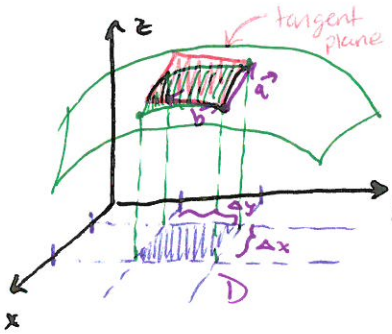
Section 15.6 - Surface Area

Vector Calc

Let S be a surface with equation $z = f(x,y)$ continuous with partial derivatives.



Idea: for a small rectangle in D, compute area of tangent plane over the rectangle to surface, Add of the areas to approx. surface area

$$\vec{a} = \Delta x\,\vec{i} + 0\vec{j} + f_x(x,y)\,\Delta x\,\vec{k}$$

$$\vec{b} = 0\hat{i} + \Delta y\,\hat{j} + f_y(x,y)\,\Delta y\,\vec{k}$$

Area of tangent rectangle $= |\vec{a}\times\vec{b}|$

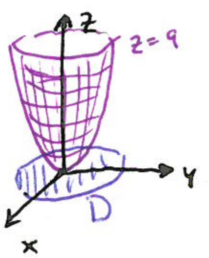
$$= \left|-f_x(x,y)\,\Delta x\Delta y\,\vec{i} + f_y(x,y)\,\Delta x\Delta y\,\vec{j} + \Delta x\Delta y\,\vec{k}\right|$$

$$= \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}\; dA$$

Summing up all tangent rectangles as $\Delta A \to 0$ gives

$$\boxed{S = \iint_D \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}\; dA}$$

Ex 1 Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.



$$D = \{(r,\theta) \mid 0 \le r \le 3,\ 0 \le \theta \le 2\pi\}$$

$$S = \int_0^{2\pi}\int_0^3 \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\; dA$$

$$\frac{\partial z}{\partial x} = 2x = 2r\cos\theta$$

$$\frac{\partial z}{\partial y} = 2y = 2r\sin\theta$$

$$= \int_0^{2\pi}\int_0^3 \sqrt{1 + 4r^2}\cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^3 \sqrt{1+4r^2}\, r\,dr = (2\pi)\left[(1+4r^2)^{3/2}\cdot\frac{2}{3}\cdot\frac{1}{8}\right]\Big|_0^3$$

$$= \frac{2\pi}{12}\left(37^{3/2} - 1\right) = \boxed{\frac{\pi}{6}\left(37^{3/2} - 1\right)}$$