Statistics 153 (Introduction to Time Series) Homework 5

Due on Wednesday April 7 by 11:59pm PDT, on Gradescope

1 Theory: Bartlett's Formula

1. Assuming its conditions are met, show that for an ARMA(p, q) process X_t with $p = q = 0$ (i.e. X_t is white noise) Bartlett's formula means that r_k is approximately Normal with mean zero and variance $1/n$, or in other words *(2 points)*

$$\sqrt{n}\begin{pmatrix}\hat{r_1}\\ \vdots \\ \hat{r_k}\end{pmatrix} \xrightarrow{d} N_k(0, I_k)$$

***This is the asymptotic result for the sample correlations of white noise covered earlier in class*

2. For the AR(1) process $X_t = \phi X_{t-1} + W_t$, calculate the asymptotic variance of $\hat{r_k}$ for $k \geq 1$. Note that, technically, the autocorrelation for AR(1) is $\rho(h) = \phi^{|h|}$, where the absolute value of h is in the exponent. *(3 points)*

2 Computation: Bartlett's Formula

Here you will use the results for the AR(1) model that you derived above. Assume $\phi = 0.7$.

1. Simulate $n = 300$ observations of the process using arima.sim(). Plot the sample ACF of this simulated stationary process. Note the blue dashed lines at $\pm 1.96/\sqrt{n}$ bars: these are the default for the acf() function and represent the distribution of r_h for white noise, aka ARMA(0,0). *(1 point)*

2. Now let's visualize the distribution of ARMA(1,0), usually simply called AR(1). Add to this plot the expectations and 95% intervals for r_1,...,r_{20} in red. Comment on the difference between the blue lines and the red lines. *(2 points)*

3 Prediction of MA(1)

Consider an invertible MA(1) model $X_t = W_t + \theta W_{t-1}$ for some i.i.d. white noise process $\{W_t\}$ with variance σ^2.

1. Derive the explicit form of the minimum mean-square error one-step prediction

$$\tilde{X}_{n+1} = E(X_{n+1} | X_n, X_{n-1}, X_{n-2}, \ldots)$$

for X_{n+1} based on the complete infinite past $X_n, X_{n-1}, X_{n-2}, \ldots$. Hint: invertible processes can be put in $AR(\infty)$ form, right?

(2 points)

2. Derive the mean squared error $E\left[(\tilde{X}_{n+1} - X_{n+1})^2\right]$. (Hint: $\tilde{X}_{n+1} = \theta W_n$, right?) (1 point)

3. Now consider the truncated estimate $\tilde{X}^n_{n+1}$, which equals $\tilde{X}_{n+1}$ but with unobserved data being set to zero, that is, $0 = X_0 = X_{-1} = \ldots$. Show that (2 points)

$$E\left[(X_{n+1} - \tilde{X}^n_{n+1})^2\right] = \sigma^2(1 + \theta^{2+2n}).$$

4. Comment on how well the truncated estimate $\tilde{X}^n_{n+1}$ works compared to $\tilde{X}_{n+1}$ (i.e. compare the MSE's of the two). (1 point)

4 Prediction of MA(q)

Consider an invertible MA(q) model $X_t = \theta(B)W_t$ for some white noise $\{W_t\}$ with variance σ^2.

1. Show that for any $m > q$ the best linear predictor of X_{n+m} based on $X_1, \dots, X_n$ is always zero. (1 point)

2. Now assume that the white noise $\{W_t\}$ is also i.i.d.. Show that for any $m > q$ the best predictor (minimum mean-square error forecast) of X_{n+m} based on the full history $X_n, X_{n-1}, X_{n-2}, \dots$ is also zero. (1 point)

5 AR(1)

Consider a causal, zero mean AR(1) model $X_t - \phi X_{t-1} = W_t$ for some independent white noise $\{W_t\}$ with variance σ^2.

1. Derive the general form of the best predictor $\tilde{X}_{n+m}$ of X_{n+m} in terms of $X_1, \dots, X_n$. (Your result should be linear, meaning the best predictor is also the best linear predictor!) *(1 point)*

2. Show that
$$E\left[(X_{n+m} - \tilde{X}_{n+m})^2\right] = \sigma^2 \frac{1 - \phi^{2m}}{1 - \phi^2}$$
Hint: First show that $X_{n+m} = \phi^m X_n + \sum_{j=0}^{m-1} \phi^j W_{n+m-j}$. *(2 points)*

3. Given $X_1, ..., X_7$ and ϕ, what is the best mean-square predictor of X_{10}? Provide reasoning for your answer. *(1 point)*