Section 14.3 - Partial Derivatives

For a function of two variables x and y, vary x while keeping y fixed say $y=b$
so we are considering a function of a single variable x

$$g(x) = f(x,b)$$

If g has a derivative at $x=a$ then we call it the partial derivative of f with respect to x at (a,b):

$$f_x(a,b) = g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

Thus $$\boxed{f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}}.$$

Similarly, the partial derivative of f wrt y at (a,b) is

$$\boxed{f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}}$$

Notation for Partial Derivatives: $z = f(x,y)$

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

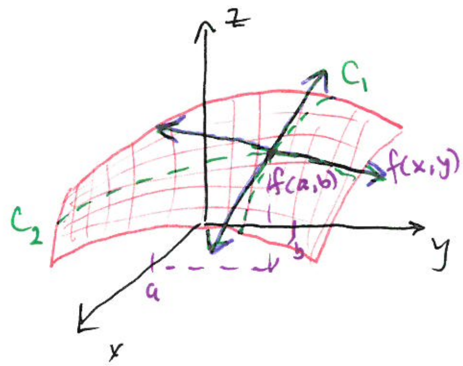
$$f_y(x,y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

$\boxed{\text{Ex.1}}$ If $f(x,y) = x^3 + x^2y^3 - 2y^2$ find $f_x(2,1)$ and $f_y(2,1)$

$f_x(x,y) = 3x^2 + 2xy^3$ $\quad f_x(2,1) = 3\cdot 2^2 + 2\cdot 2\cdot 1^3 = \boxed{16}$

$f_y(x,y) = 3x^2y^2 - 4y$ $\quad f_y(2,1) = 3\cdot 2^2\cdot 1^2 - 4\cdot 1 = \boxed{8}$

Interpretation:



$f_x(x,y)$ Slope of tangent line to C_1

$f_y(x,y)$ Slope of tangent line to C_2

$\boxed{\text{Ex 4}}$ Find $\partial z/\partial x$ and $\partial z/\partial y$ for $x^3 + y^3 + z^3 + 6xyz = 1$

$$3x^2 + 0 + 3z^2 \cdot \partial z/\partial x + 6yz + 6xy\, \partial z/\partial x = 0$$

$$\boxed{\frac{\partial z}{\partial x} = -\frac{x^2 + 2yz}{z^2 + 2xy}} \qquad \boxed{\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}}$$

Have partial derivatives for functions of more than two variables in the same way.

$\boxed{\text{Ex 5}}$ Find f_x, f_y, f_z if $f(x,y,z) = e^{xy} \ln z$

$$f_x = ye^{xy}\ln z \quad f_y = xe^{xy}\ln z \quad f_z = e^{xy}\cdot\frac{1}{z}$$

Higher Order Derivatives: Partial derivatives are functions of multiple variables so they have partial derivatives called Second partial derivatives.

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial^2 f}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial^2 f}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial^2 f}{\partial^2 y}$$

$\boxed{\text{Ex 6}}$ Find the second partial derivatives of $f(x,y) = x^3 + x^2y^3 - 2y^2$

$$f_x = 3x^2 + 2xy^3 \quad f_y = 3x^2y^2 - 4y$$

$$f_{xx} = 6x + 2y^3 \qquad f_{yy} = 6x^2y - 4$$

$$f_{xy} = 6xy^2 \qquad f_{yx} = 6xy^2$$

Clairaut's Theorem

Suppose f is defined on a disk D containing (a,b). If f_{xy} and f_{yx} are both continuous on D then $\boxed{f_{xy} = f_{yx}}$

Partial Differential Equations: equation with partial derivatives, like

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{[Laplace's Equation]}$$

Solutions are called harmonic functions → heat conduction, fluid flow, electric potential

$\boxed{\text{Ex 8}}$ Show $u = f(x,y) = e^x \sin y$ is a solution of the Laplace's Equation.

$$\frac{\partial u}{\partial x} = e^x \sin y \quad \frac{\partial^2 u}{\partial x^2} = e^x \sin y \quad \frac{\partial u}{\partial y} = e^x \cos y \quad \frac{\partial^2 u}{\partial y^2} = e^x(-\sin y)$$

$$\text{So} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \sin y - e^x \sin y = 0 \quad ✓$$