Introduction to Robotics
Analysis, Control, Applications

Solution Manual

Saeed B. Niku

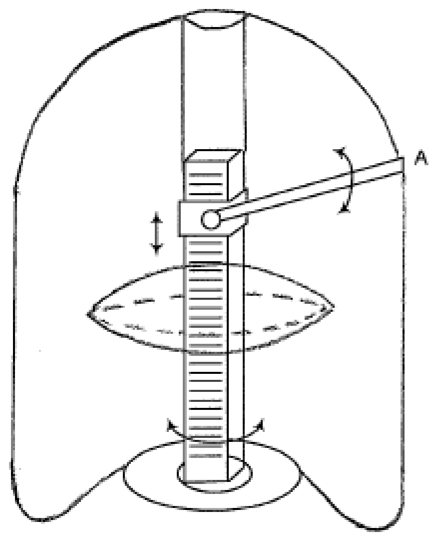
CHAPTER ONE

Problem 1.1
Draw the approximate workspace for the following robot. Assume the dimensions of the base and other parts of the structure of the robot are as shown.

Estimated student time to complete: 15-25 minutes
Prerequisite knowledge required: Text Section(s) 1.14

Solution:

The workspace shown is approximate.

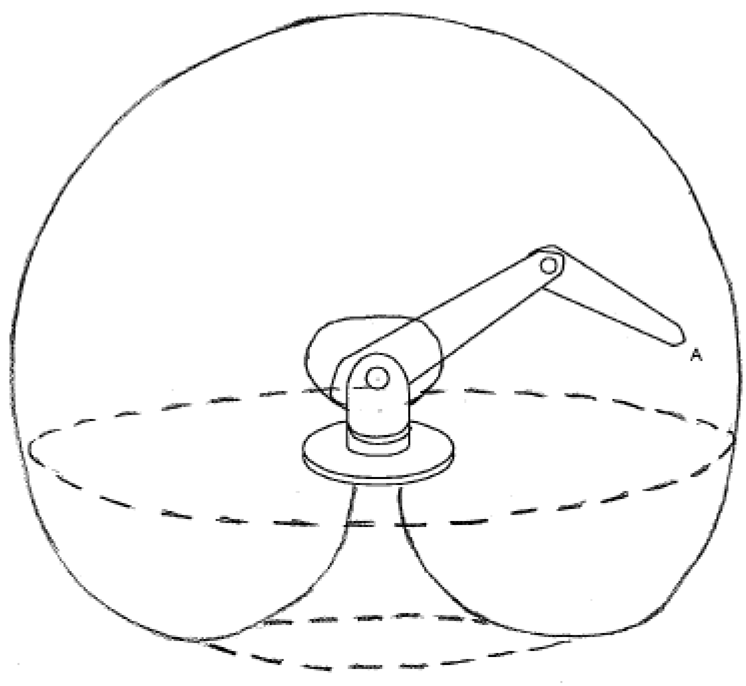


Problem 1.2
Draw the approximate workspace for the following robot. Assume the dimensions of the base and other parts of the structure of the robot are as shown.

Estimated student time to complete: 20-30 minutes
Prerequisite knowledge required: Text Section(s) 1.14

Solution:

The workspace shown is approximate.

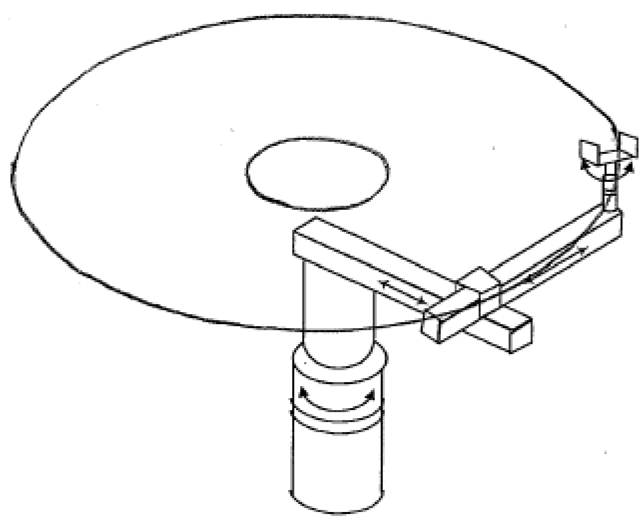


Problem 1.3
Draw the approximate workspace for the following robot. Assume the dimensions of the base and other parts of the structure of the robot are as shown.

Estimated student time to complete: 10-15 minutes
Prerequisite knowledge required: Text Section(s) 1.14

Solution:

The workspace shown is approximate.



CHAPTER TWO

Problem 2.1
Write a unit vector in matrix form that describes the direction of the cross product of $\mathbf{p} = 5\mathbf{i} + 3\mathbf{k}$ and $\mathbf{q} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$.

Estimated student time to complete: 5-10 minutes
Prerequisite knowledge required: Text Section(s) 2.4

Solution:

$$\mathbf{r} = \mathbf{p} \times \mathbf{q} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 0 & 3 \\ 3 & 4 & 5 \end{bmatrix} = \mathbf{i}\left(0-12\right) - \mathbf{j}\left(25-9\right) + \mathbf{k}\left(20-0\right) = -12\mathbf{i} - 16\mathbf{j} + 20\mathbf{k}$$

$$\lambda = \sqrt{r_x^2 + r_y^2 + r_z^2} = \sqrt{144 + 256 + 400} = 28.28$$

$$\mathbf{r} = \begin{bmatrix} \dfrac{-12}{28.28} \\ \dfrac{-16}{28.28} \\ \dfrac{20}{28.28} \end{bmatrix} = \begin{bmatrix} -0.424 \\ -0.566 \\ 0.707 \end{bmatrix}$$

Problem 2.2

A vector **p** is 8 units long and is perpendicular to vectors **q** and **r** described below. Express the vector in matrix form.

$$\mathbf{q}_{unit} = \begin{bmatrix} 0.3 \\ q_y \\ 0.4 \\ 0 \end{bmatrix} \qquad \mathbf{r}_{unit} = \begin{bmatrix} r_x \\ 0.5 \\ 0.4 \\ 0 \end{bmatrix}$$

Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 2.4

Solution:

The two vectors given are unit vectors. Therefore, each missing component can be found as:

$$q_y = \sqrt{1 - 0.09 - 0.16} = 0.866$$
$$r_x = \sqrt{1 - 0.25 - 0.16} = 0.768$$

Since **p** is perpendicular to the other two vectors, it is in the direction of the cross product of the two. Therefore:

$$\lambda_p = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.3 & 0.866 & 0.4 \\ 0.768 & 0.5 & 0.4 \end{bmatrix} = \mathbf{i}(0.346 - 0.2) - \mathbf{j}(0.12 - 0.307) + \mathbf{k}(0.15 - 0.665)$$
$$= \mathbf{i}(0.146) + \mathbf{j}(0.187) - \mathbf{k}(0.515)$$

Since **q** and **r** are not perpendicular to each other, the resulting **p** is not a unit vector. Vector **p** can be found as:

$$\lambda_p = \mathbf{i}(0.146) + \mathbf{j}(0.187) - \mathbf{k}(0.515)$$
$$|\lambda_p| = \sqrt{(0.146)^2 + (0.187)^2 + (0.515)^2} = 0.567$$
$$w = \frac{8}{0.567} = 14.1$$
$$\mathbf{p} = w(\mathbf{i}(0.146) + \mathbf{j}(0.187) - \mathbf{k}(0.515))$$
$$\mathbf{p} = \mathbf{i}(2.06) + \mathbf{j}(2.64) - \mathbf{k}(7.27)$$

Problem 2.3
Will the three vectors **p**, **q**, and **r** in Problem 2.2 form a traditional frame? If not, find the necessary unit vector **s** to form a frame between **p**, **q**, and **s**.

Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 2.4

Solution:

As we saw in Problem 2.2, since $\mathbf{q}\times\mathbf{r}$ is not a unit vector, it means that **q** and **r** and not perpendicular to each other, and therefore, they cannot form a frame. However, **p** and **q** are perpendicular to each other, and we can select **s** to be perpendicular to those two. Of course, **p** is not a unit length, therefore we use the unit vector representing it.

$$\left|\lambda_p\right| = \sqrt{(0.146)^2 + (0.187)^2 + (0.515)^2} = 0.567$$

$$w = \frac{1}{0.567} = 1.764$$

$$\mathbf{p} = w\left(\mathbf{i}(0.146) + \mathbf{j}(0.187) - \mathbf{k}(0.515)\right)$$

$$\mathbf{p} = \mathbf{i}(0.257) + \mathbf{j}(0.33) - \mathbf{k}(0.908)$$

$$\mathbf{p} = \mathbf{i}(0.257) + \mathbf{j}(0.33) - \mathbf{k}(0.908)$$

$$\mathbf{s} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.257 & 0.33 & -0.908 \\ 0.3 & 0.866 & 0.4 \end{bmatrix} = \mathbf{i}(0.918) - \mathbf{j}(0.375) + \mathbf{k}(0.124)$$

Problem 2.4

Suppose that instead of a frame, a point $P=(3,5,7)^T$ in space was translated a distance of $d=(2,3,4)^T$. Find the new location of the point relative to the reference frame.

Estimated student time to complete: 5 minutes
Prerequisite knowledge required: Text Section(s) 2.6

Solution:

As for a frame,

$$P_{new}=\begin{bmatrix}1 & 0 & 0 & 2\\ 0 & 1 & 0 & 3\\ 0 & 0 & 1 & 4\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}3\\5\\7\\1\end{bmatrix}=\begin{bmatrix}5\\8\\11\\1\end{bmatrix}$$

Problem 2.5

The following frame B was moved a distance of $d=(5,2,6)^T$. Find the new location of the frame relative to the reference frame.

$$B=\begin{bmatrix}0 & 1 & 0 & 2\\ 1 & 0 & 0 & 4\\ 0 & 0 & -1 & 6\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Estimated student time to complete: 5-10 minutes
Prerequisite knowledge required: Text Section(s) 2.6

Solution:

The transformation matrix representing the translation is used to find the new location as:

$$B_{new}=\begin{bmatrix}1 & 0 & 0 & 5\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 6\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 0 & 2\\ 1 & 0 & 0 & 4\\ 0 & 0 & -1 & 6\\ 0 & 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 7\\ 1 & 0 & 0 & 6\\ 0 & 0 & -1 & 12\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Problem 2.6
For frame *F*, find the values of the missing elements and complete the matrix representation of the frame.

$$F = \begin{bmatrix} ? & 0 & -1 & 5 \\ ? & 0 & 0 & 3 \\ ? & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Estimated student time to complete: 10 minutes
Prerequisite knowledge required: Text Section(s) 2.4

Solution:

$$F = \begin{bmatrix} n_x & 0 & -1 & 5 \\ n_y & 0 & 0 & 3 \\ n_z & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From $\mathbf{n} \times \mathbf{o} = \mathbf{a}$ $\qquad \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ n_x & n_y & n_z \\ 0 & 0 & -1 \end{bmatrix} = -\mathbf{i}$

Or: $\mathbf{i}\left(-n_y\right) - \mathbf{j}\left(-n_x\right) + \mathbf{k}\left(0\right) = -\mathbf{i}$, and therefore: $n_y = 1, \quad n_x = 0, \quad n_z = 0$

$$F = \begin{bmatrix} 0 & 0 & -1 & 5 \\ 1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 2.7
Find the values of the missing elements of frame B and complete the matrix representation of the frame.

$$B=\begin{bmatrix} 0.707 & ? & 0 & 2 \\ ? & 0 & 1 & 4 \\ ? & -0.707 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 2.4

Solution:

$$B=\begin{bmatrix} 0.707 & o_x & 0 & 2 \\ n_y & 0 & 1 & 4 \\ n_z & -0.707 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From $\mathbf{n}\times\mathbf{o}=\mathbf{a}$ $\quad \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.707 & n_y & n_z \\ o_x & 0 & 0.707 \end{bmatrix}=\mathbf{j}$

Therefore: $\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.707 & n_y & n_z \\ o_x & 0 & 0.707 \end{bmatrix}=\mathbf{j}$

And $\quad \mathbf{i}\left(0.707n_y\right)-\mathbf{j}\left(0.5-n_z o_x\right)+\mathbf{k}\left(-n_y o_x\right)=\mathbf{j} \quad \rightarrow \quad n_y=0$

From length equations: $|\mathbf{n}|=1$ or $\quad \begin{matrix} 0.707^2+n_y^2+n_z^2=1 \quad \rightarrow \quad n_z=\pm 0.707 \\ o_x^2+0.5=1 \quad \rightarrow \quad o_x=\pm 0.707 \end{matrix}$

Therefore, there are two possible acceptable solutions:

$$B=\begin{bmatrix} 0.707 & 0.707 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0.707 & -0.707 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} 0.707 & -0.707 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ -0.707 & -0.707 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 2.8
Derive the matrix that represents a pure rotation about the *y*-axis of the reference frame.

Estimated student time to complete: 10 minutes
Prerequisite knowledge required: Text Section(s) 2.6.2.

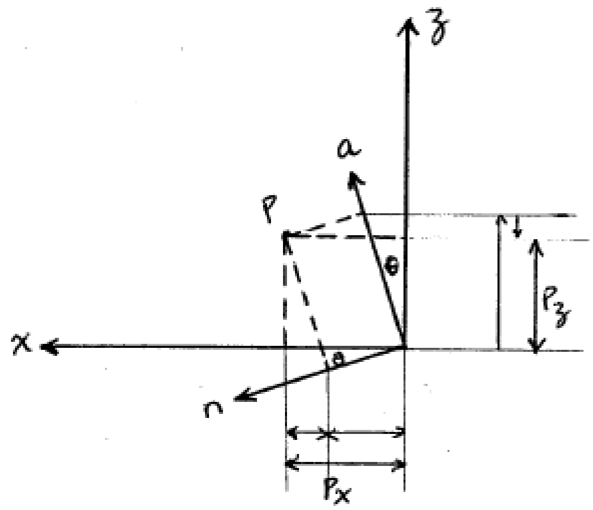
Solution:

From the figure:

$$p_x = p_n \cos\theta + p_a \sin\theta$$
$$p_y = p_o$$
$$p_z = -p_n \sin\theta + p_a \cos\theta$$

and

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} C & 0 & S \\ 0 & 1 & 0 \\ -S & 0 & C \end{bmatrix} \begin{bmatrix} p_n \\ p_o \\ p_a \end{bmatrix}$$



Problem 2.9
Derive the matrix that represents a pure rotation about the *z*-axis of the reference frame.

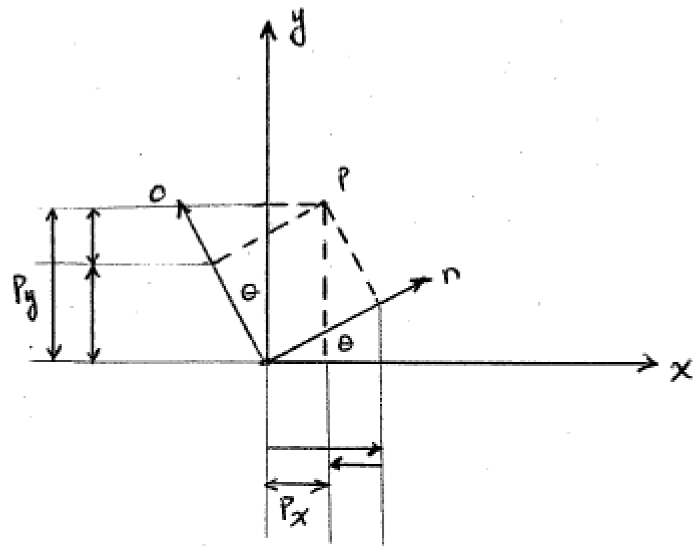
Estimated student time to complete: 10 minutes
Prerequisite knowledge required: Text Section(s) 2.6.2.

Solution:
From the Figure:

$$p_x = p_n \cos\theta - p_o \sin\theta$$
$$p_y = p_n \sin\theta + p_o \cos\theta$$
$$p_z = p_a$$

and

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_n \\ p_o \\ p_a \end{bmatrix}$$



Problem 2.10
Verify that the rotation matrices about the reference frame axes follow the required constraint equations set by orthogonality and length requirements of directional unit vectors.

Estimated student time to complete: 10 minutes.
Prerequisite knowledge required: Text Section(s) 2.4.5 and 2.6.2

Solution:

For $Rot(x,\theta)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & C & -S \\ 0 & S & C \end{bmatrix}$ we have:

$\mathbf{n}\bullet\mathbf{o}=0$ or $n_x\, o_x + n_y\, o_y + n_z\, o_z = 0$
$\mathbf{n}\bullet\mathbf{a}=0$
$\mathbf{a}\bullet\mathbf{o}=0$
$|\mathbf{n}|=1$
$|\mathbf{o}|=\sqrt{S^2+C^2}=1$
$|\mathbf{a}|=\sqrt{(-S)^2+C^2}=1$

$Rot(y,\theta)$ and $Rot(z,\theta)$ will be the same.

Problem 2.11
Find the coordinates of point $P(2,3,4)^T$ relative to the reference frame after a rotation of 45° about the *x*-axis.

Estimated student time to complete: 10 minutes
Prerequisite knowledge required: Text Section(s) 2.6.3.

Solution:

$$^{U}P = Rot(x,45)\begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 0.707 & -0.707\\0 & 0.707 & 0.707\end{bmatrix}\begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}2\\-0.707\\4.95\end{bmatrix}$$

Note that the rotation is written in a 3×3, not homogeneous form, because we are only concerned about the rotation part.

Problem 2.12
Find the coordinates of point $P(3,5,7)^T$ relative to the reference frame after a rotation of 30° about the *z*-axis.

Estimated student time to complete: 10 minutes
Prerequisite knowledge required: Text Section(s) 2.6.3.

Solution:

$$^{U}P = Rot(z,30)\begin{bmatrix}3\\5\\7\end{bmatrix} = \begin{bmatrix}0.866 & -0.5 & 0\\ 0.5 & 0.866 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}3\\5\\7\end{bmatrix} = \begin{bmatrix}0.1\\5.83\\7\end{bmatrix}$$

Note that the rotation is written in a 3×3 form not homogeneous form, because we are only concerned about the rotation part.

Problem 2.13
Find the new location of point $P(1,2,3)^T$ relative to the reference frame after a rotation of 30° about the *z*-axis followed by a rotation of 60° about the *y*-axis.

Estimated student time to complete: 10 minutes
Prerequisite knowledge required: Text Section(s) 2.6.3.

Solution:

$$^{U}P = Rot(y,60)Rot(z,30)P$$

$$^{U}P = \begin{bmatrix} 0.5 & 0 & 0.866 & 0 \\ 0 & 1 & 0 & 0 \\ -0.866 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.866 & -0.5 & 0 & 0 \\ 0.5 & 0.866 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.531 \\ 2.232 \\ 1.616 \\ 1 \end{bmatrix}$$

Problem 2.14

A point *P* in space is defined as ${}^{B}P = (5,3,4)^T$ relative to frame *B* which is attached to the origin of the reference frame *A* and is parallel to it. Apply the following transformations to frame *B* and find ${}^{A}P$. Using the 3-D grid, plot the transformations and the result and verify it. Also verify graphically that you would not get the same results if you apply the transformations relative to the current frame:

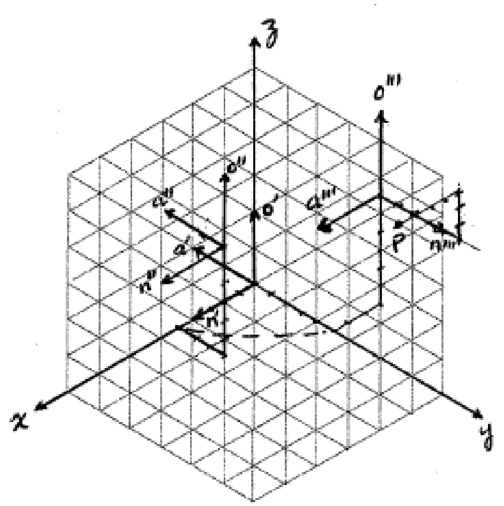
- Rotate 90° about *x*-axis, then
- Translate 3 units about *y*-axis, 6 units about *z*-axis, and 5 units about *x*-axis. Then,
- Rotate 90° about *z*-axis.

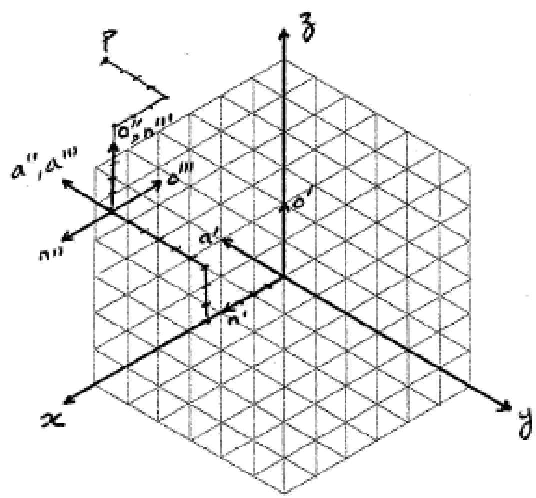
Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 2.6.3.

Solution:

$${}^{A}P = Rot(z,90)Trans(5,3,6)Rot(x,90)\,{}^{B}P$$

$${}^{A}P = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ 9 \\ 1 \end{bmatrix}$$





Problem 2.15

A point P in space is defined as ${}^{B}P=(2,3,5)^{T}$ relative to frame B which is attached to the origin of the reference frame A and is parallel to it. Apply the following transformations to frame B and find ${}^{A}P$. Using the 3-D grid, plot the transformations and the result and verify it:

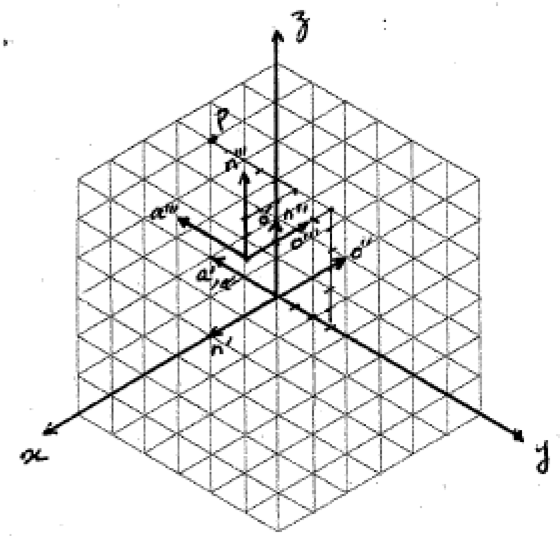
- Rotate 90° about x-axis, then
- Rotate 90° about local a-axis, then
- Translate 3 units about y-, 6 units about z-, and 5 units about x-axes.

Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 2.6.4.

Solution:

$${}^{A}P=Trans(5,3,6)Rot(x,90)Rot(a,90)\,{}^{B}P$$

$${}^{A}P=\begin{bmatrix}1&0&0&5\\0&1&0&3\\0&0&1&6\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}2\\3\\5\\1\end{bmatrix}=\begin{bmatrix}2\\-2\\8\\1\end{bmatrix}$$



Problem 2.16

A frame B is rotated 90° about the z-axis, then translated 3 and 5 units relative to the n- and o-axes respectively, then rotated another 90° about the n-axis, and finally, 90° about the y-axis. Find the new location and orientation of the frame.

$$B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 2.6.4.

Solution:

$$B_{new} = Rot(y,90)Rot(z,90)\,B\,Trans(3,5,0)Rot(n,90)$$

$$B_{new} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 6 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 2.17

The frame B of problem 2.16 is rotated 90° about the a-axis, 90° about the y-axis, then translated 2 and 4 units relative to the x- and y-axes respectively, then rotated another 90° about the n-axis. Find the new location and orientation of the frame.

$$B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 2.6.4.

Solution:

$$B_{new} = Trans(2,4,0)Rot(y,90)\,B\,Rot(a,90)Rot(n,90)$$

$$B_{new} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 2.18
Show that rotation matrices about the *y*- and the *z*-axes are unitary.

Estimated student time to complete:
Prerequisite knowledge required: Text Section(s) 2.7.

Solution:

$$A = Rot(y,\theta) = \begin{bmatrix} C & 0 & S \\ 0 & 1 & 0 \\ -S & 0 & C \end{bmatrix}$$

$$\det A = 1(C^2\theta + S^2\theta) + 0 = 1$$

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & S\theta \\ 0 & -S\theta & C\theta \end{bmatrix}$$

$$\text{adj}A = A^T_{minor} = \begin{bmatrix} C & 0 & -S \\ 0 & 1 & 0 \\ S & 0 & C \end{bmatrix} / 1 = A^{-1}$$

$$A^{-1} = A^T$$

Exact same is true for rotation about the *z*-axis.

Problem 2.19
Calculate the inverse of the following transformation matrices:

$$T_1 = \begin{bmatrix} 0.527 & -0.574 & 0.628 & 2 \\ 0.369 & 0.819 & 0.439 & 5 \\ -0.766 & 0 & 0.643 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} 0.92 & 0 & 0.39 & 5 \\ 0 & 1 & 0 & 6 \\ -0.39 & 0 & 0.92 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Estimated student time to complete: 15-20 minutes.
Prerequisite knowledge required: Text Section(s) 2.7.

Solution:

Since transformation matrices are unitary, we calculate the inverses simply by transposing the rotation part and calculating the position part by:

$$\begin{aligned} &0.527 \times 2 + 0.369 \times 5 - 0.766 \times 3 = -0.601 \\ &-0.574 \times 2 + 0.819 \times 5 - 0 \times 3 = -2.947 \\ &0.628 \times 2 + 0.439 \times 5 + 0.643 \times 3 = -5.38 \end{aligned}$$

Therefore:

$$T_1^{-1} = \begin{bmatrix} 0.527 & 0.369 & -0.766 & -0.601 \\ -0.574 & 0.819 & 0 & -2.947 \\ 0.628 & 0.439 & 0.643 & -5.38 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

And similarly,

$$T_2^{-1} = \begin{bmatrix} 0.92 & 0 & -0.39 & -3.82 \\ 0 & 1 & 0 & -6 \\ 0.39 & 0 & 0.92 & -3.79 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 2.20
Calculate the inverse of the matrix B of Problem 2.17.

Estimated student time to complete: 5 minutes
Prerequisite knowledge required: Text Section(s) 2.7.

Solution:

$$B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 2.21
Write the correct sequence of movements that must be made in order to restore the original orientation of the spherical coordinates and make it parallel to the reference frame. About what axes are these rotations supposed to be?

Estimated student time to complete: 10 minutes
Prerequisite knowledge required: Text Section(s) 2.9.3.

Solution:

$$T_{sph}(r,\beta,\gamma)Rot(o,-\beta)Rot(a,-\gamma)=\begin{bmatrix}1 & 0 & 0 & rS\beta C\gamma \\ 0 & 1 & 0 & rS\beta S\gamma \\ 0 & 0 & 1 & rC\beta \\ 0 & 0 & 0 & 1\end{bmatrix}$$

As shown, the rotations are relative to *o*- and *a*-axes in the sequence shown. These rotations must be relative to the current moving frame in order to prevent changing the position of the frame.

Problem 2.22
A spherical coordinate system is used to position the hand of a robot. In a certain situation, the hand orientation of the frame is later restored in order to be parallel to the reference frame, and the matrix representing it is described as:

$$T_{sph} = \begin{bmatrix} 1 & 0 & 0 & 3.1375 \\ 0 & 1 & 0 & 2.195 \\ 0 & 0 & 1 & 3.214 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Find the necessary values of r, β, γ to achieve this location.
- Find the components of the original matrix $\mathbf{n}, \mathbf{o}, \mathbf{a}$ vectors for the hand before the orientation was restored.

Estimated student time to complete: 20 minutes
Prerequisite knowledge required: Text Section(s) 2.9.3.

Solution:

a) The equations representing position in a spherical coordinates may be set equal to the given values as:

i. $rC\gamma S\beta = 3.1375$

ii. $rS\gamma S\beta = 2.195$

iii. $rC\beta = 3.214$

I. Assuming $S\beta$ is positive, from i and ii

$\gamma = 35^\circ$

from ii and iii $\beta = 50^\circ$
from iii *r=5* units.

II. If $S\beta$ were negative, then $\gamma = 215^\circ$

$\beta = -50^\circ$

r=5 units.

Since orientation is not specified, no more information is available to check the results.

b) For case I, substitute corresponding values of $S\beta$, $C\beta$, $S\gamma$, $C\gamma$ and *r* in spherical coordinates to get:

$$T_{sph}(r, \beta, \gamma) = T_{sph}(35, 50, 5) = \begin{bmatrix} 0.5265 & -0.5735 & 0.6275 & 3.1375 \\ 0.3687 & 0.819 & 0.439 & 2.195 \\ -0.766 & 0 & 0.6428 & 3.214 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 2.23
Suppose that a robot is made of a Cartesian and RPY combination of joints. Find the necessary RPY angles to achieve the following:

$$T = \begin{bmatrix} 0.527 & -0.574 & 0.628 & 4 \\ 0.369 & 0.819 & 0.439 & 6 \\ -0.766 & 0 & 0.643 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Estimated student time to complete: 20 minutes
Prerequisite knowledge required: Text Section(s) 2.10.1

Solution:

The position is achieved by Cartesian joints. Therefore, the robot moves 4, 6, and 9 units along the *x*-, *y*-, and *z*-axes. The orientation is achieved by RPY rotations, therefore:

$$\phi_a = ATAN2(n_y, n_x) = ATAN2(0.369, 0.527) = 35^\circ$$

or $$\phi_a = ATAN2(-n_y, -n_x) = ATAN2(-0.369, -0.527) = 215^\circ$$

For $\phi_a = 35^\circ$

$$\phi_o = ATAN2(-n_z, (n_x C\phi_a + n_y S\phi_a)) = ATAN2\left(0.766, \left(0.527 \times 0.819 + 0.369 \times 0.574\right)\right) = 50^\circ$$

For $\phi_a = 215^\circ$

$$\phi_o = ATAN2\left(0.766, -0.643\right) = 130^\circ$$

For $\phi_a = 35^\circ$

$$\begin{aligned} \phi_n &= ATAN2((-a_y C\phi_a + a_x S\phi_a), (o_y C\phi_a - o_x S\phi_a)) \\ &= ATAN2\left[\left(-0.439 \times 0.819 + 0.628 \times 0.574\right), \left(0.819 \times 0.819 + 0.574 \times 0.574\right)\right] \\ &= ATAN2\left(0, 1\right) = 0^\circ \end{aligned}$$

For $\phi_a = 215^\circ$

$$\phi_n = ATAN2\left(0, -1\right) = 180^\circ$$

Either solution is acceptable: $\begin{cases} \phi_a = 35^\circ \\ \phi_o = 50^\circ \\ \phi_n = 0^\circ \end{cases} \qquad \begin{cases} \phi_a = 215^\circ \\ \phi_o = 130^\circ \\ \phi_n = 180^\circ \end{cases}$

Problem 2.24
Suppose that a robot is made of a Cartesian and Euler combination of joints. Find the necessary Euler angles to achieve the following:

$$T = \begin{bmatrix} 0.527 & -0.574 & 0.628 & 4 \\ 0.369 & 0.819 & 0.439 & 6 \\ -0.766 & 0 & 0.643 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Estimated student time to complete: 20 minutes
Prerequisite knowledge required: Text Section(s) 2.10.2

Solution:

The position is achieved by Cartesian joints. Therefore, the robot moves 4, 6, and 9 units along the *x*-, *y*-, and *z*-axes. The orientation is achieved by Euler rotations, therefore:

$$\phi = ATAN2(a_y, a_x) \quad or \quad \phi = ATAN2(-a_y, -a_x)$$

$$\text{or} \quad \begin{cases} \phi = ATAN2(0.439, 0.628) = 35^\circ \\ \phi = ATAN2(-0.439, -0.628) = 215^\circ \end{cases}$$

$$\psi = ATAN2[(-n_x S\phi + n_y C\phi), (-o_x S\phi + o_y C\phi)]$$

$$\text{or} \quad \begin{cases} \psi = ATAN2[0,1] = 0^\circ \\ \psi = ATAN2[0,-1] = 180^\circ \end{cases}$$

$$\theta = ATAN2[(a_x C\phi + a_y S\phi), a_z)]$$

$$\text{or} \quad \begin{cases} \theta = ATAN2[(0.628 \times 0.819 + 0.439 \times 0.573), 0.643)] = 50^\circ \\ \theta = ATAN2[(0.628 \times (-0.819) + 0.439 \times (-0.573)), 0.643)] = -50^\circ \end{cases}$$

$$\text{Then} \quad \begin{cases} \phi = 35^\circ \\ \psi = 0^\circ \\ \theta = 50^\circ \end{cases} \qquad \begin{cases} \phi = 215^\circ \\ \psi = 180^\circ \\ \theta = -50^\circ \end{cases}$$

Problem 2.25
Assume that the three Euler angles used with a robot are $30^\circ, 40^\circ, 50^\circ$ respectively. Determine what angles should be used to achieve the same result if RPY is used instead.

Estimated student time to complete: 20-25 minutes
Prerequisite knowledge required: Text Section(s) 2.10

Solution:

For Euler angles:

$$\text{Euler}(\phi,\theta,\psi) = Rot(a,\phi)Rot(o,\theta), Rot(a,\psi) = Rot(a,30)Rot(o,40), Rot(a,50)$$
$$= \begin{bmatrix} 0.0435 & -0.8296 & 0.5568 & 0 \\ 0.9096 & 0.2635 & 0.3215 & 0 \\ -0.4134 & 0.4925 & 0.766 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\phi_a = ATAN2(n_y, n_x) = ATAN2(0.9096, 0.0435) = 87.26^\circ$
or $\quad \phi_a = ATAN2(-n_y, -n_x) = ATAN2(-0.9096, -0.0435) = 267.26^\circ$

$\phi_o = ATAN2(-n_z, (n_x C\phi_a + n_y S\phi_a)) = ATAN2\left(0.4134, 0.9106\right) = 24.5^\circ$
or $\quad \phi_o = ATAN2\left(0.4134, -0.9106\right) = 155.5^\circ$

$$\phi_n = ATAN2((-a_y C\phi_a + a_x S\phi_a), (o_y C\phi_a - o_x S\phi_a))$$
$$= ATAN2\left(0.5408, 0.8412\right) = 32.7^\circ$$

or $\quad \phi_n = ATAN2(-0.5408, -0.8412) = 212.7^\circ$

Problem 2.26
A frame ${}^{U}B$ was moved along its own *o*-axis a distance of 6 units, then rotated about its *n*-axis an angle of 60°, then translated about the *z*-axis for 3 units, followed by a rotation of 60° about the *z*-axis, and finally rotated about *x*-axis for 45°.

- Calculate the total transformation performed.
- What angles and movements would we have to make if we were to create the same location and orientation using Cartesian and Euler configurations?

Estimated student time to complete: 25-30 minutes
Prerequisite knowledge required: Text Section(s) 2.9- 2.10

Solution:

$${}^{U}T_B = Rot(x,45)Rot(z,60)Trans(0,0,3)Trans(0,6,0)Rot(n,60)$$

$$B_{new} = \begin{bmatrix} 0.5 & -0.433 & 0.75 & -5.196 \\ 0.6123 & -0.4355 & -0.6596 & 0 \\ 0.6123 & 0.789 & 0.0474 & 4.242 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The Cartesian translations are: $p_x = -5.196, \quad p_x = 0, \quad p_x = 4.242$. Euler angles are:

$$\phi = ATAN2(a_y, a_x) \quad or \quad \phi = ATAN2(-a_y, -a_x)$$

or $\begin{cases} \phi = ATAN2(-0.6596, 0.75) = -41.33^\circ \\ \phi = ATAN2(0.6596, -0.75) = 138.67^\circ \end{cases}$

$$\psi = ATAN2[(-n_x S\phi + n_y C\phi), (-o_x S\phi + o_y C\phi)]$$

or $\begin{cases} \psi = ATAN2[0.79, -0.613] = 127.8^\circ \\ \psi = ATAN2[-0.79, 0.613] = -52.2^\circ \end{cases}$

$$\theta = ATAN2[(a_x C\phi + a_y S\phi), a_z)]$$

or $\begin{cases} \theta = ATAN2[0.9988, 0.0474] = 87.28^\circ \\ \theta = ATAN2[-0.9988, 0.0474] = -87.28^\circ \end{cases}$

Problem 2.27
A frame ^{U}F was moved along its own *n*-axis a distance of 5 units, then rotated about its *o*-axis an angle of 60°, followed by a rotation of 60° about the *z*-axis, then translated about its *a*-axis for 3 units, and finally rotated 45° about the *x*-axis.

- Calculate the total transformation performed.
- What angles and movements would we have to make if we were to create the same location and orientation using Cartesian and RPY configurations?

Estimated student time to complete: 20-25 minutes
Prerequisite knowledge required: Text Section(s) 2.11

Solution:

$$^{U}T_B = Rot(x,45)Rot(z,60)Trans(5,0,0)Rot(o,60)Trans(0,0,3)$$

$$B_{new} = \begin{bmatrix} 0.25 & -0.866 & 0.433 & 3.8 \\ 0.918 & 0.354 & 0.177 & 3.59 \\ -0.306 & 0.354 & 0.884 & 5.71 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For Cartesian coordinates: $p_x = 3.8, \quad p_y = 3.59, \quad p_z = 5.71$

For RPY angles:

$$\phi_a = ATAN2(n_y, n_x) = ATAN2(0.918, 0.25) = 74.8^\circ$$

or $\quad \phi_a = ATAN2(-n_y, -n_x) = 254.8^\circ$

$$\phi_o = ATAN2(-n_z, (n_x C\phi_a + n_y S\phi_a)) = ATAN2\left(0.306, 0.951\right) = 17.8^\circ$$

$$\phi_o = ATAN2\left(0.306, -0.951\right) = 162.2^\circ$$

$$\phi_n = ATAN2((-a_y C\phi_a + a_x S\phi_a), (o_y C\phi_a - o_x S\phi_a))$$
$$= ATAN2\left[0.372, 0.928\right] = 21.8^\circ$$

$$\phi_n = ATAN2\left(-0.372, -0.928\right) = 201.8^\circ$$

Problem 2.28
Frames describing the base of a robot and an object are given relative to the Universe frame.

- Find a transformation ${}^{R}T_{H}$ of the robot configuration if the hand of the robot is to be placed on the object.
- By inspection, show whether this robot can be a 3-axis spherical robot, and if so, find α, β, r.
- Assuming that the robot is a 6-axis robot with Cartesian and Euler coordinates, find $p_x, p_y, p_z, \phi, \theta, \psi$.

$$ {}^{U}T_{obj} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{U}T_{R} = \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Estimated student time to complete: 20-25 minutes
Prerequisite knowledge required: Text Section(s) 2.11

Solution:

a. We want to place the hand on the part. Therefore, ${}^{R}T_{H} = {}^{U}T_{H}^{-1}\,{}^{U}T_{obj}$

$$ {}^{U}T_{obj} = {}^{U}T_{R}\,{}^{R}T_{H} = {}^{U}T_{R}\,{}^{R}T_{obj} \qquad \rightarrow \qquad {}^{R}T_{H} = {}^{U}T_{R}^{-1}\,{}^{U}T_{obj} $$

Substitute: $${}^{R}T_{H} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 5 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b. No. The 3,2 element of spherical coordinate transformation matrix should be zero. This is not.

c. $p_x = 5, \quad p_y = 1, \quad p_z = 0$

$$ \phi = ATAN2[0,-1] = 180^\circ \quad \text{or} \quad \phi = ATAN2[0,1] = 0^\circ $$
$$ \psi = ATAN2[-1(-1),0] = 90^\circ \quad \text{or} \quad \psi = ATAN2[-1(1),0] = 270^\circ $$
$$ \theta = ATAN2[-1(-1),0] = 90^\circ \quad \text{or} \quad \theta = ATAN2[-1(1),0] = 270^\circ $$

Then $$\begin{cases} \phi = 180^\circ \\ \psi = 90^\circ \\ \theta = 90^\circ \end{cases} \qquad \begin{cases} \phi = 0^\circ \\ \psi = 270^\circ \\ \theta = 270^\circ \end{cases}$$

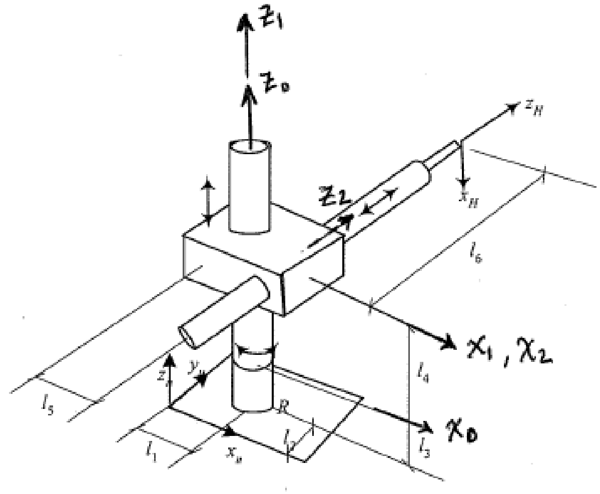
Problem 2.29
A 3-DOF robot arm has been designed for applying paint on flat walls, as shown.

- Assign coordinate frames as necessary based on the D-H representation.
- Fill out the parameters table.
- Find the ${}^{U}T_{H}$ matrix.

Estimated student time to complete: 30 minutes
Prerequisite knowledge required: Text Section(s) 2.12

Solution:

In this solution, the locations of the origins of some of the frames are arbitrary. Therefore, intermediate matrices might be different for each case. However, the final answer should be the same. Please also note that no particular reset position is specified.



#	θ	d	a	α
0-1	θ_1	l_4	0	0
1-2	0	0	l_5	-90
2-*H*	90	l_6	0	0

$$
{}^{U}T_{H} = {}^{U}T_{R}\,{}^{R}T_{H} = {}^{U}T_{R}A_1A_2A_3
$$

$$
= \begin{bmatrix} 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & l_2 \\ 0 & 0 & 1 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & l_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & l_5 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & l_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 & -C_1 & -S_1 & -S_1 l_6 + C_1 l_5 + l_1 \\ 0 & S_1 & C_1 & C_1 l_6 + S_1 l_5 + l_2 \\ -1 & 0 & 0 & l_4 + l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Problem 2.30
In the 2-DOF robot shown, the transformation matrix ${}^0T_H$ is given in symbolic form, as well as in numerical form for a specific location. The length of each link l_1 and l_2 is 1 ft. Calculate the values of θ_1 and θ_2 for the given location.

$$ {}^0T_H = \begin{bmatrix} C_{12} & -S_{12} & 0 & l_2C_{12}+l_1C_1 \\ S_{12} & C_{12} & 0 & l_2S_{12}+l_1S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.2924 & -0.9563 & 0 & 0.6978 \\ 0.9563 & -0.2924 & 0 & 0.8172 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

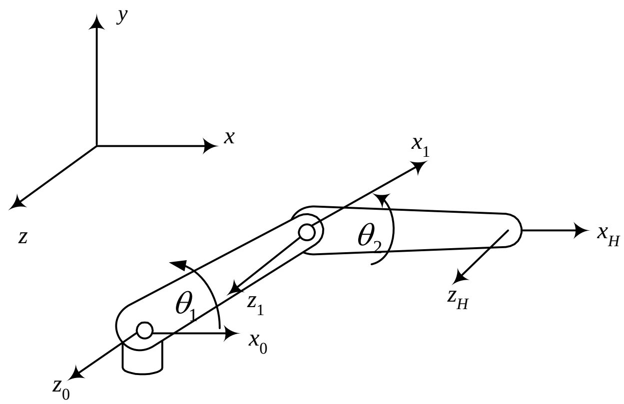


Figure P.2.30

Estimated student time to complete: 20 minutes
Prerequisite knowledge required: Text Section(s) 2.13

Solution:

This example is very simple, with only two degrees of freedom. Therefore, there is really no need to use the traditional method of inverse kinematic mentioned in Section 2.13. The joint angles can be determined as follows:

$$\theta_1+\theta_2 = ATAN2(S_{12},C_{12}) = ATAN2(0.9563,-0.2924) = 107^\circ$$

$$\begin{cases} l_2C_{12}+l_1C_1 = p_x \\ l_2S_{12}+l_1S_1 = p_y \end{cases} \rightarrow \begin{cases} C_1 = \left(p_x - l_2C_{12}\right)/l_1 \\ S_1 = \left(p_y - l_2S_{12}\right)/l_1 \end{cases} \rightarrow \theta_1 = ATAN2\left(\left(p_y-l_2S_{12}\right)/l_1, \left(p_x-l_2C_{12}\right)/l_1\right)$$

Substitute:

$$\theta_1 = ATAN2\left(\left(p_y-l_2S_{12}\right)/l_1, \left(p_x-l_2C_{12}\right)/l_1\right)$$

$$\theta_1 = ATAN2\left(\left(0.8172-0.9563\right),\left(0.6978+0.2924\right)\right) = ATAN2\left(-0.1391, 0.9902\right) = -8^\circ$$

$$\theta_2 = 107-(-8) = 115^\circ$$

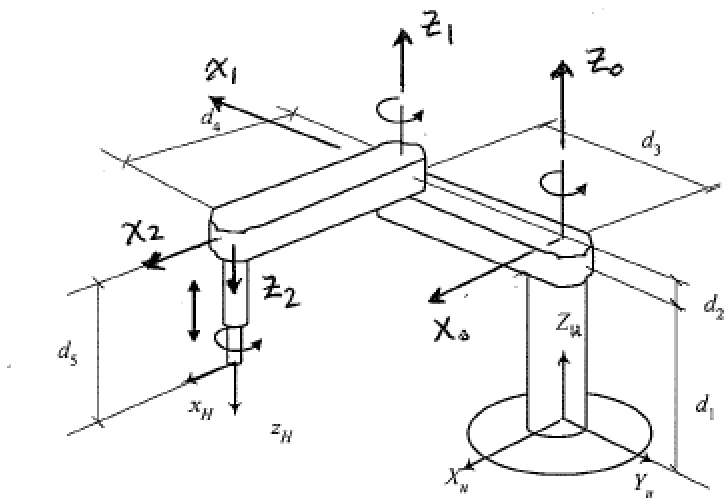
Problem 2.31
For the following SCARA-type robot:

- Assign the coordinate frames based on the D-H representation.
- Fill out the parameters table.
- Write all the *A* matrices.
- Write the ${}^{U}T_{H}$ matrix in terms of the *A* matrices.

Estimated student time to complete: 20-30 minutes
Prerequisite knowledge required: Text Section(s) 2.12

Solution:

In this solution, the locations of the origins of some of the frames are arbitrary. Therefore, intermediate matrices might be different for each case. However, the final answer should be the same. Please also note that no particular reset position is specified.



#	θ	d	a	α
0-1	θ_1	0	d_3	0
1-2	θ_2	0	d_4	180
2-*H*	θ_3	d_5	0	0

$$
{}^{U}T_{H} = {}^{U}T_{0}\,{}^{0}T_{H} = {}^{U}T_{0}A_1A_2A_3
$$

$$
= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_1+d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} C_1 & -S_1 & 0 & d_3C_1 \\ S_1 & C_1 & 0 & d_3S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} C_2 & S_2 & 0 & d_4C_2 \\ S_2 & -C_2 & 0 & d_4S_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} C_3 & -S_3 & 0 & 0 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 1 & d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

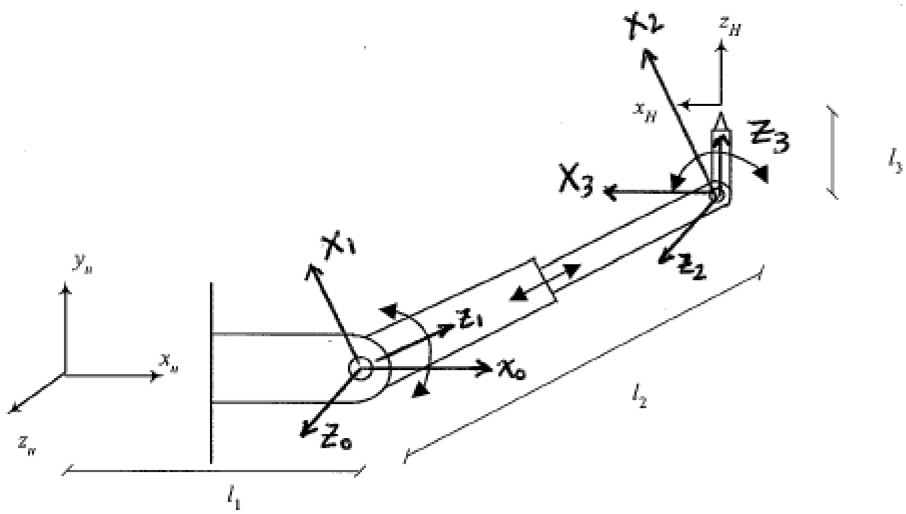
Multiplying these matrices will yield the total transformation.

Problem 2.32
A special 3-DOF spraying robot has been designed as shown:

- Assign the coordinate frames based on the D-H representation.
- Fill out the parameters table.
- Write all the *A* matrices.
- Write the ${}^{U}T_{H}$ matrix in terms of the *A* matrices.

Estimated student time to complete: 20-30 minutes
Prerequisite knowledge required: Text Section(s) 2.12

Solution:
Please note that no particular reset position is specified for this robot.



#	θ	d	a	α
0-1	$90+\theta_1$	0	0	90
1-2	0	l_2	0	-90
2-3	θ_3	0	0	90
3-*H*	0	l_3	0	0

$$ {}^{U}T_0 = \begin{bmatrix} 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_1 = \begin{bmatrix} -S_1 & 0 & C_1 & 0 \\ C_1 & 0 & S_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} $$

$$ A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_3 = \begin{bmatrix} C_3 & 0 & S_3 & 0 \\ S_3 & 0 & -C_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{U}T_H = {}^{U}T_0\, {}^{0}T_H = {}^{U}T_0 A_1 A_2 A_3 A_4 $$

The total transformation can be summarized by multiplying these matrices.

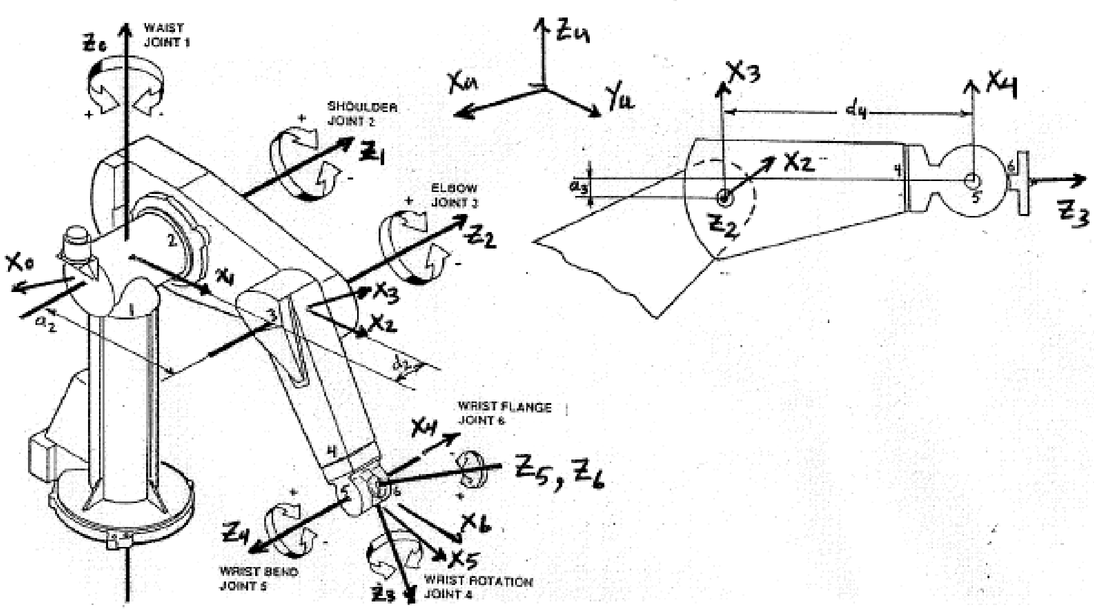
Problem 2.33
For the Unimation Puma 562, 6-axis robot shown,

- Assign the coordinate frames based on the D-H representation.
- Fill out the parameters table.
- Write all the *A* matrices.
- Find the ${}^{R}T_{H}$ matrix for the following values:
  Base height=27 in, d_2=6 in, a_2=15 in, a_3=1 in, d_4=18 in
  $\theta_1 = 0^\circ, \theta_2 = 45^\circ, \theta_3 = 0^\circ, \theta_4 = 0^\circ, \theta_5 = -45^\circ, \theta_6 = 0^\circ$

Estimated student time to complete: 30-40 minutes
Prerequisite knowledge required: Text Section(s) 2.12

Solution:

In this solution, the locations of the origins of some frames are arbitrary. Therefore, intermediate matrices might be different for each case. However, the final answer should be the same. To express the height from the table, the 27-in base height must be added to the p_z value. Please also note that no particular reset position is specified. Therefore, the results relate to a position and orientation that would correspond to the given transformations from a reset position. We assume at reset, there is 90 degrees between x_0 and x_1. Please note how frames 2, 3, and 4 relate to each other by referring to the side view of the arm.



(Reprinted with permission from Staubli Robotics.)

#	θ	D	a	α
0-1	θ_1+90	0	0	−90
1-2	θ_2	6"	15"	0
2-3	θ_3	0	1"	−90
3-4	θ_4	18"	0	90
4-5	θ_5	0	0	−90
5-6	θ_6	0	0	0

$$A_1 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0.707 & -0.707 & 0 & 10.61 \\ 0.707 & 0.707 & 0 & 10.61 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 18 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_5 = \begin{bmatrix} 0.707 & 0 & 0.707 & 0 \\ -0.707 & 0 & 0.707 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0T_6 = \begin{bmatrix} 0 & 1 & 0 & -6 \\ 1 & 0 & 0 & -1.41 \\ 0 & 0 & -1 & -24 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

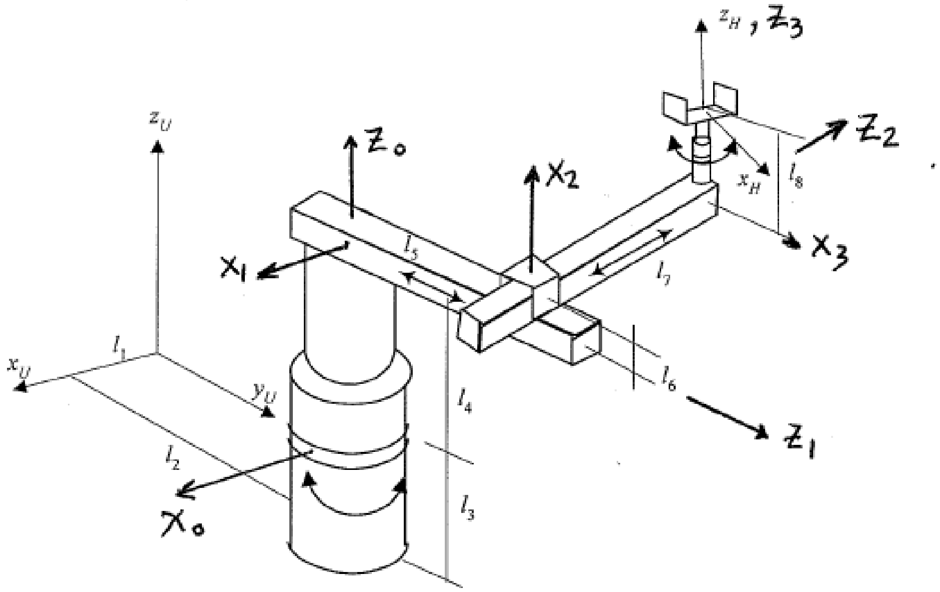
Problem 2.34
For the given 4-DOF robot:

- Assign appropriate frames for the Denavit-Hartenberg representation.
- Fill out the parameters table.
- Write an equation in terms of *A*-matrices that shows how ${}^{U}T_{H}$ can be calculated.

Estimated student time to complete: 20-30 minutes
Prerequisite knowledge required: Text Section(s) 2.12

Solution:

In this solution, the locations of the origins of some of the frames are arbitrary. Therefore, intermediate matrices might be different for each case. However, the final answer should be the same. Please also note that no particular reset position is specified.



#	θ	d	a	α
0-1	θ_1	l_4	0	−90
1-2	−90	l_5	l_6	90
2-3	90	l_7	0	90
3-*H*	θ_4	l_8	0	0

$$
{}^{U}T_{H} = {}^{U}T_{0}\,{}^{0}T_{H} = {}^{U}T_{0}A_1A_2A_3A_4 = \begin{bmatrix} 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & l_2 \\ 0 & 0 & 1 & l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} A_1A_2A_3A_4
$$

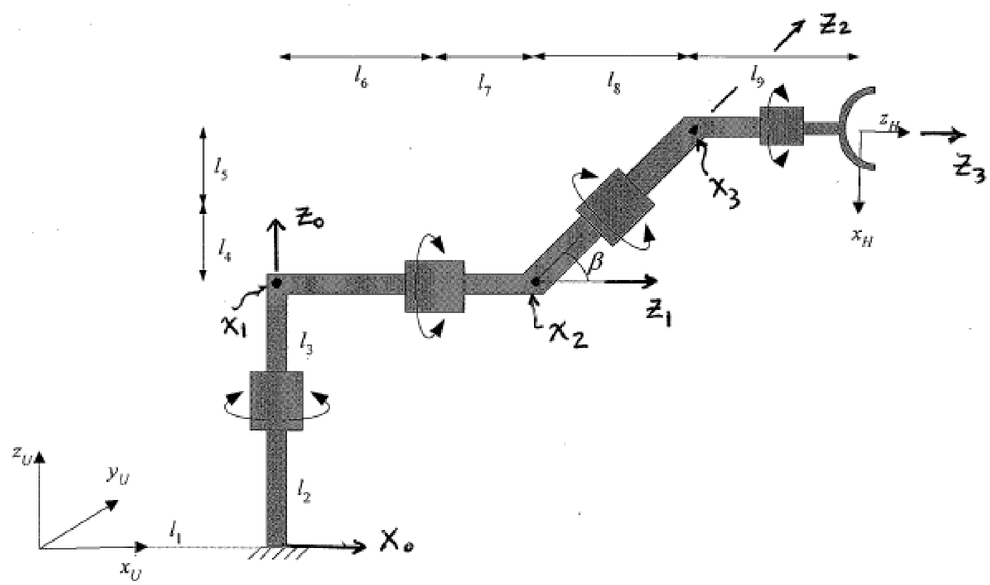
Problem 2.35
For the given 4-DOF robot designed for a specific operation:

- Assign appropriate frames for the Denavit-Hartenberg representation.
- Fill out the parameters table.
- Write an equation in terms of *A*-matrices that shows how ${}^{U}T_{H}$ can be calculated.

Estimated student time to complete: 20-30 minutes
Prerequisite knowledge required: Text Section(s) 2.12

Solution:

In this solution, the locations of the origins of some of the frames are arbitrary. Therefore, intermediate matrices might be different for each case. However, the final answer should be the same. Please also note that no particular reset position is specified.



#	θ	d	a	α
0-1	θ_1	l_2+l_3	0	−90
1-2	θ_2	l_6+l_7	0	β
2-3	θ_3	L^*	0	$-\beta$
3-H	θ_4	l_9	0	0

$$*L=(l_4+l_5)/\sin\beta$$

The total transformation for the robot can be determined from:

$${}^{U}T_{H}={}^{U}T_{0}\,{}^{0}T_{H}={}^{U}T_{0}A_1A_2A_3A_4=\begin{bmatrix}1&0&0&l_1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}A_1A_2A_3A_4$$

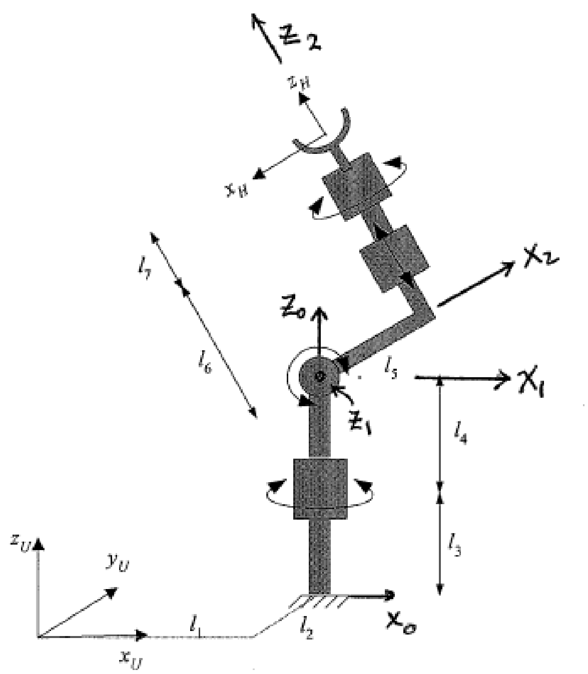
Problem 2.36
For the given specialty-designed 4-DOF robot:

- Assign appropriate frames for the Denavit-Hartenberg representation.
- Fill out the parameters table.
- Write an equation in terms of *A*-matrices that shows how ${}^{U}T_{H}$ can be calculated.

Estimated student time to complete: 20-30 minutes
Prerequisite knowledge required: Text Section(s) 2.12

Solution:

In this solution, the locations of the origins of some of the frames are arbitrary. Therefore, intermediate matrices might be different for each case. However, the final answer should be the same. Please also note that no particular reset position is specified.



#	θ	d	a	α
0-1	θ1	l3+l4	0	90
1-2	θ2	0	l5	-90
2-H	180+θ3	l6+l7	0	0

$$
{}^{U}T_{H} = {}^{U}T_{0}\,{}^{0}T_{H} = {}^{U}T_{0}A_1A_2A_3 = \begin{bmatrix} 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & l_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A_1A_2A_3
$$

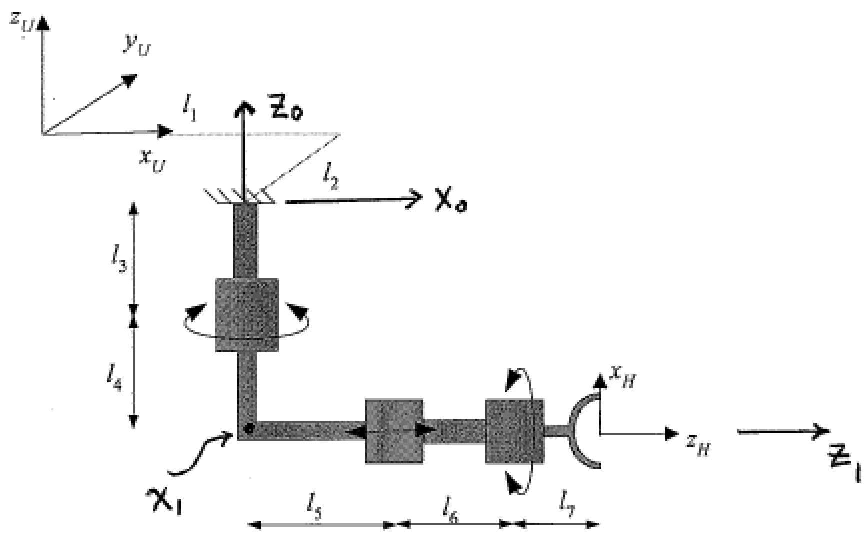
Problem 2.37
For the given 3-DOF robot:

- Assign appropriate frames for the Denavit-Hartenberg representation.
- Fill out the parameters table.
- Write an equation in terms of A-matrices that shows how ${}^{U}T_{H}$ can be calculated.

Estimated student time to complete: 20-30 minutes
Prerequisite knowledge required: Text Section(s) 2.12

Solution:

In this solution, the locations of the origins of some of the frames are arbitrary. Therefore, intermediate matrices might be different for each case. However, the final answer should be the same. Please also note that no particular reset position is specified. Please also note that there can be an additional frame between frames 1 and H. In that case, the two joint variables will be represented by two separate matrices.



#	θ	d	a	α
0-1	$-90+\theta_1$	$-l_3-l_4$	0	-90
1-H	θ_2	$l_5+l_6+l_7$	0	0

$$
{}^{U}T_{H} = {}^{U}T_{0}\,{}^{0}T_{H} = {}^{U}T_{0}A_1A_2 = \begin{bmatrix} 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & -l_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A_1A_2
$$

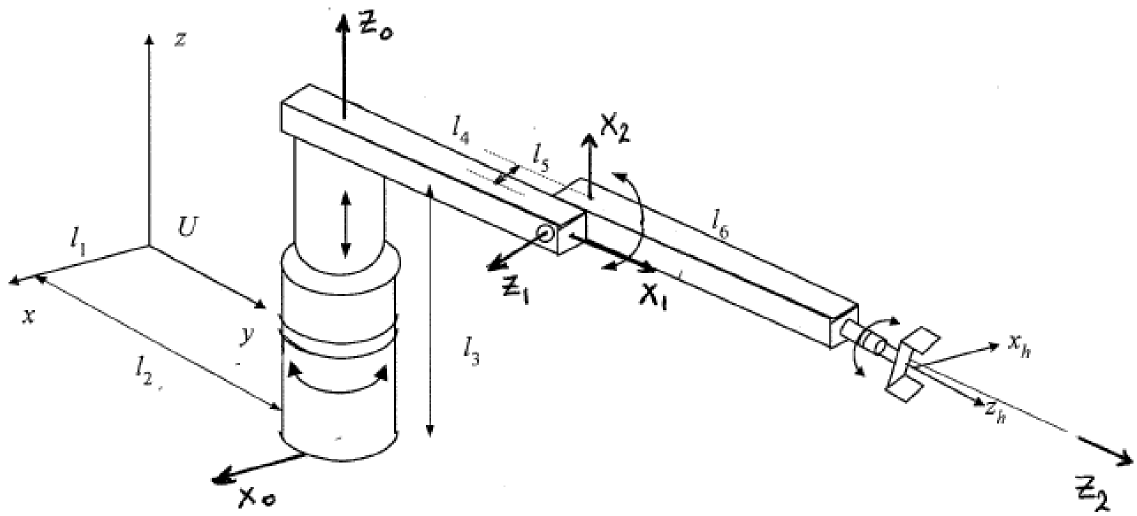
Problem 2.38
For the given 4-DOF robot:

- Assign appropriate frames for the Denavit-Hartenberg representation.
- Fill out the parameters table.
- Write an equation in terms of A matrices that shows how ${}^{U}T_{H}$ can be calculated.

Estimated student time to complete: 20-30 minutes
Prerequisite knowledge required: Text Section(s) 2.12

Solution:

In this solution, the locations of the origins of some of the frames are arbitrary. Therefore, intermediate matrices might be different for each case. However, the final answer should be the same. Please also note that no particular reset position is specified.



#	θ	d	a	α
0-1	θ_1	l_3	l_4	90
1-2	$90+\theta_2$	$-l_5$	0	90
2-*H*	θ_3	l_6	0	0

$$ {}^{U}T_{H} = {}^{U}T_{0}\,{}^{0}T_{H} = {}^{U}T_{0}A_1A_2A_3 = \begin{bmatrix} 1 & 0 & 0 & l_1 \\ 0 & 1 & 0 & l_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A_1A_2A_3 $$

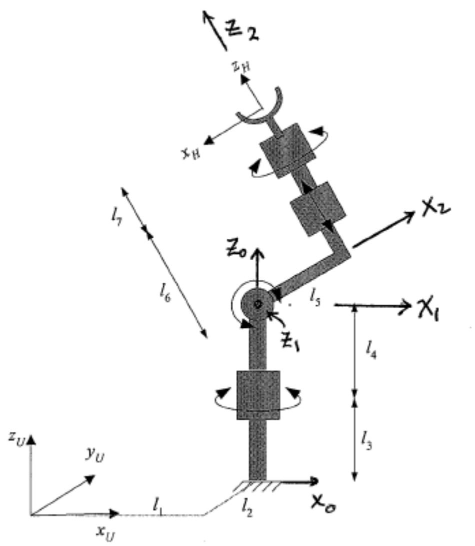
Problem 2.39

Derive the inverse kinematic equations for the robot of Problem 2.36.

Estimated student time to complete: 20-30 minutes if Problem 2.36 is already solved
Prerequisite knowledge required: Text Section(s) 2.13

Solution:

We assume all positions are made relative to the base of the robot, and therefore, ${}^{U}T_0$ is not included in the solution. Using the Denavit-Hartenberg representation and the joint parameters shown, we get:



#	θ	d	a	α
0-1	θ_1	l_3+l_4	0	90
1-2	θ_2	0	l_5	-90
2-H	$180+\theta_3$	l_6+l_7	0	0

$$A_1=\begin{bmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & l_3+l_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2=\begin{bmatrix} C_2 & 0 & -S_2 & l_5C_2 \\ S_2 & 0 & C_2 & l_5S_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_3=\begin{bmatrix} -C_3 & S_3 & 0 & 0 \\ -S_3 & -C_3 & 0 & 0 \\ 0 & 0 & 1 & l_6+l_7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{0}T_H = A_1A_2A_3 \quad \rightarrow \quad A_1^{-1}\,{}^{0}T_H = A_2A_3$$

$$\begin{bmatrix} C_1 & S_1 & 0 & 0 \\ 0 & 0 & 1 & -l_3-l_4 \\ S_1 & -C_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} C_2 & 0 & -S_2 & l_5C_2 \\ S_2 & 0 & C_2 & l_5S_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -C_3 & S_3 & 0 & 0 \\ -S_3 & -C_3 & 0 & 0 \\ 0 & 0 & 1 & l_6+l_7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiplying these matrices we get:

$$LHS=\begin{bmatrix} C_1n_x+S_1n_y & C_1o_x+S_1o_y & C_1a_x+S_1a_y & C_1p_x+S_1p_y \\ n_z & o_z & a_z & p_z-l_3-l_4 \\ S_1n_x-C_1n_y & S_1o_x-C_1o_y & S_1a_x-C_1a_y & S_1p_x-C_1p_y \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$RHS=\begin{bmatrix} -C_2C_3 & C_2S_3 & -S_2 & -S_2(l_6+l_7)+l_5C_2 \\ -S_2C_3 & S_2S_3 & C_2 & C_2(l_6+l_7)+l_5S_2 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From 3,3 or 3,4 elements we get:

$$S_1a_x-C_1a_y=0 \quad \rightarrow \quad \theta_1=\tan^{-1}\left(\frac{a_y}{a_x}\right)$$

$$S_1p_x-C_1p_y=0 \quad \rightarrow \quad \theta_1=\tan^{-1}\left(\frac{p_y}{p_x}\right)$$

From 1,3 and 2,3 elements we get:

$$\begin{cases} S_2=-(C_1a_x+S_1a_y) \\ C_2=a_z \end{cases} \quad \rightarrow \quad \theta_2=ATAN2\left(-C_1a_x-S_1a_y, a_z\right)$$

From 2,1 and 2,2 elements we get:

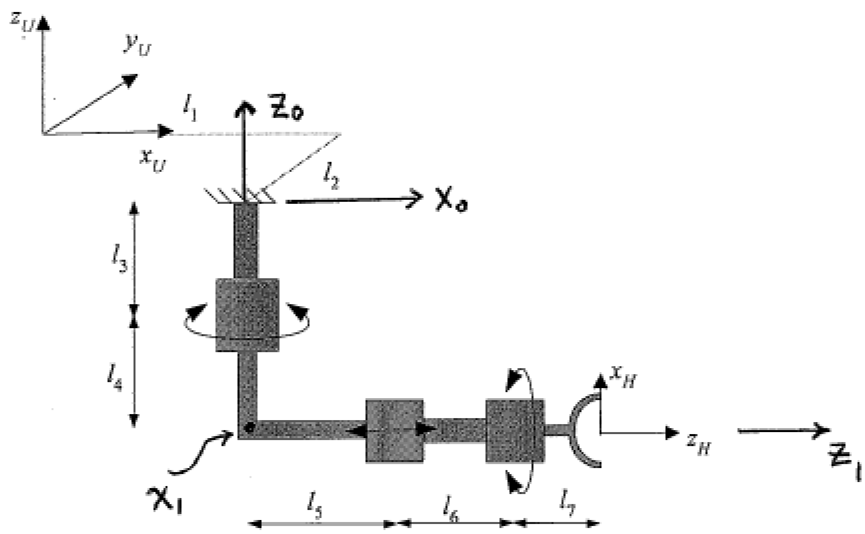
$$\begin{cases} S_3=S_1n_x-C_1n_y \\ C_3=S_1o_x-C_1o_y \end{cases} \quad \rightarrow \quad \theta_3=ATAN2\left(S_1n_x-C_1n_y, S_1o_x-C_1o_y\right)$$

Problem 2.40
Derive the inverse kinematic equations for the robot of Problem 2.37.

Estimated student time to complete: 20-30 minutes if Problem 2.37 is already solved
Prerequisite knowledge required: Text Section(s) 2.13

Solution:

We assume all positions are made relative to the base of the robot, and therefore, ${}^{U}T_{0}$ is not included in the solution. Using the Denavit-Hartenberg representation and the joint parameters shown, we get:



#	θ	d	a	α
0-1	$-90+\theta_1$	$-l_3-l_4$	0	-90
1-H	θ_2	$l_5+l_6+l_7$	0	0

$$A_1=\begin{bmatrix} S_1 & 0 & C_1 & 0 \\ -C_1 & 0 & S_1 & 0 \\ 0 & -1 & 0 & -l_3-l_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2=\begin{bmatrix} C_2 & -S_2 & 0 & 0 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 1 & l_6+l_6+l_7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{0}T_H = A_1A_2$$

$$\begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S_1 & 0 & C_1 & 0 \\ -C_1 & 0 & S_1 & 0 \\ 0 & -1 & 0 & -l_3 - l_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_2 & -S_2 & 0 & 0 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 1 & l_5 + l_6 + l_7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} S_1 C_2 & -S_1 S_2 & C_1 & C_1(l_5 + l_6 + l_7) \\ -C_1 C_2 & S_2 C_1 & S_1 & S_1(l_5 + l_6 + l_7) \\ -S_2 & -C_2 & 0 & -l_3 - l_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From 1,3 and 2,3 elements we get: $\begin{cases} S_1 = a_y \\ C_1 = a_x \end{cases} \rightarrow \quad \theta_1 = ATAN2\left(a_y, a_x\right)$

From 3,1 and 3,2 elements we get: $\begin{cases} S_2 = -n_z \\ C_2 = -o_z \end{cases} \rightarrow \quad \theta_2 = ATAN2\left(-n_z, -o_z\right)$

From 1,4 elements we get: $C_1(l_5 + l_6 + l_7) = p_x \quad \rightarrow \quad l_5 + l_6 + l_7 = \dfrac{p_x}{C_1}$

CHAPTER THREE

Problem 3.1
Suppose the location and orientation of a hand frame is expressed by the following matrix. What is the effect of a differential rotation of 0.15 radians about the *z*-axis, followed by a differential translation of [0.1, 0.1, 0.3]? Find the new location of the hand.

$$^{R}T_{H} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 3.6, 3.7.

Solution:

For $\delta z = 0.15, \quad dx = 0.1, \quad dy = 0.1, \quad dz = 0.3,$ we get:

$$\Delta = \begin{bmatrix} 0 & -0.15 & 0 & 0.1 \\ 0.15 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0.3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$d\left[{}^{R}T_{H}\right] = [\Delta]\left[{}^{R}T_{H}\right] = \begin{bmatrix} 0 & -0.15 & 0 & 0.1 \\ 0.15 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0.3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.15 & 0 & 0 & -0.95 \\ 0 & 0 & 0.15 & 0.4 \\ 0 & 0 & 0 & 0.3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[{}^{R}T_{H}\right]_{new} = \left[{}^{R}T_{H}\right]_{old} + d\left[{}^{R}T_{H}\right] = \begin{bmatrix} -0.15 & 0 & 1 & 1.05 \\ 1 & 0 & 0.15 & 7.4 \\ 0 & 1 & 0 & 5.3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 3.2
As a result of applying a set of differential motions to frame T shown, it has changed an amount dT as shown. Find the magnitude of the differential changes made ($dx, dy, dz, \delta x, \delta y, \delta z$) and the differential operator with respect to frame T.

$$T = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & -1 & 0 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad dT = \begin{bmatrix} 0 & -0.1 & -0.1 & 0.6 \\ 0.1 & 0 & 0 & 0.5 \\ -0.1 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 3.6-3.8.

Solution:

$$[dT] = [\Delta][T] \quad \rightarrow \quad [\Delta] = [dT][T]^{-1}$$

$$[\Delta] = \begin{bmatrix} 0 & -0.1 & -0.1 & 0.6 \\ 0.1 & 0 & 0 & 0.5 \\ -0.1 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 0 & -1 & 8 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -0.1 & 0.1 & 0.1 \\ 0.1 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By inspection: d = [0.1,0,0], δ = [0,0.1,0.1].

$$[dT] = [T][{}^{T}\Delta] \quad \rightarrow \quad [{}^{T}\Delta] = [T]^{-1}[dT]$$

$$[{}^{T}\Delta] = \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 0 & -1 & 8 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -0.1 & -0.1 & 0.6 \\ 0.1 & 0 & 0 & 0.5 \\ -0.1 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -0.1 & -0.1 & 0.6 \\ 0.1 & 0 & 0 & 0.5 \\ 0.1 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 3.3
Suppose the following frame was subjected to a differential translation of $d = [1 \quad 0 \quad 0.5]$ units and a differential rotation of $\delta = [0 \quad 0.1 \quad 0]$.

a. What is the differential operator relative to the reference frame?
b. What is the differential operator relative to the frame *A*?

$$A = \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Estimated student time to complete: 10 minutes
Prerequisite knowledge required: Text Section(s) 3.6-3.8

Solution:
a.

$$\Delta = \begin{bmatrix} 0 & 0 & 0.1 & 1 \\ 0 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b.

$$\left[{}^{A}\Delta\right] = [A]^{-1}[\Delta][A] = \begin{bmatrix} 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -10 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0.1 & 1 \\ 0 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.5 \\ 0 & 0.1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 3.4
The initial location and orientation of a robot's hand are given by T_1, and its new location and orientation after a change are given by T_2.
a. Find a transformation matrix Q that will accomplish this transform (in the Universe frame).
b. Assuming the change is small, find a differential operator Δ that will do the same.
c. By inspection, find a differential translation and a differential rotation that constitute this operator.

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_2 = \begin{bmatrix} 1 & 0 & 0.1 & 4.8 \\ 0.1 & 0 & -1 & 3.5 \\ 0 & 1 & 0 & 6.2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 3.6-3.8

Solution:

a.
$$T_2 = QT_1 \quad \rightarrow \quad Q = T_2T_1^{-1} = \begin{bmatrix} 1 & 0 & 0.1 & 4.8 \\ 0.1 & 0 & -1 & 3.5 \\ 0 & 1 & 0 & 6.2 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & -6 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -0.1 & 0 & 0.1 \\ 0.1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
b.
$T_2 = T_1 + dT = T_1 + \Delta T_1 = \left(\Delta + I\right)T_1 = QT_1$
Therefore: $\Delta = Q - I$

$$\text{or } \Delta = \begin{bmatrix} 0 & -0.1 & 0 & 0.1 \\ 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
c.
By inspection: d = [0.1, 0, 0.2] and δ = [0, 0, 0.1].

Problem 3.5
The hand frame of a robot and the corresponding Jacobian are given. For the given differential changes of the joints, compute the change in the hand frame, its new location, and corresponding Δ.

$$T_6 = \begin{bmatrix} 0 & 1 & 0 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{T_6}J = \begin{bmatrix} 8 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad D_\theta = \begin{bmatrix} 0 \\ 0.1 \\ -0.1 \\ 0.2 \\ 0.2 \\ 0 \end{bmatrix}$$

Estimated student time to complete: 30 minutes
Prerequisite knowledge required: Text Section(s) 3.6-3.11

Solution:

$$\left[{}^{T_6}D\right] = \left[{}^{T_6}J\right]\left[D_\theta\right] = \begin{bmatrix} 8 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0.1 \\ -0.1 \\ 0.2 \\ 0.2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.1 \\ 1 \\ 0.3 \\ 0.2 \\ 0 \end{bmatrix} = \begin{bmatrix} {}^{T_6}dx \\ {}^{T_6}dy \\ {}^{T_6}dz \\ {}^{T_6}\delta x \\ {}^{T_6}\delta y \\ {}^{T_6}\delta z \end{bmatrix}$$

Substitute in ${}^{T_6}\Delta$ to get: $${}^{T_6}\Delta = \begin{bmatrix} 0 & 0 & 0.2 & 0 \\ 0 & 0 & -0.3 & -0.1 \\ -0.2 & 0.3 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then

$$dT_6 = [T_6]\left[{}^{T_6}\Delta\right] = \begin{bmatrix} 0 & 1 & 0 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0.2 & 0 \\ 0 & 0 & -0.3 & -0.1 \\ -0.2 & 0.3 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -0.3 & -0.1 \\ 0 & 0 & 0.2 & 0 \\ 0.2 & -0.3 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[T_6]_{new} = [T_6]_{old} + [dT_6] = \begin{bmatrix} 0 & 1 & -0.3 & 9.9 \\ 1 & 0 & 0.2 & 5 \\ 0.2 & -0.3 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Delta = \left[T_6\right]\left[{}^{T_6}\Delta\right]\left[T_6\right]^{-1} = \begin{bmatrix} 0 & 1 & 0 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0.2 & 0 \\ 0 & 0 & -0.3 & -0.1 \\ -0.2 & 0.3 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & -5 \\ 1 & 0 & 0 & -10 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0.3 & -0.1 \\ 0 & 0 & -0.2 & 0 \\ -0.3 & 0.2 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 3.6
Two consecutive frames describe the old (T_1) and new (T_2) positions and orientations of the end of a 3-DOF robot. The corresponding Jacobian relative to T_1, relating to ${}^{T_1}dz, {}^{T_1}\delta x, {}^{T_1}\delta z$ is also given. Find values of joint differential motions $ds_1, d\theta_2, d\theta_3$ of the robot that caused the given frame change.

$$T_1 = \begin{bmatrix} 0 & 0 & 1 & 8 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_2 = \begin{bmatrix} 0 & 0.01 & 1 & 8.1 \\ 1 & -0.05 & 0 & 5 \\ 0.05 & 1 & -0.01 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{T_1}J = \begin{bmatrix} 5 & 10 & 0 \\ 3 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Estimated student time to complete: 30 minutes
Prerequisite knowledge required: Text Section(s) 3.6-3.11

Solution:

$$T_2 - T_1 = dT = \begin{bmatrix} 0 & 0.01 & 0 & 0.1 \\ 0 & -0.05 & 0 & 0 \\ 0.05 & 0 & -0.01 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = [\Delta][T_1] = [T_1]\left[{}^{T_1}\Delta\right]$$

$$\left[{}^{T_1}\Delta\right] = [T_1]^{-1} dT = \begin{bmatrix} 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -2 \\ 1 & 0 & 0 & -8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0.01 & 0 & 0.1 \\ 0 & -0.05 & 0 & 0 \\ 0.05 & 0 & -0.01 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -0.05 & 0 & 0 \\ 0.05 & 0 & -0.01 & 0 \\ 0 & 0.01 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, $\begin{cases} {}^{T_1}\delta_z = 0.05 \\ {}^{T_1}\delta_x = 0.01 \\ {}^{T_1}d_z = 0.1 \end{cases} \rightarrow \quad {}^{T_1}D = \begin{bmatrix} 0.1 \\ 0.01 \\ 0.05 \end{bmatrix}$

And: $\left[{}^{T_1}D\right] = \left[{}^{T_1}J\right][D_\theta] \quad \rightarrow \quad [D_\theta] = \left[{}^{T_1}J\right]^{-1}\left[{}^{T_1}D\right]$

Please see Appendix A for methods to calculate the inverse of a non-unitary matrix.
Substitute to get:

$$[D_\theta] = \left[{}^{T_1}J\right]^{-1}\left[{}^{T_1}D\right] = \begin{bmatrix} 5 & 10 & 0 \\ 3 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0.1 \\ 0.01 \\ 0.05 \end{bmatrix} = \begin{bmatrix} 0.0033 \\ 0.00833 \\ 0.0417 \end{bmatrix}$$

Problem 3.7
Two consecutive frames describe the old (T_1) and new (T_2) positions and orientations of the end of a 3-DOF robot. The corresponding Jacobian, relating to $dz, \delta x, \delta z$ is also given. Find values of joint differential motions $ds_1, d\theta_2, d\theta_3$ of the robot that caused the given frame change.

$$T_1 = \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_2 = \begin{bmatrix} -0.05 & 0 & 1 & 9.75 \\ 1 & -0.1 & 0.05 & 5.2 \\ 0.1 & 1 & 0 & 3.7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad J = \begin{bmatrix} 5 & 10 & 0 \\ 3 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Estimated student time to complete: 30 minutes
Prerequisite knowledge required: Text Section(s) 3.6-3.11

Solution:

$$T_2 - T_1 = dT = \begin{bmatrix} -0.05 & 0 & 0 & -0.25 \\ 0 & -0.1 & 0.05 & 0.2 \\ 0.1 & 0 & 0 & 0.7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[dT] = [\Delta][T_1] \quad \rightarrow \quad [\Delta] = [dT][T_1]^{-1} = \begin{bmatrix} -0.05 & 0 & 0 & -0.25 \\ 0 & -0.1 & 0.05 & 0.2 \\ 0.1 & 0 & 0 & 0.7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -3 \\ 1 & 0 & 0 & -10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -0.05 & 0 & 0 \\ 0.05 & 0 & -0.1 & 0 \\ 0 & 0.1 & 0 & 0.2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, $\begin{cases} d_z = 0.2 \\ \delta_x = 0.1 \\ \delta_z = 0.05 \end{cases} \quad \rightarrow \quad D = \begin{bmatrix} 0.2 \\ 0.1 \\ 0.05 \end{bmatrix}$

Substitute to get: $[D_\theta] = [J]^{-1}[D] = \begin{bmatrix} 5 & 10 & 0 \\ 3 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0.2 \\ 0.1 \\ 0.05 \end{bmatrix} = \begin{bmatrix} 0.0334 \\ 0.00334 \\ 0.0467 \end{bmatrix}$

Problem 3.8
A camera is attached to the hand frame T of a robot as given. The corresponding inverse Jacobian of the robot at this location is also given. The robot makes a differential motion, as a result of which, the change in the frame dT is recorded as given.

a. Find the new location of the camera after the differential motion.
b. Find the differential operator.
c. Find the joint differential motion values associated with this move.
d. Find how much the differential motions of the hand-frame (${}^{T}D$) should have been instead, if measured relative to frame T, to move the robot to the same new location as in part a.

$$T=\begin{bmatrix}0&1&0&3\\1&0&0&2\\0&0&-1&8\\0&0&0&1\end{bmatrix}\quad J^{-1}=\begin{bmatrix}1&0&0&0&0&0\\2&0&-1&0&0&0\\0&-0.2&0&0&0&0\\0&-1&0&0&1&0\\0&0&0&1&0&0\\1&0&0&0&0&1\end{bmatrix}\quad dT=\begin{bmatrix}-0.03&0&-0.1&0.79\\0&0.03&0&0.09\\0&-0.1&0&-0.4\\0&0&0&0\end{bmatrix}$$

Estimated student time to complete: 30-40 minutes
Prerequisite knowledge required: Text Section(s) 3.6-3.11

Solution:

a. $T_{new}=T_{old}+dT=\begin{bmatrix}-0.03&1&-0.1&3.79\\1&0.03&0&2.09\\0&-0.1&-1&7.6\\0&0&0&1\end{bmatrix}$

b.

$$\Delta=dT\cdot T^{-1}=\begin{bmatrix}-0.03&1&-0.1&3.79\\1&0.03&0&2.09\\0&-0.1&-1&7.6\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&1&0&-2\\1&0&0&-3\\0&0&-1&8\\0&0&0&1\end{bmatrix}=\begin{bmatrix}0&-0.03&0.1&0.05\\0.03&0&0&0\\-0.1&0&0&-0.1\\0&0&0&0\end{bmatrix}$$

c. $D=\begin{bmatrix}0.05&0&-0.1&0&0.1&0.03\end{bmatrix}^T$

$$\text{Therefore: } D_\theta=J^{-1}\cdot D=\begin{bmatrix}1&0&0&0&0&0\\2&0&-1&0&0&0\\0&-0.2&0&0&0&0\\0&-1&0&0&1&0\\0&0&0&1&0&0\\1&0&0&0&0&1\end{bmatrix}\begin{bmatrix}0.05\\0\\-0.1\\0\\0.1\\0.03\end{bmatrix}=\begin{bmatrix}0.05\\0.2\\0\\0.1\\0\\0.08\end{bmatrix}$$

d.

$$dT = T \cdot {}^{T}\Delta \quad \rightarrow \quad {}^{T}\Delta = T^{-1} \cdot dT = \begin{bmatrix} 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -0.03 & 0 & -0.1 & 0.79 \\ 0 & 0.03 & 0 & 0.09 \\ 0 & -0.1 & 0 & -0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.03 & 0 & 0.09 \\ -0.03 & 0 & -0.1 & 0.79 \\ 0 & 0.1 & 0 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$${}^{T}D = \begin{bmatrix} 0.09 & 0.79 & 0.4 & 0.1 & 0 & -0.03 \end{bmatrix}^{T}$$

Problem 3.9
A camera is attached to the hand frame T of a robot as given. The corresponding inverse Jacobian of the robot relative to the frame at this location is also given. The robot makes a differential motion, as a result of which, the change dT in the frame is recorded as given.

a. Find the new location of the camera after the differential motion.
b. Find the differential operator.
c. Find the joint differential motion values D_θ associated with this move.

$$T = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{T}J^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & -0.1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$dT = \begin{bmatrix} -0.02 & 0 & -0.1 & 0.7 \\ 0 & 0.02 & 0 & 0.08 \\ 0 & -0.1 & 0 & -0.3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Estimated student time to complete: 30-40 minutes
Prerequisite knowledge required: Text Section(s) 3.6-3.11

Solution:

a.

$$T_{new} = T_{old} + dT = \begin{bmatrix} -0.02 & 1 & -0.1 & 3.7 \\ 1 & 0.02 & 0 & 2.08 \\ 0 & -0.1 & -1 & 7.7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b.

$$\Delta = dT \cdot T^{-1} = \begin{bmatrix} -0.02 & 0 & -0.1 & 0.7 \\ 0 & 0.02 & 0 & 0.08 \\ 0 & -0.1 & 0 & -0.3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -0.02 & 0.1 & -0.06 \\ 0.02 & 0 & 0 & 0.02 \\ -0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

c.

$$^{T}\Delta = \left[T^{-1}\right]\left[\Delta\right]\left[T\right] = \left[T^{-1}\right]\left[dT\right] =$$

$$= \begin{bmatrix} 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -0.02 & 0 & -0.1 & 0.7 \\ 0 & 0.02 & 0 & 0.08 \\ 0 & -0.1 & 0 & -0.3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.02 & 0 & 0.08 \\ -0.02 & 0 & -0.1 & 0.7 \\ 0 & 0.1 & 0 & 0.3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore: $^{T}D = \begin{bmatrix} 0.08 & 0.7 & 0.3 & 0.1 & 0 & -0.02 \end{bmatrix}^{T}$

$$^{T}D = {}^{T}J \cdot D_{\theta} \quad \rightarrow \quad D_{\theta} = {}^{T}J^{-1} \cdot {}^{T}D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & -0.1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.08 \\ 0.7 \\ 0.3 \\ 0.1 \\ 0 \\ -0.02 \end{bmatrix} = \begin{bmatrix} 0.08 \\ -0.14 \\ -0.07 \\ -0.7 \\ 0.1 \\ 0.06 \end{bmatrix}$$

Problem 3.10
The hand frame T_H of a robot is given. The corresponding inverse Jacobian of the robot at this location relative to this frame is also shown. The robot makes a differential motion relative to this frame described as

$${}^{T_H}D = \begin{bmatrix} 0.05 & 0 & -0.1 & 0 & 0.1 & 0.1 \end{bmatrix}^T.$$

a. Find which joints must make a differential motion, and by how much, in order to create the indicated differential motions.
b. Find the change in the frame.
c. Find the new location of the frame after the differential motion.
d. Find how much the differential motions (given above) should have been if measured relative to the Universe, to move the robot to the same new location as in Part c.

$$T_H = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{T_H}J^{-1} = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & -0.2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Estimated student time to complete: 30 minutes
Prerequisite knowledge required: Text Section(s) 3.6-3.11

Solution:
a.

$$[D_\theta] = \left[{}^{T_H}J\right]^{-1}\left[{}^{T_H}D\right] = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & -0.2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0.05 \\ 0 \\ -0.1 \\ 0 \\ 0.1 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.2 \\ 0 \\ 0.1 \\ 0 \\ 0.15 \end{bmatrix}$$

Therefore, joints 1, 2, 4, and 6 must move as shown.

b. Substituting values from ${}^{T_H}D$ to form ${}^{T_H}\Delta$, we get:

$$dT = T_H \cdot {}^{T_H}\Delta = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -0.1 & 0.1 & 0.05 \\ 0.1 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & -0.1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & -0.1 & 0.1 & 0.05 \\ 0.1 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

c.

$$T_{H_{new}} = T_{H_{old}} + dT = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & -0.1 & 0.1 & 0.05 \\ 0.1 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.1 & 1 & 0 & 3 \\ 1 & -0.1 & 0.1 & 3.05 \\ 0.1 & 0 & -1 & 8.1 \\ 01 & 0 & 0 & 1 \end{bmatrix}$$

d.

$$dT = T_H \cdot {}^{T_H}\Delta = \Delta \cdot T_H$$

$$\Delta = dT \cdot T_H^{-1} = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & -0.1 & 0.1 & 0.05 \\ 0.1 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & -3 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0.1 & 0 & -0.3 \\ -0.1 & 0 & -0.1 & 1.15 \\ 0 & 0.1 & 0 & -0.2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, $D = \begin{bmatrix} -0.3 & 1.15 & -0.2 & 0.1 & 0 & -0.1 \end{bmatrix}^T$

Problem 3.11
The hand frame T of a robot is given. The corresponding inverse Jacobian of the robot at this location is also shown. The robot makes a differential motion described as
$D = \begin{bmatrix} 0.05 & 0 & -0.1 & 0 & 0.1 & 0.1 \end{bmatrix}^T$.

a. Find which joints must make a differential motion, and by how much, in order to create the indicated differential motions.
b. Find the change in the frame.
c. Find the new location of the frame after the differential motion.
d. Find how much the differential motions (given above) should have been, if measured relative to Frame T, to move the robot to the same new location as in Part c.

$$T = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad J^{-1} = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & -0.2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Estimated student time to complete: 30 minutes
Prerequisite knowledge required: Text Section(s) 3.6-3.11

Solution:

a.

$$D_\theta = J^{-1} \cdot D = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & -0.2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.05 \\ 0 \\ -0.1 \\ 0 \\ 0.1 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.2 \\ 0 \\ 0.1 \\ 0 \\ 0.15 \end{bmatrix}$$

Therefore, joints 1, 2, 4, and 6 must move as shown.

b. Using D given above to form Δ,

$$dT = \Delta \cdot T = \begin{bmatrix} 0 & -0.1 & 0.1 & 0.05 \\ 0.1 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & -0.1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.1 & 0 & -0.1 & 0.55 \\ 0 & 0.1 & 0 & 0.3 \\ 0 & -0.1 & 0 & -0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

c.

$$T_{new} = T_{old} + dT = \begin{bmatrix} -0.1 & 1 & -0.1 & 3.55 \\ 1 & 0.1 & 0 & 3.3 \\ 0 & -0.1 & -1 & 7.6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

d.

$${}^{T}\Delta = \left[T^{-1}\right][\Delta][T] = \left[T^{-1}\right][dT] = \begin{bmatrix} 0 & 1 & 0 & -3 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -0.1 & 0 & -0.1 & 0.55 \\ 0 & 0.1 & 0 & 0.3 \\ 0 & -0.1 & 0 & -0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.1 & 0 & 0.3 \\ -0.1 & 0 & -0.1 & 0.55 \\ 0 & 0.1 & 0 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The differential motions should have been:

$${}^{T}D = \begin{bmatrix} 0.3 & 0.55 & 0.4 & 0.1 & 0 & -0.1 \end{bmatrix}$$

Problem 3.12
Calculate the ${}^{T_6}J_{21}$ element of the Jacobian for the revolute robot of Example 2.25.

Estimated student time to complete: 20 minutes
Prerequisite knowledge required: Text Section(s) 3.10

Solution:

The robot has 6 revolute joints. From Equation 3.26:

${}^{T_6}J_{2i} = \left(-o_x p_y + o_y p_x\right)$ where for i=1 we use ${}^{o}T_6 = A_1A_2A_3A_4A_5A_6$ given in Equation 2.59.
Substitute to get:

$$\begin{aligned}{}^{T_6}J_{21} &= -\left[C_1(-C_{234}C_5C_6 - S_{234}C_6) + S_1S_5S_6\right]\times\left[S_1(C_{234}a_4 + C_{23}a_3 + C_2a_2)\right] \\ &\quad +\left[S_1(-C_{234}C_5C_6 - S_{234}C_6) - C_1S_5S_6\right]\times\left[C_1(C_{234}a_4 + C_{23}a_3 + C_2a_2)\right]\end{aligned}$$

Problem 3.13
Calculate the ${}^{T_6}J_{16}$ element of the Jacobian for the revolute robot of Example 2.25.

Estimated student time to complete: 20 minutes
Prerequisite knowledge required: Text Section(s) 3.10

Solution:

The robot has 6 revolute joints. From Equation 3.26:

${}^{T_6}J_{1i} = \left(-n_x p_y + n_y p_x\right)$ where for i=6 we use ${}^5T_6 = A_6$ given in Equation 2.57. Substitute to get:

$${}^{T_6}J_{21} = -\left[\left(C_6\right)\left(0\right)\right] + \left[\left(S_6\right)\left(0\right)\right] = 0$$

Problem 3.14
Using Equation (2.34), differentiate proper elements of the matrix to develop a set of symbolic equations for joint differential motions of a cylindrical robot and write the corresponding Jacobian.

Estimated student time to complete: 10 minutes
Prerequisite knowledge required: Text Section(s) 3.10-3.11

Solution:

For a cylindrical coordinate (without additional rotation) there can only be 3 variables. The Jacobian will be a 3×3 matrix.

$$T_{cyl}(r,\alpha,l)=\begin{bmatrix} C\alpha & -S\alpha & 0 & rC\alpha \\ S\alpha & C\alpha & 0 & rS\alpha \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then $\begin{cases} p_x = rC\alpha \\ p_y = rS\alpha \\ p_z = l \end{cases}$ and $\begin{cases} dp_x = drC\alpha - rS\alpha d\alpha \\ dp_y = drS\alpha + rC\alpha d\alpha \\ dp_z = dl \end{cases}$

Therefore, $\begin{bmatrix} dp_x \\ dp_y \\ dp_z \end{bmatrix} = \begin{bmatrix} C\alpha & -rS\alpha & 0 \\ S\alpha & rC\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} dr \\ d\alpha \\ dl \end{bmatrix}$

Problem 3.15
Using Equation (2.36), differentiate proper elements of the matrix to develop a set of symbolic equations for joint differential motions of a spherical robot and write the corresponding Jacobian.

Estimated student time to complete: 10 minutes
Prerequisite knowledge required: Text Section(s) 3.10-3.11

Solution:

For a spherical coordinate (without additional rotation) there can only be 3 variables. The Jacobian will be a 3×3 matrix.

$$T_{sph}(r,\beta,\gamma)=\begin{bmatrix} C\beta C\gamma & -S\gamma & S\beta C\gamma & rS\beta C\gamma \\ C\beta S\gamma & C\gamma & S\beta S\gamma & rS\beta S\gamma \\ -S\beta & 0 & C\beta & rC\beta \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then $\begin{cases} p_x = rS\beta.C\gamma \\ p_y = rS\beta.S\gamma \\ p_z = rC\beta \end{cases}$ and $\begin{cases} dp_x = drS\beta.C\gamma + rC\beta.C\gamma d\beta - rS\beta.S\gamma d\gamma \\ dp_y = drS\beta.S\gamma + rC\beta.S\gamma d\beta + rS\beta.C\gamma d\gamma \\ dp_z = drC\beta - rS\beta d\beta \end{cases}$

Therefore, $\begin{bmatrix} dp_x \\ dp_y \\ dp_z \end{bmatrix} = \begin{bmatrix} S\beta.C\gamma & rC\beta.C\gamma & -rS\beta.S\gamma \\ S\beta.S\gamma & rC\beta.S\gamma & rS\beta.C\gamma \\ C\beta & -rS\beta & 0 \end{bmatrix}\begin{bmatrix} dr \\ d\beta \\ d\gamma \end{bmatrix}$

Problem 3.16
For a cylindrical robot, the three joint velocities are given for a corresponding location. Find the three components of the velocity of the hand frame.

$\dot{r}$ =0.1 in/sec, $\dot{\alpha}$ =0.05 rad/sec, $\dot{l}$ =0.2 in/sec, *r*=15 in, $\alpha = 30^\circ$, *l*=10 in.

Estimated student time to complete: 10 or 20 minutes
Prerequisite knowledge required: Text Section(s) 3.10-3.11

Solution:

Referring to the solution of Problem 3.14, and dividing both sides by *dt*, substitute the given values into the equation to get:

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} = \begin{bmatrix} C\alpha & -rS\alpha & 0 \\ S\alpha & rC\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{r} \\ \dot{\alpha} \\ \dot{l} \end{bmatrix} = \begin{bmatrix} 0.866 & -7.5 & 0 \\ 0.5 & 13 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.05 \\ 0.2 \end{bmatrix} = \begin{bmatrix} -0.288 \\ 0.7 \\ 0.2 \end{bmatrix} \text{in/sec}$$

Problem 3.17
For a spherical robot, the three joint velocities are given for a corresponding location. Find the three components of the velocity of the hand frame.

$\dot{r}$ =2 in/sec, $\dot{\beta}$ =0.05 rad/sec, $\dot{\gamma}$ =0.1 rad/sec, r=20 in, $\beta = 60^\circ$, $\gamma = 30^\circ$.

Estimated student time to complete: 10 or 20 minutes
Prerequisite knowledge required: Text Section(s) 3.10-3.11

Solution:

Referring to the solution of Problem 3.15, and dividing both sides by *dt*, substitute the given values into the equation to get:

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} = \begin{bmatrix} S\beta C\gamma & rC\beta C\gamma & -rS\beta S\gamma \\ S\beta S\gamma & rC\beta S\gamma & rS\beta C\gamma \\ C\beta & -rS\beta & 0 \end{bmatrix} \begin{bmatrix} \dot{r} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} 0.75 & 8.66 & -8.66 \\ 0.433 & 5 & 15 \\ 0.5 & -17.32 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0.05 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 1.067 \\ 2.616 \\ 0.134 \end{bmatrix}$$

Problem 3.18
For a spherical robot, the three joint velocities are given for a corresponding location. Find the three components of the velocity of the hand frame.

$\dot{r}$ =1 unit/sec, $\dot{\beta}$ =1 rad/sec, $\dot{\gamma}$ =1 rad/sec, r=5 units, $\beta = 45^\circ$, $\gamma = 45^\circ$.

Estimated student time to complete: 10 or 20 minutes
Prerequisite knowledge required: Text Section(s) 3.10-3.11

Solution:

Referring to the solution of Problem 3.15, and dividing both sides by *dt*, substitute the given values into the equation to get:

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} = \begin{bmatrix} S\beta C\gamma & rC\beta C\gamma & -rS\beta S\gamma \\ S\beta S\gamma & rC\beta S\gamma & rS\beta C\gamma \\ C\beta & -rS\beta & 0 \end{bmatrix} \begin{bmatrix} \dot{r} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} 0.5 & 2.5 & -3.535 \\ 0.5 & 2.5 & 3.535 \\ 0.707 & -3.535 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.535 \\ 6.535 \\ -2.828 \end{bmatrix}$$

Problem 3.19
For a cylindrical robot, the three components of the velocity of the hand frame are given for a corresponding location. Find the required three joint velocities that will generate the given hand frame velocity.

$\dot{x} = 1$ in/sec, $\dot{y} = 3$ in/sec, $\dot{z} = 5$ in/sec, $\alpha = 45^\circ$, $r = 20$ in, $l = 25$ in

Estimated student time to complete: 10 or 25 minutes
Prerequisite knowledge required: Text Section(s) 3.12

Solution:

Referring to the solution of Problem 3.16, substitute to get:

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} = \begin{bmatrix} C\alpha & -rS\alpha & 0 \\ S\alpha & rC\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{r} \\ \dot{\alpha} \\ \dot{l} \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.707 & -14.14 & 0 \\ 0.707 & 14.14 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{r} \\ \dot{\alpha} \\ \dot{l} \end{bmatrix}$$

$$\begin{cases} 0.707\dot{r} - 14.14\dot{\alpha} = 1 \\ 0.707\dot{r} + 14.14\dot{\alpha} = 3 \\ \dot{l} = 5 \end{cases} \quad \rightarrow \quad \begin{cases} \dot{r} = 2.83 \quad \text{in/sec} \\ \dot{\alpha} = 0.071 \quad {}^\circ\text{/sec} \\ \dot{l} = 5 \quad \text{in/sec} \end{cases}$$

Please note that this problem could have been solved by calculating the inverse of the Jacobian as well.

Problem 3.20
For a spherical robot, the three components of the velocity of the hand frame are given for a corresponding location. Find the required three joint velocities that will generate the given hand frame velocity.

$$\dot{x} = 5 \text{ in/sec}, \dot{y} = 9 \text{ in/sec}, \dot{z} = 6 \text{ in/sec}, \beta = 60^\circ, r = 20 \text{ in}, \gamma = 30^\circ$$

Estimated student time to complete: 15 or 30 minutes
Prerequisite knowledge required: Text Section(s) 3.12

Solution:

Referring to the solution of Problem 3.17, substitute to get:

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} = \begin{bmatrix} S\beta C\gamma & rC\beta C\gamma & -rS\beta S\gamma \\ S\beta S\gamma & rC\beta S\gamma & rS\beta C\gamma \\ C\beta & -rS\beta & 0 \end{bmatrix} \begin{bmatrix} \dot{r} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 5 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 0.75 & 8.66 & -8.66 \\ 0.433 & 5 & 15 \\ 0.5 & -17.32 & 0 \end{bmatrix} \begin{bmatrix} \dot{r} \\ \dot{\alpha} \\ \dot{l} \end{bmatrix}$$

$$\begin{cases} 0.75\dot{r} + 8.66\dot{\beta} - 8.66\dot{\gamma} = 5 \\ 0.433\dot{r} + 5\dot{\beta} + 15\dot{\gamma} = 9 \\ 0.5\dot{r} - 17.32\dot{\beta} = 6 \end{cases} \quad \rightarrow \quad \begin{cases} \dot{r} = 10.647 \quad \text{in/sec} \\ \dot{\beta} = -0.039 \quad {}^\circ\text{/sec} \\ \dot{\gamma} = 0.306 \quad {}^\circ\text{/sec} \end{cases}$$

Please note that this problem could have been solved by calculating the inverse of the Jacobian as well.

CHAPTER FOUR

Problem 4.1
Using Lagrangian mechanics derive the equations of motion of a cart with two tires under the cart as shown:

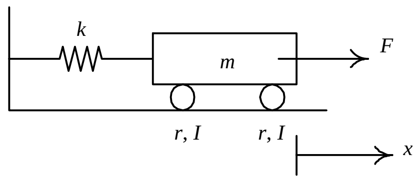
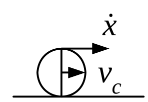


Figure P.4.1

Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 4.2

Solution:

I. Kinematics:
Since the rollers are in contact with the block at the top, the contact-point velocity is $\dot{x}$.
Therefore:



$$\dot{x} = 2r\omega \quad \rightarrow \quad \omega = \frac{\dot{x}}{2r}$$

II. Dynamics:

$$K = \frac{1}{2}m\dot{x}^2 + 2\left(\frac{1}{2}m_w v_c^2 + \frac{1}{2}I_c\omega^2\right) = \frac{1}{2}m\dot{x}^2 + \frac{m_w}{4}\dot{x}^2 + \frac{I_c}{4r^2}\dot{x}^2 = m_{eff}\dot{x}^2$$

where $\left(\frac{1}{2}m + \frac{m_w}{4} + \frac{I_c}{4r^2}\right) = m_{eff}$

$$P = \frac{1}{2}kx^2$$

Therefore, $L = K - P = m_{eff}\dot{x}^2 - \frac{1}{2}kx^2$

$$\frac{\partial L}{\partial \dot{x}} = 2m_{eff}\dot{x} \quad \rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 2m_{eff}\ddot{x}$$

$$\frac{\partial L}{\partial x} = -kx$$

$$F = 2m_{eff}\ddot{x} + kx$$

Problem 4.2
Calculate the total kinetic energy of the link *AB*, attached to a roller with negligible mass, as shown.

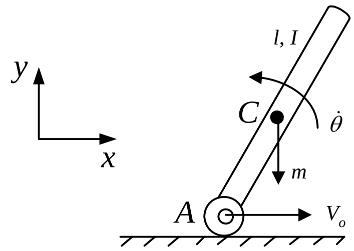


Figure P.4.2

Estimated student time to complete: 20-30 minutes
Prerequisite knowledge required: Text Section(s) 4.2

Solution:

First, find the velocity of point *C*, assuming that the roller has negligible mass too:

$$\mathbf{v}_C = \mathbf{v}_A + \mathbf{v}_{C/A} = \mathbf{v}_o + \dot{\boldsymbol{\theta}} \times \frac{\mathbf{l}}{2} = (v_o)\mathbf{i} + \dot{\theta}\mathbf{k} \times \left(\frac{l}{2}\cos\theta \mathbf{i} + \frac{l}{2}\sin\theta \mathbf{j} \right)$$

$$= (v_o)\mathbf{i} + \left(\frac{l}{2}\cos\theta \right)\dot{\theta}\, j - \left(\frac{l}{2}\sin\theta \right)\dot{\theta}\mathbf{i} = \left(v_o - \frac{l\sin\theta}{2}\dot{\theta} \right)\mathbf{i} + \frac{l\cos\theta}{2}\dot{\theta}\mathbf{j}$$

$$v_C^2 = v_o{}^2 + \frac{l^2\sin^2\theta}{4}\dot{\theta}^2 - v_o l\sin\theta\dot{\theta} + \frac{l^2\cos^2\theta}{4}\dot{\theta}^2 = v_o{}^2 + \frac{l^2}{4}\dot{\theta}^2 - v_o l\sin\theta\dot{\theta}$$

$$K = \frac{1}{2}mv_C^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}m\left(v_o{}^2 + \frac{l^2}{4}\dot{\theta}^2 - v_o l\sin\theta\dot{\theta} \right) + \frac{1}{2}I\dot{\theta}^2$$

Problem 4.3
Derive the equations of motion for the 2-link mechanism with distributed mass, as shown.

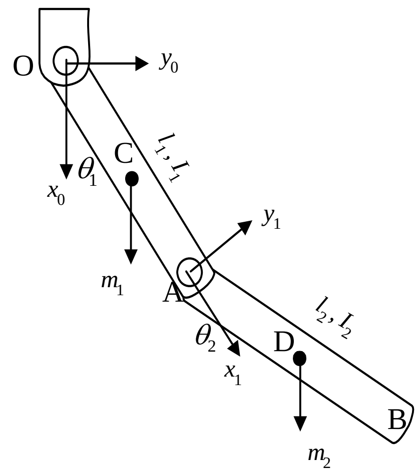


Figure P.4.3

Estimated student time to complete: 1.5-2 hours
Prerequisite knowledge required: Text Section(s) 4.2

Solution:

I. Kinematic: First we write equations describing the kinematic relationships.

$$\begin{cases} x_D = l_1 C_1 + \frac{l_2}{2} C_{12} \\ y_D = l_1 S_1 + \frac{l_2}{2} S_{12} \end{cases} \rightarrow \begin{cases} \dot{x}_D = -l_1\dot{\theta}_1 S_1 - \frac{l_2}{2}\left(\dot{\theta}_1 + \dot{\theta}_2\right) S_{12} \\ \dot{y}_D = l_1\dot{\theta}_1 C_1 + \frac{l_2}{2}\left(\dot{\theta}_1 + \dot{\theta}_2\right) C_{12} \end{cases}$$

The velocity of point D will be:

$$v_D^2 = \dot{x}_D^2 + \dot{y}_D^2 = l_1^{\,2}\dot{\theta}_1^{\,2} S_1^{\,2} + \frac{l_2^{\,2}}{4}\left(\dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2} + 2\dot{\theta}_1\dot{\theta}_2\right) S_{12}^{\,2} + l_1 l_2 \dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) S_1 S_{12}$$

$$+ l_1^{\,2}\dot{\theta}_1^{\,2} C_1^{\,2} + \frac{l_2^{\,2}}{4}\left(\dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2} + 2\dot{\theta}_1\dot{\theta}_2\right) C_{12}^{\,2} + l_1 l_2 \dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) C_1 C_{12}$$

$$= l_1^{\,2}\dot{\theta}_1^{\,2} + \frac{l_2^{\,2}}{4}\left(\dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2} + 2\dot{\theta}_1\dot{\theta}_2\right) + l_1 l_2 \dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) C_2$$

Rearrange:

$$v_D^2 = \dot{\theta}_1^{\,2}\left(l_1^{\,2} + \frac{l_2^{\,2}}{4} + l_1 l_2 C_2\right) + \dot{\theta}_2^{\,2}\left(\frac{l_2^{\,2}}{4}\right) + \dot{\theta}_1\dot{\theta}_2\left(\frac{l_2^{\,2}}{2} + l_1 l_2 C_2\right)$$

II: Kinetics:

Kinetic Energy:

$$
\begin{aligned}
K &= K_1 + K_2 = \frac{1}{2} I_O \dot{\theta}_1^2 + \frac{1}{2} I_D \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + \frac{1}{2} m_2 v_D^2 \\
&= \frac{1}{2}\left(\frac{1}{3} m_1 l_1^2\right)\dot{\theta}_1^2 + \frac{1}{2}\left(\frac{1}{12} m_2 l_2^2\right)\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + \frac{1}{2} m_2 v_D^2 \\
&= \dot{\theta}_1^2\left(\frac{1}{6} m_1 l_1^2 + \frac{1}{6} m_2 l_2^2 + \frac{1}{2} m_2 l_1^2 + \frac{1}{2} m_2 l_1 l_2 C_2\right) + \dot{\theta}_2^{\,2}\left(\frac{1}{6} m_2 l_2^2\right) + \dot{\theta}_1 \dot{\theta}_2\left(\frac{1}{3} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 C_2\right)
\end{aligned}
$$

Potential Energy:

$$
P = P_1 + P_2 = m_1 g \frac{l_1}{2}\left(1 - C_1\right) + m_2 g l_1 \left(1 - C_1\right) + m_2 g \frac{l_2}{2}\left(1 - C_{12}\right)
$$

Lagrangian:

$$
\begin{aligned}
L = K - P &= \dot{\theta}_1^2\left(\frac{1}{6} m_1 l_1^2 + \frac{1}{6} m_2 l_2^2 + \frac{1}{2} m_2 l_1^2 + \frac{1}{2} m_2 l_1 l_2 C_2\right) + \dot{\theta}_2^{\,2}\left(\frac{1}{6} m_2 l_2^2\right) + \dot{\theta}_1 \dot{\theta}_2\left(\frac{1}{3} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 C_2\right) \\
&+ \left(m_1 g \frac{l_1}{2} + m_2 g l_1\right) C_1 + m_2 g \frac{l_2}{2} C_{12} - m_1 g \frac{l_1}{2} - m_2 g l_1 - m_2 g \frac{l_2}{2}
\end{aligned}
$$

Derivatives and equations of motion:

$$
\frac{\partial L}{\partial \dot{\theta}_1} = 2\dot{\theta}_1\left(\frac{1}{6} m_1 l_1^2 + \frac{1}{6} m_2 l_2^2 + \frac{1}{2} m_2 l_1^2 + \frac{1}{2} m_2 l_1 l_2 C_2\right) + \dot{\theta}_2\left(\frac{1}{3} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 C_2\right)
$$

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) &= 2\ddot{\theta}_1\left(\frac{1}{6} m_1 l_1^2 + \frac{1}{6} m_2 l_2^2 + \frac{1}{2} m_2 l_1^2 + \frac{1}{2} m_2 l_1 l_2 C_2\right) + \ddot{\theta}_2\left(\frac{1}{3} m_2 l_2^2 + \frac{1}{2} m_2 l_1 l_2 C_2\right) \\
&\quad - \dot{\theta}_1 \dot{\theta}_2 m_2 l_1 l_2 S_2 - \frac{1}{2}\dot{\theta}_2^{\,2} m_2 l_1 l_2 S_2
\end{aligned}
$$

$$
\frac{\partial L}{\partial \theta_1} = -\left(m_1 g \frac{l_1}{2} + m_2 g l_1\right) S_1 - m_2 g \frac{l_2}{2} S_{12}
$$

Substitute to get:

$$T_1 = \left(\frac{1}{3}m_1 l_1^2 + \frac{1}{3}m_2 l_2^2 + m_2 l_1{}^2 + m_2 l_1 l_2 C_2\right)\ddot{\theta}_1 + \left(\frac{1}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right)\ddot{\theta}_2$$
$$- m_2 l_1 l_2 S_2 \dot{\theta}_1 \dot{\theta}_2 - \frac{1}{2}m_2 l_1 l_2 S_2 \dot{\theta}_2{}^2 + \left(\frac{1}{2}m_1 + m_2\right) g l_1 S_1 + \frac{1}{2}m_2 g l_2 S_{12}$$

For the second variable:

$$\frac{\partial L}{\partial \dot{\theta}_2} = \frac{1}{3}m_2 l_2^2 \dot{\theta}_2 + \dot{\theta}_1\left(\frac{1}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right)$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = \frac{1}{3}m_2 l_2^2 \ddot{\theta}_2 + \ddot{\theta}_1\left(\frac{1}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right) - \frac{1}{2}m_2 l_1 l_2 S_2 \dot{\theta}_1 \dot{\theta}_2$$
$$\frac{\partial L}{\partial \theta_2} = -\frac{1}{2}m_2 l_1 l_2 S_2 \dot{\theta}_1^2 - \frac{1}{2}m_2 l_1 l_2 S_2 \dot{\theta}_1 \dot{\theta}_2 - \frac{1}{2}m_2 g l_2 S_{12}$$

Substitute to get:

$$T_2 = \left(\frac{1}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 C_2\right)\ddot{\theta}_1 + \frac{1}{3}m_2 l_2^2 \ddot{\theta}_2 + \frac{1}{2}m_2 l_1 l_2 S_2 \dot{\theta}_1^2 + \frac{1}{2}m_2 g l_2 S_{12}$$

The result can be written in a matrix form too.

Problem 4.4
Write the equations that express U_{62}, U_{35}, U_{53}, U_{623}, and U_{534} for a 6-axis cylindrical-RPY robot in terms of the A and Q matrices.

Estimated student time to complete: 30 minutes
Prerequisite knowledge required: Text Section(s) 4.4.1.

Solution:

For joints 1 and 3: $$Q_{(prismatic)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = Q_P$$

For joints 2,4,5,6: $$Q_{(revolute)} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = Q_R$$

$$U_{ij} = \frac{\partial\, {}^0T_i}{\partial q_j} = A_1 A_2 .. Q_j A_j .. A_i \qquad j \le i$$

$$U_{ijk} = \partial U_{ij} \big/ \partial q_k$$

$$U_{62} = \frac{\partial\, {}^0T_6}{\partial \alpha} = \frac{\partial \left(A_1 A_2 A_3 A_4 A_5 A_6 \right)}{\partial \alpha} = A_1 Q_R A_2 A_3 A_4 A_5 A_6$$

$$U_{35} = \frac{\partial\, {}^0T_3}{\partial \phi_0} = \frac{\partial \left(A_1 A_2 A_3 \right)}{\partial \phi_0} = 0 \quad (i \ge j)$$

$$U_{53} = \frac{\partial\, {}^0T_5}{\partial l} = \frac{\partial \left(A_1 A_2 A_3 A_4 A_5 \right)}{\partial l} = A_1 A_2 Q_P A_3 A_4 A_5$$

$$U_{623} = \frac{\partial U_{62}}{\partial l} = \frac{\partial \left(A_1 Q_R A_2 A_3 A_4 A_5 A_6 \right)}{\partial l} = A_1 Q_R A_2 Q_P A_3 A_4 A_5 A_6$$

$$U_{534} = \frac{\partial U_{53}}{\partial q_a} = \frac{\partial \left(A_1 A_2 Q_P A_3 A_4 A_5 \right)}{\partial q_a} = A_1 A_2 Q_P A_3 Q_R A_4 A_5$$

Problem 4.5
Using Equations 4.49 to 4.54, write the equations of motion for a 3-DOF revolute robot and describe each term.

Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 4.4.4.

Solution:

We have:

$$
\begin{aligned}
T_1 &= D_{11}\ddot{\theta}_1 + D_{12}\ddot{\theta}_2 + D_{13}\ddot{\theta}_3 + I_{1(act)}\ddot{\theta}_1 + D_{111}\dot{\theta}_1^2 + D_{122}\dot{\theta}_2^2 + D_{133}\dot{\theta}_3^2 \\
&\quad + 2D_{112}\dot{\theta}_1\dot{\theta}_2 + 2D_{113}\dot{\theta}_1\dot{\theta}_3 + 2D_{123}\dot{\theta}_2\dot{\theta}_3 + D_1 \\
T_2 &= D_{21}\ddot{\theta}_1 + D_{22}\ddot{\theta}_2 + D_{23}\ddot{\theta}_3 + I_{2(act)}\ddot{\theta}_2 + D_{211}\dot{\theta}_1^2 + D_{222}\dot{\theta}_2^2 + D_{233}\dot{\theta}_3^2 \\
&\quad + 2D_{212}\dot{\theta}_1\dot{\theta}_2 + 2D_{213}\dot{\theta}_1\dot{\theta}_3 + 2D_{223}\dot{\theta}_2\dot{\theta}_3 + D_2 \\
T_3 &= D_{31}\ddot{\theta}_1 + D_{32}\ddot{\theta}_2 + D_{33}\ddot{\theta}_3 + I_{3(act)}\ddot{\theta}_3 + D_{311}\dot{\theta}_1^2 + D_{322}\dot{\theta}_2^2 + D_{333}\dot{\theta}_3^2 \\
&\quad + 2D_{312}\dot{\theta}_1\dot{\theta}_2 + 2D_{313}\dot{\theta}_1\dot{\theta}_3 + 2D_{323}\dot{\theta}_2\dot{\theta}_3 + D_3
\end{aligned}
$$

$(D)\ddot{\theta}_i$ terms are caused by angular accelerations.
$(D)\dot{\theta}_i^2$ terms represent centripetal accelerations.
$(D)\dot{\theta}_i\dot{\theta}_j$ terms are related to the Coriolis accelerations.
(D_i) terms are gravity terms.

Problem 4.6
Expand the D_{134} and D_{15} terms of Equation 4.49 in terms of their constituent matrices.

Estimated student time to complete: 1 hour
Prerequisite knowledge required: Text Section(s) 4.4

Solution:

a. From Equation 4.44: $D_{ijk} = \sum_{p=\max(i,j,k)}^{n} Trace(U_{pjk} J_p U_{pi}^T)$

For D_{134} we have i =1, j =3, k =4, p =4, n =6. Then:

$$D_{134} = \sum_{4}^{6} Trace(U_{p34} J_p U_{p1}^T) = Trace\left(U_{434} J_4 U_{41}^T\right) + Trace\left(U_{534} J_5 U_{51}^T\right) + Trace\left(U_{634} J_6 U_{61}^T\right)$$

$$U_{43} = \frac{\partial\, {}^0T_4}{\partial q_3} = \frac{\partial\left(A_1 A_2 A_3 A_4\right)}{\partial q_3} = A_1 A_2 Q_3 A_3 A_4$$

$$U_{434} = \frac{\partial U_{43}}{\partial q_4} = \frac{\partial\left(A_1 A_2 Q_3 A_3 A_4\right)}{\partial q_4} = A_1 A_2 Q_3 A_3 Q_4 A_4$$

$$U_{41} = \frac{\partial\, {}^0T_4}{\partial q_1} = \frac{\partial\left(A_1 A_2 A_3 A_4\right)}{\partial q_1} = Q_1 A_1 A_2 A_3 A_4$$

From Equation A.11 of Appendix A: $U_{41}^T = A_4^T A_3^T A_2^T A_1^T Q_1^T$

$$U_{53} = \frac{\partial\left(A_1 A_2 A_3 A_4 A_5\right)}{\partial q_3} = A_1 A_2 Q_3 A_3 A_4 A_5$$

$$U_{534} = \frac{\partial\left(A_1 A_2 Q_3 A_3 A_4 A_5\right)}{\partial q_4} = A_1 A_2 Q_3 A_3 Q_4 A_4 A_5$$

$$U_{51} = \frac{\partial\left(A_1 A_2 A_3 A_4 A_5\right)}{\partial q_1} = Q_1 A_1 A_2 A_3 A_4 A_5 \quad \rightarrow \quad U_{51}^T = A_5^T A_4^T A_3^T A_2^T A_1^T Q_1^T$$

$$U_{63} = \frac{\partial\left(A_1 A_2 A_3 A_4 A_5 A_6\right)}{\partial q_3} = A_1 A_2 Q_3 A_3 A_4 A_5 A_6$$

$$U_{634} = \frac{\partial U_{63}}{\partial q_4} = A_1 A_2 Q_3 A_3 Q_4 A_4 A_5 A_6$$

$$U_{61}^T = A_6^T A_5^T A_4^T A_3^T A_2^T A_1^T Q_1^T$$

Substitute to get:

$$\begin{aligned} D_{134} = &\, Trace\left(A_1 A_2 Q_3 A_3 Q_4 A_4 J_4 A_4^T A_3^T A_2^T A_1^T Q_1^T\right) \\ &+ Trace\left(A_1 A_2 Q_3 A_3 Q_4 A_4 A_5 J_5 A_5^T A_4^T A_3^T A_2^T A_1^T Q_1^T\right) \\ &+ Trace\left(A_1 A_2 Q_3 A_3 Q_4 A_4 A_5 A_6 J_6 A_6^T A_5^T A_4^T A_3^T A_2^T A_1^T Q_1^T\right) \end{aligned}$$

b. From Equation 4.43: $D_{ij} = \sum_{p=\max(i,j)}^{n} Trace(U_{pj} J_p U_{pi}^T)$

For D_{15} we have i =1, j =5, p =5, n =6. Then:

$$D_{15} = \sum_{5}^{6} Trace(U_{p5} J_p U_{p1}^T) = Trace\left(U_{55} J_5 U_{51}^T\right) + Trace\left(U_{65} J_6 U_{61}^T\right)$$

$$U_{55} = A_1 A_2 A_3 A_4 Q_5 A_5$$

$$U_{51} = Q_1 A_1 A_2 A_3 A_4 A_5 \quad \rightarrow \quad U_{51}^T = A_5^T A_4^T A_3^T A_2^T A_1^T Q_1^T$$

$$U_{65} = A_1 A_2 A_3 A_4 Q_5 A_5 A_6$$

$$U_{61} = Q_1 A_1 A_1 A_2 A_3 A_4 A_5 A_6 \quad \rightarrow \quad U_{61}^T = A_6^T A_5^T A_4^T A_3^T A_2^T A_1^T Q_1^T$$

Substitute to get:

$$D_{15} = Trace\left(A_1 A_2 A_3 A_4 Q_5 A_5 J_5 A_5^T A_4^T A_3^T A_2^T A_1^T Q_1^T\right)$$
$$+ Trace\left(A_1 A_2 A_3 A_4 Q_5 A_5 A_6 J_6 A_6^T A_5^T A_4^T A_3^T A_2^T A_1^T Q_1^T\right)$$

Although a large number of matrices are involved, they are all known.

Problem 4.7

An object is subjected to the following forces and moments relative to the reference frame. Attached to the object is a frame, which describes the orientation and the location of the object. Find the equivalent forces and torques acting on the object relative to the current frame.

$$B = \begin{bmatrix} 0.707 & 0.707 & 0 & 2 \\ 0 & 0 & 1 & 5 \\ 0.707 & -0.707 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad F^T = [10, 0, 5, 12, 20, 0] \text{ N, N.m}$$

Estimated student time to complete: 20-30 minutes
Prerequisite knowledge required: Text Section(s) 4.6.

Solution:

Substitute:

$$\mathbf{f} = [10, 0, 5]^T \qquad \mathbf{m} = [12, 20, 0]^T \qquad \mathbf{p} = [2, 5, 3]^T$$
$$\mathbf{n} = [0.707, 0, 0.707]^T \qquad \mathbf{o} = [0.707, 0, -0.707]^T \qquad \mathbf{a} = [0, 1, 0]^T$$

$$\mathbf{f} \times \mathbf{p} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 10 & 0 & 5 \\ 2 & 5 & 3 \end{vmatrix} = \mathbf{i}(-25) - \mathbf{j}(20) + \mathbf{k}(50)$$
$$(\mathbf{f} \times \mathbf{p}) + \mathbf{m} = -13\mathbf{i} + 50\mathbf{k}$$

$${}^B f_x = \mathbf{n} \cdot \mathbf{f} = 7.07 + 3.54 = 10.61 \quad N$$
$${}^B f_y = \mathbf{o} \cdot \mathbf{f} = 7.07 - 3.54 = 3.53 \quad N$$
$${}^B f_z = \mathbf{a} \cdot \mathbf{f} = 0$$
$${}^B m_x = \mathbf{n} \cdot [(\mathbf{f} \times \mathbf{p}) + \mathbf{m}] = -9.19 + 35.35 = 26.16 \quad Nm$$
$${}^B m_y = \mathbf{o} \cdot [(\mathbf{f} \times \mathbf{p}) + \mathbf{m}] = -9.19 - 35.35 = -44.54 \quad Nm$$
$${}^B m_z = \mathbf{a} \cdot [(\mathbf{f} \times \mathbf{p}) + \mathbf{m}] = 0$$

and

$$F = \begin{bmatrix} 10.61 \\ 3.53 \\ 0 \\ 26.16 \\ -44.54 \\ 0 \end{bmatrix}$$

Problem 4.8
In order to assemble two parts together, one part must be pushed into the other with a force of 10 lb in the *x*-axis direction, 5 lb in the *y*-direction, and be turned with a torque of 5 $\text{lb} \cdot \text{in}$ along the *x*-axis direction. The object's location relative to the base frame of a robot is described by ${}^{R}T_0$. Assuming that the two parts must be aligned together for this purpose, find the necessary forces and moments that the robot must apply to the part relative to its hand frame.

$$ {}^{R}T_0 = \begin{bmatrix} 0 & -0.707 & 0.707 & 4 \\ 1 & 0 & 0 & 6 \\ 0 & 0.707 & 0.707 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Estimated student time to complete: 20-30 minutes
Prerequisite knowledge required: Text Section(s) 4.6.

Solution:

Since the robot must be aligned with the object for this operation, then ${}^{R}T_H = {}^{R}T_O$. Therefore we have:

$$ \mathbf{f} = [10, 5, 0]^T \qquad \mathbf{m} = [5, 0, 0]^T \qquad \mathbf{p} = [4, 6, 3]^T $$
$$ \mathbf{n} = [0, 1, 0]^T \qquad \mathbf{o} = [-0.707, 0, 0.707]^T \qquad \mathbf{a} = [0.707, 0, 0.707]^T $$

$$ \mathbf{f} \times \mathbf{p} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 10 & 5 & 0 \\ 4 & 6 & 3 \end{vmatrix} = \mathbf{i}(15) - \mathbf{j}(30) + \mathbf{k}(40) $$

$$ (\mathbf{f} \times \mathbf{p}) + \mathbf{m} = 20\mathbf{i} - 30\mathbf{j} + 40\mathbf{k} $$

$$ {}^{H}f_x = \mathbf{n} \cdot \mathbf{f} = 5 \quad \textit{units} $$
$$ {}^{H}f_y = \mathbf{o} \cdot \mathbf{f} = -7.07 \quad \textit{units} $$
$$ {}^{H}f_z = \mathbf{a} \cdot \mathbf{f} = 7.07 \quad \textit{units} $$
$$ {}^{H}m_x = \mathbf{n} \cdot [(\mathbf{f} \times \mathbf{p}) + \mathbf{m}] = -30 \quad \textit{units} $$
$$ {}^{H}m_y = \mathbf{o} \cdot [(\mathbf{f} \times \mathbf{p}) + \mathbf{m}] = 14.14 \quad \textit{units} $$
$$ {}^{H}m_z = \mathbf{a} \cdot [(\mathbf{f} \times \mathbf{p}) + \mathbf{m}] = 42.42 \quad \textit{units} $$

and

$$ F = \begin{bmatrix} 5 \\ -7.07 \\ 7.07 \\ -30 \\ 14.14 \\ 42.42 \end{bmatrix} $$

CHAPTER FIVE

Problem 5.1
It is desired to have the first joint of a 6-axis robot go from an initial angle of 50° to a final angle of 80° in 3 seconds. Calculate the coefficients for a third-order polynomial joint-space trajectory. Determine the joint angles, velocities, and accelerations at 1, 2, and 3 seconds. It is assumed that the robot starts from rest, and stops at its destination.

Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 5.5.1.

Solution:

For a 3rd-order polynomial:

$$\theta(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$$
$$\dot{\theta}(t) = c_1 + 2c_2 t + 3c_3 t^2$$

Substitute boundary conditions to get:

$$\theta_i = 50 = c_0 + 0$$
$$\dot{\theta}_i = 0 = c_1 + 0$$
$$\theta_f = 80 = 50 + 9c_2 + 27c_3$$
$$\dot{\theta}_f = 0 = 6c_2 + 27c_3$$

Solve to get:

$$c_0 = 5 \qquad c_1 = 0 \qquad c_2 = 10 \qquad c_3 = -2.222$$
$$\theta(t) = 50 + 10t^2 - 2.222t^3$$
$$\dot{\theta}(t) = 20t - 6.666t^2$$
$$\ddot{\theta}(t) = 20 - 13.332t$$

$$@t{=}1 \text{ sec} \begin{cases} \theta = 57.78\ ^\circ \\ \dot{\theta} = 13.334\ ^\circ\!/\text{sec} \\ \ddot{\theta} = 6.668\ ^\circ\!/\text{sec} \end{cases} \qquad @t{=}2 \text{ sec} \begin{cases} \theta = 72.22\ ^\circ \\ \dot{\theta} = 13.34\ ^\circ\!/\text{sec} \\ \ddot{\theta} = -6.664\ ^\circ\!/\text{sec} \end{cases} \qquad @t{=}3 \text{ sec} \begin{cases} \theta = 80\ ^\circ \\ \dot{\theta} = 0\ ^\circ\!/\text{sec} \\ \ddot{\theta} = -20\ ^\circ\!/\text{sec} \end{cases}$$

Problem 5.2
It is desired to have the third joint of a 6-axis robot go from an initial angle of 20° to a final angle of 80° in 4 seconds. Calculate the coefficients for a third-order polynomial joint-space trajectory and plot the joint angles, velocities, and accelerations. The robot starts from rest, but should have a final velocity of $5^{\circ/\sec}$.

Estimated student time to complete: 20-30 minutes (with plotting)
Prerequisite knowledge required: Text Section(s) 5.5.1.

Solution:

For a 3rd-order polynomial:

$$\theta(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$$
$$\dot{\theta}(t) = c_1 + 2c_2 t + 3c_3 t^2$$

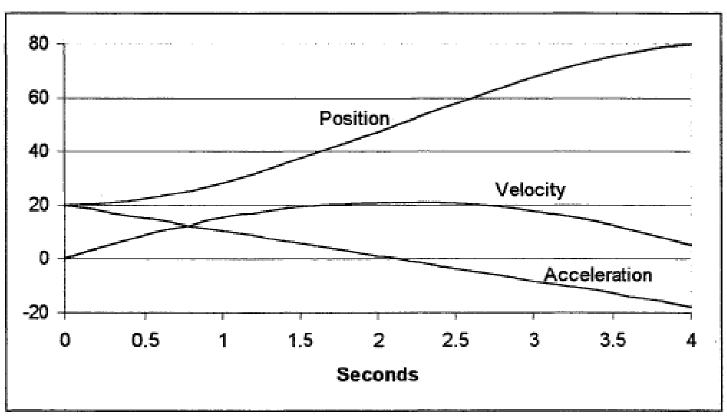
Substitute boundary conditions to get:

$$\theta_i = 20 = c_0 + 0$$
$$\dot{\theta}_i = 0 = c_1 + 0$$
$$\theta_f = 80 = 20 + 16c_2 + 64c_3$$
$$\dot{\theta}_f = 5 = 8c_2 + 48c_3$$

Solve to get:

$$c_0 = 20 \quad c_1 = 0 \quad c_2 = 10 \quad c_3 = -1.5625$$
$$\theta(t) = 20 + 10t^2 - 1.5625t^3$$
$$\dot{\theta}(t) = 20t - 4.6875t^2$$
$$\ddot{\theta}(t) = 20 - 9.375t$$

The plot is shown:



Problem 5.3
The second joint of a 6-axis robot is to go from initial angle of 20° to an intermediate angle of 80° in 5 seconds and continue to its destination of 25° in another 5 seconds. Calculate the coefficients for third-order polynomials in joint-space. Plot the joint angles, velocities, and accelerations. Assume the joint stops at intermediate points.

Estimated student time to complete: 30-40 minutes (with plotting)
Prerequisite knowledge required: Text Section(s) 5.5.1

Solution:

There are two segments to this motion. Therefore:

Segment 1

$$\begin{cases} \theta(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 \\ \dot{\theta}(t) = c_1 + 2c_2 t + 3c_3 t^2 \end{cases}$$

Segment 2

$$\begin{cases} \theta(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3 \\ \dot{\theta}(t) = b_1 + 2b_2 t + 3b_3 t^2 \end{cases}$$

There are eight unknowns; eight equations are needed. Known initial and final conditions are: θ_{1i}, θ_{1f}, θ_{2i}, θ_{2f}, $\dot{\theta}_{1i}$, $\dot{\theta}_{2f}$, and $\dot{\theta}_{1f} = \dot{\theta}_{2i}$. This results in seven equations. The eighth equation can be generated by making assumptions such as a maximum allowable acceleration or an intermediate velocity.

For this problem we will assume that the joint will come to a stop at the intermediate point. Therefore,

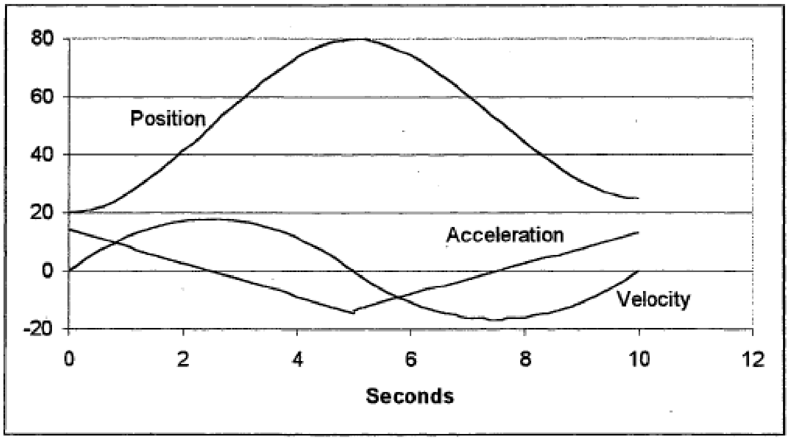
$$\boxed{1} \quad \begin{cases} 20 = c_0 + 0 \\ 0 = c_1 + 0 \\ 80 = 20 + 25c_2 + 125c_3 \\ 0 = 10c_2 + 75c_3 \end{cases} \rightarrow \begin{cases} c_0 = 20 \\ c_1 = 0 \\ c_2 = 7.2 \\ c_3 = -0.96 \end{cases}$$

$$\boxed{2} \quad \begin{cases} 80 = b_0 + 0 \\ 0 = b_1 + 0 \\ 25 = 80 + 25b_2 + 125b_3 \\ 0 = 10b_2 + 75b_3 \end{cases} \rightarrow \begin{cases} b_0 = 80 \\ b_1 = 0 \\ b_2 = -6.6 \\ b_3 = 0.88 \end{cases}$$

Substitute to get:

$$\boxed{1} \quad \begin{cases} \theta(t) = 20 + 7.2t^2 - 0.96t^3 \\ \dot{\theta}(t) = 14.4t - 2.88t^2 \\ \ddot{\theta}(t) = 14.4 - 5.76t \end{cases} \quad \text{and} \quad \boxed{2} \quad \begin{cases} \theta(t) = 80 - 6.6t^2 + 0.88t^3 \\ \dot{\theta}(t) = -13.2t + 2.64t^2 \\ \ddot{\theta}(t) = -13.2 + 5.28t \end{cases}$$

The position, velocities, and accelerations are shown:


80
60
40
20
0
-20
Position
Acceleration
Velocity
0
2
4
6
8
10
12
Seconds

Problem 5.4
A fifth-order polynomial is to be used to control the motions of the joints of a robot in joint-space. Find the coefficients of a fifth-order polynomial that will allow a joint to go from an initial angle of 0° to a final joint angle of 75° in 3 seconds, while the initial and final velocities are zero and initial acceleration and final decelerations are $10^\circ/\sec^2$.

Estimated student time to complete: 20-30 minutes (with plotting)
Prerequisite knowledge required: Text Section(s) 5.5.2

Solution:

For a 5th-order polynomial:

$$\begin{cases} \theta(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 \\ \dot{\theta}(t) = c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 \\ \ddot{\theta}(t) = 2c_2 + 6c_3 t + 12c_4 t^2 + 20c_5 t^3 \end{cases}$$

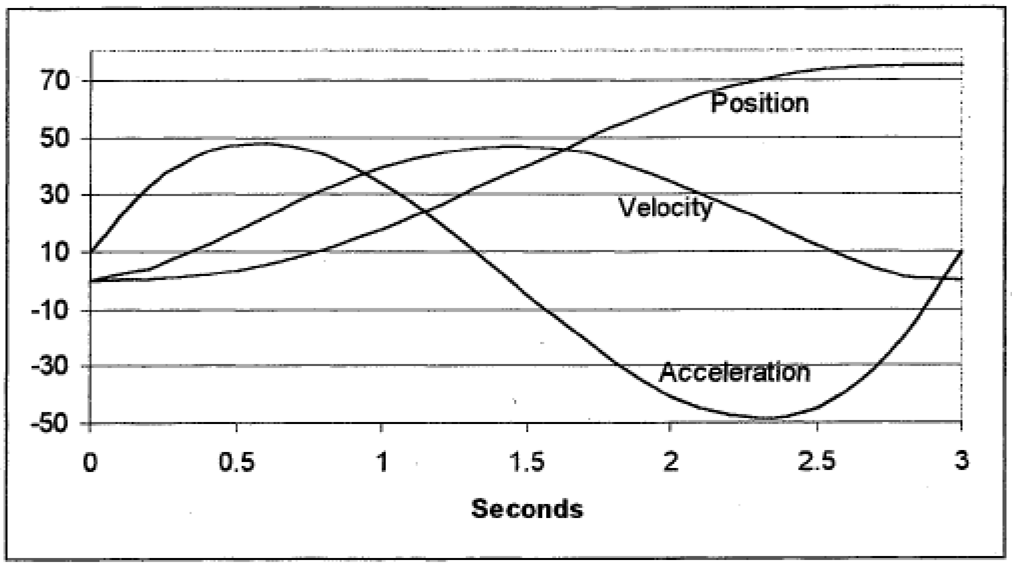
Substitute the given initial and final conditions and get:

$$\begin{cases} 0 = c_0 \\ 75 = 3c_1 + 9c_2 + 27c_3 + 81c_4 + 243c_5 \\ 0 = c_1 \\ 0 = 6c_2 + 27c_3 + 108c_4 + 405c_5 \\ 10 = 2c_2 \\ 10 = 2c_2 + 18c_3 + 108c_4 + 540c_5 \end{cases}$$

Solve to get: $c_0 = 0, \quad c_1 = 0, \quad c_2 = 5, \quad c_3 = 24.24, \quad c_4 = -13.33, \quad c_5 = 1.85$. Substitute to get:

$$\begin{cases} \theta(t) = 5t^2 + 24.44t^3 - 13.33t^4 + 1.85t^5 \\ \dot{\theta}(t) = 10t + 73.33t^2 - 53.33t^3 + 9.26t^4 \\ \ddot{\theta}(t) = 10 + 146.66t - 160t^2 + 37.04t^3 \end{cases}$$

The position, velocity, and acceleration plots are shown:


70
50
30
10
-10
-30
-50
0
0.5
1
1.5
2
2.5
3
Seconds
Position
Velocity
Acceleration

Problem 5.5

Joint 1 of a 6-axis robot is to go from an initial angle of $\theta_i = 30^\circ$ to the final angle of $\theta_f = 120^\circ$ in 4 seconds with a cruising velocity of $\omega_1 = 30^{\circ/\text{sec}}$. Find the necessary blending time for a trajectory with linear segments and parabolic blends and plot the joint positions, velocities, and accelerations.

Estimated student time to complete: 15-30 minutes (with plotting)
Prerequisite knowledge required: Text Section(s) 5.5.3

Solution:

$$t_b = \frac{\theta_i - \theta_f + \omega t_f}{\omega} = \frac{30 - 120 + 30(4)}{30} = 1 \text{ sec}$$

For $\theta = \theta_i$ to θ_A

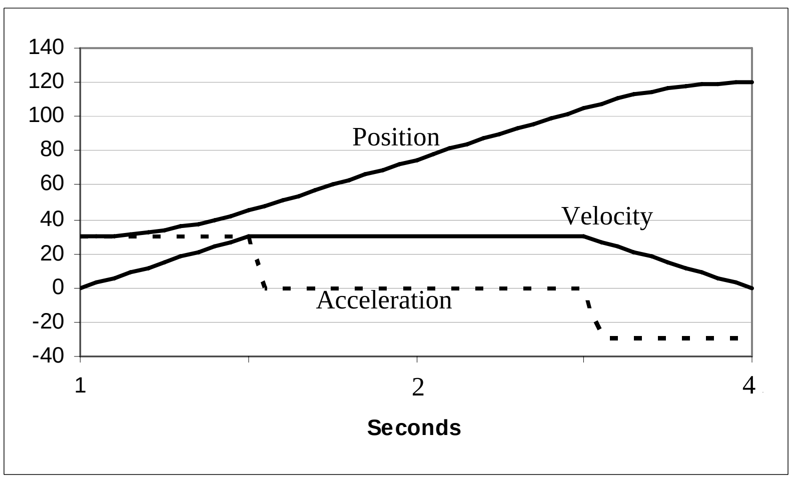
$$\begin{cases} \theta = 30 + \frac{1}{2}(30)t^2 \\ \dot{\theta} = 30t \\ \ddot{\theta} = 30 \end{cases}$$

For $\theta = \theta_A$ to θ_B

$$\begin{cases} \theta = \theta_A + 30(t-1) \\ \dot{\theta} = 30 \\ \ddot{\theta} = 0 \end{cases}$$

For $\theta = \theta_B$ to θ_f

$$\begin{cases} \theta = 120 - \frac{1}{2}(30)(4-t)^2 \\ \dot{\theta} = 30(4-t) \\ \ddot{\theta} = -30 \end{cases}$$



Problem 5.6
A robot is to be driven from an initial position through two via points before it reaches its final destination using a 4-3-4 trajectory. The positions, velocities, and time duration for the three segments for one of the joints are given below. Determine the trajectory equations and plot the position, velocity, and acceleration curves for the joint.

$$
\begin{array}{lllll}
\theta_1 = 20^\circ & \dot{\theta}_1 = 0 & \ddot{\theta}_1 = 0 & \tau_{1i} = 0 & \tau_{1f} = 1 \\
\theta_2 = 60^\circ & \tau_{2i} = 0 & \tau_{2f} = 2 & & \\
\theta_3 = 100^\circ & \tau_{3i} = 0 & \tau_{3f} = 1 & & \\
\theta_4 = 40^\circ & \dot{\theta}_4 = 0 & \ddot{\theta}_4 = 0 & &
\end{array}
$$

Estimated student time to complete: 50-60 minutes (with plotting)
Prerequisite knowledge required: Text Section(s) 5.5.5

Solution:

Using Equations 5.19 to 5.32 we get the following equations. The same can be found using the matrix equation of Equation 5.33:

$$
\begin{cases}
a_0 = 20 \\
a_1 = 0 \\
a_2 = 0 \\
b_0 = 60 \\
c_0 = 100 \\
60 = 20 + a_3 + a_4 \\
3a_3 + 4a_4 = b_1 \\
6a_3 + 12a_4 = 2b_2 \\
100 = 60 + 2b_1 + 4b_2 + 8b_3 \\
b_1 + 4b_2 + 12b_3 = c_1 \\
2b_2 + 12b_3 = 2c_2 \\
40 = 100 + c_1 + c_2 + c_3 + c_4 \\
0 = c_1 + 2c_2 + 3c_3 + 4c_4 \\
0 = 2c_2 + 6c_3 + 12c_4
\end{cases}
\quad \rightarrow \quad
\begin{cases}
a_3 = 80.95 \\
a_4 = -40.95 \\
b_1 = 79.05 \\
b_2 = -2.85 \\
b_3 = -13.34 \\
c_1 = -92.4 \\
c_2 = -82.875 \\
c_3 = 202.9 \\
c_4 = -87.62
\end{cases}
$$

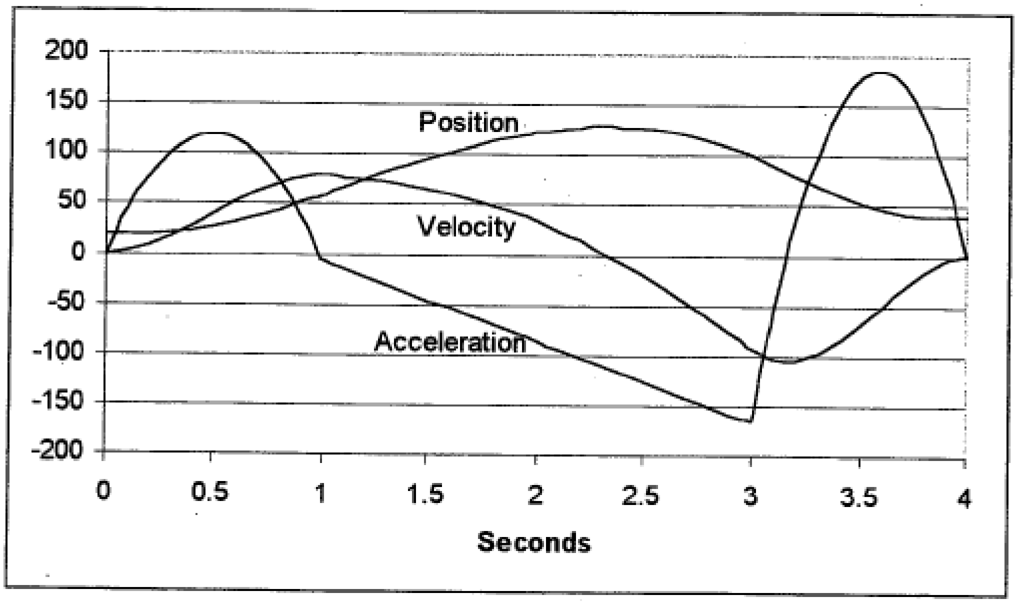
These equations can be solved by elimination or an equation solver. Substituting the coefficients into each segment we get:

$\theta_1(t) = 20 + 80.95t^3 - 40.95t^4$

$\theta_2(t) = 60 + 79.05t - 2.85t^2 - 13.34t^3$

$\theta_3(t) = 100 - 92.4t - 82.875t^2 + 202.9t^3 - 87.62t^4$

The results are plotted below.

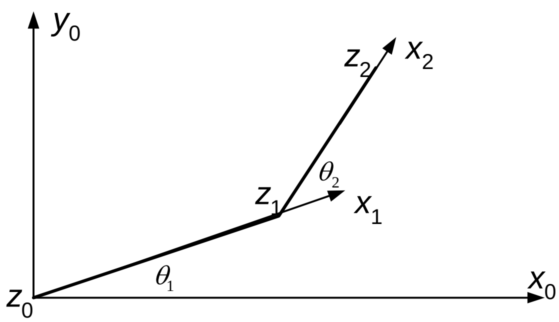


Problem 5.7
A 2-DOF planar robot is to follow a straight line in Cartesian-space between the start (2,6) and the end (12,3) points of the motion segment. Find the joint variables for the robot if the path is divided into 10 sections. Each link is 9 inches long.

Estimated student time to complete: 60-75 minutes (with plotting)
Prerequisite knowledge required: Text Section(s) 5.6

Solution:

The inverse kinematic equations can be found in different ways. We will use a simple Denavit-Hartenberg representation as follows:



#	θ	d	a	α
0-1	θ_1	0	9	0
1-2	θ_2	0	9	0

$$A_1 = \begin{bmatrix} C_1 & -S_1 & 0 & 9C_1 \\ S_1 & C_1 & 0 & 9S_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} C_2 & -S_2 & 0 & 9C_2 \\ S_2 & C_2 & 0 & 9S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_2 = A_1 A_2$$

Multiplying A_1 by A_2 and setting p_x and p_y equal to the components of position for ${}^0T_2$, we get:

$$\begin{cases} p_x = 9C_1C_2 - 9S_1S_2 + 9C_1 = 9C_{12} + 9C_1 & (1) \\ p_y = 9S_1S_2 + 9C_1C_2 + 9S_1 = 9S_{12} + 9S_1 & (2) \end{cases}$$

Squaring these equations and adding them together, and then regrouping, we get:

$$C_2 = \frac{p_x^2 + p_y^2}{162} - 1 \quad \rightarrow \quad \theta_2 = \cos^{-1}\left(\frac{p_x^2 + p_y^2}{162} - 1 \right)$$

Substitute into (1) to get:

$$C_1 = \frac{p_x/9 + S_1 S_2}{C_2 + 1} \qquad (3)$$

Substitute (3) into (2) to get:

$$S_1\left(C_2+1\right)^2 + \frac{S_2 p_x}{9} + S_1 S_2^2 = \frac{p_y}{9}\left(C_2+1\right)$$

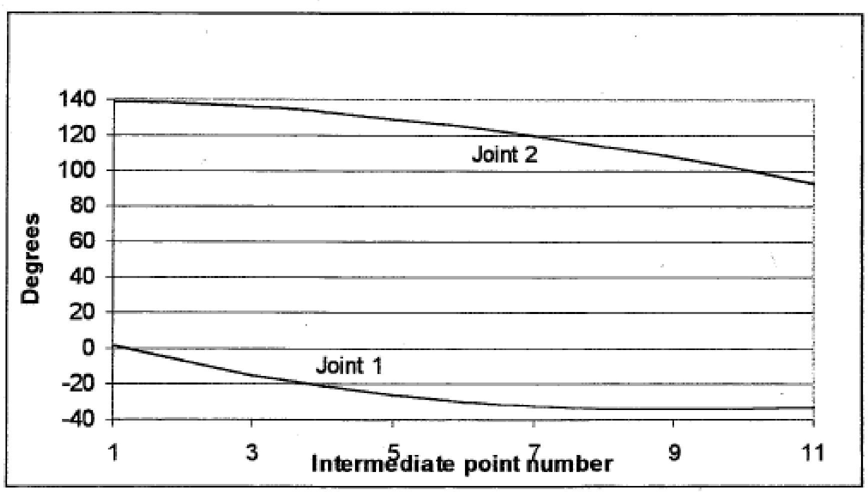
$$S_1 = \frac{p_y\left(C_2+1\right) - S_2 p_x}{18\left(C_2+1\right)} \qquad \rightarrow \quad \theta_1 = \sin^{-1}\left(\frac{p_y\left(C_2+1\right) - S_2 p_x}{18\left(C_2+1\right)}\right)$$

Next, calculate the slope of the straight line motion:

$$\frac{y-3}{x-12} = \frac{3-6}{12-2} = -0.3$$

The following table shows the coordinates of the intermediate points and the joint angles from the above equations. The joint angles are shown on the graph as well:

#	x	y	Joint 1	Joint 2
1	2	6	2.14	139
2	3	5.7	-6.79	138
3	4	5.4	-14.6	136
4	5	5.1	-21.1	133
5	6	4.8	-26.1	129
6	7	4.5	-29.7	125
7	8	4.2	-32.2	120
8	9	3.9	-33.6	114
9	10	3.6	-34	108
10	11	3.3	-33.7	101
11	12	3	-32.6	93.2



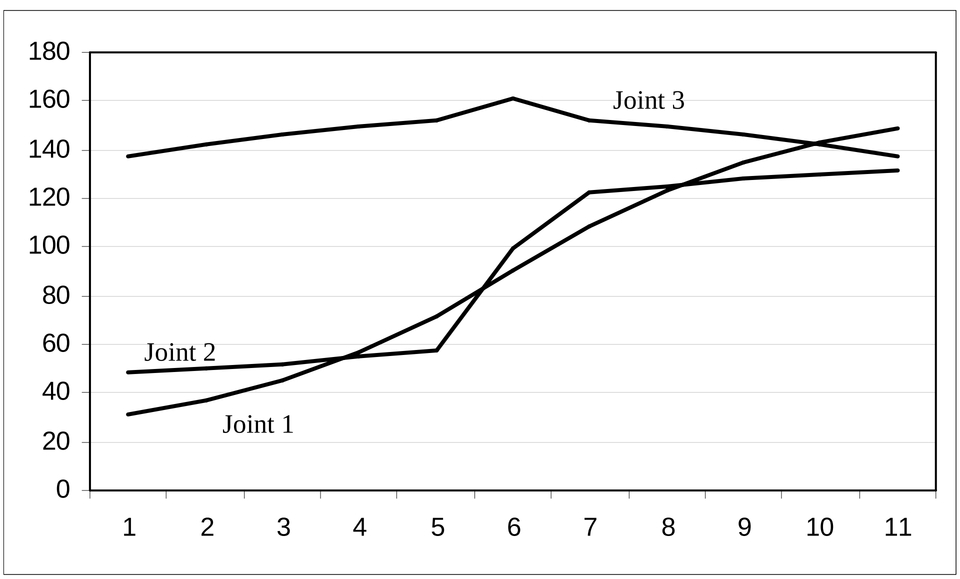
Problem 5.8
The 3-DOF robot of Example 5.7, as shown in Figure 5.18, is to move from point (3, 5, 5) to point (3, -5, 5) along a straight line, divided into 10 sections. Find the angles of the three joints for each intermediate point and plot the results.

Estimated student time to complete: 30 minutes, depending on programming expertise
Prerequisite knowledge required: Text Section(s) 5.6

Solution:

Using the inverse kinematic equations of the robot from Example 5.7, the joint angles can be found as shown. The values are also plotted:

#	P_x	P_y	P_z	*Theta 1*	*Theta 2*	*Theta 3*
1	3	5	5	30.96	48.59	137.27
2	3	4	5	36.87	49.87	142.2
3	3	3	5	45	52.04	146.45
4	3	2	5	56.31	54.87	149.8
5	3	1	5	71.57	57.51	151.98
6	3	0	5	90	99.6	160.82
7	3	-1	5	108.4	122.5	151.98
8	3	-2	5	123.7	125.1	149.8
9	3	-3	5	135	128	146.45
10	3	-4	5	143.1	130.1	142.2
11	3	-5	5	149	131.4	137.27



CHAPTER SIX

Problem 6.1
Derive the inverse Laplace transform of the following equation:

Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 6.5 and 6.6

Solution:

$$F(s)=\frac{3}{(s^2+5s+4)}=\frac{3}{(s+1)(s+4)}=\frac{a_1}{s+1}+\frac{a_2}{s+4}$$

$$a_1=\left[(s+1)\frac{3}{(s+1)(s+4)}\right]_{s=-1}=\frac{3}{3}=1$$

$$a_2=\left[(s+4)\frac{3}{(s+1)(s+4)}\right]_{s=-4}=\frac{3}{-3}=-1$$

$$F(s)=\frac{1}{s+1}-\frac{1}{s+4}$$

$$f(t)=\mathcal{L}^{-1}\frac{1}{s+1}+\mathcal{L}^{-1}\frac{-1}{s+4}=e^{-t}-e^{-4t}$$

Problem 6.2
Derive the inverse Laplace transform of the following equation:

$$F(s)=\frac{(s+6)}{s(s^2+5s+6)}$$

Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 6.5, 6.6

Solution:

$$F(s)=\frac{(s+6)}{s(s^2+5s+6)}=\frac{(s+6)}{s(s+5)\left(s+3\right)}$$

$$=\frac{a_1}{s}+\frac{a_2}{s+2}+\frac{a_3}{s+3}$$

$$a_1=\left[\left(s\right)\frac{s+6}{s\left(s+2\right)\left(s+3\right)}\right]_{s=0}=\frac{6}{6}=1$$

$$a_2=\left[\left(s+2\right)\frac{s+6}{s\left(s+2\right)\left(s+3\right)}\right]_{s=-2}=\frac{4}{-2}=-2$$

$$a_3=\left[\left(s+3\right)\frac{s+6}{s\left(s+2\right)\left(s+3\right)}\right]_{s=-3}=\frac{3}{\left(-3\right)\left(-1\right)}=1$$

$$F(s)=\frac{1}{s}+\frac{-2}{s+2}+\frac{1}{s+3}$$

$$f(t)=\mathcal{L}^{-1}\left(\frac{1}{s}+\frac{-2}{s+2}+\frac{1}{s+3}\right)=u(t)-e^{-2t}+e^{-3t}$$

Problem 6.3
Derive the inverse Laplace transform of the following equation:

$$F(s) = \frac{1}{(s+1)^2(s+2)}$$

Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 6.5, 6.6

Solution:

$$F(s) = \frac{1}{(s+1)^2(s+2)} = \frac{b_2}{(s+1)^2} + \frac{b_1}{(s+1)} + \frac{a_1}{(s+2)}$$

$$b_2 = \left[\left(s+1\right)^2 \frac{1}{(s+1)^2(s+2)} \right]_{s=-1} = 1$$

$$b_1 = \frac{d}{ds}\left[\frac{1}{(s+2)} \right]_{s=-1} = \frac{-1}{(s+2)^2} = -1$$

$$a_1 = \left[\left(s+2\right) \frac{1}{(s+1)^2(s+2)} \right]_{s=-2} = 1$$

$$F(s) = \frac{1}{(s+1)^2} - \frac{1}{(s+1)} + \frac{1}{(s+2)}$$

$$f(t) = \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2} - \frac{1}{(s+1)} + \frac{1}{(s+2)} \right) = \left(t-1\right)e^{-t} + e^{-2t}$$

Problem 6.4
Derive the inverse Laplace transform of the following equation:

$$F(s) = \frac{10}{(s+4)(s+2)^3}$$

Estimated student time to complete: 20-30 minutes
Prerequisite knowledge required: Text Section(s) 6.5-6.6

Solution:

$$F(s) = \frac{10}{(s+4)(s+2)^3} = \frac{b_3}{(s+2)^3} + \frac{b_2}{(s+2)^2} + \frac{b_1}{(s+2)} + \frac{a_1}{(s+4)}$$

$$b_3 = \left[(s+2)^3 \frac{10}{(s+4)(s+2)^3} \right]_{s=-2} = 5$$

$$b_2 = \frac{d}{ds}\left[(s+2)^3 \frac{10}{(s+4)(s+2)^3} \right]_{s=-2} = \frac{d}{ds}\left(\frac{10}{s+4} \right)\Bigg|_{s=-2} = \frac{-10}{(s+4)^2}\Bigg|_{s=-2} = \frac{-10}{4} = -2.5$$

$$b_1 = \frac{1}{2}\frac{d^2}{ds^2}\left(\frac{10}{s+4} \right)\Bigg|_{s=-2} = \frac{1}{2}\frac{d}{ds}\frac{-10}{(s+4)^2}\Bigg|_{s=-2} = \frac{1}{2}\frac{20}{(s+4)^3}\Bigg|_{s=-2} = \frac{5}{4}$$

$$a_1 = \left[(s+4) \frac{10}{(s+4)(s+2)^3} \right]_{s=-4} = \frac{-5}{4}$$

$$F(s) = \frac{5}{(s+2)^3} - \frac{5}{2(s+2)^2} + \frac{5}{4(s+2)} - \frac{5}{4(s+4)}$$

$$f(t) = \mathcal{L}^{-1}\left(\frac{5}{(s+2)^3} - \frac{5}{2(s+2)^2} + \frac{5}{4(s+2)} - \frac{5}{4(s+4)} \right) = \left(\frac{5}{2}t^2 - \frac{5}{2}t + \frac{5}{4} \right)e^{-2t} - \frac{5}{4}e^{-4t}$$

Problem 6.5
Simplify the block diagram of Figure P.6.5.

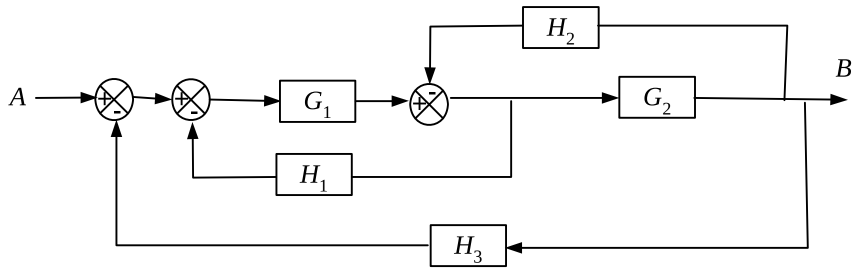
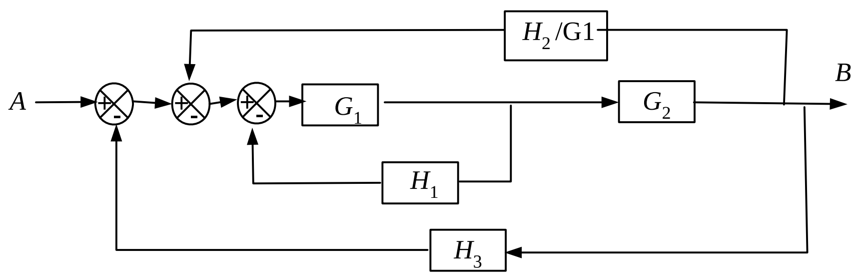


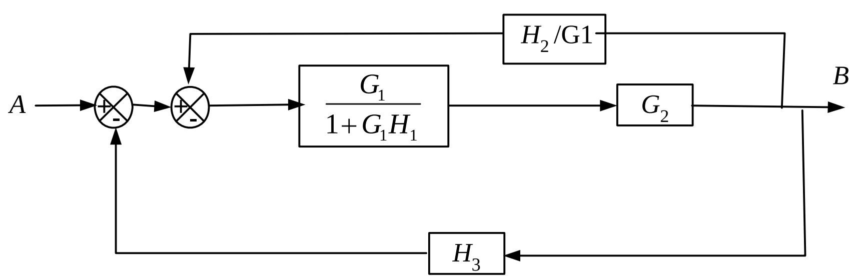
Figure P.6.5

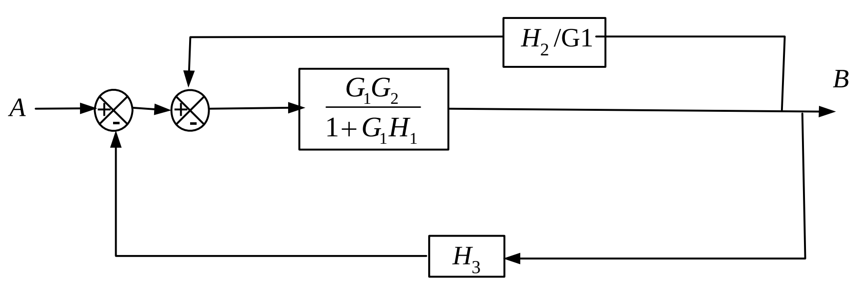
Estimated student time to complete: 30 minutes
Prerequisite knowledge required: Text Section(s) 6.8.

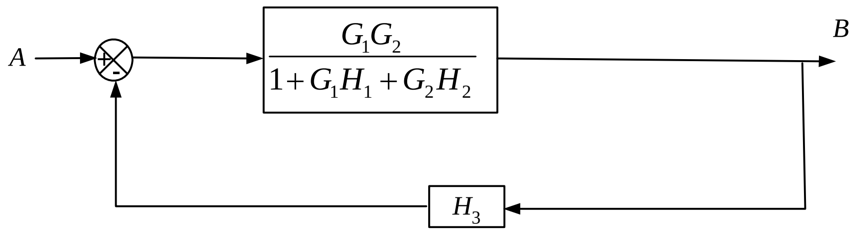
Solution:

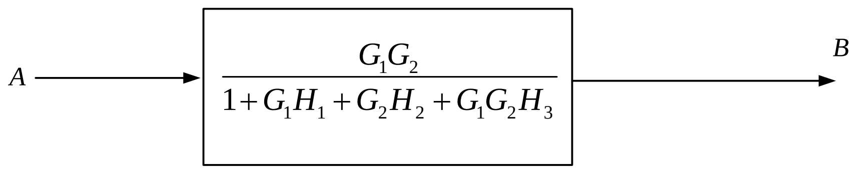
The following sequence is used to simplify the block diagram:








A
+
-
$\frac{G_1G_2}{1+G_1H_1+G_2H_2}$
B
H_3


A
$\frac{G_1G_2}{1+G_1H_1+G_2H_2+G_1G_2H_3}$
B

Problem 6.6
Simplify the block diagram of Figure P.6.6.

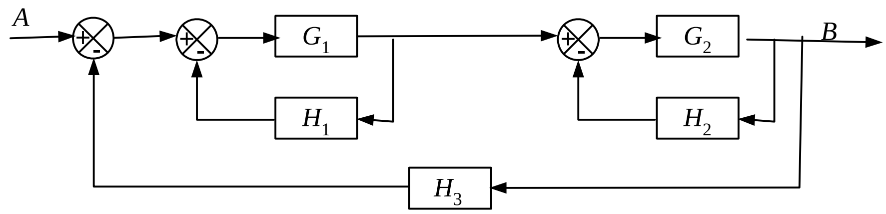
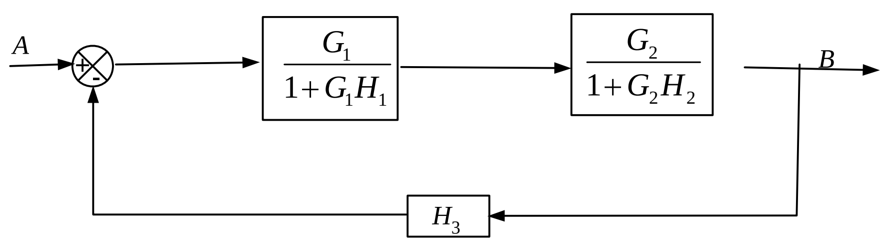


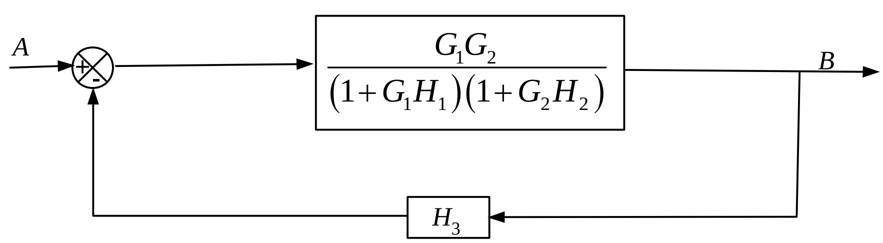
Figure P.6.6

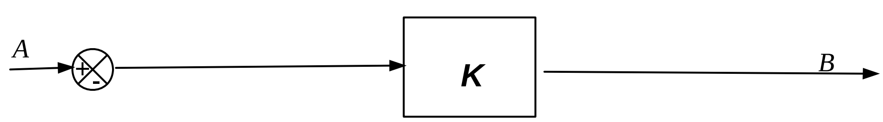
Estimated student time to complete: 30 minutes
Prerequisite knowledge required: Text Section(s) 6.8.

Solution:

The following sequence is used to simplify the block diagram:







where K is calculated as follows:

$$K = \frac{\dfrac{G_1G_2}{(1+G_1H_1)(1+G_2H_2)}}{1+\dfrac{G_1G_2H_3}{(1+G_1H_1)(1+G_2H_2)}} = \frac{G_1G_2}{(1+G_1H_1)(1+G_2H_2)+G_1G_2H_3}$$

Problem 6.7
Simplify the block diagram of Figure P.6.7.

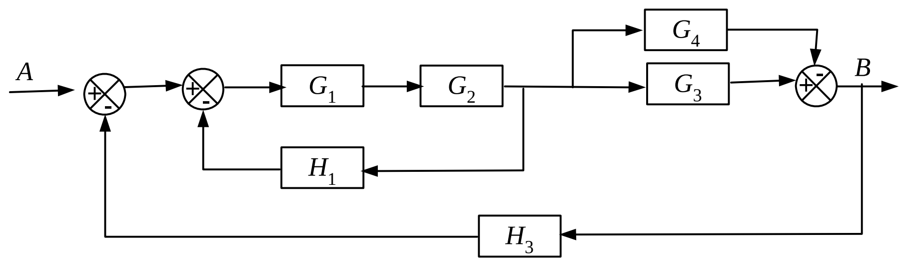
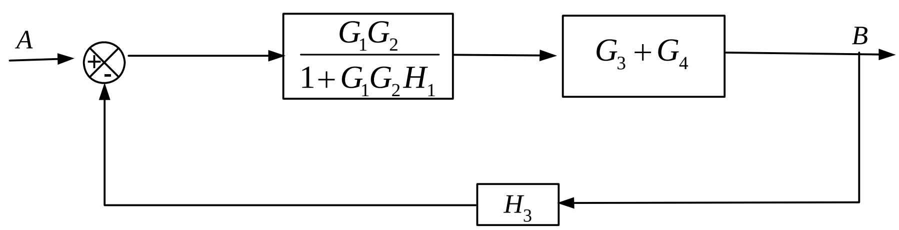


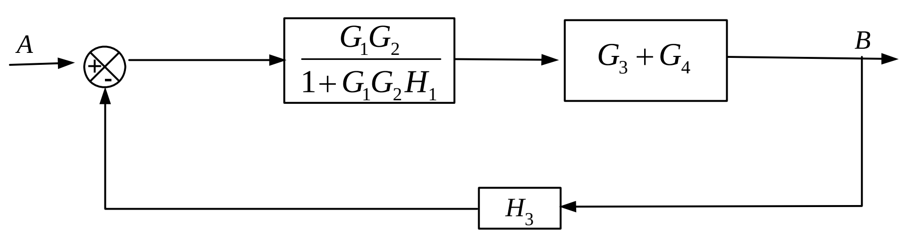
Figure P.6.7

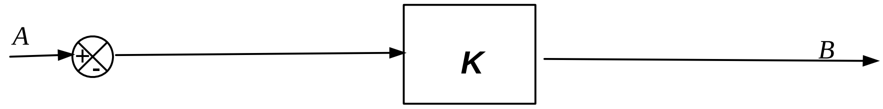
Estimated student time to complete: 20-30 minutes
Prerequisite knowledge required: Text Section(s) 6.8.

Solution:

The following sequence is used to simplify the block diagram:







where $K = \dfrac{\dfrac{G_1G_2\left(G_3+G_4\right)}{1+G_1G_2H_1}}{1+\dfrac{G_1G_2\left(G_3+G_4\right)H_3}{1+G_1G_2H_1}} = \dfrac{G_1G_2\left(G_3+G_4\right)}{1+G_1G_2H_1+G_1G_2\left(G_3+G_4\right)H_3}$

Problem 6.8
Write an equation that describes the output of the system of Figure P.6.8.

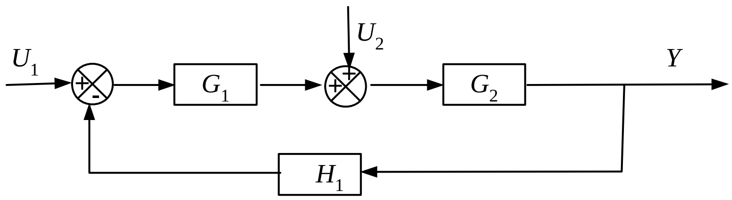


Figure P.6.8

Estimated student time to complete: 10-15 minutes
Prerequisite knowledge required: Text Section(s) 6.8.

Solution:

$$\left(\left(U_1 - YH_1\right)G_1 + U_2\right)G_2 = Y$$
$$\left(U_1G_1 - YG_1H_1 + U_2\right)G_2 = Y$$
$$U_1G_1G_2 - YG_1G_2H_1 + U_2G_2 = Y$$
$$U_1G_1G_2 + U_2G_2 = Y\left(1 + G_1G_2H_1\right)$$
$$Y = \frac{U_1G_1G_2 + U_2G_2}{1 + G_1G_2H_1}$$

Problem 6.9
Write the equations that describe the input-output relationships for Figure P.6.9.

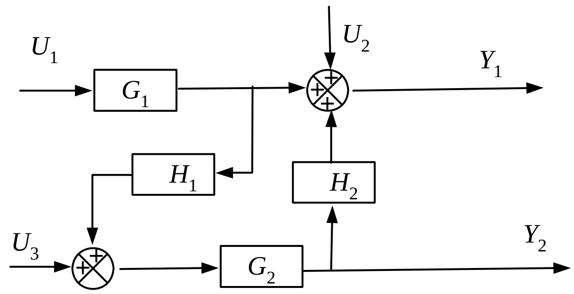


Figure P.6.9

Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 6.8.

Solution:

$$\begin{cases} U_1G_1 + U_2 + Y_2H_2 = Y_1 \\ \left(U_3 + U_1G_1H_1\right)G_2 = Y_2 \end{cases}$$

$$\begin{cases} U_1G_1 + U_2 + Y_2H_2 = Y_1 \\ U_3G_2 + U_1G_1G_2H_1 = Y_2 \end{cases}$$

Substitute Y_2 into Y_1 to get:

$$\begin{cases} Y_1 = U_1G_1 + U_2 + U_3G_2H_2 + U_1G_1G_2H_1H_2 \\ Y_2 = U_3G_2 + U_1G_1G_2H_1 \end{cases}$$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} G_1 + G_1G_2H_1H_2 & 1 & G_2H_2 \\ G_1G_2H_1 & 0 & G_2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

Problem 6.10
Sketch the root locus for the following:

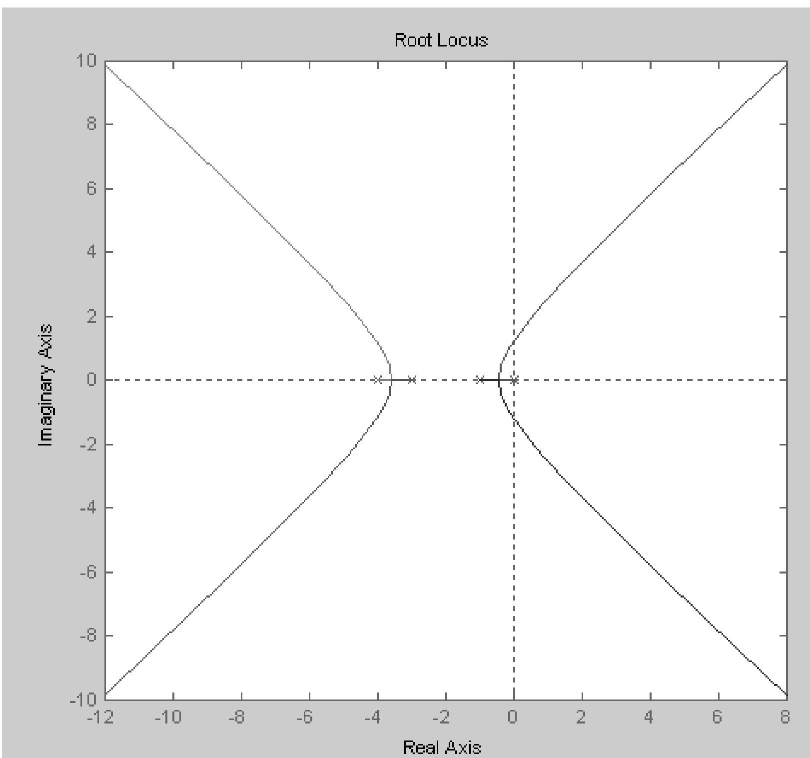
$$GH = \frac{K}{s(s+1)(s+3)(s+4)}$$

Estimated student time to complete: 20-30 minutes
Prerequisite knowledge required: Text Section(s) 6.13.

Solution:

Poles are at $p=0, \quad p=-1, \quad p=-3, \quad p=-4$
Number of asymptotes: $\alpha = 4-0=4$, angles of asymptotes: 45, 135, 225, 315.
Asymptotic center: $\sigma_A = \frac{\sum \sigma_p - \sum \sigma_Z}{\alpha} = \frac{(0-1-3-4)-0}{4} = -2$

The root locus, as drawn by MATLAB, is shown:



Problem 6.11
Sketch the root locus for the following:

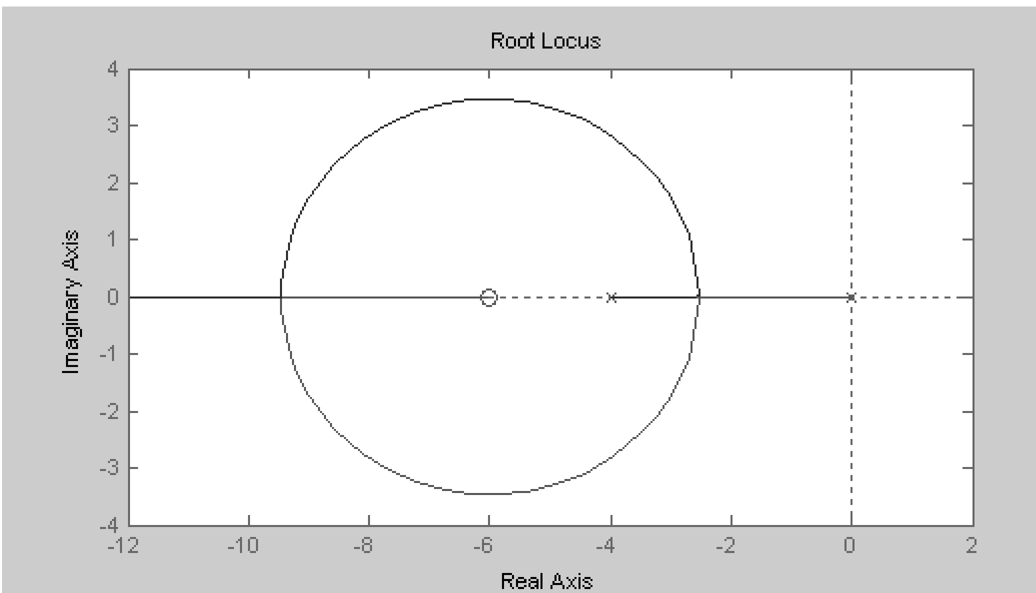
$$GH = \frac{K(s+6)}{s(s+4)}$$

Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 6.13.

Solution:

Roots are at $z = -6, \quad p = 0, \quad p = -4$
Number of asymptotes: $\alpha = 2 - 1 = 1$, angles of asymptote: 180.

The root locus, as drawn by MATLAB, is shown:



Problem 6.12
Sketch the root locus for the following:

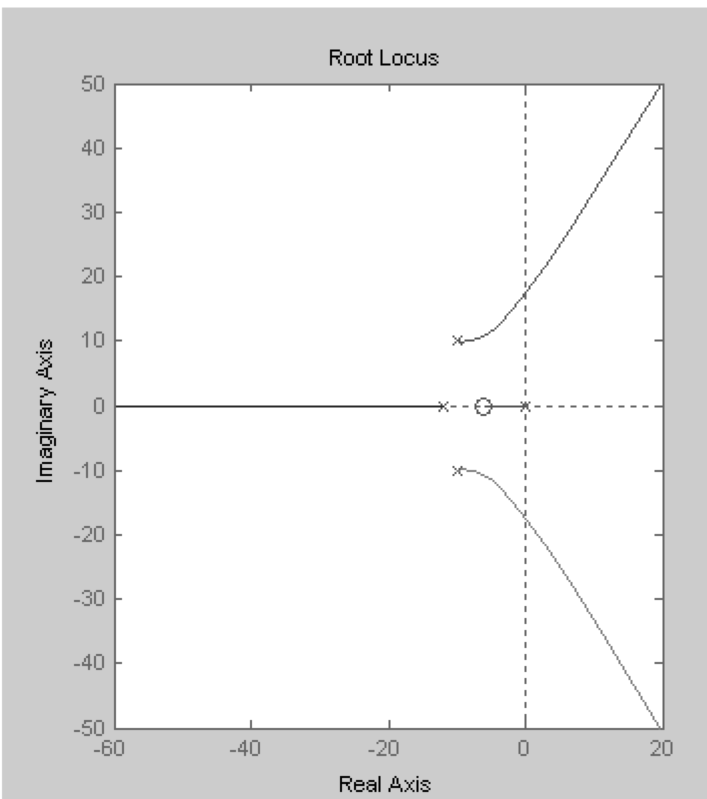
$$GH = \frac{K(s+6)}{s(s+10-j10)(s+10+j10)(s+12)}$$

Estimated student time to complete: 20-30 minutes
Prerequisite knowledge required: Text Section(s) 6.13.

Solution:

Roots are at $z=-6, \quad p=0, \quad p=-10\pm j10, \quad p=-12$
Number of asymptotes: $\alpha = 4-1=3$, angles of asymptotes: 60, 180, 300.
Asymptotic center: $\sigma_A = \frac{\sum\sigma_p - \sum\sigma_Z}{\alpha} = \frac{(0-10-10-12)-(-6)}{4} = -6.5$

The root locus, as drawn by MATLAB, is shown:



Problem 6.13
For the system of Problem 6.10, assume that two of the roots are chosen at $s = -5 \pm 2.55j$. Find the system's gain, damping ratio, and natural frequency. Show that the angle criterion is met. Can you determine from the root locus whether or not the system is stable?

Estimated student time to complete: 30 minutes
Prerequisite knowledge required: Text Section(s) 6.13, 6.14.

Solution:

$$K = \frac{\prod M_{p_i}}{\prod M_{z_i}} = \sqrt{5^2 + 2.55^2}\sqrt{4^2 + 2.55^2}\sqrt{2^2 + 2.55^2}\sqrt{1^2 + 2.55^2} = 236$$

$$\theta_1 = 180 - \tan^{-1}\left(\frac{2.55}{5}\right) = 180 - 27 = 153$$

$$\theta_2 = 180 - \tan^{-1}\left(\frac{2.55}{4}\right) = 180 - 32.52 = 147.48$$

$$\theta_3 = 180 - \tan^{-1}\left(\frac{2.55}{2}\right) = 180 - 51.9 = 128.1$$

$$\theta_4 = 180 - \tan^{-1}\left(\frac{2.55}{1}\right) = 180 - 68.59 = 111.4$$

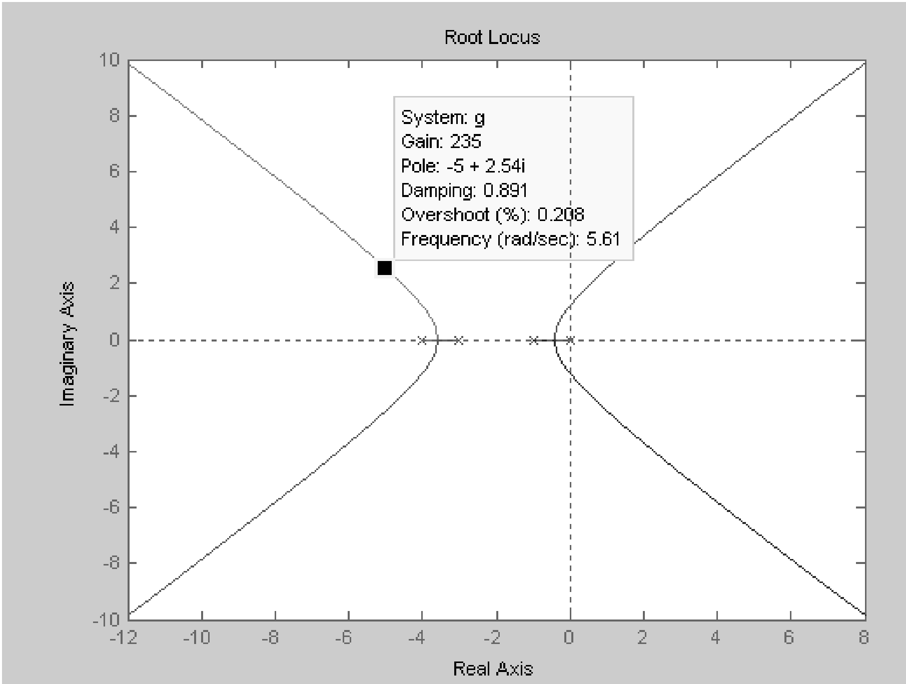
$$\sum \theta_{total} = 153 + 147.48 + 128.1 + 111.4 = 540 = 3 \times 180$$

It meets the criterion.

$$\zeta = \cos\theta = \cos\left(\tan^{-1}\frac{2.55}{5}\right) = 0.891$$

$$s = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}\,j = -0.891\omega_n \pm \omega_n\sqrt{1-0.891^2}\,j$$
$$= -5 \pm 2.55j$$
$$\omega_n = 5.61 \quad rad/\sec$$

The root have to potential to fall on the right side, and therefore, may become unstable..


Root Locus
System: g
Gain: 235
Pole: -5 + 2.54i
Damping: 0.891
Overshoot (%): 0.208
Frequency (rad/sec): 5.61
Imaginary Axis
Real Axis
10
8
6
4
2
0
-2
-4
-6
-8
-10
-12
-10
-8
-6
-4
-2
0
2
4
6
8

Problem 6.14
For the system of Problem 6.10, assume that two of the roots are chosen at $s = -4 \pm 1.24j$. Find the system's gain, damping ratio, and natural frequency.

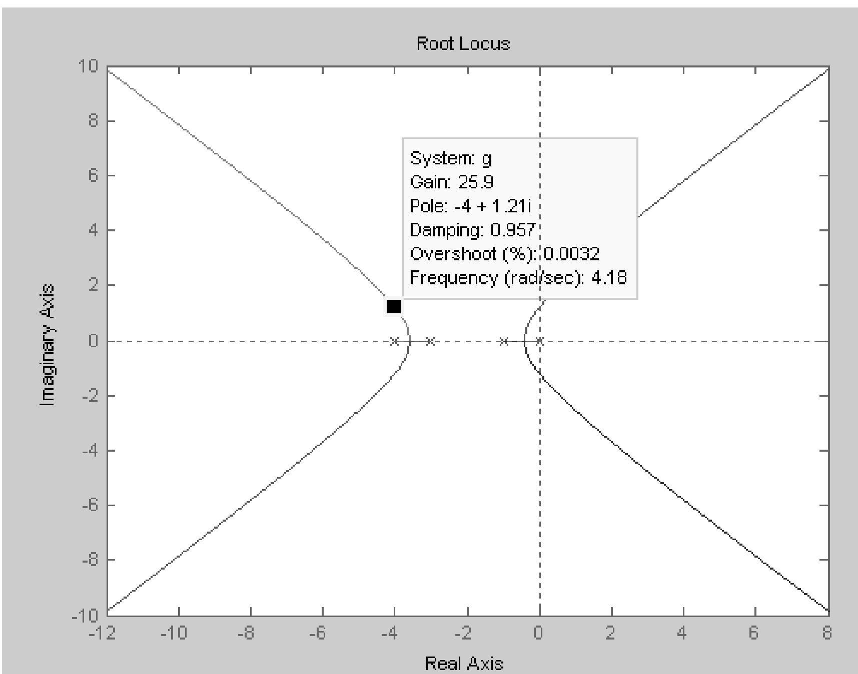
Estimated student time to complete: 30 minutes
Prerequisite knowledge required: Text Section(s) 6.13,6.14

Solution:

$$K = \frac{\prod M_{p_i}}{\prod M_{z_i}} = \sqrt{4^2 + 1.24^2}\sqrt{3^2 + 1.24^2}\sqrt{1^2 + 1.24^2}\sqrt{1.24^2} = 26.85$$

$$\zeta = \cos\theta = \cos\left(\tan^{-1}\frac{1.24}{4}\right) = 0.955$$

$$s = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}\,j = -0.891\omega_n \pm \omega_n\sqrt{1-0.955^2}\,j$$
$$= -4 \pm 1.24j$$
$$\omega_n = 4.19 \quad rad/\sec$$



Problem 6.15
For the system of Problem 6.11, assume that the roots are chosen at $s = -3 \pm 1.73j$. Find the system's gain, damping ratio, and natural frequency. Show that the angle criterion is met. Can you determine whether or not the system may become unstable as the gain changes?

Estimated student time to complete: 30 minutes
Prerequisite knowledge required: Text Section(s) 6.13, 6.14.

Solution:

$$K = \frac{\prod M_{p_i}}{\prod M_{z_i}} = \frac{\sqrt{3^2 + 1.73^2}\sqrt{1^2 + 1.73^2}}{\sqrt{3^2 + 1.73^2}} = 2$$

$$\zeta = \cos\theta = \cos\left(\tan^{-1}\frac{1.73}{3}\right) = 0.866$$

$$s = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}\,j = -0.866\omega_n \pm \omega_n\sqrt{1-0.866^2}\,j$$
$$= -3 \pm 1.73j$$
$$\omega_n = 3.46 \quad rad/\sec$$

$$\theta_{p_0} = 180 - \tan^{-1}\left(\frac{1.73}{3}\right) = 180 - 30 = 150$$

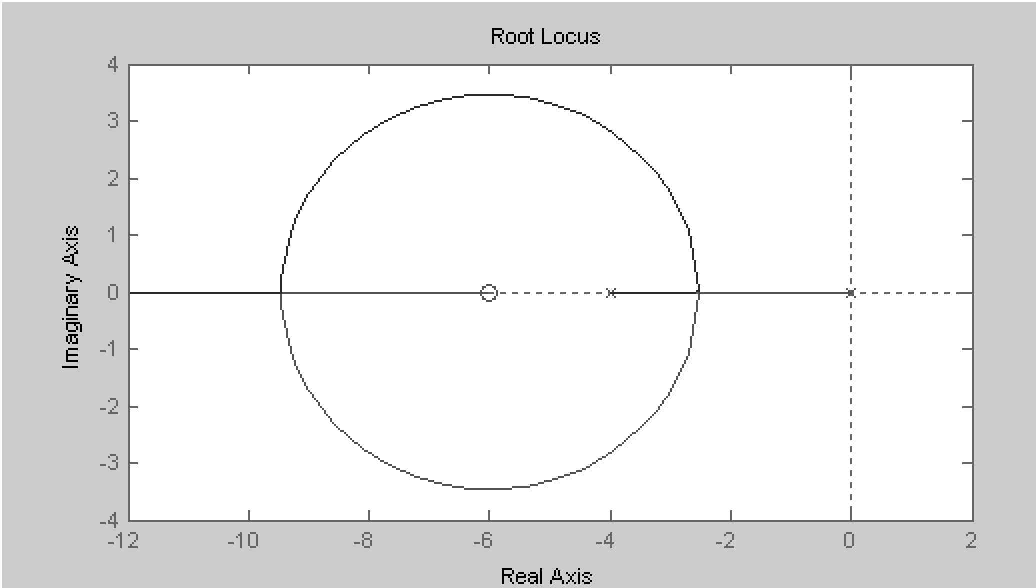
$$\theta_{p_{-4}} = 180 - \tan^{-1}\left(\frac{1.73}{1}\right) = 60$$

$$\theta_{z_6} = \tan^{-1}\left(\frac{1.73}{3}\right) = 30$$

$$\sum\theta_{total} = 150 + 60 - 30 = 180$$

It meets the criterion.

The roots are always on the left side, and therefore, the system is always stable.


Root Locus
Imaginary Axis
Real Axis
4
3
2
1
0
-1
-2
-3
-4
-12
-10
-8
-6
-4
-2
0
2

Problem 6.16
For the system of Problem 6.11, find the roots, the gain, and the steady-state error for a settling time of less that 1 second and overshoot of 4% or less.

Estimated student time to complete: 30-40 minutes
Prerequisite knowledge required: Text Section(s) 6.14

Solution:

$$GH = \frac{K(s+6)}{s(s+4)}$$

By trial and error,

$$\%OS = e^{\left(-\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100\% = 4\% \quad \rightarrow \quad \zeta \geq 0.715,$$
$$\theta = \cos^{-1}\zeta = 44.4^\circ$$

The chosen roots must be below this value, therefore choose $s = -4 \pm 2.83j$

$$\zeta\omega_n(approximate) = \frac{4}{1} = 4$$

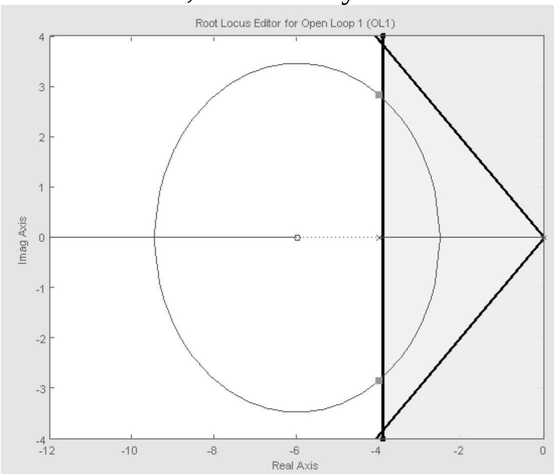
System type: 1 $\rightarrow E_{ss} = 0$

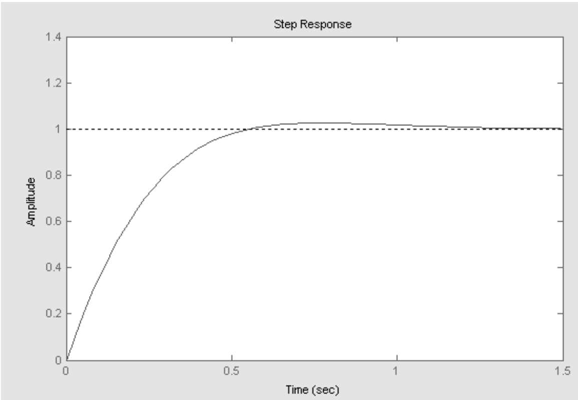
$$K = \frac{\prod M_{p_i}}{\prod M_{z_i}} = \frac{\sqrt{-4^2+2.83^2}\sqrt{2.83^2}}{\sqrt{-2^2+2.83^2}} = \frac{(4.9)(2.83)}{3.465} = 4$$

$$\theta = \tan^{-1}\frac{2.83}{4} = 35.28^\circ$$

$$\zeta_{actual} = \cos\theta = 0.816$$

The root locus, as drawn by MATLAB is shown:





Problem 6.17
For the following system, find the roots, the gain, and steady-state error for the fastest response and a settling time of less than 2 seconds and an overshoot of less than 4%.

$$GH = \frac{K}{(s+1)(s+3)}$$

Estimated student time to complete: 30-40 minutes
Prerequisite knowledge required: Text Section(s) 6.14

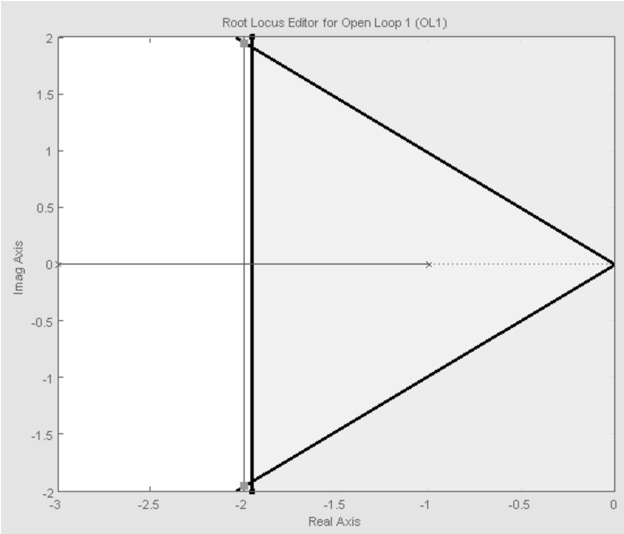
Solution:

For root locus: $\sigma_A = \dfrac{-1-3}{2} = -2$ and $\theta_A = \pm 90^\circ$

For overshoot of 4%: $\%OS = e^{\left(-\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100\% = 4\% \quad \rightarrow \quad \zeta \cong 0.72$

$$\theta = \cos^{-1}\zeta = 44^\circ \quad \text{and} \quad T_s = \frac{4}{\zeta\omega_n} = \frac{4}{2} = 2$$

The root locus and the limitations are shown:



Choose poles for fastest response: $\tan 44^\circ = \dfrac{l}{2} \quad \rightarrow \quad l = 1.93$

Therefore: $s = -2 \pm 1.93j$

$$K = \frac{\prod M_{p_i}}{\prod M_{z_i}} = \frac{\sqrt{(3-2)^2 + 1.93^2}\sqrt{(1-2)^2 + 1.93^2}}{1} = 4.72$$

System type: 0 $\quad \rightarrow \quad K_P = \lim_{s\to 0} G(s) = \lim_{s\to 0} \dfrac{4.72}{(s+1)(s+3)} = 1.57$

$$E_{ss} = \frac{1}{1+1.57} = 0.39$$

Problem 6.18
For the system of problem 6.17, select the locations and the proportional and integral gains to change it to a proportional-plus-integral system with zero steady-state error.

Estimated student time to complete: 20-30 minutes
Prerequisite knowledge required: Text Section(s) 6.15

Solution:

We add $s = 0$ at the origin and a zero near it (location to be found). Therefore,

$$GH = \frac{K(s+z)}{s(s+1)(s+3)} = \frac{K(s+0.1)}{s(s+1)(s+3)}$$

Assume $\zeta = 0.73$ for 4% *OS* even though the system is not second-order (this is close enough). Therefore,

$\theta = 43^\circ$ approximately at $s = -2 \pm 1.87j$.

$$K = \frac{\prod M_{p_i}}{\prod M_{z_i}} = \frac{\sqrt{2^2+1.87^2}\sqrt{1^2+1.87^2}\sqrt{1^2+1.87^2}}{\sqrt{1.9^2+1.87^2}} = 4.6$$

The gain can be calculated as:

$$K = \frac{K_P s + K_I}{s} = \frac{K_P\left(s + \frac{K_I}{K_P}\right)}{s} = \frac{4.6(s+0.1)}{s}$$

$K_P = 4.6$ and $K_I = 0.46$

Since this is Type-1, $E_{ss} = 0$.

Problem 6.19
For the system of Problem 6.17, we would like to improve the settling time to 1 second by adding a zero to the system, therefore a proportional-plus-derivative system. Find a proper location for the zero and the loop gain.

Estimated student time to complete: 20-30 minutes
Prerequisite knowledge required: Text Section(s) 6.16

Solution:

Although not accurate, assume that $\sigma = \frac{4}{1} = 4$. Therefore, we arbitrarily choose the poles to be at $s = -4 \pm 3.85j$.

Angle deficiency:

$$\theta_{-1} = 180 - \tan^{-1}\frac{3.85}{3} = 127.9^\circ$$

$$\theta_{-3} = 180 - \tan^{-1}\frac{3.85}{1} = 104.6^\circ$$

$$\theta_z = 127.9 + 104.6 - 180 = 52.5^\circ$$

$$\tan 52.5 = \frac{3.85}{z-4} \quad \rightarrow \quad z = 6.95$$

The loop gain is:

$$K = \frac{\prod M_{p_i}}{\prod M_{z_i}} = \frac{\sqrt{3^2 + 3.85^2}\sqrt{1^2 + 3.85^2}}{\sqrt{2.95^2 + 3.85^2}} = 4$$

$$GH = \frac{4\left(z + 6.95\right)}{\left(s+1\right)\left(s+3\right)}$$

Problem 6.20
For the system of problem 6.19, add an integrator to the system to make it into a PID system in order to achieve a zero steady-state error. Find the location of an additional zero and proportional, derivative, and integral gains.

Estimated student time to complete: 20-30 minutes
Prerequisite knowledge required: Text Section(s) 6.17

Solution:

Assume: $$GH = \frac{K\left(s+6.95\right)\left(s+z\right)}{s\left(s+1\right)\left(s+3\right)}$$

Magnitude criterion:

$$K = \frac{\prod M_{p_i}}{\prod M_{z_i}} = \frac{\sqrt{4^2+3.85^2}\sqrt{3^2+3.85^2}\sqrt{1^2+3.85^2}}{\sqrt{2.95^2+3.85^2}\sqrt{3.9^2+3.85^2}} = 4.06$$

$$G = \frac{V_a}{E} = K_P + \frac{K_I}{s} + K_D s = \frac{K_D\left(s^2 + \frac{K_P}{K_D}s + \frac{K_I}{K_D}\right)}{s}$$

$$= \frac{4.06(s+6.95)(s+0.1)}{s} = \frac{4.06(s^2+7.05s+0.695)}{s}$$

$$\begin{cases} K_D = 4.06 \\ \dfrac{K_P}{K_D} = 7.05 \\ \dfrac{K_I}{K_D} = 0.695 \end{cases} \rightarrow \begin{cases} K_D = 4.06 \\ K_P = 28.6 \\ K_I = 2.82 \end{cases}$$

CHAPTER SEVEN

Problem 7.1

A motor with a rotor inertia of 0.030 Kgm^2 and maximum torque of 12 Nm is connected to a uniformly distributed arm with a concentrated mass at its end, as shown in Figure P.7.1. Ignoring the inertia of a pair of reduction gears and viscous friction in the system, calculate the total inertia felt by the motor and the maximum angular acceleration it can develop if the gear ratio is a) 5, b) 50, c) 100. Compare the results.

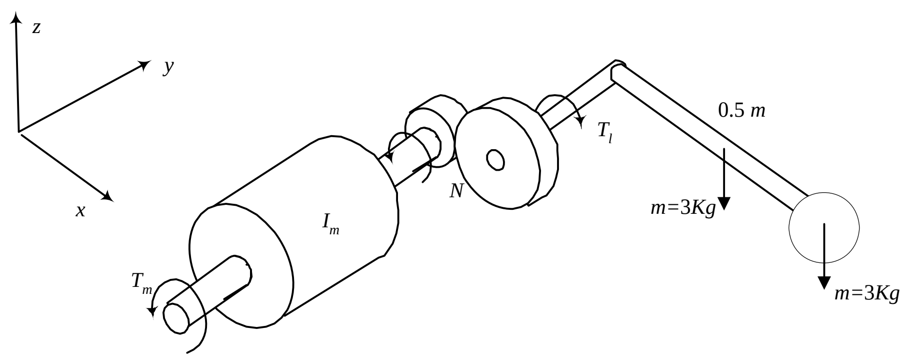


Figure P.7.1.

Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 7.2

Solution:

$$I_{load} = \frac{1}{3} m_{arm} l^2 + m_{mass} l^2 = \frac{1}{3}(3)(0.5)^2 + (3)(0.5)^2 = 1 \text{ Kgm}^2$$

a) For $N = 5$,
$$\begin{cases} I_{total} = I_{motor} + \left(\dfrac{1}{N^2}\right) I_{load} = 0.03 + \dfrac{1}{25} = 0.07 \text{ Kgm}^2 \\ \ddot{\theta}_{\max} = \dfrac{T_{\max}}{I_{total}} = \dfrac{12}{0.07} = 171 \quad r/s^2 \end{cases}$$

b) For $N = 50$,
$$\begin{cases} I_{total} = I_{motor} + \left(\dfrac{1}{N^2}\right) I_{load} = 0.03 + \dfrac{1}{2500} = 0.0304 \text{ Kgm}^2 \\ \ddot{\theta}_{\max} = \dfrac{T_{\max}}{I_{total}} = \dfrac{12}{0.0304} = 395 \quad r/s^2 \end{cases}$$

c) For $N = 100$,
$$\begin{cases} I_{total} = I_{motor} + \left(\dfrac{1}{N^2}\right) I_{load} = 0.03 + \dfrac{1}{10000} = 0.0301 \text{ Kgm}^2 \\ \ddot{\theta}_{\max} = \dfrac{T_{\max}}{I_{total}} = \dfrac{12}{0.0301} = 399 \quad r/s^2 \end{cases}$$

Problem 7.2
Repeat Problem 1, but assume that the two gears have 0.002 Kgm^2 and 0.005 Kgm^2 inertias respectively.

Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 7.2

Solution:

$$I_{load} = \frac{1}{3} m_{arm} l^2 + m_{mass} l^2 = \frac{1}{3}(3)(0.5)^2 + (3)(0.5)^2 = 1 \text{ Kgm}^2$$

$$I_{gears} = 0.002 + \frac{0.005}{N^2}$$

$$I_{total} = I_{motor} + I_{gears} + \left(\frac{1}{N^2}\right) I_{load} = 0.03 + 0.002 + \frac{0.005}{N^2} + \frac{1}{N^2} = 0.032 + \frac{1.005}{N^2}$$

a) For $N = 5$,
$$\begin{cases} I_{total} = 0.032 + \dfrac{1.005}{25} = 0.0722 \text{ Kgm}^2 \\ \ddot{\theta}_{\max} = \dfrac{T_{\max}}{I_{total}} = \dfrac{12}{0.0722} = 166 \quad r/s^2 \end{cases}$$

b) For $N = 50$,
$$\begin{cases} I_{total} = 0.032 + \dfrac{1.005}{2500} = 0.0324 \text{ Kgm}^2 \\ \ddot{\theta}_{\max} = \dfrac{T_{\max}}{I_{total}} = \dfrac{12}{0.0324} = 370 \quad r/s^2 \end{cases}$$

c) For $N = 100$,
$$\begin{cases} I_{total} = 0.032 + \dfrac{1.005}{10,000} = 0.0321 \text{ Kgm}^2 \\ \ddot{\theta}_{\max} = \dfrac{T_{\max}}{I_{total}} = \dfrac{12}{0.0321} = 374 \quad r/s^2 \end{cases}$$

Problem 7.3
The three-axis robot shown in Figure P.7.3. is powered by geared servomotors attached to the joints by worm gears. Each link is 22 cm long, made of hollow aluminum bars, each weighing 0.5 Kg. The center of mass of the second motor is 20 cm from the center of rotation. The gear ratio is 1/3 in the servomotor and 1/5 in the worm gear set. The worst case scenario for the elbow joint is when the arm is fully extended, as shown. Calculate the torque needed to accelerate both arms together, fully extended, at a rate of 90 rad/s^2. Assume the inertias of the worm gears are negligible.

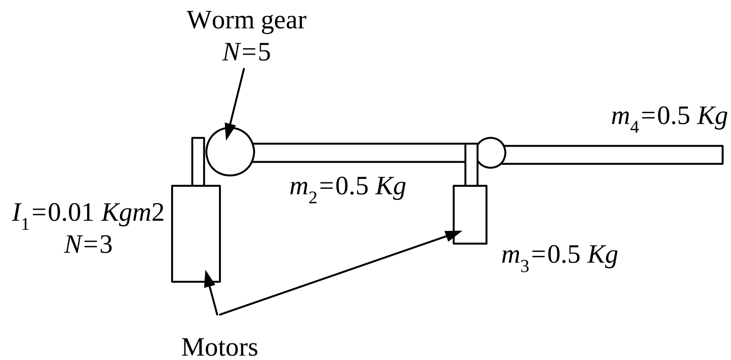


Figure P.7.3.

Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 7.2

Solution:

Total gear ratio $N = 5\times 3 = 15$

$$I_{total} = I_{rotor1} + \left(I_{arm_{1+2}} + I_{motor_2}\right)\frac{1}{N^2}$$

$$= 0.01 + \left(\frac{1}{3}\left(2m_{arm}\right)\left(2l\right)^2 + m_{motor}\left(l_{motor}\right)^2\right)$$

$$= 0.01 + \left[\frac{1}{3}(1)(0.44)^2 + (0.5)(0.2)^2\right]\frac{1}{15^2} = 0.01038$$

$$T = I\ddot{\theta} = 0.01038\times 90 = 0.934 \quad Nm$$

Problem 7.4
Repeat Problem 3, but suppose the maximum torque this motor can provide is 0.9 Nm. Therefore, a new motor must be picked. Two other motors are available, one with the inertia of 0.009 Kgm^2 and torque of 0.85 Nm, one with inertia of 0.012 Kgm^2 and torque of 1 Nm. Which one would you use?

Estimated student time to complete:15-20 minutes
Prerequisite knowledge required: Text Section(s) 7.2.

Solution:

For $N = 5\times 3 = 15$,

For case 1:

$$I_{total} = I_{rotor1} + \left(I_{arm_{1+2}} + I_{motor_2}\right)\frac{1}{N^2}$$
$$= 0.009 + \left(\frac{1}{3}\left(2m_{arm}\right)\left(2l\right)^2 + m_{motor}\left(l_{motor}\right)^2\right)$$
$$= 0.009 + \left[\frac{1}{3}(1)(0.44)^2 + (0.5)(0.2)^2\right]\frac{1}{15^2} = 0.00938$$
$$T = I\ddot{\theta} = 0.00938\times 90 = 0.844 \quad Nm$$

For case 2:

$$I_{total} = I_{rotor1} + \left(I_{arm_{1+2}} + I_{motor_2}\right)\frac{1}{N^2}$$
$$= 0.012 + \left(\frac{1}{3}\left(2m_{arm}\right)\left(2l\right)^2 + m_{motor}\left(l_{motor}\right)^2\right)$$
$$= 0.009 + \left[\frac{1}{3}(1)(0.44)^2 + (0.5)(0.2)^2\right]\frac{1}{15^2} = 0.01238$$
$$T = I\ddot{\theta} = 0.01238\times 90 = 1.114 \quad Nm$$

The smaller motor is better.

Problem 7.5
Estimate how much the torque/inertia ratio of a disk motor might be if it can go from zero to 2000 rpm in one millisecond, and compare it to the motor of Problem 1.

Estimated student time to complete: 10 minutes
Prerequisite knowledge required: Text Section(s) 7.2

Solution:

$$\dot{\theta}_2 = \frac{2000 \times 2\pi}{60} = 209.44 \quad rad/\sec$$

$$\ddot{\theta} = \frac{\dot{\theta}_2 - \dot{\theta}_1}{t_2 - t_1} = \frac{209.44 - 0}{0.001} = 209440 \quad rad/\sec^2$$

The torque/inertia ratio is:

$$T = I\ddot{\theta} \quad \rightarrow \quad \frac{T}{I} = \ddot{\theta} = 209440$$

For the motor of Problem 1:

$$T = I\ddot{\theta} \quad \rightarrow \quad \frac{T}{I} = \frac{12}{0.03} = 400 \quad rad/\sec^2$$

Problem 7.6
Using a timer circuit, design a pulse generating circuit that will deliver a range of 5-500 pulses per second to a stepper motor driver.

Estimated student time to complete:15-20 minutes
Prerequisite knowledge required: Text Section(s) 7.6.7.

Solution:

We will pick a 50% duty cycle for each signal. Therefore:

$$t_{low} = t_{high} = \frac{1}{2} t_{total}$$

t_{total} ranges between 0.002 and 0.2 seconds. Therefore $t_{low} = 0.001$ sec or $t_{low} = 0.1$ sec.

Pick $C_2 = 0.1\ \mu F$. Therefore:

$$t_{low} = 0.693 C_2 (R_2)$$
$$0.001 = 0.693\left(0.1 \times 10^{-6}\right) R_2 \quad \rightarrow \quad R_2 = 14,430 \quad \Omega$$
$$0.1 = 0.693\left(0.1 \times 10^{-6}\right) R_2 \quad \rightarrow \quad R_2 = 1,443,000 \quad \Omega$$

Therefore, a potentiometer with $R_2 = 1.5 \quad M\Omega$ enables us to achieve the desired values. For 50% duty cycle, as shown above,

$$t_{low} = t_{high}$$
$$0.693 C_2 (R_2) = 0.693 C_2 (R_H + R_2) \quad \rightarrow \quad R_H = R_1 + R_3 + R_4 = 0$$

Pick $R_1 = R_3 = R_4 = 1 \quad K\Omega$. Therefore:

$$\begin{cases} C_2 = 0.1 \quad \mu F \\ R_1 = 1 \quad K\Omega \\ R_2 = 1.5 \quad M\Omega \\ R_3 = 1 \quad K\Omega \\ R_4 = 1 \quad K\Omega \end{cases}$$

Problem 7.7
Calculate the gear ratio for a Harmonic drive if N_L=100, N_F=95, N_2=90, N_3=95.

Estimated student time to complete: 10-15 minutes
Prerequisite knowledge required: Text Section(s) 7.11

Solution:

From Equation 7.25:

$$e = \frac{\omega_L}{\omega_A} = \frac{N_2 N_L - N_F N_3}{N_2 N_L} = \frac{(90)(100)-(95)(95)}{(90)(100)} = -\frac{1}{360}$$

The gear ratio between the Arm and the Last gear (with Arm as input) is:

$$\frac{\omega_A}{\omega_L} = -360$$

The minus sign indicates that the input (Arm) and the output (Last) gears rotate in opposite directions.

Problem 7.8
Write a program to generate a variable pulse stream to drive a motor with pulse-width-modulated voltages of 1, 2, 3, 4, and 5 volts for a 5 volt input.

Estimated student time to complete: Variable depending on expertise.
Prerequisite knowledge required: Text Section(s) 7.7.

Solution:

The accuracy with which this program will work depends on the processing speed of the microprocessor. It also depends on the level of programming language and how efficiently it is written. We assume that the processor can accept and run a compiled "C" program, and that it is fast enough such that the execution of commands will not negatively affect the accuracy of the output timing. Since only five distinct output voltages of 1, 2, 3, 4, and 5 volts are desired, we only need to generate 5 variations in the timing loops. We assume the output level will be indicated by an input port. Therefore, if input port 1 is high, the desired output level is 1 volts, etc.

The following program is written in "C" language for execution by a Mini-Board™ microprocessor. It is the essentials of the program that are important here, not the actual program and the way it is written. This program must be modified for use for any other specific microprocessor with its own unique programming syntax and requirements.

In this program, the command "motor(1,n)" indicates that the output port 1 is turned on at $V_{CC}\frac{\mathrm{n}}{15}$. If the input voltage is 5 volts, for n=15, the output is 5 volts, etc. The command "msleep(m)" indicates a stop in execution for m milliseconds. "Off(1)" means output port 1 is turned off.

```
/************************************************************/
/*              Developing variable voltages with           */
/*                                                          */
/*                  Pulse Width Modulation                  */
/*                                                          */
/************************************************************
#define WANT_MOTORS
#include <mboard.h>
#include <mbintsvc.h>

void
main ()
{
int input1, input2,input3,input4,input5;

    while (1){
    input1=digital(1);
    if(input1==1){
    motor(1,15); msleep(1);
    off(1);      msleep(4);
    }
    input2=digital(2) ;
    if(input2==1){
    motor(1,15); msleep(2);
    off(1);      msleep(3);
    }
     input3=digital(3)   ;
    if(input3==1){
    motor(1,15); msleep(3);
    off(1);      msleep(2);
    }
     input4=digital(4)    ;
    if(input4==1){
    motor(1,15); msleep(4);
    off(1);      msleep(1);
    }
     input5=digital(5)     ;
    if(input5==1){
    motor(1,15);}
    if(input5==0){ off(1);}

   }
}
```

Problem 7.9
Write a program to generate a sinusoidal pulse-width-modulated output for a constant input voltage.

Estimated student time to complete: Variable depending on expertise.
Prerequisite knowledge required: Text Section(s) 7.7.

Solution:

The accuracy with which this program will work depends on the processing speed of the microprocessor. It also depends on the level of programming language and how efficiently it is written. We assume that the processor can accept and run a compiled "C" program, and that it is fast enough such that the execution of commands will not negatively affect the accuracy of the output timing. Since a sinusoidal out put is desired for a 5 volt input voltage, we need to generate a variable control timing for the pulse width modulation. We calculate the sine value for each degree between zero and 90 degrees. This allows for approximately 10 pulses per second.

The following program is written in "C" language for execution by a Mini-Board™ microprocessor. It is the essentials of the program that are important here, not the actual program and the way it is written. This program must be modified for use for any other specific microprocessor with its own unique programming syntax and requirements.

Since Mini-Board can generate voltages in both polarities, the program can generate a full sine wave. The program must be modified accordingly if the microprocessor cannot do this.

In this program, the command "motor(1,n)" indicates that the output port 1 is turned on at $V_{CC}\frac{\text{n}}{15}$. If the input voltage is 5 volts, for n=15, the output is 5 volts, etc. The command "msleep(m)" indicates a stop in execution for m milliseconds. "Off(1)" means output port 1 is turned off.

```
/*********************************************************/
/*                 Creating a sine function with         */
/*                                                       */
/*                 Pulse Width Modulation                */
/*                                                       */
/*********************************************************/

#define WANT_MOTORS
#include <mboard.h>
#include <mbintsvc.h>
#include <math.h>
void
main ()
{
int timeon, timeoff,a;
a=0;

    while (1)
    {timeoff=abs(100*sin(a));

    timeon=100-timeoff;
    if(a<=90)
    {motor(1,15);      msleep(timeon);
     off(1);           msleep(timeoff);}
    if(90<a<180)
     {motor(1,-15);    msleep(timeon);
      off(1);          msleep(timeoff);}
      a=a+1;
      if(a==180)  {(a=0);}
    }
}
```

Problem 7.10
If you have access to a microprocessor and electronic components such as transistors, make an H-bridge and write a control program to drive a motor in either direction or to brake it. Be mindful of the problems associated with an H-bridge's transistors turning on and off at inappropriate times.

Estimated student time to complete:
Prerequisite knowledge required: Text Section(s)

Solution:

The following program is written in "C" language for execution by a Mini-Board™ microprocessor. It is the essentials of the program that are important here, not the actual program and the way it is written. This program must be modified for use for any other specific microprocessor with its own unique programming syntax and requirements.

In this program, the command "motor(1,n)" indicates that the output port 1 is turned on at $V_{CC}\frac{n}{15}$. If the input voltage is 5 volts, for n=15, the output is 5 volts, etc. The command "msleep(m)" indicates a stop in execution for m milliseconds. "Off(1)" means output port 1 is turned off.

It is assumed that an input to the microprocessor equal to 1 indicates rotation in the CCW direction, 2 indicates CW rotation, and a 0 means the motor should be braked.

```
/**********************************************************/
/*                    H-Bridge Controller                  */
/**********************************************************/
.
.
main( )
{
int direction
        while(1)
        {direction = digital(0);              /* motor rotation direction input */
        if (direction = = 1) {msleep (1);     /* CCW rotation */
        motor (1,15);   motor (4,15);}
        off (2); off (3);

        if (direction = = 2) {msleep (1);     /* CW rotation */
        motor (2,15);  motor (3,15);}
        off (1); off (4);

        if (direction = = 0) {msleep (1);     /* brake the motor */
        motor (3,15);   motor (4,15);}
                            off (1); off (2);   }}
```

CHAPTER EIGHT

There are no problems in Chapter 8.

CHAPTER NINE

Problem 9.1
Calculate the necessary memory requirement for a still color image from a camera with 10 megapixels at:

- 8-bits per pixel (256 levels)
- 16-bits per pixel (65536 levels)

Estimated student time to complete: 5 minutes
Prerequisite knowledge required: Text Section(s) 9.5

Solution:

For an 8-bit per pixel system, each pixel requires 1 byte. The total memory needed is 10 megabytes.

For a 16-bit per channel system, each pixel requires 2 bytes. The total memory needed is 20 megabytes.

The assumption here is that the pixel count given is the total number for all three colors. The actual memory is greatly influenced by the format in which the image is saved. A bitmap image needs much more memory than JPEG.

Problem 9.2
Consider the pixels of an image, with values as shown in Figure P.9.2, as well as a convolution mask with the given values. Calculate the new values for the given pixels.

4	7	1	8
3	2	6	6
5	3	2	3
8	6	5	9

1	0	1
1	1	1
1	0	1

Figure P.9.2

Estimated student time to complete: 5-10 minutes
Prerequisite knowledge required: Text Section(s) 9.10

Solution:

The mask will only affect the middle pixels. The remaining pixels on the perimeter will not be affected. Depending on how a new image is formatted, the pixels on the perimeter may be set to zero or the current values.

The new values for the image, assuming that the old file is copied into a new file, are:

4	7	1	8
3	3.3	5	6
5	4.6	4.4	3
8	6	5	9

Problem 9.3
Consider the pixels of an image, with values as shown in Figure P.9.3, as well as a convolution mask with the given values. Calculate the new values for the given pixels. Substitute 0 for negative grey levels. What conclusion can you make from the result?

5	5	5	7	7	8
5	5	5	8	8	9
10	10	10	10	10	10
5	5	5	9	9	8
8	8	8	9	9	10

1	0	1
0	-4	0
1	0	1

Figure P.9.3

Estimated student time to complete: 5-10 minutes
Prerequisite knowledge required: Text Section(s) 9.10.

Solution:

The mask will only affect the middle pixels. The remaining pixels on the perimeter will not be affected. Depending on how a new image is formatted, the pixels on the perimeter may be set to zero or the current values.

The new values for the image, assuming that the old file is copied into a new file, are:

5	5	5	7	7	8
5	10	12	0	3	9
10	0	0	0	0	10
5	16	17	1	3	8
8	8	8	9	9	10

This can be an indication of an edge.

Problem 9.4
Repeat Problem 9.3, but substitute the absolute value of negative grey levels. What conclusion can you make from the result?

Estimated student time to complete: 5 -10 minutes
Prerequisite knowledge required: Text Section(s) 9.10

Solution:

The mask will only affect the middle pixels. The remaining pixels on the perimeter will not be affected. Depending on how a new image is formatted, the pixels on the perimeter may be set to zero or the current values.

The new values for the image, assuming that the old file is copied into a new file, are:

5	5	5	7	7	8
5	10	12	0	3	9
10	20	13	13	6	10
5	16	17	1	3	8
8	8	8	9	9	10

When absolute values of negative pixels are shown, edges are no longer discernable.

Problem 9.5

Repeat Problem 9.3, but apply the mask of Figure P.9.5 and compare your results with Problem 9.3. Which one is better?

-1	0	-1
0	4	0
-1	0	-1

Figure P.9.5

Estimated student time to complete: 5 -10 minutes
Prerequisite knowledge required: Text Section(s) 9.10

Solution:

The mask will only affect the middle pixels. The remaining pixels on the perimeter will not be affected. Depending on how a new image is formatted, the pixels on the perimeter may be set to zero or the current values.

The new values for the image, assuming that the old file is copied into a new file, are:

5	5	5	7	7	8
5	0	0	0	0	9
10	20	13	13	6	10
5	0	0	0	0	8
8	8	8	9	9	10

This mask also shows the edge, but unlike the one in problem 9.3., the pixels are brighter than the neighboring pixels.

Problem 9.6
Repeat Problem 9.3, but apply the mask of Figure P.9.6 and compare your results with Problem 9.3. Which one is better?

0	1	0
1	-4	1
0	1	0

Figure P.9.6

Estimated student time to complete: 5 -10 minutes
Prerequisite knowledge required: Text Section(s)9.10

Solution:

The mask will only affect the middle pixels. The remaining pixels on the perimeter will not be affected. Depending on how a new image is formatted, the pixels on the perimeter may be set to zero or the current values.

The new values for the image, assuming that the old file is copied into a new file, are:

5	5	5	7	7	8
5	5	8	0	2	9
10	0	0	0	0	10
5	8	12	0	0	8
8	8	8	9	9	10

At least for this image, the effect is similar to the mask in problem 9.3.

Problem 9.7

Repeat Problem 9.3, but apply the mask of Figure P.9.7 and compare your results with Problem 9.3. Which one is better?

1	1	1
1	-8	1
1	1	1

Figure P.9.7

Estimated student time to complete: 5 -10 minutes
Prerequisite knowledge required: Text Section(s) 9.10

Solution:

The mask will only affect the middle pixels. The remaining pixels on the perimeter will not be affected. Depending on how a new image is formatted, the pixels on the perimeter may be set to zero or the current values.

The new values for the image, assuming that the old file is copied into a new file, are:

5	5	5	7	7	8
5	15	20	0	5	9
10	0	0	0	0	10
5	24	29	0	3	8
8	8	8	9	9	10

At least for this image, the effect is similar to Problem 9.3, but darker.

Problem 9.8

An image is represented by values shown below.

a) Find the value of pixel 2c when mask 1 is applied.

b) Find the value of pixel 3b when mask 2 is applied.

c) Find the values of pixels 2b and 3c when a 3×3 median filter is applied.

d) Find the area of the major object that results when a threshold of 4.5 is applied based on a +4-connectivity (start at the first on-pixel).

	a	b	c	d	e
1	4	9	6	2	6
2	4	3	7	4	6
3	6	2	6	5	3
4	1	6	5	9	2
5	2	8	4	4	7

1	1	1
1	1	1
1	1	1

Mask 1

0	1	0
1	-4	1
0	1	0

Mask 2

Figure P.9.8.

Estimated student time to complete: 10 -15 minutes
Prerequisite knowledge required: Text Section(s) 9.10 and 9.17.

Solution:

The mask will only affect the middle pixels. The remaining pixels on the perimeter will not be affected. Depending on how a new image is formatted, the pixels on the perimeter may be set to zero or the current values.

The new values for the image, assuming that the old file is copied into a new file, are:

a) For 2c:
$R = (9\times1 + 6\times1 + 2\times1 + 3\times1 + 7\times1 + 4\times1 + 2\times1 + 6\times1 + 5\times1) / 9 = 4.9$ (or 5)

b) For 3b:
$R = (4\times0 + 3\times1 + 7\times0 + 6\times1 + 2\times -4 + 6\times1 + 1\times0 + 6\times1 + 5\times0) = 13$

c) For 2b: 2,3,4,4,6,6,6,7,9. The median is 6.
For 3c: 2,3,4,5,5,6,6,7,9. The median is 5.

d) The resulting image after the application of the threshold is:

	a	b	c	d	e
1		X	X		X
2			X		X
3	X		X	X	
4		X	X	X	
5		X			X

Applying the +4 connectivity rule starting at the first on-pixel results in: 1b, 1c, 2c, 3c, 3d, 4c, 4d, 4b, 5b, for a total of 9 pixels.

Problem 9.9
An image is represented by values shown below.

a. Find the value of pixel 3b when mask 1 is applied.
b. Find the values of pixels 2b, 2c, 2d when mask 2 is applied.
c. Find the value of pixel 3c when a 5×5 median filter is applied.
d. Find the area of the major object that results when a threshold of 4.5 is applied based on a $\times$4-connectivity (start at the first on-pixel).

	a	b	c	d	e
1	8	9	6	2	5
2	4	6	2	4	6
3	6	7	5	6	5
4	1	10	5	9	4
5	2	8	4	3	2

0	1	0
1	4	1
0	1	0

Mask 1

1	1	1
-2	-2	-2
1	1	1

Mask 2

Figure P.9.9.

Estimated student time to complete: 10 -15 minutes
Prerequisite knowledge required: Text Section(s) 9.10 and 9.17.

Solution:

The mask will only affect the middle pixels. The remaining pixels on the perimeter will not be affected. Depending on how a new image is formatted, the pixels on the perimeter may be set to zero or the current values.

The new values for the image, assuming that the old file is copied into a new file, are:

a) For 3b:
$R = (4\times0 + 6\times1 + 2\times0 + 6\times1 + 7\times4 + 5\times1 + 1\times0 + 10\times1 + 5\times0)/8 = 6.875$ (or 7)

b) For 2b:
$R = (8\times1 + 9\times1 + 6\times1 + 4\times\text{-}2 + 6\times\text{-}2 + 2\times\text{-}2 + 6\times1 + 7\times1 + 5\times1) = 17$

For 2c:
$R = (9\times1 + 6\times1 + 2\times1 + 6\times\text{-}2 + 2\times\text{-}2 + 4\times\text{-}2 + 7\times1 + 5\times1 + 6\times1) = 11$

For 2d:
$R = (6\times1 + 2\times1 + 5\times1 + 2\times\text{-}2 + 4\times\text{-}2 + 6\times\text{-}2 + 5\times1 + 6\times1 + 5\times1) = 5$

c) For 3c: 1,2,2,2,2,3,4,4,4,4,5,5,5,5,6,6,6,6,6,7,8,8,9,9,10. The median is 5.

d) The resulting image after the application of the threshold is:

	a	b	c	d	e
1	X	X	X		X
2		X			X
3	X	X	X	X	X
4		X	X	X	
5		X			

Applying the $\times 4$ connectivity rule starting at the first on-pixel results in: 1a, 2b, 3c, 1c, 3a, 4d, 4b, 3e, for a total of 8 pixels.

Problem 9.10
Write a computer program for the application of a 3×3 averaging convolution mask unto a 15×15 image. Refer to the note on page 415 for more information.

Estimated student time to complete: Variable depending on the expertise
Prerequisite knowledge required: Text Section(s) 9.10

Solution:

The following program is written in "C" language. It is the essentials of the programming that are important here, not the syntax. You may use the proper syntax for the program you are using. This program may be written in countless other languages, both graphical and scientific.

In this program, the assumption is that the image input file is read from a data file or it is entered after initializing the array.

```
/*******************************************************/
/*                                                     */
/*                3x3 Convolution Mask                 */
/*                Averaging filter                     */
/*                                                     */
/*                    1   1   1                        */
/*                    1   1   1                        */
/*                    1   1   1                        */
/*                                                     */
/*******************************************************/
#include <stdio.h>
main( )
{

int  p [14][14] = {0};
int  new [14][14] = {0};
int  I , J , sum ;

/* You will need to either read the pixel values by a read command, or */
/* you will have to declare the values into the array.                 */

while (1) ;

        for  ( I = 1;  I <= 13; I = I+1 )

        {for  ( J = 1;  J <= 13; J = J+1 )

        {sum = p [I-1][J+1] + p [I][J+1] + p [I+1][J+1] +
               p [I-1][J] +    p [I][J]    + p [I+1][J]     +
               p [I-1][J-1] + p [I][J-1]  + p [I+1][J-1] ;

        sum = sum / 9 ;
        new [I] [J] = sum ; }}

        return;

}
```

Problem 9.11
Write a computer program for the application of a 5×5 averaging convolution mask unto a 15×15 image. Refer to the note on page 415 for more information.

Estimated student time to complete: Variable depending on the expertise
Prerequisite knowledge required: Text Section(s) 9.10

Solution:

The following program is written in "C" language. It is the essentials of the programming that are important here, not the syntax. You may use the proper syntax for the program you are using. This program may be written in countless other languages, both graphical and scientific.

In this program, the assumption is that the image input file is read from a data file or it is entered after initializing the array.

```
/*********************************************************/
/*                                                       */
/*                5x5 Convolution Mask                   */
/*                Averaging filter                       */
/*                                                       */
/*                  1  1  1  1  1                        */
/*                  1  1  1  1  1                        */
/*                  1  1  1  1  1                        */
/*                  1  1  1  1  1                        */
/*                  1  1  1  1  1                        */
/*                                                       */
/*********************************************************/
#include <stdio.h>
main( )
{
int  p [14][14] = {0};
int  new [14][14] = {0};
int  I , J , K ,L , sum ;

/* You will need to either read the pixel values by a read command, or  */
/* you will have to declare the values into the array.                  */

while (1) ;

        for  ( I = 2;  I <= 12; I ++ )

        {for  ( J = 2;  J <= 12; J++ )

                {for ( K= I-2; K<=I+2; K++)
                {for ( L= J-2; L<=J+2; L++)

                {sum = sum + p [K][L];}}
                return;

        sum = sum / 25 ;
        new [I] [J] = sum ;
        sum=0;  }}
        return;
}
```

Problem 9.12
Write a computer program for the application of a 3×3 high pass convolution mask unto a 15×15 image for edge detection. Refer to the note on page 415 for more information.

Estimated student time to complete: Variable depending on expertise
Prerequisite knowledge required: Text Section(s) 9.10

Solution:

The following program is written in "C" language. It is the essentials of the programming that are important here, not the syntax. You may use the proper syntax for the program you are using. This program may be written in countless other languages, both graphical and scientific.

In this program, the assumption is that the image input file is read from a data file or it is entered after initializing the array.

```
/************************************************************/
/*                                                          */
/*                  3x3 Convolution Mask                    */
/*                  High Pass  filter                       */
/*                                                          */
/*                     -1   0   -1                          */
/*                      0   4    0                          */
/*                     -1   0   -1                          */
/*                                                          */
/************************************************************/
#include <stdio.h>
main( )
{

int  p [14][14] = {0};
int  new [14][14] = {0};
int  I , J , sum ;

/* You will need to either read the pixel values by a read command, or  */
/* you will have to declare the values into the array.                  */

while (1) ;

        for  ( I = 1;  I <= 13; I = I+1 )

        {for  ( J = 1;  J <= 13; J = J+1 )

        {sum = -p [I-1][J+1] - p [I+1][J+1] + 4 * p [I][J] -p [I-1][J-1]
                - p [I+1][J-1] ;

        sum = sum / 9 ;
        new [I] [J] = sum ;}}


        return;
}
```

Problem 9.13
Write a computer program for the application of an $n \times n$ convolution mask unto a $k \times k$ image. Refer to the note on page 415 for more information. You should write the routine such that the user can choose the size of the mask and the values of each mask cell individually.

Estimated student time to complete: Variable depending on expertise
Prerequisite knowledge required: Text Section(s) 9.10

Solution:

The following program is written in "C" language. It is the essentials of the programming that are important here, not the syntax. You may use the proper syntax for the program you are using. This program may be written in countless other languages, both graphical and scientific.

In this program, the assumption is that the image input file is read from a data file or it is entered after initializing the array.

```
/**********************************************************/
/*                                                        */
/*              (n x n) Convolution Mask                  */
/*                  (k x k) Image                         */
/*                                                        */
/**********************************************************/
#include <stdio.h>
main( )
{

int  p [256][256] = {0};          /* maximum size of the image */
int  new [256][256] = {0};
int m [11][11]={0};               /* maximum size of the mask */
int  I , J , sum , n , k , R , C , D, E , nI , nJ , Norm  ;

/* You will need to either read the pixel values by a read command, or  */
/* you will have to declare the values into the array.                  */
/* You may either declare the initial values of the mask cells in the   */
/* declaration statement, or read the values in from a data file.      */
/* n is the desired size of the mask. k is the desired size of the image. */

D= (n – 1 ) / 2 ;
Norm = 0 ;

while (1) ;

        for  ( nI = 0;  nI<=n-1 ;n I = n I+1 )

        {for  ( nJ = 0;  nJ <= n-1; nJ = nJ+1 )

        {Norm = Norm + m [nI][nJ];  }}

                return;
        if (Norm ==0)
        { Norm=1; }
        for  ( I = 1;  I <= k-1; I = I+1 )

        {for  ( J = 1;  J <= k-1; J = J+1 )


                {for ( R= I-D; R<=I+D; R++)
                {for ( C= J-D; C<=J+D; C++)

                {sum = sum + p [R][C] * m [R] [C] ;}}

                return;

        sum = sum / Norm ;
        new [I] [J] = sum ;
        sum=0;  }}


        return;
}
```

Problem 9.14
Write a computer program that will perform the Left-Right search routine for a 15×15 image. Refer to the note on page 415 for more information.

Estimated student time to complete: Variable depending on expertise
Prerequisite knowledge required: Text Section(s) 9.13

Solution:

The following program is written in “C” language. It is the essentials of the programming that are important here, not the syntax. You may use the proper syntax for the program you are using. This program may be written in countless other languages, both graphical and scientific.

In this program, the assumption is that the image input file is read from a data file or it is entered after initializing the array.

```
/*************************************************************************/
/*                                                                       */
/*                      L-R Edge Detection Routine                       */
/*                                                                       */
/*************************************************************************/
#include <stdio.h>
main( )
{

int  p [15][15] = {0};              /* maximum size of the image */
int Edge[15][15] = {0};

int  ER , EC, R, C , PR , PC;       /* Edge, picture, and previous row and column indices. */

/* You will need to either read the pixel values by a read command, or you will have to    */
/* declare the values into the array.  The image must be binary; only 0's and 1's are       */
/* are allowed. When an edge is found, the pixels belonging to the edge will be             */
/* placed in a file called Edge.                                                            */

while (1)

/* Search for an edge */

        for  ( R = 0;  R <= 14; R++ )

        {for  ( C = 0;  C <= 14; C++ )

        {if p[R][C]=1
             {Edge[ER][EC] = 1; }    /* Edge starts */
              PR=R ; PC=C -1 ;   /*  signifies the direction of movement */
             {break; } }
        return;

/* search for the other pixels in the edge */
        Search:

        if (p[R][C]=1 {Edge[ER][EC] = 1; goto Left;}
        goto Right;

/* Turning Left */
        Left:

        if (PC<C) {PC = C; C = C ; PR = R ; R = R-1 ;}
        if (PC>C) {PC = C; C = C ;  PR = R; R = R+1;}
        if (PR<R) {PC = C; C = C+1 ; PR = R; R = R ;}
        if (PR>R) {PC = C; C = C -1 ; PR = R; R = R ;}
        goto Search;

        Right:

        if (PC<C) {PC = C; C = C ; PR = R ; R = R+1 ;}
        if (PC>C) {PC = C; C = C ;  PR = R; R = R - 1;}
        if (PR<R) {PC = C; C = C - 1 ; PR = R; R = R ;}
        if (PR>R) {PC = C; C = C +1 ; PR = R; R = R ;}
        goto Search;

}
```

Problem 9.15
Using the Left-Right search technique, find the outer edge of the object in Figure P.9.15:

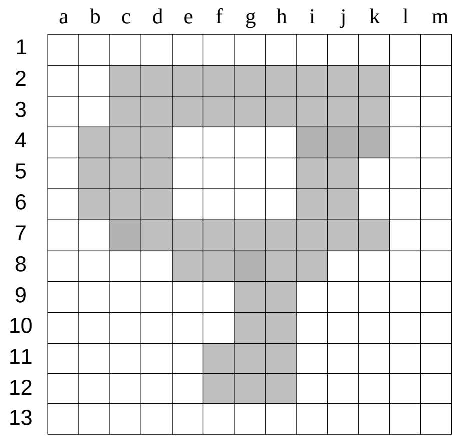
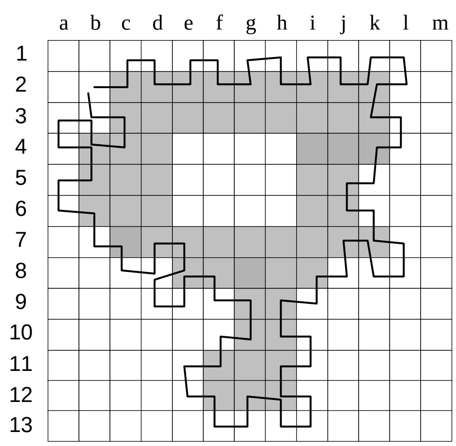


Figure P.9.15

Estimated student time to complete: 5-10 minutes
Prerequisite knowledge required: Text Section(s) 9.13

Solution:



The edge consists of: 2c, 2d, 2e, 2f, 2g, 2h, 2i, 2j, 2k, 3k, 4k, 5j, 6j, 7k, 7j, 8i, 9h, 10h, 11h, 12h, 12g, 12f, 11f, 10g, 9g, 8f, 8e, 7e, 7d, 7c, 6b, 5b, 4b, 4c, 3c.

Problem 9.16
The *x* and *y* coordinates of 5 points are given as (2.5, 0), (4,2), (5,4), (7,6), and (8.5,8). Using the Hough transform, determine which of these points form a line and find its slope and intercept.

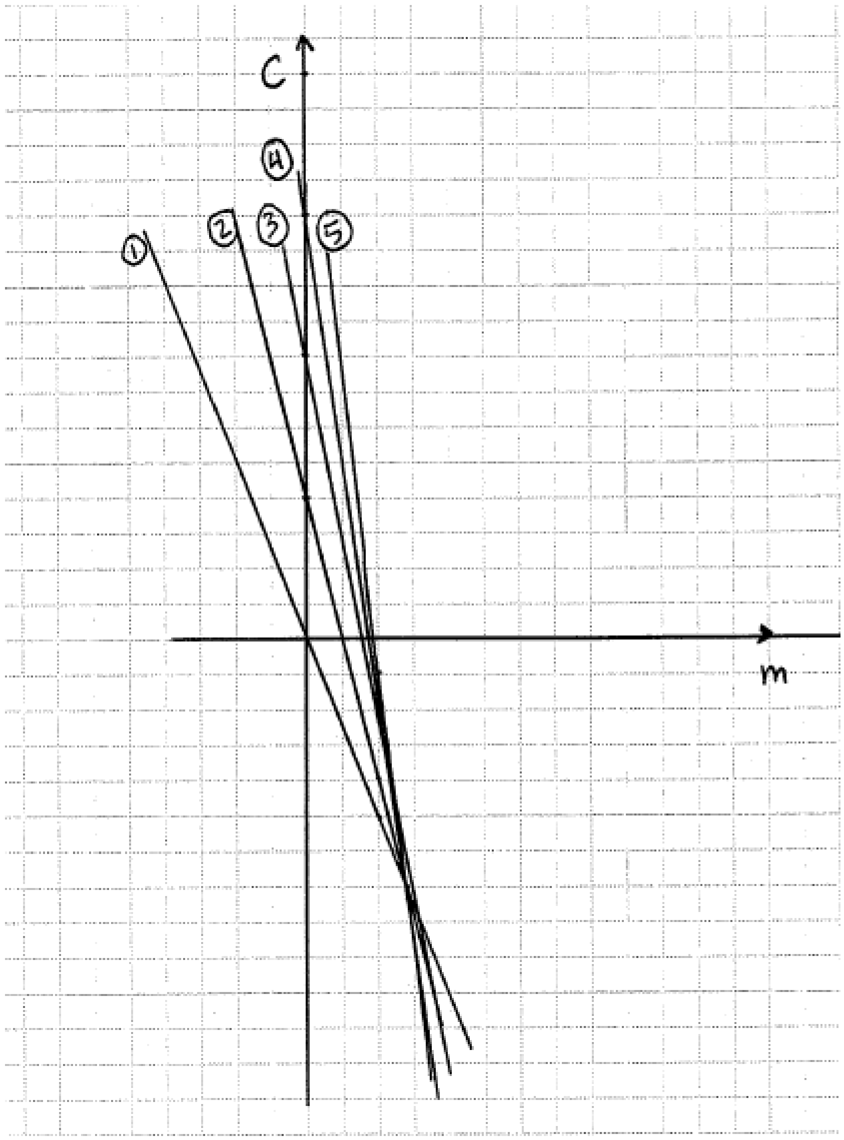
Estimated student time to complete: 20-25 minutes
Prerequisite knowledge required: Text Section(s) 9.15.

Solution:

y	x	x,y	m,c
0	2.5	$0=2.5m+c$	$c= -2.5m+0$
2	4	$2=4m+c$	$c= -4m+2$
4	5	$4=5m+c$	$c= -5m+4$
6	7	$6=7m+c$	$c= -7m+6$
8	8.5	$8=8.5m+c$	$c= -8.5m+8$

The following graph shows the lines and where they intersect. Line 3 (for point 5,4) does not intersect the remaining lines.

The approximate slope and intercept of the line on which the other four points lay are 1.4 and -3.5.


C
4
1
2
3
5
m

Problem 9.17
Write a computer program that will perform a region growing operation based on +4 connectivity. The routine should start at the 1,1 corner pixel, search for a nucleus, grow a region with a chosen index number, and after finishing that region, must continue searching for another nuclei until all object pixels have been checked. Refer to the note on page 415 for more information.

Estimated student time to complete: Variable depending on expertise.
Prerequisite knowledge required: Text Section(s) 9.17

Solution:

The following program is written in “C” language. It is the essentials of the programming that are important here, not the syntax. You may use the proper syntax for the program you are using. This program may be written in countless other languages, both graphical and scientific.

In this program, the assumption is that the image input file is read from a data file or it is entered after initializing the array.

In this program, when a nucleus is found, all the pixels connected to it based on a +4-connectivity that are “on” (have a value in them and therefore, are part of the image not the background) are placed in a stack by changing the value of the cell form zero to 1 in an array of 256x256. The cell’s value is later checked until all are found and the cell’s are all zero. Each corresponding pixel’s value is changed from 1 to the region number. At the end of execution, the original image is changed to as many regions as necessary. Since there will be no cells (pixels) left with a value of 1, execution ends. All segments will be 2 and larger in value.

```
/**********************************************************/
/*                                                        */
/*                 Region Growing Routine                 */
/*                                                        */
/**********************************************************/

#include <stdio.h>

main( )
{

int  p [256][256] = {0};          /* maximum size of the image */
int stack[256][256] = {0};

int  I , J , Region , R , C , sI , sJ , Count  ;

/* You will need to either read the pixel values by a read command, or  */
/* you will have to declare the values into the array.                  */
/* The image must be binary; only 0's and 1's are allowed.              */
/* When a region starts, the pixels belonging to the region will take the */
/* region numbers, starting with number 2.    */

while (1)

Region=1;

/* Search for a nucleus */

          Nucleus:
          Count =1;
          Region = Region + 1;

          for  ( I = 1;  I <= 255; I++ )

          {for  ( J = 1;  J <= 255; J++ )

          {if p[I][J]=1 {R=I; C=J; {break;} } } }    /* p[C][J] is the nucleus */
          return;

/* search for connectivity  and set the stack index values */

          Search:

          if (p[R][C+1]=1) {stack[R][C+1]=1;}
          if (p[R][C -1]=1) {stack[R][C -1]=1;}
          if (p[R+1][C]=1) {stack[R+1][C]=1;}
          if (p[R -1][C]=1) {stack[R -1][C]=1;}
/* Assign the pixel to a region number */

          p[R][C]= Region;

/* Find the next pixel in the stack  */

          for  ( sI = 1; sI <= 255; sI++ )

          {for  ( sJ = 1;  sJ <= 255; sJ++ )

          {if stack[sI][sJ] = 1 {R=sI; C=sJ; stack[sI][sJ]=0; {break;} } }}

          Count = 0;
          return;

          if  (Count = 0) {goto Nucleus;}

          goto Search ;

}
```

Problem 9.18
Write a computer program that will perform a region growing operation based on $\times 4$ connectivity. The routine should start at the 1,1 corner pixel, search for a nucleus, grow a region with a chosen index number, and after finishing that region, must continue searching for another nuclei until all object pixels have been checked. Refer to the note on page 415 for more information.

Estimated student time to complete: Variable depending on expertise.
Prerequisite knowledge required: Text Section(s) 9.17

Solution:

Please see the solution for Problem 9.17. The program for growing a region based on $\times 4$ connectivity is very similar to that program, except that the search will follow the $\times 4$ sequence given in Equation 9.7.

Problem 9.19
Using +4 connectivity logic and starting from 1a pixel, write the sequence of pixels in correct order that will be detected by a region growing routine for the object in Figure P.9.19:

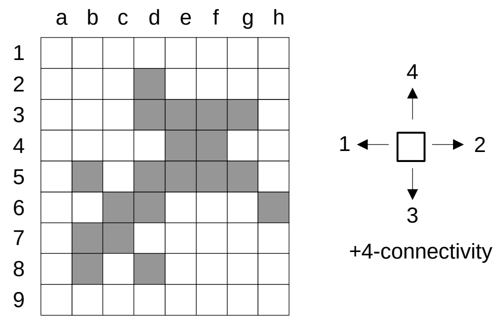


Figure P.9.19

Estimated student time to complete: 10 minutes
Prerequisite knowledge required: Text Section(s) 9.17

Solution:

The sequence of pixels detected is: 2d, 3d, 3e, 3f, 4e, 3g, 4f, 5e, 5f, 5d, 5g, 6d, 6c, 7c, 7b, 8b.

The remaining pixels are not connected to this region and will not be detected.

Problem 9.20
Using $\times 4$ connectivity logic and starting from 1a pixel, write the sequence of pixels in correct order that will be detected by a region growing routine for the object in Figure P.9.20:

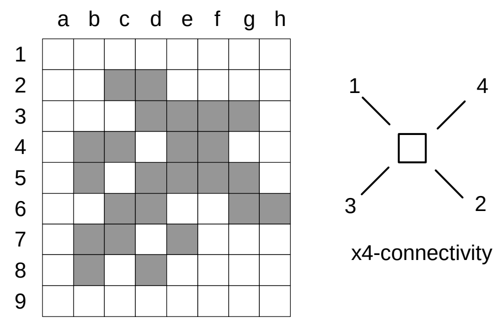


Figure P.9.20

Estimated student time to complete: 10 minutes
Prerequisite knowledge required: Text Section(s) 9.17

Solution:

The sequence of pixels detected is: 2c, 3d, 4e, 4c, 5f, 5d, 3f, 4b, 6g, 6c, 7b.

The remaining pixels are not connected to this region and will not be detected.

Problem 9.21
Find the union between the two objects in Figure P.9.21.

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	1	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

0	0	0	0	0	0
0	1	0	0	0	0
0	1	0	0	0	0
0	1	1	1	1	0
0	0	0	0	1	0
0	0	0	0	1	0
0	0	0	0	0	0

Figure P.9.21

Estimated student time to complete: 10-15 minutes
Prerequisite knowledge required: Text Section(s) 9.18

Solution:

The union is shown:

0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	0	0	0	0	0	0
0	0	0	1	1	1	2	0	0	0	0	0	0
0	0	0	1	2	3	5	3	2	1	0	0	0
0	0	0	0	0	0	2	2	2	2	0	0	0
0	0	0	0	0	0	1	1	1	2	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0

Problem 9.22
Apply a single-pixel erosion based on 8-connectivity on the image of Figure P.9.22.

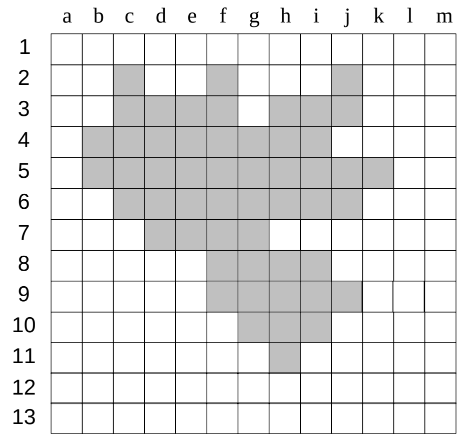
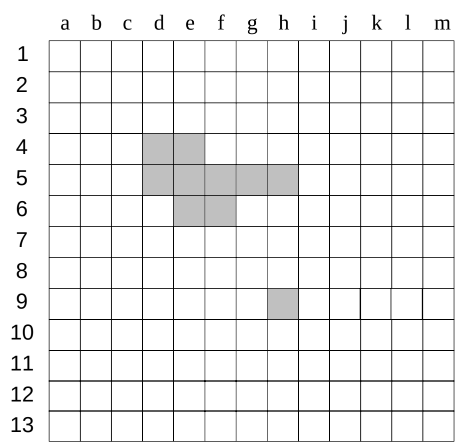


Figure P.9.22

Estimated student time to complete: 10-20 minutes
Prerequisite knowledge required: Text Section(s) 9.18

Solution:

The result is shown:

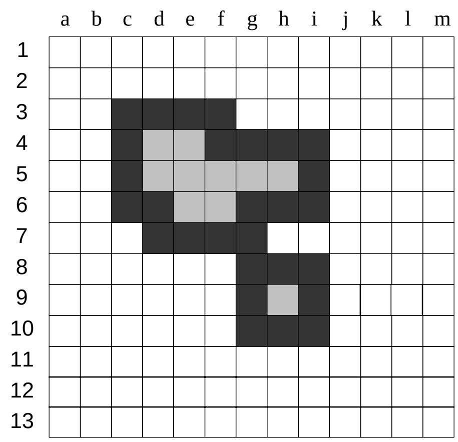


Problem 9.23
Apply a one-pixel dilation to the result of Problem 9.22 and compare your result to Figure P.9.22.

Estimated student time to complete: 15-25 minutes
Prerequisite knowledge required: Text Section(s)9.18

Solution:

The result is shown. As you notice, the image is different from the original.



Problem 9.24
Apply an open operation to Figure P.9.24.

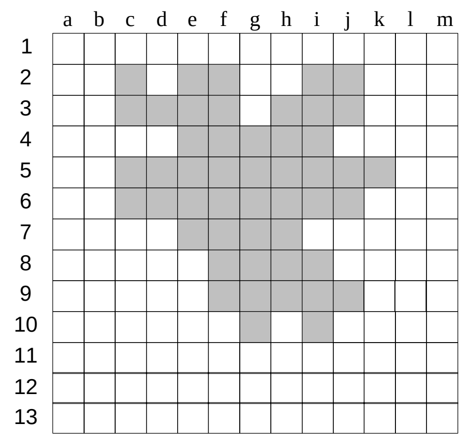
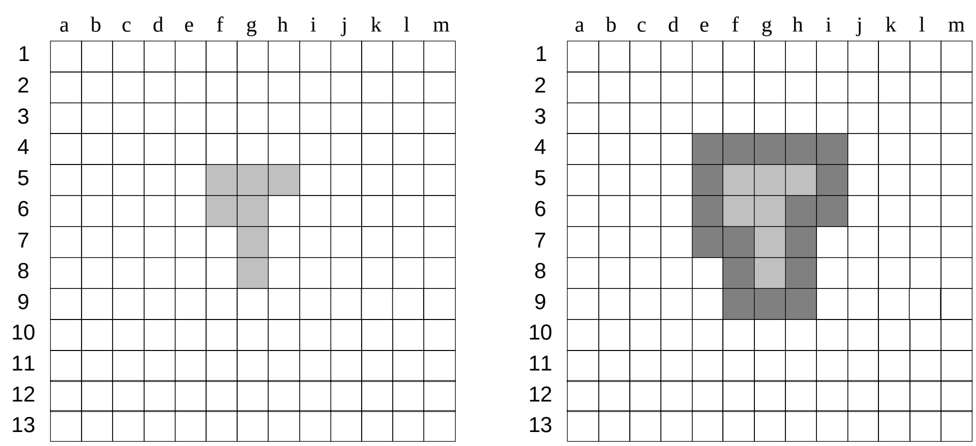


Figure P.9.24

Estimated student time to complete: 10-20 minutes
Prerequisite knowledge required: Text Section(s) 9.18

Solution:

The result is shown:

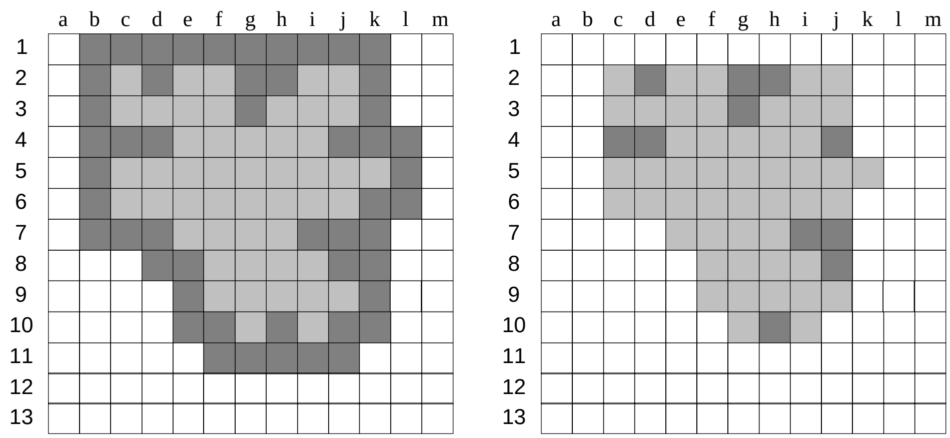


Problem 9.25
Apply a close operation to Figure P.9.24.

Estimated student time to complete: 10-20 minutes
Prerequisite knowledge required: Text Section(s) 9.18

Solution:

The result is shown:



Problem 9.26
Apply a skeletonization operation to Figure P.9.26.

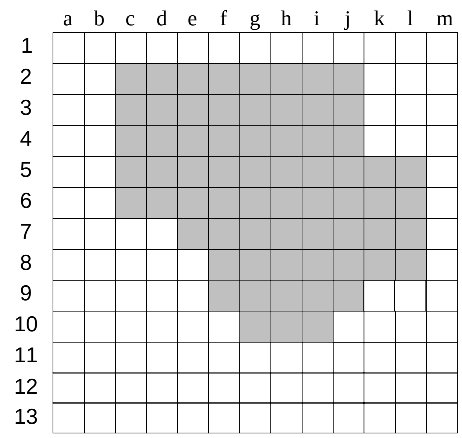
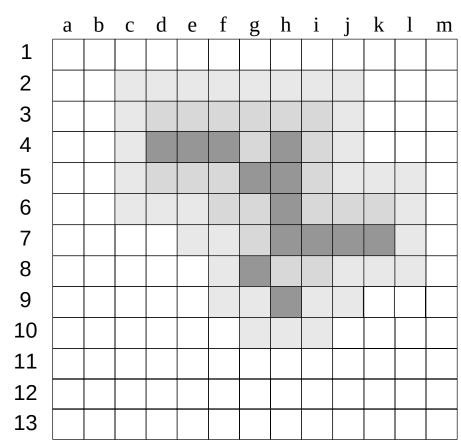


Figure P.9.26

Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 9.18

Solution:

The result is shown:



Problem 9.27

Write a computer program in which different moments of an object in an image can be calculated. The program should ask you for moment indices. The results may be reported to you in a new file, or may be stored in memory. Refer to the note on page 415 for more information.

Estimated student time to complete: Variable depending on expertise
Prerequisite knowledge required: Text Section(s) 9.21.

Solution:

The following program is written in “C” language. It is the essentials of the programming that are important here, not the syntax. You may use the proper syntax for the program you are using. This program may be written in countless other languages, both graphical and scientific.

In this program, the assumption is that the image input file is read from a data file or it is entered after initializing the array.

```
/**********************************************************/
/*                                                        */
/*              Moment Calculations Routine               */
/*                                                        */
/**********************************************************/

#include <stdio.h>
#include <math.h>
main( )
{

int  p [256][256] = {0};           /* maximum size of the image */
int Moment [4][4]={0};             /* moment indices as high as 3*/
int  I , J , a , b, x, y, xorigin, yorigin ;

/* You will need to either read the pixel values from a data file by      */
/* a read command, or you will have to declare the values into the array.*/
/* The image must be binary; only 0's and 1's are allowed.                */
/* You should also read in the x and y of the origin about which the      */
/* moments are calculated.  You must input the a and b indices from       */
/* screen or read from a data file.                                       */

while (1)

        for  ( I = 1; I <= 255; I++ )

        {for  ( J = 1;  J <= 255; J++ )

        {
        if p[I][J]=1
        {x = I-xorigin;
         y = J-yorigin;
         Moment[a][b]=moment[a][b]+pow (x,a) * pow (y,b);}
        }}
        return;

}
```

Problem 9.28
Calculate the $M_{0,2}$ moment for the result of Problem 9.8d based on +4-connectivity.

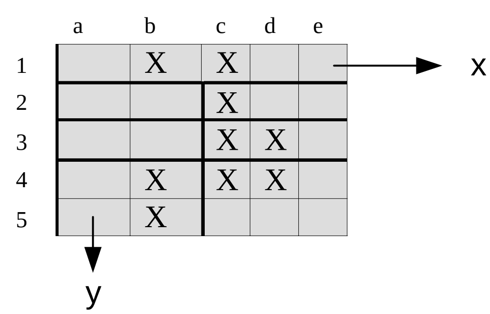
Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 9.21.

Solution:

The resulting image after the application of the threshold is:

	a	b	c	d	e
1		X	X		X
2			X		X
3	X		X	X	
4		X	X	X	
5		X			X

Applying the +4 connectivity rule starting at the first on-pixel results in the following area:



The moment, calculated relative to the first row and column of pixels with whole-number distances is:

$$M_{0,2} = \sum x^0 y^2 = \sum y^2 = 1(1)^2 + 2(2)^2 + 3(3)^2 + 1(4)^2 = 52$$

Problem 9.29
For the binary image of a key in Figure P.9.29, calculate the following:

- Perimeter, based on the Left-Right search technique.
- Thinness, based on $P^2/Area$.
- Center of gravity
- Moment $M_{0,1}$ about the origin (pixel 1,1) and about the lowest pixel of a rectangular box around the key (2,2).

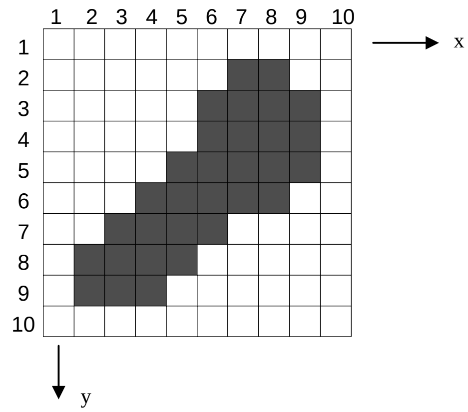
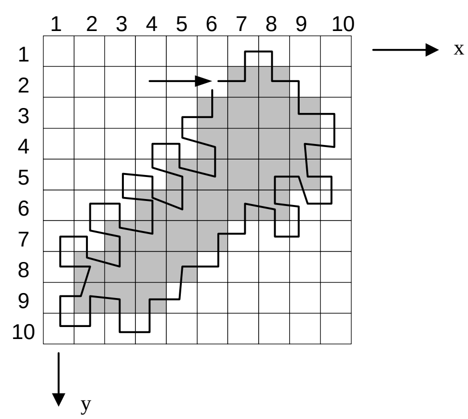


Figure P.9.29

Estimated student time to complete: 15-20 minutes
Prerequisite knowledge required: Text Section(s) 9.13, 9.21.

Solution:



The left-right search result is shown. The perimeter based on the pixels on the search line is $P = 23$.

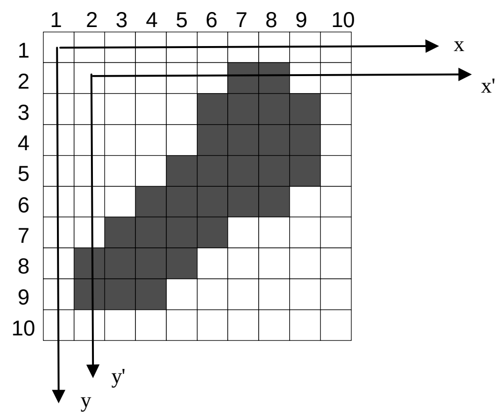
The area may be calculated by $M_{0,0}$, by region growing or by counting.

Thinness: $\dfrac{P^2}{Area} = \dfrac{23^2}{31} = 17$

The 1st moments are:

$$M_{1,0} = \sum x^1 y^0 = \sum x = 2(1) + 3(2) + 4(3) + 4(4) + 5(5) + 5(6) + 5(7) + 3(8) = 150$$
$$M_{0,1} = \sum x^0 y^1 = \sum y = 2(1) + 4(2) + 4(3) + 5(4) + 5(5) + 4(6) + 4(7) + 3(8) = 143$$

Center of gravity is: $\bar{x} = \dfrac{M_{1,0}}{M_{0,0}} = \dfrac{150}{31} = 4.84$ and $\bar{y} = \dfrac{M_{0,1}}{M_{0,0}} = \dfrac{143}{31} = 4.61$



$$M_{0,1} @(1,1) = \sum x^0 y^1 = 143$$
$$M_{0,1} @(2,2) = \sum x^0 y^1 = 4(1) + 4(2) + 5(3) + 5(4) + 4(5) + 4(6) + 3(7) = 112$$

Problem 9.30
Using moment equations, calculate $M_{0,2}$ and $M_{2,0}$ about the centroidal axes of the part in Figure P.9.30.

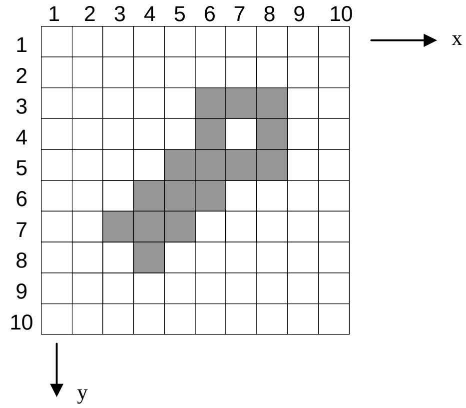


Figure P.9.30

Estimated student time to complete: 20-30 minutes
Prerequisite knowledge required: Text Section(s) 9.21

Solution:

The moment values for the image are:

$$M_{0,0} = \sum x^0 y^0 = 16(1) = 16$$
$$M_{1,0} = \sum x^1 y^0 = \sum x = 1(2) + 3(3) + 3(4) + 4(5) + 2(6) + 3(7) = 76$$
$$M_{0,1} = \sum x^0 y^1 = \sum y = 3(2) + 2(3) + 4(4) + 3(5) + 3(6) + 1(7) = 68$$
$$\overline{x} = \frac{M_{1,0}}{M_{0,0}} = \frac{76}{16} = 4.75 \quad \text{and} \quad \overline{y} = \frac{M_{0,1}}{M_{0,0}} = \frac{68}{16} = 4.25$$
$$M_{2,0} = \sum x^2 y^0 = \sum x^2 = 1(2)^2 + 3(3)^2 + 3(4)^2 + 4(5)^2 + 2(6)^2 + 3(7)^2 = 398$$
$$M_{0,2} = \sum x^0 y^2 = \sum y^2 = 3(2)^2 + 2(3)^2 + 4(4)^2 + 3(5)^2 + 3(6)^2 + 1(7)^2 = 326$$

The second moments about the centroids can be calculates as follows:

$$M_{2,0} @ \overline{x} = M_{2,0} - M_{0,0} \left(\overline{x}\right)^2 = 398 - 16(4.75)^2 = 37$$
$$M_{0,2} @ \overline{y} = M_{0,2} - M_{0,0} (\overline{y})^2 = 326 - 16(4.25)^2 = 37$$

Problem 9.31
Using moment equations, calculate $M_{0,2}$ and $M_{2,0}$ about the centroidal axes of the part in Figure P.9.31.

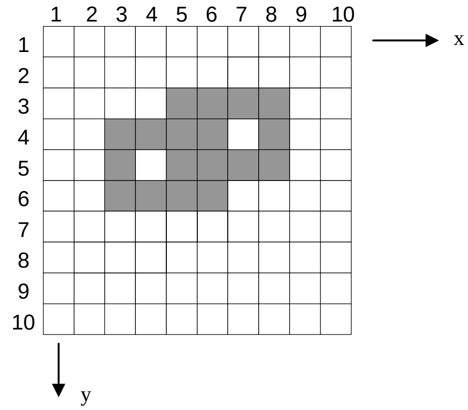


Figure P.9.31

Estimated student time to complete: 20 -30 minutes
Prerequisite knowledge required: Text Section(s) 9.21

Solution:

The moment values for the image are:

$$M_{0,0} = \sum x^0 y^0 = 18(1) = 18$$
$$M_{1,0} = \sum x^1 y^0 = \sum x = 3(2) + 2(3) + 4(4) + 4(5) + 2(6) + 3(7) = 81$$
$$M_{0,1} = \sum x^0 y^1 = \sum y = 4(2) + 5(3) + 5(4) + 4(5) = 63$$
$$\bar{x} = \frac{M_{1,0}}{M_{0,0}} = \frac{81}{18} = 4.5 \quad \text{and} \quad \bar{y} = \frac{M_{0,1}}{M_{0,0}} = \frac{63}{18} = 3.5$$
$$M_{2,0} = \sum x^2 y^0 = \sum x^2 = 3(2)^2 + 2(3)^2 + 4(4)^2 + 4(5)^2 + 2(6)^2 + 3(7)^2 = 413$$
$$M_{0,2} = \sum x^0 y^2 = \sum y^2 = 4(2)^2 + 5(3)^2 + 5(4)^2 + 4(5)^2 = 241$$

The second moments about the centroids can be calculates as follows:

$$M_{2,0} @ \bar{x} = M_{2,0} - M_{0,0}\left(\bar{x}\right)^2 = 413 - 18(4.5)^2 = 48.5$$
$$M_{0,2} @ \bar{y} = M_{0,2} - M_{0,0}(\bar{y})^2 = 241 - 18(3.5)^2 = 20.5$$

CHAPTER TEN

Problem 10.1
Develop a fuzzy inference system for a robot, where the force exerted at the hand and the velocity of the hand are the inputs and the % power to the actuators is the output.

Estimated student time to complete:--------
Prerequisite knowledge required: Text Section(s) 10.3-10.9

Solution:

As in any other design problem, the solution is not unique. For example, different inputs and outputs may be selected, different fuzzy sets may be chosen, the ranges selected may be different, and alternate rules may be assigned. Therefore, not only the solution is not unique, the result will also differ. This suggested solution may only be used as a guide.

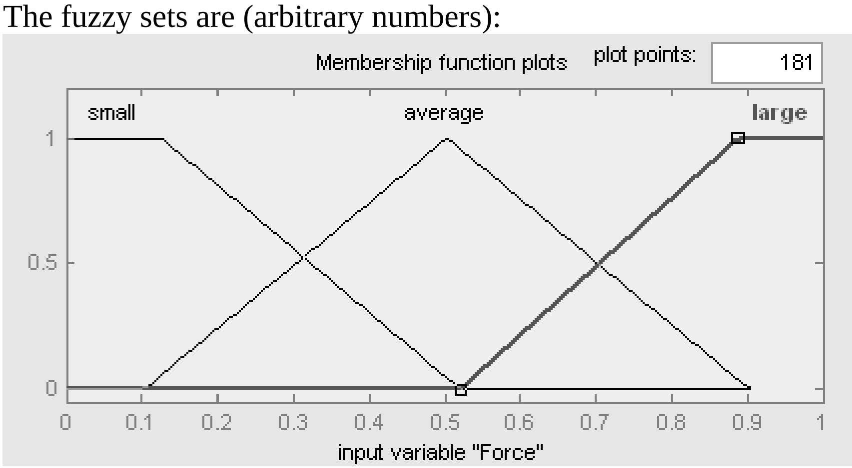
This solution is implemented on the MATLAB Fuzzy Toolbox. You may use this or any other available system.

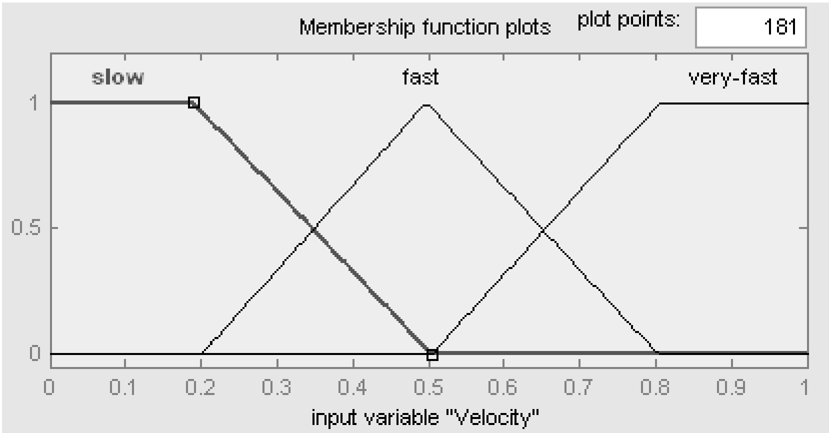
In this solution, the inputs, outputs, the sets, and the rules are as follows:

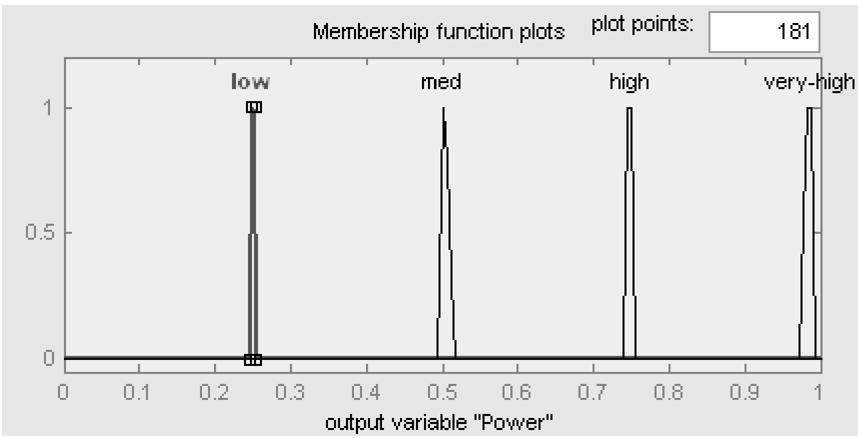
Inputs: Force; Small, Average, Large
Velocity; Slow, Fast, Fast, Very-Fast

Outputs: Power; Low, Medium, High, Very-High

The fuzzy sets are (arbitrary numbers):



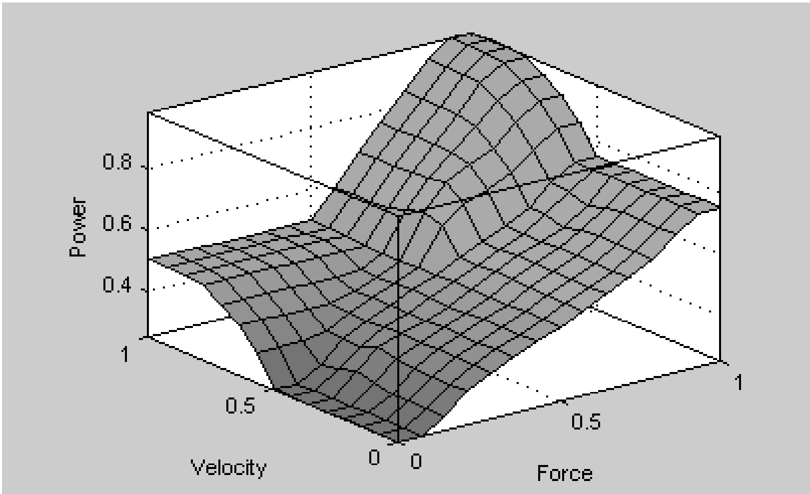




The rules are:

1. If (Force is small) and (Velocity is slow) then (Power is low) (1)
2. If (Force is small) and (Velocity is fast) then (Power is low) (1)
3. If (Force is small) and (Velocity is very-fast) then (Power is med) (1)
4. If (Force is average) and (Velocity is slow) then (Power is med) (1)
5. If (Force is average) and (Velocity is fast) then (Power is med) (1)
6. If (Force is average) and (Velocity is very-fast) then (Power is med) (1)
7. If (Force is large) and (Velocity is slow) then (Power is high) (1)
8. If (Force is large) and (Velocity is fast) then (Power is high) (1)
9. If (Force is large) and (Velocity is very-fast) then (Power is very-high) (1)

The resulting output is:



Changing the rules or the membership functions will change this result.

Problem 10.2
Develop a fuzzy inference system for a washing machine. The inputs are how dirty the fabrics are and how much clothes are being washed, and the output is the wash time.

Estimated student time to complete:--------
Prerequisite knowledge required: Text Section(s) 10.3-10.9

Solution:

As in any other design problem, the solution is not unique. For example, different inputs and outputs may be selected, different fuzzy sets may be chosen, the ranges selected may be different, and alternate rules may be assigned. Therefore, not only the solution is not unique, the result will also differ. This suggested solution may only be used as a guide.

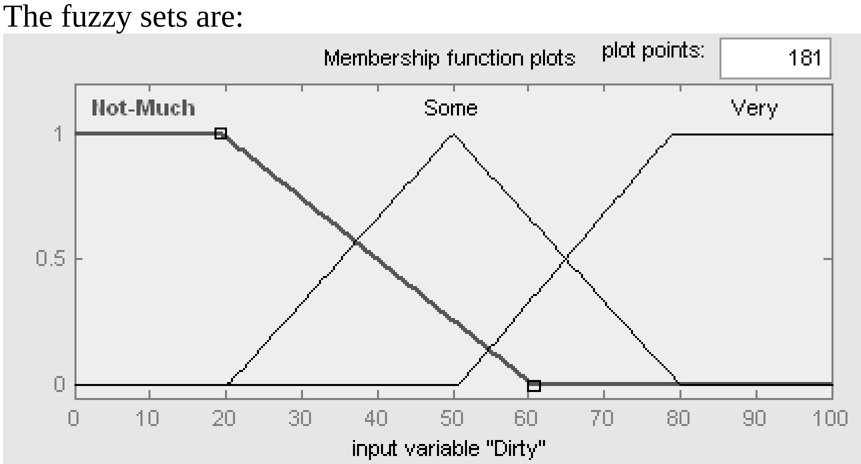
This solution is implemented on the MATLAB Fuzzy Toolbox. You may use this or any other available system.

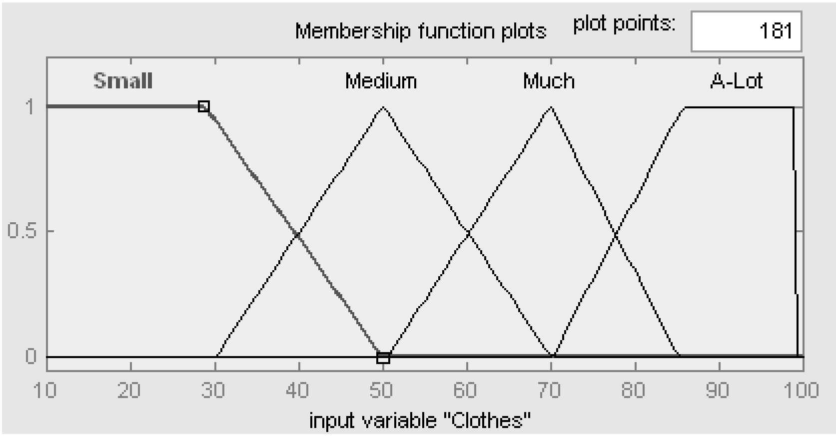
In this solution, the inputs, outputs, the sets, and the rules are as follows:

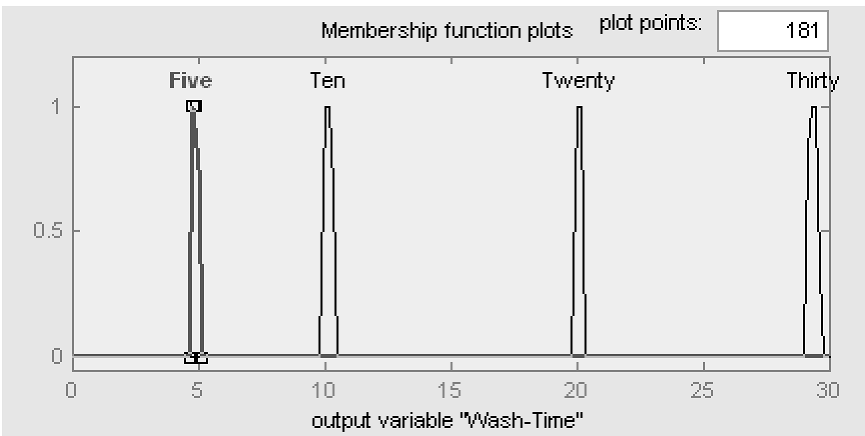
Inputs: Dirty; Not-Much, Some, Very.
Clothes; Small, Med, Much, A-Lot.

Outputs: Wash-Time; Five, Ten, Twenty, Thirty

The fuzzy sets are:



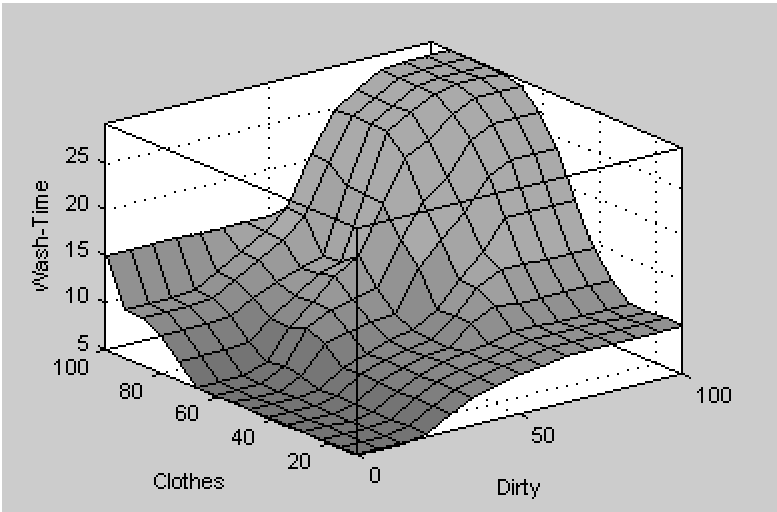




The rules are:

1. If (Dirty is Not-Much) and (Clothes is Small) then (Wash-Time is Five) (1)
2. If (Dirty is Not-Much) and (Clothes is Medium) then (Wash-Time is Five) (1)
3. If (Dirty is Not-Much) and (Clothes is Much) then (Wash-Time is Five) (1)
4. If (Dirty is Not-Much) and (Clothes is A-Lot) then (Wash-Time is Ten) (1)
5. If (Dirty is Some) and (Clothes is Small) then (Wash-Time is Ten) (1)
6. If (Dirty is Some) and (Clothes is Medium) then (Wash-Time is Ten) (1)
7. If (Dirty is Some) and (Clothes is Much) then (Wash-Time is Twenty) (1)
8. If (Dirty is Some) and (Clothes is A-Lot) then (Wash-Time is Twenty) (1)
9. If (Dirty is Very) and (Clothes is Small) then (Wash-Time is Ten) (1)
10. If (Dirty is Very) and (Clothes is Medium) then (Wash-Time is Twenty) (1)
11. If (Dirty is Very) and (Clothes is Much) then (Wash-Time is Thirty) (1)
12. If (Dirty is Very) and (Clothes is A-Lot) then (Wash-Time is Thirty) (1)

The resulting output is:



Changing the rules or the membership functions will change this result.

Problem 10.3
Develop a fuzzy inference system for a barbecue. The inputs may be the thickness of the steak and how cooked or rare it is desired to be. The output may be the temperature of the flame and/or the time of cooking.

Estimated student time to complete:
Prerequisite knowledge required: Text Section(s) 10.3-10.9

Solution:

As in any other design problem, the solution is not unique. For example, different inputs and outputs may be selected, different fuzzy sets may be chosen, the ranges selected may be different, and alternate rules may be assigned. Therefore, not only the solution is not unique, the result will also differ. This suggested solution may only be used as a guide.

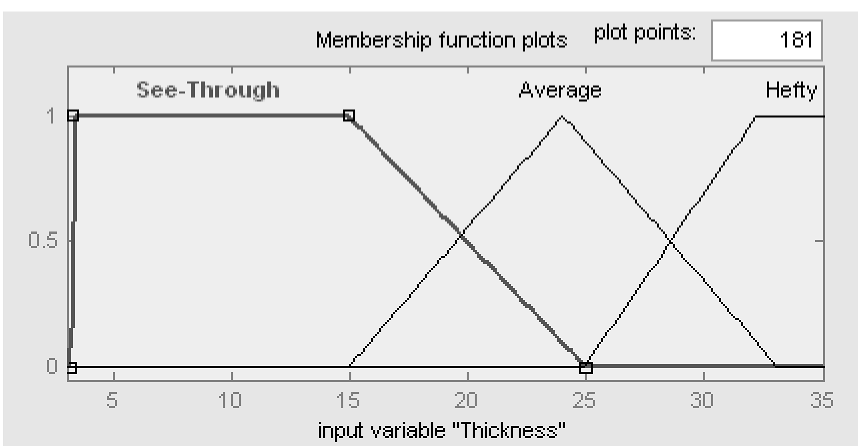
This solution is implemented on the MATLAB Fuzzy Toolbox. You may use this or any other available system.

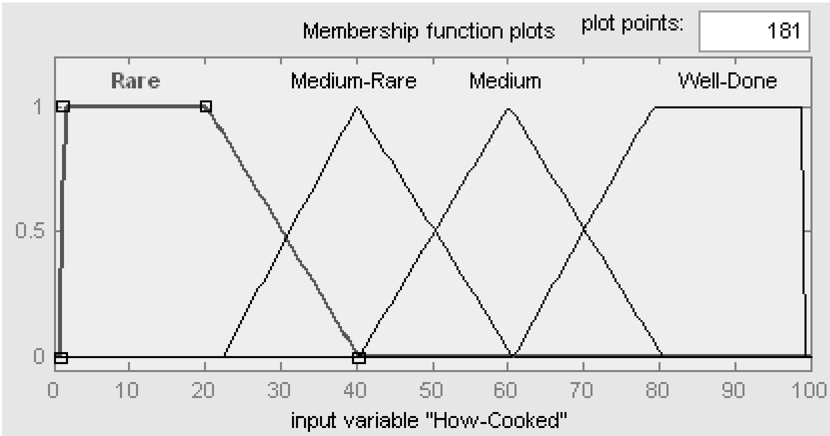
In this solution, the inputs, outputs, the sets, and the rules are as follows:

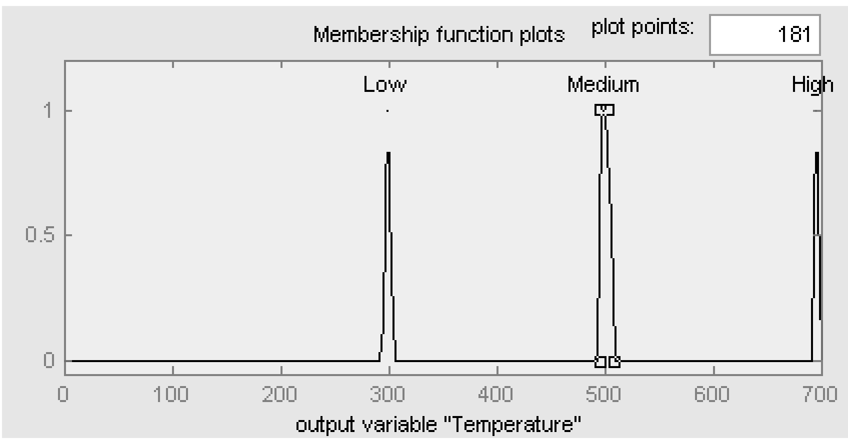
Inputs: Thickness; See-Through, Average, Hefty
How Cooked; Rare, Medium-Rare, Medium, Well-Done

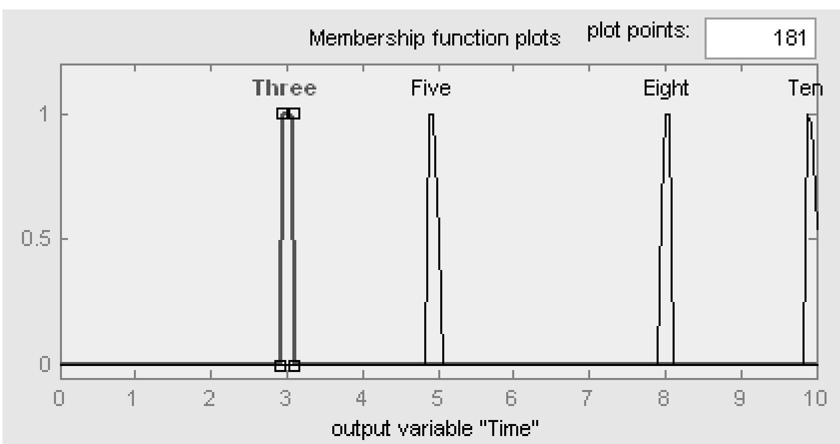
Outputs: Temp; Low, Medium, High
Time; Three, Five, Eight, Ten

The fuzzy sets are:





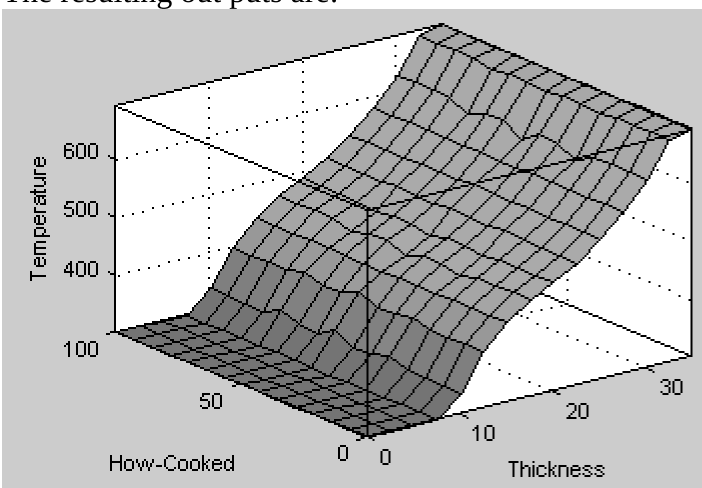




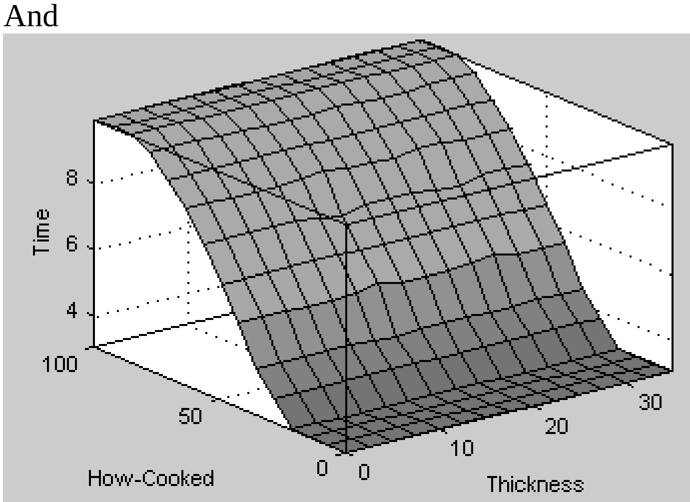
The rules are:

1. If (Thickness is See-Through) and (How-Cooked is Rare) then (Temperature is Low)(Time is Three) (1)
2. If (Thickness is See-Through) and (How-Cooked is Medium-Rare) then (Temperature is Low)(Time is Five) (1)
3. If (Thickness is See-Through) and (How-Cooked is Medium) then (Temperature is Low)(Time is Eight) (1)
4. If (Thickness is See-Through) and (How-Cooked is Well-Done) then (Temperature is Low)(Time is Ten) (1)
5. If (Thickness is Average) and (How-Cooked is Rare) then (Temperature is Medium)(Time is Three) (1)
6. If (Thickness is Average) and (How-Cooked is Medium-Rare) then (Temperature is Medium)(Time is Five) (1)
7. If (Thickness is Average) and (How-Cooked is Medium) then (Temperature is Medium)(Time is Eight) (1)
8. If (Thickness is Average) and (How-Cooked is Well-Done) then (Temperature is Medium)(Time is Ten) (1)
9. If (Thickness is Hefty) and (How-Cooked is Rare) then (Temperature is High)(Time is Three) (1)
10. If (Thickness is Hefty) and (How-Cooked is Medium-Rare) then (Temperature is High)(Time is Five) (1)
11. If (Thickness is Hefty) and (How-Cooked is Medium) then (Temperature is High)(Time is Eight) (1)
12. If (Thickness is Hefty) and (How-Cooked is Well-Done) then (Temperature is High)(Time is Ten) (1)

The resulting out puts are:



And



Changing the rules or the membership functions will change this result.

Problem 10.4
Develop a fuzzy inference system for an automatic gearbox. The inputs are the speed of the car and the load on the engine, and the output is the gear ratio of the transmission.

Estimated student time to complete:
Prerequisite knowledge required: Text Section(s) 10.3-10.9

Solution:

As in any other design problem, the solution is not unique. For example, different inputs and outputs may be selected, different fuzzy sets may be chosen, the ranges selected may be different, and alternate rules may be assigned. Therefore, not only the solution is not unique, the result will also differ. This suggested solution may only be used as a guide.

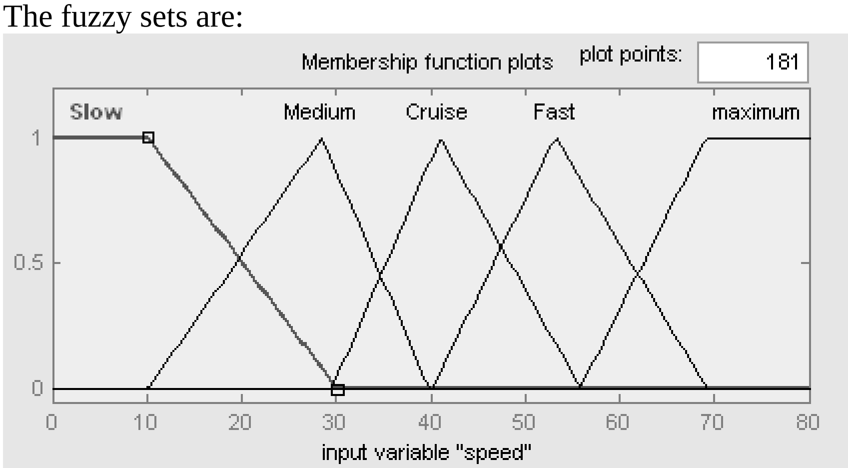
This solution is implemented on the MATLAB Fuzzy Toolbox. You may use this or any other available system.

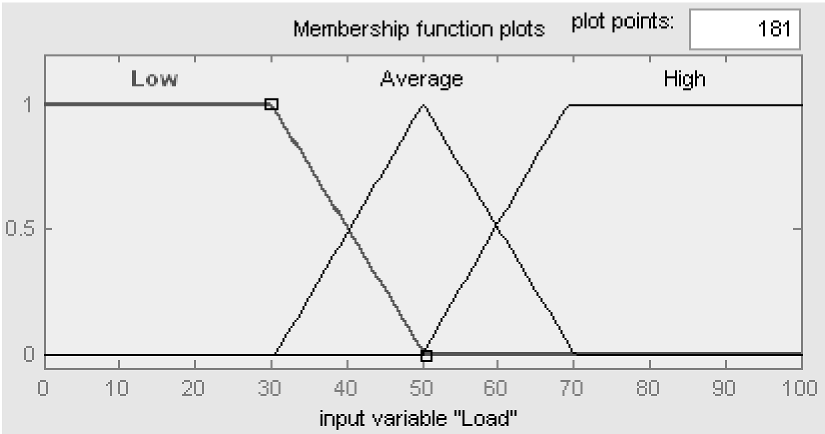
In this solution, the inputs, outputs, the sets, and the rules are as follows:

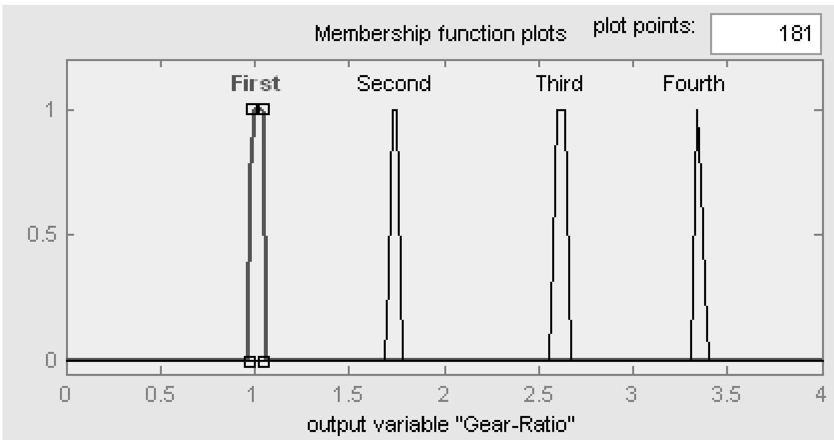
Inputs: Speed; Slow, Medium, Cruise, Fast, Maximum
Load; Low, Average, High

Outputs: Gear-Ratio; Fourth, Third, Second, First

The fuzzy sets are:



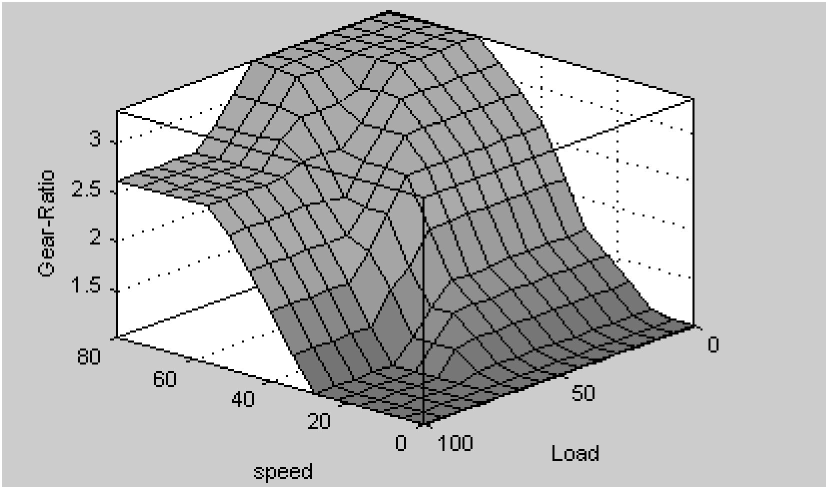




The rules are:

1. If (speed is Slow) and (Load is Low) then (Gear-Ratio is First) (1)
2. If (speed is Slow) and (Load is Average) then (Gear-Ratio is First) (1)
3. If (speed is Slow) and (Load is High) then (Gear-Ratio is First) (1)
4. If (speed is Medium) and (Load is Low) then (Gear-Ratio is Second) (1)
5. If (speed is Medium) and (Load is Average) then (Gear-Ratio is Second) (1)
6. If (speed is Medium) and (Load is High) then (Gear-Ratio is First) (1)
7. If (speed is Cruise) and (Load is Low) then (Gear-Ratio is Third) (1)
8. If (speed is Cruise) and (Load is Average) then (Gear-Ratio is Third) (1)
9. If (speed is Cruise) and (Load is High) then (Gear-Ratio is Second) (1)
10. If (speed is Fast) and (Load is Low) then (Gear-Ratio is Fourth) (1)
11. If (speed is Fast) and (Load is Average) then (Gear-Ratio is Third) (1)
12. If (speed is Fast) and (Load is High) then (Gear-Ratio is Third) (1)
13. If (speed is maximum) and (Load is Low) then (Gear-Ratio is Fourth) (1)
14. If (speed is maximum) and (Load is Average) then (Gear-Ratio is Fourth) (1)
15. If (speed is maximum) and (Load is High) then (Gear-Ratio is Third) (1)

The resulting out put is:



Changing the rules or the membership functions will change this result.

Problem 10.5
Develop a fuzzy logic system for a vision system in which the inputs are the intensities of the three colors of red, green, and blue (RGB) in a color image and the output is the relationship of the combination to the colors of the rainbow.

Estimated student time to complete:
Prerequisite knowledge required: Text Section(s) 10.3-10.9

Solution:

As in any other design problem, the solution is not unique. For example, different inputs and outputs may be selected, different fuzzy sets may be chosen, the ranges selected may be different, and alternate rules may be assigned. Therefore, not only the solution is not unique, the result will also differ. This suggested solution may only be used as a guide.

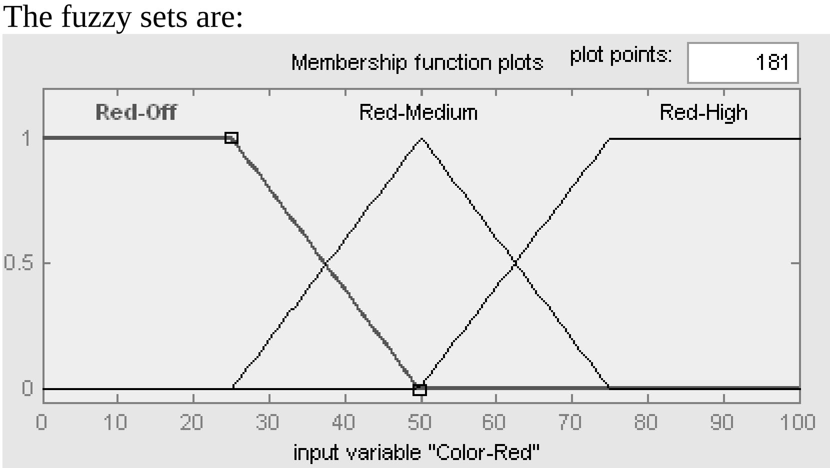
This solution is implemented on the MATLAB Fuzzy Toolbox. You may use this or any other available system.

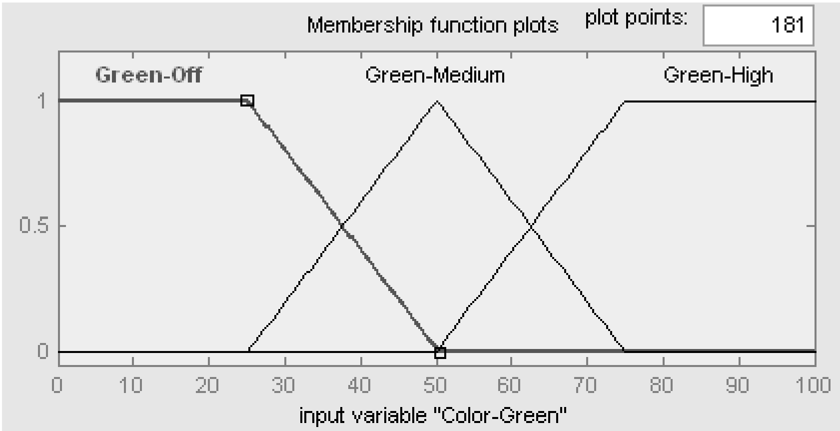
In this solution, the inputs, outputs, the sets, and the rules are as follows:

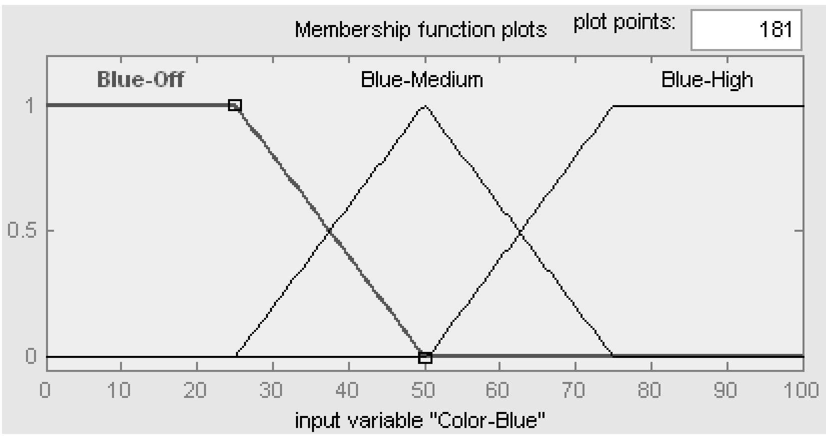
Inputs: Color-Red; Red-Off, Red-Medium, Red-High
Color-Green; Green-Off, Green-Medium, Green-High
Color-Blue; Blue-Off, Blue-Medium, Blue-High

Outputs: Color; Black, Red, Orange, Yellow, Green, Cyan, Blue, Violet, White

The fuzzy sets are:



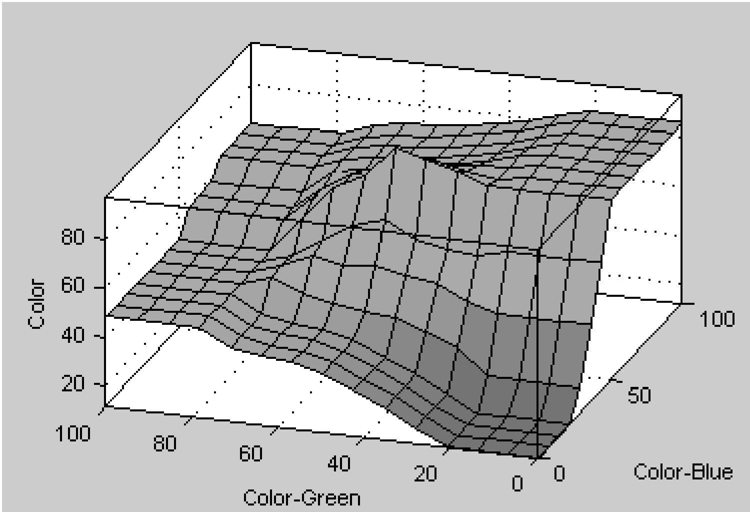


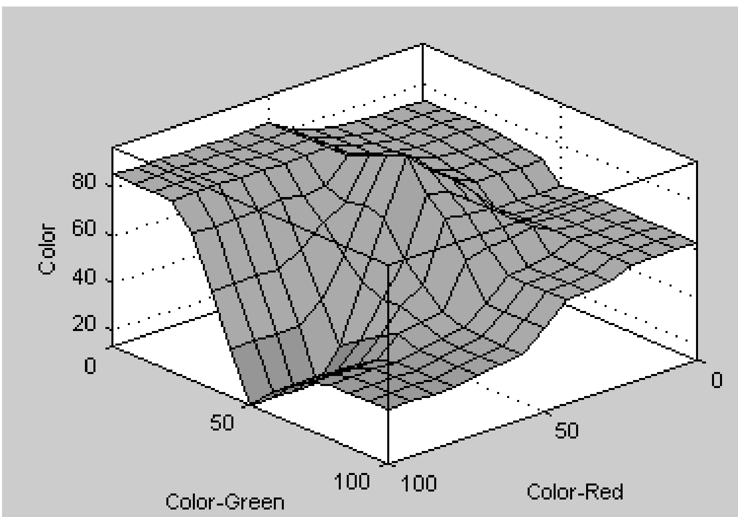


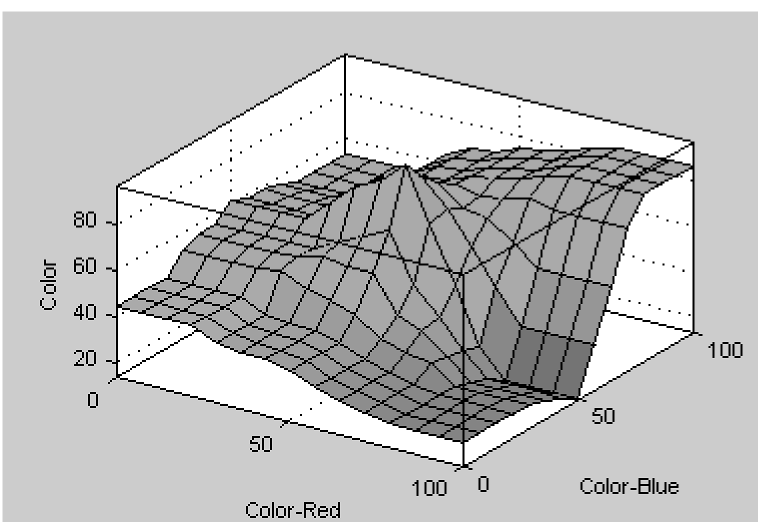
The rules are:

1. If (Color-Red is Red-Off) and (Color-Green is Green-Off) and (Color-Blue is Blue-Off) then (Color is Black) (1)
2. If (Color-Red is Red-Off) and (Color-Green is Green-Off) and (Color-Blue is Blue-Medium) then (Color is Blue) (1)
3. If (Color-Red is Red-Off) and (Color-Green is Green-Off) and (Color-Blue is Blue-High) then (Color is Blue) (1)
4. If (Color-Red is Red-Off) and (Color-Green is Green-Medium) and (Color-Blue is Blue-Off) then (Color is Green) (1)
5. If (Color-Red is Red-Off) and (Color-Green is Green-Medium) and (Color-Blue is Blue-Medium) then (Color is Cyan) (1)
6. If (Color-Red is Red-Off) and (Color-Green is Green-Medium) and (Color-Blue is Blue-High) then (Color is Green) (1)
7. If (Color-Red is Red-Off) and (Color-Green is Green-High) and (Color-Blue is Blue-Off) then (Color is Green) (1)
8. If (Color-Red is Red-Off) and (Color-Green is Green-High) and (Color-Blue is Blue-Medium) then (Color is Cyan) (1)
9. If (Color-Red is Red-Off) and (Color-Green is Green-High) and (Color-Blue is Blue-High) then (Color is Cyan) (1)
10. If (Color-Red is Red-Medium) and (Color-Green is Green-Off) and (Color-Blue is Blue-Off) then (Color is Red) (1)
11. If (Color-Red is Red-Medium) and (Color-Green is Green-Off) and (Color-Blue is Blue-Medium) then (Color is Violet) (1)
12. If (Color-Red is Red-Medium) and (Color-Green is Green-Off) and (Color-Blue is Blue-High) then (Color is Violet) (1)
13. If (Color-Red is Red-Medium) and (Color-Green is Green-Medium) and (Color-Blue is Blue-Off) then (Color is Yellow) (1)
14. If (Color-Red is Red-Medium) and (Color-Green is Green-Medium) and (Color-Blue is Blue-Medium) then (Color is White) (1)
15. If (Color-Red is Red-Medium) and (Color-Green is Green-Medium) and (Color-Blue is Blue-High) then (Color is Blue) (1)
16. If (Color-Red is Red-Medium) and (Color-Green is Green-High) and (Color-Blue is Blue-Off) then (Color is Green) (1)
17. If (Color-Red is Red-Medium) and (Color-Green is Green-High) and (Color-Blue is Blue-Medium) then (Color is Green) (1)
18. If (Color-Red is Red-Medium) and (Color-Green is Green-High) and (Color-Blue is Blue-High) then (Color is Cyan) (1)
19. If (Color-Red is Red-High) and (Color-Green is Green-Off) and (Color-Blue is Blue-Off) then (Color is Red) (1)
20. If (Color-Red is Red-High) and (Color-Green is Green-Off) and (Color-Blue is Blue-Medium) then (Color is Violet) (1)
21. If (Color-Red is Red-High) and (Color-Green is Green-Off) and (Color-Blue is Blue-High) then (Color is Violet) (1)
22. If (Color-Red is Red-High) and (Color-Green is Green-Medium) and (Color-Blue is Blue-Off) then (Color is Orange) (1)
23. If (Color-Red is Red-High) and (Color-Green is Green-Medium) and (Color-Blue is Blue-Medium) then (Color is Red) (1)
24. If (Color-Red is Red-High) and (Color-Green is Green-Medium) and (Color-Blue is Blue-High) then (Color is Violet) (1)
25. If (Color-Red is Red-High) and (Color-Green is Green-High) and (Color-Blue is Blue-Off) then (Color is Yellow) (1)
26. If (Color-Red is Red-High) and (Color-Green is Green-High) and (Color-Blue is Blue-Medium) then (Color is Yellow) (1)
27. If (Color-Red is Red-High) and (Color-Green is Green-High) and (Color-Blue is Blue-High) then (Color is White) (1)

The resulting out put is:







Changing the rules or the membership functions will change this result.

Problem 10.6
Develop a fuzzy inference system for grading a robotics course. The inputs are your effort level in the course and your exam grade, and the output is your letter grade.

Estimated student time to complete:
Prerequisite knowledge required: Text Section(s) 10.3-10.9

Solution:

As in any other design problem, the solution is not unique. For example, different inputs and outputs may be selected, different fuzzy sets may be chosen, the ranges selected may be different, and alternate rules may be assigned. Therefore, not only the solution is not unique, the result will also differ. This suggested solution may only be used as a guide.

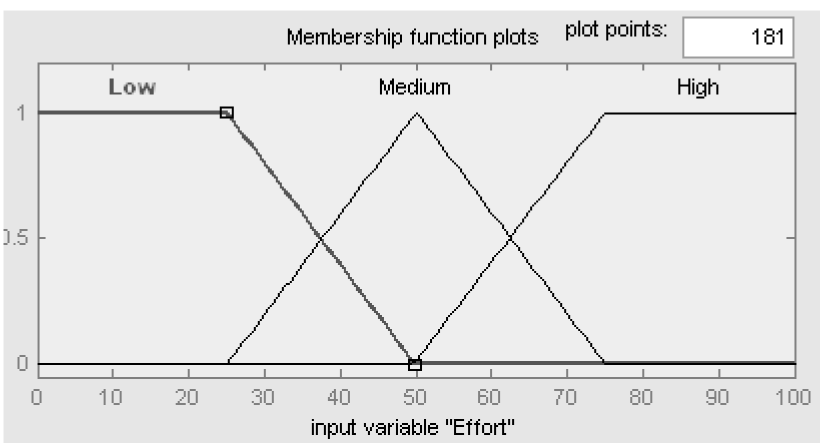
This solution is implemented on the MATLAB Fuzzy Toolbox. You may use this or any other available system.

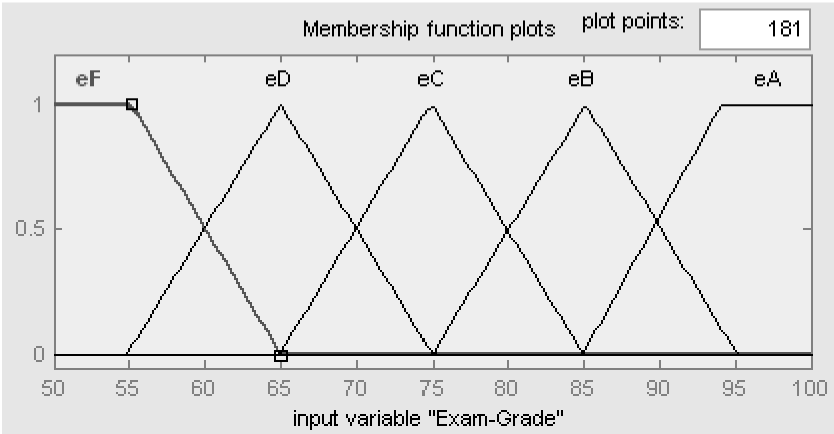
In this solution, the inputs, outputs, the sets, and the rules are as follows:

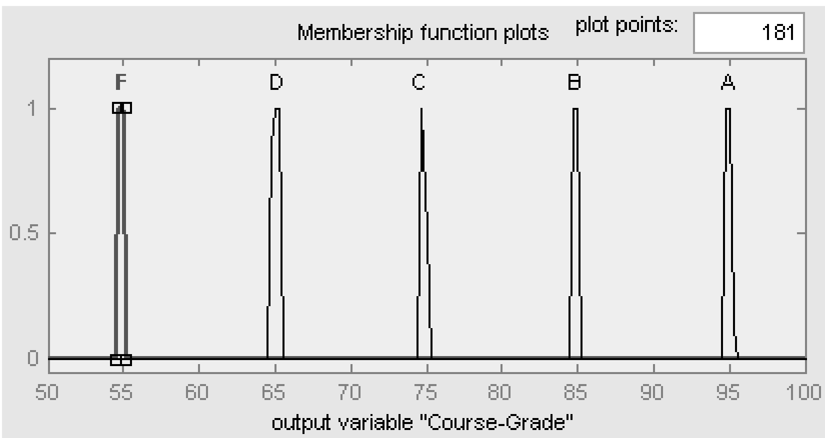
Inputs: Effort; Low, Medium, High
Exam-Grade; eA, eB, eC, eD, eF

Outputs: Course-Grade; A, B, C, D, F

The fuzzy sets are:



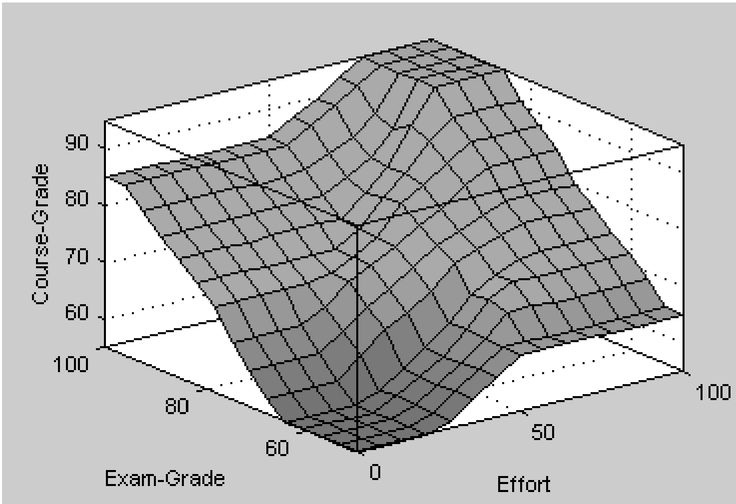




The rules are:

1. If (Effort is Low) and (Exam-Grade is eF) then (Course-Grade is F) (1)
2. If (Effort is Low) and (Exam-Grade is eD) then (Course-Grade is F) (1)
3. If (Effort is Low) and (Exam-Grade is eC) then (Course-Grade is D) (1)
4. If (Effort is Low) and (Exam-Grade is eB) then (Course-Grade is C) (1)
5. If (Effort is Low) and (Exam-Grade is eA) then (Course-Grade is B) (1)
6. If (Effort is Medium) and (Exam-Grade is eF) then (Course-Grade is D) (1)
7. If (Effort is Medium) and (Exam-Grade is eD) then (Course-Grade is D) (1)
8. If (Effort is Medium) and (Exam-Grade is eC) then (Course-Grade is C) (1)
9. If (Effort is Medium) and (Exam-Grade is eB) then (Course-Grade is C) (1)
10. If (Effort is Medium) and (Exam-Grade is eA) then (Course-Grade is B) (1)
11. If (Effort is High) and (Exam-Grade is eF) then (Course-Grade is D) (1)
12. If (Effort is High) and (Exam-Grade is eD) then (Course-Grade is C) (1)
13. If (Effort is High) and (Exam-Grade is eC) then (Course-Grade is B) (1)
14. If (Effort is High) and (Exam-Grade is eB) then (Course-Grade is A) (1)
15. If (Effort is High) and (Exam-Grade is eA) then (Course-Grade is A) (1)

The resulting out put is:



Changing the rules or the membership functions will change this result.

APPENDICES

Problem A.1
Show that the determinant of a matrix can be calculated by picking any row or column.

Estimated student time to complete: 10-15 minutes
Prerequisite knowledge required: Text Section(s) A.1

Solution:

For an arbitrary matrix:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\begin{aligned}
\det A &= a(ei - fh) - b(di - fg) + c(dh - eg) \\
&= aei - afh - bdi + bfg + cdh - ceg \\
\det A &= -d(bi - ch) + e(ai - cg) - f(ah - bg) \\
&= -dbi + dch + eai - ecg - fah + fbg \\
\det A &= -b(di - fg) + e(ai - cg) - h(af - cd) \\
&= -bdi + bfg + eai - ecg - haf + hcd
\end{aligned}$$

They are all the same.

Problem A.2
Calculate the determinant of the following (4 × 4) matrix.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Estimated student time to complete: 5 minutes
Prerequisite knowledge required: Text Section(s) A.1.

Solution:

$$\det A = 1\big(1(1-0)-0+0\big)-1\big(0-2(3-1)-0\big)+0-0 = 1+4 = 5$$

Problem A.3
Calculate the inverse of the following matrix using method 1:

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

Estimated student time to complete: 5-10 minutes
Prerequisite knowledge required: Text Section(s) A.1.

Solution:

$$\det B = 1(3-0) - 0 + 2(-2) = 3 - 4 = -1$$

$$B^T = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

$$adjB = \begin{bmatrix} 3 & -3 & -2 \\ 0 & -1 & 0 \\ -2 & 2 & 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} -3 & 3 & 2 \\ 0 & 1 & 0 \\ 2 & -2 & -1 \end{bmatrix}$$

Problem A.4
Calculate the inverse of the following matrix using method 2:

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

Estimated student time to complete: 10-15 minutes
Prerequisite knowledge required: Text Section(s) A.1.

Solution:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{cases} x_{11} + x_{31} = 1 \\ 2x_{21} + x_{31} = 0 \\ 3x_{11} + x_{21} = 0 \end{cases} \rightarrow \begin{cases} x_{11} = 1/7 \\ x_{31} = 6/7 \\ x_{21} = -3/7 \end{cases}$$

$$\begin{cases} x_{12} + x_{32} = 0 \\ 2x_{22} + x_{32} = 1 \\ 3x_{12} + x_{22} = 0 \end{cases} \rightarrow \begin{cases} x_{12} = -1/7 \\ x_{22} = 3/7 \\ x_{32} = 1/7 \end{cases}$$

$$\begin{cases} x_{31} + x_{33} = 0 \\ 2x_{23} + x_{33} = 0 \\ 3x_{13} + x_{23} = 1 \end{cases} \rightarrow \begin{cases} x_{13} = 2/7 \\ x_{23} = 1/7 \\ x_{33} = -2/7 \end{cases}$$

$$C^{-1} = \begin{bmatrix} \frac{1}{7} & -\frac{1}{7} & \frac{2}{7} \\ \frac{-3}{7} & \frac{3}{7} & \frac{1}{7} \\ \frac{6}{7} & \frac{1}{7} & \frac{-2}{7} \end{bmatrix}$$