Lecture Notes for
A Geometrical Introduction to Robotics and Manipulation

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Chapter 4 Manipulator Dynamics

Section 4.1

4.1 Introduction

Definition: Dynamics
Physical laws governing the motions of bodies and aggregates of bodies.

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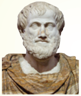
- **A short history:**



"Everything happens for a reason."

Aristotle (384 BC - 322 BC)

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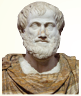

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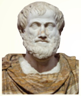

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Laws of motion.

I. Newton (1642 - 1726)

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4.1 Introduction

Laws of motion from particles to rigid bodies.

L. Euler (1707 - 1783)

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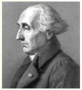

Calculus of Variation and the Principles of least action.

J. Lagrange (1736 - 1813)

4.1 Introduction

Laws of motion from particles to rigid bodies.

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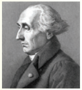

Calculus of Variation and the Principles of least action.

J. Lagrange (1736 - 1813)

Quaternions and Hamilton's Principle.

W. Hamilton (1805 - 1865)

† End of Section †

Section 4.2

4.2 Lagrange Equations

□ A simple example:

Newton's Equation:

$$\boxed{\begin{aligned} m\ddot{x} &= F_x \\ m\ddot{y} &= F_y - mg \end{aligned}}$$

Momentum: $P_x = m\dot{x}$

$$P_y = m\dot{y}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_x = F_x, \frac{\mathrm{d}}{\mathrm{d}t}P_y = F_y - mg$$

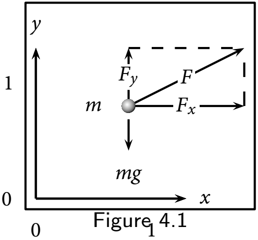

Figure 4.1

Lagrangian Equation:

4.2 Lagrange Equations

□ A simple example:

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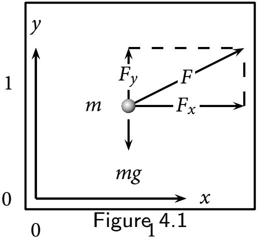

Figure 4.1

$$\Longleftrightarrow$$

Lagrangian Equation:

$$\boxed{\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= F_x \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} &= F_y \end{aligned}}$$

Lagrangian function:

$$L = T - V, P_x = \frac{\partial L}{\partial \dot{x}}, P_y = \frac{\partial L}{\partial \dot{y}}$$

Kinetic energy:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

Potential energy:

$$V = mgy$$

□ Generalization to multibody systems:

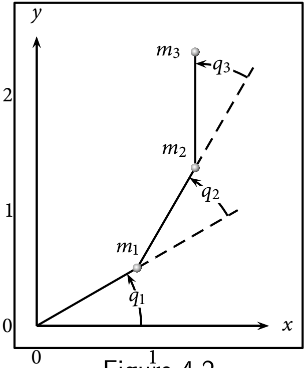

Figure 4.2

$q_i, i = 1, \ldots, n$: generalized coordinates

Kinetic energy:

$$T = T(q, \dot{q})$$

Potential energy:

$$V = V(q)$$

Lagrangian:

$$L(q, \dot{q}) = T(q, \dot{q}) - V(q)$$

$\tau_i, i = 1, \ldots, n$: external force on q_i

Lagrangian Equation:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \tau_i, i = 1, \ldots, n}$$

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◇ Example: Pendulum equation

Generalized coordinate:

$$\theta \in S^1$$

Kinematics:

$$x = l\sin\theta, y = -l\cos\theta$$

$$\dot{x} = l\cos\theta\cdot\dot{\theta}, \dot{y} = l\sin\theta\cdot\dot{\theta}$$

Kinetic energy:

$$T(\theta,\dot{\theta}) = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}ml^2\dot{\theta}^2$$

Potential energy:

$$V = mgl(1-\cos\theta)$$

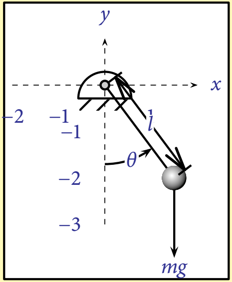

Figure 4.3

Lagrangian function:

$$L = T - V = \frac{1}{2}ml^2\dot{\theta} - mgl(1-\cos\theta), \Rightarrow \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta}, \frac{\partial L}{\partial\theta} = -mgl\sin\theta$$

Equation of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = \tau \Rightarrow ml^2\ddot{\theta} + mgl\sin\theta = \tau$$

◇

◇ Example: A spherical pendulum

Generalized coordinate:

$$r(\theta,\phi) = \begin{bmatrix} l\sin\theta\cos\phi \\ l\sin\theta\sin\phi \\ -l\cos\theta \end{bmatrix}$$

Kinetic energy:

$$T = \frac{1}{2}m\|\dot{r}\|^2 = \frac{1}{2}ml^2(\dot{\theta}^2 + (1-\cos^2\theta)\dot{\phi}^2)$$

Potential energy:

$$V = -mgl\cos\theta$$

Lagrangian function:

$$L(q,\dot{q}) = \frac{1}{2}ml^2(\dot{\theta} + (1-\cos^2\theta)\dot{\phi}^2) + mgl\cos\theta$$

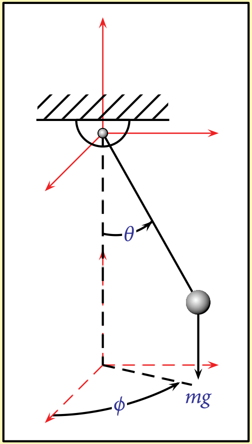

Figure 4.4

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4.2 Lagrange's Equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial \dot{\theta}} = \dfrac{\mathrm{d}}{\mathrm{d}t}(ml^2\dot{\theta}) = ml^2\ddot{\theta}, \\ \dfrac{\partial L}{\partial \theta} = ml^2 \sin\theta\cos\theta\dot{\phi}^2 - mgl\sin\theta \end{cases}$$

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial \dot{\phi}} = \dfrac{\mathrm{d}}{\mathrm{d}t}(ml^2\sin^2\theta\dot{\phi}) = ml^2\sin^2\theta\ddot{\phi} + 2ml^2\sin\theta\cos\theta\dot{\theta}\dot{\phi}, \\ \dfrac{\partial L}{\partial \phi} = 0 \end{cases}$$

$$\begin{bmatrix} ml^2 & 0 \\ 0 & ml^2 s_\theta^2 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} + \begin{bmatrix} -ml^2 s_\theta c_\theta \dot{\phi}^2 \\ 2ml^2 s_\theta c_\theta \dot{\theta}\dot{\phi} \end{bmatrix} + \begin{bmatrix} mgls_\theta \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

◇

☐ Kinetic energy of a rigid body:

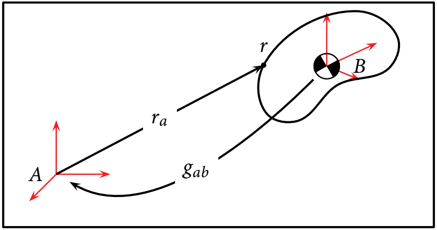

Figure 4.5

Volume occupied by the body:	V
Mass density:	$\rho(r)$
Mass:	$m = \int_V \rho(r)\mathrm{d}V$
Mass center:	$\bar{r} \triangleq \frac{1}{m}\int_V \rho(r) r \mathrm{d}V$
Relative to frame at the mass center:	$\bar{r} = 0$

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4.2 Lagrange's Equations

Kinetic energy (In A-frame):

$$
\begin{aligned}
T &= \frac{1}{2}\int_V \rho(r)\|\dot{p}+\dot{R}r\|^2 \mathrm{d}V = \frac{1}{2}\int_V \rho(r)(\|\dot{p}\|^2 + 2\dot{p}^T\dot{R}r + \|\dot{R}r\|^2)\mathrm{d}V \\
&= \frac{1}{2}m\|\dot{p}\|^2 + \underbrace{\dot{p}^T\dot{R}\int_V \rho(r) r \mathrm{d}V}_{=0} + \frac{1}{2}\int_V \rho(r)\|\dot{R}r\|^2 \mathrm{d}V
\end{aligned}
$$

4.2 Lagrange's Equations

Kinetic energy (In A-frame):

$$T = \frac{1}{2}\int_V \rho(r)\|\dot{p} + \dot{R}r\|^2 \mathrm{d}V = \frac{1}{2}\int_V \rho(r)(\|\dot{p}\|^2 + 2\dot{p}^T\dot{R}r + \|\dot{R}r\|^2)\mathrm{d}V$$

$$= \frac{1}{2}m\|\dot{p}\|^2 + \underbrace{\dot{p}^T\dot{R}\int_V \rho(r)r\mathrm{d}V}_{=0} + \boxed{\frac{1}{2}\int_V \rho(r)\|\dot{R}r\|^2\mathrm{d}V}$$

$$\frac{1}{2}\int_V \rho(r)\|\dot{R}r\|^2\mathrm{d}V$$

$$= \frac{1}{2}\int \rho(r)\|R^T\dot{R}r\|^2\mathrm{d}V = \frac{1}{2}\int \rho(r)\|\hat{\omega}r\|^2\mathrm{d}V = \frac{1}{2}\int \rho(r)\|\hat{r}\omega\|^2\mathrm{d}V$$

$$= \frac{1}{2}\int \rho(r)(-\omega^T\hat{r}^2\omega)\mathrm{d}V = \frac{1}{2}\omega^T\left(-\int \rho(r)\hat{r}^2\mathrm{d}V\right)\omega \triangleq \frac{1}{2}\omega^T\mathcal{I}\omega$$

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4.2 Lagrange's Equations

where

$$\mathcal{I} = -\int \rho(r)\hat{r}^2 \mathrm{d}V \triangleq \begin{bmatrix} \mathcal{I}_{xx} & \mathcal{I}_{xy} & \mathcal{I}_{xz} \\ \mathcal{I}_{xy} & \mathcal{I}_{yy} & \mathcal{I}_{yz} \\ \mathcal{I}_{xz} & \mathcal{I}_{yz} & \mathcal{I}_{zz} \end{bmatrix}$$

is the *Inertia tensor* with

$$\mathcal{I}_{xx} = \int \rho(r)(y^2 + z^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z, \quad \mathcal{I}_{xy} = -\int \rho(r)xy\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$T = \frac{1}{2}m\|\dot{p}\|^2 + \frac{1}{2}(\omega^b)^T\mathcal{I}\omega^b = \frac{1}{2}m\|R^T\dot{p}\|^2 + \frac{1}{2}(\omega^b)^T\mathcal{I}\omega^b$$

$$= \frac{1}{2}(V^b)^T \underbrace{\begin{bmatrix} mI & 0 \\ 0 & \mathcal{I} \end{bmatrix}}_{M^b} V^b$$

M^b: Generalized inertia matrix in B-frame. ♢

◇ Example: M^b for a rectangular object

$$\rho = \frac{m}{l\omega h}$$

$$\begin{aligned}
\mathcal{I}_{xx} &= \int_V \rho(y^2+z^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{\omega}{2}}^{\frac{\omega}{2}} \int_{-\frac{l}{2}}^{\frac{l}{2}} (y^2+z^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \rho\left(\frac{1}{12}(l\omega^3 h + l\omega h^3)\right) = \frac{m}{12}(\omega^2+h^2)
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{xy} &= -\int_V \rho xy\mathrm{d}V = -\rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{\omega}{2}}^{\frac{\omega}{2}} \int_{-\frac{l}{2}}^{\frac{l}{2}} xy\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= -\rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{\omega}{2}}^{\frac{\omega}{2}} \frac{y}{2}x^2\Big|_{-\frac{l}{2}}^{\frac{l}{2}} \mathrm{d}y\mathrm{d}z = 0
\end{aligned}$$

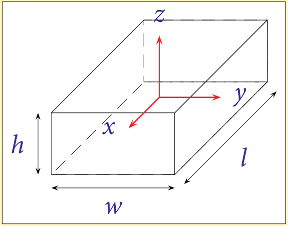

Figure 4.6

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4.2 Lagrange's Equations

$$\mathcal{I} = \begin{bmatrix} \frac{m}{12}(w^2+h^2) & 0 & 0 \\ 0 & \frac{m}{12}(l^2+h^2) & 0 \\ 0 & 0 & \frac{m}{12}(w^2+l^2) \end{bmatrix}, M^b = \begin{bmatrix} mI_{3\times 3} & 0 \\ 0 & \mathcal{I} \end{bmatrix}$$

4.2 Lagrange's Equations

$$\mathcal{I} = \begin{bmatrix} \frac{m}{12}(w^2+h^2) & 0 & 0 \\ 0 & \frac{m}{12}(l^2+h^2) & 0 \\ 0 & 0 & \frac{m}{12}(w^2+l^2) \end{bmatrix}, M^b = \begin{bmatrix} mI_{3\times 3} & 0 \\ 0 & \mathcal{I} \end{bmatrix}$$

- M^b **under change of frames:**

$$\hat{V}_1 = g_1^{-1}\cdot\dot{g}_1, \quad T = \frac{1}{2}V_1^T M_1^b V_1$$

$$V_1 = \mathrm{Ad}_{g_0} V_2$$

$$T = \frac{1}{2}(\mathrm{Ad}_{g_0} V_2)^T M_1^b (\mathrm{Ad}_{g_0} V_2)$$

$$= \frac{1}{2} V_2^T \mathrm{Ad}_{g_0}^T M_1^b \mathrm{Ad}_{g_0} V_2 \triangleq \frac{1}{2} V_2^T M_2^b V_2$$

$$\boxed{M_2^b = \mathrm{Ad}_{g_0}^T M_1^b \mathrm{Ad}_{g_0}}$$

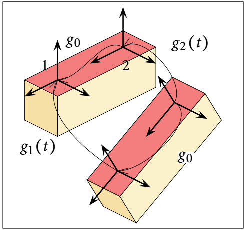

Figure 4.7

4.2 Lagrange's Equations

⋄ Example: Dynamics of a 2-dof planar robot

$$\mathcal{I}_i = \begin{bmatrix} \mathcal{I}_{xx_i} & 0 & 0 \\ 0 & \mathcal{I}_{yy_i} & 0 \\ 0 & 0 & \mathcal{I}_{zz_i} \end{bmatrix}, i = 1, 2$$

$$T(\theta, \dot{\theta}) = \frac{1}{2} m_1 \|v_1\|^2 + \frac{1}{2} \omega_1^T \mathcal{I}_1 \omega_1 + \frac{1}{2} m_2 \|v_2\|^2 + \frac{1}{2} \omega_2^T \mathcal{I}_2 \omega_2$$

$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \qquad \omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

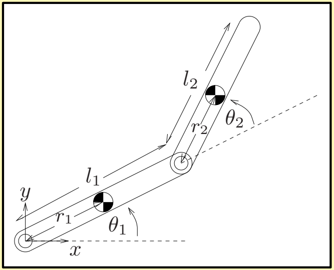

Figure 4.8

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4.2 Lagrange's Equations

$P_i = \begin{bmatrix} \bar{x}_i \\ \bar{y}_i \\ 0 \end{bmatrix}$: Mass center

γ_i : Distance from joint i to mass center

Change of Coordinates:

$$\begin{cases} \bar{x}_1 = r_1 c_1 \\ \bar{y}_1 = r_1 s_1 \end{cases} \Rightarrow \begin{cases} \dot{\bar{x}}_1 = -r_1 s_1 \dot{\theta}_1 \\ \dot{\bar{y}}_1 = r_1 c_1 \dot{\theta}_1 \end{cases}$$

$$\begin{cases} \bar{x}_2 = l_1 c_1 + r_2 c_{12} \\ \bar{y}_2 = l_1 s_1 + r_2 s_{12} \end{cases} \Rightarrow \begin{cases} \dot{\bar{x}}_2 = -(l_1 s_1 + r_2 s_{12})\dot{\theta}_1 - r_2 s_{12} \dot{\theta}_2 \\ \dot{\bar{y}}_1 = (l_1 c_1 + r_2 c_{12})\dot{\theta}_1 + r_2 c_{12} \dot{\theta}_2 \end{cases}$$

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4.2 Lagrange's Equations

Kinetic energy:

$$T(\theta,\dot\theta) = \frac{1}{2}m_1(\dot{\bar{x}}_1^2+\dot{\bar{y}}_1^2)+\frac{1}{2}\mathcal{I}_{z_1}\dot\theta_1^2+\frac{1}{2}m_2(\dot{\bar{x}}_2^2+\dot{\bar{y}}_2^2)+\frac{1}{2}\mathcal{I}_{z_2}(\dot\theta_1^2+\dot\theta_2^2)$$

$$= \frac{1}{2}[\dot\theta_1 \;\; \dot\theta_2]\underbrace{\begin{bmatrix}\alpha+2\beta c_2 & \delta+\beta c_2\\ \delta+\beta c_2 & \delta\end{bmatrix}}_{M(\theta)}\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix}$$

$$\alpha = \mathcal{I}_{z_1}+\mathcal{I}_{z_2}+m_1r_1^2+m_2(l_1^2+r_2^2),$$

$$\beta = m_2l_1r_2,\delta = \mathcal{I}_{z_2}+m_2r_2^2, L = T$$

Equation of motion:

$$M(\theta)\begin{bmatrix}\ddot\theta_1\\ \ddot\theta_2\end{bmatrix}+\begin{bmatrix}-\beta s_2\dot\theta_2 & -\beta s_2(\dot\theta_1+\dot\theta_2)\\ \beta s_2\dot\theta_1 & 0\end{bmatrix}\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix}=\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix}$$

† End of Section †

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Section 4.3

□ Dynamics of an open-chain manipulator:

Definition:

L_i: frame at mass center of link i, $g_{sl_i}(\theta) = e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_i\theta_i} g_{sl_i}(o)$

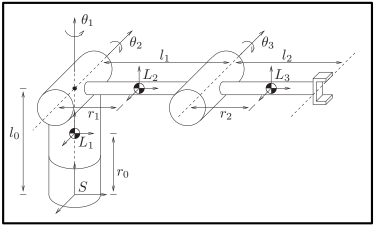

Figure 4.9

$$V^b_{sl_i} = J^b_{sl_i}(\theta)\dot{\theta} = [\xi_1^\dagger \;\; \xi_2^\dagger \;\; \cdots \;\; \xi_i^\dagger \;\; 0 \;\cdots\; 0] \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_i \\ \dot{\theta}_{i+1} \\ \vdots \\ \dot{\theta}_n \end{bmatrix} = J_i(\theta)\dot{\theta}$$

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4.3 Dynamics of Open-chain Manipulators

$$\xi_j^\dagger = \mathrm{Ad}^{-1}(e^{\hat{\xi}_{j+1}\theta_{j+1}} \cdots e^{\hat{\xi}_i\theta_i} g_{sl_i}(0))\xi_j, \; j \le i$$

$$T_i(\theta,\dot{\theta}) = \frac{1}{2}(V_{sl_i}^b)^T M_i^b V_{sl_i}^b = \frac{1}{2}\dot{\theta}^T J_i^T(\theta) M_i^b J_i(\theta)\dot{\theta}$$

$$T(\theta) = \sum_{i=1}^n T_i(\theta,\dot{\theta}) = \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta},$$

$$M(\theta) = \sum_i J_i^T(\theta) M_i^b J_i(\theta) = \frac{1}{2}\sum_{i,j=1}^n M_{ij}(\theta)\dot{\theta}_i\dot{\theta}_j$$

$h_i(\theta)$: Height of L_i, $\quad V_i(\theta) = m_i g h_i(\theta), \quad V(\theta) = \sum_{i=1} m_i g h_i(\theta)$

Lagrange's Equation:

$$\boxed{\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = \tau_i, \; i = 1,\ldots,n, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}_i} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{j=1}^n M_{ij}\dot{\theta}_j\right) = \sum_{j=1}^n M_{ij}\ddot{\theta}_j + \dot{M}_{ij}\dot{\theta}_j \end{aligned}}$$

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4.3 Dynamics of Open-chain Manipulators

$$\frac{\partial L}{\partial \theta_i} = \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial M_{kj}}{\partial \theta_i}\dot{\theta}_k\dot{\theta}_j - \frac{\partial V}{\partial \theta_i}, \qquad \dot{M}_{ij} = \sum_k \frac{\partial M_{ij}}{\partial \theta_k}\dot{\theta}_k$$

$$\Rightarrow \sum_{j=1}^{n} M_{ij}\ddot{\theta}_j + \sum_{j,k=1}^{n}\left(\frac{\partial M_{ij}}{\partial \theta_k}\dot{\theta}_j\dot{\theta}_k - \frac{1}{2}\frac{\partial M_{kj}}{\partial \theta_i}\dot{\theta}_k\dot{\theta}_j\right) + \frac{\partial V}{\partial \theta_i} = \tau_i$$

$$\Rightarrow \sum_{j=1}^{n} M_{ij}\ddot{\theta}_j + \sum_{j,k=1}^{n} \Gamma_{ijk}\dot{\theta}_k\dot{\theta}_j + \frac{\partial V}{\partial \theta_i} = \tau_i$$

$$\Gamma_{ijk} \triangleq \frac{1}{2}\left(\frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{ik}}{\partial \theta_j} - \frac{\partial M_{kj}}{\partial \theta_i}\right)$$

$\dot{\theta}_i \cdot \dot{\theta}_j, i \neq j:$ Coriolis term, $\dot{\theta}_i^2:$ Centrifugal term

4.3 Dynamics of Open-chain Manipulators

$$\frac{\partial L}{\partial \theta_i} = \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial M_{kj}}{\partial \theta_i}\dot\theta_k\dot\theta_j - \frac{\partial V}{\partial \theta_i}, \qquad \dot M_{ij} = \sum_k \frac{\partial M_{ij}}{\partial \theta_k}\dot\theta_k$$

$$\Rightarrow \sum_{j=1}^{n} M_{ij}\ddot\theta_j + \sum_{j,k=1}^{n}\left(\frac{\partial M_{ij}}{\partial \theta_k}\dot\theta_j\dot\theta_k - \frac{1}{2}\frac{\partial M_{kj}}{\partial \theta_i}\dot\theta_k\dot\theta_j\right) + \frac{\partial V}{\partial \theta_i} = \tau_i$$

$$\Rightarrow \sum_{j=1}^{n} M_{ij}\ddot\theta_j + \sum_{j,k=1}^{n}\Gamma_{ijk}\dot\theta_k\dot\theta_j + \frac{\partial V}{\partial \theta_i} = \tau_i$$

$$\Gamma_{ijk} \triangleq \frac{1}{2}\left(\frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{ik}}{\partial \theta_j} - \frac{\partial M_{kj}}{\partial \theta_i}\right)$$

$\dot\theta_i \cdot \dot\theta_j, i \neq j:$ Coriolis term, $\qquad \dot\theta_i^2:$ Centrifugal term

Define: $\quad C_{ij}(\theta,\dot\theta) = \sum_{k=1}^{n}\Gamma_{ijk}\dot\theta_k = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{ik}}{\partial \theta_j} - \frac{\partial M_{kj}}{\partial \theta_i}\right)\dot\theta_k$

$$\Rightarrow \boxed{M(\theta)\ddot\theta + C(\theta,\dot\theta)\dot\theta + N(\theta) = \tau}$$

Property 1:

1. $M(\theta) = M^T(\theta), \dot{\theta}^T M(\theta)\dot{\theta} \geq 0, \dot{\theta}^T M(\theta)\dot{\theta} = 0 \Leftrightarrow \dot{\theta} = 0$
2. $\dot{M} - 2C \in \mathbb{R}^{n\times n}$ is skew symmetric

4.3 Dynamics of Open-chain Manipulators

Property 1:

1. $M(\theta) = M^T(\theta), \dot{\theta}^T M(\theta)\dot{\theta} \geq 0, \dot{\theta}^T M(\theta)\dot{\theta} = 0 \Leftrightarrow \dot{\theta} = 0$
2. $\dot{M} - 2C \in \mathbb{R}^{n \times n}$ is skew symmetric

Proof :

$$
\begin{aligned}
(\dot{M} - 2C)_{ij} &= \dot{M}_{ij} - 2C_{ij}(\theta) \\
&= \sum_{k=1}^{n} \frac{\partial M_{ij}}{\partial \theta_k}\dot{\theta}_k - \frac{\partial M_{ij}}{\partial \theta_k}\dot{\theta}_k - \frac{\partial M_{ik}}{\partial \theta_j}\dot{\theta}_k + \frac{\partial M_{kj}}{\partial \theta_i}\dot{\theta}_k \\
&= \sum_{k=1}^{n} \frac{\partial M_{kj}}{\partial \theta_i}\dot{\theta}_k - \frac{\partial M_{ik}}{\partial \theta_j}\dot{\theta}_k
\end{aligned}
$$

Switching i and j shows $(\dot{M} - 2C)^T = -(\dot{M} - 2C)$ □

◇ Example: Planar 2-DoF Robot (continued)

$$
\begin{aligned}
m_{11}(\theta) &= \alpha + 2\beta\cos\theta_2, m_{22} = \delta \\
m_{12}(\theta) &= m_{21}(\theta) = \delta + \beta\cos\theta_2 \\
c_{11}(\theta,\dot{\theta}) &= -\beta\sin\theta_2\cdot\dot{\theta}_2, c_{12}(\theta,\dot{\theta}) = -\beta\sin\theta_2(\dot{\theta}_1+\dot{\theta}_2) \\
c_{21}(\theta,\dot{\theta}) &= \beta\sin\theta_2\cdot\dot{\theta}_1, c_{22}(\theta,\dot{\theta}) = 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{111} &= \frac{1}{2}(\frac{\partial M_{11}}{\partial\theta_1} + \frac{\partial M_{11}}{\partial\theta_1} - \frac{\partial M_{11}}{\partial\theta_1}) = \frac{1}{2}\frac{\partial M_{11}}{\partial\theta_1} = 0 \\
\Gamma_{112} &= \frac{1}{2}(\frac{\partial M_{11}}{\partial\theta_2} + \frac{\partial M_{12}}{\partial\theta_1} - \frac{\partial M_{21}}{\partial\theta_1}) = \frac{1}{2}\frac{\partial M_{11}}{\partial\theta_2} = -\beta\sin\theta_2 \\
\Gamma_{121} &= \frac{1}{2}(\frac{\partial M_{12}}{\partial\theta_1} + \frac{\partial M_{11}}{\partial\theta_2} - \frac{\partial M_{12}}{\partial\theta_1}) = \frac{1}{2}\frac{\partial M_{11}}{\partial\theta_2} = -\beta\sin\theta_2 \\
\Gamma_{122} &= \frac{1}{2}(\frac{\partial M_{12}}{\partial\theta_2} + \frac{\partial M_{12}}{\partial\theta_2} - \frac{\partial M_{22}}{\partial\theta_1}) = \frac{\partial M_{12}}{\partial\theta_2} - \frac{1}{2}\frac{\partial M_{22}}{\partial\theta_1} = -\beta\sin\theta_2
\end{aligned}
$$

(Continues next slide)

4.3 Dynamics of Open-chain Manipulators

$$\Gamma_{211} = \frac{1}{2}\left(\frac{\partial M_{21}}{\partial \theta_1} + \frac{\partial M_{21}}{\partial \theta_1} - \frac{\partial M_{11}}{\partial \theta_2}\right) = \frac{\partial M_{21}}{\partial \theta_1} - \frac{1}{2}\frac{\partial M_{11}}{\partial \theta_2} = \beta \sin \theta_2$$

$$\Gamma_{212} = \frac{1}{2}\left(\frac{\partial M_{21}}{\partial \theta_2} + \frac{\partial M_{22}}{\partial \theta_1} - \frac{\partial M_{21}}{\partial \theta_2}\right) = \frac{1}{2}\frac{\partial M_{22}}{\partial \theta_1} = 0$$

$$\Gamma_{221} = \frac{1}{2}\left(\frac{\partial M_{22}}{\partial \theta_1} + \frac{\partial M_{21}}{\partial \theta_2} - \frac{\partial M_{12}}{\partial \theta_2}\right) = \frac{1}{2}\frac{\partial M_{22}}{\partial \theta_1} = 0$$

$$\Gamma_{222} = \frac{1}{2}\left(\frac{\partial M_{22}}{\partial \theta_2} + \frac{\partial M_{22}}{\partial \theta_2} - \frac{\partial M_{22}}{\partial \theta_2}\right) = \frac{1}{2}\frac{\partial M_{22}}{\partial \theta_2} = 0$$

$$\begin{aligned}
\dot{M} - 2C &= \begin{bmatrix} -2\beta \sin\theta_2 \cdot \dot{\theta}_2 & -\beta \sin\theta_2 \cdot \dot{\theta}_2 \\ -\beta \sin\theta_2 \cdot \dot{\theta}_2 & 0 \end{bmatrix} \\
&\quad - \begin{bmatrix} -2\beta \sin\theta_2 \cdot \dot{\theta}_2 & -2\beta \sin\theta_2 (\dot{\theta}_1 + \dot{\theta}_2) \\ 2\beta \sin\theta_2 \cdot \dot{\theta}_1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \beta \sin\theta_2 (2\dot{\theta}_1 + \dot{\theta}_2) \\ -\beta \sin\theta_2 (2\dot{\theta}_1 + \dot{\theta}_2) & 0 \end{bmatrix} \Leftarrow \text{skew-symmetric}
\end{aligned}$$

$\diamondsuit$

◇ Example: Dynamics of a 3-dof robot

$$\xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \xi_2 = \begin{bmatrix} 0 \\ -l_0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \xi_3 = \begin{bmatrix} 0 \\ -l_0 \\ l_1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$g_{sl_1}(0) = \begin{bmatrix} I & \begin{pmatrix} 0 \\ 0 \\ r_0 \end{pmatrix} \\ 0 & 1 \end{bmatrix},$$

$$g_{sl_2}(0) = \begin{bmatrix} I & \begin{pmatrix} 0 \\ r_1 \\ l_0 \end{pmatrix} \\ 0 & 1 \end{bmatrix},$$

$$g_{sl_3}(0) = \begin{bmatrix} I & \begin{pmatrix} 0 \\ l_1 + r_2 \\ l_0 \end{pmatrix} \\ 0 & 1 \end{bmatrix}$$

$$M_i = \left[\begin{array}{ccc|ccc} m_i & 0 & 0 & & & \\ 0 & m_i & 0 & & 0 & \\ 0 & 0 & m_i & & & \\ \hline & & & \mathcal{I}_{x_i} & 0 & 0 \\ & 0 & & 0 & \mathcal{I}_{y_i} & 0 \\ & & & 0 & 0 & \mathcal{I}_{z_i} \end{array}\right]$$

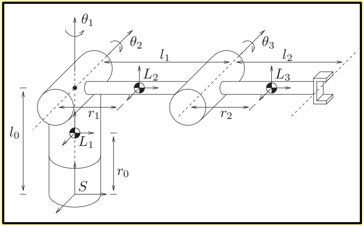

Figure 4.9

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4.3 Dynamics of Open-chain Manipulators

m_i: The mass of the object

$\mathcal{I}_{x_i}$: The moment of inertia about the x axis

$$
\begin{aligned}
\Gamma_{112} &= (\mathcal{I}_{y_2} - \mathcal{I}_{z_2} - m_2 r_1^2) c_2 s_2 + (\mathcal{I}_{y_2} - \mathcal{I}_{z_3}) c_{23} s_{23} - t(l_1 s_2 + r_2 s_{23}) \\
\Gamma_{113} &= (\mathcal{I}_{y_3} - \mathcal{I}_{z_3}) c_{23} s_{23} - t r_2 s_{23} \\
\Gamma_{121} &= (\mathcal{I}_{y_2} - \mathcal{I}_{z_2} - m_2 r_1^2) c_2 s_2 + (\mathcal{I}_{y_3} - \mathcal{I}_{z_3}) c_{23} s_{23} - t(l_1 s_2 + r_2 s_{23}) \\
\Gamma_{131} &= (\mathcal{I}_{y_3} - \mathcal{I}_{z_3}) c_{23} s_{23} - t r_2 s_{23} \\
\Gamma_{211} &= (\mathcal{I}_{z_2} - \mathcal{I}_{y_2} + m_2 r_1^2) c_2 s_2 + (\mathcal{I}_{z_3} - \mathcal{I}_{y_3}) c_{23} s_{23} + t(l_1 s_2 + r_2 s_{23}) \\
\Gamma_{223} &= -l_1 m_3 r_2 s_3 \\
\Gamma_{232} &= -l_1 m_3 r_2 s_3 \\
\Gamma_{233} &= -l_1 m_3 r_2 s_3 \\
\Gamma_{311} &= (\mathcal{I}_{z_3} - \mathcal{I}_{y_3}) c_{23} s_{23} + t r_2 s_{23} \\
\Gamma_{322} &= l_1 m_3 r_2 s_3
\end{aligned}
$$

where $t = m_3(l_1 c_2 + r_2 c_{23})$.

(Continues next slide)

4.3 Dynamics of Open-chain Manipulators

$$N(\theta,\dot{\theta}) = \frac{\partial V}{\partial \theta}, V(\theta) = m_1 g h_1(\theta) + m_2 g h_2(\theta) + m_2 g h_3(\theta)$$

$$g_{sl_i}(\theta) = e^{\hat{\xi}_1\theta_1}\dots e^{\hat{\xi}_i\theta_i} g_{sl_i}(0) \Rightarrow$$

$$h_1(\theta) = r_0,\ h_2(\theta) = l_0 - r_1 \sin\theta, h_3(\theta) = l_0 - l_1 \sin\theta_2 - r_2 \sin(\theta_2+\theta_3)$$

$$J_1 = J^b_{sl_1}(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$J_2 = J^b_{sl_2}(\theta) = \begin{bmatrix} -r_1 c_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -r_1 & 0 \\ 0 & -1 & 0 \\ -s_2 & 0 & 0 \\ c_2 & 0 & 0 \end{bmatrix}$$

$$J_3 = J^b_{sl_3}(\theta) = \begin{bmatrix} -l_2 c_2 - r_2 c_{23} & 0 & 0 \\ 0 & l_1 s_3 & 0 \\ 0 & -r_2 - l_1 c_3 & -r_2 \\ 0 & -1 & -1 \\ -s_{23} & 0 & 0 \\ c_{23} & 0 & 0 \end{bmatrix}$$

(Continues next slide)

4.3 Dynamics of Open-chain Manipulators

$$M(\theta) = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} = J_1^T M_1 J_1 + J_2^T M_2 J_2 + J_3^T M_3 J_3$$

$$M_{11} = I_{y2} s_2^2 + I_{y3} s_{23}^2 + I_{z1} + I_{z2} c_2^2 + I_{z3} c_{23}^2 + m_2 r_1^2 c_2^2 + m_3 (l_1 c_2 + r_2 c_{23})^2$$

$$M_{12} = M_{13} = M_{21} = M_{31} = 0$$

$$M_{22} = I_{x2} + I_{x3} + m_2 l_1^2 + M_2 r_1^2 + m_3 r_2^2 + 2 m_3 l_1 r_2 c_3$$

$$M_{23} = I_{x3} + m_3 r_2^2 + m_3 l_1 r_2 c_3$$

$$M_{32} = I_{x3} + m_3 r_2^2 + m_3 l_1 r_2 c_3$$

$$M_{33} = I_{x_3} + m_3 r_2^2$$

$$C_{ij}(\theta, \dot\theta) = \sum_{k=1}^{n} \Gamma_{ijk} \dot\theta_k = \frac{1}{2} \sum_{k=1}^{n} \left(\frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{ik}}{\partial \theta_j} - \frac{\partial M_{kj}}{\partial \theta_i} \right) \dot\theta_k \qquad \diamond$$

□ Additional Properties of the dynamics in terms of POE:

Define:

$$A_{ij} = \begin{cases} \mathrm{Ad}^{-1}_{e^{\hat{\xi}_{j+1}\theta_{j+1}}\cdots e^{\hat{\xi}_i\theta_i}} & i > j \\ I & i = j \\ 0 & i < j \end{cases}$$

$$J_i(\theta) = \mathrm{Ad}_{g^{-1}_{sl_i}(0)}\left[A_{i1}\xi_1 \cdots A_{ii}\xi_i \;\; 0 \cdots 0\right]$$

$$M_i' = \mathrm{Ad}^T_{g^{-1}_{sl_i(0)}} M_i \mathrm{Ad}_{g^{-1}_{sl_i(0)}} \quad (\text{ intertia of } i^{th} \text{ link in } S)$$

☐ Additional Properties of the dynamics in terms of POE:

Define:

$$A_{ij} = \begin{cases} \mathrm{Ad}^{-1}_{e^{\hat{\xi}_{j+1}\theta_{j+1}} \cdots e^{\hat{\xi}_i\theta_i}} & i > j \\ I & i = j \\ 0 & i < j \end{cases}$$

$$J_i(\theta) = \mathrm{Ad}_{g^{-1}_{sl_i}(0)} \left[A_{i1}\xi_1 \cdots A_{ii}\xi_i \ \ 0 \cdots 0 \right]$$

$$M'_i = \mathrm{Ad}^T_{g^{-1}_{sl_i}(0)} M_i \mathrm{Ad}_{g^{-1}_{sl_i}(0)} \quad (\text{ intertia of } i^{th} \text{ link in } S)$$

Property 2:

$$M_{ij}(\theta) = \sum_{l=\max(i,j)}^{n} \xi_i^T A_{li}^T M'_l A_{lj} \xi_j, \quad C_{ij}(\theta, \dot{\theta}) = \frac{1}{2} \sum_{k=1}^{n} \left(\frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{ik}}{\partial \theta_j} - \frac{\partial M_{kj}}{\partial \theta_i} \right) \dot{\theta}_k$$

where

$$\frac{\partial M_{ij}}{\partial \theta_k} = \sum_{l=\max(i,j)}^{n} \left([A_{k-1,i}\xi_i, \xi_k]^T A_{lk}^T M'_l A_{lj}\xi_j + \xi_i^T A_{li}^T M'_l A_{lk} [A_{k-1,j}\xi_j, \xi_k] \right)$$

† End of Section †

Section 4.4

4.4 Coordinate Invariant Algorithms

☐ Newton-Euler equations in spatial frame:

Newton's Equation:

$$f^s = \frac{d}{dt}(m\dot{p}) = m\ddot{p}$$

Spatial angular momentum:

$$\mathcal{I}^s \cdot \omega^s = R(\mathcal{I} \cdot \omega^b) = \underbrace{R \cdot \mathcal{I} \cdot R^T}_{\mathcal{I}^s} \cdot \omega^s$$

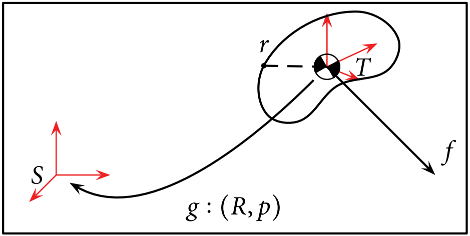

Figure 4.10

(Continues next slide)

4.4 Coordinate Invariant Algorithms

$$
\begin{aligned}
\tau^s &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{I}^s\omega^s) = \frac{\mathrm{d}}{\mathrm{d}t}(R\mathcal{I}R^T\omega^s) = \mathcal{I}^s\dot{\omega}^s + \dot{R}\mathcal{I}R^T\omega^s + R\mathcal{I}\dot{R}^T\omega^s \\
&= \mathcal{I}^s\dot{\omega}^s + \underbrace{\dot{R}R^T}_{\hat{\omega}^s}\mathcal{I}^s\omega^s - R\mathcal{I}R^T\hat{\omega}^s\omega^s = \mathcal{I}^s\dot{\omega}^s + \omega^s \times (\mathcal{I}^s\omega^s)
\end{aligned}
$$

□ Transform equations to body frame:

$$
\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{p}) = \frac{\mathrm{d}}{\mathrm{d}t}\left(mRv^b\right) = m\dot{R}v^b + mR\dot{v}^b, R^T f^s = mR^T\dot{R}v^b + m\dot{v}^b
$$

$$
\Rightarrow f^b = m\omega^b \times v^b + m\dot{v}^b,
$$

$$
\tau^b = R^T\tau^s = R^T\frac{\mathrm{d}}{\mathrm{d}t}(R\mathcal{I}\omega^b) = \mathcal{I}\dot{\omega}^b + \omega^b \times \mathcal{I}\omega^b
$$

$$
\Rightarrow \underbrace{\begin{bmatrix} mI & 0 \\ 0 & \mathcal{I} \end{bmatrix}}_{M^b}\underbrace{\begin{bmatrix} \dot{v}^b \\ \dot{\omega}^b \end{bmatrix}}_{V^b} + \begin{bmatrix} \omega^b \times mv^b \\ \omega^b \times \mathcal{I}\omega^b \end{bmatrix} = \begin{bmatrix} f^b \\ \tau^b \end{bmatrix} = F^b \qquad (*)
$$

(Continues next slide)

4.4 Coordinate Invariant Algorithms

Define:

$$[\ ,\ ] : se(3) \times se(3) \mapsto se(3), [\hat{\xi}_1, \hat{\xi}_2] \triangleq \hat{\xi}_1\hat{\xi}_2 - \hat{\xi}_2\hat{\xi}_1$$

if

$$[\hat{\xi}_1, \hat{\xi}_2] = \begin{bmatrix} \hat{\omega}_i & v_i \\ 0 & 0 \end{bmatrix}, i = 1, 2$$

then

$$\hat{\xi} = \begin{bmatrix} (\omega_1 \times \omega_2)^\wedge & \hat{\omega}_1 v_2 - \hat{\omega}_2 v_1 \\ 0 & 0 \end{bmatrix} = \mathrm{ad}_{\xi_i} \cdot \xi_2$$

where

$$\mathrm{ad}_{\xi_1} = \begin{bmatrix} \hat{\omega}_1 & \hat{v}_1 \\ 0 & \hat{\omega}_1 \end{bmatrix}$$

It is straightforward computation to see that:

$$(*) \Leftrightarrow M^b \dot{V}^b - \mathrm{ad}^T_{V^b} M^b V^b = F^b$$

Property 3:

$$\mathrm{Ad}_g[\hat{\xi}_1, \hat{\xi}_2] = [\mathrm{Ad}_g\hat{\xi}_1, \mathrm{Ad}_g\hat{\xi}_2] \Rightarrow \mathrm{Ad}_g \mathrm{ad}_{\hat{\xi}_1} = \mathrm{ad}_{\mathrm{Ad}_g\hat{\xi}_1}\mathrm{Ad}_g$$

4.4 Coordinate Invariant Algorithms

☐ Coordinate invariance of Newton-Euler equations:

$$M_1\dot{V}_1 - \mathrm{ad}^T_{V_1} M_1 V_1 = F_1$$

$$\begin{cases} V_1 = \mathrm{Ad}_{g_0} V_2, \\ F_1 = \mathrm{Ad}^T_{g_0^{-1}} F_2 \end{cases} \Rightarrow$$

$$F_1 = (\mathrm{Ad}^T_{g_0})^{-1} F_2 = (\mathrm{Ad}^{-1}_{g_0})^T F_2$$

$$M_1 = \mathrm{Ad}^{-T}_{g_0} M_2 \mathrm{Ad}^{-1}_{g_0}$$

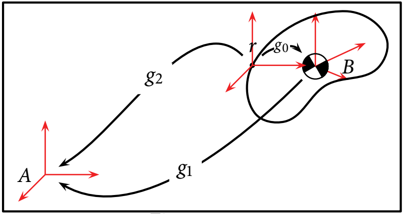

Figure 4.11

$$\Rightarrow \mathrm{Ad}^{-T}_{g_0} M_2 \mathrm{Ad}^{-1}_{g_0} \mathrm{Ad}_{g_0} \dot{V}_2 - \mathrm{ad}^T_{(\mathrm{Ad}_{g_0} \dot{V}_2)} \mathrm{Ad}^{-T}_{g_0} M_2 \mathrm{Ad}^{-1}_{g_0} \mathrm{Ad}_{g_0} V_1 = \mathrm{Ad}^{-T}_{g_0} F_2$$

Since $\mathrm{ad}_{\mathrm{Ad}_{g_0} V} = \mathrm{Ad}_{g_0} \mathrm{ad}_V \mathrm{Ad}^{-1}_{g_0}$ by Property 3, we have

$$\boxed{M_2 \dot{V}_2 - \mathrm{ad}^T_{V_2} M_2 V_2 = F_2}$$

4.4 Coordinate Invariant Algorithms

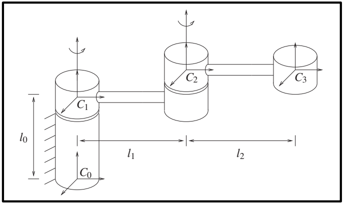

Figure 4.12

C_i: Frame fixed to link i, located along the i^{th} axis

F_i: Generalized force link $i-1$ exerting on link i, expressed in C_i

τ_i: Joint torque of link i

$g_{i-1,i}$: Transformation of C_i relative to C_{i-1}

$$g_{i-1,i}(\theta_i) = e^{\hat{\xi}'_i \theta_i} \cdot g_{i-1,i}(0) = g_{i-1,i}(0) e^{\hat{\xi}_i \theta_i}$$

$\xi_i = \mathrm{Ad}_{g^{-1}_{i-1,i}(0)} \cdot \xi'_i$: i^{th} axis in C_i frame.

$$\xi_i = \begin{cases} \begin{bmatrix} 0 \\ z_i \end{bmatrix} : & \text{Revolute joint.} \\ \begin{bmatrix} z_i \\ 0 \end{bmatrix} : & \text{Prismatic joint.} \end{cases}$$

4.4 Coordinate Invariant Algorithms

$$\Rightarrow g_{i-1,i}^{-1} \cdot \dot{g}_{i-1,i} = \hat{\xi}_i \cdot \dot{\theta}_i$$

M_i: Moment of inertia in C_i

$$M_i = \begin{bmatrix} m_i I & -m_i \hat{r}_i \\ m_i \hat{r}_i & \mathcal{I}_i - m_i \hat{r}_i^2 \end{bmatrix}$$

m_i: Mass of link i

$\mathcal{I}_i$: inertia tensor

$$g_i = g_{i-1} g_{i-1,i}$$

$$\hat{V}_i = g_i^{-1} \cdot \dot{g}_i = g_{i-1,i}^{-1} \hat{V}_{i-1} g_{i-1,i} + \hat{\xi}_i \dot{\theta}_i$$

$$V_i = \mathrm{Ad}_{g_{i-1,i}^{-1}} V_{i-1} + \xi_i \dot{\theta}_i$$

$$\begin{aligned} \dot{\hat{V}}_i &= \dot{g}_{i-1,i}^{-1} \hat{V}_{i-1} g_{i-1,i} + g_{i-1,i}^{-1} \hat{V}_{i-1} \dot{g}_{i-1,i} + g_{i-1,i}^{-1} \dot{\hat{V}}_{i-1} g_{i-1,i} + \hat{\xi}_i \ddot{\theta}_i \\ &= -\dot{g}_{i-1,i}^{-1} g_{i-1,i} g_{i-1,i}^{-1} \hat{V}_{i-1} g_{i-1,i} + g_{i-1,i}^{-1} \hat{V}_{i-1} g_{i-1,i} g_{i-1,i}^{-1} \dot{g}_{i-1,i} \\ &\quad + g_{i-1,i}^{-1} \dot{\hat{V}}_{i-1} g_{i-1,i} + \hat{\xi}_i \ddot{\theta}_i \end{aligned}$$

(Continues next slide)

$$= -\hat{\xi}_i \dot{\theta}_i (\mathrm{Ad}_{g_{i-1,i}^{-1}} V_{i-1})^\wedge + (\mathrm{Ad}_{g_{i-1,i}^{-1}} V_{i-1})^\wedge \hat{\xi}_i \dot{\theta}_i + (\mathrm{Ad}_{g_{i-1,i}^{-1}} \dot{V}_{i-1})^\wedge + \hat{\xi}_i \ddot{\theta}_i$$

$$\Rightarrow \dot{V}_i = \xi_i \ddot{\theta}_i + \mathrm{Ad}_{g_{i-1,i}^{-1}} \dot{V}_{i-1} - \mathrm{ad}_{\xi_i \dot{\theta}_i} (\mathrm{Ad}_{g_{i-1,i}^{-1}} V_{i-1})$$

□ Forward Recursion (kinematics):

$$\begin{cases} \text{init.}: V_0 = 0, \dot{V}_0 = \begin{bmatrix} g \\ 0 \end{bmatrix} (\text{gravity vector}) \\ g_{i-1,i} = g_{i-1,i}(0) e^{\hat{\xi}_i \theta_i} \\ V_i = \mathrm{Ad}_{g_{i-1,i}^{-1}} V_{i-1} + \xi_i \dot{\theta}_i \\ \dot{V}_i = \xi_i \ddot{\theta}_i + \mathrm{Ad}_{g_{i-1,i}^{-1}} \dot{V}_{i-1} - \mathrm{ad}_{\xi_i \dot{\theta}_i} (\mathrm{Ad}_{g_{i-1,i}^{-1}} V_{i-1}) \end{cases}$$

□ Backward Recursion (inverse dynamics):

F_{n+1}: End-effector wrench, $g_{n,n+1}$: transform from tool frame to C_n

$$F_i = \mathrm{Ad}^T_{g_{i,i+1}^{-1}} \cdot F_{i+1} + M_i \dot{V}_i - \mathrm{ad}^T_{V_i} \cdot M_i V_i$$

$$\tau_i = \xi_i^T \cdot F_i$$

4.4 Coordinate Invariant Algorithms

$$V_1 = \mathrm{Ad}_{g_{0,1}^{-1}} \cdot V_0 + \xi_1 \dot{\theta}_1$$

$$F_n = \mathrm{Ad}^T_{g^{-1}_{n,n+1}} \cdot F_{n+1} + M_n \dot{V}_n - \mathrm{ad}^T_{V_n} \cdot (M_n V_n)$$

Define:

$$V = \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix} \in \mathbb{R}^{6n\times 1}, \dot{\theta} = \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} \in \mathbb{R}^n, \xi = \begin{bmatrix} \xi_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \xi_n \end{bmatrix} \in \mathbb{R}^{6n\times n}$$

$$F = \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix} \in \mathbb{R}^{6n\times 1}, \tau = \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix} \in \mathbb{R}^n, P_0 = \begin{bmatrix} \mathrm{Ad}_{g_{0,1}^{-1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{6n\times 6}$$

$$P_t = \begin{bmatrix} 0 \cdots 0 & \mathrm{Ad}_{g^{-1}_{n,n+1}} \end{bmatrix} \in \mathbb{R}^{6\times 6n}$$

4.4 Coordinate Invariant Algorithms

$$
\begin{aligned}
V_1 &= \mathrm{Ad}_{g_{0,1}^{-1}} \cdot V_0 + \xi_1 \dot\theta_1 \\
V_2 - \mathrm{Ad}_{g_{1,2}^{-1}} V_1 &= \xi_2 \dot\theta_2 \\
&\vdots \\
V_n - \mathrm{Ad}_{g_{n-1,n}^{-1}} V_{n-1} &= \xi_n \dot\theta_n
\end{aligned}
$$

$$
\Rightarrow \underbrace{\begin{bmatrix} I & 0 & \cdots & 0 \\ -\mathrm{Ad}_{g_{1,2}^{-1}} & I & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & -\mathrm{Ad}_{g_{n-1,n}^{-1}} & I \end{bmatrix}}_{G^{-1}} \underbrace{\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}}_{V}
$$

$$
= \underbrace{\begin{bmatrix} \mathrm{Ad}_{g_{0,1}^{-1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{P_0} V_0 + \underbrace{\begin{bmatrix} \xi_1 & & & \\ & \xi_2 & & \\ & & \ddots & \\ & & & \xi_n \end{bmatrix}}_{\xi} \underbrace{\begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \vdots \\ \dot\theta_n \end{bmatrix}}_{\dot\theta}
$$

Thus $V = G P_0 V_0 + G \xi \dot\theta$

4.4 Coordinate Invariant Algorithms

where

$$G = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ \mathrm{Ad}_{g_{1,2}^{-1}} & I & 0 & \cdots & 0 \\ \mathrm{Ad}_{g_{1,3}^{-1}} & \mathrm{Ad}_{g_{2,3}^{-1}} & I & \ddots & \vdots \\ \vdots & \vdots & \ddots & I & 0 \\ \mathrm{Ad}_{g_{1,n}^{-1}} & \mathrm{Ad}_{g_{2,n}^{-1}} & \cdots & \mathrm{Ad}_{g_{n-1,n}^{-1}} & I \end{bmatrix} \in \mathbb{R}^{6n\times 6n}$$

$$\begin{aligned} \dot{V}_1 &= \xi_1\ddot{\theta}_1 + \mathrm{Ad}_{g_{0,1}^{-1}}\dot{V}_0 - \mathrm{ad}_{\xi_1\dot{\theta}_1}(\mathrm{Ad}_{g_{0,1}^{-1}}V_0) \\ \dot{V}_2 - \mathrm{Ad}_{g_{1,2}^{-1}}\dot{V}_1 &= \xi_2\ddot{\theta}_2 - \mathrm{ad}_{\xi_2\dot{\theta}_2}(\mathrm{Ad}_{g_{1,2}^{-1}}V_1) \\ \dot{V}_n - \mathrm{Ad}_{g_{n-1,n}^{-1}}\dot{V}_{n-1} &= \xi_n\ddot{\theta}_n - \mathrm{ad}_{\xi_n\dot{\theta}_n}(\mathrm{Ad}_{g_{n-1,n}^{-1}}V_{n-1}) \end{aligned}$$

(Continues next slide)

4.4 Coordinate Invariant Algorithms

$$\underbrace{\begin{bmatrix} I & 0 & \cdots & 0 \\ -\mathrm{Ad}_{g_{1,2}^{-1}} & I & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & -\mathrm{Ad}_{g_{n-1,n}^{-1}} & I \end{bmatrix}}_{G^{-1}} \begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \vdots \\ \dot{V}_n \end{bmatrix} = \underbrace{\begin{bmatrix} \mathrm{Ad}_{g_{0,1}^{-1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{P_0} \dot{V}_0 + \underbrace{\begin{bmatrix} \xi_1 & & \\ & \xi_2 & \\ & & \ddots \\ & & & \xi_n \end{bmatrix}}_{\xi} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \vdots \\ \ddot{\theta}_n \end{bmatrix} +$$

$$\underbrace{\begin{bmatrix} -\mathrm{ad}_{\xi_1\dot{\theta}_1} & 0 & \cdots & 0 \\ 0 & -\mathrm{ad}_{\xi_2\dot{\theta}_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\mathrm{ad}_{\xi_n\dot{\theta}_n} \end{bmatrix}}_{\mathrm{ad}_{\xi\dot{\theta}}} \begin{bmatrix} Ad_{g_{0,1}^{-1}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} V_0 +$$

$$\begin{bmatrix} -\mathrm{ad}_{\xi_1\dot{\theta}_1} & 0 & \cdots & 0 \\ 0 & -\mathrm{ad}_{\xi_2\dot{\theta}_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\mathrm{ad}_{\xi_n\dot{\theta}_n} \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ \mathrm{Ad}_{g_{1,2}^{-1}} & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \mathrm{Ad}_{g_{n-1,n}^{-1}} & 0 \end{bmatrix}}_{\Gamma} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}$$

(Continues next slide)

4.4 Coordinate Invariant Algorithms

Thus

$$\dot{V} = G \cdot \xi \ddot{\theta} + G \cdot P_0 \dot{V}_0 + G \cdot \mathrm{ad}_{\xi \dot{\theta}} P_0 V_0 + G \cdot \mathrm{ad}_{\xi \dot{\theta}} \Gamma V$$

Finally the backward recursion:

$$\begin{aligned}
F_n &= \mathrm{Ad}^T_{g^{-1}_{n,n+1}} F_{n+1} + M_n \dot{V}_n - \mathrm{ad}^T_{V_n} \cdot (M_n V_n) \\
F_{n-1} &= \mathrm{Ad}^T_{g^{-1}_{n-1,n}} F_n + M_{n-1} \dot{V}_{n-1} - \mathrm{ad}^T_{V_{n-1}} \cdot (M_{n-1} V_{n-1}) \\
&\vdots \\
F_1 &= \mathrm{Ad}^T_{g^{-1}_{1,2}} F_2 + M_1 \dot{V}_1 - \mathrm{ad}^T_{V_1} (M_1 V_1)
\end{aligned}$$

(Continues next slide)

4.4 Coordinate Invariant Algorithms

$$\Rightarrow \begin{bmatrix} I & -\mathrm{Ad}^T_{g^{-1}_{1,2}} & 0 & 0 \\ 0 & I & \ddots & 0 \\ \vdots & \ddots & \ddots & -\mathrm{Ad}^T_{g^{-1}_{n-1,n}} \\ 0 & \cdots & 0 & I \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix} = \underbrace{\begin{bmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_n \end{bmatrix}}_{M} \begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \vdots \\ \dot{V}_n \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{Ad}^T_{g^{-1}_{n,n+1}} \end{bmatrix}}_{P_t^T} F_{n+1} +$$

$$\underbrace{\begin{bmatrix} -\mathrm{ad}^T_{V_1} & & \\ & \ddots & \\ & & -\mathrm{ad}^T_{V_n} \end{bmatrix}}_{\mathrm{ad}^T_V} \begin{bmatrix} M_1 & & \\ & \ddots & \\ & & M_n \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix}$$

4.4 Coordinate Invariant Algorithms

$$\Rightarrow \begin{bmatrix} I & -\mathrm{Ad}^T_{g^{-1}_{1,2}} & 0 & 0 \\ 0 & I & \ddots & 0 \\ \vdots & \ddots & \ddots & -\mathrm{Ad}^T_{g^{-1}_{n-1,n}} \\ 0 & \cdots & 0 & I \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix} = \underbrace{\begin{bmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_n \end{bmatrix}}_{M} \begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \vdots \\ \dot{V}_n \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{Ad}^T_{g^{-1}_{n,n+1}} \end{bmatrix}}_{P_t^T} F_{n+1} +$$

$$\underbrace{\begin{bmatrix} -\mathrm{ad}^T_{V_1} & & \\ & \ddots & \\ & & -\mathrm{ad}^T_{V_n} \end{bmatrix}}_{\mathrm{ad}^T_V} \begin{bmatrix} M_1 & & \\ & \ddots & \\ & & M_n \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix}$$

$$\boxed{F = G^T M \dot{V} + G^T P_t^T \underbrace{F_{n+1}}_{F_t} + G^T \cdot \mathrm{ad}^T_V M V}$$

(Continues next slide)

4.4 Coordinate Invariant Algorithms

$$\tau = \xi^T \cdot F$$

$$\Rightarrow \begin{aligned} \tau &= \xi^T G^T M \dot{V} + \xi^T G^T P_t^T F_t + \xi^T G^T \cdot \mathrm{ad}_V^T M V \\ &= \xi^T G^T M (G \xi \ddot{\theta} + G P_0 \dot{V}_0 + G \cdot \mathrm{ad}_{\xi \dot{\theta}} P_0 V_0 + G \cdot \mathrm{ad}_{\xi \dot{\theta}} \Gamma V) + \\ & \xi^T G^T P_t^T F_t + \xi^T G^T \cdot \mathrm{ad}_V^T M V \\ &= \xi^T G^T M G \xi \ddot{\theta} + \xi^T G^T M G P_0 \dot{V}_0 + \xi^T G^T M G \cdot \mathrm{ad}_{\xi \dot{\theta}} \Gamma V + \\ & \xi^T G^T P_t^T F_t + \xi^T G^T \cdot \mathrm{ad}_V^T M V \end{aligned}$$

4.4 Coordinate Invariant Algorithms

$$\tau = \xi^T \cdot F$$

$$\Rightarrow \begin{aligned} \tau &= \xi^T G^T M \dot{V} + \xi^T G^T P_t^T F_t + \xi^T G^T \cdot \mathrm{ad}_V^T M V \\ &= \xi^T G^T M (G\xi\ddot{\theta} + G P_0 \dot{V}_0 + G \cdot \mathrm{ad}_{\xi\dot{\theta}} P_0 V_0 + G \cdot \mathrm{ad}_{\xi\dot{\theta}} \Gamma V) + \\ &\xi^T G^T P_t^T F_t + \xi^T G^T \cdot \mathrm{ad}_V^T M V \\ &= \xi^T G^T M G \xi \ddot{\theta} + \xi^T G^T M G P_0 \dot{V}_0 + \xi^T G^T M G \cdot \mathrm{ad}_{\xi\dot{\theta}} \Gamma V + \\ &\xi^T G^T P_t^T F_t + \xi^T G^T \cdot \mathrm{ad}_V^T M V \end{aligned}$$

Finally we get (recall that $V_0 = 0$, we have $V = G\xi\dot{\theta}$):

$$\overbrace{\xi^T G^T M G \xi}^{M(\theta)} \ddot{\theta} + \overbrace{\xi^T G^T (M G \cdot \mathrm{ad}_{\xi\dot{\theta}} \Gamma + \mathrm{ad}_V^T M) G \xi}^{C(\theta,\dot{\theta})} \dot{\theta} +$$
$$\underbrace{\xi^T G^T M G P_0 \dot{V}_0}_{\phi(\theta)} + \underbrace{\xi^T G^T P_t^T}_{J_t^T(\theta)} F_t = \tau$$

4.4 Coordinate Invariant Algorithms

$$\boxed{M(\theta)\ddot{\theta} + C(\theta,\dot{\theta}) + \phi(\theta) + J_t^T(\theta)F_t = \tau}$$

$$\begin{aligned} M(\theta) &= \xi^T G^T M G \xi \\ C(\theta,\dot{\theta}) &= \xi^T G^T (M G \operatorname{ad}_{\xi\dot{\theta}} \Gamma + \operatorname{ad}_V^T M) G \xi \dot{\theta} \\ \phi(\theta) &= \xi^T G^T M G P_0 \dot{V}_0 \\ J_t &= P_t G \xi \end{aligned}$$

$$M = \begin{bmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_n \end{bmatrix}$$

Property 4:

$$\Gamma^n = 0,$$

$$G = (I - \Gamma)^{-1} = I + \Gamma + \Gamma^2 + \cdots + \Gamma^{n-1}$$

$$I + \Gamma G = G$$

† End of Section †

4.4 Coordinate Invariant Algorithms

□ Square root factorization of M:

Definition: Articulated body inertia algorithm (Featherstone)

$$
\begin{aligned}
\hat{M}_n &= M_n, b_n = -\mathrm{ad}_{V_n}^T (M_n V_n) \\
F_i &= \hat{M}_i \dot{V}_i + b_i, i = n, \ldots, 1 \\
b_i &= b_i(V_i, V_{i+1}, \xi_{i+1}, \hat{M}_{i+1}, \tau_{i+1}) \text{ (bias force)}
\end{aligned}
$$

$$
\left\{
\begin{aligned}
F_i &= \mathrm{Ad}_{g_{i,i+1}^{-1}}^T F_{i+1} + M_i \dot{V}_i - \mathrm{ad}_{V_i}^T M_i V_i \\
F_{i+1} &= \hat{M}_{i+1} \dot{V}_{i+1} + b_{i+1} \\
\dot{V}_{i+1} &= \mathrm{Ad}_{g_{i,i+1}^{-1}} \dot{V}_i + \xi_{i+1} \ddot{\theta}_{i+1} - \mathrm{ad}_{\xi_{i+1}\dot{\theta}_{i+1}} \mathrm{Ad}_{g_{i,i+1}^{-1}} V_i
\end{aligned}
\right. \Rightarrow
$$

(Continues next slide)

□ Square root factorization of M:

$$F_{i+1} = \hat{M}_{i+1}(\mathrm{Ad}_{g_{i,i+1}^{-1}} \dot{V}_i + \xi_{i+1}\ddot{\theta}_{i+1} - \mathrm{ad}_{\xi_{i+1}\dot{\theta}_{i+1}} \mathrm{Ad}_{g_{i,i+1}^{-1}} V_i) + b_{i+1} \Rightarrow$$

$$\tau_{i+1} = \xi_{i+1}^T F_{i+1} = \xi_{i+1}^T(\hat{M}_{i+1}(\mathrm{Ad}_{g_{i,i+1}^{-1}} \dot{V}_i + \xi_{i+1}\ddot{\theta}_{i+1} - \mathrm{ad}_{\xi_{i+1}\dot{\theta}_{i+1}} \mathrm{Ad}_{g_{i,i+1}^{-1}} V) + b_{i+1}) \Rightarrow$$

$$\ddot{\theta}_{i+1} = \frac{\tau_{i+1} + \xi_{i+1}^T(\hat{M}_{i+1}(-\mathrm{Ad}_{g_{i,i+1}^{-1}} \dot{V}_i + \mathrm{ad}_{\xi_{i+1}\dot{\theta}_{i+1}} \mathrm{Ad}_{g_{i,i+1}^{-1}} V_i) - b_{i+1})}{\xi_{i+1}^T \hat{M}_{i+1} \xi_{i+1}}$$

$$\Rightarrow$$

$$F_i = \mathrm{Ad}_{g_{i,i+1}^{-1}}^T(\hat{M}_{i+1}(\mathrm{Ad}_{g_{i,i+1}^{-1}} \dot{V}_i +$$

$$\xi_{i+1}\left(\frac{\tau_{i+1} + \xi_{i+1}^T(\hat{M}_{i+1}(-\mathrm{Ad}_{g_{i,i+1}^{-1}} \dot{V}_i + \mathrm{ad}_{\xi_{i+1}\dot{\theta}_{i+1}} \mathrm{Ad}_{g_{i,i+1}^{-1}} V_i) - b_{i+1})}{\xi_{i+1}^T \hat{M}_{i+1} \xi_{i+1}}\right)$$

$$- \mathrm{ad}_{\xi_{i+1}\dot{\theta}_{i+1}} \mathrm{Ad}_{g_{i,i+1}^{-1}} V_i) + b_{i+1}) + M_i \dot{V}_i - \mathrm{ad}_{V_i}^T M_i V_i$$

$$\Rightarrow$$

(Continues next slide)

□ Square root factorization of M:

$$\hat{M}_i = M_i + \mathrm{Ad}^T_{g^{-1}_{i,i+1}} \hat{M}_{i+1} \mathrm{Ad}_{g^{-1}_{i,i+1}} - \frac{\mathrm{Ad}^T_{g^{-1}_{i,i+1}} \hat{M}_{i+1} \xi_{i+1} \xi^T_{i+1} \hat{M}_{i+1} \mathrm{Ad}_{g^{-1}_{i,i+1}}}{\xi^T_{i+1} \hat{M}_{i+1} \xi_{i+1}}$$

$$b_i = \mathrm{Ad}^T_{g^{-1}_{i,i+1}} b_{i+1} - \mathrm{ad}^T_{V_i} M_i V_i - \mathrm{Ad}^T_{g^{-1}_{i,i+1}} \hat{M}_{i+1} \mathrm{ad}_{\xi_{i+1}\dot{\theta}_{i+1}} \mathrm{Ad}_{g^{-1}_{i,i+1}} V_i + \frac{\mathrm{Ad}^T_{g^{-1}_{i,i+1}} \hat{M}_{i+1} \xi_{i+1} (\tau_{i+1} - \xi^T_{i+1} b_{i+1} + \xi^T_{i+1} \hat{M}_{i+1} \mathrm{ad}_{\xi_{i+1}\dot{\theta}_{i+1}} \mathrm{Ad}_{g^{-1}_{i,i+1}} V_i)}{\xi^T_{i+1} \hat{M}_{i+1} \xi_{i+1}}$$

$$\Rightarrow \hat{M} = M + \Gamma^T \hat{M} \Gamma - \Gamma^T \hat{M} \xi (\xi^T \hat{M} \xi)^{-1} \xi^T \hat{M} \Gamma$$

$$\Rightarrow M = \hat{M} - \Gamma^T \hat{M} \Gamma + \Gamma^T \hat{M} \xi (\xi^T \hat{M} \xi)^{-1} \xi^T \hat{M} \Gamma$$

$$\Rightarrow M(\theta) = \xi^T G^T (\hat{M} - \Gamma^T \hat{M} \Gamma + \Gamma^T \hat{M} \xi (\xi^T \hat{M} \xi)^{-1} \xi^T \hat{M} \Gamma) G \xi$$

$$\Rightarrow M(\theta) = \xi^T G^T \hat{M}^T \xi (\xi^T \hat{M} \xi)^{-1} \xi^T \hat{M} G \xi \triangleq W W^T$$

$$\boxed{W = \xi^T G^T \hat{M}^T \xi (\xi^T \hat{M} \xi)^{-\frac{1}{2}}}$$

† End of Section †

Chapter 4 Manipulator Dynamics
Introduction
Lagrange's Equations
Dynamics of Open-chain Manipulators
Coordinate Invariant Algorithms
Lagrange's Equations with Constraints
References

Section 4.5

Definition: Holonomic constraints

Let $q = (q_1, \ldots, q_n) \in E$, Ambient Space. A *holonomic constraint* is a set of constraint equations:

$$h_i(q) = 0, i = 1, \ldots, k$$

If $\{\mathrm{d}h_i(q)\}_{i=1}^{k}$ is linearly independent, then $Q = h^{-1}(0)$ is a manifold of dim $m \triangleq n - k$.

$$T_q Q : \{V \in T_q E | \mathrm{d}h_i \cdot V = 0, \forall i = 1, \ldots, k\} \subset T_q E :$$

subspace of permissible velocities.

(Continues next slide)

$$T_q^*Q^\perp : \{f \in T_q^*E | \langle f, v\rangle = 0, \forall V \in T_qQ\}$$
$$= \mathrm{span}\{\mathrm{d}h_1, \ldots, \mathrm{d}h_k\} \subset T_q^*E :$$

the subspace of constraint forces (Annihilator of T_qQ).

$$\dim(T_qQ) = m, \dim(T_q^*Q^\perp) = n - m = k$$

$$T_q^*Q^\perp : \{f \in T_q^*E | \langle f, v \rangle = 0, \forall V \in T_qQ\}$$
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the subspace of constraint forces (Annihilator of T_qQ).

$$\dim(T_qQ) = m, \dim(T_q^*Q^\perp) = n - m = k$$

Definition: Constraint forces

$$\Gamma = \frac{\partial h}{\partial q}^T \cdot \lambda$$

$\lambda \in \mathbb{R}^k$: the vector of relative magnitudes of constraint forces.

Definition: Pfaffian Constraints

A Pfaffian constraint has the form:

$$A(q)\dot{q} = 0, A(q) \in \mathbb{R}^{k\times n}$$

Given a Pfaffian constraint,

$$\Delta_q = \{v_q \in T_qE | A(q) \cdot v_q = 0\} \subset T_qE :$$

distribution of permissible velocities.

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Given a Pfaffian constraint,

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distribution of permissible velocities.

Definition:
$A(q)\dot{q} = 0$ is holonomic (or integrable) iff Δ_q is an involutive distribution, iff

$$\exists h_i : E \mapsto \mathbb{R}, i = 1, \ldots, k \text{ s.t. } \Delta_q = T_q Q, Q = h^{-1}(0)$$

§ Constraint forces:

$$\Gamma = A^T(q) \cdot \lambda, \lambda \in \mathbb{R}^k$$

§ Constraint forces:

$$\Gamma = A^T(q) \cdot \lambda, \lambda \in \mathbb{R}^k$$

§ Kinetic energy:

$$T(q, \dot{q}) = \frac{1}{2}\dot{q}^T \cdot M(q) \cdot \dot{q}$$

§ Constraint forces:

$$\Gamma = A^T(q) \cdot \lambda, \lambda \in \mathbb{R}^k$$

§ Kinetic energy:

$$T(q, \dot{q}) = \frac{1}{2}\dot{q}^T \cdot M(q) \cdot \dot{q}$$

§ Potential energy:

$$V(q)$$

§ Constraint forces:

$$\Gamma = A^T(q) \cdot \lambda, \lambda \in \mathbb{R}^k$$

§ Kinetic energy:

$$T(q, \dot{q}) = \frac{1}{2}\dot{q}^T \cdot M(q) \cdot \dot{q}$$

§ Potential energy:

$$V(q)$$

§ Lagrangian:

$$L(q, \dot{q}) = T(q, \dot{q}) - V(q)$$

Lagrange's equations with constraints:

$$M(q)\ddot{q} + C(q, \dot{q}) + N(q) + A^T(q)\lambda = F$$

□ Lagrange's equations with constraints:

$$M(q)\ddot{q} + C(q,\dot{q}) + N(q) + A^T(q)\lambda = F$$

□ Explicit solution for constraint forces:

$$A(q)\ddot{q} + \dot{A}(q)\dot{q} = 0$$

$$(AM^{-1}A^T)\lambda = AM^{-1}(F - C - N) + \dot{A}\dot{q}$$

$$\lambda = (AM^{-1}A^T)^{-1}(AM^{-1}(F - C - N) + \dot{A}\dot{q})$$

◇ Example:

$$x^2 + y^2 = l^2$$

$$\underbrace{\begin{bmatrix} x & y \end{bmatrix}}_{A(q)} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = 0$$

$$L(q, \dot{q}) = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy$$

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ mg \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} \lambda = 0$$

$$\lambda = (AM^{-1}A^T)^{-1}(AM^{-1}(F - C - N) + \dot{A}\dot{q})$$

$$= -\frac{m}{l^2}(gy + \dot{x}^2 + \dot{y}^2)$$

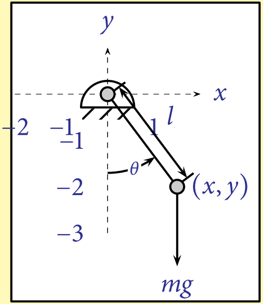


Figure 4.13

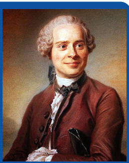
$$\text{Tension } = \left\| \begin{bmatrix} x \\ y \end{bmatrix} \lambda \right\| = \frac{mg}{l} y + \frac{m}{l}(\dot{x}^2 + \dot{y}^2)$$

□ Lagrange-d'Alembert formulation:

Given the Pfaffian constraint $A(q)\dot{q} = 0$ and virtual displacement $\delta q \in \mathbb{R}^k$, we have:

Theorem 1 (D'alembert Principle):
Forces of constraints do no virtual work!

$$(A^T(q)\lambda) \cdot \delta q = 0 \text{ for } A(q)\delta q = 0$$



1717–1783

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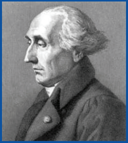
$$\left(A^T(q)\lambda\right) \cdot \delta q = 0 \text{ for } A(q)\delta q = 0$$

1717–1783

Theorem 2 (Lagrange-d'Alembert Equation):

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} - \tau\right) \cdot \delta q = 0, A(q)\delta q = 0$$



1736-1813

Let $A(q) = [A_1(q) \;\; A_2(q)]$, and $A_2(q) \in \mathbb{R}^{k\times k}$ is invertible, then $\delta q_1 \in \mathbb{R}^{n-k}$ are free variables:

$$
\begin{aligned}
&\delta q_2 = -A_2^{-1}(q)A_1(q)\delta q_1 \\
&\Rightarrow \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} - \tau\right)\cdot \delta q \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} - \tau_1\right)\cdot \delta q_1 + \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} - \tau_2\right)\cdot \delta q_2 \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} - \tau_1\right)\cdot \delta q_1 + \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} - \tau_2\right)\cdot (-A_2^{-1}A_1)\delta q_1
\end{aligned}
$$

(Continues next slide)

As $\delta q_1 \in \mathbb{R}^{n-k}$ is free,

$$\boxed{\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} - \tau_1\right) - A_1^T A_2^{-T}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} - \tau_2\right) = 0}$$

Lagrange-d'Alembert equation

◇ Example: Dynamics of a rolling disk

Pfaffian constraint:

$$\begin{cases} \dot{x} - \rho\cos\theta\dot{\phi} = 0 \\ \dot{y} - \rho\sin\theta\dot{\phi} = 0 \end{cases}$$

$$\Rightarrow A(q)\dot{q} = \begin{bmatrix} 1 & 0 & 0 & -\rho\cos\theta \\ 0 & 1 & 0 & -\rho\sin\theta \end{bmatrix}\dot{q} = 0$$

$$L(q,\dot{q}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}\mathcal{I}_1\ddot{\theta}^2 + \frac{1}{2}\mathcal{I}_2\dot{\phi}^2$$

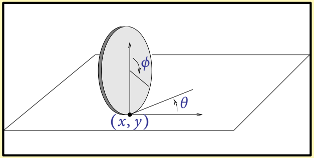


Figure 4.14

Lagrange-d'Alembert equation:

$$\left(\begin{bmatrix} m & & & \\ & m & & \\ & & \mathcal{I}_1 & \\ & & & \mathcal{I}_2 \end{bmatrix}\ddot{q} - \begin{bmatrix} 0 \\ 0 \\ \tau_\theta \\ \tau_\phi \end{bmatrix}\right)\cdot\delta q = 0$$

where $A(q)\cdot\delta q = 0$.

(Continues next slide)

As

$$\begin{cases} \delta x = \rho\cos\theta \cdot \delta\phi \\ \delta y = \rho\sin\theta \cdot \delta\phi \end{cases}$$

the equation of motion is:

$$\left(\begin{bmatrix} 0 & 0 \\ m\rho c_\theta & m\rho s_\theta \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & \mathcal{I}_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} - \begin{bmatrix} \tau_\theta \\ \tau_\phi \end{bmatrix} \right) \cdot \begin{bmatrix} \delta\theta \\ \delta\phi \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ m\rho c_\theta & m\rho s_\theta \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & \mathcal{I}_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} \tau_\theta \\ \tau_\phi \end{bmatrix}$$

As

$$\begin{cases} \ddot{x} = \rho\cos\theta \cdot \ddot{\phi} - \rho\sin\theta \cdot \dot{\theta}\dot{\phi} \\ \ddot{y} = \rho\sin\theta \cdot \ddot{\phi} + \rho\cos\theta \cdot \dot{\theta}\dot{\phi} \end{cases}$$

$$\Rightarrow \begin{bmatrix} \mathcal{I}_1 & 0 \\ 0 & \mathcal{I}_2 + m\rho^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} \tau_\theta \\ \tau_\phi \end{bmatrix}$$

Solve for $(\theta(t), \phi(t))$, and then solve for $(x(t), y(t))$ from:

$$\begin{cases} \dot{x} = \rho\cos\theta \cdot \dot{\phi} \\ \dot{y} = \rho\sin\theta \cdot \dot{\phi} \end{cases} \Leftarrow \text{1st order differential equation}$$

□ Nature of nonholonomic constraints:

Consider $q = (r, s) \in \mathbb{R}^2 \times \mathbb{R}$ with Pfaffian constraint,

$$\dot{s} + a^T(r)\dot{r} = 0, a(r) \in \mathbb{R}^2$$

Lagrangian:

$$L = L(r, \dot{r}, \dot{s})$$

and **constrained Lagrangian:**

$$L_c(r, \dot{r}) = L(r, \dot{r}, -a^T(r)\dot{r})$$

$\Rightarrow$ **Lagrange's equation:**

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_c}{\partial \dot{r}_i} - \frac{\partial L_c}{\partial r_i} = 0, i = 1, 2$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}_i} - a_i(r)\frac{\partial L}{\partial \dot{s}}\right) - \left(\frac{\partial L}{\partial r_i} - \frac{\partial L}{\partial \dot{s}}\sum_j \frac{\partial a_j}{\partial r_i}\dot{r}_j\right) = 0$$

(Continues next slide)

$$\Rightarrow \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot r_i} - \frac{\partial L}{\partial r_i}\right) - a_i(r)\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot s} - \frac{\partial L}{\partial s}\right) = \underbrace{\frac{\partial L}{\partial \dot s}\left(\dot a_i(r) - \sum_j \frac{\partial a_j}{\partial r_i}\dot r_j\right)}_{\neq 0} (*)$$

If the constraint is holonomic, i.e.

$$a_i(r) = \frac{\partial h}{\partial r_i} \text{ for some } h : E \mapsto \mathbb{R}$$

then RHS (right hand side) of $(*)$ equals

$$\frac{\partial L}{\partial \dot s}\left(\sum_j \frac{\partial^2 h}{\partial r_i \partial r_j}\dot r_j - \sum_j \frac{\partial^2 h}{\partial r_j \partial r_i}\dot r_j\right) = 0$$

□ Metric,duality and orthogonality on T_qE:

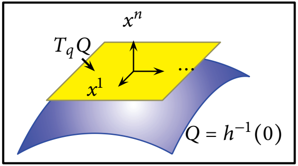


Figure 4.15

$$\mathcal{K} \triangleq \frac{1}{2} \ll \dot{q}, \dot{q} \gg_M = \frac{1}{2}\dot{q}^T M(q)\dot{q}$$

$$T_qQ^\perp = \{V_1 \in T_qE | \ll V_1, V_2 \gg_M \triangleq V_1^T M V_2 = 0, \forall V_2 \in T_qQ\}$$

$$T_q^*Q^\perp = \{f \in T_q^*E | \langle f, V\rangle = 0, \forall V \in T_qQ\}:$$ constraint forces

$$T_qE = T_qQ \oplus T_qQ^\perp, T_q^*E = T_q^*Q \oplus T_q^*Q^\perp$$

□ Metric,duality and orthogonality on T_qE:

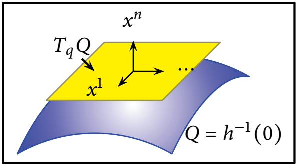


Figure 4.15

$$\mathcal{K} \triangleq \frac{1}{2} \ll \dot{q}, \dot{q} \gg_M = \frac{1}{2}\dot{q}^T M(q)\dot{q}$$

$$T_qQ^\perp = \{V_1 \in T_qE \mid \ll V_1, V_2 \gg_M \triangleq V_1^T M V_2 = 0, \forall V_2 \in T_qQ\}$$

$$T_q^*Q^\perp = \{f \in T_q^*E \mid \langle f, V\rangle = 0, \forall V \in T_qQ\} : \quad \text{constraint forces}$$

$$T_qE = T_qQ \oplus T_qQ^\perp, T_q^*E = T_q^*Q \oplus T_q^*Q^\perp$$

Definition:

$$M^\flat : T_qE \mapsto T_q^*E, \langle M^\flat V_1, V_2\rangle = V_1^T M V_2 = \ll V_1, V_2 \gg_M$$

$$M^\sharp : T_q^*E \mapsto T_qE, M^\sharp = {M^\flat}^{-1}$$

4.5 Lagrange's Equations with Constraints

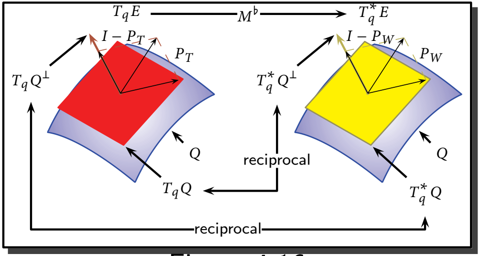


Figure 4.16

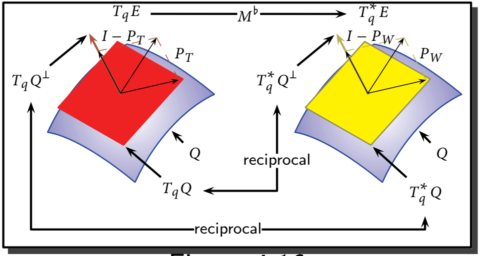


Figure 4.16

Property 5:
Under the basis $\frac{\partial}{\partial q_i}$ and $dq_i, i = 1, \ldots, n$ of T_qE and T_q^*E respectively, the matrix representation of $M^\flat$ and $M^\sharp$ is M and M^{-1} respectively.

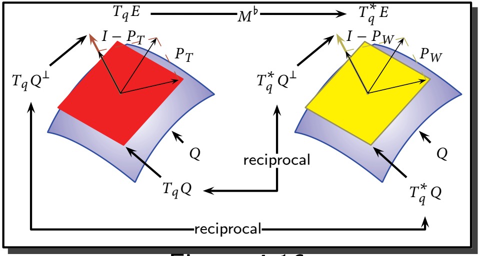


Figure 4.16

Property 5:
Under the basis $\frac{\partial}{\partial q_i}$ and $dq_i, i = 1, \ldots, n$ of T_qE and T_q^*E respectively, the matrix representation of $M^\flat$ and $M^\sharp$ is M and M^{-1} respectively.

Property 6:

$$M^\sharp(T_q^*Q) = T_qQ$$

$$M^\sharp(T_q^*Q^\perp) = T_qQ^\perp$$

Given

$$h : E \mapsto \mathbb{R}^k, m = n - k$$

$$h_* \triangleq T_q h : T_q E \mapsto T_{h(q)}\mathbb{R}^k$$

$$h^* \triangleq T_q^* h : T_{h(q)}^* \mathbb{R}^k \mapsto T_q^* E$$

we have:

Property 7:

$$\ker h_* = T_q Q, h_*(T_q Q^\perp) = T_{h(q)}\mathbb{R}^k, h^*(T_{h(q)}^*\mathbb{R}^k) = T_q^* Q^\perp$$

$$\begin{tikzcd}
T_q^* E \arrow[d, "M^\sharp"'] & T_q^* \mathbb{R}^k \arrow[l, "h^*"'] \arrow[d, "M_2^\sharp"] \\
T_q E \arrow[r, "h_*"'] & T_{h(q)}\mathbb{R}^k
\end{tikzcd}$$

$$M_2^\sharp = h_* \circ M^\sharp \circ h^*$$

4.5 Lagrange's Equations with Constraints

Lemma 1:

The map $(I - P_\omega) : T_q^* E \mapsto T_q^* Q^\perp$ given by

$$(I - P_\omega) = h^* \circ M_2^\flat \circ h_* \circ M^\sharp$$

is a well-defined projection map, with the property:

$$(I - P_\omega)f_1 = 0, \forall f_1 \in T_q^* Q$$
$$(I - P_\omega)f_2 = f_2, \forall f_2 \in T_q^* Q^\perp$$

Lemma 1:
The map $(I - P_\omega) : T_q^* E \mapsto T_q^* Q^\perp$ given by

$$(I - P_\omega) = h^* \circ M_2^\flat \circ h_* \circ M^\sharp$$

is a well-defined projection map, with the property:

$$(I - P_\omega) f_1 = 0, \forall f_1 \in T_q^* Q$$
$$(I - P_\omega) f_2 = f_2, \forall f_2 \in T_q^* Q^\perp$$

Proof :
Given $f_1 \in T_q^* Q, M^\sharp(f_1) \in T_q Q = \ker h_*$, then $(I - P_\omega) f_1 = 0$. For $f_2 \in T_q^* Q^\perp, \exists \lambda \in \mathbb{R}^{n-m}$ s.t. $f_2 = h^* \lambda$, and

$$(I - P_\omega) f_2 = h^* M_2^\flat h_* M^\sharp h^* \lambda = h^* \lambda = f_2$$

thus $P_\omega : T_q^* E \mapsto T_q^* Q$ is a well-defined projection map. Similarly,

$$P_T : T_q E \mapsto T_q Q, P_T = I - M^\sharp h^* M_2^b h_*$$

and

$$(I - P_T) : T_q E \mapsto T_q Q^\perp$$

are projection maps. $\square$

Lemma 2:

$$P_\omega M = M P_T$$

$$P_\omega h^* = h_* P_T = 0$$

$$P_T = P_\omega^T$$

Lemma 2:

$$P_\omega M = M P_T$$
$$P_\omega h^* = h_* P_T = 0$$
$$P_T = P_\omega^T$$

For nonholonomic constraints:

$$h_* \leftarrow A(q)$$
$$h^* \leftarrow A^*(q)$$
$$T_q Q \leftarrow \Delta_q$$
$$T_q^* Q^\perp \leftarrow \text{span}\{a_i(q), i = 1, \ldots, k\}$$

application in hybrid velocity/force control.

□ Lagrange's equations of motion:

$$M(q)\ddot{q} + C(q,\dot{q}) + N = \tau + A^T(q)\lambda \Rightarrow$$

$$\lambda = (AM^{-1}A^T)^{-1}AM^{-1}(M\ddot{q} + (C + N - \tau))$$

□ Lagrange's equations of motion:

$$M(q)\ddot{q} + C(q,\dot{q}) + N = \tau + A^T(q)\lambda \Rightarrow$$
$$\lambda = (AM^{-1}A^T)^{-1}AM^{-1}(M\ddot{q} + (C + N - \tau))$$

Define $P_\omega = I - A^T(AM^{-1}A^T)^{-1}AM^{-1}$, then:

$$P_\omega M\ddot{q} + P_\omega C + P_\omega N = P_\omega \tau$$

Denote $\tilde{C} = P_\omega C, \tilde{N} = P_\omega N, \tilde{\tau} = P_\omega \tau$, $P_\omega M\ddot{\theta} \triangleq \tilde{M}\ddot{\theta}$ is the intertia force in T_q^*Q.

□ Lagrange's equations of motion:

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Define $P_\omega = I - A^T(AM^{-1}A^T)^{-1}AM^{-1}$, then:

$$P_\omega M\ddot{q} + P_\omega C + P_\omega N = P_\omega \tau$$

Denote $\tilde{C} = P_\omega C, \tilde{N} = P_\omega N, \tilde{\tau} = P_\omega \tau,\ P_\omega M\ddot{\theta} \triangleq \tilde{M}\ddot{\theta}$ is the intertia force in $T_q^* Q$.

Definition: Dynamics in $T_q^* Q$

$$\tilde{M}\ddot{\theta} + \tilde{C} + \tilde{N} = \tilde{\tau}$$

(Continues next slide)

Similarly

$$(I - P_\omega)(M\ddot{q} + C + N) = (I - P_\omega)\tau + A^T\tau$$

Let

$$P_T = I - M^{-1}A^T(AM^{-1}A^T)^{-1}A$$

then since $P_\omega M = MP_T$, we have:

Definition: Dynamics in $T_q^*Q^\perp$

$$M(I - P_T)(\ddot{q} + M^{-1}C) = (I - P_\omega)(\tau - N) + A^T\lambda$$

Chapter 4 Manipulator Dynamics
Introduction
Lagrange's Equations
Dynamics of Open-chain Manipulators
Coordinate Invariant Algorithms
Lagrange's Equations with Constraints
References

Similarly

$$(I - P_\omega)(M\ddot{q} + C + N) = (I - P_\omega)\tau + A^T\tau$$

Let

$$P_T = I - M^{-1}A^T(AM^{-1}A^T)^{-1}A$$

then since $P_\omega M = MP_T$, we have:

Definition: Dynamics in $T_q^* Q^\perp$

$$M(I - P_T)(\ddot{q} + M^{-1}C) = (I - P_\omega)(\tau - N) + A^T\lambda$$

□ Geometric Interpretation:

$$\nabla \leftrightarrow M$$

$$M\ddot{q} + C + N = \tau + A^T\lambda \Leftrightarrow M\nabla_{\dot{q}}\dot{q} = \tau - N + A^T\lambda$$

$$\tilde{\nabla} \leftrightarrow \text{ induced metric on } T_qQ$$

$$S : TQ \otimes TQ \mapsto N(Q) : 2^{\text{nd}} \text{ fundamental form}$$

$$TQ: \text{ tangent vector field}$$

$$N(Q): \text{ normal vector field}$$

$$\nabla_X Y = \tilde{\nabla}_X Y + S(X, Y)$$

$$M \underbrace{(I - P_T)(\ddot{q} + M^{-1}C)}_{S(\dot{q},\dot{q})} = (I - P_\omega)(\tau - N) + A^T\lambda$$

$MS(\dot{q}, \dot{q})$: centrifugal force due to curvature of Q in E

$$\nabla_X Y = \tilde{\nabla}_X Y + S(X, Y)$$

$$M \underbrace{(I - P_T)(\ddot{q} + M^{-1}C)}_{S(\dot{q},\dot{q})} = (I - P_\omega)(\tau - N) + A^T \lambda$$

$MS(\dot{q}, \dot{q})$: centrifugal force due to curvature of Q in E

Definition: Hybrid position/force control

$$M\tilde{\nabla}_{\dot{q}}\dot{q} = \tilde{\tau} - \tilde{N}$$

$$MS(\dot{q}, \dot{q}) = (I - P_\omega)(\tau - N) + A^T \lambda$$

◇ Example: Dynamics of a Spherical Pendulum

$$K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}\dot{q}^T M \dot{q}$$

$$q = (x, y, z)^T, M = mI$$

$$h(q) = q^T q - r^2 = 0$$

$$A = (x, y, z), M_2^\sharp = AM^\sharp A^T = r^2/m$$

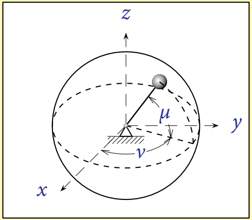


Figure 4.17

$$P_\omega = \frac{1}{r^2}\begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -yz & x^2 + z^2 & -yz \\ -zx & -zy & x^2 + y^2 \end{bmatrix}$$

$$I - P_\omega = \frac{1}{r^2}\begin{bmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{bmatrix}$$

(Continues next slide)

$$P_T = P_\omega^T = P_\omega$$

(μ, ν): Spherical coordinates

$$q = (r\cos\mu\cos\nu, r\cos\mu\sin\nu, r\sin\mu)^T$$

$$\begin{aligned}
\tilde{\nabla}_{\dot q}\dot q &= P_T(\nabla_{\dot q}\dot q) \\
&= \begin{bmatrix} -r\sin\mu\cos\nu & -r\sin\nu \\ -r\sin\mu\sin\nu & r\cos\nu \\ r\cos\mu & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\
S(\dot q, \dot q) &= (I - P_T)(\nabla_{\dot q}\dot q) \\
&= (-\dot\mu^2 - \cos^2\mu\dot\nu^2)\begin{bmatrix} r\cos\mu\cos\nu \\ r\cos\mu\sin\nu \\ r\sin\mu \end{bmatrix}
\end{aligned}$$

where

$$\begin{aligned}
v_1 &= \ddot\mu + \sin\mu\cos\mu\dot\nu^2 \\
v_2 &= \cos\mu\ddot\nu - 2\sin\mu\dot\mu\dot\nu
\end{aligned}$$

Control Algorithm:

1. holonomic constraints:

$\tilde{q}$: coordinates of Q

$$q = \psi(\tilde{q}) \Rightarrow \dot{q} = J \cdot \dot{\tilde{q}}$$

$$\tau = MJ(\ddot{\tilde{q}}_d - K_v \dot{\tilde{e}} - K_p \tilde{e}) + C_1 + N + A^T(-\lambda_d + K_I \int (\lambda - \lambda_d))$$

Control Algorithm:

1. holonomic constraints:

   $\tilde{q}$: coordinates of Q

   $q = \psi(\tilde{q}) \Rightarrow \dot{q} = J \cdot \dot{\tilde{q}}$

   $$\tau = MJ(\ddot{\tilde{q}}_d - K_v \dot{\tilde{e}} - K_p \tilde{e}) + C_1 + N + A^T(-\lambda_d + K_I \int (\lambda - \lambda_d))$$

2. nonholonomic constraints:

   Let $J(q) \in \mathbb{R}^{n \times m}$ be s.t. $AJ = 0$. Write $\dot{q} = J \cdot u$ for some u

   $$\tau = MJ(\dot{u}_d - K_p(u - u_d)) + M\dot{J}u + C + N + A^T(-\lambda_d + K_I \int (\lambda - \lambda_d))$$

◇ Example: 6-DoF manipulator on a sphere with frictionless point contact

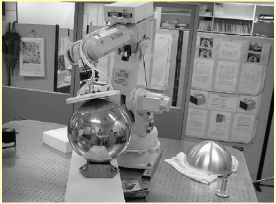
- Contact constraint:

$$v_z = 0 \Leftrightarrow [0\ 0\ 1\ 0\ 0\ 0]\mathrm{Ad}_{g_{fl_f}^{-1}} V_{of} = 0$$

$\Rightarrow$ Holonomic constraint:

$\eta = (\alpha_o^T, \alpha_f^T, \psi)$: Parametrization of Q

$$P_\omega = \mathrm{diag}(1,1,0,1,1,1)$$



- Newton-Euler Equations of motion:

$$M\dot{V}_{of} - \mathrm{ad}_{V_{of}}^T M V_{of} = F_m + G + A^T\lambda$$

$$V_{of} = \begin{bmatrix} R_\psi M_o & -M_f & 0 \\ 0 & 0 & 0 \\ R_\psi R_o K_o M_o & -R_o K_f M_f & 0 \\ -T_o M_o & -T_f M_f & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_o \\ \dot{\alpha}_f \\ \dot{\psi} \end{bmatrix} \triangleq J\dot{\eta}$$

$$MJ\ddot{\eta} + C_1 = F_m + G + A^T\lambda \qquad (*)$$

(Continues next slide)

4.5 Lagrange's Equations with Constraints

2 Unified geometric approach (Liu G.F., Li Z.X.)

$P_\omega(*)$:

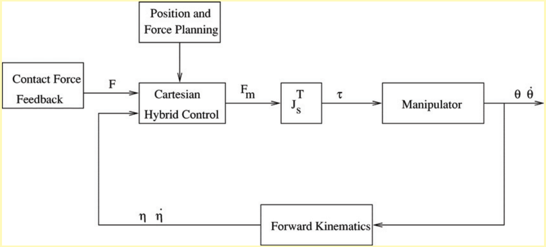
$$\tilde{M}\ddot{\eta} + \tilde{C}_1 = B_1 F_m + B_1 G$$

$$-\phi_3 - \lambda = b_2 F_m + b_2 G$$

$$\hat{F} = [f_1 \quad f_2 \quad f_4 \quad f_5 \quad f_6]^T = \tilde{M}(\ddot{\eta}_d - K_v \dot{\tilde{e}} - K_p \tilde{e}) + \tilde{C}_1 - B_1 G$$

$$f_3 = -\phi_3 - \lambda_d + K_I \int (\lambda - \lambda_d) - b_2 G$$

$$\tau = J_s^T F_m$$



◇ Example: 6-DoF manipulator rolling on a sphere

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\dot{A}_1(q)} V_{l_o l_f} = 0$$

$$f_c = A_1^T \lambda, \lambda \in \mathbb{R}^4$$

$$\begin{bmatrix} \omega_x \\ \omega_y \end{bmatrix} = -R_0(K_f + R_\psi K_o R_\psi) M_f \dot{\alpha}_f$$

$$\begin{aligned} V_{of} &= \mathrm{Ad}_{g_{fl_f}} \cdot V_{l_o l_f} \\ &= \mathrm{Ad}_{g_{fl_f}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -R_o(K_f + R_\psi K_o R_\psi) M_f \\ o \end{bmatrix} \dot{\alpha}_f \\ &\triangleq J_f \dot{\alpha}_f \end{aligned}$$

$\mathrm{span}\{J_f\}$: Not involutive

(Continues next slide)

4.5 Lagrange's Equations with Constraints

2 Unified geometric approach (Liu G.F., Li Z.X.)

$$M J_f \ddot{\alpha}_f + (M\dot{J}_f \dot{\alpha}_f - \mathrm{ad}^T_{J_f \dot{\alpha}_f} M J_f \dot{\alpha}_f) = F_m + G + A^T \lambda$$

$$F_m = M J_f(\ddot{\alpha}_{fd} - K_p(\dot{\alpha}_f - \dot{\alpha}_{fd})) + (M\dot{J}_f \dot{\alpha}_f - \mathrm{ad}^T_{J_f \dot{\alpha}_f} M J_f \dot{\alpha}_f)$$
$$+ A^T(-\lambda_d + \int (\lambda - \lambda_d)) - G$$

◊

$$M J_f \ddot{\alpha}_f + (M \dot{J}_f \dot{\alpha}_f - \mathrm{ad}^T_{J_f \dot{\alpha}_f} M J_f \dot{\alpha}_f) = F_m + G + A^T \lambda$$

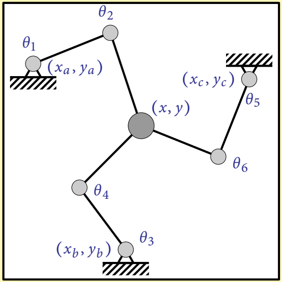
$$F_m = M J_f (\ddot{\alpha}_{fd} - K_p(\dot{\alpha}_f - \dot{\alpha}_{fd})) + (M \dot{J}_f \dot{\alpha}_f - \mathrm{ad}^T_{J_f \dot{\alpha}_f} M J_f \dot{\alpha}_f) + A^T(-\lambda_d + \int (\lambda - \lambda_d)) - G$$

◇

◇ Example: Redundant parallel manipulator

$$\theta = (\theta_1, \ldots, \theta_6) \in E$$
$$\theta_a = (\theta_1, \theta_3, \theta_5)$$
$$\theta_p = (\theta_2, \theta_4, \theta_6)$$

$$H(\theta) = \begin{bmatrix} x_a + lc_1 + lc_{12} - x_b - lc_3 - lc_{34} \\ y_a + ls_1 + ls_{12} - y_b - ls_3 - ls_{34} \\ x_a + lc_1 + lc_{12} - x_c - lc_5 - lc_{56} \\ y_a + ls_1 + ls_{12} - y_c - ls_5 - ls_{56} \end{bmatrix} = 0$$



where $c_{ij} = \cos(\theta_i + \theta_j)$, $s_{ij} = \sin(\theta_i + \theta_j)$.

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$$M_i(\theta) \in \mathbb{R}^{2\times 2} : i^{th} \text{ chain}$$

$$M(\theta) = \operatorname{diag}(M_1(\theta), \ldots, M_3(\theta))$$

$$M(\theta)\ddot{\theta} + C + N = \tau + A^T\lambda$$

If all joints are actuated, we can achieve:

Position control of end-effector
+
internal grasping force

As $\tau_2, \tau_4, \tau_6 = 0$,

$$\tilde{\theta} \in \mathbb{R}^2 : \text{ local parametrization of } Q = H^{-1}(0)$$

$$\theta = \psi(\tilde{\theta}) : \text{ embedding of } Q \text{ in } E$$

$$\dot{\theta} = J\dot{\tilde{\theta}}$$

(Continues next slide)

4.5 Lagrange's Equations with Constraints

Given $P_\omega : T^*_\theta E \mapsto T^*_\theta Q$, the dynamics in $T^*_\theta Q$ is given by:

$$
\begin{aligned}
P_\omega M J \ddot{\tilde{\theta}} + P_\omega (C_1 + N) &= P_\omega \tau \\
\tilde{\tau} &= (\tau_1, \tau_3, \tau_5) \\
\tilde{P}_\omega &= (P_1, P_3, P_5) \\
\hat{\tau} &= \hat{P}_\omega \tilde{\tau} = P_\omega \tau \in \mathbb{R}^6 \\
\hat{\tau} &= P_\omega M J (\ddot{\tilde{\theta}}_d - K_v \dot{\tilde{e}} - K_p \tilde{e}) + P_\omega (C_1 + N)
\end{aligned}
$$

$\diamond$

□ Gauge-invariant Formulation (Aghili):

◇ **Square root factorization of inertia matrix:**

$$M = WW^T \text{(square root factorization)}$$

$$\begin{cases} v \triangleq W^T \dot{q} \in \mathbb{R}^n \\ u \triangleq W^{-1} \tau \in \mathbb{R}^n \end{cases} \qquad T = \frac{1}{2} \dot{q}^T M \dot{q} = \frac{1}{2} v^T v$$

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◇ **Square root factorization of inertia matrix:**

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$$\begin{cases} v \triangleq W^T \dot{q} \in \mathbb{R}^n \\ u \triangleq W^{-1}\tau \in \mathbb{R}^n \end{cases} \qquad T = \frac{1}{2}\dot{q}^T M \dot{q} = \frac{1}{2} v^T v$$

◇ **Lagrange's Equation:**

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = \tau \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(Wv) - \frac{\partial v^T}{\partial q} v = \tau \Rightarrow$$

$$W\dot{v} + \dot{W}v - \frac{\partial v^T}{\partial q} v = \tau \Rightarrow \dot{v} + W^{-1}(\dot{W} - \frac{\partial v^T}{\partial q})v = W^{-1}\tau = u$$

Define $\Gamma \triangleq W^{-1}(\dot{W} - \frac{\partial v^T}{\partial q})$, then:

$$\boxed{\dot{v} + \Gamma v = u}$$

(Continues next slide)

□ Gauge-invariant Formulation (Aghili):

◇ **Change of coordinates:**

$$\bar{v} = V^T v, \bar{u} = V^T u, V \in U(n) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(V\bar{v}) + \Gamma(V\bar{v}) = V\bar{u} \Rightarrow$$

$$V\dot{\bar{v}} + \dot{V}\bar{v} + \Gamma V\bar{v} = V\bar{u} \Rightarrow \dot{\bar{v}} + V^T(\Gamma + \dot{V}V^T)V\bar{v} = \bar{u}$$

$$VV^T = I \Rightarrow \dot{V}V^T + V\dot{V}^T = 0 \Rightarrow \dot{V}V^T = -(\dot{V}V^T)^T =: -\Omega \Rightarrow$$

$$\bar{\Gamma} \triangleq V^T(\Gamma - \Omega)V, \Rightarrow \boxed{\dot{\bar{v}} + \bar{\Gamma}\bar{v} = \bar{u}}$$

□ Gauge-invariant Formulation (Aghili):

◇ **Change of coordinates:**

$$\bar{v} = V^T v, \bar{u} = V^T u, V \in U(n) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(V\bar{v}) + \Gamma(V\bar{v}) = V\bar{u} \Rightarrow$$

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◇ **Pfaffian constraint:**

$$A(q)\dot{q} = 0, A(q) \in \mathbb{R}^{m\times n}, \Lambda \triangleq AW^{-T} \Rightarrow \Lambda v = 0,$$

□ Gauge-invariant Formulation (Aghili):

◇ **Change of coordinates:**

$$\bar{v} = V^T v, \bar{u} = V^T u, V \in U(n) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(V\bar{v}) + \Gamma(V\bar{v}) = V\bar{u} \Rightarrow$$

$$V\dot{\bar{v}} + \dot{V}\bar{v} + \Gamma V\bar{v} = V\bar{u} \Rightarrow \dot{\bar{v}} + V^T(\Gamma + \dot{V}V^T)V\bar{v} = \bar{u}$$

$$VV^T = I \Rightarrow \dot{V}V^T + V\dot{V}^T = 0 \Rightarrow \dot{V}V^T = -(\dot{V}V^T)^T =: -\Omega \Rightarrow$$

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◇ **Pfaffian constraint:**

$$A(q)\dot{q} = 0, A(q) \in \mathbb{R}^{m\times n}, \Lambda \triangleq AW^{-T} \Rightarrow \Lambda v = 0,$$

◇ **Lagrange's equation with constraint:**

$$\boxed{\dot{v} + \Gamma v = u + \Lambda^T\lambda}$$

(Continues next slide)

□ Gauge-invariant Formulation (Aghili):

◊ **SVD of Λ:**

$$\boxed{\Lambda = U\Sigma V^T, \bar{v} = V^T v, \bar{u} = V^T u, \bar{\Lambda} \triangleq \Lambda V}$$

where $\bar{\Lambda}\bar{v} = 0$ and:

$$\sigma = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}, S = \operatorname{diag}(\sigma_1, \ldots, \sigma_r), \sigma_1 \geq \cdots \geq \sigma_r, r \leq m$$

$$\begin{cases} U = [U_1 \ \ U_2], U_1 \in \mathbb{R}^{m\times r}, U_2 \in \mathbb{R}^{m\times(m-r)} \\ V = [V_1 \ \ V_2], V_1 \in \mathbb{R}^{n\times r}, V_2 \in \mathbb{R}^{n\times(n-r)} \end{cases} \Rightarrow$$

$$\bar{\Lambda} = [\Lambda_r \ \ 0_{m\times(n-r)}], \Lambda_r \triangleq U_1 S, \bar{\Lambda}\bar{v} = 0 \Rightarrow$$

$$\bar{v} = \begin{bmatrix} 0_{r\times 1} \\ v_r \end{bmatrix}, v_r \triangleq V_2^T v = V_2^T W^T \dot{q}$$

(Continues next slide)

□ Gauge-invariant Formulation (Aghili):

$$\bar{u} = \begin{bmatrix} u_o \\ u_r \end{bmatrix}, \begin{cases} u_o = V_1^T W^{-1} \tau \\ u_r = V_2^T W^{-1} \tau \end{cases}$$

$$\bar{\Gamma}_{ij} \triangleq V_i^T (\Gamma - \Omega) V_j, i, j = 1, 2, \Gamma_r \triangleq \bar{\Gamma}_{22}, \Gamma_o \triangleq \bar{\Gamma}_{12}$$

□ Gauge-invariant Formulation (Aghili):

$$\bar{u} = \begin{bmatrix} u_o \\ u_r \end{bmatrix}, \begin{cases} u_o = V_1^T W^{-1} \tau \\ u_r = V_2^T W^{-1} \tau \end{cases}$$

$$\bar{\Gamma}_{ij} \triangleq V_i^T (\Gamma - \Omega) V_j, i, j = 1, 2, \Gamma_r \triangleq \bar{\Gamma}_{22}, \Gamma_o \triangleq \bar{\Gamma}_{12}$$

◇ **Decoupled equation of motion/constrained force:**

$$\boxed{\begin{aligned} \dot{v}_r + \Gamma_r v_r &= u_r \\ \Gamma_o v_r &= u_o + \Gamma_r^T \lambda \end{aligned}}$$

□ Gauge-invariant Formulation (Aghili):

$$\bar{u} = \begin{bmatrix} u_o \\ u_r \end{bmatrix}, \begin{cases} u_o = V_1^T W^{-1} \tau \\ u_r = V_2^T W^{-1} \tau \end{cases}$$

$$\bar{\Gamma}_{ij} \triangleq V_i^T (\Gamma - \Omega) V_j, i, j = 1, 2, \Gamma_r \triangleq \bar{\Gamma}_{22}, \Gamma_o \triangleq \bar{\Gamma}_{12}$$

◇ **Decoupled equation of motion/constrained force:**

$$\boxed{\begin{aligned} \dot{v}_r + \Gamma_r v_r &= u_r \\ \Gamma_o v_r &= u_o + \Gamma_r^T \lambda \end{aligned}}$$

◇ **Combined equation of motion (Kane's equation):**

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} q \\ v_r \end{bmatrix} = \begin{bmatrix} W^{-T} V_2 \\ -\Gamma_r \end{bmatrix} v_r + \begin{bmatrix} 0 \\ I \end{bmatrix} u_r}$$

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□ Gauge-invariant Formulation (Aghili):

◇ **Composite error vector ε**

$$q = q(\theta), \theta \in \mathbb{R}^{n-r} : \text{generalized coordinate (of } Q \subset \mathbb{R}^n\text{)}$$

$$v_r = B(\theta)\dot{\theta}, B(\theta) \triangleq V_2^T W^T J, J \triangleq \frac{\partial q}{\partial \theta}, \tilde{\theta} \triangleq \theta - \theta_d, \tilde{v}_r \triangleq v_r - v_{r_d} \Rightarrow$$

$$\varepsilon \triangleq \tilde{v}_r + BK_p\tilde{\theta} = B(\dot{\tilde{\theta}} + K_p\tilde{\theta}) : \text{ composite error}$$

$$s \triangleq v_{r_d} - BK_p\tilde{\theta} = v_r - \varepsilon = B(\dot{\theta}_d - K_p\tilde{\theta}), \tilde{\lambda} \triangleq \lambda - \lambda_d$$

□ Gauge-invariant Formulation (Aghili):

◊ **Composite error vector ε**

$$q = q(\theta), \theta \in \mathbb{R}^{n-r} : \text{generalized coordinate (of } Q \subset \mathbb{R}^n)$$

$$v_r = B(\theta)\dot{\theta}, B(\theta) \triangleq V_2^T W^T J, J \triangleq \frac{\partial q}{\partial \theta}, \tilde{\theta} \triangleq \theta - \theta_d, \tilde{v}_r \triangleq v_r - v_{r_d} \Rightarrow$$

$$\varepsilon \triangleq \tilde{v}_r + BK_p\tilde{\theta} = B(\dot{\tilde{\theta}} + K_p\tilde{\theta}) : \text{ composite error}$$

$$s \triangleq v_{r_d} - BK_p\tilde{\theta} = v_r - \varepsilon = B(\dot{\theta}_d - K_p\tilde{\theta}), \tilde{\lambda} \triangleq \lambda - \lambda_d$$

◊ **Hybrid position/force control**

$$\boxed{\begin{aligned} u_r &= \dot{s} + \Gamma_r s - K_d \varepsilon \\ u_o &= -\Lambda_r^T \lambda_d + \Gamma_o v_r \end{aligned}}$$

Note: integration term $K_I \int (\lambda - \lambda_d)$ is missing from u_o.

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Gauge-invariant Formulation (Aghili):

$$\dot{v}_r + \Gamma_r v_r = u_r = \frac{\mathrm{d}}{\mathrm{d}t}(v_r - \varepsilon) + \Gamma_r(v_r - \varepsilon) - K_d\varepsilon \Rightarrow \dot{\varepsilon} = -(\Gamma_r + K_d)\varepsilon$$

$$\Lambda_r^T \lambda + \Gamma_o v_r = u_o = -\Lambda_r^T \lambda_d + \Gamma_o v_r \Rightarrow \Lambda_r^T \tilde{\lambda} + K_I \int \tilde{\lambda} = 0$$

□ Gauge-invariant Formulation (Aghili):

$$\dot{v}_r + \Gamma_r v_r = u_r = \frac{\mathrm{d}}{\mathrm{d}t}(v_r - \varepsilon) + \Gamma_r (v_r - \varepsilon) - K_d \varepsilon \Rightarrow \dot{\varepsilon} = -(\Gamma_r + K_d)\varepsilon$$

$$\Lambda_r^T \lambda + \Gamma_o v_r = u_o = -\Lambda_r^T \lambda_d + \Gamma_o v_r \Rightarrow \Lambda_r^T \tilde{\lambda} + K_I \int \tilde{\lambda} = 0$$

◇ **Equivalence to the Geometric approach**

$$\boxed{\begin{aligned} P_\omega &= W V_2 V_2^T W^{-1} \\ I - P_\omega &= W V_1 V_1^T W^{-1} \end{aligned}} \Rightarrow \begin{aligned} V^T W^{-1} P_\omega \tau &= \begin{bmatrix} 0 \\ V_2^T W^{-1} \tau \end{bmatrix} = \begin{bmatrix} 0 \\ u_r \end{bmatrix} \\ V^T W^{-1} (I - P_\omega) \tau &= \begin{bmatrix} V_1^T W^{-1} \tau \\ 0 \end{bmatrix} = \begin{bmatrix} u_o \\ 0 \end{bmatrix} \end{aligned}$$

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□ Gauge-invariant Formulation (Aghili):

1. **Geometric approach** (K_p, K_I, K_d):

$$u = \underbrace{\begin{bmatrix} V_1^T W^{-1} A^T(-\lambda_d + K_I \int(\lambda - \lambda_d)) \\ V_2^T W^T J(\ddot{\theta}_d - K_d \dot{\tilde{\theta}} - K_p \tilde{\theta}) \end{bmatrix}}_{fb} + \underbrace{V^T W^{-1} C_1}_{ff}$$

2. **Gauge-invariant formulation** $(\tilde{K}_p, \tilde{K}_d)$:

$$u = \begin{bmatrix} u_o \\ u_r \end{bmatrix} = \underbrace{\begin{bmatrix} V_1^T W^{-1} A^T(-\lambda_d) \\ V_2^T W^T J(\ddot{\theta}_d - K_d' \dot{\tilde{\theta}} - K_p' \tilde{\theta}) \end{bmatrix}}_{fb} + \underbrace{C_1'}_{ff} \Rightarrow$$

$$K_d' \triangleq \tilde{K}_p + (V_2^T W^T J)^{-1} \tilde{K}_d (V_2^T W^T J)$$

$$K_p' \triangleq (V_2^T W^T J)^{-1} \tilde{K}_d (V_2^T W^T J) \tilde{K}_p$$

† End of Section †

Section 4.6

4.6 References

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