6/26/18 Lecture Notes: Waves on the Half-Line and Random Tidbits

Fun Fact: $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$

<u>Proof</u> Let $A = \int_{-\infty}^{\infty} e^{-x^2}dx$

$$A^2 = \int_{-\infty}^{\infty} e^{-x^2}dx \cdot \int_{-\infty}^{\infty} e^{-y^2}dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}dx\,dy$$

Polar Coordinates

$x^2+y^2=r^2$

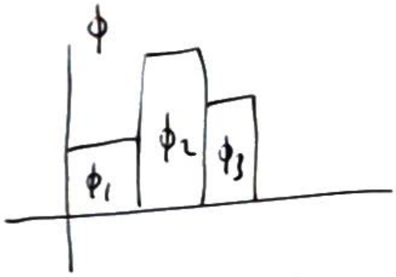
$dx\,dy = r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^{\infty} re^{-r^2}dr\,d\theta$$

$$\int_0^{\infty} re^{-r^2}dr = \frac{-e^{-r^2}}{2}\Big|_{r=0}^{\infty} = \frac{1}{2}$$

$$= \int_0^{2\pi} \frac{1}{2}\,d\theta = \pi$$

Additional (and optional) way of understanding yesterday's lecture:



$u = u_1 + u_2 + u_3$

$u = \sum_K u_K$

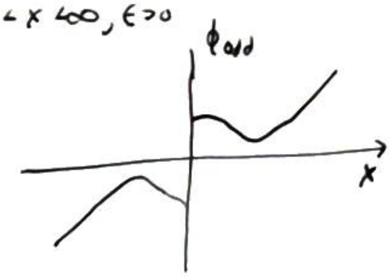
Office Hour Location: Switch to 110 Barrows?

Last time: Want to solve

$v_t = kv_{xx} \quad 0 < x < \infty,\ t > 0$

$v(x,0) = \phi(x)$

$v(0,t) = 0$



Solve

$u_t = ku_{xx} \quad -\infty < x < \infty,\ t > 0$

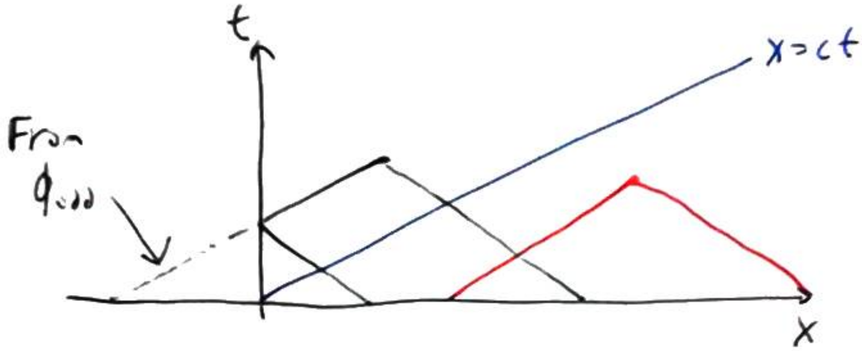
$u(x,0) = \phi_{odd}(x)$

Waves on the Half-Line

$$v_{tt} - c^2 v_{xx} = 0 \qquad 0 < x < \infty,\ t > 0$$
$$v(x,0) = \phi(x) \quad v_t(x,0) = \psi(x)$$
$$v(0,t) = 0$$

Like last time, $\phi \to \phi_{odd}$, $\psi \to \psi_{odd}$ on $-\infty < x < \infty$

D'Alembert's Formula $\to v(x,t) = \frac{1}{2}\left[\phi_{odd}(x+ct) + \phi_{odd}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{odd}(y)\,dy$



If $x \geq ct$, $x+ct, x-ct \geq 0$, and

$$v(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy$$

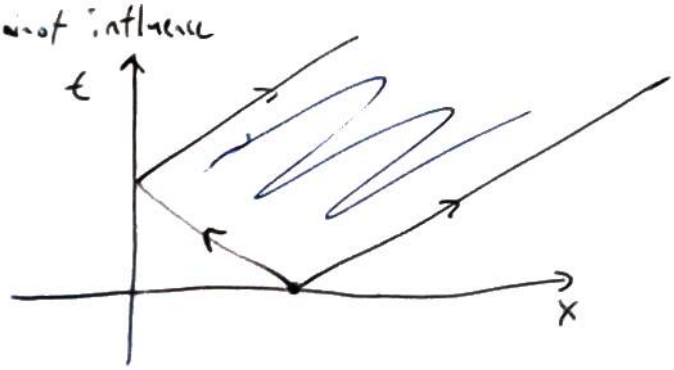
If $x < ct$, $x-ct < 0$, $x+ct > 0$

$$\phi_{odd}(x-ct) = -\phi(ct-x)$$

$$v(x,t) = \frac{1}{2}\left[\phi(x+ct) - \phi(ct-x)\right] + \frac{1}{2c}\int_{0}^{x+ct} \psi(y)\,dy - \frac{1}{2c}\int_{x-ct}^{0} \psi(-y)\,dy$$

$$= \frac{1}{2}\left[\phi(x+ct) - \phi(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(y)\,dy$$

Domain of influence



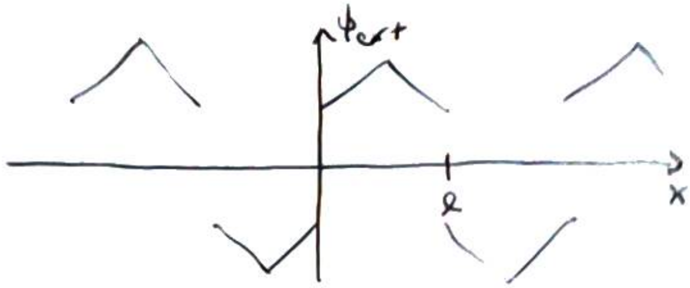
Finite Interval

$u_{tt} = c^2 u_{xx} \quad 0 < x < \ell,\ t > 0$

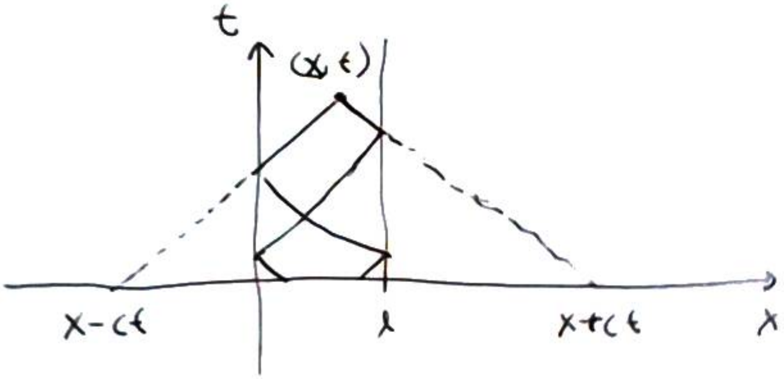
$u(x,0) = \phi(x) \quad u_t(x,0) = \psi(x)$

$u(0,t) = 0 = u(\ell, t)$

Strategy: Extend ϕ (and ψ) to be odd around $x=0$ and $x=\ell$



Solution: $u(x,t) = \frac{1}{2}\left[\phi_{ext}(x+ct) + \phi_{ext}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{ext}(y)\,dy$



Coming Next Week: "Better" Method called "Separation of Variables"

- break down u, ϕ into frequencies

Coming This Week: Solving $\mathcal{L}u = f(x,t)$