x - y + z = 1 \ \longrightarrow$

⑤ $x^2+y^2=1 \ \rightarrow\ \frac{1}{\mu^2} + \frac{25}{4}\cdot\frac{1}{\mu^2} = 1$

$\mu = \pm \sqrt{29}/2$

$x = \mp 2/\sqrt{29}$

$y = \pm 5/\sqrt{29}$

$z = 1 \pm 7/\sqrt{29}$

$f\left(\frac{\mp 2}{\sqrt{29}}, \frac{\pm 5}{\sqrt{29}}, 1 \pm \frac{7}{\sqrt{29}}\right)$

$= 3 \pm \sqrt{29}$

max is

$$\boxed{3 + \sqrt{29}}$$