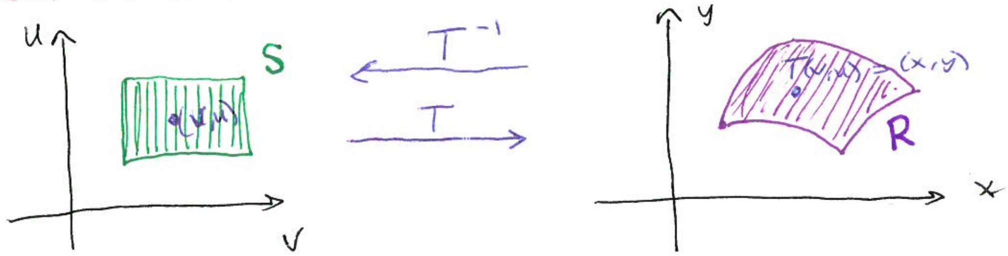
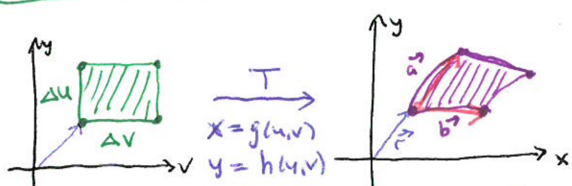
Section 15.10 - Change of variables

★ Can be useful to create your own Coordinate System. But we need an easier way to change between the two.



- T is called a <u>Transformation</u>: $x = g(u,v)$, $y = h(u,v)$; $u = k(x,y)$, $v = \ell(x,y)$
- point (u,v) <u>maps</u> to (x,y), (x,y) is the <u>image</u> of (v,u)
- If no two points have the same image then T is <u>One-to-One</u>
- If T is One-to-One then T has an inverse T^{-1}

How This affects a double Integral:



Goal approx. area of purple shape with parallelogram

$\vec{r}(u,v) = g(u,v)\vec{i} + h(u,v)\vec{j}$

$|\vec{a}\times\vec{b}| = |\vec{r}_u\Delta u \times \vec{r}_v \Delta v| = |\vec{r}_u \times \vec{r}_v|\,\Delta u \Delta v$

$$|\vec{r}_u \times \vec{r}_v| = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} k$$

The Jacobian of the transformation T given by $x = g(u,v)$ and $y = h(u,v)$ is

$$\boxed{\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}}$$

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

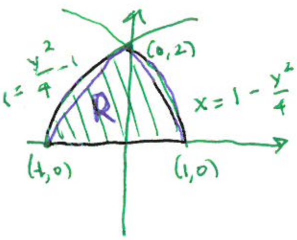
<u>Change of variables</u>: T a transformation with continuous first-order partials, whose Jacobian is nonzero, who maps region S(uv) onto a region R (x,y). f Continuous on R and T one-to-one, except on the boundary of S then

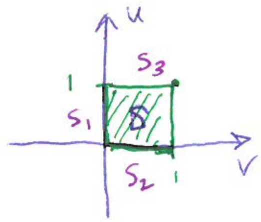
$$\boxed{\iint_R f(x,y)\,dA = \iint_S f(x(u,v),\, y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$$

Polar: $x = r\cos\theta$ $\quad y = r\sin\theta \quad \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r > 0$

So $\iint_R f(x,y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$.

Ex 2 Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate $\iint_R y\, dA$ where R is bounded by the x-axis, and parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geq 0$.





$y = 0 \Rightarrow u \text{ or } v = 0$

$-1 \leq x \leq 0 \Rightarrow u = 0 \quad S_1$

$0 \leq x \leq 1 \Rightarrow v = 0 \quad S_2$

$y^2 = 4 - 4x \quad 0 \leq x \leq 1 \Rightarrow 4u^2v^2 = 4 - 4u^2 + 4v^2$

$4u^2(v^2+1) = 4(v^2+1)$

$u^2 = 1 \quad S_3$

$y^2 = 4 + 4x \quad -1 \leq x \leq 0$

$\Rightarrow 4u^2v^2 = 4 + 4u^2 - 4v^2$

$4v^2(u^2+1) = 4(u^2+1)$

$v^2 = 1 \quad S_4$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} 2u & 2v \\ -2v & 2u \end{vmatrix} = 4u^2 + 4v^2 > 0$$

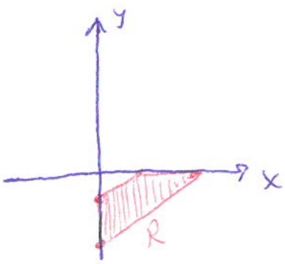
$$\iint_R y\, dA = \int_0^1 \int_0^1 2uv \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$$

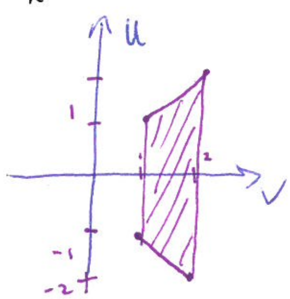
$$= \int_0^1 \int_0^1 2uv(4u^2 + 4v^2)\, du\, dv$$

$$= 8\int_0^1 \frac{u^4}{4}v + \frac{u^2}{2}v^3 \Big|_0^1 dv$$

$$= 8\int_0^1 \frac{1}{4}v + \frac{1}{2}v^3\, dv = 8\left[\frac{1}{8}v^2 + \frac{1}{8}v^4\right]\Big|_0^1 = \boxed{2}$$

Ex 3 Evaluate $\iint_R e^{\frac{(x+y)}{(x-y)}}\, dA$ where R is the trapezoid region with vertices $(0,-1), (0,-2), (2,0), (1,0)$





$u = x + y \quad v = x - y$ since $e^{u/v}$ easy to integrate

$x = \frac{1}{2}(u+v) \quad y = \frac{1}{2}(u-v)$

$x = 0 \quad -2 \leq y \leq -1 \Rightarrow u = -v$

$y = 0 \quad 1 \leq x \leq 2 \Rightarrow u = v$

$y = x - 1 \quad 0 \leq x \leq 1 \Rightarrow v = 1$

$y = x - 2 \quad 0 \leq x \leq 2 \Rightarrow v = 2$

$$\iint_R e^{\frac{x+y}{x-y}}\, dA = \iint_S e^{\frac{u}{v}} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}\right| = \left|-\frac{1}{2}\right|$$

$$= +\frac{1}{2}\int_1^2 \int_{-v}^{v} e^{u/v}\, du\, dv = +\frac{1}{2}\int_1^2 v(e^{v/v} - e^{-v/v})\, dv = +\frac{1}{2}(e^1 - e^{-1})\left(\frac{4-1}{2}\right) = \boxed{\frac{3}{4}(e - e^{-1})}$$