Section 12.3 - The Dot Product

Possible to multiply 2 vectors? Useful quantity?

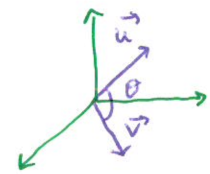
Dot Product: $\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\vec{b} = \langle b_1, b_2, b_3 \rangle$
(Scalar Product)

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad \text{(Scalar)}$$

★ Defined for V^n

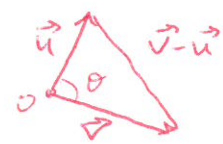
Properties of Dot Product:

1. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$ 2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ 3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
4. $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$ 5. $\vec{0} \cdot \vec{a} = 0$



$\boxed{\text{Theorem 3}}$ If θ is the angle between $\vec{v}$ and $\vec{u}$ then $\vec{v} \cdot \vec{u} = |\vec{v}| \cdot |\vec{u}| \cos\theta$

Proof apply Law of Cosines to triangle



★ If two vectors are parrallel then $\theta = 0$ or π

$\boxed{\text{Corollary 6}}$ $\cos\theta = \dfrac{\vec{v} \cdot \vec{u}}{|\vec{v}| \cdot |\vec{u}|}$

Orthogonal (Perpendicular) Vectors : if angle between is $\pi/2$

$\boxed{\text{Corollary 7}}$ $\vec{v}$ and $\vec{u}$ are orthogonal if and only if $\vec{v} \cdot \vec{u} = 0$.

$\boxed{\text{Example 3}}$ Find the angle between $\vec{a} = \langle 2, 2, -1 \rangle$ and $\vec{b} = \langle 5, -3, 2 \rangle$.

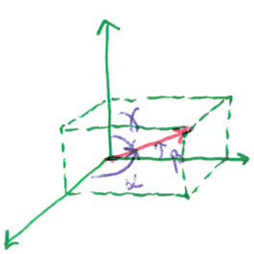
$|\vec{a}| = \sqrt{4+4+1} = 3$ $\quad |\vec{b}| = \sqrt{25+9+4} = \sqrt{38}$

$$\vec{a} \cdot \vec{b} = 10 + (-6) + (-2) = 2$$

$$\cos\theta = \frac{2}{3\sqrt{38}} \quad \text{hence} \quad \theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \approx \boxed{1.46 \text{ or } 84^\circ}$$

Direction Angles

The angles $\vec{a}$ makes with the positive x, y, z axes



$$\boxed{\cos\alpha = \frac{\vec{a} \cdot \vec{i}}{|\vec{a}| \cdot |\vec{i}|} = \frac{a_1}{|\vec{a}|} \quad \cos\beta = \frac{\vec{a} \cdot \vec{j}}{|\vec{a}| \cdot |\vec{j}|} = \frac{a_2}{|\vec{a}|} \quad \cos\gamma = \frac{\vec{a} \cdot \vec{k}}{|\vec{a}| \cdot |\vec{k}|} = \frac{a_3}{|\vec{a}|}}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a_1^2 + a_2^2 + a_3^2}{|\vec{a}|^2} = 1$$

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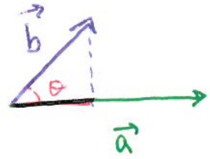
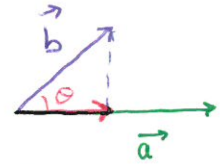
Vector Calc

Projections



Scalar projection of $\vec{b}$ onto $\vec{a}$



Vector projection of $\vec{b}$ onto $\vec{a}$

$$\text{Comp}_{\vec{a}} \vec{b} = |\vec{b}| \cos\theta = |\vec{b}| \cdot \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$$

$$\boxed{\text{Comp}_a b = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}}$$

$$\text{Proj}_{\vec{a}} \vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$\boxed{\text{Proj}_{\vec{a}} \vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2} \cdot a}$$

Example 6 Find the scalar and vector projections of $\vec{u} = \langle 1, 1, 2 \rangle$ onto $\vec{v} = \langle -2, 3, 1 \rangle$.

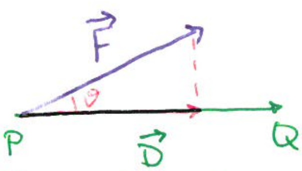
$$\text{Comp}_{\vec{v}} \vec{u} = \frac{\vec{v}\cdot\vec{u}}{|\vec{v}|} = \frac{-2+3+2}{\sqrt{14}} = \boxed{\frac{3}{\sqrt{14}}}$$ length of projection

$$\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{v}\cdot\vec{u}}{|\vec{v}|^2} \cdot \vec{v} = \frac{3}{14} \vec{v} = \boxed{\left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle}$$ Vector of projection

Applications

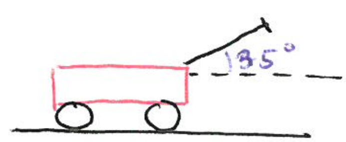
$\boxed{W = fd}$

Constant force vector $\vec{F}$ not in direction of $\vec{D}$



Work: $W = (|\vec{F}| \cos\theta)\,|\vec{D}| = \vec{F}\cdot\vec{D}$ $\boxed{\text{by Theorem 3}}$

Example 7 A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 35° above the horizon.



$$W = \vec{F}\cdot\vec{D} = |\vec{F}|\cdot|\vec{D}| \cos 35^\circ$$
$$= 70 \cdot 100 \cos 35^\circ$$
$$\approx \boxed{5734 \text{ Joules}}$$