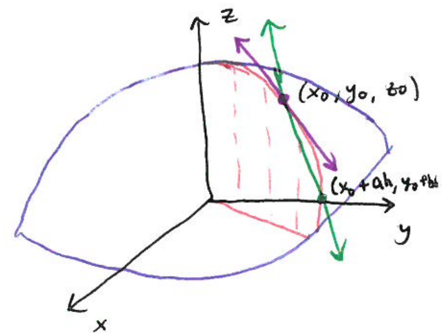
★ **Directional Derivative** - slope of the tangent line in the direction of $\vec{u} = \langle a, b \rangle$



For a function f at (x_0, y_0) in the direction of a unit vector $\vec{u} = \langle a, b \rangle$ is

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if the limit exists.

Theorem 3 If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\vec{u} = \langle a, b \rangle$

$$\boxed{D_{\vec{u}} f(x,y) = a f_x(x,y) + b f_y(x,y)}$$

Proof: Let $g(h) = f(x + ha, y + hb)$ then $g'(0) = D_{\vec{u}} f(x, y)$

and $g'(h) = \frac{\partial f}{\partial x} \frac{dx}{dh} + \frac{\partial g}{\partial y} \cdot \frac{dy}{dh} = a f_x(x,y) + b f_y(x,y) = g'(0)$ ■

Ex 2 Find the directional derivative if $f(x,y) = x^3 - 3xy + 4y^2$ and $\vec{u}$ is the unit vector given by $\theta = \pi/6$. Find $D_{\vec{u}} f(1, 2)$.

$\vec{u} = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$ $\quad D_{\vec{u}} f(x,y) = \boxed{\frac{\sqrt{3}}{2}(3x^2 - 3y) + \frac{1}{2}(-3x + 8y)}$

$$D_{\vec{u}} f(1, 2) = \boxed{\frac{-3\sqrt{3}}{2} + \frac{13}{2}}$$

Note: $D_{\vec{u}} f(x,y) = \langle f_x, f_y \rangle \cdot \vec{u} = \boxed{\nabla f \cdot \vec{u}}$ Given on Tests/Quizzes

Definition - If f is a function of two variables x and y then the gradient of f is a vector function ∇f given by

$$\boxed{\nabla f = \langle f_x, f_y \rangle}$$

Directional Derivative and Gradient extended to 3 or more variables:

$\nabla f(x_1, x_2, \ldots, x_n) = \langle f_{x_1}, f_{x_2}, \ldots, f_{x_n} \rangle$ $\quad D_{\vec{u}} f(x_1, \ldots, x_n) = \nabla f \cdot \vec{u}$

Ex $f(x,y,z) = y \ln(x^2 + z)$ find ∇f and $D_{\vec{u}} f$ in the direction of $\vec{v} = \langle 1, -1, 1 \rangle$ at $(0, 5, 1)$

$\nabla f = \langle \frac{2x}{x^2+z}, \ln(x^2+z), \frac{1}{x^2+z} \rangle$ $\quad \vec{u} = \frac{\sqrt{3}}{3} \vec{v}$ $\quad \nabla f(0,5,1) = \langle 0, 0, 1 \rangle$

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} \Big|_{(0,5,1)} = \boxed{\sqrt{3}/3}$$

Maximizing the Directional Derivative - Direction of max rate of change

Theorem 15 f differentiable function, then the maximum value of the directional derivative $D_{\vec{u}} f(\vec{x})$ is $|\nabla f(\vec{x})|$ and it occurs when $\vec{u}$ is in the direction of $\nabla f(\vec{x})$.

Proof: $D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f| |\vec{u}| \cos\theta = |\nabla f| \cos\theta$

maximized when $\theta = 0°$

Ex 7 Suppose that the temperature at a point (x, y, z) in space is given by

$$T(x,y,z) = \frac{80}{1+x^2+2y^2+3z^2} \quad \text{in } °C$$

and x, y, z in meters. In which direction does the temp increase fastest at $(1,1,-2)$ and what is the max rate of increase?

$$\nabla T = \left\langle \frac{-160x}{(1+x^2+2y^2+3z^2)^2}, \frac{-320y}{(1+x^2+2y^2+3z^2)^2}, \frac{-480z}{(1+x^2+2y^2+3z^2)^2} \right\rangle$$

$$\nabla T(1,1,-2) = \frac{-160}{16^2}\langle 1, 2, -6\rangle = \boxed{-\frac{5}{8}\langle 1,2,-6\rangle \text{ Direction of fastest increase}}$$

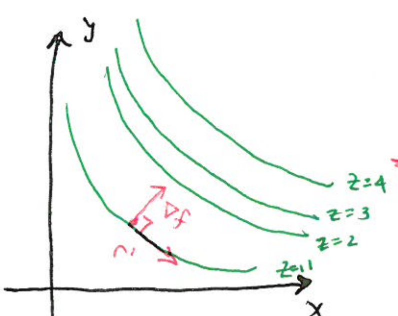
$$|\nabla T(1,1,-2)| = \boxed{\frac{5}{8}\sqrt{41} \ °C/m \quad \text{max rate of increase}}$$

$$\approx 4 \ °C/m$$

Tangent Plane to a level Surface:

$F(x,y,z) = K$ C Curve on S

$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$



$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial F}{\partial z}\cdot\frac{dz}{dt} = 0 \Rightarrow \boxed{\nabla F \cdot r'(t) = 0}$$

★ Thus the gradient vector is perpendicular to the tangent vector

Direction of Normal Line = gradient vector

Tangent plane to $F(x,y,z) = k$ at (x_0, y_0, z_0) is

$$\boxed{F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0}$$

Ex. 8 Find the equations of the tangent plane and normal line at $(-2,1,-3)$ to $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$.

$F_x(-2,1,-3) = -1 \quad F_y(-2,1,-3) = 2 \quad F_z(-2,1,-3) = -\frac{2}{3}$

tangent plane: $\boxed{-1(x+2) + 2(y-1) - \frac{2}{3}(z+3) = 0}$

Normal line: $\boxed{x = -2 - t \quad y = 1 + 2t \quad z = -3 - \frac{2}{3}t}$