arc length $\longrightarrow$ line integrals

Surface Area $\longrightarrow$ Surface integrals

$\vec{r}(u,v) = \langle x, y, z \rangle \quad (u,v) \in D$

Volume of rectangular Prism:

$= \text{Area of Base} \times \text{height}$

$= \Delta S \times f(x,y,z)$

$= |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v \times f(\vec{r}(u,v))$

Surface integral of f over the surface S:

$$\boxed{\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, dA}$$

$\boxed{\text{Ex 1}}$ Compute the surface integral $\iint_S x^2 dS$ where S is the unit sphere.

$\vec{r}(\varphi,\theta) = \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi \rangle \quad 0 \le \varphi \le \pi \quad 0 \le \theta \le 2\pi$

$\vec{r}_\varphi = \langle \cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi \rangle$

$\vec{r}_\theta = \langle -\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0 \rangle$

$|\vec{r}_\varphi \times \vec{r}_\theta| = |\langle -\sin^2\varphi\cos\theta, -\sin^2\varphi\sin\theta, \sin\varphi\cos\varphi \rangle|$

$= \sin\varphi$

$$\iint_S x^2 dS = \int_0^\pi \int_0^{2\pi} \sin^2\varphi \cos^2\theta \cdot \sin\varphi \, d\theta \, d\varphi$$

$$= \int_0^{2\pi} \cos^2\theta \, d\theta \cdot \int_0^\pi \sin\varphi(1-\cos^2\varphi)\, d\varphi$$

$$= \left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi} \cdot \left(-\cos\varphi + \frac{1}{3}\cos^3\varphi\right)\Big|_0^\pi$$

$$= \pi \cdot \left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) = \boxed{\frac{4\pi}{3}}$$

Section 16.7 – Surface Integrals   Vector Calc

Application: Surface S a thin sheet with density ρ

$$m = \iint_S \rho \, dS \qquad (\bar{x}, \bar{y}, \bar{z}) = \frac{1}{m}\left(\iint_S x\rho\, dS,\ \iint_S y\rho\, dS,\ \iint_S z\rho\, dS\right)$$

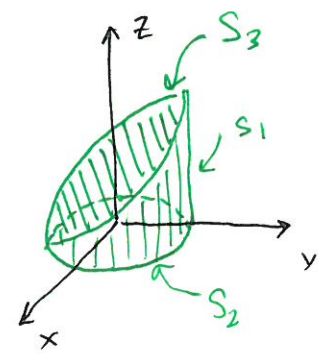
Graphs: Any surface S with $z = g(x,y)$ can be regarded as parametric

$$x = x \qquad y = y \qquad z = g(x,y)$$

then $\vec{r}_x = \langle 1, 0, g_x \rangle \qquad \vec{r}_y = \langle 0, 1, g_y \rangle$

then $$\boxed{\iint_S f(x,y,z)\, dS = \iint_D f(x,y,g(x,y))\sqrt{g_x^2 + g_y^2 + 1}\, dA}$$

$\boxed{\text{Ex 3}}$ Evaluate $\iint_S z\, dS$ where S is the surface whose sides S_1 is given by $x^2+y^2=1$, whose base S_2 is $x^2+y^2 \le 1$, in the $z=0$ plane, and whose top S_3 is the plane $z = 1+x$ above S_2.



$$\iint_S z\, dS = \iint_{S_1} z\, dS + \iint_{S_2} z\, dS + \iint_{S_3} z\, dS$$

S_1: $x = \cos\theta \quad y = \sin\theta \quad z = z$

$0 \le \theta \le 2\pi \qquad 0 \le z \le 1 + x = 1 + \cos\theta$

S_2: $x = \cos\theta \quad y = \sin\theta \quad z = 0$

$0 \le \theta \le 2\pi$

$$\iint_{S_2} z\, dS = \iint_{S_2} 0\, dS = 0$$

S_3: $x = x, \quad y = y, \quad z = x + 1$

$-1 \le x \le 1, -1 \le y \le 1,$

$$\iint_{S_1} z\, dS = \int_0^{2\pi}\int_0^{1+\cos\theta} z\, dz\, d\theta = \frac{1}{2}\int_0^{2\pi} 1 + 2\cos\theta + \cos^2\theta\, d\theta = \frac{1}{2}\left[\theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{2}\frac{\sin 2\theta}{2}\right]_0^{2\pi} = \boxed{\frac{3\pi}{2}}$$

$+$

$$\iint_{S_3} z\, dS = \iint_D (x+1)\sqrt{2}\, dA = \int_0^{2\pi}\sqrt{2}\int_0^1 (r\cos\theta + 1)\, r\, dr\, d\theta = \int_0^{2\pi}\sqrt{2}\left(\frac{1}{2} + \frac{1}{3}\cos\theta\right) d\theta = \boxed{\sqrt{2}\pi}$$