Section 12.4 - The Cross Product

Cross Product: $\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\quad \vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \times \vec{b} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle$$

Recall Determinant of a 3x3 matrix:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Example 1 $\vec{a} = \langle 1, 3, 4 \rangle$ $\quad \vec{b} = \langle 2, 7, -5 \rangle$ Find $\vec{a} \times \vec{b}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \vec{k}$$

$$= (-15 - 28)\vec{i} - (-5 - 8)\vec{j} + (7 - 6)\vec{k}$$

$$= \boxed{-43\vec{i} + 13\vec{j} + \vec{k}}$$

Theorem 8 The vector $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

Proof: $(\vec{a} \times \vec{b}) \cdot \vec{a} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle \cdot \langle a_1, a_2, a_3 \rangle$

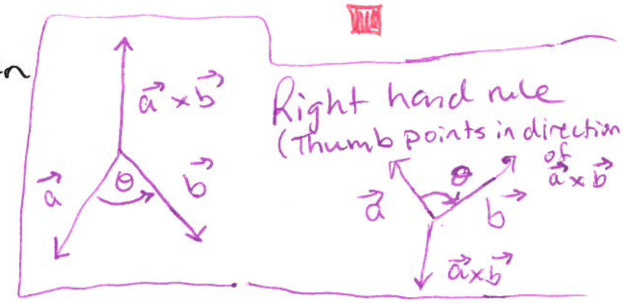
$$= a_1\cancel{a_2}b_3 - \cancel{a_1}a_3b_2 + \cancel{a_2}a_3b_1 - \cancel{a_1}a_2b_3 + a_1\cancel{a_3}b_2 - a_2\cancel{a_3}b_1$$

$$= 0$$

Theorem 9 If θ is the angle between $\vec{a}$ and $\vec{b}$ then

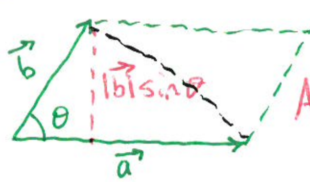
$$|a \times b| = |a|\,|b| \sin\theta$$

Proof see Book



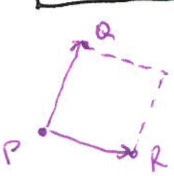
Corollary 10 Two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $a \times b = 0$.

★ The length of the cross product $\vec{a} \times \vec{b}$ is equal to the area of the Parallelogram determined by $\vec{a}$ and $\vec{b}$.



Area $= |\vec{b}| \cdot |\vec{a}| \sin\theta$
$= |\vec{a} \times \vec{b}|$

Example 4 Find the area of the triangle with vertices $P(1,4,6)$, $Q(-2,5,-1)$ and $R(1,-1,1)$



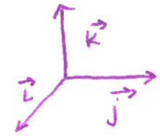
$\vec{PQ} = \langle -3, 1, -7 \rangle$
$\vec{PR} = \langle 0, -5, -5 \rangle$

$$\frac{1}{2} |\vec{PQ} \times \vec{PR}| = \frac{1}{2} |\langle -40, -15, 15 \rangle| = \boxed{\frac{5}{2}\sqrt{82}}$$

Section 12.4 - The Cross Product

$\vec{i} \times \vec{j} = \vec{k}$ $\qquad$ $\vec{j} \times \vec{k} = \vec{i}$ $\qquad$ $\vec{k} \times \vec{i} = \vec{j}$

$\vec{j} \times \vec{i} = -\vec{k}$ $\qquad$ $\vec{k} \times \vec{j} = -\vec{i}$ $\qquad$ $\vec{i} \times \vec{k} = -\vec{j}$



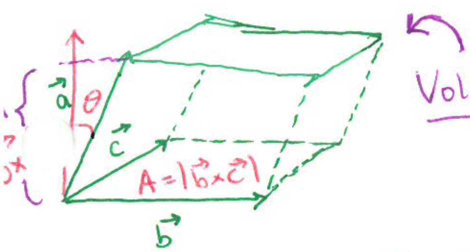
★ Cross Product is NOT commutative and NOT associative

$$\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a} \qquad (\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$$

Theorem 11

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
6. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b}) \cdot \vec{c}$

Scalar Triple Product: $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$



Volume of a Parallelepiped

$$\boxed{V = |\vec{a} \cdot (\vec{b} \times \vec{c})|}$$

$$V = A \cdot h = |\vec{b} \times \vec{c}| \cdot |a| \cos\theta = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

Example 5 Use the scalar triple product to show $a = \langle 1, 4, 7 \rangle$, $b = \langle 2, -1, 4 \rangle$, $c = \langle 0, -9, 18 \rangle$ are coplanar. (All in the same plane, i.e. parallelepiped has volume 0)

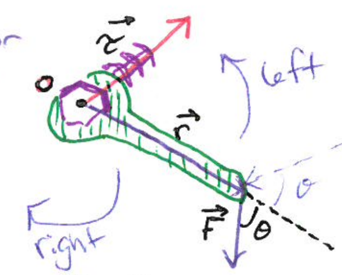
$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = 1(18) - 4(36) - 7(-18) = \boxed{0}$$

Application

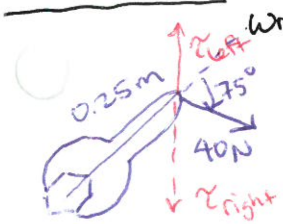
Force acting on a rigid body given by a position vector $\vec{r}$ $\vec{F}$

The torque $\vec{\tau}$ (relative to the origin) is $\vec{\tau} = \vec{r} \times \vec{F}$

★ Measures the tendency of the body to rotate about the origin.



Example 6 A bolt is tightened by applying a 40-N force to a 0.25m wrench. Find the magnitude of the torque about the center of the bolt.



$$|\tau| = |\vec{r} \times \vec{F}| = |\vec{r}||\vec{F}| \sin\theta$$
$$= (0.25)(40) \sin 75° \approx 9.66 \text{ Nm}$$