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Math 3012 Combinatorics

1. For a set $[1, 2, 3, ..., n]$, any number up to n can be listed zero, one or two times in its corresponding multiset, three total. Thus, the number of possibilities is 3^n.

2. Two cases must be considered in this problem; if the last number in a string is or isn't a zero. If the first digit is a zero, then the second number must be a one or a two. These two choices allow for a $2a_{n-2}$ relation to describe the next term $(n-1)$. If that digit isn't a zero, then the relation $2a_{n-1}$ applies as there are two options to describe the nth term. The initial values are for the two most basic sets. The empty string a_0 has a value of 1 because there is only one way to represent length zero, and the string a_1 has a value of 3 because there are three possibilites (1,0,2) to describe the next one.

3. (a) The relation $k\binom{n}{k}$ describes if out of every n objects, k are selected out into a group α, and then another object was selected out of α. This can be equivalently stated as first choosing that special object from a list of n objects, and then choosing the rest of α afterward: The number of objects as well as the cardinality of α are now one smaller, and there are n ways of choosing that special object: $n\binom{n-1}{k-1}$.

   (b) $k\binom{n}{k} = n\binom{n-1}{k-1} \rightarrow \frac{k*n!}{(n-k)!k!} = \frac{n*(n-1)!}{(n-1-k+1)!(k-1)!} \rightarrow \frac{n!}{(n-k)!(k-1)!} = \frac{n!}{(n-k)!(k-1)!}$

4. As zeroes can be placed anywhere in a ternary string, we can choose where to place them. If we choose k spots out of an n-length string to put zeroes, then that is described by $\binom{n}{k}$. Now we are only left with one and two, and going by the logic that either a one or a two can go in a spot, then there are 2^x ways this can happen. x in this case is $n-k$. Thus by going through all the slots in a string, $\displaystyle\sum_{k=0}^{n}\binom{n}{k}2^{n-k}$ can be derived. This is equal to starting over, and putting either a 0, 1 or 2 into the string instead of putting all zeroes first, which has 3^n possible ways. Thus, those are equal.

5. This can be computed by solving as if every path was allowed and removing the unwanted areas. In order to make this problem more manageable, all points have been translated by (-3,-2) in order for the bottom-left corner of the area to be at the origin. The total amount of paths is $\binom{21}{9}$. Next, the unwanted parts are solved for so as to realize them in their entirety. For the first avoided point, (1,3), all paths that lead to it can be described by $\binom{4}{1}$ and all that lead away to the end can be described by $\binom{17}{8}$. These can be multiplied together to obtain the full range of paths through that point. A similar process with the second point (3,7) gives $\binom{10}{3}\binom{11}{6}$, and the relation becomes $\binom{21}{9} - \binom{4}{1}\binom{17}{8} - \binom{10}{3}\binom{11}{6}$. However, duplicate points must be accounted for. This can be done by finding the paths that went through both points to the end, or, the origin to (1,3) multiplied by the paths from (1,3) to (3,7), and from (3,7) to the end. Thus, the solution is $\binom{21}{9} - \binom{4}{1}\binom{17}{8} - \binom{10}{3}\binom{11}{6} + \binom{4}{1}\binom{6}{2}\binom{11}{6}$.

6. There are two separate conditions that have to be met in order to solve this problem. The first is when $24 < x_4$ and the second when $x_4 \leq 39$. This can be done by taking

two different combinations for each one. What is similar in both is that the reqirements for all x to be greater than a number are made positive by subtracting that number minus one from the total, as that is included by definition. For the second as well, 39 should be subtracted as it provides a complement of $x_4 > 0$. This produces an answer of $\binom{453}{3} - \binom{438}{3}$.

7.

(a) 2, 8, 14, 20, 26, 32, 38,...

Each number is six greater than the previous number. Thus, in order to define the above sequence, then the recursive relation $x_n = x_{n-1} + 6$ can be used, provided an initial condition of 2 for x_0.

(b) 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$,...

Each number is half that of the previous number. Thus $x_n = \frac{x_{n-1}}{2}$ can be used with the initial x_0 equal to 8.

(c) 1, 3, 7, 15, 31, 63, 127,...

Each number is one less than two to the power of the number's location in the sequence. Thus the recursive relation $x_n = 2x_{n-1} + 1$ with initial condition $x_0 = 1$.