FTC: $\int_a^b F'(x) = F(b) - F(a)$ where F' continuous on $[a,b]$
$F(x) = \int_a^x F'(x)\,dx$ with $\frac{dF}{dx} = F'(x)$.

Can think of ∇f as a kind of derivative

FTC

(FTC I) **Theorem 2** C smooth curve given by $\vec{r}(t)$, $a \le t \le b$, f differentiable, ∇f continuous on C then $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

Proof: $\int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f'(r(t)) \cdot \vec{r}\,'(t)\,dt$

$= \int_a^b \left(\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t} \right) dt$

$= \int_a^b \frac{d}{dt}\left(f(r(t))\right) dt = f(r(b)) - f(r(a))$

★ Theorem 2 also holds for piecewise smooth curves

Definition - A vector field $\vec{F}$ is called a conservative vector field if it is the gradient field of some function f. If f exists such that $\nabla f = \vec{F}$ then we say f is a potential function for $\vec{F}$.

★ recall that in general $\int_{C_1} \vec{F} \cdot d\vec{r} \ne \int_{C_2} \vec{F} \cdot d\vec{r}$ even if C_1 and C_2 have the same initial and terminal points.

However, by theorem 2 $\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$ when ∇f is continuous

Thus the line integral of a conservative vector field only depends on the initial and terminal points or independent of path.

Definition - $\vec{F}$ a continuous vector field with domain D,
$\int_C \vec{F} \cdot d\vec{r}$ is independent of path if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for any two paths C_1, C_2 in D having the same initial and terminal points.

Section 16.3 - Fundamental Theorem for Line Integrals

Vector Calc

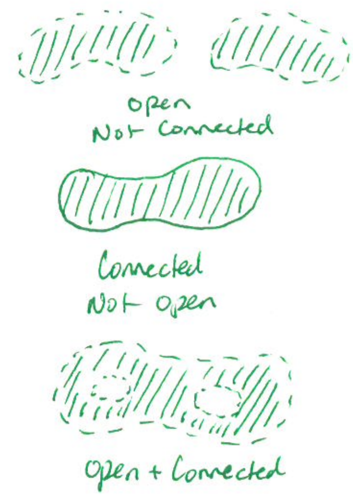
Definition - a curve C is called closed if its terminal point coincides with its initial point, $\vec{r}(b) = \vec{r}(a)$.

★ If $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D and C is a closed curve in D then $\int_C \vec{F}\cdot d\vec{r} = 0$.

$\boxed{\text{Theorem 3}}$ $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D iff $\int_C \vec{F}\cdot d\vec{r} = 0$ for every closed path C in D.

$\Rightarrow$ Only vector fields that are independent of path are conservative.

Definition - D is open if for all points P in D there is a disk with center P that lies completely in D. D is connected if for every two points in D can be joined by a path that lies in D.



General FTC II

$\boxed{\text{Theorem 4}}$ $\vec{F}$ continuous on open connected region D

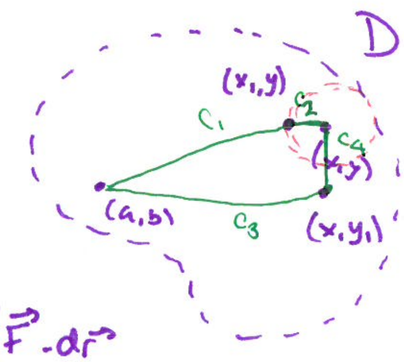
FTC II

If $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D, then $\vec{F}$ is a conservative vector field on D. That is there exists a function f such that $\nabla f = \vec{F}$.

Proof: Assume $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D.

Show: there is f such that $\nabla f = \vec{F}$.

Let $f(x,y) = \int_{(a,b)}^{(x,y)} \vec{F}\cdot d\vec{r}$ with (a,b) in D.



Since D is open there is an open disk around (x,y)

Choose point (x_1,y) and (x,y_1) in the disk.

$$f(x,y) = \int_{(a,b)}^{(x_1,y)} \vec{F}\cdot d\vec{r} + \int_{(x_1,y)}^{(x,y)} \vec{F}\cdot d\vec{r} \Rightarrow \frac{\partial}{\partial x} f(x,y) = 0 + \frac{\partial}{\partial x}\int_{C_2} \vec{F}\cdot d\vec{r}$$

$\vec{F} = \langle P, Q\rangle$ then $\quad \frac{\partial}{\partial x} f(x,y) = P(x,y)$ since $\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_2} P\,dx + Q\,dy$

Similarly $f(x,y) = \int_{(a,b)}^{(x,y_1)} \vec{F}\cdot d\vec{r} + \int_{(x,y_1)}^{(x,y)} \vec{F}\cdot d\vec{r} \Rightarrow \frac{\partial}{\partial y} f(x,y) = 0 + \frac{\partial}{\partial x}\int_{C_4} \vec{F}\cdot d\vec{r} = Q(x,y)$

Thus $\vec{F} = \langle P, Q\rangle = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle$ ■