We consider a resource allocation problem of the form

$$\max_{w \in \mathcal{W}} \min_{r \in \mathcal{E}} r^T w$$

where $\mathcal{W} := \{w \in \mathbb{R}^n_+ \ : \ w_1 + \ldots + w_n = 1\}$, and

$$\mathcal{E} := \{\hat{r} + Du \ : \ \|u\|_2 \leq 1\}.$$

Here, $\hat{r} \in \mathbb{R}^n$ and $D = \mathbf{diag}(\sigma_1, \ldots, \sigma_n)$ are given, with $\sigma \in \mathbb{R}^n$ $\sigma > 0$.

The above problem appears when trying to allocate resources to various revenue-generating processes (which could be ads, financial investments, physical sensors, etc). The revenue vector r is unknown but bounded, and the goal is to maximize the worst-case total revenue $\min_{r \in \mathcal{E}} r^T w$.

(a) Describe the shape of $\mathcal{E}$ in simple geometrical terms.

(b) Use the Cauchy-Schwartz inequality to prove that

$$\min_{r \in \mathcal{E}} r^T w = \hat{r}^T w - \|Dw\|_2.$$

(c) Express the problem as an SOCP in standard format, involving the variable w and one extra scalar variable.

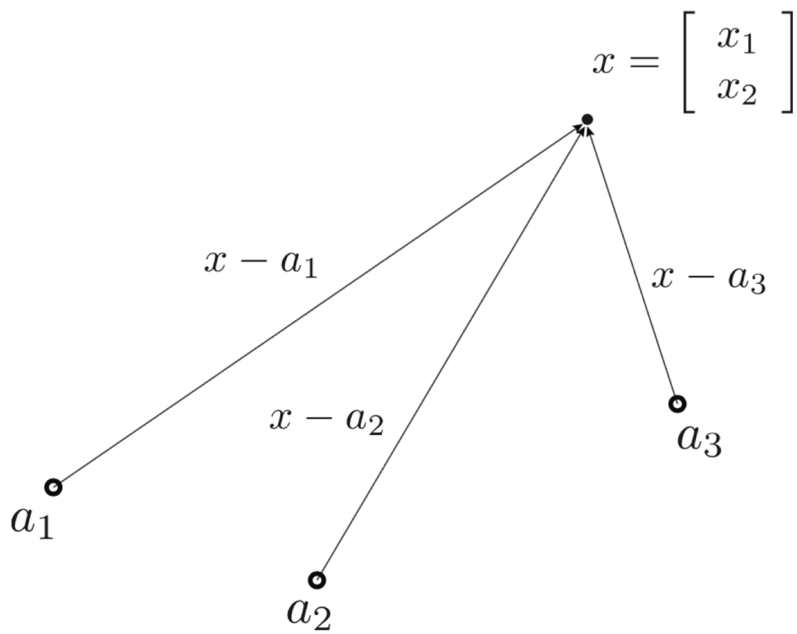
Exercise 3.5 (A lower bound on the rank) Let $A \in \mathbb{S}^n_+$ be a symmetric, positive semi-definite matrix.

1. Show that the trace, $\operatorname{trace} A$, and the Frobenius norm, $\|A\|_F$, depend only on its eigenvalues, and express both in terms of the vector of eigenvalues.

2. Show that

$$(\operatorname{trace} A)^2 \leq \operatorname{rank}(A) \|A\|_F^2.$$

Exercise 4.4 (Trilateration) Trilateration is a method for determining the position of a point, given the distances to known control points (anchors). Trilateration can be applied to many different areas such as geographic mapping, seismology, navigation (e.g., GPS systems), etc.



In Figure 4.6, the coordinates of the three anchor points $a_1, a_2, a_3 \in \mathbb{R}^2$ are known, and the distances from point $x = [x_1 \ x_2]^\top$ to the anchors are measured as d_1, d_2, d_3. The unknown coordinates of x are related to the distance measurements by three *nonlinear* equations

$$\|x - a_1\|_2^2 = d_1^2, \quad \|x - a_2\|_2^2 = d_2^2, \quad \|x - a_3\|_2^2 = d_3^2.$$

Derive a formulation of the problem in the form of a system of *linear* equations $Ax = y$, whose solutions contain the solution set of the above nonlinear system. Here A is a suitable matrix (to be determined) that depends on the anchors' positions, and y is a suitable vector (to be determined), that depend on the anchors' positions and on the measured distances.

Exercise 5.4 (Quadratic inequalities) Consider the set defined by the following inequalities

$$(x_1 \geq x_2 - 1 \text{ and } x_2 \geq 0) \text{ or } (x_1 \leq x_2 - 1 \text{ and } x_2 \leq 0).$$

1. Draw the set.

2. Show that it can be described as a single quadratic inequality of the form $q(x) = x^\top A x + 2b^\top x + c \leq 0$, for matrix $A = A^\top \in \mathbb{R}^{2,2}$, $b \in \mathbb{R}^2$ and $c \in \mathbb{R}$ which you will determine.

Exercise 5 (Projection on a hyperplane.) Consider the hyperplane $\{z \in \mathbb{R}^n \ : \ a^\top z = b\}$, $a \neq 0$, and a point $y \in \mathbb{R}^n$.

(a) (10 pts.) Determine the Euclidean projection of y onto the hyperplane.

(b) (5 pts.) Determine the Euclidean distance between y and its projection on the hyperplane.

Exercise 6 (Properties of dyad.) Let $x, y \in \mathbb{R}^n$, both not identical to the zero vector, and $A = xy^\top \in \mathbb{R}^{n,n}$.

(a) (5 pts.) Determine an eigenvalue and an eigenvector of A.

(b) (5 pts.) We know that A has rank one. Write a proof of this fact.

(c) (5 pts.) What is the dimension of $\mathcal{N}(A)$?

(d) (5 pts.) Compute a singular value decomposition of A and write it in compact form.

4. (10 points) *Control problem.* We consider a dynamical system with one input and one output. described by a linear, time-invariant relationship:

$$y(t) = h_0 u(t) + h_1 u(t-1) + \ldots + h_m u(t-m), \quad t = 0, 1, 2, \ldots,$$

where $m > 0$ is called the *order* of the system, and $h = (h_0, \ldots, h_m)$ is called the *impulse response.*

Consider an input design problem, where we seek to find a sequence of inputs which tracks a given desired output signal $y_{\text{des}}(t)$ over a given time horizon $\{0, \ldots, T\}$,with $T > m$ given. Specifically, assuming that $u(t) = 0$ when $t < 0$, we seek inputs $u(0), \ldots, u(T)$ should be such that

- The peak deviation between $y(t)$ and $y_{\text{des}}(t)$

  $$\max_{0 \le t \le T} |y(t) - y_{\text{des}}(t)|$$

  is minimized.

- The inputs should be zero for $t < 0$, $t > M$, where $M < T$ is given.
- The inputs should satisfy amplitude and slew rate constraints:

$$|u(t)| \leq U, \quad |u(t+1) - u(t)| \leq S,$$

  where $U > 0$, $S > 0$ are given.

(a) Show that the problem can be formulated as a linear program.

(b) Assume that we replace the amplitude constraint on u with a upper bound on the average energy: $E(u) \leq E_{\max}$, where $E_{\max} > 0$ is given, and

$$E(u) := \frac{1}{T} \sum_{t=0}^{T} u(t)^2.$$

Formulate the resulting problem as an SOCP.