Math 110 Professor K. A. Ribet

Midterm Exam September 26, 2002

Please put away all books, calculators, digital toys, cell phones, pagers, PDAs, and other electronic devices. You may refer to a single 2-sided sheet of notes. Please write your name on each sheet of paper that you turn in. Don't trust staples to keep your papers together.

The symbol "$\mathbf{R}$" denotes the field of real numbers. In this exam, "0" was used to denote the vector space $\{0\}$ consisting of the single element 0.

These solutions were written by Ken Ribet. Sorry if they're a little terse. If you have a question about the grading of your problem, see Ken Ribet for problems 1–2 and Tom Coates for 3–4.

1. (9 points) Let $T : \mathbf{R}^4 \to \mathbf{R}^3$ be the linear transformation whose matrix with respect to the standard bases is $A = \begin{pmatrix} -1 & -1 & 7 & 5 \\ 1 & 0 & -5 & -3 \\ 0 & -1 & 2 & 2 \end{pmatrix}$. (In the book's notation, $T = L_A$.)

Find bases for (i) the null space and (ii) the range of T.

The null space consists of quadruples (x, y, z, w) satisfying three equations of which the first two are $-x-y-7z+5w = 0$ and $x-5z-3w = 0$. It turns out that the third equation is the sum of the first two, so we can forget it. (If you don't notice this circumstance, you'll still get the right answer.) Replace the first equation by the sum of the first two, leaving the second alone. We get the two equations

$$\begin{aligned} -y + 2z + 2w &= 0 \\ x \qquad - 5z - 3w &= 0. \end{aligned}$$

The interpretation is that z and w can be chosen freely, and then x and y are determined by z and w. If we take $z = 1$, $w = 0$, we get the solution $(5, 2, 1, 0)$. With the reverse choice, we get the solution $(3, 2, 0, 1)$. These form a basis for the null space.

Once we know that the null space has dimension 2, we deduce that the range has dimension 2 as well. In fact, it consists of the space of triples (a, b, c) with $c = a + b$. A basis would be the set containing $(1, 0, 1)$ and $(0, 1, 1)$. Of course, there are other correct answers: bases aren't unique! Let me stress that $R(T)$ lies in 3-space. If your answer consists of vectors in $\mathbf{R}^4$, you've messed up.

Note from Ribet: An answer that just has a bunch of numbers with no explanation as to what is going on is very unlikely to receive full credit. You need to tell the reader (me, in this case) what you are doing.

2. (9 points) Let V be a vector space over a field F. Suppose that $v_1, \ldots, v_n$ are elements of V and that $w_1, \ldots, w_n, w_{n+1}$ lie in the span of $\{ v_1, \ldots, v_n \}$. Show that the set $\{ w_1, \cdots, w_{n+1} \}$ is linearly dependent.

Let W be the span of $\{ v_1, \ldots, v_n \}$. Then W is generated by n elements, so its dimension d is at most n (for example, by Theorem 1.9 on page 42). If the vectors w_i were linearly independent, the set $\{ w_1, \cdots, w_{n+1} \}$ could be extended to a basis of W. This is impossible because all bases of W have d elements.

3. (10 points) Let W_1 and W_2 be subspaces of a finite-dimensional F-vector space V. Recall that $W_1 \times W_2$ denotes the set of pairs (w_1, w_2) with $w_1 \in W_1$, $w_2 \in W_2$. This product comes equipped with a natural addition and scalar multiplication:

$$(w_1, w_2) + (w_1', w_2') := (w_1 + w_1', w_2 + w_2'), \qquad a(w_1, w_2) := (aw_1, aw_2).$$

This addition and scalar multiplication make $W_1 \times W_2$ into an F-vector space. *(There was no requirement or expectation that students verify this point.)*

(1) Check that the map

$$T : W_1 \times W_2 \to V, \quad (w_1, w_2) \mapsto w_1 + w_2$$

is a linear transformation.

This is fairly routine. For example, $T(a(w_1, w_2)) = T(aw_1, aw_2) = aw_1 + aw_2 = a(w_1 + w_2) = aT(w_1, w_2)$. A similar computation shows that T of a sum is the sum of the T's.

(2) Prove that $N(T) = 0$ if and only if $W_1 \cap W_2 = 0$.

The space $N(T)$ consists of pairs (w_1, w_2) with $w_1 + w_2 = 0$. Hence w_2 is completely determined by w_1 as its negative. The wrinkle is that w_1 has to be in W_1 while $w_2 = -w_1$ has to be in W_2. Thus w_1 has to be in both W_1 and W_2, i.e., in the intersection of the two spaces. The null space $N(T)$ is in 1-1 correspondence with $W_1 \cap W_2$, with $(w_1, w_2) \in N(T)$ corresponding to $w_1 \in W_1 \cap W_2$ and $w \in W_1 \cap W_2$ corresponding to $(w, -w) \in N(T)$. In particular, $N(T) = 0$ if and only if $W_1 \cap W_2 = 0$.

(3) Show that $\dim(W_1 \cap W_2) \geq \dim W_1 + \dim W_2 - \dim V$.

Consider $T : W_1 \times W_2 \to V$. We have

$$\dim(W_1 \times W_2) = \dim N(T) + \dim R(T) = \dim(W_1 \cap W_2) + \dim R(T).$$

Here, I have used the fact that the identification between $N(T)$ and $W_1 \cap W_2$ that we discussed above is a linear identification—one that respects addition and scalar multiplication. Hence it preserves dimension. For this problem, we have to use another fact, namely that $\dim(W_1 \times W_2))$ is the sum of the dimensions of W_1 and W_2. This follows from the *fact that we get a basis for the product by taking the union of* $\beta_1 \times \{0\}$ *and* $\{0\} \times \beta_2$*, where the* β_i *are bases of the* W_i *(*$i = 1, 2$*). This fact gives*

$$\dim(W_1 \times W_2) = \dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim R(T) \le \dim(W_1 \cap W_2) + \dim V.$$

The desired inequality follows.

4. (12 points) Label the following statements as being *true* or *false*. For each statement, explain your answer

(1) The span of the empty set is the empty vector space.

There is no empty vector space! The span of the empty set is $\{0\}$*.*

(2) If v is a vector in a vector space V that has more than two elements, then V is spanned by the set $S = \{\, w \in V \mid w \neq v \,\}$.

The span of S *contains* S*, so it contains all* w *different from* v*. Does it contain* v *as well? Yes, indeed: choose* $w \in V$ *different from 0 and* v*; this choice is possible because* V *has more than two elements. Write* $v = (v - w) + w$*. The vectors* $v - w$ *and* w *are both in* S*: neither is* v*. Hence* v *is in the span of* S*. Our conclusion is that the span of* S *contains all of* V*, so the assertion is true.*

(3) Suppose that $T : V \to W$ is a linear transformation between finite-dimensional $\mathbf{R}$-vector spaces. If $\dim V > \dim W$ and w lies in the range of T, then there are infinitely many $v \in V$ such that $T(v) = w$.

If w *lies in the range of* T*, then there is some* $v \in V$ *such that* $T(v) = w$*. Fix this* v*, and notice that* $T(v') = w$ *if and only if* $T(v' - v) = 0$*. Thus the set of* v' *mapping to* w *is the set of* $v + x$ *where* x *runs over* $N(T)$*. Thus there are infinitely many elements of* V *that map to* w *if and only if* $N(T)$ *is infinite. Since we are working over the field of real numbers, which is an infinite field,* $N(T)$ *is infinite if and only if it is non-zero. Were* $N(T)$ *zero, we would have* $\dim R(T) = \dim V$*. However,* $\dim R(T)$ *is at most* $\dim W$*, which is less than* $\dim V$*. Accordingly,* $N(T)$ *must be non-zero. The assertion is true.*

(4) Suppose that $T : V \to W$ is a linear transformation between finite-dimensional $\mathbf{R}$-vector spaces. If $\dim V < \dim W$ and w lies in the range of T, then there is exactly one $v \in V$ such that $T(v) = w$.

The assertion being made is not necessarily true; thus the best answer to the question is "false." For example, V *could be 1-dimensional,* W *could be 10-dimensional,* T *could be identically 0, and* w *could be 0. The set of* v *mapping to* w *would then be the entire vector space* V*, which is infinite in the situation that we're contemplating.*

Math 110 Professor K. A. Ribet

Midterm Exam February 22, 2005

This exam was an 80-minute exam. It began at 12:40PM. There were 4 problems, for which the point counts were 12, 8, 8 and 8. The maximum possible score was 36.

Please put away all books, calculators, electronic games, cell phones, pagers, .mp3 players, PDAs, and other electronic devices. You may refer to a single 2-sided sheet of notes. Explain your answers in full English sentences as is customary and appropriate. Your paper is your ambassador when it is graded. At the conclusion of the exam, please hand in your paper to your GSI.

According to the GSIs, there were 136 people at the exam. The median score is somewhere between 16.5 and 20. The average is something like 19.2. Here is a quick summary of how many papers were in different ranges of scores:

32.5–36: 5
28.5–32: 8
24.5–28: 21
20.5–24: 23
16.5–20: 35
12.5–16: 27
8.5–12: 13
4.5–8: 3
0–4: 1

If you want to know how your score might convert into a letter grade, you can recall the statement that I made about final grades in Fall, 2002: "I awarded 61 grades as follows: 15 As, 20Bs, 13Cs, 12Ds and 1F. Those students who took the course on a P/NP basis had their letter grades converted into either P or NP. I awarded Fs to those students who were signed up for the course but did not take the final exam. I believe that all of them left the course early on in the semester and had forgotten to drop the class."

1. *Label the following statements as TRUE or FALSE, giving a short explanation (e.g., a proof or counterexample). There are six parts to this problem, which* *continues on page 3.*

a. *If A and B are $n \times n$ matrices over F, then $AB = 0$ if and only if $BA = 0$.*

This looks severely FALSE to me. If I'm not mistaken, $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is non-zero and the product of these matrices taken in the other order is 0.

b. *If A and B are $n \times n$ matrices over the field of real numbers, then $AB = 2I_n$ if and only if $BA = 2I_n$.*

TRUE because we're saying that A is the left inverse of $\frac{1}{2}B$. See §2.4, problem 9 and my solution to it (on the web page). This works only because there is a $\frac{1}{2}$ in our field. If 2 were equal to 0 in our field, then we'd be back in part (b) of this question and the answer would be FALSE. If F were given as a field and there were no hypothesis about the invertibility of the number 2, then the answer would again be FALSE for the same reason.

c. *If $x, y \in V$ and $a, b \in F$, then $ax + by = 0$ if and only if x is a scalar multiple of y or y is a scalar multiple of x.*

This is complete nonesense; I was trying to imitate the spirit of our authors' questions. For example, a and b could both be 0. This one is FALSE.

d. *For $A \in M_{m\times n}(F)$, $[L_A]_\beta^\gamma = A$ if β and γ are the standard bases of F^n and F^m.*

TRUE, basically by definition of L_A and by the definition of how you compute the matrix of a linear transformation.

e. *Every vector space over F is isomorphic to F^n for some $n \geq 1$.*

This is FALSE because the vector space might be $\{0\}$, for instance. The vector space could also be infinite-dimensional.

f. *If v is a non-zero vector in a finite-dimensional vector space V, then there is a linear form $f \in V^*$ such that $f(v) = 1$.*

TRUE as explained in the proof of the lemma on page 122 of the book and in my lecture last week.

2. *Suppose that $T : V \to V$ is a linear transformation on an F-vector space V. Prove that $T^2 = 0$ if and only if the range of T is contained in the null space of T.*

This question was put on the exam in order to give you a hint for the question that follows. To say that the range is contained in the null space is to say that T vanishes on all vectors of the form Tv with $v \in V$, i.e., that $T(Tv) = 0$ for all v. Thus the range is contained in the null space if and only if $T^2(v) = 0$ for all $v \in V$, which is the statement that $T^2 = 0$.

Assume that V is finite-dimensional and that $T^2 = 0$. Prove that $\text{nullity}(T) \geq (\dim V)/2$.

Under the assumption $T^2 = 0$, the nullity of T is at least as big as the rank of T, since the range of T is contained in the null space of T. Since the rank plus the nullity sum up to $\dim V$, twice the nullity is at least as big as $\dim V$.

3. *Let $V = P(\mathbf{R})$ be the vector space of polynomials with real coefficients. For each $i \geq 0$, let $f_i \in \mathcal{L}(V, \mathbf{R})$ be the linear transformation that maps a polynomial $p(x)$ to the value $p^{(i)}(0)$ of its ith derivative at 0. (The 0th derivative of p is p itself.) Show that the linear functionals $f_0, f_1, f_2, \ldots$ are linearly independent. [Evaluate linear combinations of the f_i on powers of x.]*

It's important here to understand what is meant by the linear independence of an infinite set of elements of a vector space (which in this case we take to be $\mathcal{L}(V, \mathbf{R})$): it means that there is no non-trivial linear relation among a finite set of the elements. To say that the infinite set $\{f_0, f_1, f_2, \ldots\}$ is linearly independent is to say that a linear combination $a_1 f_1 + \cdots a_n f_n$ (with $n \geq 1$) can vanish only when all of the coefficients a_i are 0. Assume that we have $0 = a_1 f_1 + \cdots a_n f_n$. For each $i = 1, 2, \ldots, n$, evaluate $a_1 f_1 + \cdots a_n f_n$ at x^i, as suggested by the hint. The point is that $f_j(x^i)$ is $i!$ if $j = i$ and 0 otherwise. We thus get $a_i i!$ when we evaluate $a_1 f_1 + \cdots a_n f_n$ at x^i. Since this value is 0 because $0 = a_1 f_1 + \cdots a_n f_n$, we obtain $a_i = 0$ on dividing by $i!$.

Show that the linear transformation $g : p(x) \mapsto p(1)$ is not in the span of the f_i.

This is similar to the first part of the problem. Suppose that g is in the span; then it's equal to $a_1 f_1 + \cdots a_n f_n$ for some choice of n and some coefficients a_i. If we exaluate $a_1 f_1 + \cdots a_n f_n$ on x^{n+1}, we get 0. If we evaluate g on x^{n+1}, we get 1. Since 1 and 0 are not equal, we have a contradiction.

4. *Let $T : V \to W$ be a linear transformation between F-vector spaces. Suppose that $x_1, \ldots, x_r$ are linearly independent elements of $N(T)$ and that $v_1 \ldots, v_s$ are vectors in V such that $T(v_1), \ldots, T(v_s)$ are linearly independent. Show that the $r + s$ vectors $x_1, \ldots, x_r; v_1, \ldots, v_s$ are linearly independent.*

This question was inspired by my solution to problem 16 in §2.2. Suppose that we have a linear dependence relation among the vectors $x_1, \ldots, x_r$, $v_1, \ldots, v_s$: $0 = \sum a_i x_i + \sum b_j v_j$. Take T of both sides; we get $0 = \sum b_j T(v_j)$ because $T(x_i) = 0$ for all i. Because the v_j are linearly independent, all b_j are 0. Thus we have $0 = \sum a_i x_i$. Since the x_i are linearly independent, the a_i are 0 as well.

Math 110 — Professor K. A. Ribet

Midterm Exam — September 25, 2008

This was an 80-minute exam, 3:40–5PM. There were 30 points on the test, with two questions being worth 7 points and two being worth 8 points. The explanations that follow are intended to communicate the main points of each problem but might be a little skeletal. (They're more like extended hints than complete solutions.)

1. *Let $\mathcal{P}(\mathbf{R})$ be the real vector space consisting of all polynomials with coefficients in $\mathbf{R}$. Let $D : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ be the linear map $f(x) \mapsto f'(x)$ that takes a polynomial to its derivative.*

(**a.**) *Describe the null space and the range of D.*

The null space is the space of polynomials with derivative $= 0$. Those are the constant polynomials. The range is all of $\mathcal{P}(\mathbf{R})$ since every polynomial is the derivative of some polynomial. (We know how to integrate.)

(**b.**) *Find a subspace U of $\mathcal{P}(\mathbf{R})$ such that $\mathcal{P}(\mathbf{R}) = U \oplus \operatorname{null} D$.*

We can take U to be the space of polynomials whose constant terms are 0. The intersection of U and $\operatorname{null} D$ is $\{0\}$ because the only constant in U is the polynomial 0. The sum of U and the null space is all of $\mathcal{P}(\mathbf{R})$ because each polynomial is the sum of its constant term and a polynomial whose constant term is 0.

2. *Let $T : V \to W$ be a linear map between $\mathbf{F}$-vector spaces.*

(**a.**) *Suppose that $\operatorname{null} T = \{0\}$ and that $(v_1, \dots, v_n)$ is a linearly independent list in V. Show that $(Tv_1, \dots, Tv_n)$ is linearly independent in W.*

I think that this is a good problem because it requires knowledge of the definitions and some proof-writing skills. To show that $(Tv_1, \dots, Tv_n)$ is linearly independent in W, we start with the equation $0 = a_1Tv_1 + a_2Tv_2 + \cdots + a_nTv_n$ and seek to show that the coefficients a_i are all 0. Because T is linear, we can rewrite the equation as

$$0 = T(a_1v_a + \cdots + a_nv_n).$$

Since $\operatorname{null} T = \{0\}$, the vector $a_1v_a + \cdots + a_nv_n$ is 0. Since $(v_1, \dots, v_n)$ is a linearly independent list in V, all a_i are 0.

(**b.**) *Assume that $(Tv_1, \dots, Tv_n)$ is linearly independent in W for all linearly independent lists $(v_1, \dots, v_n)$ in V. Show that $\operatorname{null} T = \{0\}$.*

The assumption is that T sends independent lists to independent lists—there is no restriction on the size of the lists. In particular, it sends independent lists of length 1 to independent lists of length 1. Note that (v) is linearly independent if and only if v is non-zero! Hence if v is non-zero, (v) is linearly independent; thus (Tv) is linearly independent, so Tv is non-zero. Therefore the null space of T is $\{0\}$, which s what we wanted to prove.

3. *Let V be an* $\mathbf{F}$*-vector space such that* $\dim U \leq 4$ *for all finite-dimensional subspaces U of V. Prove that V is finite-dimensional and that its dimension is at most 4.*

This is kind of a silly problem (sorry). If V is *not* finite-dimensional, then there are independent lists $(v_1, \ldots, v_n)$ in V of arbitrary length. (We proved this in homework and discussed the proof in office hours a lot. This is a good place to cite a homework problem; alternatively, you could recapitulate the proof that there are such lists.) If $(v_1, \ldots, v_n)$ is linearly independent in V, its span is a subspace of V of dimension n. By the assumption of the problem, n can be at most 4, so we have a contradiction.

4. *Let $S : V \to W$ and $T : W \to V$ be linear maps between finite-dimensional* $\mathbf{F}$*-vector spaces. Suppose that TS is the identity map on V.*

(**a.**) *Prove that T is surjective (onto) and that S is injective (1-1).*

This has little to do with linear algebra: it's just a fact about composed functions. The hypothesis is that $T(S(v)) = v$ for all v in V. We see that the range of T is all of v because each v is T of something, namely v is $T(S(v))$. In a similar vein, if $S(v) = S(v')$, we get $v = v'$ by applying T to both sides of the equation $S(v) = S(v')$. Hence S is indeed 1-1.

(**b.**) *Show that we have* $\dim V \leq \dim W$.

We can use, for instance, the fact that the dimension of V is the sum of the nullity of S and the rank of S. Since S is 1-1, the nullity is 0; thus $\dim V = \operatorname{rank} S$. Since the range of S is a subspace of W, the dimension of the range is at most the dimension of W. In other words, as we discussed in class, $\operatorname{rank} S \leq \dim W$. We thus have the desired inequality $\dim V \leq \dim W$.

Math 110

First Midterm Examination
February 16, 2010
2:10–3:30 PM, 10 Evans Hall

Please put away all books, calculators, and other portable electronic devices—anything with an ON/OFF switch. You may refer to a single 2-sided sheet of notes. For numerical questions, *show your work* but do not worry about simplifying answers. For proofs, write your arguments in complete sentences that explain what you are doing. Remember that your paper becomes your only representative after the exam is over.

Problem	Your score	Possible points
1		5 points
2		12 points
3		6 points
4		7 points
Total:		30 points

1. In $\mathbf{R}^3$, express $(3, 18, -11)$ as a linear combination of $(1, 2, 3)$, $(-2, 0, 3)$ and $(2, 4, 1)$.

This was a standard numerical problem of the type that most of you know how to do. The coefficients are: $-49/5$, 3 and $47/5$. I apologize for the fractions: I intended the answers to be whole numbers and must have mistyped.

2. Label each of the following statements as TRUE or FALSE. Along with your answer, provide an informal proof or an explanation. For false statements, an explicit counterexample might work best. In interpreting the statements, take v to be a vector, a to be a scalar, β to be a basis of V, etc., etc.

Each part was worth 2 points. We gave out one point for the correct T/F answer and one point for the explanation.

a. If $av = v$, then either $a = 1$ or $v = 0$.

This is true, but a lot of you didn't give a good reason. Since $v = 1 \cdot v$, as proved in class, the equation $av = v$ may be written $(a - 1) \cdot v = 0$. If the scalar $a - 1$ is non-zero, we may divide by it (i.e., multiply by its inverse) and get $v = 0$. In other words, if a isn't 1, v is 0. This means that we have $a = 1$ or $v = 0$, or both.

b. If A and B are real 3×3 matrices, the formula $T(M) = AM - MB$ defines a linear map $\mathsf{M}_{3\times 3}(\mathbf{R}) \to \mathsf{M}_{3\times 3}(\mathbf{R})$.

This is true because of the distribution relations for matrix multiplication. For example, to see that $T(M + M') = T(M) + T(M')$, we have to expand out $A(M + M') + (M + M')B$ and rearrange terms.

c. If V is spanned by a set of 6 distinct vectors, all bases of V have exactly 6 vectors.

This is false. For example, the 1-dimensional $\mathbf{R}$-vector space $\mathbf{R} = \mathbf{R}^1$ is spanned by the 6 distinct elements 1, 2, 3, 4, 5 and 6, but all bases of this space have one element!

d. If W is a subspace of a finite-dimensional vector space V and $w_1, \ldots, w_m$ form an ordered basis of W, then every basis of V includes $w_1, \ldots, w_m$.

This, again, is silly. Take W to be the subspace of $\mathbf{R}^2$ generated by $(1, 0)$, so that $(1, 0)$ is a basis of W. You can find lots of bases of $\mathbf{R}^2$ that do not contain $(1, 0)$. One such basis consists of $(1, 1)$ and $(0, 1)$.

e. In $\mathcal{L}(F^6, F^4)$, one may find linear transformations T for which the dimensions of $\mathsf{N}(T)$ are 2, 3, 4, 5 and 6.

This is true. Just make up 6×4 matrices of 0s and 1s with exactly i linearly independent columns, where i takes each of the values 0, 1, 2, 3 and 4.

f. If $m = \dim(V)$ and $n = \dim(W)$, then $[T]_\beta^\gamma$ is an $n \times m$ matrix. (Here T is a linear map $V \to W$.)

Well, this is just true, by definition of $[T]_\beta^\gamma$. There was a nearly identical question in the homework, but the HW answer was "false." I exchanged m and n and made the statement true instead!

3. Suppose that V is an F-vector space with at least three vectors. Let w be a vector in V. Prove that V is spanned by the set $S = \{\, v \in V \mid v \neq w \,\}$.

This caused a lot of trouble, sorry. The span of S certainly contains all vectors in S. There's only one vector in V that isn't in S, namely w. Therefore, to prove that the span of S is all of V, we just have to prove that w is in the span! For every v in the vector space, we have

$$w = (w - v) + v.$$

This will write w in the span of S provided that the two summands v and $w - v$ are in S. To have v in S, we need to have $v \neq w$. To have $w - v \in S$, we need to have $v \neq 0$. Since V has more than two vectors, there is a $v \in V$ different from both 0 and w. Take such a v and we're home.

4. Let $f(x)$ be a polynomial of degree n with real coefficients. Prove that the $n+1$ polynomials

$$f(x), f'(x), f''(x), \ldots, f^{(n)}(x)$$

are linearly independent. Conclude that they span $\mathsf{P}_n(\mathbf{R})$.

For the first statement, there are several correct proofs. One way to proceed is to realize that the last (i.e., nth) derivative is a non-zero constant because $f(x)$ has degree n. Hence the set consisting of the last vector $f^{(n)}(x)$ is linearly independent. The vector $f^{(n-1)}(x)$ then has degree 1, so it can't be a multiple of the vector $f^{(n)}(x)$. Thus the two vectors $f^{(n)}(x)$ and $f^{(n-1)}(x)$ form a linearly independent set. We proceed in this manner, incrementing the number of vectors in the set that we are proving to be linearly independent. At one stage we have seen that $f^{(n)}(x), f^{(n-1)}(x), \ldots, f^{(n-i)}(x)$ make a linearly independent set and ask whether the larger set $f^{(n)}(x), f^{(n-1)}(x), \ldots, f^{(n-i)}(x), f^{(n-i-1)}(x)$ is also linearly independent. If not, then $f^{(n-i-1)}(x)$ will be a linear combination of $f^{(n)}(x), f^{(n-1)}(x), \ldots, f^{(n-i)}(x)$. You can see that this is impossible because the degree of $f^{(n-i-1)}(x)$ is larger than the degrees of the polynomials $f^{(n)}(x), f^{(n-1)}(x), \ldots, f^{(n-i)}(x)$.

Maybe a better way to proceed is to start with

$$a_0 f(x) + \cdots + a_n f^{(n)}(x) = 0 \tag{1}$$

and to prove in turn that each of the a_i is 0. Note that all derivatives of $f(x)$ after the nth derivative are 0. If we differentiate (1) n times, all terms but the first disappear; we get $a_0 f^{(n)}(x) = 0$. Since the nth derivative is non-zero (it's in fact a non-zero constant), we get $a_0 = 0$. Hence the first term in (1) is really $a_1 f'(x)$. Now differentiate (1) $n-1$ times instead of n times; we get $a_1 f^{(n)}(x) = 0$, so $a_1 = 0$. We continue in this fashion, knocking off the terms one by one. At the end of the game, (1) has only one term left: it reads $a_n f^{(n)}(x) = 0$. We get $a_n = 0$, so all coefficients are 0, as was required.

For the second statement, which was worth two points, you just have to say that the polynomials $f(x), f'(x), f''(x), \ldots, f^{(n)}(x)$ are known to be linearly independent by the first part. There are $n+1$ of them, and $n+1$ is the dimension of $\mathsf{P}_n(\mathbf{R})$. By a corollary to the theorem that many of you knew the number of, the linearly independent set

$$\{ f(x), f'(x), f''(x), \ldots, f^{(n)}(x) \}$$

is actually a basis of $\mathsf{P}_n(\mathbf{R})$.

MATH 110, solutions to the mock midterm.

1. Consider the vector space $P(\mathbb{R})$ and the subsets V consisting of those vectors (polynomials) f for which:

(a) f has degree 3,

(b) $2f(0) = f(1)$,

(c) $f(t) \geq 0$ whenever $t \geq 0$,

(d) $f(t) = f(1-t)$ for all t.

In which of these cases is V a subspace of $P(\mathbb{R})$?

Solution. (a) This is not a subspace of $P(\mathbb{R})$. Indeed, adding or subtracting two polynomials of exact degree 3 may result in a polynomial of a smaller degree, e.g., if $f(x) = x^3 - 3x$ and $g(x) = x^3$, then $g(x) - g(x) = 3x$, and $\deg(g - f) = 1$. (Another observation that shows that V is not a subspace is that the zero polynomial 0 is not in V.)

(b) This is a subspace of $P(\mathbb{R})$. The zero polynomial satisfies the defining condition $2f(0) = f(1)$. Also, if $2f(0) = f(1)$ and $2g(0) = g(1)$, then

$$2(\alpha f + \beta g)(0) = \alpha f(0) + \beta g(0) = \alpha f(1) + \beta g(1) = (\alpha f + \beta g)(1)$$

for all α, $\beta \in \mathbb{R}$. So V is a subspace of $P(\mathbb{R})$.

(c) This is not a subspace, the (additive) inverse of a function that is nonnegative over $\mathbb{R}_+$ is nonpositive over $\mathbb{R}_+$. E.g., if $f(x) = x^2$, then $f \in V$, but $-f \notin V$. Thus V is not a subspace of $P(\mathbb{R})$.

(d) This is a subspace of $P(\mathbb{R})$. Indeed, the zero polynomial satisfies the condition $f(t) = f(1-t)$. If two functions, f and g satisfy this condition, then so are all their linear combinations $\alpha f + \beta g$, i.e.,

$$(\alpha f + \beta g)(t) = \alpha f(t) + \beta g(t) = \alpha f(1-t) + \beta g(1-t) = (\alpha f + \beta g)(1-t).$$

So V is a subspace of $P(\mathbb{R})$.

2.

$$\text{Let } A = \begin{bmatrix} 0 & 2 & 3 \\ -1 & 3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 0 \\ 6 & 4 \\ -4 & 6 \end{bmatrix}, \quad v = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad w = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}.$$

Do the products Aw, $B^t v^t$, vAw exist? Evaluate those that do. Is the set $\{A, B^t\}$ linearly independent?

Solution. The matrix A is of size 2×3 whereas w is of size 3×1. Since the inner dimensions agree, the product Aw exists. The sizes of the pair B^t, v^t are exactly the same, so the product $B^t v^t$ exists as well. Now, the product Aw is of size 2×1, but the size of v is 3×1, so the inner dimensions 1 and 2 disagree, and the product $vAw = v(Aw)$ does not exist. The products Aw and $B^t v^t$ are equal to

$$Aw = \begin{bmatrix} 4 \\ -7 \end{bmatrix}, \qquad B^t v^t = (vB)^t = \begin{bmatrix} -2 & 26 \end{bmatrix}^t = \begin{bmatrix} -2 \\ 26 \end{bmatrix}.$$

The set $\{A, B^t\}$ is linearly independent: Assume $\alpha A + \beta B^t = 0_{2\times 3}$, i.e.,

$$\alpha \begin{bmatrix} 0 & 2 & 3 \\ -1 & 3 & -2 \end{bmatrix} + \beta \begin{bmatrix} -2 & 6 & 4 \\ 0 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the $(1,1)$ entry is zero, we conclude that $\beta = 0$, and since the $(2,1)$ entry is zero, we get $\alpha = 0$. So, the matrices A and B^t are linearly independent.

3.

$$\text{Let } A = \begin{bmatrix} 0 & 2 \\ 2 & -2 \end{bmatrix}, \quad \beta = \{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \}.$$

Find the representation $[L_A]_\beta$, the dual basis β^*, and the matrix $[(L_A)^t]_{\beta^*}$.

Solution. The map L_A acts on β as follows :

$$L_A\beta = \{ \begin{bmatrix} 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \}.$$

Since

$$\begin{bmatrix} 2 \\ -4 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ 0 \end{bmatrix} = - \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

we conclude

$$[L_A]_\beta = \begin{bmatrix} -3 & -1 \\ -1 & 1 \end{bmatrix}.$$

The dual basis for β is

$$\mathbf{f}_1(x, y) = -x/2 + y/2, \qquad \mathbf{f}_2(x, y) = x/2 + y/2,$$

and by the theorem on duals of linear maps,

$$[(L_A)^t]_\beta = \begin{bmatrix} -3 & -1 \\ -1 & 1 \end{bmatrix}.$$

4. Let

$$A \;:\; P_n(\mathbb{R}) \to P_n(\mathbb{R}) \;:\; (Af)(t) := f(t+1).$$

Prove that

$$A = I + \frac{D}{1!} + \frac{D^2}{2!} + \cdots + \frac{D^n}{n!},$$

where D is the differentiation operator on $P_n(\mathbb{R})$.

Proof. Since any polynomial is infinitely differentiable, we can apply Taylor's formula of any order to expand $f(t+1)$ around the point t. By using Taylor's formula of order n, we obtain

$$f(t+1) = f(t) + \frac{f'(t)}{1!} + \frac{f''(t)}{2!} + \cdots + \frac{f^{(n)}}{n!} + \frac{f^{(n+1)}(\xi)}{(n+1)!}, \tag{1}$$

where $\xi \in (t, t+1)$. But it f is a polynomial of degree n or smaller, then $f^{(n+1)}(\xi) = 0$ for any point $\xi \in \mathbb{R}$. So the last term in (1) disappears and we see that

$$Af = (I + \frac{D}{1!} + \cdots + \frac{D^n}{n!})f \quad \text{for all } f \in P_n(\mathbb{R}). \qquad \square$$

5. Let $m < n$ and let $\mathbf{f}_1, \ldots, \mathbf{f}_m$ be linear functionals on an n-dimensional space V. Prove that there exists a nonzero vector $x \in V$ such that $\mathbf{f}_j x = 0$ for all $j = 1, \ldots, m$. What does this result say about solutions of linear equations?

Proof. Without loss of generality we may assume that the set $\{\mathbf{f}_j : j = 1, \ldots, m\}$ is linearly independent, since the result for linearly dependent functionals will follow from the result for linearly independent functionals.

So, assuming independence, complete $\{\mathbf{f}_j : j = 1, \ldots, m\}$ to a basis $\{\mathbf{f}_j : j = 1, \ldots, n\}$ of V^*, and consider its dual basis $\{u_j : j = 1, \ldots, n\}$ of V. Since $\mathbf{f}_j u_i = \delta_{ij}$, we see that a nonzero vector $x := u_n$ satisfies the required condition $\mathbf{f}_j x = 0$ for all $j = 1, \ldots, m$. $\square$

Since the action of a linear functional on $\mathbb{R}^n$ is realized as multiplication from the left by a row vector, this result says the following: Any linear homogeneous system with fewer equations than unknowns has a nontrivial solution.

6. Reduce the matrix

$$\begin{bmatrix} 1 & -1 & 4 & 3 & -2 & -2 \\ 0 & 2 & 0 & 1 & 1 & 3 \\ -1 & 3 & -4 & -2 & 3 & 5 \end{bmatrix}$$

to its reduced row echelon form. Show all steps.

Solution. To create 0 in position $(3, 1)$, add the first row to the third row. This gives

$$\begin{bmatrix} 1 & -1 & 4 & 3 & -2 & -2 \\ 0 & 2 & 0 & 1 & 1 & 3 \\ 0 & 2 & 0 & 1 & 1 & 3 \end{bmatrix}.$$

To create zero in position $(3, 2)$, subtract the second row from the third to get

$$\begin{bmatrix} 1 & -1 & 4 & 3 & -2 & -2 \\ 0 & 2 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

To make the $(2,2)$ entry equal to 1, divide the second row by 2 to get

$$\begin{bmatrix} 1 & -1 & 4 & 3 & -2 & -2 \\ 0 & 1 & 0 & 1/2 & 1/2 & 3/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally, to create 0 in position $(1,2)$, add the second row to the first row to obtain

$$\begin{bmatrix} 1 & 0 & 4 & 7/2 & -3/2 & -1/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & 3/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This matrix is in reduced row echelon form.

Math 110 - Fall 2003 - Haiman
Midterm 1 Solutions

1. (5 points each) Determine whether each of the following assertions is *true* or *false*. Give a brief explanation for each answer (full proof is not required).

(a) If a linear transformation $T\colon V \to W$ between finite-dimensional vector spaces is 1-to-1, then $\dim(V) \leq \dim(W)$.

True. Since $\text{nullity}(T)=0$, $\dim(V) = \text{rank}(T)$ by the dimension theorem. And $\text{rank}(T) \leq \dim(W)$ since $R(T) \subseteq W$.

(b) If V and W are finite-dimensional vector spaces such that $\dim(V) \leq \dim(W)$, and $T\colon V \to W$ is a linear transformation, then T is 1-to-1.

False. A counterexample is the zero map $T: \mathbb{R}^n \to \mathbb{R}^n$ for any $n>0$.

(c) The set of vectors (x_1, x_2, x_3, x_4) which satisfy $x_1 = x_4$ and $x_2 = x_3$ is a subspace of $\mathbb{R}^4$.

True. The simplest reason is that part (d) is also true.

(d) The set of vectors in part (c) is the nullspace of a linear transformation from $\mathbb{R}^4$ to some vector space over $\mathbb{R}$.

True. It's the nullspace of $T: \mathbb{R}^4 \to \mathbb{R}^2$ defined by $T((x_1, x_2, x_3, x_4)) = (x_1 - x_4, x_2 - x_3)$. (It's easy to check that T is linear, but you need not do so to get full credit on the problem.)

(e) The set of vectors in part (c) is the nullspace of a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}$ (in other words, a linear functional).

False. Since T in part (d) is onto, it has $\text{rank}(T)=2$, and therefore its nullspace, which is the set of vectors in (c), has dimension 2. But the nullspace of any linear $S: \mathbb{R}^4 \to \mathbb{R}$ has dimension ≥ 3 by dimension theorem.

(f) $\mathbb{Q}^n$ is a subspace of the vector space $\mathbb{R}^n$ over $\mathbb{R}$. ($\mathbb{Q}$ denotes the field of rational numbers.)

False. Not closed under scalar multiplication by irrational scalars

(g) $\mathbb{Q}^n$ is a subspace of $\mathbb{R}^n$ considered as a vector space over $\mathbb{Q}$ (with the usual addition, and multiplication by rational scalars).

True. Clearly closed under addition, and closed under scalar multiplication since $(ax_1, \dots, ax_n) \in \mathbb{Q}^n$ if a and all x_i are rational.

2. Let S be the following subset of $P(\mathbb{R})$:

$$S = \{f(x) = x^5 + x^2,\ g(x) = x^5 + 2,\ h(x) = x^3,\ j(x) = x^2 - 2\}$$

(a) (30 points) Find a subset of S which is a basis of $\mathrm{Span}(S)$ and prove that your answer is correct.

There are three possible correct answers: any subset consisting of $h(x)$ and two elements from $\{f(x), g(x), j(x)\}$. I'll prove that $B = \{f(x), g(x), h(x)\}$ is a basis. The proof for the other bases is similar.

First we'll show $\mathrm{Span}(B) = \mathrm{Span}(S)$. Since $B \subseteq S$, $\mathrm{Span}(B) \subseteq \mathrm{Span}(S)$, and to prove $\mathrm{Span}(S) \subseteq \mathrm{Span}(B)$, since $\mathrm{Span}(B)$ is a subspace, it's enough to prove $S \subseteq \mathrm{Span}(B)$. Thus we only need to show that $j(x)$ is in $\mathrm{Span}(B)$, which is true because

$$j(x) = f(x) - g(x).$$

Now we'll show B is linearly independent. Suppose a, b, c are scalars such that

$$af(x) + bg(x) + ch(x) = 0$$

(identically as polynomials). The left-hand side is

$$(a+b)x^5 + cx^3 + ax^2 + 2b.$$

For this to be the 0 polynomial we must have $a = b = c = 0$.

(b) (5 points) Find $\dim(\mathrm{Span}(S))$.

$$\dim(\mathrm{Span}(S)) = |B| = 3.$$

3. (30 points) Let $T\colon V \to W$ be a linear transformation. Prove that if T is 1-to-1, and $v_1, \ldots, v_k \in V$ are linearly independent, then $T(v_1), \ldots, T(v_k)$ are linearly independent.

Suppose $a_1 T(v_1) + a_2 T(v_2) + \ldots + a_k T(v_k) = 0$.

Since T is linear, the left-hand side is equal to

$$T(a_1 v_1 + a_2 v_2 + \ldots + a_k v_k).$$

Since T is 1-to-1, the fact that this is zero implies

$$a_1 v_1 + a_2 v_2 + \ldots + a_k v_k = 0.$$

Finally, since $v_1, \ldots, v_k$ are linearly independent, this implies that all coefficients a_i are zero.

Hence $T(v_1), \ldots, T(v_k)$ are linearly independent.