Kepler's Planetary Laws Project

1. Find and write down Kepler's three laws of planetary motion.

   (a) A planet revolved around the sun in an elliptical orbit with the sun at one focus.

   (b) The line joining the sun to a planet sweeps out equal areas in equal times.

   (c) The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

2. In Isaac Newton's book, *Principia Mathematic 1687*, he showed that Kepler's three laws of planetary motion where consequences of two of his own Laws. Research which of his two laws were used. State the two laws.

   (a) **Newton's Second Law of Motion:** *The rate of change of momentum of a body is directly proportional to the force applied, and this change in momentum takes place in the direction of the applied force.*

   (b) **Law of Universal Gravitation:** *Every particle attracts every other particle in the universe with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centers.*

   Set up for proving Kepler's Law:
   Since the gravitational force of the sun on a planet is so much larger than forces exerted by other celestial bodies, we can safely ignore all bodies in the universe except the sun and the one planet revolving about it. Using a coordinate system with the sun at the origin, let $\mathbf{r}(t)$ be the position vector of the planet relative to the sun.

3. State Newton's Laws from part 2 in vector form, labeling constants.

   (a) **Newton's Second law of Motion:**

   $$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}$$

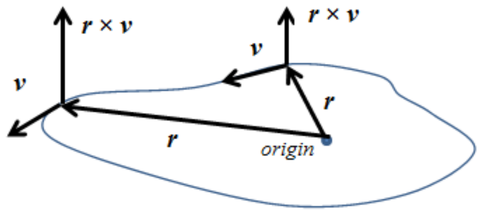
   where $\mathbf{v} = \mathbf{r}'$, $\mathbf{a} = \mathbf{v}' = \mathbf{r}''$, and m is the constant mass of the object.

   (b) **Law of Universal Gravitaion:**

   $$\mathbf{F} = \frac{GMm}{r^2}\left(-\frac{\mathbf{r}}{r}\right) = -\frac{GMm}{r^2}\mathbf{u}$$

   where G is the gravitational constant, m and M are the masses of the planet and the sun, $r = |\mathbf{r}|$, and $\mathbf{u} = \dfrac{\mathbf{r}}{r}$.

4. For Kepler's first law, we begin by showing the planet moves in one plane. How can this be shown using vectors? (Hint: Draw a picture of a curve in the xy-plane, label vectors you know, and ask what would allow you to conclude the position of the planet is always in the plane.)



By showning that the vector $\mathbf{r}\times\mathbf{v}$ *doesn't change. That is that* $\frac{d}{dt}(\mathbf{r}\times\mathbf{v}) = \mathbf{0}$.

5. Prove what you stated in 4. (Hint: You will need to write the acceleration in terms of the position.)

$$\mathbf{a} = \frac{\mathbf{F}}{m} = -\frac{GMm}{mr^3}\mathbf{r} = -\frac{GM}{r^3}\mathbf{r}$$

$$\begin{aligned}\frac{d}{dt}(\mathbf{r}\times\mathbf{v}) &= \mathbf{v}\times\mathbf{v} + \mathbf{r}\times\mathbf{a}\\ &= \mathbf{r}\times\mathbf{a}\\ &= \mathbf{r}\times -\frac{GM}{r^3}\mathbf{r}\\ &= -\frac{GM}{r^3}(\mathbf{r}\times\mathbf{r})\\ &= \mathbf{0}\end{aligned}$$

6. How can you conclude that the orbit of the planet is in a plane?

   Because $\mathbf{r}$ *and* $\mathbf{v}$ *are not parallel,* $\mathbf{r}\times\mathbf{v}\neq\mathbf{0}$ *thus as* $\frac{d}{dt}(\mathbf{r}\times\mathbf{v}) = \mathbf{0}$ *this means that* $\mathbf{r}\times\mathbf{v}$ *is constant hence the orbit of the planet is in a plane.*

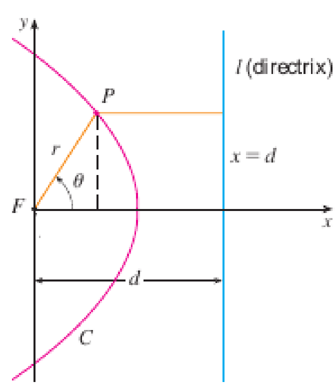
   Next we prove that the planet's orbit is an ellipse. It is convenient to choose a coordinate axes so that the planet is moving in the xy-plane. In order to show the path is an ellipse you need to know the standard form of an ellipse and the polar form involving the eccentricity constant, e, a fixed positive number.

7. What is the standard form of an ellipse with axis length $2a$ along the x-axis and axis length $2b$ along the y-axis?

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

   where (h,k) *is the center of the ellipse.*

8. Let F be the focus, l the directix and e the eccentricity. Show that an ellise is the set of all points with $\frac{|PF|}{|Pl|} = e$ if $e < 1$. (Hint: You will need to rewrite the equation so that r and θ are eliminated then complete the square to put into the standard form.)



$|PF| = r$ and $|Pl| = d - r\cos\theta$ then

$$r = e(d - r\cos\theta) \qquad (\dagger)$$

so

$$r = \frac{ed}{1 + e\cos\theta} \qquad (\dagger\dagger)$$

Squaring both sides of $(\dagger)$ gives:

$$r^2 = e^2(d^2 - 2rd\cos\theta + r^2\cos^2\theta)$$

Converting to Cartiesian coordinates gives:

$$x^2 + y^2 = e^2d^2 - 2e^2dx + e^2x^2$$

Complete the square to put in the standard form of an ellipse:

$$\begin{aligned}
(x^2 - e^2x^2 + 2e^2dx) + y^2 &= e^2d^2 \\
(1 - e^2)\left(x^2 + \frac{e^2d}{1-e^2}x + \frac{e^4d^2}{(1-e^2)^2}\right) + y^2 &= e^2d^2 + \frac{e^4d^2}{1-e^2} \\
(1-e^2)\left(x + \frac{e^2d}{1-e^2}\right)^2 + y^2 &= \frac{e^2d^2}{1-e^2} \\
\frac{(1-e^2)^2}{e^2d^2}\left(x + \frac{e^2d}{1-e^2}\right)^2 + \left(\frac{1-e^2}{e^2d^2}\right)y^2 &= 1
\end{aligned}$$

Since $e < 1$ this gives that $1 - e^2 > 0$. Letting $a^2 = \frac{e^2d^2}{(1-e^2)^2}$ and $b^2 = \frac{e^2d^2}{1-e^2}$ results in the standard form of an ellipse:

$$\frac{\left(x + \frac{e^2d}{1-e^2}\right)^2}{a^2} + \frac{y^2}{b^2} = 1$$

Note that $(1-e^2)^2 \leq (1-e^2)$ hence $a^2 = \dfrac{e^2d^2}{(1-e^2)^2} \geq \dfrac{e^2d^2}{1-e^2} = b^2$ so $2a$ is the major axis length.

9. Now we see the form that we need to get $r = |\mathbf{r}|$ into to prove it's an ellipse. Show that $\mathbf{a} \times (\mathbf{r} \times \mathbf{v}) = GM\mathbf{u}'$ where $\mathbf{u}$ is a unit vector in the direction of $\mathbf{r}$.

$$\begin{aligned}
\mathbf{a} \times (\mathbf{r} \times \mathbf{v}) &= \mathbf{a} \times (r\mathbf{u} \times (r\mathbf{u})') \\
&= \mathbf{a} \times (r\mathbf{u} \times r\mathbf{u}' + r\mathbf{u} \times r'\mathbf{u}) \\
&= r^2\left(-\frac{GM}{r^2}\right)\mathbf{u} \times (\mathbf{u} \times \mathbf{u}') \\
&= -GM[(\mathbf{u}\cdot\mathbf{u}')\mathbf{u} - (\mathbf{u}\cdot\mathbf{u})\mathbf{u}'] \\
&= -GM\left[(\mathbf{u}\cdot\mathbf{u}')\mathbf{u} - (\mathbf{u}\cdot\mathbf{u})\mathbf{u}'\right] \\
&= -GM[(\mathbf{u}\cdot\mathbf{u}')\mathbf{u} - \mathbf{u}'] \\
&= GM\mathbf{u}'
\end{aligned}$$

Since $|\mathbf{u}| = 1$ it follows that $\mathbf{u}\cdot\mathbf{u}' = 0$:

$$0 = \frac{d}{dt}|\mathbf{u}|^2 = \frac{d}{dt}(\mathbf{u}\cdot\mathbf{u}) = \mathbf{u}\cdot\mathbf{u}' + \mathbf{u}'\cdot\mathbf{u}$$

10. Let $\mathbf{h} = \mathbf{r} \times \mathbf{v}$. Show $\mathbf{v} \times \mathbf{h} = GM\mathbf{u} + \mathbf{c}$ with $\mathbf{c}$ in the xy-plane. (Hint: Differential $\mathbf{v} \times \mathbf{h}$ and then integrate.)

$$(\mathbf{v} \times \mathbf{h})' = \mathbf{a} \times \mathbf{h} + \mathbf{v} \times \mathbf{h}' = GM\mathbf{u}' + \mathbf{0}$$

Thus $\mathbf{v} \times \mathbf{h} = GM\mathbf{u} + \mathbf{c}$ *by integrating where* $\mathbf{c}$ *is a constant vector.*

11. Show $\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = h^2$ and that $\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = GMr + rc\cos\theta$ where $h = |\mathbf{h}|$ and $c = |\mathbf{c}|$.

$$\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = \mathbf{r} \cdot (GM\mathbf{u} + \mathbf{c}) = GM\mathbf{r} \cdot \mathbf{u} + \mathbf{r} \cdot \mathbf{c} = GMr + rc\cos\theta$$

$$\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{h} = \mathbf{h} \cdot \mathbf{h} = h^2$$

12. Conclude you have the polar equation of an ellipse. This proves Kelper's first law.

From 11. we have: $h^2 = GMr + rc\cos\theta$ *and solving for* r *gives:*

$$r = \frac{h^2}{GM + c\cos\theta}$$

Letting $e = \dfrac{c}{GM}$*:*

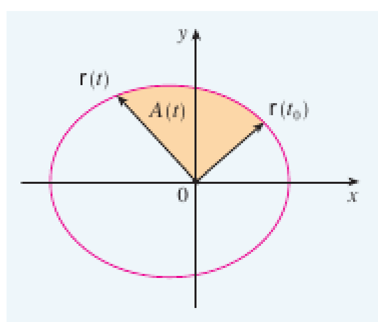
$$r = \frac{h^2/GM}{1 + e\cos\theta}$$

Letting $d = \frac{h^2}{c}$ *gives:*

$$r = \frac{ed}{1 + e\cos\theta}$$

This is equation (††) *from 8, the polar form of an ellipse. Hence the planet's orbit is an ellipse.*

13. Next to prove Kepler's second law let $A(t)$ be the area swept out by the position $\mathbf{r}(t)$ over $[t_0, t]$as shown in the figure. Use polar coordinates to express $\mathbf{r}(t)$ in terms of $r = |\mathbf{r}|$ and θ.



$$\mathbf{r}(t) = \langle r\cos\theta, r\cos\theta\rangle$$

14. What needs to be shown to conclude Kepler's second law?

    The rate of change of the area with respect to time is constant or $\frac{dA}{dT} = c$.

15. Show that $h = r^2 \frac{d\theta}{dt}$. Where $h = |\mathbf{h}| = |\mathbf{r} \times \mathbf{v}|$.

$$\begin{aligned}
\mathbf{h} = \mathbf{r} \times \mathbf{v} &= r^2 \mathbf{u} \times \mathbf{u}' \\
&= r^2 \langle \cos\theta, \sin\theta, 0\rangle \times \langle -\sin\theta, \cos\theta, 0\rangle \frac{d\theta}{dt} \\
&= r^2 \frac{d\theta}{dt} \langle 0, 0, \cos^2\theta + \sin^2\theta\rangle = r^2 \frac{d\theta}{dt}\mathbf{k}
\end{aligned}$$

    Hence $h = \left| r^2 \frac{d\theta}{dt}\mathbf{k} \right| = r^2 \frac{d\theta}{dt}$.

16. Show that:

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

    .

$$\lim_{x \to t} \Delta A(t) = \frac{1}{2}|\mathbf{h}| dt = \frac{r^2}{2} d\theta$$

    Thus

$$\frac{dA}{dt} = \lim_{x \to t} \frac{\Delta A}{\Delta t} = \frac{r^2/2 d\theta}{dt} = \frac{r^2}{2}\frac{d\theta}{dt}$$

17. Conlude Kepler's second law.

    So

$$\frac{dA}{dt} = \frac{r^2}{2}\frac{d\theta}{dt} = \frac{1}{2}h$$

    and as $h = |\mathbf{h}|$ *with* $\frac{d}{dt}\mathbf{h} = 0$ *it follows that* h *is constant and hence* $\frac{dA}{dt}$ *is constant.*

    Lastly, to prove Kepler's third law let T be the period of the planet about the sun and let the major and minor axes lengths be $2a$ and $2b$ respectively.

18. Show $T = \frac{2\pi ab}{h}$.

    By letting D *be the region of the elliptical path and* $C = \partial D$*, the orbit path of the planet, we have by Green's Theorem:*

$$\begin{aligned}
A(T) &= \int\int_D dA = \frac{1}{2}\int_C x\ dy - y\ dx \\
&= \frac{1}{2}\int_0^{2\pi} a\cos\theta(b\cos\theta) - b\sin\theta(-a\cos\theta)\ d\theta \\
&= \frac{1}{2}\int_0^{2\pi} ab\ d\theta = \pi ab \qquad (*)
\end{aligned}$$

Also by 17 $A(t) = \frac{1}{2}ht$ *so we have combining with* $(*)$ *that:*

$$\pi ab = \frac{h}{2}T \qquad \textit{thus} \qquad T = \frac{2\pi ab}{h}$$

19. Show $\frac{h^2}{GM} = \frac{b^2}{a}$. (Hint: You will need to look at your work in 8 and 12.)

From 12 we have that $h^2 = dc = deGM$ *and from 8 we have that*

$$a^2 = \frac{e^2d^2}{(1-e^2)^2} \qquad \textit{and} \qquad b^2 = \frac{e^2d^2}{1-e^2}$$

hence

$$\frac{b^2}{a} = ed = \frac{h^2}{GM}$$

20. Show $T^2 = \frac{4\pi^2}{GM}a^3$ and conclude Kepler's third law.

$$T^2 = \frac{4\pi^2a^2b^2}{h^2} = 4\pi^2a^2\frac{b^2}{h^2} = 4\pi^2a^2\frac{a}{GM} = \frac{\pi^2(2a)^3}{2GM}$$

Since $\frac{\pi^2}{2GM}$ *is constant this gives the square of the period* T *of the orbit is proportional to the cube of the length of the major axis* $2a$.