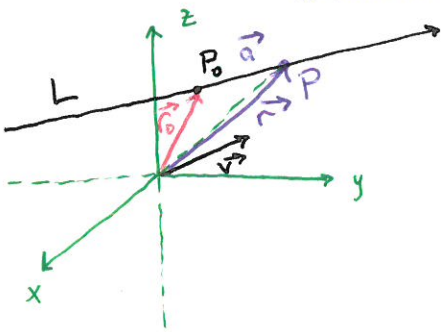
Section 12.5 - Equations of Lines and Planes

Line in 2D

Need point and slope
(direction)

Line in 3D

Need point and a vector
(direction)



Given P_0 and $\vec{v}$

Vector Equation: $\vec{r} = \vec{r_0} + \vec{a}$

$$\boxed{\vec{r} = \vec{r_0} + t\vec{v}}$$

As t varies the line L is traced out by the tip of $\vec{r}$.

Parametric Equations: $\vec{v} = \langle a, b, c\rangle$ $\quad r_0 = \langle x_0, y_0, z_0\rangle$

$$\boxed{x = x_0 + at \quad , \quad y = y_0 + bt \quad , \quad z = z_0 + ct}$$

Symmetric Equations: (Eliminate parameter t)

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

Example 2 (a) Find parametric equations of the line through $A(2,4,-3)$ and $B(3,-1,1)$.
(b) At what point does this line intersect the xy-plane?

(a)

$\vec{r_0} = \langle 2, 4, -3\rangle \qquad \vec{v} = \langle 3-2, -1-4, 1+3\rangle = \langle 1, -5, 4\rangle$

$$\boxed{x = 2 + t \qquad y = 4 - 5t \qquad z = -3 + 4t}$$

(b) xy-plane $z = 0$ so $t = \frac{3}{4}$ hence $\boxed{x = \frac{11}{4}, \ y = \frac{1}{4}, \ z = 0}$

★ **Line Segment:** From $\vec{r_0}$ to $\vec{r_1}$ is given by

$$\vec{r}(t) = (\vec{r_0}) + t(\vec{r_1} - \vec{r_0}) = (1-t)\vec{r_0} + t\vec{r_1} \qquad 0 \le t \le 1$$

Skew Lines: Do not intersect and are not parallel.

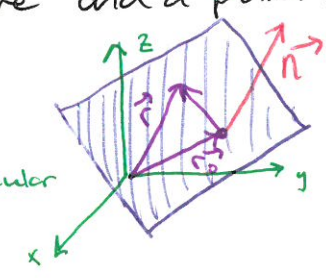
Plane: Direction need a vector perpendicular to the plane and a point.

$\vec{n}$ normal vector (perpendicular to plane)

$\vec{r_0}$ vector to initial point $\qquad \vec{r} - \vec{r_0}$ vector on plane

$\vec{r}$ vector to arbitrary point

Need $\boxed{\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0}$ to be perpendicular



Vector Equation of Plane: $\vec{n}\cdot(\vec{r}-\vec{r_0})=0$

Scalar Equation of Plane: $\vec{n}=\langle a,b,c\rangle$, $\vec{r}=\langle x,y,z\rangle$, $\vec{r_0}=\langle x_0,y_0,z_0\rangle$

$$\boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$$

Linear Equation: $d=-ax_0-by_0-cz_0$ $\boxed{ax+by+cz+d=0}$

Example 5 Find an equation of the plane that passes through the points $P(1,3,2)$, $Q(3,-1,6)$, and $R(5,2,0)$.

$\vec{PQ}=\vec{a}=\langle 2,-4,4\rangle$ $\vec{PR}=\vec{b}=\langle 4,-1,-2\rangle$

$$\vec{n}=\vec{a}\times\vec{b}=\begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix}=12\vec{i}+20\vec{j}+14\vec{k}$$

$$12(x-1)+20(y-3)+14(z-2)=0$$

$$\boxed{6x+10y+7z=50}$$

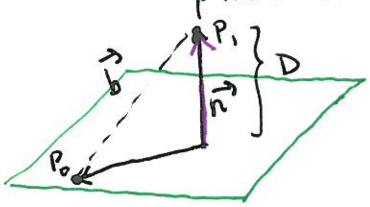
Example 7 Find the angle between the planes $x+y+z=1$ and $x-2y+3z=1$.

$\vec{n_1}=\langle 1,1,1\rangle$ $\vec{n_2}=\langle 1,-2,3\rangle$ $\cos\theta=\dfrac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}=\dfrac{2}{\sqrt{42}}$

$$\boxed{\theta\approx 72^\circ}$$

★ Note two planes are parallel if their normal vectors are parallel.

Example 8 Find a formula for the distance D from a point $P_1(x_1,y_1,z_1)$ to the plane $ax+by+cz+d=0$.



$P_0(x_0,y_0,z_0)$ any point on the plane

$\vec{b}=\langle x_1-x_0,\ y_1-y_0,\ z_1-z_0\rangle$

normal vector to plane $\langle a,b,c\rangle$

$D=|\text{Comp}_n b|$ Scalar projection of $\vec{b}$ onto $\vec{n}$

Since P_0 is in the plane $ax_0+by_0+cz_0=-d$

$$=\frac{|\vec{n}\cdot\vec{b}|}{|\vec{n}|}=\frac{|a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)|}{\sqrt{a^2+b^2+c^2}}$$

$$=\boxed{\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}}$$