Chapter 14 Review — Partial Derivatives

Level Curves: $K = f(x,y)$ $\quad \lim_{(x,y)\to(a,b)} f(x,y) = L$ if $f(x,y)$ approaches L for any path $(x,y)\to(a,b)$. (14.1)

★ Two different limits on two different paths then Limit DNE.

f is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ (14.2)

$$f_x = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \qquad f_y = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

★ Treat y - constant ★ Treat x - constant (14.3)

Clairaut's Theorem: f defined on $D \ni (a,b)$ f_{xy}, f_{yx} continuous then $f_{xy} = f_{yx}$.

tangent plane: $z = f(x,y)$ $\quad z - z_0 = f_x(x-x_0) + f_y(y-y_0)$

Thm: f_x, f_y exist near (a,b) + Continuous then f is differentiable at (a,b). (14.4)

Differential $z=f(x,y)$: $\quad dz = f_x\,dx + f_y\,dy$

Chain Rule: (14.5)

① $z = f(x(t), y(t))$ $\quad \dfrac{dz}{dt} = \dfrac{\partial f}{\partial x}\cdot\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\cdot\dfrac{dy}{dt}$

② $z = f(x(s,t), y(s,t))$ $\quad \dfrac{\partial z}{\partial t} = \dfrac{\partial f}{\partial x}\cdot\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial y}{\partial t}$

Directional Derivative: $D_{\vec{u}} f = f_x a + f_y b$ where $\vec{u} = \langle a,b\rangle$ unit vector (14.6)

Gradient: $\nabla f = \langle f_x, f_y\rangle$ so $D_{\vec{u}} f = \nabla f \cdot \vec{u}$

Thm: max of $D_{\vec{u}} f$ is $|\nabla f|$ in direction of ∇f

Second Derivative Test: $f_x = f_y = 0$ at (a,b) (14.7)

$$D = f_{xx}(a,b)\, f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

a) $D > 0$
- $f_{xx} > 0 \Rightarrow$ min
- $f_{xx} < 0 \Rightarrow$ max

b) $D < 0 \Rightarrow$ saddle point

c) $D = 0$ Inconclusive.

Chapter 15 Review — Multiple Integrals

$f(x,y) > 0$, R a region in xy-plane $\quad V = \iint_R f\,dA$ - Volume under f over R (15.1)

Fubini's Theorem:

$R = [a,b] \times [c,d]$
f continuous

$$\iint_R f\,dA = \int_a^b \int_c^d f\,dy\,dx = \int_c^d \int_a^b f\,dx\,dy \qquad (15.2)$$

① $D = \{(x,y) \mid a < x < b,\ g_1 < y < g_2\}$

$$\iint_D f\,dA = \int_a^b \int_{g_1}^{g_2} f\,dy\,dx$$

② $D = \{(x,y) \mid c < y < d,\ h_1 < x < h_2\}$

$$\iint_D f\,dA = \int_c^d \int_{h_1}^{h_2} f\,dx\,dy$$

(15.3)

Polar Coords: $r^2 = x^2 + y^2 \quad x = r\cos\theta \quad y = r\sin\theta$

$$dA = r \cdot dr\,d\theta$$

$$\iint_D f\,dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) \cdot r\,dr\,d\theta$$

(15.4)

Surface Area:

S: $z = f(x,y)$ $\qquad A(S) = \iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1}\,dA$ (15.6)

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{h_1}^{h_2} f\,dz\right] dA \qquad (15.7)$$

Cylindrical Coords: $x = r\cos\theta \quad y = r\sin\theta \quad z = z$ (15.8)

$$dV = dz \cdot r\,dr\,d\theta$$

Sphereical Coords: $x = \rho\sin\varphi\cos\theta \quad y = \rho\sin\varphi\sin\theta \quad z = \rho\cos\varphi$

$$\rho^2 = x^2 + y^2 + z^2$$

$$dV = \rho^2 \sin\varphi\,d\rho\,d\theta\,d\varphi$$

(15.9)

Chapter 14 & 15 Practice

① $f(x,y) = x^2 + 2x - y$

a) find all first and second partials

$f_x = 2x+2 \quad f_y = -1 \qquad f_{xx} = 2 \quad f_{xy} = 0 \quad f_{yy} = 0$

b) find the gradient $\quad \nabla f = \langle f_x, f_y \rangle = \langle 2x+2, -1 \rangle$

c) what types of graphs are the level curves of f?

$K = x^2 + 2x - y$

$y = x^2 + 2x + K \leftarrow$ Parabolas upwards

$= (x+1)^2 + K - 2 \quad$ Vertex $(-1, K-2)$

② Find critical points of $f(x,y) = x^3 - 12x + y^2$ and classify them using second derivative test.

$0 = \nabla f = \langle 3x^2 - 12, 2y \rangle$

$x = \pm 2 \quad y = 0$

$D = f_{xx} f_{yy} - (f_{xy})^2 = 6x \cdot 2 - 0^2 = 12x$

$(2,0) \quad D(2,0) > 0 \quad f_{xx}(2,0) > 0 \Rightarrow \boxed{\text{local min}}$

$(-2,0) \quad D(-2,0) < 0 \Rightarrow \boxed{\text{Saddle point}}$

③ If $z = f(x,y)$ and $x = g(r,s)$ and $y = h(r,s)$ find $\frac{\partial z}{\partial s}$

$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}}$$

④ Compute by changing order $\int_0^1 \int_x^1 y^2 \sin(xy)\, dy\, dx$

$= \int_0^1 \int_0^y y^2 \sin(xy)\, dx\, dy = \int_0^1 -y^2 \frac{\cos(y^2)}{y} + y\, dy$

$= -\frac{\sin(y^2)}{2} + \frac{y^2}{2} \Big|_0^1 = \boxed{-\frac{\sin(1)}{2} + \frac{1}{2}}$

Chapter 15 Practice

⑤ $\iint_D x\,dA$ D: first quadrant $x^2+x^2=1$ $x^2+y^2=2$

$$= \int_0^{\pi/2}\int_1^{\sqrt{2}} r\cos\theta \cdot r\,dr\,d\theta = \int_0^{\pi/2} \cos\theta\,d\theta \int_1^{\sqrt{2}} r^2\,dr$$

$$= \sin\theta \Big|_0^{\pi/2} \cdot \frac{r^3}{3}\Big|_1^{\sqrt{2}}$$

$$= \boxed{\frac{(\sqrt{2})^3}{3} - \frac{1}{3}}$$

⑥ $\iiint_E z\,dV$ E: $y=0$, $z=0$, $x+y=2$, in $y^2+z^2=1$ if first octant

$$= \int_0^{\pi/2}\int_0^1\int_0^{2-y} z\,dx\,r\,dr\,d\theta = \int_0^{\pi/2}\int_0^1\int_0^{2-y} r^2\sin\theta\,dx\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^1 (2-r\cos\theta)\,r^2\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{2}{3}\sin\theta - \frac{1}{4}\cos\theta\sin\theta\,d\theta$$

$$= -\frac{2}{3}\cos\theta - \frac{1}{4}\frac{\sin^2\theta}{2}\Big|_0^{\pi/2}$$

$$= \boxed{-\frac{1}{8} + \frac{2}{3}}$$