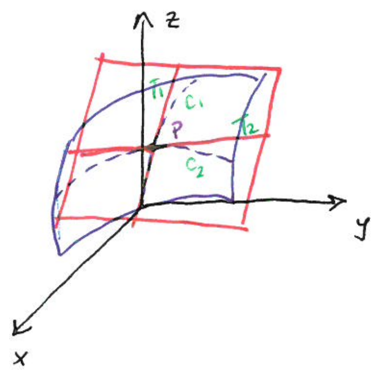
Section 14.4 - Tangent Planes and Linear Approximations



Let S be a surface with $z=f(x,y)$, where f has
f has continuous first partial derivatives.
Let $P(x_0, y_0, z_0)$ be a point on S. So P lies on C_1 and C_2.
Let T_1 and T_2 be the tangent lines to C_1 and C_2 at P.
Then, the <u>tangent plane</u> to S at P is the plane containing the tangent lines T_1 and T_2.

T_1: Line in plane $y=y_0$, slope $f_x(x_0,y_0)$: $z-z_0 = f_x(x_0,y_0)(x-x_0)$

T_2: Line in plane $x=x_0$, slope $f_y(x_0,y_0)$: $z-z_0 = f_y(x_0,y_0)(y-y_0)$

<u>Equation of plane:</u> $Ax+By+Cz=0$

$$z-z_0 = -\frac{A}{C}(x-x_0) - \frac{B}{C}(y-y_0)$$

$y=y_0 \Rightarrow T_1$
$x=x_0 \Rightarrow T_2$

tangent plane to $z=f(x,y)$ at $P(x_0,y_0,z_0)$

$$\boxed{z-z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

$\boxed{\text{Ex 1}}$ Find the tangent plane to $z=2x^2+y^2$ at $(1,1,3)$.

$f_x(x,y) = 4x$ $\quad f_x(1,1)=4$

$f_y(x,y) = 2y$ $\quad f_y(1,1)=2$

tangent plane to $z=f(x,y)$ at $(1,1,3)$ is

$$\boxed{z-3 = 4(x-1)+2(y-1)}$$

<u>Linear Approximation:</u>

$z = f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)+z_0 = L(x,y) \approx f(x,y)$ for (x,y) near (x_0,y_0).

<u>Linearization of f at (a,b):</u> $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

<u>Linear approx (tangent Plane Approx.) of f at (a,b):</u> $f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$\boxed{\text{Theorem 8}}$ If the partial derivatives f_x and f_y exist near (a,b) and are continuous at (a,b) then f is differentiable at (a,b).

$\boxed{\text{E...}}$ Show that $f(x,y) = xe^{xy}$ is differentiable at $(1,0)$ and find its linearization, use it to approx $f(1.1,-0.1)$.

$f_x(x,y) = e^{xy} + xye^{xy}$
$f_y(x,y) = x^2e^{xy}$

Both continuous. So f is differentiable by Theorem 8

$f_x(1,0)=1$
$f_y(1,0)=1$

$L(x,y) = 1+(x-1)+y$

$L(1.1,-0.1) = 1+0.1-0.1 = 1 \approx f(1.1,-0.1)$

Differentials (Important for implicit differentiation and integration)

One variable: the differential dx defined to be any real number (independent variable)

$y = f(x)$ then $dy = f'(x)\,dx$ $\left[dx = \Delta x \quad \text{but } \Delta y = f(x+\Delta x) - f(x) \approx dy \right]$

Two variables: the differentials dx, dy independent variables, can be any real number

$z = f(x,y)$ then the differential of z (total differential) is defined by

$\left[\begin{array}{l} dx = \Delta x \\ dy = \Delta y \end{array} \quad \Delta z = f(x+\Delta x, y+\Delta y) - f(x,y) \approx \right]$ $\boxed{dz = f_x(x,y)\,dx + f_y(x,y)\,dy}$

Ex 5 The base radius and height of a right circular cone are measured as 10cm and 25cm, with a possible error of 0.1 cm in each. Use differentials to estimate the max error in calculating the volume of the cone.

$$V = \frac{1}{3}\pi r^2 \cdot h \qquad |\Delta r| = |dr| \leq 0.1$$

$$dV = \frac{2}{3}\pi r h\,dr + \frac{1}{3}\pi r^2 dh \qquad |\Delta h| = |dh| \leq 0.1$$

Largest error when $dr = 0.1$ and $dh = 0.1$

$$dV = \frac{2}{3}\pi(10)(25)(0.1) + \frac{1}{3}\pi(10)^2(0.1) = \boxed{20\pi \text{ cm}^3 \approx 63\text{cm}^3}$$

Functions of Three or More Variables: $w = f(x,y,z)$

Linear approx.: $L(x,y,z) = f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z\,(z-z_0)$

Increment of w: $\Delta w = f(x+\Delta x, y+\Delta y, z+\Delta z)$

Differential of w: $dw = f_x(x,y,z)\,dx + f_y(x,y,z)\,dy + f_z(x,y,z)\,dz$