Section 16.2 - Line Integrals in Space & Vector fields — Vector Calc

C a curve in space: $x = x(t)$, $y = y(t)$, $z = z(t)$, $a \le t \le b$

Line integral of f on C wrt arc length: $\boxed{\int_C f(x,y,z)\,ds = \int_C f(x(t),y(t),z(t))\sqrt{\left(\frac{\partial x}{\partial t}\right)^2+\left(\frac{\partial y}{dt}\right)^2+\left(\frac{\partial z}{\partial x}\right)^2}\,dt}$

$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ $\qquad = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$

When $f(x,y,z)=1$ we get the length of C $= \int_C ds = \int_a^b |\vec{r}\,'(t)|\,dt = L$.

Also have line integrals of f on C wrt to x, y, z:

$$\int_C f(x,y,z)\,dx \qquad \int_C f(x,y,z)\,dy \qquad \int_C f(x,y,z)\,dz$$

Notation for line integrals in the plane: $\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$.

$\boxed{\text{Example 6}}$ Evaluate $\int_C y\,dx + z\,dy + x\,dz$, where C consists of the line segment C_1 from $(2,0,0)$ to $(3,4,5)$ followed by C_2 from $(3,4,5)$ to $(3,4,0)$.

Parametrization of C_1: $\vec{r}_1(t) = \langle 2,0,0\rangle + t\langle 1,4,5\rangle$

$x = t+2$, $y = 4t$, $z = 5t$ $\qquad 0 \le t \le 1$

$dx = dt$ $\qquad dy = 4dt$ $\qquad dz = 5dt$

Parametrization of C_2: $\vec{r}_2(t) = \langle 3,4,5\rangle + t\langle 0,0,-5\rangle$

$x = 3$, $y = 4$, $z = -5t+5$ $\qquad 0 \le t \le 1$

$dx = 0$ $\qquad dy = 0$ $\qquad dz = -5dt$

$$\int_C y\,dx + z\,dy + x\,dz = \int_{C_1} y\,dx + z\,dy + x\,dz + \int_{C_2} \underbrace{x\,dx + z\,dy + x\,dz}$$

$$= \int_0^1 (4t)\,dt + 5t(4dt) + (t+2)\,5dt + 3(-5dt)$$

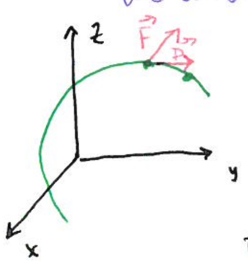
$$= \int_0^1 29t - 5\,dt = \frac{29}{2} - 5 = \boxed{\frac{19}{2}}$$

Line Integrals of Vector Fields

Work done by force $f(x)$ from (a,b) was $\int_a^b f(x)\,dx$

Now suppose $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a continuous force field on $\mathbb{R}^3$

We wish to compute work to move a particle along a curve C.



$W = \vec{F}\cdot\vec{D} = \vec{F}\cdot\vec{T}(t)\cdot\Delta s$ where $\vec{T}(t)$ is the unit tangent vector

$$\boxed{W = \int_C \vec{F}\cdot\overbrace{\vec{T}\,ds}^{d\vec{r}} \quad \text{work}} = \int_a^b \vec{F}(\vec{r}(t))\cdot\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\,|\vec{r}\,'(t)|\cdot dt = \boxed{\int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt}$$

Def. $\vec{F}$ continuous on smooth C: $\vec{r}(t)$ then $\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_C \vec{F}\cdot\vec{T}\,ds$

$\boxed{\text{Example 7}}$ Find the work done by the force field $\vec{F}(x,y) = x^2\vec{i} - xy\vec{j}$ in moving a particle along $\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le \pi/2$.

$$\vec{F}(\vec{r}(t)) = \langle \cos^2 t, -\cos t \sin t \rangle \qquad \vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \langle \cos^2 t, -\cos t \sin t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_0^{\pi/2} -2\cos^2 t \sin t \, dt$$

$$= +2\frac{\cos^3 t}{3} \Big|_0^{\pi/2} = \boxed{-\frac{2}{3}}$$

★Note: $\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T} \, ds$ and $\int_{-C} \vec{F} \cdot (-\vec{T}) \, ds = \int_C \vec{F} \cdot \vec{T} \, ds$

$\boxed{\text{Orientation change}}$ and $\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$

$$\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \langle P, Q, R \rangle \cdot \overbrace{\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle}^{\vec{r}\,'(t)} dt$$

$$= \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt} \right) dt$$

$$= \boxed{\int_C P\,dx + Q\,dy + R\,dz}$$