CS 70 Discrete Mathematics and Probability Theory
Spring 2018 Nithin Raghavan

HW 1

Sundry

I did not work with anyone else in a group for this assignment. I worked alone.

I certify that all solutions are entirely in my words and that I have not looked at another students solutions. I have credited all external sources in this write up.
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1 Propositional Practice

Convert the following English sentences into propositional logic and the following propositions into English. State whether or not each statement is true with brief justification.

(a) There is a real number which is not rational.

$$\exists x \in \mathbb{R}, x \notin \mathbb{Q}$$

This is true. Proof by example: the existence of $\sqrt{2}$ (proved in lecture)

(b) All integers are natural numbers or are negative, but not both.

$$\forall x \in \mathbb{Z}, (x \in \mathbb{N} \vee x < 0) \wedge \neg(x \in \mathbb{N} \wedge x < 0)$$

This is true. The definition of 'natural number' is an integer which is nonnegative, and thus an integer which is natural cannot be negative.

(c) If a natural number is divisible by 6, it is divisible by 2 or it is divisible by 3.

$$\forall x \in \mathbb{N}, 6 \mid n \implies 3 \mid n \vee 2 \mid n$$

This is true. The fundamental theorem of arithmetic implies a number divisible by 6 is divisible by both its prime factors 2 and 3.

(d) $(\forall x \in \mathbb{R})\ (x \in \mathbb{C})$
This states 'Every real number is a complex number'. This is true. Let $x \in \mathbb{R}\ \therefore x = x + 0i \in \mathbb{C}$.

(e) $(\forall x \in \mathbb{Z})\ ((2 \mid x \vee 3 \mid x) \implies 6 \mid x)$
This states 'If an integer is divisible by 2 or 3, it is divisible by 6.' This is not true.
Proof by counterexample: $2 \mid 4, 6 \nmid 4$

(f) $(\forall x \in \mathbb{N})\ ((x > 7) \implies ((\exists a, b \in \mathbb{N})\ (a + b = x)))$
This states 'If a natural number x is greater than seven, there exist two natural numbers such that their sum equals x. This is true.
Proof by example: $x = 10, a = 1, b = 9, a + b = 1 + 9 = 10 = x$.

2 Miscellaneous Logic

(a) Let the statement, $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}\ G(x, y)$, be true for predicate $G(x, y)$.

For each of the following statements, decide if the statement is certainly true, certainly false, or possibly true.

(i) $G(3, 4)$
Possibly true. $G(x, y) := x > y$ is False but $G(x, y) := x < y$ is True.

(ii) $\forall x \in \mathbb{R}\ G(x, 3)$
Possibly true. $G(x, y) := x + y = 5$ is False but $G(x, y) := x - y \in \mathbb{R}$ is True.

(iii) $\exists y\ G(3, y)$
Certainly true. $\forall x \exists y \in \mathbb{R}\ G(x, y) \implies G(3, y)$.

(iv) $\forall y\ \neg G(3, y)$
Certainly false. $\neg \exists y G(3, y) \equiv \forall y \neg G(3, y)$.

(v) $\exists x\ G(x, 4)$
Possibly true. $G(x, y) := x = y$ is True but $G(x, y) := y = 5$ is False.

(b) Give an expression using terms involving $\lor, \land$ and $\neg$ which is true if and only if exactly one of X, Y, and Z are true. (Just to remind you: $(X \land Y \land Z)$ means all three of X,Y,Z are true, $(X \lor Y \lor Z)$ means at least one of X,Y and Z is true.)
$(((X \lor Y) \land \neg(X \land Y)) \lor Z) \land \neg(((X \lor Y) \land \neg(X \land Y)) \land Z) \land (\neg(X \land Y) \land \neg(X \land Z) \land \neg(Y \land Z))$

3 Prove or Disprove

(a) $\forall n \in \mathbb{N}$, if n is odd then $n^2 + 2n$ is odd.
$k \in \mathbb{N} \implies n = 2k + 1$
$\therefore (k+1)^2 + 2(k+1) = 4k^2 + 8k + 3$.
All terms except the last are even; even + even = even, even + odd = odd. Thus, $n^2 + 2n$ is odd if n is odd. QED

(b) $\forall x, y \in \mathbb{R}$, $\min(x, y) = (x + y - |x - y|)/2$.
If for $a, b \in \mathbb{R}$, $|a| = a$ for a positive and $-a$ for a negative, then $|a - b|$ indicates the positive separation between a and b.
$x > y \implies |x-y| = x-y, \therefore (x+y-x+y)/2 = 2y/2 = y$, the smaller number.
$x < y \implies |x-y| = y-x, \therefore (x+y-y+x)/2 = 2x/2 = x$, the smaller number. QED

(c) $\forall a, b \in \mathbb{R}$ if $a + b \leq 10$ then $a \leq 7$ or $b \leq 3$.
Contrapositive: $a > 7 \land b > 3 \implies a + b > 10$. Adding these two equations together gives $a + b > 7 + 3 = 10$. QED

(d) $\forall r \in \mathbb{R}$, if r is irrational then $r + 1$ is irrational.
Proof by contradiction: If $r + 1 \in \mathbb{Q}$, then take $r + 1 = p/q$, with $p, q \in \mathbb{Z}$ reduced. $r = (r + 1) - 1 = \frac{p}{q} - 1 = \frac{p-q}{q}$. since $p, q \in \mathbb{Z}$, $k = p - q \in \mathbb{Z}$, and $r = \frac{k}{q} \in \mathbb{Q}$. This contradicts original $r \notin \mathbb{Q}$, so $r + 1$ must be irrational. QED

(e) $\forall n \in \mathbb{N}$, $10n^2 > n!$.
False by counterexample ($n = 10$). QED

4 Preserving Set Operations

Define the image of a set X to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. Define the inverse image of a set Y to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which A and B are sets. By doing so, you will show that inverse images preserve set operations, but images typically do not.

Hint: For sets X and Y, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $\forall x,\ x \in X \implies x \in Y$.

1. $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.
   $x \in f^{-1}(A \cup B) \implies f(x) \in A \cup B \implies f(x) \in A \vee f(x) \in B$
   $\implies x \in f^{-1}(A) \vee x \in f^{-1}(B) \implies x \in f^{-1}(A) \cup f^{-1}(B)$, by definition of the union of two sets.

2. $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.
   $x \in f^{-1}(A \cap B) \implies f(x) \in A \cap B \implies f(x) \in A \wedge f(x) \in B$
   $\implies x \in f^{-1}(A) \wedge x \in f^{-1}(B) \implies x \in f^{-1}(A) \cap f^{-1}(B)$, by definition of the intersection of two sets.

3. $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.
   $x \in f^{-1}(A \setminus B) \implies f(x) \in A \setminus B \implies f(x) \in A \wedge f(x) \notin B$
   $\implies x \in f^{-1}(A) \wedge x \notin f^{-1}(B) \implies x \in f^{-1}(A) \setminus f^{-1}(B)$, by definition of the disjunction of two sets.

4. $f(A \cup B) = f(A) \cup f(B)$.
   $x \in f(A \cup B) \implies f^{-1}(x) \in A \cup B \implies f^{-1}(x) \in A \vee f^{-1}(x) \in B$
   $\implies x \in f(A) \vee x \in f(B) \implies x \in f(A) \cup f(B)$, by definition of the union of two sets.

5. $f(A \cap B) \subseteq f(A) \cap f(B)$, and give an example where equality does not hold.
   $x \in f(A \cap B) \implies f^{-1}(x) \in A \cap B \implies f^{-1}(x) \in A \wedge f^{-1}(x) \in B$
   $\implies x \in f(A) \wedge x \in f(B) \implies x \in f(A) \cap f(B)$, however the converse is not necessarily true. For example, consider $f(x) = x^2$, $A = \{-3, -4\}$, $B = \{3, 4\}$, $x = 9$. x is not within $f(A \cap B) = \{\}$ but it is within $f(A) \cap f(B) = \{9, 16\}$.

6. $f(A \setminus B) \supseteq f(A) \setminus f(B)$, and give an example where equality does not hold.
   $x \in f(A) \setminus f(B) \implies x \in f(A) \wedge x \notin f(B)$
   $\implies f^{-1}(x) \in A \wedge f^{-1}(x) \notin B \implies f^{-1}(x) \in A \setminus B \implies x \in f(A \setminus B)$,

however the converse is not necessarily true. For example, consider $f(x) = x^2$, $A = \{0, -1\}$, $B = \{1, 0\}$, $x = 1$. x is within $f(A \setminus B) = \{1\}$ but it is not within $f(A) \setminus f(B) = \{\}$.

5 Hit or Miss?

State which of the proofs below is correct or incorrect. For the incorrect ones, please explain clearly where the logical error in the proof lies. Simply saying that the claim or the induction hypothesis is false is *not* a valid explanation of what is wrong with the proof. You do not need to elaborate if you think the proof is correct.

(a) **Claim:** For all positive numbers $n \in \mathbb{R}$, $n^2 \geq n$.

Proof. The proof will be by induction on n.
Base Case: $1^2 \geq 1$. It is true for $n = 1$.
Inductive Hypothesis: Assume that $n^2 \geq n$.
Inductive Step: We must prove that $(n+1)^2 \geq n+1$. Starting from the left hand side,

$$\begin{aligned}(n+1)^2 &= n^2 + 2n + 1 \\ &\geq n + 1.\end{aligned}$$

Therefore, the statement is true. □

This proof is false. Induction cannot be used to prove this statement, because it implies that there is a smallest possible number against which a base case could be determined; in the reals, there is no such number. Adding by one also neglects all the cases $\mathbb{R} \setminus \mathbb{Z}$.

(b) **Claim:** For all negative integers n, $-1 - 3 - \cdots + (2n+1) = -n^2$.

Proof. The proof will be by induction on n.
Base Case: $-1 = -(-1)^2$. It is true for $n = -1$.
Inductive Hypothesis: Assume that $-1 - 3 - \cdots + (2n+1) = -n^2$.
Inductive Step: We need to prove that the statement is also true for

$n-1$ if it is true for n, that is, $-1-3-\cdots+(2(n-1)+1) = -(n-1)^2$. Starting from the left hand side,

$$\begin{aligned} -1-3-\cdots+(2(n-1)+1) &= (-1-3-\cdots+(2n+1)) + (2(n-1)+1) \\ &= -n^2 + (2(n-1)+1) \quad \text{(Inductive Hypothesis)} \\ &= -n^2 + 2n - 1 \\ &= -(n-1)^2. \end{aligned}$$

Therefore, the statement is true. □

This proof is correct.

(c) **Claim:** For all nonnegative integers n, $2n = 0$.

Proof. We will prove by strong induction on n.
Base Case: $2 \times 0 = 0$. It is true for $n = 0$.
Inductive Hypothesis: Assume that $2k = 0$ for all $0 \leq k \leq n$.
Inductive Step: We must show that $2(n+1) = 0$. Write $n+1 = a+b$ where $0 < a, b \leq n$. From the inductive hypothesis, we know $2a = 0$ and $2b = 0$, therefore,

$$2(n+1) = 2(a+b) = 2a + 2b = 0 + 0 = 0.$$

The statement is true. □

This proof is false. This is because the inductive hypothesis is wrong, and that $2k = 0$ cannot be assumed for all $0 \leq x \leq n$ if $n > 0$ (e.g. it cannot be assumed out of the base case). For example, if $n = 1$, $2 * n = 2 \neq 0$.

6 Badminton Ranking

A team of n ($n \geq 2$) badminton players held a tournament, where every person plays with every other person exactly once, and there are no ties. Prove by induction that after the tournament, we can arrange the n players in a sequence, so that every player in the sequence has won against the person immediately to the right of him.

Proof. $n \geq 2$ players $(P_1, P_2, ..., P_n)$ can be arranged in a sequence so every player in the sequence has won against the person immediately to the right of him.
Base Case: $n = 2$. If P_1 wins against P_2, the sequence is P_1P_2. Else, if P_2 wins against P_1, the sequence is P_2P_1. Valid.
Inductive Step: Assume for $n = \beta + 1$ that all n up to $n = \beta$ is valid. Then, arrange the first β players in a valid, correct sequence (labelled $P_aP_b...P_\beta$, to denote that it could be any order of $P_1, P_2, ..., P_n$). Consider $P_{\beta+1}$. If $P_{\beta+1}$ won against P_a, then place him at the beginning. If $P_{\beta+1}$ won against P_b, then place $P_{\beta+1}$ in between P_a and P_b. Continue to iterate; if $P_{\beta+1}$ won over P_β, then place him at the penultimate position, else place him at the end. Thus at any step, a valid sequence consisting from P_1 to $P_{\beta+1} = P_n$ can be made. $\square$