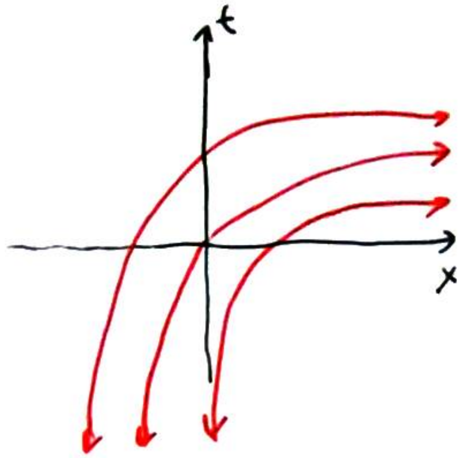
8/6/18 Lecture Notes: Shock Waves

Review: Chapter 1 Characteristics

solve $u_t + e^{x+t} u_x = 0 \qquad u(x,0) = \phi(x)$

$\nabla u \cdot \langle 1, e^{x+t} \rangle = 0$



$u(x,t)$ is constant along curves $x(t)$ satisfying

$$\frac{dx}{dt} = e^{x+t} \longrightarrow e^{-x} dx = e^{t} dt$$

$$e^{-x} + e^{t} = C$$

So, $u(x,t) = f(C) = f(e^{-x} + e^{+t})$ (curves given by $x(t) = -\log(C - e^t)$)

$u(x,0) = f(e^{-x}+1) = \phi(x)$

$s = e^{-x} + 1$

$s - 1 = e^{-x}$

$x = -\log(s-1) \rightarrow f(s) = \phi(-\log(s-1))$

Thus, $u(x,t) = \phi\left(-\log(e^{-x} + e^{t} - 1)\right)$.

What is new here?

Today: Solve $u_t + a(u) u_x = 0 \quad -\infty < x < \infty,\ t > 0$ This is NON-linear!

$u(x,0) = \phi(x)$

By "characteristics reasoning," find curves $x(t)$ with

$$\frac{dx}{dt} = a(u).$$

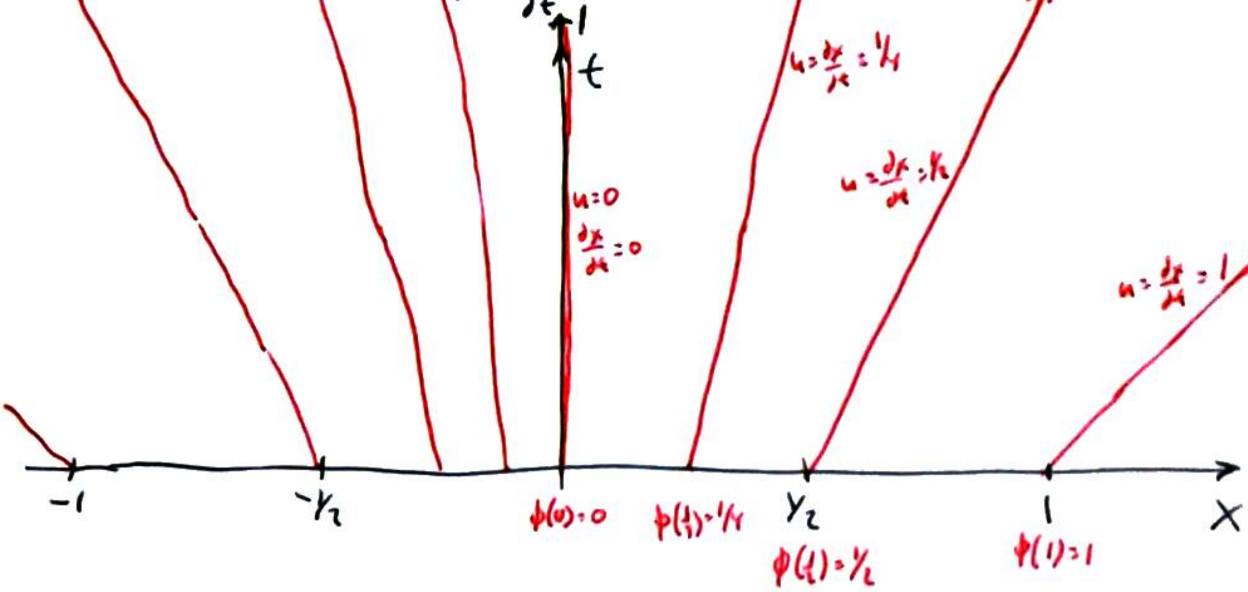
Check: $\underbrace{\frac{d}{dt}\left[u(x(t),t)\right]}_{\text{change in } u \text{ along characteristic}} = u_x \cdot \overset{a(u)}{\overset{\shortparallel}{\frac{dx}{dt}}} + u_t = 0$

Since u is constant on characteristics, $\frac{dx}{dt} = a(c)$ = constant,

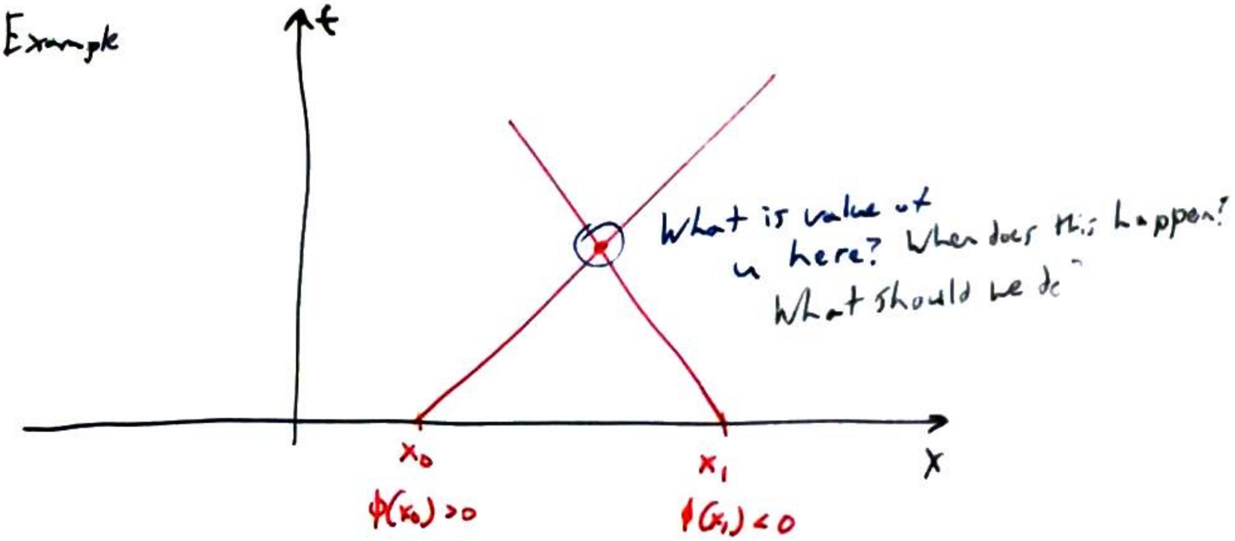
so characteristics are lines!

Example $u_t + u u_x = 0 \qquad \phi(x) = x$ Graph solution with level curves

characteristics satisfy $\frac{dx}{dt} = u$



Example



<u>3 Options</u>

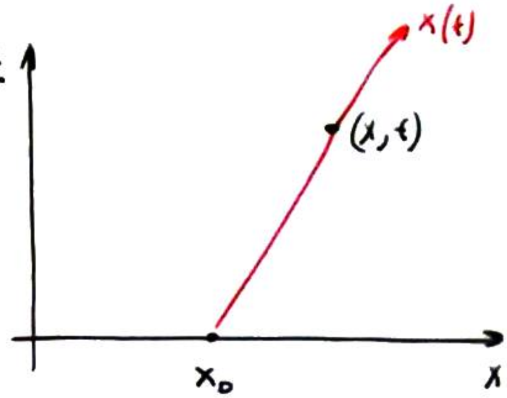
1) Choose increasing $\phi(x)$
2) Just say solution only guaranteed in small range $0 \le t \le T$ (local well-posedness)
3) Allow for discontinuous solutions.

Option 1: Nope. You can't choose ϕ (ϕ chooses you!)

Option 2: Example: $u_t + uu_x = 0$

$\phi(x) = x^2$

$$\overbrace{\frac{dx}{dt} = u}^{\text{both constant}} \text{ on } x(t)$$



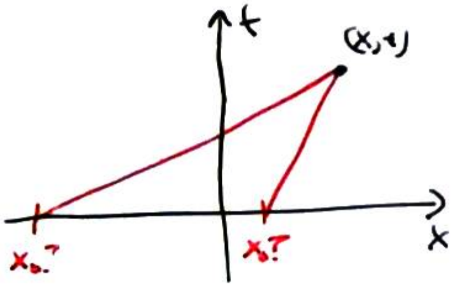
$$\boxed{\frac{x - x_0}{t} = \phi(x_0)}$$ this holds for any $\phi(x) = u(x,0)$

For $\phi(x) = x^2$, $x - x_0 = t x_0^2$

If $t \neq 0$, $x_0 = \frac{-1 \pm \sqrt{1+4tx}}{2t}$, $u(x,t) = \left(\frac{-1 \pm \sqrt{1+4tx}}{2t}\right)^2$

$$= \frac{1 \mp 2\sqrt{1+4tx} + 1 + 4tx}{4t^2}$$

$$= \frac{1 + 2tx \mp \sqrt{1+4tx}}{2t^2}$$



Which one?

Key info: Want solution continuous as $t \to 0$

$$\lim_{t\to 0} \frac{1 + 2tx - \sqrt{1+4tx}}{2t^2} = \lim_{t\to 0} \frac{2x - 2x(1+4tx)^{-1/2}}{4t} = \lim_{t\to 0} \frac{4x^2(1+4tx)^{-3/2}}{4} = x^2$$

L'Hospital's Rule

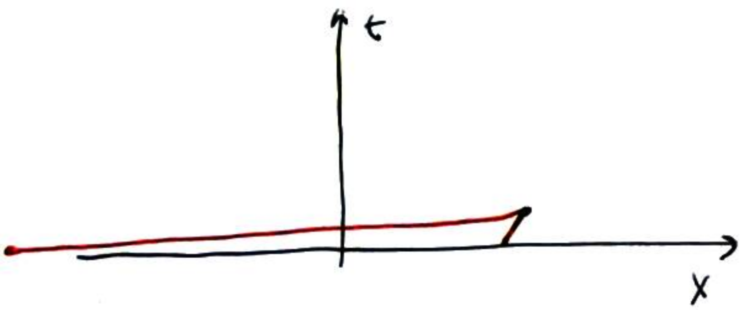
This is the right limit!

No finite limit as $t \to 0$ for $\frac{1 + 2tx + \sqrt{1+4tx}}{t}$, Taking $x_0 = \frac{-1 - \sqrt{1+4tx}}{2t}$,

$x_0 \to -\infty$

$u \to \infty$, so

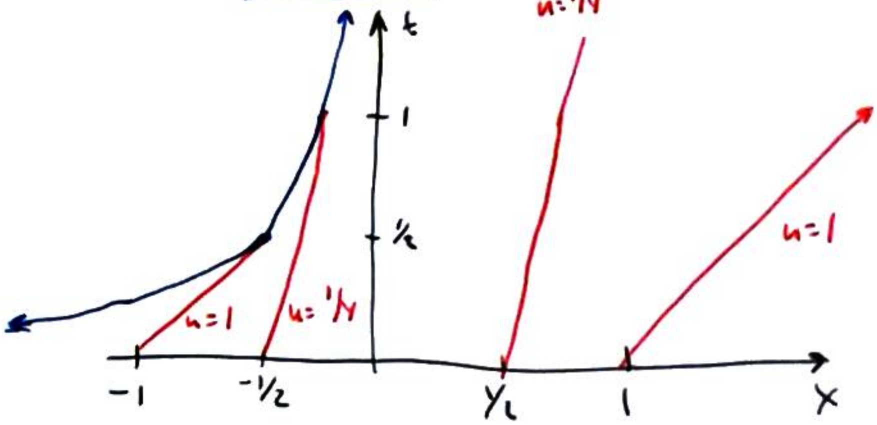
take $x_0 = \frac{-1 + \sqrt{1+4tx}}{2t}$



Solution valid as long as $\sqrt{1+4tx}$ defined

$1+4tx \ge 0$

$\boxed{tx \ge -1/4}$



Break

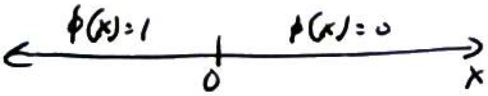
Generally, $u(x,t)=\phi(x_0)$, where (when solving $u_t + a(u)\,u_x = 0$, $u(x,0)=\phi(x)$)

$x - x_0 = t\,a(\phi(x_0))$

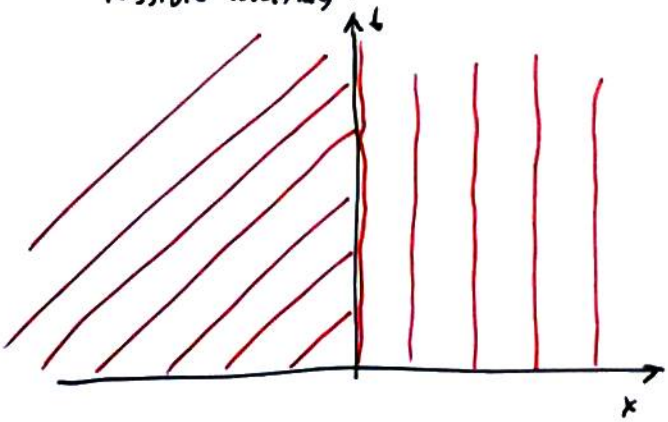
Characteristics don't intersect if $a(\phi(z)) \le a(\phi(w))$ for $z \le w$ (still can't guarantee this)

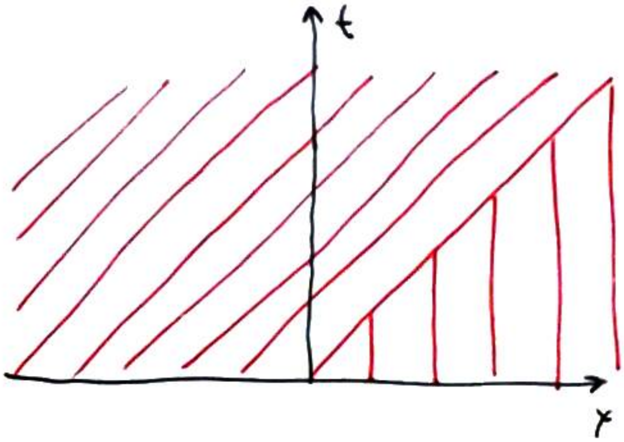
Option 3: Discontinuous solutions/Jumps

Example: $u_t + u u_x = 0$



Possible solutions:





Which one is best? Practice voting since midterms only 3 months away

Previous Problem: Discontinuous solutions to wave equation

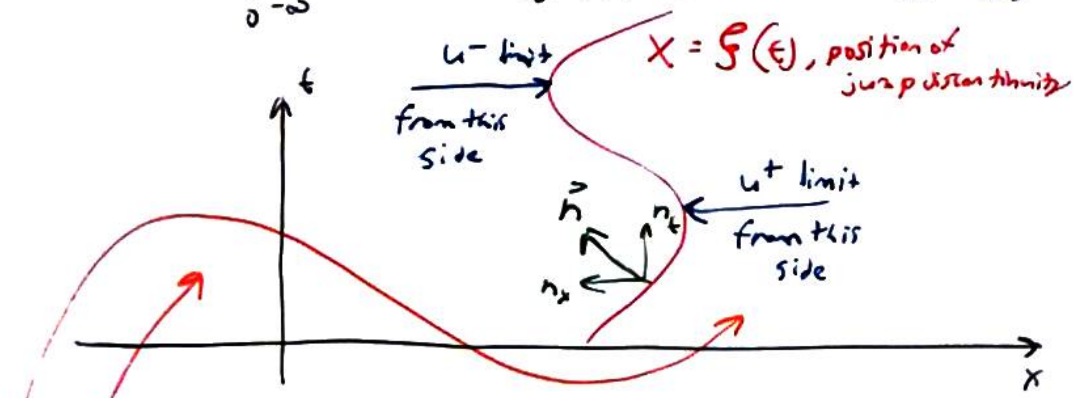
Solution: Distributions

Current Problem: Discontinuous solutions to nonlinear transport equation.

Solution: Distributions!

transfer derivatives to Ψ through IBP

Want: $\int_0^\infty \int_{-\infty}^\infty [u\Psi_t + A(u)\Psi_x]\,dx\,dt = 0$ for all $\Psi \in C_0^\infty$ (*), where $u_t + a(u)u_x = u_t + [A(u)]_x = 0$



Split integral in 2 pieces, apply Green's theorem

$$\int_0^\infty \int_{-\infty}^{\xi(t)} u\Psi_t + A(u)\Psi_x\,dx\,dt = \underbrace{\cancel{\int_{\text{at infinity}} \Psi}}^{0,\ \Psi\in C_0^\infty} + \int_{x=\xi(t)} u^-\Psi n_t + A(u^-)\Psi n_x\,d\ell$$

$$+\quad \int_0^\infty \int_{\xi(t)}^{\infty} u\Psi_t + A(u)\Psi_x\,dx\,dt = \underbrace{\cancel{\int_{\text{at infinity}} \Psi}}_{0} - \int_{x=\xi(t)} u^+\Psi n_t + A(u^+)\Psi n_x\,d\ell$$

$$0 = \int_{x=\xi(t)} (u^-\Psi n_t + A(u^-)\Psi n_x) - (u^+\Psi n_t + A(u^+)\Psi n_x)\,d\ell$$

Since solution (*)

Ψ arbitrary $\longrightarrow$ $u^+ n_t + A(u^+) n_x = u^- n_t + A(u^-) n_x$

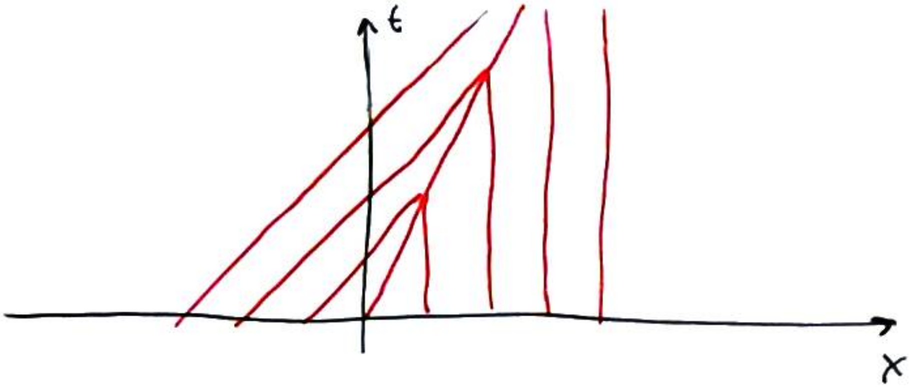
$$\boxed{\frac{A(u^+) - A(u^-)}{u^+ - u^-} = \frac{-n_t}{n_x} := s(t)} \text{ — like } \frac{dx}{dt}$$

Rankine–Hugoniot formula

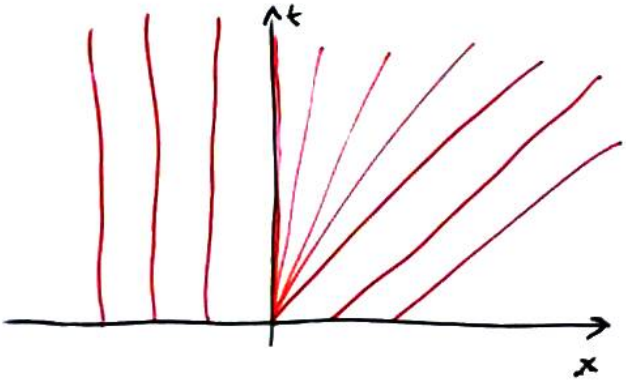
Example $u_t + u u_x = 0$

$u_t + \left(\frac{1}{2}u^2\right)_x = 0$ so $A(u) = \frac{1}{2}u^2$

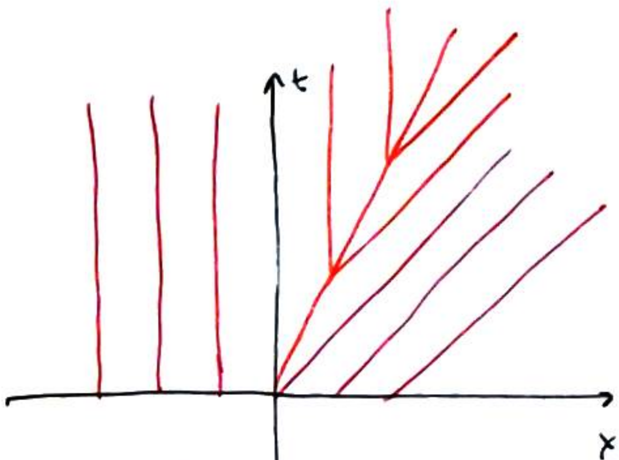
If $\phi(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$, $s = \dfrac{A(u^+) - A(u^-)}{u^+ - u^-} = \dfrac{0 - 1/2}{0 - 1} = 1/2$



Example $\phi(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$



Continuous



Shock wave

Which is correct?

Both satisfy Rankine-Hugoniot formula

Physically, want entropy condition $\boxed{a(u^-) > s > a(u^+)}$ for solutions' discontinuities, so ~~Shock wave~~
Continuous wins