Define triple integrals for functions of three variables

<u>Simple Case</u> : Over a rectangular box

$$B = \{(x,y,z) \mid a \le x \le b,\ c \le y \le d,\ e \le z \le f\}$$

$$\Delta V = \Delta x \, \Delta y \, \Delta z$$

The <u>triple Integral</u> of $f(x,y,z)$ over B is

$$\boxed{\iiint_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i, y_j, z_k)\,dV}$$

<u>Fubini's Theorem for triple integrals</u>: f continuous on $B = [a,b]\times[c,d]\times[e,f]$

$$\iiint_B f(x,y,z)\,dV = \int_e^f\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz$$

<u>General Case</u> :

① $E = \{(x,y,z) \mid a \le x \le b,\ h_1(x) \le y \le h_2(x),\ u_1(x,y) \le z \le u_2(x,y)\}$

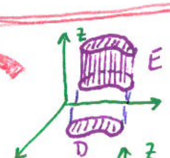
$$\boxed{\iiint_E f(x,y,z)\,dV = \int_a^b\int_{h_1(x)}^{h_2(x)}\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx}$$

② $E = \{(x,y,z) \mid c \le y \le d,\ g_1(y) \le x \le g_2(y),\ u_1(x,y) \le z \le u_2(x,y)\}$

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_c^d\int_{h_1(x)}^{h_2(x)}\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy}$$

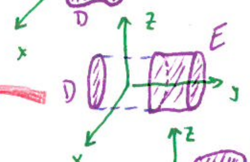
Note

$$\iiint_E f(x,y,z)\,dV = \iint_D\left[\int_{z=u_1(x,y)}^{z=u_2(x,y)} f(x,y,z)\,dz\right]dA$$



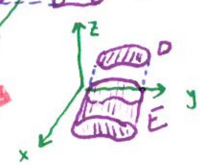
Type I : D in xy-plane

$$= \iint_D\left[\int_{y=u_1(x,z)}^{y=u_2(x,z)} f(x,y,z)\,dy\right]dA$$



Type II : D in xz-plane

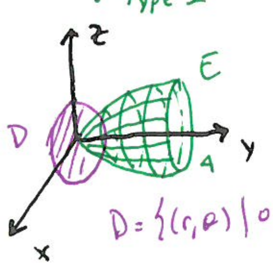
$$= \iint_D\left[\int_{x=u_1(y,z)}^{x=u_2(y,z)} f(x,y,z)\,dx\right]dA$$



Type III : D in yz-plane

Ex. 3 Evaluate $\iiint_E \sqrt{x^2+z^2}\,dV$, where E is the region bounded by the paraboloid $y = x^2+z^2$ and the plane $y=4$.

• Type II



$$\iiint_E \sqrt{x^2+z^2}\,dV = \iint_D\left[\int_{x^2+z^2}^{4}\sqrt{x^2+z^2}\,dy\right]dA = \iint_D (4-x^2-z^2)\sqrt{x^2+z^2}\,dA$$

$$= \int_0^{2\pi}\int_0^2 (4-r^2)\cdot r^2\,dr\,d\theta = 2\pi\left[\frac{4(2)^3}{3} - \frac{(2)^5}{5}\right] = \boxed{\frac{128\pi}{15}}$$

$D = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

Section 15.7 - Triple Integrals

Ex 4 Express the integral $\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\, dz\, dy\, dx$ as an integral with $dx\, dz\, dy$.

$E = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le y\}$

$0 \le \sqrt{y} \le x$

$E = \{(x,y,z) \mid 0 \le y \le 1,\ 0 \le z \le y,\ \sqrt{y} \le x \le 1\} \Rightarrow \boxed{\int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\, dx\, dz\, dy}$

Applications of Triple Integrals:

E is the domain of $f(x,y,z)$ so if $f(x,y,z) = 1$ then $\boxed{\iiint_E 1\, dV = V(E)}$

Mass of a solid object with density function $\rho(x,y,z)$ occupying E:

$$m = \iiint_E \rho(x,y,z)\, dV \qquad M_{yz} = \iiint_E x\, \rho(x,y,z)\, dV \qquad M_{xz} = \iiint_E y\, \rho(x,y,z)\, dV$$

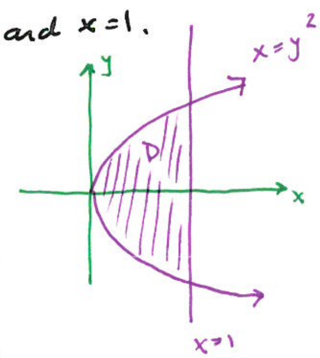
$$M_{xy} = \iiint_E z\, \rho(x,y,z)\, dV$$

Center of mass: $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right)$

$$I_x = \iiint_E (y^2+z^2)\rho(x,y,z)\, dV \qquad I_y = \iiint_E (x^2+z^2)\rho(x,y,z)\, dV$$

$$I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\, dV$$

Ex 6 Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$, and $x = 1$.



$E = \{(x,y,z) \mid -1 \le y \le 1,\ y^2 \le x \le 1,\ 0 \le z \le x\}$

$$m = \int_{-1}^1 \int_{y^2}^1 \int_0^x \rho\, dV = \int_{-1}^1 \int_{y^2}^1 \rho x\, dx\, dy = \int_{-1}^1 \rho \tfrac{1}{2}(1 - y^4)\, dy$$

$$= \frac{\rho}{2}\left(\left(y - \frac{y^5}{5}\right)\right)\Big|_{-1}^1 = \frac{\rho}{2}\left(\frac{4}{5} - \left(-\frac{4}{5}\right)\right) = \boxed{\frac{4\rho}{5}}$$

By symmetry $M_{xz} = \boxed{0}$ $\quad M_{yz} = \int_{-1}^1 \int_{y^2}^1 \int_0^x \rho x\, dz\, dx\, dy = \int_{-1}^1 \int_{y^2}^1 \rho x^2\, dx\, dy$

$$= \frac{\rho}{3}\int_{-1}^1 (1 - y^6)\, dy = \frac{\rho}{3}\left(\frac{6}{7} - \left(-\frac{6}{7}\right)\right) = \boxed{\frac{4\rho}{7}}$$

$$M_{xy} = \int_{-1}^1 \int_{y^2}^1 \int_0^x \rho z\, dz\, dx\, dy = \int_{-1}^1 \int_{y^2}^1 \frac{\rho}{2} x^2\, dx\, dy = \boxed{\frac{2\rho}{7}}$$

Center of mass $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right) = \boxed{\left(\frac{5}{7}, 0, \frac{5}{14}\right)}$