Section 13.4 - Motion in Space

- tangent, normal vectors and curvature used in physics

Particle moving in space with position $\vec{r}(t)$,

$$\frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$ approx. direction it moves

Velocity Vector $\vec{v}(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$ = tangent vector at t

Speed $|\vec{v}(t)| = |\vec{r}\,'(t)|$

Acceleration vector $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$

Example 3 A moving particle starts at an initial position $\vec{r}(0) = \langle 0,0,0 \rangle$ with initial velocity $\vec{v}(0) = i - j + k$. It's acceleration is $\vec{a}(t) = 4ti + 6tj + k$. Find its velocity and position at time t.

$\vec{a}(t) = \vec{v}\,'(t)$ so $\vec{v}(t) = \int \vec{a}(t)\,dt$

$= 2t^2 i + 3t^2 j + tk + C$

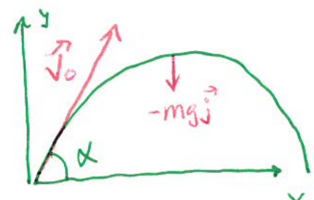
$C = \vec{v}(0) = \langle 1, -1, 1 \rangle$ so $\boxed{\vec{v}(t) = (2t^2+1)i + (3t^2-1)j + (t+1)k}$

$\vec{r}(t) = \int \vec{v}(t)\,dt = \left(\frac{2}{3}t^3 + t\right)i + (t^3 - t)j + (t^2/2 + t)k + C$

$C = \vec{r}(0) = \langle 0,0,0 \rangle$ so $\boxed{\vec{r}(t) = \left(\frac{2}{3}t^3 + t\right)i + (t^3 - t)j + \left(\frac{t^2}{2} + t\right)k}$

Newton's Second Law of Motion: $\vec{F}(t) = m\vec{a}(t)$ (force $\vec{F}$ acting on object of mass m producing acceleration $\vec{a}$.)

Example 5 A projectile is fired with angle of elevation α and initial velocity $\vec{v}_0$. Assume air resistance is negligible and only external force is due to gravity. Find $\vec{r}(t)$ and α that maximizes horizontal range.



$\vec{F} = m\vec{a} = -mg\vec{j} \Rightarrow \vec{a} = -g\vec{j}$

$\vec{v}(t) = -gt\vec{j} + \vec{v}_0$ $\quad \vec{r}(t) = -\frac{1}{2}gt^2\vec{j} + t\vec{v}_0$ since $\vec{r}(0) = \vec{0}$

$\vec{v}_0 = |\vec{v}_0|\cos\alpha\, i + |\vec{v}_0|\sin\alpha\, j$ So parametric Equations:

$\boxed{x = v_0 \cos\alpha\, t}$ $\quad \boxed{y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2}$

Horizontal distance: $y = 0$ $\quad t = \frac{2v_0 \sin\alpha}{g}$ $\quad d = v_0\cos\alpha \cdot \frac{2v_0\sin\alpha}{g} = \frac{v_0^2 \sin 2\alpha}{g}$ $\quad \alpha = \frac{\pi}{4}$

Tangential and Normal Componets of Acceleration

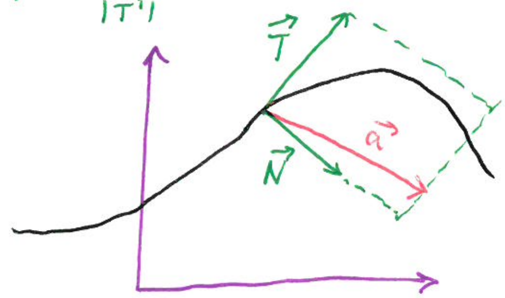
- useful to resolve acceleration into two components, direction of $\vec{T}$ and $\vec{N}$

$\vec{T} = \frac{\vec{r}'}{|\vec{r}'|} = \frac{\vec{v}}{v} \Rightarrow \vec{v} = v\vec{T}$ so $\vec{a} = \vec{v}' = v'\vec{T} + v\vec{T}'$

$K = \frac{|\vec{T}'|}{|\vec{r}'|} = \frac{|\vec{T}'|}{v}$ so $|\vec{T}'| = vK$ and $\vec{N} = \frac{\vec{T}'}{|\vec{T}'|}$

so $\vec{T}' = |\vec{T}'|\vec{N} = vK\vec{N}$ thus

$$\boxed{\vec{a} = v'\vec{T} + v^2K\vec{N}}$$



Example 7 A particle moves with position $\vec{r}(t) = \langle t^2, t^2, t^3 \rangle$
Find the tangential and normal components of acceleration.

$\vec{v} = \langle 2t, 2t, 3t^2 \rangle \qquad v = |\vec{v}| = \sqrt{8t^2 + 9t^4}$

$\vec{a} = \langle 2, 2, 6t \rangle \qquad v' = \frac{1}{2}(8t^2 + 9t^4)^{-1/2} \cdot (16t + 36t^3)$

$K = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{|\langle 6t^2, -6t^2, 0 \rangle|}{v^3} = \frac{6\sqrt{2}t^2}{v^3}$

$\vec{a} = v'\vec{T} + \frac{6\sqrt{2}t^2}{v}\vec{N}$

$$\boxed{a_T = \frac{8t + 18t^3}{\sqrt{8t^2 + 9t^4}} \qquad a_N = \frac{6\sqrt{2}t^2}{\sqrt{8t^2 + 9t^4}}}$$

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.

2. The line joining the sun to a planet sweeps out equal areas in equal times.

3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

Proof of first Law:

Sun at origin

Consequences of Newton's Laws:

- Second Law of motion: $\vec{F} = m\vec{a}$
- Law of Gravitation: $\vec{F} = -\frac{GMm}{r^3}\vec{r}$

$\vec{F}$: force on planet
m: mass of planet
M: mass of sun
G: Gravitational Constant
$\vec{r}$: position vector of planet
$r = |\vec{r}|$

Section 13.4 – Motion in Space

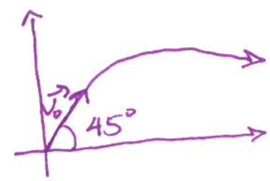
Vector Calc

$\boxed{\#22}$ Show that if a particle moves with constant speed, then the velocity and acceleration vectors are orthogonal.

$|\vec{r}\,'(t)| = |\vec{v}(t)| = C$ WTS: $\vec{v}(t) \cdot \vec{a}(t) = 0$ or $\vec{r}\,'(t) \cdot \vec{r}\,''(t) = 0$

$$0 = \frac{d}{dt}\left(\vec{v}(t) \cdot \vec{v}(t)\right) = 2\vec{v}\,'(t) \cdot \vec{v}(t) = 2v(t) \cdot \vec{a}(t) \quad \blacksquare$$

$\boxed{\#25}$ A ball is thrown at an angle of 45° to the ground. If the ball lands 90m away, what was the initial speed?



Find $|\vec{v_0}|$.

$$\vec{r}(t) = \left\langle |\vec{v_0}| \cos 45° t,\ |\vec{v_0}| \sin 45° t - \frac{gt^2}{2} \right\rangle$$

① $90 = |\vec{v_0}| \frac{\sqrt{2}}{2} t$ ② $0 = \left(|\vec{v_0}| \frac{\sqrt{2}}{2} - \frac{gt}{2}\right) t$

$$t = \frac{|\vec{v_0}|\sqrt{2}}{g}$$

$$90 = \frac{|\vec{v_0}|^2}{g} \Rightarrow \boxed{|\vec{v_0}| = \sqrt{90/g}}$$

$\boxed{\#45}$ The position of a spaceship is

$$\vec{r}(t) = (3+t)\vec{i} + (2+\ln t)\vec{j} + \left(7 - \frac{4}{t^2+1}\right)\vec{k}$$

and the coordinates of a space station are (6, 4, 9). The Captain wants the spaceship lined up with the space station so it can coast in with the engines off. When should he turn the engines off?

Direction + Distance in the direction of velocity = ⟨6, 4, 9⟩

$$\vec{r}(t) + s \cdot \vec{v}(t) = \langle 6, 4, 9 \rangle$$

$$3 + t + s = 6, \quad 2 + \ln t + \frac{s}{t} = 4, \quad 7 - \frac{4}{t^2+1} + \frac{4s \cdot 2t}{(t^2+1)^2} = 9$$

$s = 3 - t$

$2 + \ln t + \frac{3-t}{t} = 4$

Possible rational roots: ±1, ±3

$\boxed{t=1}$ works and

$2 + \ln(1) + \frac{3-1}{2} = 4$ ✓

$7(t^2+7)^2 - 4(t^2+1) + (12-4t)(2t) = 9(t^2+1)^2$

$-4t^2 - 4 + 24t - 8t^2 = 2t^4 + 4t^2 + 2$

$2t^4 + 16t^2 - 24t + 6 = 0$

$t^4 + 8t^2 - 12t + 3 = 0$