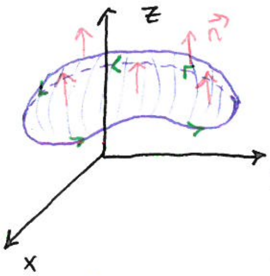
- Higher dimension version of Green's Theorem

Green's Theorem - relating double integral to line integral over boundary of D

Stoke's Theorem - relating surface integral to line integral over boundary of S

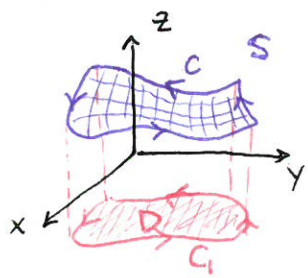
* The orientation of S induces the positive orientation of its boundary curve C.



★ Walking in the positive direction around C (head in the direction of $\vec{n}$) then S is always on your left.

Stoke's Theorem - S oriented piecewise-smooth surface, bounded by a simple, closed, piecewise-smooth curve C with positive orientation $\vec{F}$ a vector field, Components have continuous partials on $\mathbb{R}^3$ open region containing S then

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}}$$



Proof (Special case): $S: z = g(x,y) \quad (x,y)\in D$

D simple plane region with boundary C_1 corresponding to the boundary of S, C.

$$\vec{F} = \langle P, Q, R\rangle \qquad \operatorname{Curl}\vec{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

$$\iint_S \operatorname{Curl}\vec{F}\cdot d\vec{S} = \iint_D \operatorname{Curl}\vec{F}\cdot \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right\rangle dA$$

$$= \iint_D \underline{\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right)\frac{\partial z}{\partial x}} + \underline{\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\frac{\partial z}{\partial y}} + \underline{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)} dA$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \langle P, Q, R\rangle \cdot \langle dx, dy, dz\rangle$$

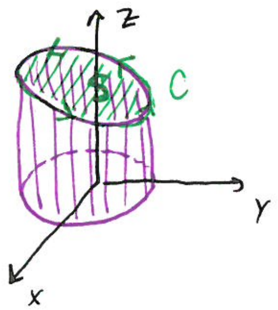
$$= \int_C P\,dx + Q\,dy + R\,dz = \int_C P\,dx + Q\,dy + R\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right)$$

$$= \int_C \left(P + \frac{\partial z}{\partial x}R\right)dx + \left(Q + \frac{\partial z}{\partial y}R\right)dy$$

[Green's Theorem] $$= \iint_D \frac{\partial}{\partial x}\left(Q + \frac{\partial z}{\partial y}R\right) - \frac{\partial}{\partial y}\left(P + \frac{\partial z}{\partial x}R\right) dA$$

$$= \iint_D \underline{\frac{\partial Q}{\partial x}} + \cancel{\frac{\partial^2 z}{\partial x\partial y}R} + \underline{\frac{\partial z}{\partial y}\frac{\partial R}{\partial x}} + \underline{\frac{\partial Q}{\partial z}\cdot\frac{\partial z}{\partial x}} + \cancel{\frac{\partial z}{\partial y}\frac{\partial R}{\partial z}\frac{\partial z}{\partial x}} - \underline{\frac{\partial P}{\partial y}} - \underline{\frac{\partial P}{\partial z}\frac{\partial z}{\partial y}} - \cancel{\frac{\partial^2 z}{\partial x\partial y}R} - \underline{\frac{\partial z}{\partial x}\frac{\partial R}{\partial y}} - \cancel{\frac{\partial z}{\partial x}\frac{\partial R}{\partial z}\frac{\partial z}{\partial y}}$$

Ex. 1 Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = \langle -y^2, x, z^2\rangle$ and C is the curve of intersection of $y+z=2$ and $x^2+y^2=1$ (Orient C to be counterclockwise when viewed from above)



$S: z = 2-y$

$\frac{\partial z}{\partial x} = 0 \quad \frac{\partial z}{\partial y} = -1$

$D: \begin{array}{ll} x = r\cos\theta & y = r\sin\theta \\ 0 \le r \le 1 & 0 \le \theta \le 2\pi \end{array}$

$\text{Curl}\,\vec{F} = \langle 0, 0, 1+2y\rangle$

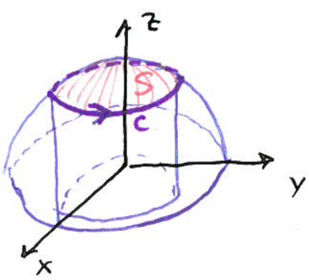
$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \langle 0,0,1+2y\rangle \cdot d\vec{S}$$

$$= \iint_D 1+2y\, dA$$

$$= \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} + \frac{2}{3}\sin\theta\, d\theta = \boxed{\pi}$$

Ex 2 Use Stoke's Theorem to compute the integral $\iint_S \text{Curl}\,\vec{F}\cdot d\vec{S}$ where $\vec{F} = \langle xz, yz, xy\rangle$, $S: x^2+y^2+z^2=4$ and $x^2+y^2=1$ above xy-plane.



$$\iint_S \text{Curl}\,\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle \sqrt{3}\cos\theta, \sqrt{3}\sin\theta, \cos\theta\sin\theta\rangle \cdot d\vec{r}$$

$\vec{r}(\theta) = \langle \cos\theta, \sin\theta, \sqrt{3}\rangle \quad 0 \le \theta \le 2\pi$

$$= \int_0^{2\pi} \langle \sqrt{3}\cos\theta, \sqrt{3}\sin\theta, \cos\theta\sin\theta\rangle \cdot \langle -\sin\theta, \cos\theta, 0\rangle\, d\theta$$

$$= \int_0^{2\pi} 0\, d\theta = \boxed{0}$$

$\vec{v}\cdot\vec{T}$ **Component of $\vec{v}$ in the direction of the unit tangent $\vec{T}$**

$$\int_C \vec{v}\cdot d\vec{r} = \int_C \vec{v}\cdot\vec{T}\, ds$$

Closer $\vec{v}$ is to the direction of $\vec{T}$, $\vec{v}\cdot\vec{T}$ is larger so $\int_C \vec{v}\cdot d\vec{r}$ measures the tendency of the fluid to move around C, called the Circulation of $\vec{v}$ around C.