Updated Derivation of Coupled LLE-GPE for a BEC in a Driven Cavity: Special Case of Plane Waves and Homogeneous BEC

Adapted from original derivation

May 27, 2025

Contents

1 Derivation of Coupled LLE-GPE for a BEC in a Driven Cavity

1.1 Introduction and Physical System

This document outlines the derivation of the coupled equations describing the dynamics of a Bose-Einstein Condensate (BEC) interacting with an optical field within a driven optical cavity. The model is based on the setup described in "Cavity Theory.pdf" [cite: 1, 7], where a far red-detuned optical pump drives a unidirectional ring cavity containing a BEC. The interaction between the light and atoms is dispersive. This version of the derivation specifically considers the simplification arising from assuming a plane wave optical field and a transversely homogeneous BEC.

1.2 Starting Point: General Coupled Equations

The general goal is to derive a set of coupled partial differential equations. For systems with transverse spatial variations, these are (from "Cavity Theory.pdf" [cite: 8] and "Model Handbook (1).pdf (Sec. 10.1)" [cite: 305]):

For the BEC field ψ:

$$\partial_\tau \psi = \frac{\alpha_\psi}{\kappa}\left[i\nabla_\perp^2\psi - i(s|F|^2 - 2\beta_{dd}|F|^2|\psi|^2 + \beta_{col}|\psi|^2 - iL_3|\psi|^4)\psi\right] \tag{1}$$

For the optical field F:

$$\partial_\tau F = -(1+i\theta)F + i\alpha_F \nabla_\perp^2 F - i\frac{2L}{Tk_L w_s^2}(s|\psi|^2 - \beta_{dd}|\psi|^4)F + F_P \tag{2}$$

Here, $\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the transverse Laplacian.

1.3 Evolution Equation for the Optical Field F (LLE-type)

The evolution equation for the intracavity optical field F is derived by considering the cavity dynamics under the mean-field approximation and the interaction with the BEC.

1.3.1 Cavity Dynamics and Mean-Field Approach

For a high-finesse optical cavity, the mean-field approximation allows us to describe the evolution of the slowly varying envelope of the optical field. Over one round trip time $t_R = \mathcal{L}/c$, the field undergoes changes due to:

- Cavity losses (contributing $-F$ in normalized form).
- External pumping, F_P.
- Cavity detuning, θ (contributing $-i\theta F$).
- Diffraction, represented by the transverse Laplacian $\nabla_\perp^2 F$ scaled by a coefficient α_F. This term arises from the paraxial wave equation.

Combining these effects in a normalized form (scaled time $\tau = (cT)/(2\mathcal{L})t_{lab}$) leads to the standard LLE backbone.

1.3.2 Simplification for Plane Wave Optical Field

In the special case of a **plane wave optical field**, the field amplitude F does not vary in the transverse plane (x, y). Therefore, the transverse Laplacian of the optical field envelope F is zero:

$$\nabla_\perp^2 F = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) F = 0 \tag{3}$$

As a consequence, the diffraction term $i\alpha_F \nabla_\perp^2 F$ in Equation (2) vanishes.

1.3.3 Interaction with BEC

The BEC acts as a nonlinear optical medium. The term $-i\frac{2L}{Tk_L w_s^2}(s|\psi|^2 - \beta_{dd}|\psi|^4)F$ in Eq. (2) accounts for the phase shift induced by the BEC.

1.3.4 Final Equation for Optical Field (Plane Wave Case)

Combining the cavity dynamics with the BEC interaction term, and applying the plane wave simplification ($\nabla_\perp^2 F = 0$), yields the Lugiato-Lefever type equation for the optical field F:

$$\partial_\tau F = -(1+i\theta)F - i\frac{2L}{Tk_L w_s^2}(s|\psi|^2 - \beta_{dd}|\psi|^4)F + F_P \tag{4}$$

where $\beta_F = \frac{2L}{Tk_L w_s^2}$ [cite: 313].

1.4 Evolution Equation for the BEC Field ψ (GPE-type)

The evolution of the BEC wavefunction is described by a Gross-Pitaevskii type equation, modified to include the interaction with the optical field F.

1.4.1 Gross-Pitaevskii Formalism

Standard GPE terms include:

- Kinetic energy: $i\nabla_\perp^2\psi$, arising from the BEC's spatial variation.
- Interatomic interactions: $\beta_{col}|\psi|^2\psi$.
- Three-body losses: $-iL_3|\psi|^4\psi$.

1.4.2 Simplification for Homogeneous BEC

In the special case of a **transversely homogeneous BEC**, the condensate wavefunction ψ is constant in the transverse plane (x, y). Therefore, the transverse Laplacian of ψ is also zero:

$$\nabla_\perp^2\psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi = 0 \tag{5}$$

As a consequence, the kinetic energy term $i\frac{\alpha_\psi}{\kappa}\nabla_\perp^2\psi$ in Equation (1) vanishes.

1.4.3 Interaction with Optical Field

The BEC atoms experience an optical dipole potential. The term $-i(s|F|^2 - 2\beta_{dd}|F|^2|\psi|^2)\psi$ in Eq. (1) represents this optical dipole force.

1.4.4 Temporal Scaling

The temporal dynamics of the atomic field are scaled by α_ψ/κ.

1.4.5 Final Equation for BEC Field (Homogeneous Case)

Combining these elements, and applying the homogeneous BEC simplification ($\nabla_\perp^2\psi = 0$), gives the GPE for the BEC field ψ:

$$\partial_\tau\psi = -\frac{i\alpha_\psi}{\kappa}\left(s|F|^2 - 2\beta_{dd}|F|^2|\psi|^2 + \beta_{col}|\psi|^2 - iL_3|\psi|^4\right)\psi \tag{6}$$

This can also be written as:

$$\partial_\tau\psi = \frac{\alpha_\psi}{\kappa}\left[-i(s|F|^2 - 2\beta_{dd}|F|^2|\psi|^2 + \beta_{col}|\psi|^2)\psi - L_3|\psi|^4\psi\right]$$

1.5 Full Coupled System for Plane Waves and Homogeneous BEC

The full system for a plane wave optical field and a homogeneous BEC is described by the coupled Equations (6) and (4):

$$\partial_\tau\psi = -\frac{i\alpha_\psi}{\kappa}\left(s|F|^2 - 2\beta_{dd}|F|^2|\psi|^2 + \beta_{col}|\psi|^2 - iL_3|\psi|^4\right)\psi \tag{7}$$

$$\partial_\tau F = -(1+i\theta)F - i\frac{2L}{Tk_L w_s^2}(s|\psi|^2 - \beta_{dd}|\psi|^4)F + F_P \tag{8}$$

1.6 Definition of Parameters and Scalings

The parameters are defined and scaled as follows (consistent with "Cavity Theory.pdf" and "Model Handbook (1).pdf Sec. 10.1.1" [cite: 309]):

- ψ: Complex amplitude of the BEC field.
- F: Complex amplitude of the intracavity optical field.
- F_P: Amplitude of the optical pump field.
- τ: Scaled time, $\tau = \frac{cT}{2\mathcal{L}}t_{lab}$.
- $\alpha_\psi = \hbar/(m_a w_s^2)$: Parameter related to atomic kinetic energy.

- $\kappa = (cT)/(2\mathcal{L})$: Cavity decay rate parameter.
- $s = \pm 1$: Sign of atom-light detuning.
- $\beta_{dd} = 2/(3k_L^2 w_s^2)$: Parameter for higher-order dipole-dipole effects.
- β_{col}: Parameter for interatomic scattering strength.
- L_3: Parameter for three-body losses.
- θ: Cavity detuning parameter.
- $\alpha_F = (2\mathcal{L})/(k_L w_s^2 T)$: Diffraction parameter for the optical field (Note: this term is zero in the plane wave case as $\nabla_\perp^2 F = 0$, but the definition is retained for completeness of parameter listing from the original document).
- L: Length of the BEC medium.
- T: Mirror transmittivity.
- k_L: Optical wavenumber.
- w_s: Characteristic optical beam waist for scaling.
- The coefficient $\frac{2L}{Tk_L w_s^2}$ in Eq. (4) is equivalent to β_F from "Model Handbook (1).pdf" [cite: 313].

1.7 Important Considerations for Coefficients in the Plane Wave Limit

Several coefficients in these equations, as defined in "Task_2_derivation.pdf", depend on a characteristic beam waist w_s:

- $\alpha_F = (2\mathcal{L})/(k_L w_s^2 T)$
- $\alpha_\psi = \hbar/(m_a w_s^2)$
- The interaction coefficient $\beta_F = \frac{2L}{Tk_L w_s^2}$ (explicitly present in Eq. (4)).
- $\beta_{dd} = 2/(3k_L^2 w_s^2)$

While the terms involving $\alpha_F \nabla_\perp^2 F$ and $\alpha_\psi \nabla_\perp^2 \psi$ have disappeared from the simplified equations, the definitions of β_F (or the explicit coefficient $\frac{2L}{Tk_L w_s^2}$) and β_{dd} still formally contain w_s.

For a true plane wave (which is infinite in transverse extent), the concept of a beam waist w_s is not directly applicable. In a practical scenario where a "plane wave" is an approximation for a very wide beam, w_s might be considered very large.

Therefore, when using these simplified equations for a plane wave and homogeneous BEC system, one must carefully reconsider how these coefficients are defined or scaled. The fundamental physical interaction strengths they represent will still be present, but their formulation might need to be adapted to remove the dependence on w_s or to reinterpret w_s as a relevant transverse system size or normalization length. For instance, the intensity of the plane wave would be a key parameter. The "Model Handbook" referenced in the original document might provide more general definitions or guidance for this limit.

1.8 Summary of Key Approximations (for the general case, with notes for plane wave simplification)

The derivation of the coupled LLE-GPE system relies on several approximations:

1. Mean-field approximation for the optical cavity.
2. Adiabatic elimination of atomic excited state.
3. Slowly Varying Envelope Approximation (SVEA) for the optical field.
4. Paraxial approximation for the optical field. (For plane waves, this is satisfied in the limit of an infinitely broad beam; the transverse Laplacian terms then vanish).

5. Static atomic medium.
6. Pancake-shaped BEC (longitudinally).
7. Neglect of optical absorption.
8. Neglect of optical saturation.