Section 16.5 - Curl and divergence

Vector Calc

★ Two operations of vector fields – used in fluid flow, electricity, magnetism

Curl: $\vec{F} = \langle P, Q, R \rangle$ on $\mathbb{R}^3$ with partials of P, Q, R existing then

$$\boxed{\text{Curl } \vec{F} = \nabla \times \vec{F}} \quad \text{where } \nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

Curl vector associated to rotation
Particle rotates about axis in direction of $\text{Curl}(\vec{F})$

Resembles differentiation, gives a vector field

Theorem 3 If f is a function of 3 variables with continuous second order partial derivatives then $\text{Curl}(\nabla f) = \vec{0}$.

Proof: $\text{Curl}(\nabla f) = \nabla \times (\nabla f) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\vec{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\vec{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\vec{k}$

$= \vec{0}$ by Clairaut's Theorem ■

Since a conservative vector field has $\vec{F} = \nabla f$ then

★ If $\vec{F}$ is conservative then $\text{Curl}(\vec{F}) = 0$.

Get a partial converse of this:

Theorem 4 (partial converse of Thm 3) If $\vec{F}$ is defined on all $\mathbb{R}^3$, components have continuous partials and $\text{Curl}(\vec{F}) = 0$ then $\vec{F}$ is conservative.

Also say $\vec{F}$ is irrotational.

Ex 3 a) Show $\vec{F} = \langle y^2 z^3, 2xyz^3, 3xy^2 z^2 \rangle$ is conservative

b) Find f so that $\nabla f = \vec{F}$.

a) $\text{Curl } \vec{F} = (6xyz^2 - 6xyz^2)\vec{i} + (3y^2z^2 - 3y^2z^2)\vec{j} + (2yz^3 - 2yz^3)\vec{k} = 0$ ✓

b) ① $f_x = y^2 z^3 \Rightarrow f(x,y,z) = \underline{y^2 z^3 x} + \cancel{g_1(y,z)}^{0} + \cancel{h_1(y)}^{0} + \cancel{k_1(z)}^{0} + C$

② $f_y = 2xyz^3 \Rightarrow f(x,y,z) = \underline{xy^2 z^3} + \cancel{g_2(x,z)}^{0} + \cancel{h_2(x)}^{0} + \cancel{k_2(z)}^{0} + C$

③ $f_z = 3xy^2 z^2 \Rightarrow f(x,y,z) = \underline{xy^2 z^3} + \cancel{g_3(x,y)}^{0} + \cancel{h_3(x)}^{0} + \cancel{k_3(y)}^{0} + C$

$$\boxed{f(x,y,z) = xy^2 z^3 + C}$$

Divergence: $\vec{F} = \langle P, Q, R \rangle$ on $\mathbb{R}^3$ with partials existing then

$$\text{div}(\vec{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \boxed{\nabla \cdot \vec{F}}$$

Resembles differentiation, gives a scalar field

Section 16.5 - Curl and Divergence

Theorem 11 $\vec{F} = \langle P, Q, R \rangle$ on $\mathbb{R}^3$, P, Q, R have continuous second partials then

$$\text{div curl } \vec{F} = 0$$

properties of cross product

Proof: $\text{div curl}(\vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = (\nabla \times \nabla) \cdot \vec{F} = \vec{0} \cdot \vec{F} = 0$ ∎

Example 5 Show $\vec{F} = \langle xz, xyz, -y^2 \rangle$ can't be written as the curl of another vector field that is $\vec{F} \neq \text{curl } \vec{G}$.

$\text{div}(\vec{F}) = z + xy + 0 \neq 0$ for all $\mathbb{R}^3$

but $\text{div}(\text{curl } \vec{G}) = 0$ by theorem 11 ⨯⨯.

★ divergence of $\vec{F}$ velocity of a fluid measures the tendency of the fluid to diverge from a point (x, y, z). If $\text{div } \vec{F} = 0$ then $\vec{F}$ is incompressible.

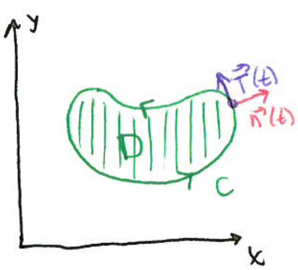
Laplace Operator: $\text{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \underbrace{\nabla^2 f}_{\text{abbreviation}}$

Laplace Equation: $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$

$\nabla^2 \vec{F} = \langle \nabla^2 P, \nabla^2 Q, \nabla^2 R \rangle$

Green's Theorem (rewritten) $F = \langle P, Q \rangle$ $\text{Curl}(\vec{F}) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \iint_D \text{Curl}(\vec{F}) \cdot \vec{k} \, dA}$$



$\boxed{\oint_C \vec{F} \cdot \vec{n} \, ds} = \int_a^b (\vec{F} \cdot \vec{n})(t) |\vec{r}\,'(t)| \, dt$

$\vec{T} = \langle x', y' \rangle / |r'(t)|$

$\vec{n} = \langle y', -x' \rangle / |r'(t)|$

$= \int_a^b P(x,y) \cdot y' \, dt - Q(x,y) \, x' \, dt$

$= \int_C P \, dy - Q \, dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$

$= \boxed{\iint_D \text{div}(\vec{F}) \, dA}$