Vector Review - Chapter 12

Chapter 12 - Vectors and Space

- $|\langle x,y,z\rangle| = \sqrt{x^2+y^2+z^2}$
- $\langle x,y,z\rangle \circ \langle a,b,c\rangle = ax+by+cz$
- $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$
- $\vec{a}\perp\vec{b}$ iff $\vec{a}\cdot\vec{b}=0$

(12.2 - 12.3)

- $\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}$
- $\vec{a}\times\vec{b} \perp \vec{a}$ and $\vec{b}$
- $|a\times b| = |a||b|\sin\theta$ = area of parallelogram

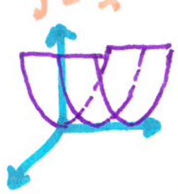
(12.4)

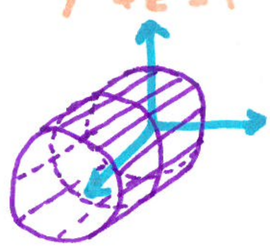
- $V = |a\circ(b\times c)|$
- Line : $\vec{r}(t) = \vec{r_0} + t(\vec{r_1} - \vec{r_0})$
  Segment: $0 \le t \le 1$
- Plane : $\vec{n}\circ\langle x-x_1, y-y_1, z-z_1\rangle = 0$
  $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

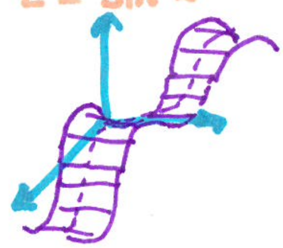
(12.5)

Surfaces: (12.6)

① Cylinders (equations with one free variable)

Ex. $y=x^2$ 

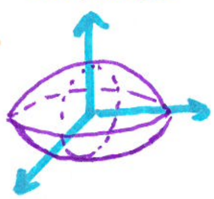
$y^2+z^2=1$ 

$z=\sin x$ 

② Quadric Surfaces

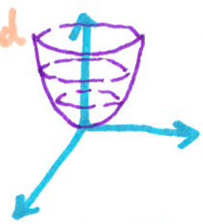
Ellipsoid

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$



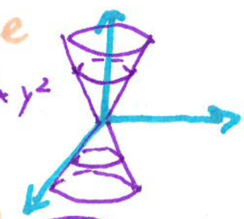
Elliptic Paraboloid

$z=x^2+y^2$



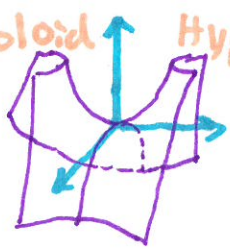
Cone

$z^2=x^2+y^2$



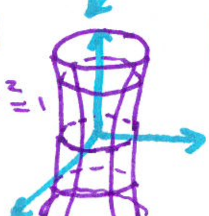
Hyperbolic Paraboloid

$z=y^2-x^2$



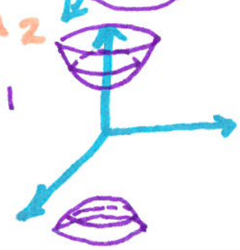
Hyperboloid 1

$x^2+y^2-z^2=1$



Hyperboloid 2

$-x^2-y^2+z^2=1$



Chapter 12 Practice Problems

① Find the equation of the plane through $A(0,1,-1)$, $B(1,0,-1)$, $C(-1,1,0)$.

$\vec{AB} = \langle 1,-1,0\rangle \quad \vec{BC} = \langle -2,1,1\rangle \quad \vec{n} = \vec{AB}\times\vec{BC} = \langle -1,-1,-1\rangle$

$$\boxed{-1(x-0)-1(y-1)-1(z+1)=0}$$

② Parametrize the plane with normal vector $\langle 0,1,1\rangle$ through the point $(1,2,3)$.

$$\boxed{x(t) = 1+0t \qquad y(t) = 2+t \qquad z(t) = 3+t}$$

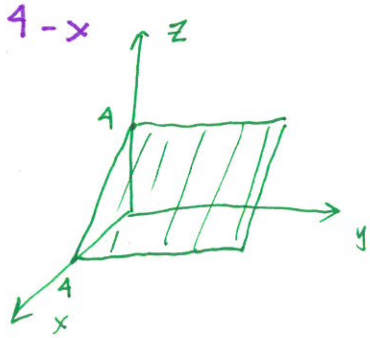
③ Find the angle between the two planes above. Are they orthogonal?

$\vec{n}_1 = \langle -1,-1,-1\rangle \quad \vec{n}_2 = \langle 0,1,1\rangle \quad \cos\theta = \dfrac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \dfrac{-2}{\sqrt{3}\sqrt{2}}$

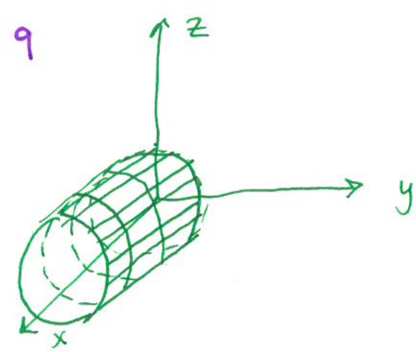
$\boxed{\theta = \cos^{-1}\left(-\frac{2\sqrt{6}}{6}\right)} \neq \frac{\pi}{2}$ thus they are $\boxed{\text{not orthogonal}}$

④ Sketch

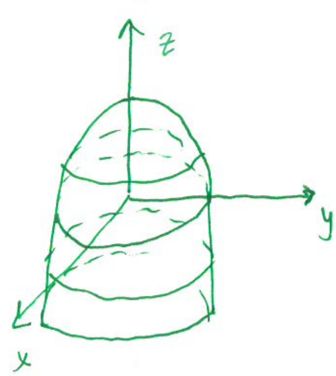
$z = 4 - x$



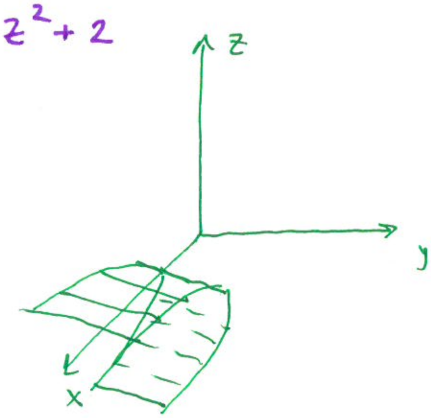
$y^2 + z^2 = 9$



$z = 4 - x^2 - y^2$



$x = z^2 + 2$



Chapter 13 - Vector Functions

Vector Function: $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ (13.1)

$$\left.\begin{aligned}
&\bullet \lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle \\
&\bullet \vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle \\
&\bullet \int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle
\end{aligned}\right\} \quad (13.2)$$

$C: \vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

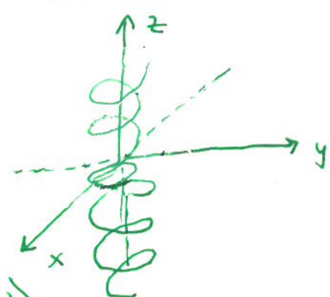
$$\left.\begin{aligned}
\bullet\ \text{Arc length of } C \text{ on } [a,b] &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt \\
&= \int_a^b |\vec{r}\,'(t)|\,dt
\end{aligned}\right\} \quad (13.3)$$

$\vec{x}(t)$ - position $\quad \vec{x}\,'(t) = \vec{v}(t)$ - velocity $\quad \vec{x}\,''(t) = \vec{a}(t)$ acceleration (13.4)

Chapter 13 Practice Problems

① Sketch $\vec{r}(t) = \langle \cos t, \sin t, -t \rangle$

$x^2 + y^2 = 1 \qquad z = -t$



② Find $\lim_{t\to 0} \left\langle \frac{\sin t}{t}, t^2, \ln(t+1) \right\rangle$

$$= \left\langle \lim_{t\to 0} \frac{\sin t}{t}, \lim_{t\to 0} t^2, \lim_{t\to 0} \ln(t+1) \right\rangle$$

$$= \left\langle \lim_{t\to 0} \frac{\cos t}{1}, 0, \ln\left(\lim_{t\to 0} t+1\right) \right\rangle = \langle 1, 0, \ln(1) \rangle = \boxed{\langle 1, 0, 0 \rangle}$$

③ Find the derivative of $\vec{r}(t) = \langle \sin^2 t, e^{4t} + 1, 3t^4 + t^2 \rangle$

$$\boxed{\vec{r}\,'(t) = \langle 2\sin t \cdot \cos t, 4e^{4t}, 12t^3 + 2t \rangle}$$

④ Find the arc length of $\vec{r}(t) = \langle \sin t^2, \cos t^2, 2x \rangle$ for $0 \le t \le 1$.

$$L = \int_0^1 \sqrt{(2t\cos t^2)^2 + (2t\sin t^2)^2 + (2)^2}\,dt$$

$$= \int_0^1 \sqrt{4t^2 + 4}\,dt = 2\int_0^1 \sqrt{t^2+1}\,dt \approx \boxed{2.294}$$