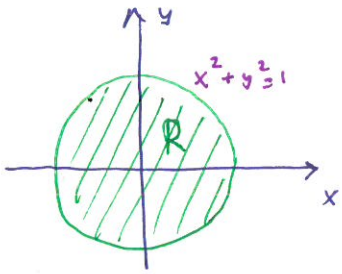
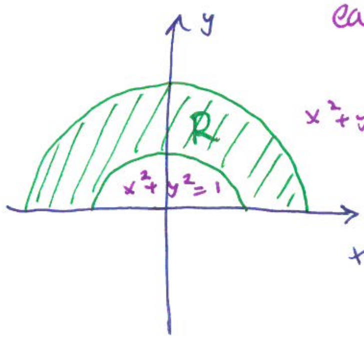
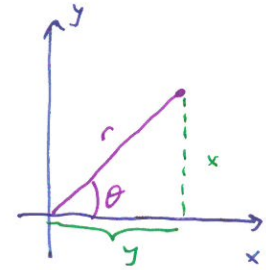
Section 15.4 - Double Integrals in Polar Coordinates

Regions that are circular in nature - hard to describe in Cartesian Coords but easy in Polar Coords.



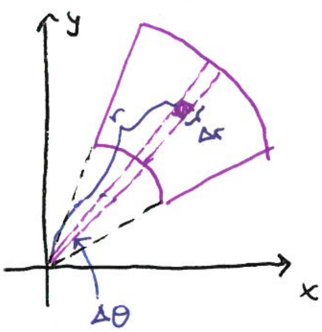


$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ x^2+y^2 &= r^2 \end{aligned}}$$



$R = \{(r,\theta) \mid |r| \le 1,\ 0 \le \theta \le 2\pi\}$ $\qquad$ $R = \{(r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$

Called a polar rectangle



$dA = dx\,dy$ $\qquad$ $dw = r \cdot d\theta$

$\Delta A = \text{width} \times \Delta r$ $\qquad$ $\text{width} = 2\pi r \cdot \dfrac{d\theta}{2\pi} = r\,d\theta$

$$\boxed{dA = r\,dr\,d\theta}$$

Change to Polar Coords in a Double Integral:

- f continuous on a polar rectangle R

$R = \{(r,\theta) \mid 0 \le a \le r \le b,\ \alpha \le \theta \le \beta\}$ where $0 \le \beta - \alpha \le 2\pi$

Then,
$$\boxed{\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

Ex 1 Evaluate $\iint_R (3x+4y^2)\,dA$, where R is the region in the upper half-plane bounded by $x^2+y^2=1$ and $x^2+y^2=4$.

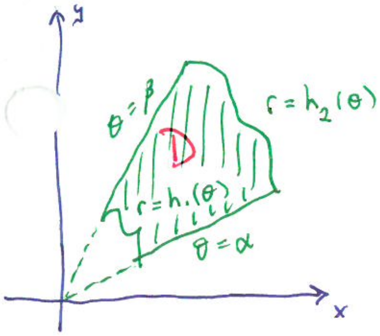
$$\iint_R (3x+4y^2)\,dA = \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\,dr\,d\theta$$

$$= \int_0^\pi \left(r^3\cos\theta + r^4\sin^2\theta \right)\Big|_1^2 \,d\theta$$

$$= \int_0^\pi (7\cos\theta + 15\sin^2\theta)\,d\theta$$

$$= \overset{0}{\cancel{7\sin\theta}} + \int_0^\pi \frac{15}{2}(1-\cos 2\theta)\,d\theta$$

$$= \left(\frac{15}{2}\theta - \frac{15}{2}\overset{0}{\cancel{\frac{\sin(2\theta)}{2}}} \right)\Big|_0^\pi = \boxed{\frac{15}{2}\pi}$$



- f continuous on a polar region

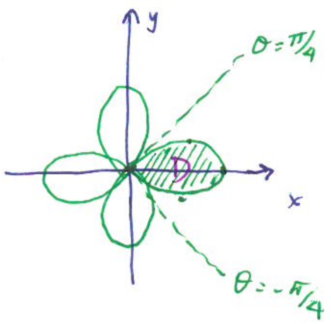
$$D=\{(r,\theta)\mid \alpha\le\theta\le\beta,\ h_1(\theta)\le r\le h_2(\theta)\}$$

then
$$\iint_D f(x,y)\,dA=\int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

- $f(x,y)=1$, $h_1(\theta)=0$, $h_2(\theta)=h(\theta)$ then get the area of D:

$$A(D)=\int_\alpha^\beta \frac{1}{2}[h(\theta)]^2\,d\theta=\int_\alpha^\beta\int_0^{h(\theta)} r\cdot 1\,dr\,d\theta$$

$\boxed{\text{Ex 3}}$ Use a double integral to find the area enclosed by one loop of the four-leaved rose curve $r=\cos 2\theta$.



$$A(D)=\int_{-\pi/4}^{\pi/4}\int_0^{\cos 2\theta} 1\cdot r\,dr\,d\theta=\int_{-\pi/4}^{\pi/4}\frac{1}{2}(\cos 2\theta)^2\,d\theta$$

$$=2\cdot\frac{1}{2}\int_0^{\pi/4}\frac{1}{2}(1+\cos 4\theta)\,d\theta$$

$$=\frac{1}{2}\left[\theta+\frac{\sin 4\theta}{4}\right]\Big|_0^{\pi/4}=\boxed{\frac{\pi}{8}}$$

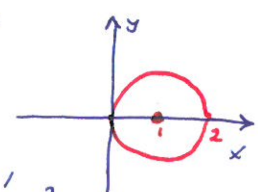
$\boxed{\text{Ex 4}}$ Find the volume of the solid that lies under $z=x^2+y^2$ above the xy-plane, and inside the cylinder $x^2+y^2=2x$.

$(x-1)^2+y^2=1 \qquad 0\le x\le 2 \qquad -1\le y\le 1$

$$r^2\cos^2\theta-2r\cos\theta+\cancel{1}+r^2\sin^2\theta=\cancel{1}$$

$$2r\cos\theta=r^2$$

$$r=2\cos\theta$$



$D=\{(r,\theta)\mid -\pi/2\le\theta\le\pi/2,\ 0\le r\le 2\cos\theta\}$

$$V=\iint_D (z)\,dA=\iint_D (x^2+y^2)\,dA$$

$$=\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3\,dr\,d\theta=\int_{-\pi/2}^{\pi/2}\frac{r^4}{4}\Big|_0^{2\cos\theta}d\theta$$

$$=\int_{-\pi/2}^{\pi/2}4\cos^4\theta\,d\theta=2\int_0^{\pi/2}4\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$=2\int_0^{\pi/2}(1+2\cos 2\theta+\cos^2 2\theta)\,d\theta$$

$$=\left(2\theta+\frac{8\cancel{\sin 2\theta}}{2}\right)\Big|_0^{\pi/2}+\int_0^{\pi/2}(1+\cos 4\theta)\,d\theta$$

$$=\pi+\left(\theta+\frac{\cancel{\sin 4\theta}}{4}\right)\Big|_0^{\pi/2}=\boxed{\frac{3}{2}\pi}$$