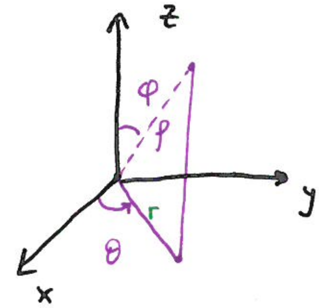
Section 15.9 - Spherical Coordinates

Spherical Coordinates - useful for triple integrals over regions involving spheres or cones

(ρ, θ, φ)

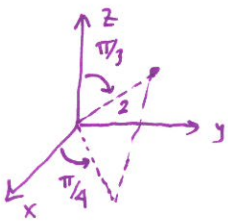


$\rho \geq 0$ like radius and $0 \leq \varphi \leq \pi$

$$\boxed{\begin{array}{ll} z = \rho\cos\varphi & r = \rho\sin\varphi \\ x = \rho\sin\varphi\cos\theta & y = \rho\sin\varphi\sin\theta \end{array}}$$

$$\boxed{\rho^2 = x^2 + y^2 + z^2}$$

Ex 1 plot $(2, \pi/4, \pi/3)$ and find rectangular coords



$4 = x^2 + y^2 + z^2$

$x = 2\sin(\pi/3)\cos(\pi/4) = 2\left(\frac{\sqrt{3}}{2}\right)\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{2}$

$y = 2\sin(\pi/3)\sin(\pi/4) = \frac{\sqrt{6}}{2}$

$z = 2\cos(\pi/3) = 1$

$$\boxed{\left(\sqrt{6}/2, \sqrt{6}/2, 1\right)}$$

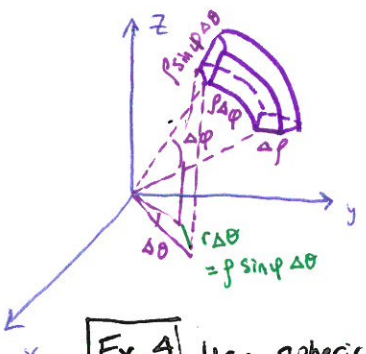
Ex 2 Convert $(0, 2\sqrt{3}, -2)$ to spherical coords.

$\rho^2 = 0^2 + (2\sqrt{3})^2 + (-2)^2 = 16 \Rightarrow \rho = 4$

Note: In Q4 below xy plane $\quad \cos\varphi = \frac{-2}{4} = -\frac{1}{2} \Rightarrow \varphi = \frac{2\pi}{3}$

$0 = 4\cdot\sin\varphi\cos\theta \Rightarrow \cos\theta = 0$ so $\theta = \pi/2$

$$\boxed{\left(4, \pi/2, \tfrac{2\pi}{3}\right)}$$

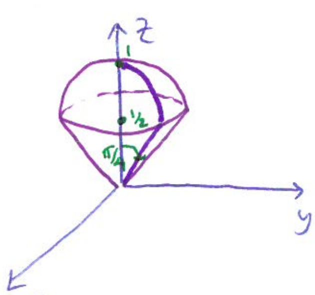


$dV = \rho\sin\varphi\cdot\rho\, d\rho\, d\theta\, d\varphi \qquad E = \{(\rho, \theta, \varphi) \mid a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \varphi \leq d\}$

$$\boxed{\iiint_E f\, dV = \int_c^d \int_\alpha^\beta \int_a^b f\cdot\rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi}$$

Ex. 4 Use spherical coords to find the volume of the solid above $z = \sqrt{x^2+y^2}$ and below $x^2+y^2+z^2 = z$.

$\rho^2 = \rho\cos\varphi \Rightarrow \rho = \cos\varphi \qquad z = \rho\sin\varphi$



$x^2 + y^2 + (z - \frac{1}{2})^2 = (1/2)^2, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \varphi \leq \pi/4, \quad 0 \leq \rho \leq \cos\varphi$

$$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\varphi} \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta = 2\pi\int_0^{\pi/4} \frac{\rho^3}{3}\sin\varphi\Big|_0^{\cos\varphi} d\varphi$$

$$= \frac{2\pi}{3}\int_0^{\pi/4}\cos^3\varphi\sin\varphi\, d\varphi = \frac{2\pi}{3}\left[-\frac{\cos^4\varphi}{4}\Big|_0^{\pi/4}\right] = \frac{2\pi}{3}\left[\frac{1}{4} - \frac{1}{16}\right] = \boxed{\frac{\pi}{8}}$$