1. *A CNF formula is satisfiable $\Leftrightarrow$ it can be made all-negative clause free by switching signs of the variables.*

   ($\Rightarrow$) If a CNF formula is satisfiable, we have a satisfying assignment. We switch the sign of those variables which are assigned `FALSE` by that assignment. Now each clause has a literal which is assigned to `TRUE`. If that is a positive literal then that clause is not a all-negative clause both in original and new formula. Else that was a negative clause, it is switched in the new formula hence that clause has a positive literal in the new formula. Hence the new formula is all-negative clause free.

   ($\Leftarrow$) Suppose a CNF formula can be made all-negative clause free by switching signs. Then in the new formula, each clause has at least one positive variable. We will assign `TRUE` to those variables, and that would make the formula satisfiable. Now, take that assignment, and change the assigned value to `FALSE` for those variables whose signs are switched. That would be a satisfying assignment for original formula because each clause in new formula has a `TRUE` literal. If that was a positive variable in original clause, that is `TRUE` in original clause too. Else if that was a negative variable in original clause, it is switched in the new formula. So assigning to `FALSE` would make it `TRUE` in the original clause.

2. *Let $R = \{12\bar{3}, 23\bar{4}, 341, 4\bar{1}2, \bar{1}\bar{2}3, \bar{2}\bar{3}4, \bar{3}\bar{4}\bar{1}, \bar{4}1\bar{2}\}$ and $R' = R \setminus \{\bar{4}1\bar{2}\}$. Show that R is unsatisfiable but $\{4, \bar{1}, 2\}$ makes R' satisfiable.*

   (a)

   $$\begin{aligned} R(\bar{1}, 2, 3, 4) &= R(1, 3, 2, 4) \\ R(1, \bar{2}, 3, 4) &= R(1, 2, 4, 3) \\ R(1, 2, \bar{3}, 4) &= R(1, 4, 3, 2) \\ R(1, 2, 3, \bar{4}) &= R(2, 1, 3, 4) \end{aligned}$$

   So no matter how I switch the variables, the new set of clauses is a new instance of R where the switched variable is not switched and other variables are permuted. So there is no way to make it all-negative clause free. Hence R is not satisfiable.

   (b) $R = R' \cup \{\bar{4}1\bar{2}\}$ is unsatisfiable. Hence $R' \Rightarrow \neg\{\bar{4}1\bar{2}\} = \{4, \bar{1}, 2\}$. Clearly $\{4, \bar{1}, 2\} \Rightarrow R'$. So, that is a satisfying assignment.

3. **Exercise 10.** *Show that every satisfiability problem with m clauses and n variables can be transformed into an equivalent monotonic problem with $m + n$ clauses and $2n$ variables, in which the first m clauses have only negative literals, and the last n clauses are binary with two positive literals.*

   Suppose F is a satisfiability problem with variables $\{1, \ldots, n\}$. We add additional $\{1', \ldots, n'\}$ variables. By intuition, v' and v has dual of same variable.

   Now we create a new formula F' replacing each occurrence of v with $\bar{v}'$ in F for $v \in \{1, \ldots, n\}$.

   Then we add n more clauses in F' $\{v', v\}$ for $v \in \{1, \ldots, n\}$.

   Then F' is a satisfiability problem with $m + n$ clauses and $2n$ variables. Now I will show, F and F' are equi-satisfiable.

   If F has a satisfying assignment σ. Then we extend σ to a satisfying assignment σ' for F' such that $\sigma'(v) = \sigma(v)$ and $\sigma'(v') = \overline{\sigma(v)}$ for $v \in \{1, \ldots, n\}$.

   Now suppose F' has a satisfying assignment σ'. We create a satisfying assignment $\sigma = \sigma'\downarrow_{1\ldots n}$ for F. This will correct because, if $\sigma'(v) = \overline{\sigma'(v')}$, then the dual nature is consistent. But if $\sigma'(v) = \sigma'(v')$ then both are `TRUE` since they both can not be `FALSE` ($\{v, v'\}$ is a clause in F'), then $\bar{v}$ and $\bar{v}'$ were `FALSE` in the first m clauses of F'. So m clauses are satisfied without v and v''s contribution. So I can assign any value to v in original F.

   Exercise 26. *Prove that Sinz's clauses in the below enforce the cardinality constraint $x_1 + \ldots + x_n \leq r$. Hint: Show that they imply $s_j^k = 1$ whenever $x_1 + \ldots + x_{j+k-1} \geq k$.*

   (a) $(\bar{s}_j^k \vee s_{j+1}^k)$ for $1 \leq j < (n - r)$ and $1 \leq k \leq r$

   (b) $(\bar{x}_{j+k} \vee \bar{s}_j^k \vee s_j^{k+1})$ for $1 \leq j < (n - r)$ and $0 \leq k \leq r$ where $\bar{s}_j^k$ is omitted when $k = 0$ and s_j^{k+1} is omitted when $k = r$.

Suppose $x_1 + \ldots + x_{j+k-1} \geq k$ *i.e.* there are at least k many $i < j + k$ such that $x_i = 1$. Then I will show s_j^k is TRUE.

I will prove by induction.

Base case: If $k = 1$, then there is at least one $x_i = 1$. (b) says $(\bar{x}_i \vee s_i^1)$. So s_i^1 is TRUE. Then I can apply (a) finite times and have s_j^1 to be TRUE (since $i < j + 1$).

Inductive step: If $x_1 + \ldots + x_{j+k-1} \geq k$. Suppose $x_l = 1$ such that there is no $x_i = 1$ with $l < i < j + k$. Then, $x_1 + \ldots x_{l-1} \geq (k-1)$. By induction hypothesis, s_{l-k+1}^{k-1} is TRUE. Then by applying (b) on $(\bar{x}_l \vee \bar{s}_{l-k+1}^{k-1} \vee s_{l-k+1}^k)$ we have s_{l-k+1}^k to be TRUE. Then applying (a) finite times we have s_j^k to be TRUE.

So whenever $x_1 + \ldots + x_{j+k-1} \geq k$, s_j^k is TRUE.

Now, if $x_{r+j} = 0$ for $1 \leq j \leq (n - r)$ then of course $x_1 + \ldots + x_n \leq r$. Else there exists $x_{l+r} = 1$ such that there is no $x_{i+r} = 1$ with $l < i \leq (n - r)$ then (b) says $(x_{l+r} \vee \bar{s}_l^r)$ or x_{l+r} implies $x_1 + \ldots + x_{l+r-1} \ngeq r$ or $x_1 + \ldots + x_{l+r} \leq r$. Since $x_{i+r} = 0$ with $l < i \leq (n - r)$, that inequality becomes $x_1 + \ldots + x_n \leq r$.

Hence every satisfying assignment for Sinz's clauses ensures $x_1 + \ldots + x_n \leq r$.