8/7/18 Lecture Notes: Special Lecture with Review

Review: Solve $u_t + a(u)u_x = 0 \quad -\infty < x < \infty,\ t>0$

$u(x,0) = \phi(x)$

Characteristics $x(t)$ satisfy

$\dfrac{dx}{dt} = a(u)$, so $u(x,t) = a(\phi(x_0))$, where x_0 given by

$$\frac{x - x_0}{t} = \phi(x_0)$$

Problem: Sometimes characteristics intersect

Solution: Allow for discontinuous solutions where

$$\frac{A(u^+) - A(u^-)}{u^+ - u^-} = s = \frac{dx}{dt} \quad \text{along jump discontinuity}$$

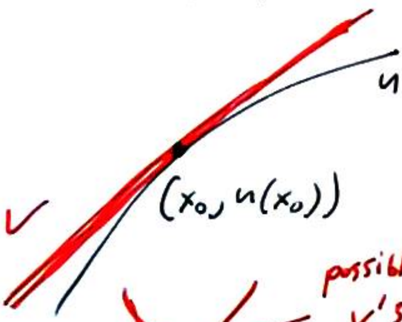
New: Viscosity solutions

Solve $u_t + H(u_x, x) = 0$ Hamilton-Jacobi equation

$u(x,0) = g(x)$

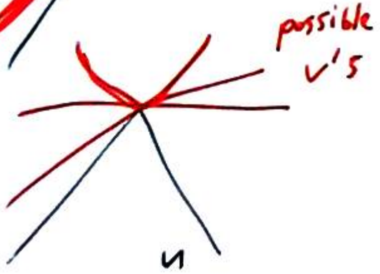
- Use method of characteristics
- Curves may intersect

How to formulate nonsmooth solutions?



v tangent to u at x_0 means $u-v$ has local max at x_0 (for u concave)

$u_x = v_x$ at x_0



- $u-v$ has local max at x_0
- still makes sense for rough u
- sub in all v_x's for u_x

Defn u is a viscosity solution if for all $v \in C_0^\infty$,

$u-v$ has local max at $(x_0,t_0) \Rightarrow v_t(x_0,t_0) + H(v_x(x_0,t_0), x_0) \le 0$

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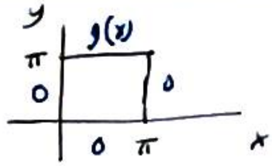
Theorem: Existence and uniqueness of viscosity solutions.

Review: Solve $\Delta u = 0$ $\quad 0 < x, y < \pi$

u on ∂D $\longrightarrow$



Separation of variables: $u(x,y) = X(x)\,Y(y)$

$$\rightarrow \underbrace{X'' + \lambda X = 0}\quad \underbrace{Y'' - \lambda Y = 0}$$

1 homogeneous BC

$X_n(x) = \sin nx$ $\qquad$ $Y_n(y) = \sinh ny$

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin nx \cdot \sinh ny,$$

$$g(x) = \sum_{n=1}^{\infty} \underline{A_n \sinh(n\pi)} \cdot \sin nx \quad \text{(to determine } A_n\text{)}$$

Fourier coefficients

New: $\Delta u = 0$ in $0 < x, y, z < \pi$ w/ $u(\pi, y, z) = g(y,z)$

all other BC $= 0$

Can show $u(x,y,z) = \sum \sum A_{mn} \sinh(x \cdot \sqrt{m^2+n^2}) \sin my \sin nz$, where

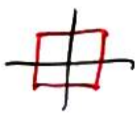
$$g(y,z) = \sum \sum A_{mn} \sinh(\pi\sqrt{m^2+n^2}) \sin my \sin nz$$

Motivates 2D Fourier series: $f(x,y) \overset{?}{=} \sum_n \sum_m c_{mn} e^{inx} e^{imy}$, where

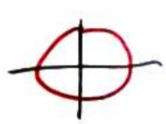
$$c_{mn} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\, e^{-i(nx+my)}\, dx\, dy$$

Convergence: If $\sum\sum |c_{mn}| < \infty$, series converges absolutely $\rightarrow$ order doesn't matter

Else: Method 1: $S_N^1 = \sum_{n=-N}^{N} \sum_{m=-N}^{N} c_{mn} e^{i(nx+my)}$



Method 2: $S_N^2 = \sum_{n^2+m^2 \le N^2} c_{mn} e^{i(nx+my)}$



Q: If $\int |f|^p \, dx\,dy < \infty$, does

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x,y) - S_N(x,y)|^p \, dx\,dy \to 0 \text{ as } N \to \infty?$$

A: Method 1: $1 < p < \infty$

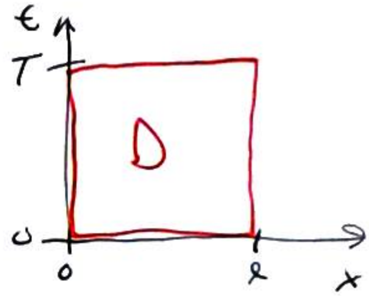
Method 2: $p = 2$ only (proof basically won Fields medal)

Review: If $\Delta u = 0$, then

Mean-Value Property: $u(x_0)$ = average value of u on any sphere/circle centered at x_0

$\Longrightarrow$ Strong Maximum Principle: If D is a connected, bounded, open set, and the max of u in D occurs in its interior, then u is constant on D.

Heat Equation: If $u_t = k u_{xx}$, then weak maximum principle holds on



New: Is there a strong version, an MVP?

Tangent: Why spheres for Δ MVP?

- surfaces where $v(x)$ is constant

Let $E(x,t;r) = \{(y,s) \mid s \leq t,\ S(x-y, t-s) \geq 1/r\}$ be the heat ball centered at (x,t).

Heat MVP: If $u_t = u_{xx}$,

$$u(x,t) = \frac{1}{4r} \iint_{E(x,t;r)} u(y,s) \frac{(x-y)^2}{(t-s)^2} \, dy\,ds.$$

$\Longrightarrow$ Strong Maximum Principle