Optical Wave Equation Derivations

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Contents

1 Derivation of the Gaussian Beam from the Paraxial Wave Equation

This section derives the Gaussian beam solution from the Helmholtz equation using the paraxial approximation.

1.1 Start with the Helmholtz Equation

The scalar wave equation for a monochromatic wave of angular frequency ω in a homogeneous medium is the Helmholtz equation:

$$\nabla^2 U + k^2 U = 0 \tag{1}$$

where:

- $U(x, y, z)$ is the complex amplitude of the field.
- $k = \frac{2\pi}{\lambda} = \frac{\omega}{c}$ is the wavenumber.
- ∇^2 is the Laplacian operator.

1.2 Apply the Paraxial Approximation

Assume the beam primarily propagates along the z-axis. The field U is expressed as:

$$U(x, y, z) = \psi(x, y, z)e^{ikz} \tag{2}$$

where $\psi(x, y, z)$ is the complex envelope function that varies slowly with z and transversely.

1.3 Substitute into the Helmholtz Equation

Substituting Eq. (2) into Eq. (1), the Laplacian of U is:

$$\nabla^2 U = e^{ikz}\left(\nabla_\perp^2 \psi + 2ik\frac{\partial \psi}{\partial z} - k^2\psi + \frac{\partial^2 \psi}{\partial z^2}\right) \tag{3}$$

where $\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the transverse Laplacian. This leads to:

$$\nabla_\perp^2 \psi + 2ik\frac{\partial \psi}{\partial z} + \frac{\partial^2 \psi}{\partial z^2} = 0 \tag{4}$$

1.4 Neglect the Second Derivative in z

In the paraxial approximation, ψ varies slowly with z, so $\left|\frac{\partial^2 \psi}{\partial z^2}\right| \ll \left|2k\frac{\partial \psi}{\partial z}\right|$. Neglecting this term yields the paraxial wave equation:

$$\nabla_\perp^2 \psi + 2ik\frac{\partial \psi}{\partial z} = 0 \tag{5}$$

This equation is a fundamental starting point, related to the scaled paraxial wave equation $\partial_\zeta A = i\nabla_\perp^2 A$ presented in the Model Handbook for light in free space[cite: 77].

1.5 Assume a Gaussian Beam Solution

A Gaussian beam solution is assumed:

$$\psi(x, y, z) = A(z)\exp\left(-\frac{x^2 + y^2}{w^2(z)}\right)\exp(i\phi(x, y, z)) \tag{6}$$

where $A(z)$ is the amplitude, $w(z)$ is the beam radius (spot size), and $\phi(x, y, z)$ is the phase.

1.6 Substitute into the Paraxial Wave Equation and Solve

Substituting Eq. (6) into Eq. (5) and separating real and imaginary parts leads to expressions for the beam parameters:

Beam Width Evolution:

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_0}\right)^2} \tag{7}$$

where w_0 is the waist radius at $z = 0$, and $z_0 = \frac{\pi w_0^2}{\lambda} = \frac{k w_0^2}{2}$ is the Rayleigh range.

Phase Term:

$$\phi(x, y, z) = k \frac{x^2 + y^2}{2R(z)} - \arctan\left(\frac{z}{z_0}\right) \tag{8}$$

where $R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]$ is the radius of curvature of the wavefronts. The term $-\arctan(z/z_0)$ is the Gouy phase.

1.7 Final Gaussian Beam Solution

Combining these, the Gaussian beam solution (with $A(z) = A_0 \frac{w_0}{w(z)}$ for power conservation) is:

$$U(x, y, z) = A_0 \frac{w_0}{w(z)} \exp\left(-\frac{x^2 + y^2}{w^2(z)}\right) \exp\left(ikz + ik\frac{x^2 + y^2}{2R(z)} - i \arctan\left(\frac{z}{z_0}\right)\right) \tag{9}$$

This describes the propagation, spreading, wavefront curvature, and phase evolution of a Gaussian beam under the paraxial approximation.

2 Derivation of the Kerr Wave Equation from Maxwell's Equations

This section derives the wave equation for an electric field in a Kerr nonlinear medium from Maxwell's equations.

2.1 Maxwell's Equations (Source-Free, Non-Magnetic)

For no free charges ($\rho = 0$) or currents ($J = 0$), in a non-magnetic medium ($B = \mu_0 H$), Maxwell's equations are:

$$\nabla \times E = -\mu_0 \frac{\partial H}{\partial t} \tag{10}$$

$$\nabla \times H = \frac{\partial D}{\partial t} \tag{11}$$

$$\nabla \cdot D = 0 \tag{12}$$

$$\nabla \cdot B = 0 \tag{13}$$

2.2 Kerr Nonlinear Constitutive Relation

The electric displacement D includes linear and cubic (Kerr) polarization:

$$D = \epsilon_0 E + P_L + P_{NL} = \epsilon_0(1 + \chi^{(1)})E + \epsilon_0 \chi^{(3)} |E|^2 E \tag{14}$$

where $\chi^{(1)}$ is the linear susceptibility and $\chi^{(3)}$ is the third-order Kerr susceptibility. Let $n_0^2 = 1 + \chi^{(1)}$ be the square of the linear refractive index. Then:

$$D = \epsilon_0 n_0^2 E + \epsilon_0 \chi^{(3)} |E|^2 E \tag{15}$$

2.3 Deriving the Wave Equation

Taking the curl of Faraday's law (Eq. 10) and substituting Ampère-Maxwell's law (Eq. 11):

$$\nabla \times (\nabla \times E) = -\mu_0 \frac{\partial^2 D}{\partial t^2} \tag{16}$$

Using the vector identity $\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E$. From $\nabla \cdot D = 0$, if the linear part of the medium is homogeneous and the nonlinearity is not too strong such that $\nabla \cdot (\epsilon_0 \chi^{(3)} |E|^2 E)$ can be considered small, then $\nabla \cdot E \approx 0$. This simplifies Eq. (16) to:

$$-\nabla^2 E = -\mu_0 \frac{\partial^2 D}{\partial t^2} \quad \Rightarrow \quad \nabla^2 E = \mu_0 \frac{\partial^2 D}{\partial t^2} \tag{17}$$

2.4 Insert the Kerr Term

Substitute Eq. (15) into Eq. (17) and use $c^2 = 1/(\mu_0 \epsilon_0)$:

$$\nabla^2 E = \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} [n_0^2 E + \chi^{(3)} |E|^2 E] \tag{18}$$

$$\nabla^2 E = \frac{n_0^2}{c^2} \frac{\partial^2 E}{\partial t^2} + \frac{\chi^{(3)}}{c^2} \frac{\partial^2}{\partial t^2} (|E|^2 E) \tag{19}$$

2.5 Final Kerr Wave Equation

Rearranging Eq. (19) gives the wave equation in a Kerr medium:

$$\nabla^2 E - \frac{n_0^2}{c^2} \frac{\partial^2 E}{\partial t^2} - \frac{\chi^{(3)}}{c^2} \frac{\partial^2}{\partial t^2} (|E|^2 E) = 0 \tag{20}$$

The terms represent:

- $\nabla^2 E$: Spatial evolution (diffraction/dispersion).
- $\frac{n_0^2}{c^2} \frac{\partial^2 E}{\partial t^2}$: Linear wave propagation.
- $\frac{\chi^{(3)}}{c^2} \frac{\partial^2}{\partial t^2} (|E|^2 E)$: Nonlinear Kerr effect.

Further approximations (e.g., slowly varying envelope, paraxial) on this equation lead to models like the Nonlinear Schrödinger Equation (NLSE) [cite: 81, 85] or the Lugiato-Lefever Equation (LLE).

3 Derivation of the Lugiato-Lefever Equation (LLE) for a Kerr Medium in a Cavity

This section derives the Lugiato-Lefever Equation (LLE) for an optical field in a Kerr nonlinear medium within a driven optical cavity, starting from Eq. (20) and applying mean-field approximations.

3.1 Starting Point: The Nonlinear Wave Equation in a Kerr Medium

We begin with the Kerr wave equation (Eq. 20):

$$\nabla^2 E - \frac{n_0^2}{c^2} \frac{\partial^2 E}{\partial t^2} = \frac{\chi^{(3)}}{c^2} \frac{\partial^2}{\partial t^2} (|E|^2 E) \tag{21}$$

3.2 Slowly Varying Envelope Approximation (SVEA) and Paraxial Approximation

Assume a linearly polarized field propagating along z with carrier frequency ω_0 and wavenumber $k_0 = n_0\omega_0/c$:

$$E(x, y, z, t) = F(x, y, z, t)e^{i(k_0 z - \omega_0 t)} + c.c. \tag{22}$$

where F is the slowly varying complex envelope. Applying SVEA (neglecting $\partial^2 F/\partial z^2$, $\partial^2 F/\partial t^2$) and considering only the fundamental frequency component of the nonlinear term ($\partial^2(|E|^2 E)/\partial t^2 \approx -\omega_0^2 |F|^2 F e^{i(k_0 z - \omega_0 t)}$), Eq. (21) transforms into the Non-Linear Schrödinger Equation (NLSE) in a frame moving at group velocity:

$$\frac{\partial F}{\partial z} = \frac{i}{2k_0}\nabla_\perp^2 F + i\gamma|F|^2 F \tag{23}$$

where $\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ describes diffraction, and $\gamma = \frac{\omega_0 \chi^{(3)}}{2n_0 c}$ is the nonlinear coefficient.

3.3 Cavity Dynamics and Mean-Field Approximation

Consider the Kerr medium (length L_{med}) in a ring cavity (total length $\mathcal{L}$, round-trip time $t_R = \mathcal{L}/c$). The field $F(t)$ evolves to $F(t + t_R)$ after one round trip:

$$F(t + t_R) = \sqrt{1 - T_{mirr}}e^{-i\theta_{phys}} F_{out}(t) + \sqrt{T_{mirr}} F_{P,ext} \tag{24}$$

where T_{mirr} is input/output mirror transmissivity (cavity loss), θ_{phys} is the physical phase detuning per round trip (e.g., $\theta_{phys} = (\omega_P - \omega_c)t_R$, with ω_P pump frequency, ω_c cavity resonance), $F_{P,ext}$ is the external pump field, and $F_{out}(t)$ is the field after passing through the Kerr medium. For a short medium ($L_{med} \ll z_0$, and small nonlinear effects per pass):

$$F_{out}(t) \approx F(t) + L_{med}\left(\frac{i}{2k_0}\nabla_\perp^2 F(t) + i\gamma|F(t)|^2 F(t)\right) \tag{25}$$

In the mean-field limit ($T_{mirr} \ll 1$, $\theta_{phys} \ll 1$, F changes slowly over t_R, small nonlinear phase shift per pass): $F(t + t_R) - F(t) \approx t_R \frac{\partial F}{\partial t_{lab}}$, $\sqrt{1 - T_{mirr}} \approx 1 - T_{mirr}/2$, $e^{-i\theta_{phys}} \approx 1 - i\theta_{phys}$. Substituting these into the map equation and keeping terms linear in small quantities:

$$t_R \frac{\partial F}{\partial t_{lab}} = -\left(\frac{T_{mirr}}{2} + i\theta_{phys}\right) F + i\frac{L_{med}}{2k_0}\nabla_\perp^2 F + i\gamma L_{med}|F|^2 F + \sqrt{T_{mirr}} F_{P,ext} \tag{26}$$

3.4 Normalization and Final LLE Form

Introduce scaled time $\tau = \frac{T_{mirr}}{2t_R} t_{lab}$ (consistent with Model Handbook [cite: 154] where $T \equiv T_{mirr}$). Thus $\frac{\partial}{\partial t_{lab}} = \frac{T_{mirr}}{2t_R}\frac{\partial}{\partial \tau}$. Substituting and dividing by $T_{mirr}/2$:

$$\frac{\partial F}{\partial \tau} = \frac{2\sqrt{T_{mirr}}}{T_{mirr}} F_{P,ext} - \left(1 + i\frac{2\theta_{phys}}{T_{mirr}}\right) F + i\frac{L_{med}}{T_{mirr} k_0}\nabla_\perp^2 F + i\frac{2\gamma L_{med}}{T_{mirr}}|F|^2 F \tag{27}$$

This is the Lugiato-Lefever Equation, matching the Model Handbook's Eq. (29) (for $\sigma_{sat} = 0$)[cite: 153]:

$$\partial_\tau F = F_P - (1 + i\theta)F + i\alpha\nabla_\perp^2 F + i\beta_K |F|^2 F \tag{28}$$

Comparing coefficients:

- **Normalized Pump (F_P)**: $F_P = \frac{2}{\sqrt{T_{mirr}}} F_{P,ext}$.
- **Cavity Detuning (θ)**: $\theta = \frac{2\theta_{phys}}{T_{mirr}}$. If $\theta_{phys} = (\omega_P - \omega_c)t_R$, then $\theta = \frac{2(\omega_P - \omega_c)\mathcal{L}}{cT_{mirr}}$. The Model Handbook's $\theta = \frac{2(\omega_c - \omega_P)\mathcal{L}}{cT}$ [cite: 175] has the opposite sign convention for frequency difference.

- **Diffraction Coefficient (α)**: The derived coefficient for the physical Laplacian $\nabla^2_{\perp,phys}$ is $\frac{L_{med}}{T_{mirr}k_0}$. The Model Handbook uses scaled transverse coordinates $(\xi, \eta) = \frac{\sqrt{2}(x,y)}{w_s}$[cite: 15], so $\nabla^2_{\perp,phys} = \frac{2}{w_s^2}\nabla^2_{\perp,scaled}$. Thus, the coefficient α for $\nabla^2_{\perp,scaled}$ is $\alpha = \frac{2L_{med}}{T_{mirr}k_0 w_s^2}$. This matches the structure $\alpha = \frac{2\mathcal{L}'}{k_L w_s^2 T}$ in the Model Handbook[cite: 155], where $k_L \equiv k_0$ and $\mathcal{L}' \equiv L_{med}$.
- **Kerr Nonlinearity (β_K)**: $\beta_K = \frac{2\gamma L_{med}}{T_{mirr}} = \frac{2L_{med}}{T_{mirr}}\left(\frac{\omega_0 \chi^{(3)}}{2n_0 c}\right) = \frac{L_{med}\omega_0\chi^{(3)}}{T_{mirr}n_0 c}$. The sign matches for self-focusing ($\chi^{(3)} > 0 \implies \beta_K > 0$) as in the Model Handbook[cite: 153].

3.5 Summary of Approximations

Key approximations for the LLE include:

1. Source-free, non-magnetic, isotropic Kerr medium.
2. Scalar field.
3. Slowly Varying Envelope Approximation (SVEA) in space and time.
4. Paraxial approximation for transverse dynamics.
5. Mean-field cavity approximation: high finesse ($T_{mirr} \ll 1$), slow field evolution per round trip, small detuning ($\theta_{phys} \ll 1$), and small nonlinear effects per pass.

References

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