6/28/18 Lecture Notes: Waves With a Source

Book makes Y cancelling mistakes, Look at HW Solutions

Solve: $u_{tt} - c^2 u_{xx} = f(x,t)$ $\quad -\infty < x < \infty,\ t > 0$

$u(x,0) = \phi(x)$ — Think about like extra force on string at (x,t)

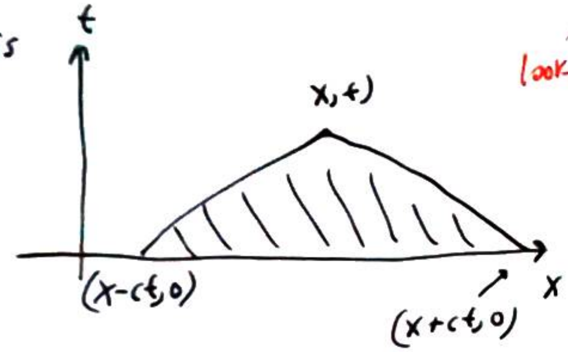
$u_t(x,0) = \psi(x)$

Theorem

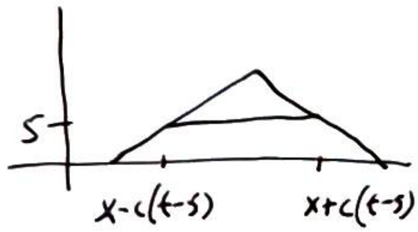
$$u(x,t) = \underbrace{\tfrac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \tfrac{1}{2c}\int_{x-ct}^{x+ct} \psi}_{\text{homogeneous solution}} + \underbrace{\tfrac{1}{2c}\iint_\Delta f}_{\text{nonhomogeneous solution}},$$

look at this, assuming $\phi = \psi = 0$

where Δ is



Note: $\iint_\Delta f = \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds$



- can check $u(x,0) = u_t(x,0) = 0$

Proof 1 Characteristic Coordinates (Proof of generalized result is often done same way as original)

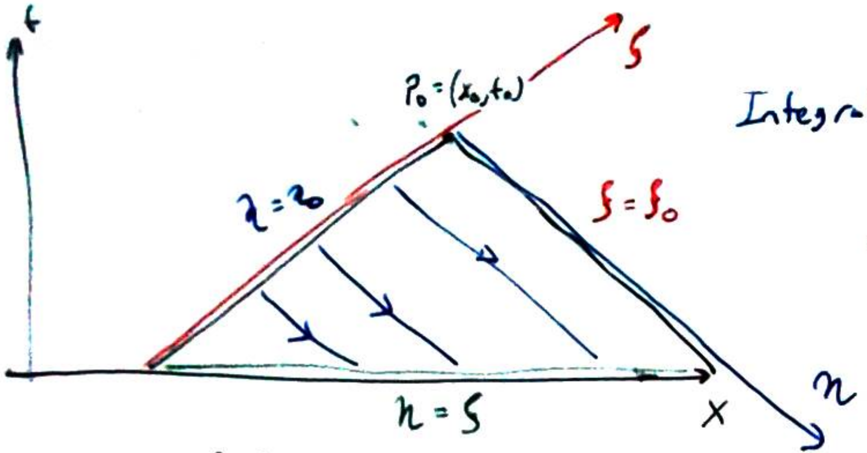
$\xi = x + ct \quad \eta = x - ct$

as before

$u_{tt} - c^2 u_{xx} = -4c^2 u_{\xi\eta} = f\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\right)$ ← change of coordinates

$u = \frac{1}{4c^2}\iint f\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\right) d\xi\, d\eta + \cancel{g(x+ct) + h(x-ct)}$

→ pick particular region, i.e. Δ



Integrate $\partial\eta$ first

$u(x,0)=0 \Rightarrow u(\xi,\xi)=0$

$u_t(x,0)=0 \Rightarrow c u_\xi(\xi,\xi)-c u_\eta(\xi,\xi)=0$

$(\partial_t = c\partial_\xi - c\partial_\eta)$

$u_x(x,0)=0 \Rightarrow u_\xi(\xi,\xi)+u_\eta(\xi,\xi)=0$

$$u(P_0)=\frac{+1}{4c^2}\int_{\eta_0}^{\xi_0}\int_{\eta_0}^{\xi} f\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2c}\right)d\eta\, d\xi$$ ~ Why?

$$u(x_0,t_0)=\frac{+1}{4c^2}\iint_D f(x,t)\, J\, dx\, dt, \text{ where}$$

$$J=\det\begin{pmatrix}\partial\eta/\partial x & \partial\eta/\partial t\\ \partial\xi/\partial x, & \partial\xi/\partial t\end{pmatrix}=\det\begin{pmatrix}1 & -c\\ 1 & c\end{pmatrix}=2c$$

$$u(x_0,t_0)=\frac{1}{2c}\iint_D f(x,t)\,dx\,dt$$

$$\int_{\eta_0}^{\xi} u_{\xi\eta}\, d\eta = \cancel{u_\xi(\xi,\xi)} - u_\xi(\xi,\eta_0) = \frac{-1}{4c^2}\int_{\eta_0}^{\xi} f\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2c}\right)d\eta$$

0

Integrate $d\xi$

$$u(\xi_0,\eta_0)-\cancel{u(\eta_0,\eta_0)}=\int_{\eta_0}^{\xi_0} u_\xi(\xi,\eta_0)\,d\xi=\frac{1}{4c^2}\int_{\eta_0}^{\xi_0}\int_{\eta_0}^{\xi} f\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2c}\right)d\eta\,d\xi$$

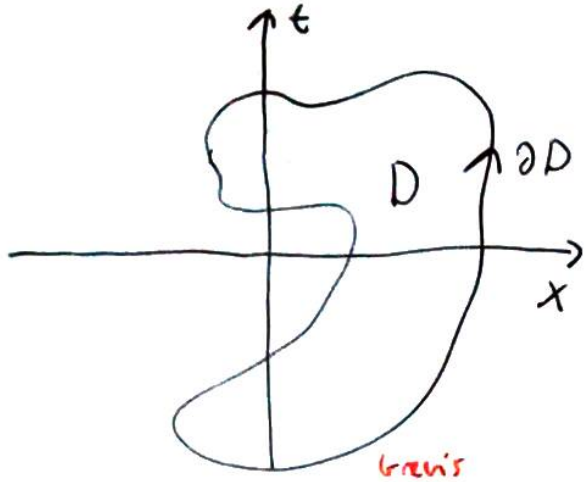
Variant: Solve system

$u_t + c u_x = v \qquad v_t - c v_x = f \qquad$ HW 3.4 #6

(made through factoring)

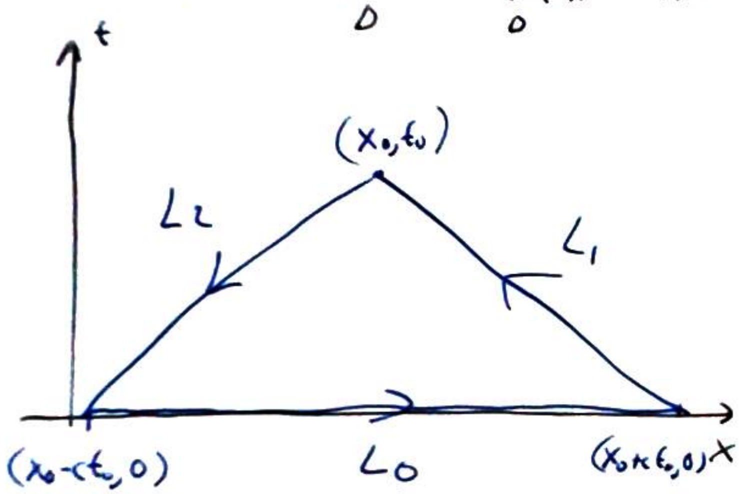
Proof 2: Green's Theorem

$$\iint_D (P_x - Q_t)\,dx\,dt = \int_{\partial D} P\,dt + Q\,dx$$



Take solution $u(x,t)$.

$$\iint_D f\,dx\,dt = \iint_D (u_{tt} - c^2 u_{xx})\,dx\,dt \overset{\text{Green's Theorem}}{=} \int_{L_0+L_1+L_2} -c^2 u_x\,dt - u_t\,dx$$



L_0: $\displaystyle\int_{L_0} -\cancel{c^2 u_x}\,dt - u_t\,dx = \underline{-\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx}$

L_1: $dx + c\,dt = 0$, so

$$\int_{L_1} -c^2 u_x\,dt - u_t\,dx = \int_{L_1} c u_x\,dx + c u_t\,dt = \int_{L_1} c\,du = \underline{cu(x_0,t_0)} - \underline{c\phi(x_0+ct_0)}$$

L_2: $dx - c\,dt = 0$, so

$$\int_{L_2} -c^2 u_x\,dt - u_t\,dx = \int_{L_2} -cu_x - cu_t\,dt = -c\int_{L_2} u = -\underline{c\phi(x_0-ct_0)} + \underline{cu(x_0,t_0)}$$

Putting it together,

$$\iint_{\Delta} f\,dx\,dt = 2cu(x_0, t_0) - c[\phi(x_0+ct_0) + \phi(x_0-ct_0)] - \int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx$$

(just rearrange from here) ∎

Well-Posedness

1) Existence - Check formula (or work backwards)
2) Uniqueness - Must be formula by derivation (also use energy methods)
3) Stability

By formula, $|u(x,t)| \le \max_x |\phi| + \frac{1}{2c}\max_x |\psi| \cdot \underbrace{2ct} + \frac{1}{2c}\max |f| \cdot \underbrace{ct^2}$

"amount being integrated over"

If $\|w\| = \max_x |w(x)|$, $\|w\|_T = \max_{\substack{x \\ 0\le t\le T}} |w(x,t)|$, then,

$$\|u_1 - u_2\|_T \le \|\phi_1 - \phi_2\| + T\|\psi_1 - \psi_2\| + \frac{T^2}{2}\|f_1 - f_2\|_T$$

Source on a Half-Line

$v_{tt} - c^2 v_{xx} = f(x,t)$

$v(x,0) = \phi(x)$ $v_t(x,0) = \psi(x)$ $v(0,t) = h(t)$

Solution (to be found on HW 3.4 #12)

$$v(x,t) = \phi \text{ term} + \psi \text{ term} + h\left(t - \frac{x}{c}\right) + \frac{1}{2c}\iint_D f,$$

D = domain of dependence