Compare the behavior, as $(x,y) \to (0,0)$

$$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2} \quad \text{and} \quad g(x,y) = \frac{x^2-y^2}{x^2+y^2}$$

See Table 1 and 2 $f(x,y) \to 1$ and $g(x,y) \to$ DNE

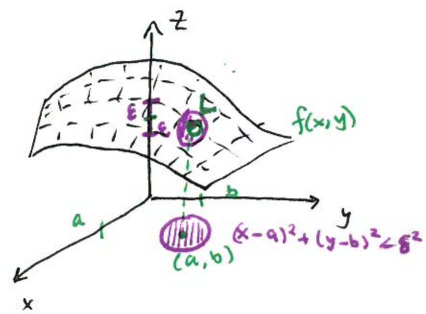
We say $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 1$ and $\lim\limits_{(x,y)\to(0,0)} g(x,y) = \text{DNE}$

- Let f be a function of two variables whose domain includes points arbitrarily close to (a,b). We say

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t. if $(x,y) \in D$ and $0 < (x-a)^2 + (y-b)^2 < \delta^2$

then $|f(x,y) - L| < \varepsilon$.



★ The idea is that no matter what path on $f(x,y)$ is taken towards $(x,y) \to (a,b)$, must have $f(x,y) \to L$.

★ If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ on path C_1 but $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ on path C_2 and $L_1 \neq L_2$ then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ DNE.

Ex 1 Show that $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ DNE.

C_1: $(x,y) \to (0,0)$ along x-axis : $y = 0$ so $f(x,0) = \frac{x^2}{x^2} = 1$ for $x \neq 0$

So $f(x,y) \to 1$ on C_1

C_2: $(x,y) \to (0,0)$ along y-axis : $x = 0$ so $f(0,y) = \frac{-y^2}{y^2} = -1$ for $y \neq 0$

So $f(x,y) \to -1$ on C_2

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x)$ DNE.

Ex 3 $f(x,y) = \frac{xy^2}{x^2+y^4}$, does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist? Paths on x, y-axis $f(x,y) \to 0$

C_m: $(x,y) \to (0,0)$ along $y = mx$: $f(x,mx) = \frac{x^3m^2}{x^2+m^4x^4} = \frac{xm^2}{1+m^4x^2}$ for $x \neq 0$ $f(x,y) \to 0$

C : $(x,y) \to (0,0)$ along $x = y^2$: $f(y^2,y) = \frac{y^4}{y^4+y^4} = \frac{1}{2}$ for $x \neq 0$ $f(x,y) \to \frac{1}{2}$

Thus $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \text{DNE}$.

$\boxed{\text{Ex 4}}$ Find $\lim\limits_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ if it exists. Similar to Ex 3 but C: $f(x,y)\to 0 = L$

Let $\varepsilon > 0$. Need to find $\delta > 0$ so that if

$$0 < \sqrt{x^2+y^2} < \delta \text{ then } \left|\frac{3x^2y}{x^2+y^2} - 0\right| < \varepsilon \qquad \frac{3x^2|y|}{x^2+y^2} < \varepsilon$$

Since $x^2 \leq x^2+y^2 \Rightarrow \frac{x^2}{x^2+y^2} \leq 1 \Rightarrow \frac{3x^2|y|}{x^2+y^2} \leq 3|y| \leq 3\sqrt{y^2} \leq 3\sqrt{y^2+x^2} < \varepsilon$

Let $\delta = \varepsilon/3$ then for all $(x,y) \in D$ with $0 < x^2+y^2 < \delta^2$ then

$$\left|\frac{3x^2}{x^2+y^2} - 0\right| \leq 3|y| \leq 3\sqrt{y^2+x^2} < \varepsilon \quad \blacksquare$$

Hence $\lim\limits_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$.

- A function f of two variables is continuous at (a,b) if $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.
  We say f is continuous on D if f is continuous at every point (a,b) in D.

Theorem 2: $\lim\limits_{(x,y)\to(a,b)} x = a \qquad \lim\limits_{(x,y)\to(a,b)} y = b \qquad \lim\limits_{(x,y)\to(a,b)} c = c$

↳ All polynomials of two variables are continuous.

$\boxed{\text{Ex 5}}$ $\lim\limits_{(x,y)\to(1,2)} x^2y^3 - x^3y^2 + 3x + 2y = (1)^2(2)^3 - (1)^3(2)^2 + 3(1) + 2(2) = \boxed{11}$

$\boxed{\text{Ex 8}}$ $f(x,y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ Is $f(x,y)$ continuous?

Only issue is when $x^2+y^2=0$ which is $(x,y)=(0,0)$.

But by Ex 4 $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$ so f is continuous.

- If f is defined on $D \subset \mathbb{R}^n$ then $\lim\limits_{\vec{x}\to\vec{a}} f(\vec{x}) = L$ means $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t. if $\vec{x} \in D$ and $0 < |\vec{x} - \vec{a}| < \delta$ then $|f(\vec{x}) - L| < \varepsilon$.

- f is continuous at $\vec{x_0}$ if $\lim\limits_{\vec{x}\to\vec{x_0}} f(\vec{x}) = f(\vec{x_0})$.