Math 126 **Name:** ______________________

Summer 2018

Midterm

7/12/2018

Time Limit: 110 Minutes

This exam contains 6 questions. Total of points is 75.

For this exam, you are allowed a handwritten cheat sheet consisting of one side of an 8 1/2 by 11 piece of paper.

The last sheet of paper is intentionally blank so that you may use it to finish your solutions *if clearly marked where the first part of your solution stops.*

Grade Table (for instructor use only)

Question	Points	Score
1	10	
2	10	
3	15	
4	15	
5	15	
6	10	
Total:	75	

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1. (10 points) Consider the following equation:

$$u_{xx} = 0, \tag{1}$$

where $u = u(x, y)$ is a function of two variables.

(a) (3 points) Use three terms from this class (like linear versus nonlinear, for instance) to classify the above PDE.

(b) (4 points) Find the general solution to (1).

(c) (3 points) Give an example of conditions (such as initial or boundary) under which (1) has exactly one solution.

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2. (10 points) Consider the function $u(x, y) = x^2 - y^2$.

   (a) (2 points) Show that $\Delta u = 0$.

   (b) (4 points) State the (weak) maximum principle for the Laplacian.

   (c) (4 points) Show directly that u satisfies the maximum principle on the domain $D = [0, 1] \times [0, 1]$.

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3. (15 points) Consider the damped wave equation

$$u_{tt} - c^2 u_{xx} + r u_t = 0, \tag{2}$$

where $r > 0$.

(a) (2 points) Define an energy $E(t)$ for (2).

(b) (5 points) Show that $E(t)$ is nonincreasing.

(c) (8 points) Use parts (a) and (b) to prove uniqueness of solutions to

$$\begin{aligned} u_{tt} - c^2 u_{xx} + r u_t = f(x,t) \qquad & -\infty < x < \infty, t > 0 \\ u(x,0) = \phi(x), u_t(x,0) = \psi(x) \qquad & \infty < x < \infty \end{aligned}$$

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4. (15 points) (a) (5 points) Using the formula for the solution $u(x,t)$ to

$$\begin{aligned} u_t - ku_{xx} &= f(x,t) && -\infty < x < \infty, t > 0 \\ u(x,0) &= \phi(x), && \infty < x < \infty \end{aligned}$$

show that $u(x,t)$ is odd in x provided that $\phi(x)$ and $f(x,t)$ are odd in x. (Note: $u(x,t)$ being odd in x means $u(-x,t) = -u(x,t)$ for all x and t.)

(b) (5 points) Repeat part (a) by showing that $-u(-x,t)$ is also a solution to the same PDE and initial conditions. Why does this suffice to show that $u(x,t)$ is odd?

(c) (5 points) Explain, in about 3 to 5 sentences, how to use the reflection method to solve

$$\begin{aligned} u_t - ku_{xx} &= f(x,t) && 0 < x < \infty, t > 0 \\ u(x,0) &= \phi(x), && \infty < x < \infty \\ u(0,t) &= 0 && t > 0. \end{aligned}$$

Be sure to include why the fact you proved in parts (a) and (b) is useful.

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5. (15 points) Using separation of variables, solve

$$
\begin{aligned}
u_t &= k u_{xx} && -\pi < x < \pi \\
u(-\pi, t) = u(\pi, t) &= 0 && t > 0 \\
u(x, 0) &= \phi(x) && -\pi < x < \pi.
\end{aligned}
$$

If a part was done exactly the same way in class, feel free to jump to its end, though do so at your own risk. Also, feel free to cite a theorem or two about eigenvalues to slightly shorten your calculations.

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6. (10 points) Take for granted that the sine series of x on the interval $(0, \pi)$ is

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - ...\right)$$

(a) (4 points) State the solution to

$$\begin{aligned} u_{tt} &= c^2 u_{xx} & -\pi < x < \pi \\ u(-\pi, t) &= u(\pi, t) = 0 & t > 0 \\ u(x, 0) &= x, u_t(x, 0) = 0 & -\pi < x < \pi. \end{aligned}$$

(b) (6 points) Determine the pointwise convergence of the above Fourier sine series, that is, if it converges and if so, to what function. Fully justify each claim.

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