7/18/18 Lecture Notes: Green's 2nd Identity and Green's Functions

Last time: $$\left(\iint_{\partial D} v \frac{\partial u}{\partial n}\, dS = \iiint_D \nabla v \cdot \nabla u \, dx + \iiint_D v \Delta u \, dx\right)$$

Swapping u, v: $$\left(\iint_{\partial D} u \frac{\partial v}{\partial n}\, dS = \iiint_D \nabla u \cdot \nabla v \, dx + \iiint_D u \Delta v \, dx\right)$$

This time:

$$\boxed{\iiint_D u\Delta v - v \Delta u \, dx = \iint_{\partial D} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) dS \quad (*)}$$

Green's 2nd Identity

In 1D: $$\int_a^b -X_1'' X_2 + X_1 X_2'' \, dx = \left(-X_1' X_2 + X_1 X_2'\right)\Big|_a^b$$

Idea: If $\Delta u = 0$ in D, and $\Delta v = \delta(x)$, then

$(*) \rightarrow \quad u(0) = \iint_{\partial D} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) dS$ [Can't actually do until δ defined in Chapter 12]

Representation Formula:

$$u(x_0) = \iint_{\partial D} \left[-u(x) \frac{\partial}{\partial n}\left(\frac{1}{|x - x_0|}\right) + \frac{1}{|x - x_0|} \frac{\partial u}{\partial n}\right] \frac{dS}{4\pi}$$

<u>Pf</u> Take $x_0 = 0$ for simplification

Want $v(x)$ to satisfy $\Delta v = 0$ away from 0

: have singularity at 0

Pick $v(x) = \frac{-1}{4\pi} \frac{1}{|x|}$ Recall radial harmonic functions in 3D are $u = -c_1/r + c_2$

Issue: $v(0)$ undefined, so replace D with D_ε (cut out ball of radius ε)

$$\iiint_{D_\varepsilon} u \underbrace{\cancel{\Delta v}}_{0} - v \underbrace{\cancel{\Delta u}}_{0}\, dx = \iint_{\partial D_\varepsilon} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS$$

$$0 = -\frac{1}{4\pi}\iint_{\partial D_\varepsilon} u\cdot\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{\partial u}{\partial n}\cdot\frac{1}{r}\, dS$$

$$= \underbrace{-\frac{1}{4\pi}\iint_{\partial D} u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{\partial u}{\partial n}\cdot\frac{1}{r}\, dS}_{\text{what we're looking for}} - \frac{1}{4\pi}\iint_{r=\varepsilon}\left[u\cdot\frac{\partial}{\partial r}\left(\frac{1}{r}\right) - \frac{\partial u}{\partial r}\cdot\frac{1}{r}\right] dS$$

$$\frac{1}{4\pi}\iint_{\partial D} u\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{\partial u}{\partial n}\cdot\frac{1}{r}\, dS = \frac{1}{4\pi}\left[\iint_{r=\varepsilon} u\cdot\frac{1}{\varepsilon^2}\, dS + \iint_{r=\varepsilon}\frac{1}{\varepsilon}\frac{\partial u}{\partial r}\, dS\right]$$

$\varepsilon \to 0$

$$= \quad u(0) \quad + \quad 0$$

since u cts. + Area $= 4\pi\varepsilon^2$

since $1/\varepsilon$ makes this term $O(\varepsilon)$

In 2D:

$$u(x_0) = \frac{1}{2\pi}\int_{\partial D}\left[u(x)\frac{\partial}{\partial n}\left(\log|x-x_0|\right) - \frac{\partial u}{\partial n}\log|x-x_0|\right] dS$$

Reminder: $\log|x|$ is radial harmonic function in 2D

Green's Functions:

Recall $$u(x_0) = \iint_{\partial D}\left\{-u(x)\frac{\partial}{\partial n}\Big(\underbrace{\frac{1}{|x-x_0|}}_{v}\Big) + \underbrace{\frac{1}{|x-x_0|}}_{v}\frac{\partial u}{\partial n}\right\}\frac{dS}{4\pi}$$

Idea: Replace v with something equal to zero on ∂D to simplify formula.

Definition: The <u>Green's function</u> for the domain D at the point $x_0 \in D$ is a function $G(x)$ defined on D such that

i) $G(x)$ has continuous 2nd derivatives and $\Delta G = 0$ in D, except at $x = x_0$.

ii) $G(x) = 0$ for $x \in \partial D$

iii) $G(x) + 1/(4\pi|x-x_0|)$ is finite at x_0, has continuous 2nd derivatives everywhere, and is harmonic at x_0.

Sometimes denote $G(x) = G(x, x_0)$

<u>Theorem</u> If $G(x, x_0)$ is a Green's function for D, then the solution to the Dirichlet problem
$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = g & \text{on } \partial D \end{cases}$$
is
$$u(x_0) = \iint_{\partial D} g(x) \frac{\partial G(x, x_0)}{\partial n} \, dS$$

<u>Pf</u> We COULD repeat proof of Representation Formula, but it's easier to use it via this trick.

$$u(x_0) = \iint_{\partial D} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS, \quad \text{where } v = \frac{-1}{4\pi|x-x_0|}$$

Let $H(x) = G(x, x_0) - v(x)$ be function from (iii).

$\Delta H = \Delta u = 0$ in D, so Green's 2nd Identity $\rightarrow$

$$+ \quad 0 = \iint_{\partial D} \left(u \frac{\partial H}{\partial n} - H \frac{\partial u}{\partial n} \right) dS$$

$$u(x_0) = \iint_{\partial D} \left(u \frac{\partial G}{\partial n} - \frac{\partial u}{\partial n} \cancelto{0}{G} \right) dS$$

$\blacksquare$

After the break:

Theorem: The solution to
$$\begin{cases} \Delta u = f & \text{in } D \\ u = h & \text{on } \partial D \end{cases}$$
is
$$u(x_0) = \iint_{\partial D} h(x) \frac{\partial G(x, x_0)}{\partial n} \, dS + \iiint_D f(x) \, G(x, x_0) \, dx$$

Is this analogous to anything else? (Other nonhomogeneous solutions?)

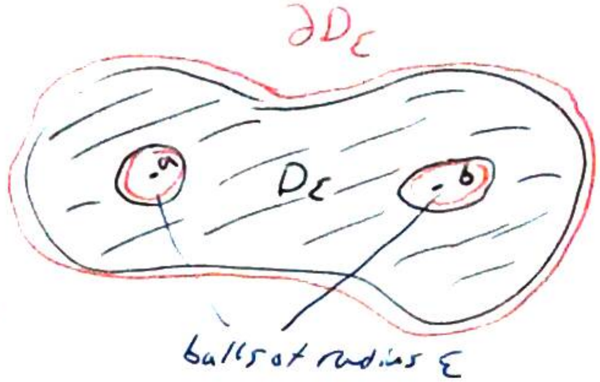
Think - Pair - Share: Prove theorem

Symmetry

Theorem: Green's functions $G(x, x_0)$ satisfy $G(a,b) = G(b,a)$

<u>Pf</u> Let $\underset{\text{singular at } a}{u(x) = G(x,a)}$, $\underset{\text{singular at } b}{v(x) = G(x,b)}$.

Apply Green's 2nd identity on D_ε =



Green's functions 0 on ∂D

$$\iiint_{D_\varepsilon} \cancel{u \Delta v}_{0} - \cancel{v \Delta u}_{0}\, dx = \iint_{\partial D} \cancel{u \frac{\partial v}{\partial n}}_{0} - \cancel{v \frac{\partial u}{\partial n}}_{0}\, dS + \underbrace{\iint_{|x-a|=\varepsilon} u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\, dS}_{A_\varepsilon} + \underbrace{\iint_{|x-b|=\varepsilon} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) dS}_{B_\varepsilon}$$

Sub $r = |x-a|$

$u(x) = G(x,a) = \frac{-1}{4\pi r} + H$, $\quad 0 = A_\varepsilon + B_\varepsilon \quad$ (for all $\varepsilon > 0$)

H is function from (iii)

$$A_\varepsilon = \iint_{r=\varepsilon} \left(\frac{-1}{4\pi r} + H \right) \frac{\partial v}{\partial n} - v \underbrace{\frac{\partial}{\partial n} \left(\frac{-1}{4\pi r}\right.}_{\substack{\text{only term} \\ \text{with } 1/\varepsilon^2}} \left. + H \right) \underbrace{r^2}_{= \varepsilon^2 \to 0 \text{ quickly}} \sin\theta \, d\theta \, d\phi \to v(a)$$

Similarly, $B_\varepsilon \to -u(b)$, so $v(a) = u(b)$

$$G(a,b) = G(b,a)$$

<u>Summary</u> Given a domain D, solve <u>all</u> Dirichlet problems with just one special function $G(x, x_0)$ (whose definition alludes to δ function).

Next time: Find actual Green's functions for
- half-plane
- sphere

using symmetries of the domain