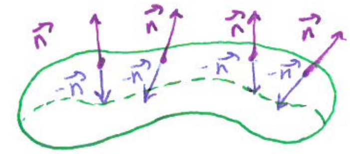
Section 16.7 - Surface Integrals

★ Make möbius strip, two colors one each side

★ Problem: Möbius Strip only has one side
Neither a top nor bottom - nonorientable



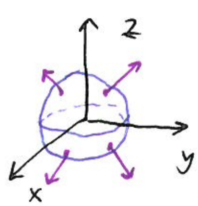
To define surface integrals of vector fields need orientable surfaces (two sided)

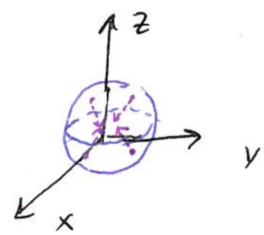
Definitions - S is orientable if there is a unit normal vector $\vec{n}$ at every point so that $\vec{n}$ varies continuously over S. The choice of $\vec{n}$ or $-\vec{n}$ provides S with an orientation.

★ Choose $\vec{n}$ with $\vec{k}$ component positive to denote positive orientation.

Unit normal vector: $\vec{n} = (1 + g_x^2 + g_y^2)^{-1/2} \langle -g_x, -g_y, 1 \rangle$

$z = g(x,y)$ (upward normal vector - natural orientation)



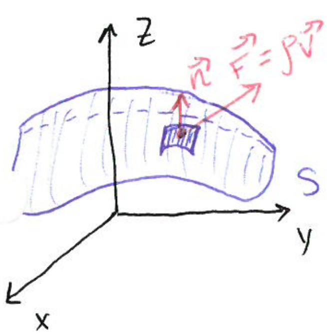


For a closed surface (boundary of a solid region)
Positive orientation point outwards, negative orientation points inwards.

Surface Integrals of Vector Fields

fluid with density ρ $\left(\frac{\text{mass}}{\text{distance}^3}\right)$ and velocity field $\vec{V}$ $\left(\text{distance}/\text{time}\right)$ flowing through S

rate of flow (mass per time per area) $\rho \vec{V} = \vec{F}$



mass per time crossing S in direction of $\vec{n}$

$(\rho\vec{v} \cdot \vec{n})\,\Delta(S)$

$\vec{F}$ continuous, defined on an oriented surface S, unit normal $\vec{n}$ then

Surface integral of $\vec{F}$ over S: $\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS}$
(called flux of F accross S)

Section 16.7 - Surface Integrals

Vector Calc

Note: $\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot \frac{\vec{r_u}\times\vec{r_v}}{|\vec{r_u}\times\vec{r_v}|}\,dS \overset{\text{Definition of surface integral}}{=} \iint_D \vec{F}(r(u,v))\cdot \frac{\vec{r_u}\times\vec{r_v}}{|\vec{r_u}\times\vec{r_v}|}\,|\vec{r_u}\times\vec{r_v}|\,dA$

$$\boxed{= \iint_D \vec{F}\cdot(\vec{r_u}\times r_v)\,dA}$$

Ex. 4 Find the flux of the vector field $\vec{F} = \langle z, y, x\rangle$ across the sphere

$S:\ x^2+y^2+z^2=1$

$\vec{r}(\varphi,\theta) = \langle \sin\varphi\cos\theta,\ \sin\varphi \sin\theta,\ \cos\varphi\rangle \qquad 0\le\varphi\le\pi \qquad 0\le\theta\le 2\pi$

$\vec{r_\varphi}\times\vec{r_\theta} = \langle \sin^2\varphi\cos\theta,\ \sin^2\varphi\sin\theta,\ \sin\varphi\cos\varphi\rangle$

$\iint_S \vec{F}\cdot d\vec{s} = \int_0^{\pi}\int_0^{2\pi} \langle \cos\varphi,\ \sin\varphi \sin\theta,\ \sin\varphi\cos\theta\rangle\cdot\langle \sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\varphi\rangle\, d\theta d\varphi$

$= \int_0^{\pi}\int_0^{2\pi} \cancelto{0}{\cos\varphi \sin^2\varphi\cos\theta} + \sin^3\varphi\sin^2\theta + \cancelto{0}{\sin^2\varphi\cos\varphi\cos\theta}\, d\theta\, d\varphi$

$= \int_0^{\pi}\sin^3\varphi\,d\varphi \int_0^{2\pi}\sin^2\theta\,d\theta = \boxed{\frac{4\pi}{3}}$ from yesterday's work

$= \int_0^{\pi}\sin\varphi - \sin\varphi\cos^2\varphi\,d\varphi \cdot \int_0^{2\pi} \frac{1-\cos 2\theta}{2}\,d\theta$

$= \left(-\cos\varphi + \frac{\cos^3\varphi}{3}\Big|_0^{\pi}\right)\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\Big|_0^{2\pi}\right)$

$= \left(1+\frac{1}{3}\right)\pi = \boxed{\frac{4\pi}{3}}$

Note: $F\cdot(\vec{r_u}\times\vec{r_v}) = \langle P, Q, R\rangle\cdot\langle g_x, g_y, 1\rangle = -Pg_x - Qg_y + R$

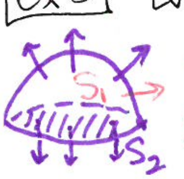
If S given by $z = g(x,y)$ then $\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_D (-Pg_x - Qg_y + R)\,dA}$

S_2

$P=y \quad Q=x \quad z=0$

$g_x=0 \quad g_y=0$

Ex 5 Evaluate $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = \langle y, x, z\rangle$ $S:\ z = 1-x^2-y^2$ and $z=0$



$P=y \quad Q=x \quad R=1-x^2-y^2$

$g_x = -2x \quad g_y = -2y$

$\iint_S \vec{F}\cdot d\vec{S} = \iint_D 2xy + 2xy + 1 - x^2 - y^2\,dA$

$= \int_0^{2\pi}\int_0^1 (4r^3\cos\theta\sin\theta + r - r^3)\,dr\,d\theta = \boxed{\frac{\pi}{2}}$