Math 126 Pre-Final Discussion Problems

1 Pre-Midterm

1. Consider the equation $u_x + 2xu_y = 0$, as discussed in class.
   a) Find all solutions given that $u(0, y) = y^3$.
   b) Find all solutions given that $u(x, 0) = x$?
   c) Find all solutions given that $u(0, 0) = 1$.

2. Consider the Dirichlet problem on $0 \le x \le \pi, t \ge 0$ for the heat equation $u_t = u_{xx}$ with $u(x, 0) = \phi(x), u(0, t) = u(\pi, t)$. Recall from Math 54 that the solution to this problem is of the form

$$u(x,t) = \sum_{n=0}^{\infty} c_n \sin(nx) e^{-n^2 t},$$

where c_n are chosen appropriately so that $u(x, 0) = \phi(x)$.

Compute the energy of such solutions and check that it decreases over time. Then, choose some VERY simple cases (like perhaps $e^{-t} \sin x$) in which you can check that the maximum principle and stability hold.

3. (Exercise 3.4.2) Solve $u_{tt} = c^2 u_{xx} + e^{ax}$ with $u(x, 0) = 0$ and $u_t(x, 0) = 0$.

4. Suppose that

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l}$$

and

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}.$$

Use Euler's formula to find a formula for c_n in terms of A_n and B_n and a formula for A_n and B_n in terms of c_n.

5. $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ is a geometric series. In what senses does in converge in the interval $-1 < x < 1$? Check all three types.

6. Let $f(x)$ be a twice differentiable function on the interval $-\pi < x < \pi$ where $f''(x)$ is piecewise continuous.
   a) If A_n and B_n are the Fourier coefficients of $f(x)$, find a formula for them in terms of A_n'' and B_n'', the Fourier coefficients of $f''(x)$.
   b) Use part a) to show that $|A_n|, |B_n| \le M/n^2$ for some $M > 0$.
   c) Use part b) to show that the Fourier series of $f(x)$ converges uniformly to it. (Note: our hypothesis here is a little stronger than the one in the book, but this makes the proof simpler and more to the point.)

2 Post-Midterm

7. Prove the Minimum Principle for Laplace's equation.

8. Solve the Dirichlet problem for the Laplacian on the circle with boundary conditions $u(a, \theta) = 2 - 2\sin 2\theta + \cos\theta$.

9. Find and prove Green's identities for functions of two variables. (Hint: Sum derivatives as before, but you may need to use Green's theorem instead of the divergence theorem, since that's the one that applies in

two dimensions.)

10. Verify that both the Green's functions we talked about in class today are in fact symmetric by swapping $\mathbf{x}$ and $\mathbf{x_0}$ in their formulas.

11. Consider the wave equation $u_{tt} = u_{xx}$ on the interval $0 \le x \le 4$ with boundary conditions $u(0,t) = 0 = u_x(4,t)$. Take $\Delta x = \Delta t = 1$ (is this stable?) and solve the wave equation forward in time when
a) $\phi = (0,0,2,0,0)$ and $\psi = 0$.
b) $\phi = 0$ and $\psi = (0,0,2,0,0)$.

12. "Descend" from two dimensions to one as follows. Let $u_{tt} = c^2 u_{xx}$ with initial data $\phi(x) = 0$ and general $\psi(x)$. Think of $u(x,t)$ as a solution of the two-dimensional equation that happens not to depend on y. Plug it into the formula for solutions to the wave equation in 2 dimensions and carry out the integration.

13. a) Prove that $\delta(a^2 - r^2) = \delta(a-r)/2a$ for $a > 0, r > 0$.
b) Deduce that the three-dimensional Riemann function for the wave equation for $t > 0$ is

$$S(\mathbf{x}, t) = \frac{1}{2\pi c}\delta(c^2t^2 - |\mathbf{x}|^2).$$

14. Use the technique of Fourier transforms to solve the PDE $u_{tt} = c^2 u_{xx}$ with initial conditions $u(x,0) = 0$ and $u_t(x,0) = \psi(x)$, where $\psi(x)$ is arbitrary. (Hint: we did something similar for the heat equation in class.)