★ Showing $\int_C \vec{F} \cdot d\vec{r}$ is path independent is a bit much to show $\vec{F}$ is conservative

<u>Goal</u>: want an easier condition

Suppose $\vec{F} = \langle P, Q \rangle$ is conservative
where P, Q have <u>continuous first order partial derivatives</u> (for Clairaut's)
Then there is a function f with $\nabla f = \vec{F}$,

$$P = \frac{\partial f}{\partial x} \quad \text{and} \quad Q = \frac{\partial f}{\partial y}$$

So by <u>Clairaut's Theorem</u>

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

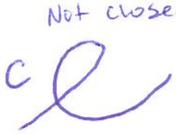
Theorem 5 If $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ is conservative where P, Q have continuous first-order partials then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

★ Want a converse of this! Get it but only for special types of regions.

<u>Definition</u> - A <u>simple curve</u> is a curve that doesn't intersect itself.
A <u>closed curve</u> is a curve with $\vec{r}(a) = \vec{r}(b)$.
A <u>simply-connected</u> region D is a connected region such that every simple closed curve in D encloses only points in D.

C

Simple
Not closed

C

Not simple
Not closed

C

Not simple
closed

C

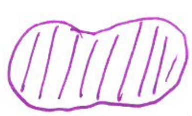
Simple + closed

D

Not Simply-Connected

D

Not Connected
$\Rightarrow$ Not Simply-Connected

D

Simply Connected.

Theorem 6 (Partial Converse of Theorem 5)

$\vec{F} = \langle P, Q \rangle$ a vector field on an open simply-connected region D, P, Q have continuous first partials with

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D$$

Then $\vec{F}$ is conservative.

Example 3 Determine if $\vec{F}(x,y) = \langle 3 + 2xy, x^2 - 3y^2 \rangle$ is conservative.

$P(x,y) = 3 + 2xy \quad Q(x,y) = x^2 - 3y^2$ on $D = \mathbb{R}^2$ open simply connected

$$\frac{\partial P}{\partial y} = 2x \; = \; \frac{\partial Q}{\partial x} = 2x \Rightarrow \vec{F} \text{ is conservative.}$$

Example 4 (a) $\vec{F}(x,y) = \langle 3+2xy,\ x^2-3y^2 \rangle$ find f so that $\nabla f = \vec{F}$

(b) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where C is given by

$$\vec{r}(t) = \langle e^t \sin t,\ e^t \cos t \rangle \quad 0 \le t \le \pi$$

(a) $f_x(x,y) = 3+2xy \qquad f = \int f_x(x,y)\,dx = 3x + x^2y + g(y)$

$f_y(x,y) = x^2 - 3y^2 \qquad f = \int f_y(x,y)\,dy = x^2y - y^3 + h(x)$

$g'(y) = -3y^2 \qquad h'(x) = 3$

$g(y) = y^3 + K \qquad h(x) = 3x + K$

$\boxed{f(x,y) = 3x + x^2y - y^3 + K}$ potential function for $\vec{F}$

(b) $\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$ by FTC

$= f(\vec{r}(\pi)) - f(\vec{r}(0))$

$= f(0, -e^{\pi}) - f(0,1)$

$= \boxed{e^{3\pi} + 1}$

Conservation of Energy:

$\vec{F}$ force fields moves object (of mass m) along $C : \vec{r}(t)$

Newton's Second Law: $\vec{F}(\vec{r}(t)) = m \cdot \vec{r}''(t)$

Work done $W = \int_C \vec{F} \cdot d\vec{r} = \int_a^b m\vec{r}''(t) \cdot r'(t)\,dt = \int_a^b m \frac{d}{dt} |r'(t)|^2 dt$

$= \frac{m}{2} |\vec{r}'(t)|^2 \Big|_a^b = \frac{m}{2}\left[|\vec{r}'(b)|^2 - |\vec{r}'(a)|^2 \right]$ = Change in Kinetic energy $K(B) - K(A)$

Note: $\frac{1}{2} m |\vec{r}'(t)|^2 = \frac{1}{2} m |v(t)|^2$ is Kinetic energy of the object

$\vec{F}$ conservative, $\vec{F} = \nabla f$ potential energy of an object at a point is defined by

$P(x,y,z) = -f(x,y,z) \Rightarrow \vec{F} = -\nabla P$

$W = \int_C \vec{F} \cdot dr = -\int_C \nabla P \cdot dr = P(A) - P(B)$

Thus $\boxed{P(A) + K(A) = P(B) + K(B)}$: sum of potential and kinetic energy remains constant.