Prime Numbers

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8
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8.1 Chebyshev's Theorems

Let $\pi(x)$ denote the number of prime numbers not exceeding x, that is,

$$\pi(x) = \sum_{p \leq x} 1$$

is the counting function for the set of primes. Euclid proved that there are infinitely many primes, or, equivalently,

$$\lim_{x \to \infty} \pi(x) = \infty.$$

A classical problem in number theory is to understand the distribution of prime numbers. This problem is still fundamentally unsolved, even though we know many beautiful results about the growth of $\pi(x)$ as x tends to infinity. In this chapter we shall show that the order of magnitude of $\pi(x)$ is $x/\log x$. In Chapter 9 we shall prove the prime number theorem, which states that $\pi(x)$ is asymptotic to $x/\log x$, that is,

$$\lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1.$$

We introduce the *Chebyshev functions*

$$\vartheta(x) = \sum_{p \leq x} \log p = \log \prod_{p \leq x} p$$

and

$$\psi(x) = \sum_{p^k \leq x} \log p.$$

For example,

$$\vartheta(10) = \log 2 + \log 3 + \log 5 + \log 7$$

and

$$\psi(10) = 3\log 2 + 2\log 3 + \log 5 + \log 7.$$

The functions $\vartheta(x)$ and $\psi(x)$ count the primes $p \leq x$ and prime powers $p^k \leq x$, respectively, with weights $\log p$. Clearly,

$$\vartheta(x) \leq \psi(x).$$

If $p^k \leq x$, then $k \leq [\log x / \log p]$, and so

$$\begin{aligned} \psi(x) &= \sum_{\substack{p^k \leq x \\ k \geq 1}} \log p = \sum_{p \leq x} \left(\sum_{\substack{p^k \leq x \\ k \geq 1}} 1 \right) \log p = \sum_{p \leq x} \left[\frac{\log x}{\log p} \right] \log p \\ &\leq \sum_{p \leq x} \log x = \pi(x) \log x. \end{aligned}$$

Chebyshev proved that the functions $\vartheta(x)$ and $\psi(x)$ have order of magnitude x and that $\pi(x)$ has order of magnitude $x/\log x$.

Before proving these theorems, we need two results about binomial coefficients. The first lemma states that for fixed n, the sequence of binomial coefficients $\binom{n}{k}$ is *unimodal* in the sense that it is increasing for $k \leq n/2$ and decreasing for $k \geq n/2$. In the second lemma we apply the binomial theorem to obtain upper and lower bounds for the *middle binomial coefficient* $\binom{2n}{n}$.

Lemma 8.1 *Let $n \geq 1$ and $1 \leq k \leq n$. Then*

$$\begin{aligned} \binom{n}{k-1} &< \binom{n}{k} && \text{if and only if } k < \tfrac{n+1}{2}, \\ \binom{n}{k-1} &> \binom{n}{k} && \text{if and only if } k > \tfrac{n+1}{2}, \\ \binom{n}{k-1} &= \binom{n}{k} && \text{if and only if } n \text{ is odd and } k = \tfrac{n+1}{2}. \end{aligned}$$

Proof. Consider the ratio

$$r(k) = \frac{\binom{n}{k}}{\binom{n}{k-1}} = \frac{\frac{n!}{k!(n-k)!}}{\frac{n!}{(k-1)!(n-k+1)!}} = \frac{(k-1)!(n-k+1)!}{k!(n-k)!} = \frac{n-k+1}{k}.$$

Then $r(k) > 1$ if and only if $k < (n+1)/2$, and $r(k) < 1$ if and only if $k > (n+1)/2$. □