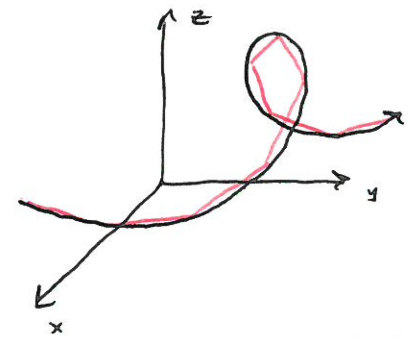
Section 13.3 - Arc Length and Curvature

Arc length of a curve with parametric equations (see 10.2)

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

So for vector functions:

$$\boxed{L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt = \int_a^b |\vec{r}\,'(t)|\, dt}$$



and the arc length function is:

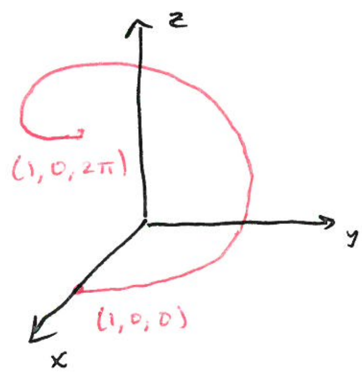
$$\boxed{s(t) = \int_a^t |\vec{r}\,'(u)|\, du}$$

★ Length of a space curve is the limit of lengths of inscribed polygons.

Example 1 Find the length of the arc of the circular helix with vector equation

$\vec{r}(t) = \cos t\, \vec{i} + \sin t\, \vec{j} + t\vec{k}$ from

$(1,0,0)$ to $(1,0,2\pi)$
$t=0$ $\quad$ $t=2\pi$



$\vec{r}\,'(t) = -\sin t\, \vec{i} + \cos t\, \vec{j} + \vec{k}$

$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$$L = \int_0^{2\pi} |\vec{r}\,'(t)|\, dt = \boxed{\sqrt{2}\, 2\pi}$$

Parametrize a curve with respect to arc length:

- arc length is natural from the shape of the curve
- doesn't depend on a particular coordinate system

Idea: $s(t)$ is a function of t, solve for $t(s)$, t a function of arc length s

Then $\vec{r}(t) = \vec{r}(t(s))$ so if $s = 3$ then

★ $\vec{r}(t(3))$ is the position vector of the point 3 units from start.

Example 2 Reparametrize the helix $\vec{r}(t) = \cos t\, \vec{i} + \sin t\, \vec{j} + t\vec{k}$ with respect to arc length measured from $(1,0,0)$ in the direction of increasing t.

$s(t) = \int_0^t |\vec{r}\,'(t)|\, dt = \int_0^t \sqrt{2}\, dt = \sqrt{2}\, t$ thus $t = s/\sqrt{2}$

$$\boxed{r(t(s)) = \cos\left(\frac{s}{\sqrt{2}}\right)\vec{i} + \sin\left(\frac{s}{\sqrt{2}}\right)\vec{j} + \frac{s}{\sqrt{2}}\vec{k}}$$

Section 13.3 - Arc Length and Curvature Vector Calc

- A parameterization $\vec{r}(t)$ is <u>smooth</u> on I if $\vec{r}'(t)$ is continuous on I and $\vec{r}'(t) \neq 0$ on I.
- A curve is <u>smooth</u> if it has a smooth parameterization.

★ Curve has no sharp corners or cusps.

- <u>Curvature</u> of a curve is the measure of how quickly the curve is changing direction at a point.

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

Curvature is $\kappa = \left|\frac{d\vec{T}}{ds}\right|$ where $\vec{T}$ is the unit tangent vector.

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\cdot\frac{ds}{dt} \Rightarrow \boxed{\kappa = \left|\frac{d\vec{T}/dt}{ds/dt}\right|} \text{ and } \frac{ds}{dt} = |\vec{r}'(t)|$$

Example 3 Show that the curvature of a circle of radius a is $1/a$.

$$\vec{r}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}$$

$$\vec{r}'(t) = -a\sin t\,\vec{i} + a\cos t\,\vec{j} \qquad |\vec{r}'(t)| = a$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{a} = -\sin t\,\vec{i} + \cos t\,\vec{j} \qquad |\vec{T}'(t)| = 1$$

So $\kappa = \frac{|d\vec{T}/dt|}{|r'(t)|} = \frac{1}{a}$ ✓

Theorem 10

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

Proof: $\vec{T} = \frac{\vec{r}'}{|\vec{r}'|}$ $\quad |\vec{r}'| = ds/dt$ so $\quad \vec{r}' = |\vec{r}'|\vec{T} = \frac{ds}{dt}\vec{T}$

So $r'' = \frac{d^2s}{dt^2}T + \frac{ds}{dt}T'$ $\qquad T \times T = 0$ we get

$$r' \times r'' = \left(\frac{ds}{dt}\right)^2 (T \times T') \qquad |r' \times r''| = \left(\frac{ds}{dt}\right)^2 \cancel{|T|}^{1}\,|T'|$$

$$\kappa = \frac{|T'|}{|r'|} = \frac{|r' \times r''| / \left(\frac{ds}{dt}\right)^2}{|r'|} = \frac{|r' \times r''|}{|r'|^3} \quad ■$$

Example 4 Find the curvature of $\vec{r}(t) = \langle t, t^2, t^3\rangle$ at $(0,0,0)$.

$r'(t) = \langle 1, 2t, 3t^2\rangle$ $\quad r'(0) = \langle 1, 0, 0\rangle$

$r''(t) = \langle 1, 2, 6t\rangle$ $\quad r''(0) = \langle 1, 2, 0\rangle$

$$\kappa(0) = \frac{|r'(0) \times r''(0)|}{|r'(0)|^3} = \frac{|\langle 0, 0, 2\rangle|}{1} = \boxed{2}$$

Special case : Plane curve $y = f(x)$

$\vec{r}(x) = x\vec{i} + f(x)\vec{j}$ $|r'(x)| = \sqrt{1 + f'^2(x)}$

$|r'(x) \times r''(x)| = |f''(x)|$ $\boxed{K(x) = \dfrac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}}$

Normal and Binormal Vectors :

Since $|\vec{T}| = 1$ we have that $\vec{T}'$ is orthogonal to $\vec{T}$

Unit normal Vector $\boxed{\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|}}$

Binormal vector $\boxed{\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)}$ $\perp$ to both $\vec{T}$ and $\vec{N}$.

Normal plane of a Curve C at a point P - is the plane containing $\vec{N}$ and $\vec{B}$ and P.

★ Normal vector is $r'(t)$ at P

Example 7 Find the equation of the normal plane to

$\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}$ at $P(0, 1, \pi/2)$

And the normal and binormal vectors.

$\vec{T}(t) = \dfrac{r'(t)}{|r'(t)|} = \dfrac{\langle -\sin t, \cos t, 1\rangle}{\sqrt{2}}$ $\vec{N}(t) = \dfrac{T'(t)}{|T'(t)|} = \boxed{\dfrac{\langle -\cos t, -\sin t, 0\rangle \cancel{/\sqrt{2}}}{\cancel{1/\sqrt{2}}}}$

$\vec{B}(t) = \vec{T} \times \vec{N} = \boxed{\frac{1}{\sqrt{2}}\langle \sin t, -\cos t, 1\rangle}$

$r'(\pi/2) = \langle -1, 0, 1\rangle$ So the normal plane is

$-x + z + d = 0$ $\Rightarrow$ $\boxed{-x + z - \frac{\pi}{2}}$