Homework 0: Linear Algebra Review

EECS/BioE C106A/206A
Introduction to Robotics

Due: September 1, 2020

Note: Problems marked [bonus] will not be graded, but you are highly encouraged to attempt them.

Problem 0. Syllabus Quiz

Complete the `Syllabus Quiz` assignment on Gradescope. This will be part of your grade for Homework 0.

Problem 1. Orthogonal Matrices

Let $\mathbf{R}$ be an $n \times n$ matrix, and let $r_i \in \mathbb{R}^n$ be the i-th column of $\mathbf{R}$. $\mathbf{R}$ is said to be *orthogonal*, if for any $i \neq j$, the vectors r_i and r_j are orthogonal to each other, and each r_i is unit length. We then also say that the vectors $\{r_1, ..., r_n\}$ form an *orthonormal basis* for $\mathbb{R}^n$.

(a) Show that a square matrix $\mathbf{A}$ is orthogonal if and only if $\mathbf{A}^T\mathbf{A} = \mathbf{I}$. *Hint: Consider writing the (i, j)th entry of $\mathbf{A}^T\mathbf{A}$ in terms of dot products of the columns of $\mathbf{A}$.*

(b) Let $\mathbf{R}$ be an orthogonal $n \times n$ matrix and let u be an $n-$dimensional vector. Show that $||\mathbf{R}u|| = ||u||$. In other words, show that $\mathbf{R}$ preserves norms when it acts on vectors. *Hint: Use the fact that for the standard euclidean norm, $||u||^2 = u^T u$.*

(c) Show that if $\mathbf{R}$ is an orthogonal matrix, then $\det(\mathbf{R}) = \pm 1$. *Hint: Take the determinant on both sides of the equation $\mathbf{R}^T\mathbf{R} = \mathbf{I}$.*

Problem 2. The Matrix Exponential: Algebraic Properties

Recall that for a scalar $a \in \mathbb{R}$, we can write its exponential e^a as a Taylor series that converges for any a:

$$e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!} = 1 + a + \frac{a^2}{2!} + \cdots \tag{1}$$

We can similarly use an infinite series to *define* the exponential of a square real $n \times n$ *matrix* $\mathbf{A}$:

$$e^{\mathbf{A}} = \sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!} = \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \cdots \tag{2}$$

where by convention we take $\mathbf{A}^0$ to be the identity matrix for any square matrix $\mathbf{A}$. The result is also an $n \times n$ matrix. As it turns out, this infinite series converges absolutely for every matrix $\mathbf{A}$. So we use this series to define the *matrix exponential function* $e^{\mathbf{A}}$.

The matrix exponential shows up all over the place in the study of rigid body motion and dynamical systems, especially in the solutions to vector differential equations, as we shall see. We will make heavy use of the matrix exponential in this class. In this problem, you will use the infinite series representation in equation (2) to derive some of the fundamental algebraic properties of this function which will prove very useful in our study of rigid body kinematics.

(a) Show that $e^{\mathbf{0}} = \mathbf{I}$. i.e. the exponential of the zero matrix is the identity matrix.

(b) Show that $(e^{\mathbf{A}})^T = e^{(\mathbf{A}^T)}$.

(c) Let g be any invertible square matrix of the same size as $\mathbf{A}$. Show that $e^{g\mathbf{A}g^{-1}} = ge^{\mathbf{A}}g^{-1}$.

Hint: Start by showing that for all n, $(g\mathbf{A}g^{-1})^n = g\mathbf{A}^n g^{-1}$.

(d) Show that if λ is an eigenvalue of $\mathbf{A}$ then e^{λ} is an eigenvalue of $e^{\mathbf{A}}$.

Hint: Use the series expansion. Show that if v is an eigenvector of $\mathbf{A}$ with eigenvalue λ then it is also an eigenvector of $e^{\mathbf{A}}$ with eigenvalue e^{λ}. i.e. show that $e^{\mathbf{A}}v = e^{\lambda}v$.

Remark: In fact, a suitable converse of the above statement is also true, though more difficult to prove. We can conclude that if the eigenvalues of $\mathbf{A}$ (possibly repeated) are $\lambda_1, ..., \lambda_n$ then the eigenvalues of $e^{\mathbf{A}}$ are exactly $e^{\lambda_1}, ..., e^{\lambda_n}$.

(e) [**Bonus**] Using the previous part, show that $\det(e^{\mathbf{A}}) = e^{\operatorname{tr}\mathbf{A}}$. Conclude that the exponential of any matrix is always invertible.

Hint: What is the relationship between the eigenvalues of a matrix, its determinant and its trace? Also use the remark from the previous part.

Remark: In fact, the inverse of $e^{\mathbf{A}}$ is simply $e^{-\mathbf{A}}$.