7/19/18 Lecture Notes: Green's Functions for the Half-Plane and Sphere

Last time: If $G(x, x_0)$

i) has continuous 2nd derivatives and is harmonic in D except at x_0

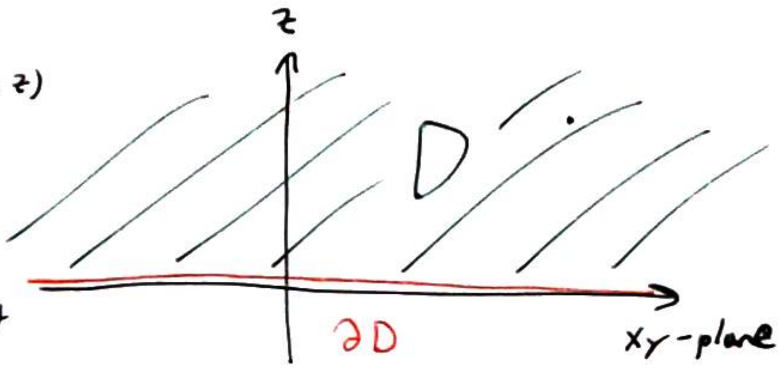
ii) is 0 on ∂D

iii) satisfies $G(x) + \frac{1}{4\pi}\frac{1}{|x-x_0|}$ is finite and harmonic at x_0,

then $u(x_0) = \iint_{\partial D} u(x)\, \frac{\partial G(x, x_0)}{\partial n}\, dS.$

<u>Half-Space</u>: $D = \{z > 0\}$ $\quad \vec{x} = (x, y, z)$

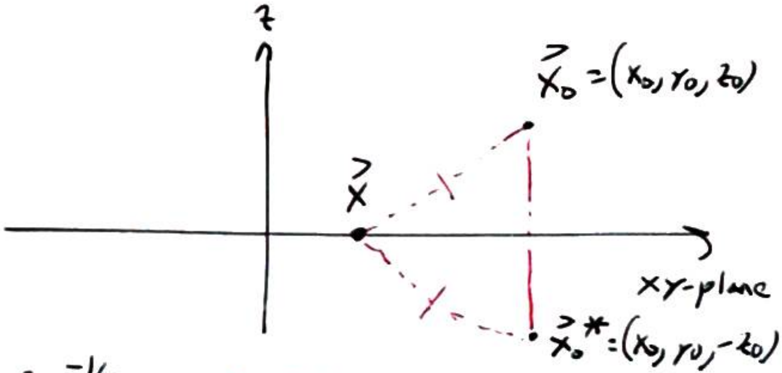
"BC at infinity" functions and derivatives $\to 0$ as $|\vec{x}| \to \infty$



$v(x) = \frac{-1}{4\pi|\vec{x}-\vec{x}_0|}$ almost works, but doesn't satisfy (ii)

Reflection Method:

To get (ii), pick



$|\vec{x} - \vec{x}_0| = |\vec{x} - \vec{x}_0^*| \to$

$G = 0$ on ∂D

$$G(\vec{x}, \vec{x}_0) = \frac{-1}{4\pi|\vec{x}-\vec{x}_0|} + \frac{1}{4\pi|\vec{x}-\vec{x}_0^*|}$$

$$= \frac{-1}{4\pi}\left[(x-x_0)^2 + (y-y_0)^2 + \underline{(z-z_0)}^2\right]^{-1/2}$$

$$+ \frac{1}{4\pi}\left[(x-x_0)^2 + (y-y_0)^2 + \underline{(z+z_0)}^2\right]^{-1/2}$$

This IS a Green's function!

For formula: $\frac{\partial G}{\partial n} = \frac{-\partial G}{\partial z} = \frac{1}{4\pi}\cdot\frac{-1}{\cancel{2}}\cdot\frac{\cancel{2}(z-z_0)}{\left[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right]^{3/2}} + \frac{1}{4\pi}\frac{z+z_0}{\left[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right]^{3/2}}$

$$= \frac{1}{4\pi}\left(\frac{z+z_0}{|\vec{x}-\vec{x}_0^*|^3} - \frac{z-z_0}{|\vec{x}-\vec{x}_0|^3}\right) \xrightarrow{z=0} \frac{1}{2\pi}\frac{z_0}{|\vec{x}-\vec{x}_0|^3}$$

Conclusion: $u(\vec{x}_0) = \frac{z_0}{2\pi}\iint_{\partial D} \frac{h(\vec{x})}{|\vec{x}-\vec{x}_0|^3}\,\partial S$ boundary values for Dirichlet problem

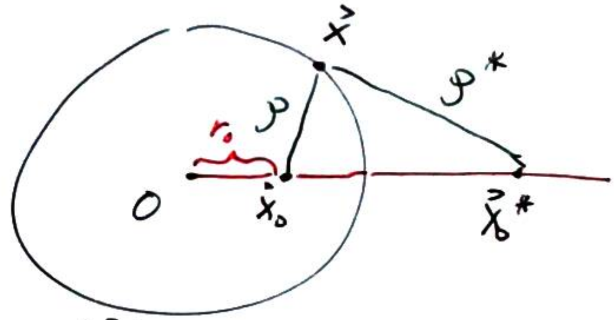
Convolution?

$$= \frac{z_0}{2\pi}\iint \left\{(x-x_0)^2+(y-y_0)^2+z_0^2\right\}^{-3/2} h(x,y)\,dx\,dy$$

<u>The Sphere</u>: $D = \{|\vec{x}| < a\}$

Pick $\vec{x}_0^{\,*}$ so $|\vec{x}_0|\cdot|\vec{x}_0^{\,*}| = a^2$;

that is, $\vec{x}_0^{\,*} = \frac{a^2 \vec{x}_0}{|\vec{x}_0|^2}$

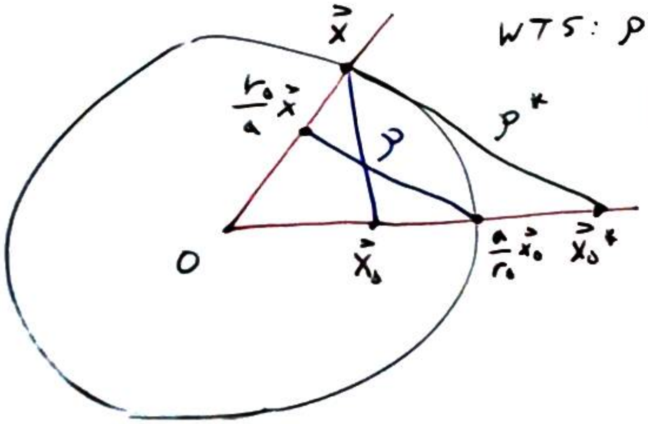


Notation: $\rho = |\vec{x}-\vec{x}_0|$, $\rho^* = |\vec{x}-\vec{x}_0^{\,*}|$, $r_0 = |\vec{x}_0|$

Claim: $G(\vec{x},\vec{x}_0) = \frac{-1}{4\pi\rho} + \frac{a}{|\vec{x}_0|}\frac{1}{4\pi\rho^*}$ is Green's function for sphere (ρ, ρ^* functions of $\vec{x}$)

<u>Pf</u> (i), (iii) hold as before since $\frac{1}{4\pi\rho^*}$ harmonic on D

(ii)



WTS: $\rho = \frac{r_0}{a}\rho^*$ on ∂D

Proof by similar triangles

✓

Alternate form: $G(\vec{x},\vec{x}_0) = \frac{-1}{4\pi|\vec{x}-\vec{x}_0|} + \frac{1}{4\pi\left|\frac{r_0}{a}\vec{x} - \frac{a}{r_0}\vec{x}_0\right|}$

note: $G(\vec{x},0) = \frac{-1}{4\pi|\vec{x}|} + \frac{1}{4\pi a}$ Can take limit as $\vec{x}_0 \to 0$ OR just observe by radial symmetry

To solve Dirichlet problem, need $\frac{\partial G}{\partial n}$.

$\rho^2 = (x-x_0)^2+(y-y_0)^2+(z-z_0)^2$

$\rho\nabla\rho = \langle 2(x-x_0), 2(y-y_0), 2(z-z_0)\rangle$ Implicit Differentiation

$\nabla\rho = \frac{2(\vec{x}-\vec{x}_0)}{\rho}$ Similarly, $\nabla\rho^* = \frac{2(\vec{x}-\vec{x}_0^{\,*})}{\rho^*}$

Recall $G(\vec{x},\vec{x}_0) = \dfrac{-1}{4\pi\rho} + \dfrac{a}{|\vec{x}_0|}\dfrac{1}{4\pi\rho^*}$, so by the Chain Rule

$$\nabla G = \frac{\vec{x}-\vec{x}_0}{4\pi\rho^3} - \frac{a}{r_0}\frac{\vec{x}-\vec{x}_0^*}{4\pi(\rho^*)^3}$$

since $\rho^* = \frac{a}{r_0}\rho$ on sphere

$$= \frac{1}{4\pi\rho^3}\left[\vec{x}-\vec{x}_0 - \left(\frac{r_0}{a}\right)^2\vec{x} + \underbrace{\left(\frac{r_0}{a}\right)^2\vec{x}_0^*}_{\vec{x}_0}\right]$$

$$= \frac{1}{4\pi\rho^3}\left[\vec{x}\right]\left(1-\left(\frac{r_0}{a}\right)^2\right)$$

sub back in non-* terms

$$\frac{\partial G}{\partial n} = \vec{n}\cdot\nabla G = \frac{\vec{x}}{a}\cdot\nabla G = \frac{a^2-r_0^2}{4\pi a\rho^3}.$$

boundary values for Dirichlet problem

So, $$u(\vec{x}_0) = \frac{a^2-|\vec{x}_0|^2}{4\pi a}\iint_{|\vec{x}|=a}\frac{h(\vec{x})}{|\vec{x}-\vec{x}_0|^3}\,dS$$

In spherical coordinates,

$$u(r_0,\theta_0,\phi_0) = \frac{a(a^2-r_0^2)}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{h(\theta,\phi)}{(a^2+r_0^2-2ar_0\cos\psi)^{3/2}}\sin\theta\,d\theta\,d\phi$$

angle between $\vec{x}, \vec{x}_0$

In 2D,

$$G(\vec{x},\vec{x}_0) = \frac{1}{2\pi}\log\rho - \frac{1}{2\pi}\log\left(\frac{r_0}{a}\rho^*\right), \text{ so}$$

$$u(x_0) = \frac{a^2-|\vec{x}_0|^2}{2\pi a}\int_{|\vec{x}|=a}\frac{h(\vec{x})}{|\vec{x}-\vec{x}_0|^2}\,ds \quad \text{Poisson's formula!}$$