7/31/18 Lecture Notes: Back to Green's Functions

Last time: Associate function $f(x)$ with $\phi \mapsto \int f(x)\phi(x)\,dx$, a continuous, linear rule.

- Any such rule is a distribution
- The derivative of a distribution always exists and is a distribution

E.g. $\delta(x) \leftrightarrow \phi \mapsto \phi(0)$ $\left(= \int \delta(x)\phi(x)\,dx\right)$

<u>Green's Functions</u> (in 3D)

Claim: $\Delta v = \delta(x)$, where $v = \frac{-1}{4\pi r}$

<u>Pf</u> Green's 2nd Identity:

$$\iiint_D (u\,\Delta v - v\,\Delta u)\,d\vec{x} = \iint_{\partial D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS$$

$u(0)$ 0 if Δu rep formula   In representation formula proof

Even if $\Delta u \neq 0$, $\iiint_D u\,\Delta v\,d\vec{x} = u(0)$, so $\Delta v = \delta(x)$

Claim: $\Delta G = \delta(\vec{x}-\vec{x}_0) = \delta_{\vec{x}_0}$ in D solution is Green's function
$G = 0$ on ∂D

Check: ii) obvious $G = 0$ on ∂D
i) obvious $\Delta G = 0$ away from $\vec{x}_0$
iii) $\Delta(G - v) = \Delta G - \Delta v = 0$ (If v centered at $\vec{x}_0$)

Alternatively, plug u, G into Green's 2nd with $\Delta u = 0$ in D
$u = h$ on ∂D

$$\iiint_D u\,\Delta G - \cancel{G\,\Delta u}^{\,0}\,d\vec{x} = \iint_{\partial D} u\frac{\partial G}{\partial n} - \cancel{G}^{\,0}\frac{\partial u}{\partial n}\,dS$$

$$u(\vec{x}_0) = \iint_{\partial D} h\frac{\partial G}{\partial n}\,dS$$

Heat Equation

Claim: If $S(\vec{x},t) = \frac{1}{(4\pi kt)^{3/2}} e^{-|\vec{x}|^2/4kt}$, then

a) $S_t = k\Delta S$ in $-\infty < x,y,z < \infty$, $t>0$

b) $S(\vec{x},0) = \delta(\vec{x})$ at $t=0$

Pf a) Already done

b) We showed as $t \to 0$,

$$\iiint S(\vec{x}-\vec{x}_0,t)\phi(\vec{x}_0)\,d\vec{x}_0 \to \phi(\vec{x}) = \iiint \delta(\vec{x}-\vec{x}_0)\phi(\vec{x}_0)\,d\vec{x}_0$$

Wave Equation

Let $S(\vec{x},t)$ be solution to

$$S_{tt} = c^2 \Delta S$$
$$S(\vec{x},0) = 0$$
$$S_t(\vec{x},0) = \delta(x)$$

Why $\delta(\vec{x})$ for ψ, not ϕ?

Claim: Solution to $u_{tt} = c^2 \Delta u$, $u(\vec{x},0) = \phi(\vec{x})$, $u_t(\vec{x},0) = \psi(\vec{x})$ is $u(\vec{x},t) = \int S(\vec{x}-\vec{y},t)\psi(\vec{y})\,d\vec{y} + \frac{\partial}{\partial t}\int S(\vec{x}-\vec{y},t)\phi(\vec{y})\,d\vec{y}$ look familiar?

Pf $u_{tt} = c^2\Delta u$ since u made from solutions

$$u(\vec{x},0) = \int \underbrace{S(\vec{x}-\vec{y},0)}_{0}\psi(\vec{y})\,d\vec{y} + \int \underbrace{S_t(\vec{x}-\vec{y},t)}_{\delta(\vec{x}-\vec{y})}\phi(\vec{y})\,d\vec{y} = \phi(\vec{x})$$

$$u_t(\vec{x},0) = \int \underbrace{S_t(\vec{x}-\vec{y},0)}_{\delta(\vec{x}-\vec{y})}\psi(\vec{y})\,d\vec{y} + \int \underbrace{S_{tt}(\vec{x}-\vec{y},0)}_{\Delta S(\vec{x}-\vec{y},0) = 0}\phi(\vec{y})\,d\vec{y} = \psi(\vec{x})$$

What is S?

1D: $u(x,t) = \frac{1}{2}[\underbrace{\phi(x+ct)}_{0} + \underbrace{\phi(x-ct)}_{0}] + \frac{1}{2c}\int_{x-ct}^{x+ct} \underbrace{\psi(s)}_{\delta(s)}\,ds$

$$S(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct}\delta(s)\,ds = \begin{cases} \frac{1}{2c} & \text{if } -ct < x < ct \\ 0 & \text{else} \end{cases}$$

sgn pronounced "signum"

$$= \frac{1}{2c} H(c^2t^2 - x^2)\,\mathrm{sgn}(t), \quad \mathrm{sgn}(t) = \begin{cases} 1 & t>0 \\ -1 & t<0 \end{cases}$$

3D: $u(\vec{x},t_0) = \frac{1}{4\pi c^2 t_0} \iint_S \cancel{\psi(\vec{x})}\, dS + \frac{\partial}{\partial t_0}\left[\frac{1}{4\pi c^2 t_0} \iint_S \cancel{\phi(\vec{x})}\, dS\right]$ Sorry for multiple uses of S!

$\delta(\vec{x})$ $\quad 0$

$$S(\vec{x},t) = \frac{1}{4\pi c^2 t}\delta(ct-|\vec{x}|)\,\mathrm{sgn}(t)$$ (almost matches 1D result)

Q: Is $\delta(ct-|\vec{x}|) = \delta(c^2t^2-|\vec{x}|^2)$? Both δ's supported where $|\vec{x}|=ct$.

Example: $\delta(2x)$: $\int \delta(2x)\phi(x)\,dx = \int \delta(y)\,\phi\left(\frac{y}{2}\right)\frac{dy}{2} = \frac{1}{2}\phi(0)$

$y=2x$
$dy = 2dx$

By change of coordinates,

$$S(\vec{x},t) = \frac{1}{2\pi c}\delta(c^2t^2-|\vec{x}|^2)$$ Notice a pattern now?

2D: $S(\vec{x},t) = \begin{cases} \frac{1}{2\pi c}(c^2t^2-|\vec{x}|^2)^{-1/2} & |\vec{x}|<ct \\ 0 & |\vec{x}|>ct \end{cases}$

- Can read off Huygen's principle (or lack thereof) by support of distribution with $c^2t^2-|\vec{x}|^2$

- This distribution has "degree" $\frac{1-d}{2}$ in dimension d, explaining difference between even and odd dimensions.

Next time: No more rephrasing. Solve PDE faster.