8/1/18 Lecture Notes: The Fourier Transform

Complex Fourier series:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/\ell}, \quad \text{with } c_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(y) e^{-in\pi y/\ell}\, dy$$

Works great on $[-\ell,\ell]$, but many PDE on $(-\infty,\infty)$. Can we take $\ell \to \infty$?

Sub $k = n\pi/\ell$ $(\pi/\ell \sim \Delta k)$

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \underbrace{\int_{-\ell}^{\ell} f(y) e^{-iky}\, dy}_{c_n}\, e^{ikx} \underbrace{\frac{\pi}{\ell}}_{\Delta k}$$

$\Delta k \to 0$
$\ell \to \infty$
$n \to$ cts. variable

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} f(y) e^{-iky}\, dy\right]}\, e^{ikx}\, dk \quad ☺$$

like c_n coefficients,
amount of f coming from
frequency k

Convergence nontrivial, sometimes only works in distributional sense.

Define the <u>Fourier transform</u> of $f(x)$ as

$$\boxed{F(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx}$$

frequency variable

Then, by ☺, $f(x) = \int_{-\infty}^{\infty} F(k)\, e^{ikx} \frac{dk}{2\pi}$ (Inverse Fourier transform)

coefficients k-th frequency

Example: $f(x) = e^{-x^2/2}$

$$F(k) = \int_{-\infty}^{\infty} e^{-x^2/2 - ikx}\, dx = \int_{-\infty}^{\infty} e^{-(x+ik)^2/2} e^{i^2k^2/2}\, dx$$

$$= e^{-k^2/2} \int_{-\infty}^{\infty} e^{-y^2/2}\, dy \qquad y = x + ik$$

$$= \sqrt{2\pi}\, e^{-k^2/2}$$

Example $f(x) = \delta(x)$

$$F(k) = \int_{-\infty}^{\infty} \delta(x) e^{-ikx}\,dx = e^{-ik\cdot 0} = 1 \text{ (the constant function)}$$

With a partner: Find $F(k)$, where $f(x) = e^{-a|x|}$ $(a>0)$.

If done early, try $f(x) = 1$

Example Chart

$f(x)$	$F(k)$
$e^{-x^2/2}$	$\sqrt{2\pi}\, e^{-k^2/2}$
$\delta(x)$	1
$e^{-a\lvert x\rvert}$	$\frac{2a}{a^2+k^2}$ $(a>0)$
1	$2\pi\delta(k)$
$H(a-\lvert x\rvert)$	$\frac{2}{k}\sin ak$
$H(x)$	$\pi\delta(k) + 1/ik$
$\mathrm{sgn}(x) = H(x) - H(-x)$	$2/ik$

Rules: What is Fourier transform of $f'(x)$?

$$\int_{-\infty}^{\infty} f'(x) e^{-ikx}\,dx \overset{IBP}{=} \underbrace{\cancel{f(x)}}_{0} e^{-ikx}\Big|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-ik)e^{-ikx}\,dx$$

$$= ik\,F(k)$$

In class, we always use the definition of $F(k)$, but you can use these results

Rule Chart if $f(x) \to F(k)$

Function	Fourier Transform	
$f'(x)$	$ikF(k)$	exchange multiplication by
$x\,f(x)$	$i\,dF/dk$	power with differentiation
$f(x-a)$	$e^{-iak}F(k)$	exchange translation
$e^{iax}f(x)$	$F(k-a)$	and modulation
$af(x)+bg(x)$	$aF(k)+bG(k)$	– linearity
$f(ax)$	$\frac{1}{\lvert a\rvert}F(k/a)$	– dilation

With a partner. Find Fourier transform of $f(x-a)$

If time, $e^{iax}f(x)$

Some PDE Solutions:

Heat: $u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\,\phi(y)\,dy$

Wave: $u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\,\psi(y)\,dy + \frac{\partial}{\partial t}\int_{-\infty}^{\infty} S(x-y,t)\,\phi(y)\,dy$

Convolution $f * g(x) := \int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy$

Meaning: Continuous limit of $\sum_k f(x-y_k)\,a_k$

If X, Y independent random variables with distribution functions $f(x)$ and $g(x)$ respectively, then $(f*g)(x)$ is the distribution function of $X+Y$.

Fourier transform of $f*g$:

$$\int (f*g)(x)e^{-ikx}dx = \iint f(x-y)g(y)\,dy\,e^{-ikx}dx \qquad z=x-y \;\; \& \;\; dz=dx$$

Check all limits $-\infty$ to ∞, preserved under change of coordinates

$$= \iint f(z)e^{-ik(y+z)}dz\,g(y)\,dy$$

$$= \int f(z)e^{-ikz}dz \cdot \int g(y)e^{-iky}dy$$

$$= F(k)\cdot G(k)$$

Recall: If $f(x)$ has Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ (and $\|f\|_2 < \infty$)

$$\int_{-\pi}^{\pi}|f(x)|^2dx = \int_{-\pi}^{\pi} f(x)\bar{f}(x)dx = \sum_m \sum_n c_n \bar{c}_m \int_{-\pi}^{\pi} e^{i(n-m)x}dx = \sum_n |c_n|^2\cdot 2\pi$$

For the Fourier transform,

$$\int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty}|F(k)|^2\frac{dk}{2\pi}$$

More generally, $\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx = \int_{-\infty}^{\infty} F(k)\overline{G(k)}\frac{dk}{2\pi}$ (Parseval's relation)

<u>Heisenberg Uncertainty Principle</u>

x - position $\qquad f(x)$ wave function

k - momentum $\qquad |f(x)|^2$ probability distribution function

Choose frame so average value of x, k both 0.

variances $\begin{cases} \sigma_x^2 = \int_{-\infty}^{\infty} x^2|f(x)|^2dx \\ \sigma_k^2 = \int_{-\infty}^{\infty} k^2|F(k)|^2dk \end{cases}$ $\qquad$ HUP: $\sigma_x\cdot\sigma_k \geq \frac{1}{2}$

In physics: One cannot simultaneously determine position and momentum of a particle.

Pf $\sigma_x \cdot \sigma_k = \left[\int_{-\infty}^{\infty} |x f(x)|^2 dx\right]^{1/2} \left[\int_{-\infty}^{\infty} |ik F(k)|^2 dk\right]^{1/2}$

$\downarrow$ Parseval's

$= \quad '' \quad \left[\int_{-\infty}^{\infty} |f'(x)|^2 dx\right]^{1/2}$

$\geq \left|\int_{-\infty}^{\infty} x f(x) f'(x) dx\right|$ by Cauchy-Schwarz

$$\int_{-\infty}^{\infty} x \underbrace{f(x) f'(x)\, dx}_{\frac{1}{2} d[f(x)]^2} = \underbrace{\cancel{\frac{1}{2} x f(x)^2 \Big|_{x=-\infty}^{\infty}}}_{0} - \int_{-\infty}^{\infty} \frac{1}{2} f(x)^2 dx = -1/2$$

Fourier Transform in 3D: $f(x, y, z)$ has Fourier transform

$$F(k_1, k_2, k_3) = \iiint f(\vec{x}) e^{-i \vec{x} \cdot \vec{k}} d\vec{x}$$ (same as doing 1D F.T. in x, then y, then z)

Inverse: $f(x) = \iiint F(\vec{k}) e^{i \vec{x} \cdot \vec{k}} d\vec{k} / (2\pi)^3$

Tomorrow Preview: Solve $S_t = S_{xx}$, $S(x,0) = \delta(x)$ $\xrightarrow{\text{F.T.}}$ $\dfrac{\partial \hat{S}}{\partial t} = -k^2 \hat{S}$, $\hat{S}(k,0) = 1$

Solve ODE in $t \rightarrow \hat{S}(k,t) = e^{-k^2 t}$

$\downarrow$ invert

$$S(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$$