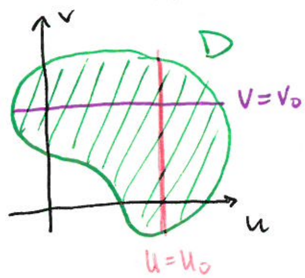
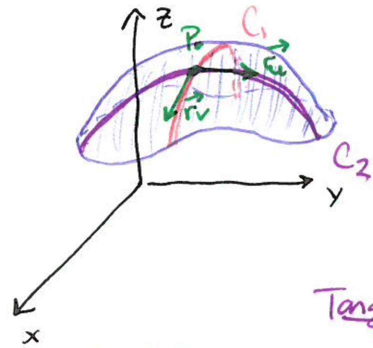
Section 16.6 - Parametric Surfaces and their Areas

Vector Calc

Tangent Planes:

$S: \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$





Tangent plane contains $\vec{r_u}$ and $\vec{r_v}$

$\vec{r_u} = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\rangle \qquad \vec{r_v} = \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\rangle$

normal to tangent plane $= \boxed{\vec{r_u} \times \vec{r_v}}$ if $\vec{r_u} \times \vec{r_v} \neq 0$ then we say S is Smooth

Ex 9 Find the tangent plane to the surface with parametric equations

$$x = u^2, \quad y = v^2, \quad z = u + 2v \quad \text{at } (1, 1, 3)$$

$\vec{r_u} = \langle 2u, 0, 1\rangle \qquad \vec{r_v} = \langle 0, 2v, 2\rangle$

$x = 1 \quad y = 1 \quad z = 3$

$\Rightarrow u = 1 \quad v = 1$

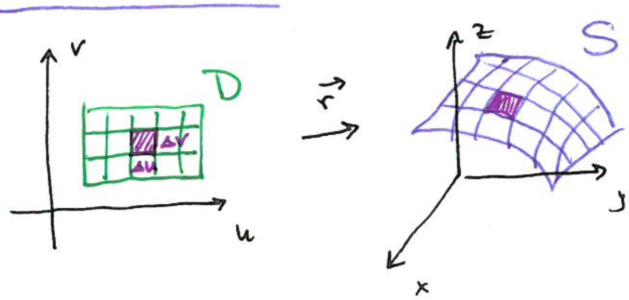
$\vec{n} = \vec{r_u} \times \vec{r_v} = \langle -2v, -4u, 4uv\rangle$

$\vec{n}(1,1) = \langle -2, -4, 4\rangle$

tangent plane: $0 = \langle x-1, y-1, z-3\rangle \langle -2, -4, 4\rangle$

$$\boxed{0 = -2(x-1) - 4(y-1) + 4(z-3)}$$

Surface Area:



Area of rectangle:

Area tangent plane Scaled by $\Delta u \cdot \Delta v$

$= |\vec{r_u} \times \vec{r_v}| \Delta u \Delta v$

S smooth given by $\vec{r}(u,v) = \langle x, y, z\rangle$ $(u,v) \in D$ covered only once then the surface area of S is

$$\boxed{A(S) = \iint_D |\vec{r_u} \times \vec{r_v}| \, dA}$$

Ex 10 Find the surface area of a sphere of radius a.

Parametrization: $x = a\sin\varphi\cos\theta$ $\quad y = a\sin\varphi\sin\theta$ $\quad z = a\cos\varphi$

$0 \le \varphi \le \pi$ $\qquad 0 \le \theta \le 2\pi$

$$\vec{r}_\varphi = \langle a\cos\varphi\cos\theta,\ a\cos\varphi\sin\theta,\ -a\sin\varphi\rangle$$

$$\vec{r}_\theta = \langle -a\sin\varphi\sin\theta,\ a\sin\varphi\cos\theta,\ 0\rangle$$

$$|\vec{r}_\varphi \times \vec{r}_\theta| = |\langle a^2\sin^2\varphi\cos\theta,\ a^2\sin^2\varphi\sin\theta,\ a^2\sin\varphi\cos\varphi\rangle|$$

$$= a^2\sin\varphi$$

$$A(S) = \iint_D a^2\sin\varphi\, dA = \int_0^{2\pi}\int_0^{\pi} a^2\sin\varphi\, d\varphi\, d\theta = \boxed{4\pi a^2}$$

Surface Area of the graph of a Function: $z = f(x,y)$

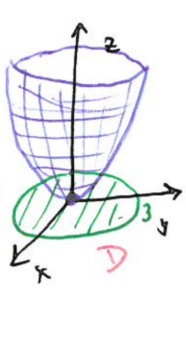
Parametrization: $x = x,\quad y = y \quad z = f(x,y)$

$$\vec{r}_x = \langle 1, 0, f_x\rangle \qquad \vec{r}_y = \langle 0, 1, f_y\rangle$$

$$|\vec{r}_x \times \vec{r}_y| = |\langle -f_x, -f_y, 1\rangle| = \sqrt{f_x^2 + f_y^2 + 1}$$

$$\boxed{A(S) = \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA}$$

Ex 11 Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.



$$A(S) = \iint_D \sqrt{4x^2 + 4y^2 + 1}\, dA = \int_0^{2\pi}\int_0^{3} \sqrt{4r^2+1}\, r\, dr\, d\theta$$

$$= \frac{2\pi}{8}\cdot\frac{2}{3}(1+4r^2)^{3/2}\Big|_0^3$$

$$= \boxed{\frac{\pi}{6}\left[(37)^{3/2} - 1\right]}$$