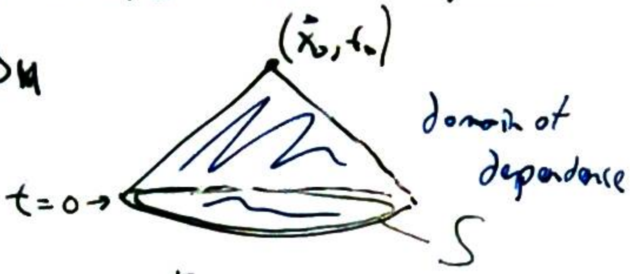
7/26/18 Lecture Notes: Solutions to 3D + 2D Wave Equation

Last time: $u_{tt} = c^2 \Delta u$




This time: Prove Kirchoff's Formula

$$\boxed{u(\vec{x}_0, t_0) = \frac{1}{4\pi c^2 t_0} \iint_S \psi(\vec{x})\, dS + \frac{\partial}{\partial t_0}\left[\frac{1}{4\pi c^2 t_0} \iint \phi(\vec{x})\, dS\right]}$$

Where $u_{tt} = \Delta u$

$u(\vec{x}, 0) = \phi(\vec{x})$ $\qquad S = \{ |\vec{x} - \vec{x}_0| = c t_0 \} \subseteq \mathbb{R}^3_{\vec{x}}$

$u_t(\vec{x}, 0) = \psi(\vec{x})$ $\qquad$ can think of as occuring at $t=0$

Kirchoff's formula + definition of $S \Longrightarrow$ Huygen's principle: Waves in 3D travel at exactly speed $c$

Proof of Kirchoff's Formula (method of spherical means)

Ideas: 1) The terms above are averages over spheres.

2) $u(\vec{x}_0, t_0)$ should be influenced by all points of $S$ equally.

Simplifications: $c = 1$ changing units/variables

$\vec{x}_0 = 0$ translation invariance

Let $\bar{u}(r,t) =$ average value of $u$ on sphere of radius $r$ at time $t$.

$$\bar{u}(r,t) = \frac{1}{4\pi r^2} \iint_{|\vec{x}|=r} u(\vec{x},t)\, dS = \frac{1}{4\pi} \int_0^{2\pi}\int_0^{\pi} u(\vec{x},t) \sin\theta \, d\theta\, d\phi$$

Want To Show: $(\bar{u})_{tt} = (\bar{u})_{rr} + \frac{2}{r}(\bar{u})_r$ (LOL) (i.e. find PDE for $\bar{u}$)

By the Divergence Theorem,

$$\iiint_{|\vec{x}|\le r} u_{tt}\, d\vec{x} = \iiint_{|\vec{x}|\le r} \Delta u\, d\vec{x} = \iint_{|\vec{x}|=r} \frac{\partial u}{\partial n}\, dS$$

In spherical coordinates,

$$\iiint_{|\vec{x}|\le r} u_{tt}\, d\vec{x} = \int_0^r \rho^2 \left[ \int_0^{2\pi}\int_0^{\pi} u_{tt} \sin\theta\, d\theta\, d\phi \right] d\rho = \int_0^{2\pi}\int_0^{\pi} \frac{\partial u}{\partial r}\, r^2 \sin\theta\, d\theta\, d\phi$$

$$\int_0^r \rho^2 (\bar{u})_{tt}\, d\rho = r^2 \frac{\partial \bar{u}}{\partial r} \quad \partial/\partial r$$

$$\boxed{r^2 \bar{u}_{tt} = r^2 \bar{u}_{rr} + 2r\bar{u}_r}$$

To solve (LOL), replace $v(r,t) = r\bar{u}(r,t)$, so

$$v_r = r\bar{u}_r + \bar{u} \qquad v_{tt} = r\bar{u}_{tt}$$

$$v_{rr} = r\bar{u}_{rr} + 2\bar{u}_r$$

← reintroduce $c^2$ because why not

$$\begin{array}{l} v(r,0) = r\bar{u}(r,0) = r\bar{\phi}(r) \\ v_t(r,0) = ru_t(r,0) = r\bar{\psi}(r) \end{array} \longrightarrow \begin{array}{l} v_{tt} = c^2 v_{rr} \\ v(0,t) = 0 \end{array} \quad 0 < r < \infty$$

Wave Equation on the Half-Line!

2 Formulas – Know to use one for $r \le ct$ because we already established domain of dependence last time

Solution: $$v(r,t) = \frac{1}{2c}\int_{ct-r}^{ct+r} \underbrace{s\bar{\psi}(s)}_{v_t(r,0)}\, ds + \frac{\partial}{\partial t}\left[\frac{1}{2c}\int_{ct-r}^{ct+r} \underbrace{s\bar{\phi}(s)}_{v(r,0)}\, ds\right]$$

Can check that 2nd term gives usual $\frac{1}{2}\{\phi(r+ct) - \phi(ct-r)\}$

Solve for u: $$u(\vec{0},t) = \bar{u}(0,t) = \lim_{r\to 0} \frac{v(r,t)}{r}$$

$$= \lim_{r\to 0} \frac{v(r,t) - v(0,t)}{r} = v_r(0,t)$$

$$v_r = \frac{1}{2c}\left[(ct+r)\bar{\psi}(ct+r) + (ct-r)\bar{\psi}(ct-r)\right] + \frac{1}{2c}\frac{\partial}{\partial t}\left[(ct+r)\bar{\phi}(ct+r) + (ct-r)\bar{\phi}(ct-r)\right]$$

$$u(\vec{0},t) = v_r(0,t) = t\,\overline{\Psi}(ct) + \frac{\partial}{\partial t}\left[t\,\overline{\Phi}(ct)\right]$$

$$= \frac{1}{4\pi c^2 t}\iint_{|\vec{x}|=ct} \psi(\vec{x})\,dS + \frac{\partial}{\partial t}\left\{\frac{1}{4\pi c^2 t}\iint_{|\vec{x}|=ct} \phi(\vec{x})\,dS\right\}$$ ✓

In 2D: BAD Idea: Repeat whole process and get stuck finding PDE for $\overline{u}$

GOOD Idea: Use 3D formula on solution independent of $z$.

(If $\phi, \psi$ independent of $z$, so is $u$ by Kirchhoff's formula)

Solve $u_{tt} = c^2 \Delta u$ in 3D    No dependence on $z$, so write

$u(x,y,z,0) = \phi(x,y)$    $u = u(x,y,t)$

$u_t(x,y,z,0) = \psi(x,y)$

$$\frac{1}{4\pi c^2 t}\iint_{|\vec{x}-\cancel{\vec{x}_0}|=ct} \psi(x,y)\,dS = \frac{1}{2\pi c^2 t}\iint_{\text{half sphere}} \psi(x,y)\,dS$$ parametrize in $x,y$

take $\vec{x}_0 = \vec{0}$ for simplicity

$$= \frac{1}{2\pi c^2 t}\iint_{x^2+y^2 \le c^2t^2} \psi(x,y)\left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{1/2} dx\,dy$$

$$1 + \left(\frac{-x}{z}\right)^2 + \left(\frac{-y}{z}\right)^2 = \frac{x^2+y^2+z^2}{z^2} = \frac{c^2t^2}{c^2t^2 - x^2 - y^2}$$

$$u(x_0,y_0,t_0) = \iint_D \frac{\psi(x,y)}{\left[c^2t_0^2 - (x-x_0)^2 - (y-y_0)^2\right]^{1/2}}\frac{dx\,dy}{2\pi c} + \frac{\partial}{\partial t_0}\left[\iint_D \frac{\phi(x,y)}{\left[c^2t_0^2 - (x-x_0)^2 + (y-y_0)^2\right]^{1/2}}\frac{dx\,dy}{2\pi c}\right]$$

where $D = \left\{(x-x_0)^2 + (y-y_0)^2 \le c^2 t_0^2\right\}$

Huygen's Principle fails in 2D (and actually all odd dimensions)