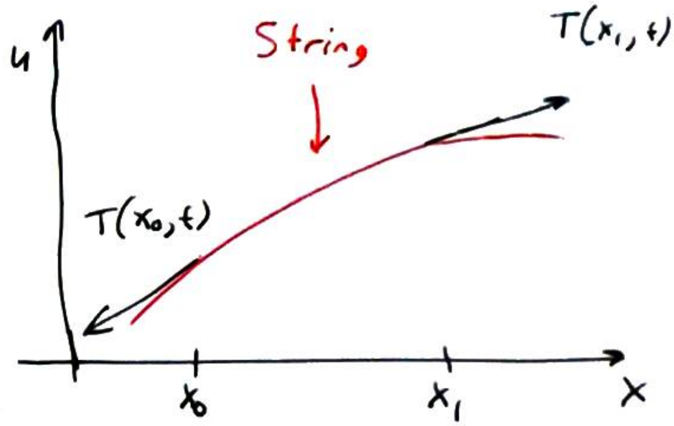
7/25/18 Lecture Notes: The Wave Equation in Higher Dimensions

Recall in 1D:



$\vec{F} = m\vec{a}$ in horizontal direction $\longrightarrow$ T constant

$\vec{F} = m\vec{a}$ in vertical direction $\longrightarrow$

$$\frac{Tu_x}{\sqrt{1+u_x^2}}\Bigg|_{x_0}^{x_1} \overset{\text{small perturbations}}{\simeq} Tu_x\Big|_{x_0}^{x_1} = \int_{x_0}^{x_1} \rho u_{tt}\, dx \quad \partial/\partial x_1$$

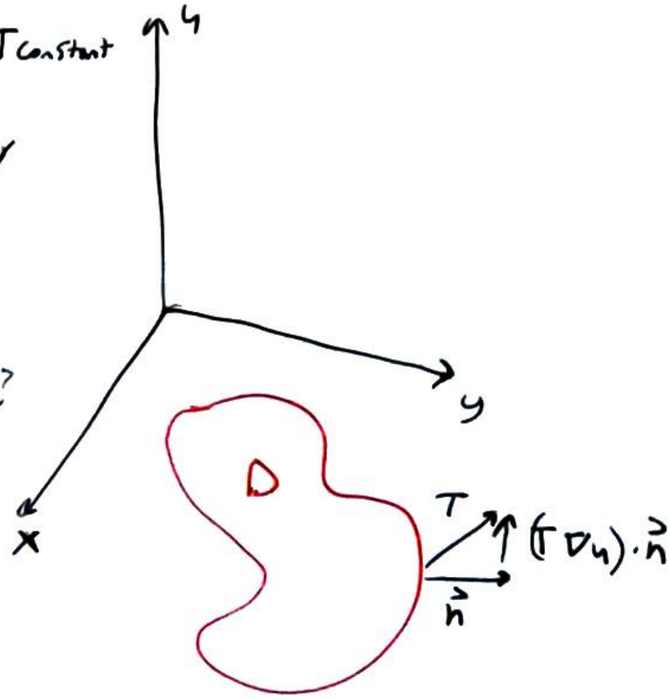
$$Tu_{xx} = \rho u_{tt}, \text{ or}$$
$$u_{tt} = c^2 u_{xx}$$

In 2D: Vibrating Drumhead

$\vec{F} = m\vec{a}$ in horizontal directions $\rightarrow$ T constant

$$\int_{\partial D} (T\nabla u)\cdot\vec{n}\, dS = \iint_D \rho u_{tt}\, dx\, dy$$

Q: How to relate values on boundary to integral over interior?



A: Green's Theorem

"Usual" form: $\int_{\partial D} p\,dx + q\,dy = \iint_D (q_x - p_y)\,dx\,dy$

Set $\vec{f} = (q, -p)$ | $\vec{n} = \left(\frac{dy}{ds}, -\frac{dx}{ds}\right)$

Other form: $\int_{\partial D} \vec{f}\cdot\vec{n}\,ds = \iint_D \underbrace{\nabla\cdot\vec{f}}_{\operatorname{div}\vec{f}}\,dx\,dy$

If $\vec{f} = \nabla u$,

$$\iint_D \rho u_{tt}\,dx\,dy = \int_{\partial D} (T\nabla u)\cdot\vec{n}\,ds = \iint_D T\nabla\cdot(\nabla u)\,dx\,dy$$

Since D arbitrary $\qquad = \iint_D T\Delta u\,dx\,dy$

$$\rho u_{tt} = T\Delta u$$

$$\boxed{u_{tt} = c^2\Delta u}$$ (same in 3D with 3D Laplacian)

Energy

Let $E(t) = \frac{1}{2}\iiint_{\mathbb{R}^3} (u_t)^2 + c^2|\nabla u|^2\,dV$. ∇ just in spatial variables $\vec{x} = (x, y, z)$, not t (In 1D, $E(t) = \int_{-\infty}^{\infty} \rho u_t^2 + T u_x^2\,dx$.)

Is energy conserved? $\frac{dE}{dt} = \iiint u_t u_{tt} + c^2[u_x u_{xt} + u_y u_{yt} + u_z u_{zt}]\,dV$

$$\iint_{\mathbb{R}^2}\left[\int_{-\infty}^{\infty} u_x u_{xt}\,dx\right]dA = \iint_{\mathbb{R}^2}\left[\cancel{u_x u_t\Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} u_{xx} u_t\right]dA = \iiint -u_{xx}u_t\,dV$$

IBP   decay

$$\frac{dE}{dt} = \iiint u_t u_{tt} + c^2[-u_{xx}u_t - u_{yy}u_t - u_{zz}u_t]\,dV$$

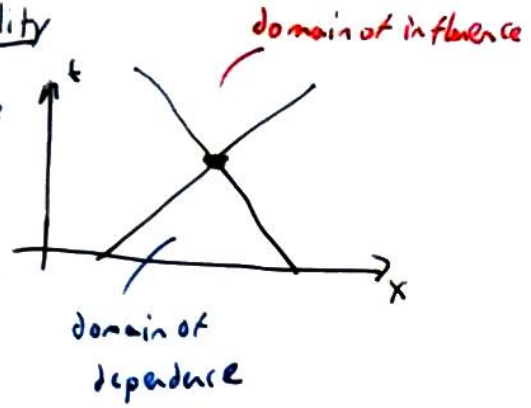
$$= \iiint u_t(u_{tt} - c^2\Delta u)\,dV = 0.$$

Question for the class: Is energy conserved in 2D?   Break

In groups: Prove uniqueness of solutions to IVP $\quad u_{tt} = c^2\Delta u$, $u(\vec{x}, 0) = \phi(\vec{x})$, $u_t(\vec{x}, 0) = \psi(\vec{x})$

Causality

1D:

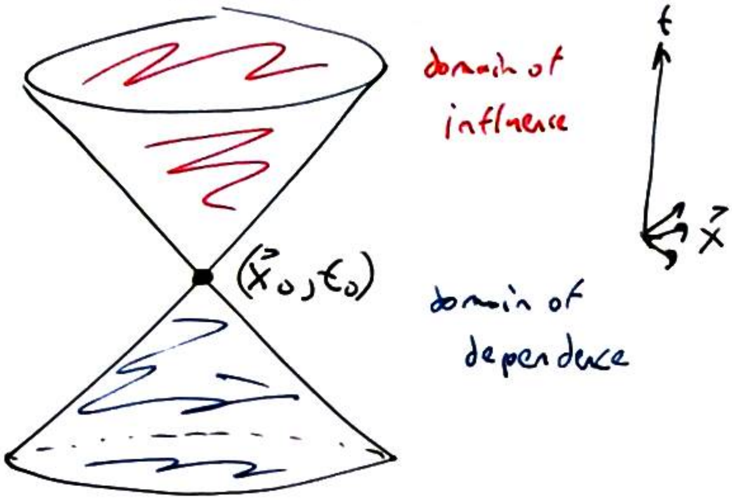


3D: Light cone (What you call a cone on an NSF application)

$$\{|\vec{x}-\vec{x}_0| < c|t-t_0|\}$$

or

$$r < c|t-t_0|$$



$$\phi(t,x,y,z) = -c^2(t-t_0)^2 + (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = 0 \text{ on light cone}$$

Unit normal vectors: $\vec{n} = \pm \dfrac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$ grad in all 4 variables, distinguish from ∇ in space

$$\text{grad}\,\phi = 2\left(x-x_0,\ y-y_0,\ z-z_0,\ -c^2(t-t_0)\right)$$

$$\vec{n} = \pm \frac{\left(x-x_0,\ y-y_0,\ z-z_0,\ -c^2(t-t_0)\right)}{\left[c^4(t-t_0)^2 + \underbrace{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}_{r^2}\right]^{1/2}}$$

$r = \pm c|t-t_0|$ on cone's surface

$$\vec{n} = \pm \frac{c}{\sqrt{c^2+1}}\left(\frac{x-x_0}{cr}, \frac{y-y_0}{cr}, \frac{z-z_0}{cr}, -\frac{(t-t_0)}{|t-t_0|}\right)$$

To prove [cone] is actually domain of dependence:

Suppose $v_{tt} = c^2\Delta v$ $\quad$ $w_{tt} = c^2 \Delta w$

$v(\vec{x},0) = \phi_1(\vec{x})$ $\quad$ $w(\vec{x},0) = \phi_2(\vec{x})$ $\quad$ and $\quad$ $\phi_1 = \phi_2 = \phi$ on B = bottom of [cone]

$v_t(\vec{x},0) = \psi_1(\vec{x})$ $\quad$ $w_t(\vec{x},0) = \psi_2(\vec{x})$ $\quad\quad\quad$ $\psi_1 = \psi_2 = \psi$

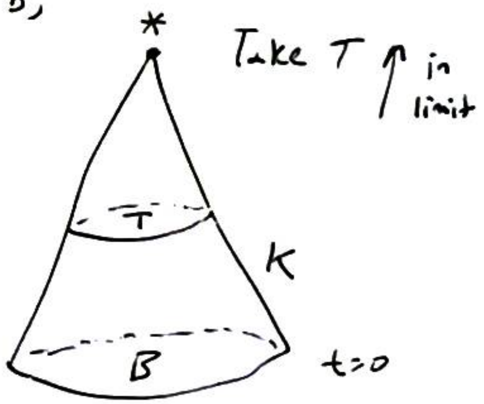
We want to show $v(*) = w(*)$, where $*$ = top of [cone]

– i.e., solution at $*$ depends only on data at B

Taking $u = v - w$, so $\quad u_{tt} = c^2 \Delta u$ $\quad$ it suffices to show $u(*) = 0$.

$u(\vec{x},0)$ is 0 on B

$u_t(\vec{x},0)$ is 0 on B,



Idea: Show energy on $T \le$ energy on B (and take $T \to *$)

Energy is integral on boundary, so use 4D divergence theorem

pick these to make divergence 0

Let $\vec{U} = \left(\frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2, \;\cdot\;, \;\cdot\;, \;\cdot\;\right)$

matches energy on B, T

$$\frac{d}{dt}\left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\right) = \underbrace{u_t u_{tt}}_{=\, u_t(c^2 u_{xx} + c^2 u_{yy} + c^2 u_{zz})} + c^2 u_x u_{xt} + c^2 u_y u_{yt} + c^2 u_z u_{zt}$$

X-terms

$$c^2\left[u_x u_{xt} + u_t u_{xx}\right] = \frac{\partial}{\partial x}\left[c^2 u_t u_x\right]$$

Pick $\vec{U} = \left(\frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2, -c^2 u_t u_x, -c^2 u_t u_y, -c^2 u_t u_z\right)$, so

- $\operatorname{div} \vec{U} = 0$
- $\vec{U} \cdot (1,0,0,0) =$ energy

4D divergence theorem $\Rightarrow \iiint\limits_{B \cup K \cup T} \vec{u} \cdot \hat{n} \, \partial V = 0$

On T, $\hat{n} = (1, 0, 0, 0)$, so $\iiint\limits_T \vec{u} \cdot \hat{n} \, \partial V = \iiint\limits_T \frac{1}{2} u_t^2 + \frac{1}{2} c^2 |\nabla u|^2 \, \partial x \, \partial y \, \partial z$

On B, $\hat{n} = (-1, 0, 0, 0)$, so $\iiint\limits_B \vec{u} \cdot \hat{n} \, \partial V = -\iiint\limits_B \frac{1}{2} \psi^2 + \frac{1}{2} c^2 |\nabla \phi|^2 \, \partial x \, \partial y \, \partial z$

outward normal $\qquad$ $t - t_0 < 0$ $\qquad$ from ∇u factor out from $\hat{r}$

On K, $\hat{n} = + \frac{c}{\sqrt{c^2+1}} \left(\frac{x - x_0}{cr}, \frac{y - y_0}{cr}, \frac{z - z_0}{cr}, 1 \right)$, so $\iiint\limits_K \vec{u} \cdot \hat{n} \, \partial V$ is

$$\frac{c}{\sqrt{c^2+1}} \iiint\limits_K \left[\frac{1}{2} u_t^2 + \frac{1}{2} c^2 |\nabla u|^2 + \frac{x - x_0}{cr} (-c^2 u_t u_x) + \frac{y - y_0}{cr} (-c^2 u_t u_y) + \frac{z - z_0}{cr} (-c^2 u_t u_z) \right] \partial V$$

Side note: $u_r = \hat{r} \cdot \nabla u$, where $\hat{r} = \frac{\vec{x} - \vec{x}_0}{|\vec{x} - \vec{x}_0|} = \left(\frac{x - x_0}{r}, \frac{y - y_0}{r}, \frac{z - z_0}{r} \right)$

$$= \frac{c}{\sqrt{c^2+1}} \iiint\limits_K \frac{1}{2} u_t^2 + \frac{1}{2} c^2 |\nabla u|^2 - c u_t u_r \, \partial V$$

$$= \frac{c}{\sqrt{c^2+1}} \iiint\limits_K \frac{1}{2} \underbrace{(u_t - c u_r)^2}_{\geq 0} + \frac{1}{2} c^2 \underbrace{\left(|\nabla u|^2 - u_r^2 \right)}_{\geq 0} \partial V \geq 0$$

Therefore, $\iiint\limits_T \frac{1}{2} u_t^2 + \frac{1}{2} c^2 |\nabla u|^2 \, \partial V \leq \iiint\limits_B \frac{1}{2} \psi^2 + \frac{1}{2} c^2 |\nabla \phi|^2 \, \partial V = 0$,

so $u \equiv 0$ on $T \longrightarrow u = 0$ at $*$

Talk about Huygen's Principle, next time if time