7/17/18 Lecture notes: Math 53 Review + Green's First Identity

Let $\vec{f}(x,y,z) = (f_1, f_2, f_3)$ be a vector field on $\mathbb{R}^3$

The <u>divergence</u> of $\vec{f}$ is $\nabla \cdot \vec{f} = \frac{\partial}{\partial x} f_1 + \frac{\partial}{\partial y} f_2 + \frac{\partial}{\partial z} f_3$ ← scalar-valued function

Divergence Theorem: If D is a bounded domain with C^1 boundary, then

$$\iiint_D \nabla \cdot \vec{f} \, dV = \iint_{\partial D} \vec{f} \cdot \vec{n} \, dS .$$

↑ outward normal vector to surface

Basic form: $\int_{\text{domain}} \text{derivative} = \int_{\text{boundary}} \text{original function}$

E.g. Fundamental Theorem of Calculus

$$\int_a^b f'(t)\, dt = f(t)\Big|_{t=a}^{b}$$

In 1D, FTC, product rule → Integration by parts

In 3D, Divergence theorem, some type of product rule → ???

Green's 1st Identity in 1D

Product rule → $(v u_x)_x = v_x u_x + v u_{xx}$ $(*)_x$

$$\int_a^b (v u_x)_x \, dx = \int_a^b v_x u_x + v u_{xx} \, dx$$

FTC

$$\boxed{v u_x \Big|_{x=a}^{b} = \int_a^b v_x u_x + v u_{xx} \, dx}$$

In 3D, start at $(*)$

$(*)_x + (*)_y + (*)_z \to$

$$(vu_x)_x + (vu_y)_y + (vu_z)_z = v_x u_x + v_y u_y + v_z u_z + v(u_{xx} + u_{yy} + u_{zz})$$

$$\iiint_D \nabla\cdot(v\nabla u) = \iiint_D \nabla v\cdot\nabla u + v\Delta u$$

$\Big\downarrow$ divergence theorem

Green's 1st Identity

$$\iint_{\partial D} v\nabla u\cdot\vec{n}\,dS = \boxed{\iint_{\partial D} v\frac{\partial u}{\partial n}\,dS = \iiint_D \nabla v\cdot\nabla u\,dV + \iiint_D v\Delta u\,dV} \quad (\dagger)$$

Neumann Problem

Suppose $\Delta u = f(x)$ in D

$\partial u/\partial n = h(x)$ on ∂D

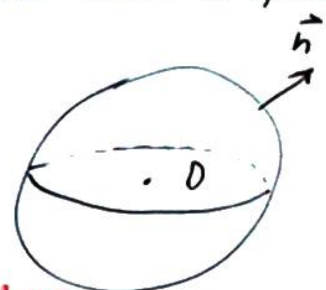
Taking $v \equiv 1$ in $(\dagger)$,

$$\iint_{\partial D} h(x)\,dS = 0 + \iiint_D f(x)\,dV$$

Existence: Must choose compatible f and h

(On that note: Uniqueness. If u a solution, so is $u+C$.)

Mean Value Property: If $\Delta u = 0$, the average value of u over a sphere is the value at the center.



Pf Pick D = ball of radius a centered at 0.

In spherical coordinates, $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial r}$, so

taking $v \equiv 1$, $\Delta u = 0$

$$\iint_{\partial D} v\frac{\partial u}{\partial n}\,dS = \int_0^{2\pi}\int_0^{\pi} u_r(a,\theta,\phi)\,\cancel{a^2}\sin\theta\,d\theta\,d\phi = 0 \div a^2 4\pi$$

$*$ In physics notation

$$\partial/\partial r\underbrace{\left[\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(r,\theta,\phi)\sin\theta\,d\theta\,d\phi\right]}_{\text{average value of } u \text{ on sphere : constant. What value?}} = 0$$

$$\text{As } r\to 0,\ \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(0)\sin\theta\,d\theta\,d\phi = u(0)$$

Average value of u on $S = \frac{1}{\text{Area}(S)} \iint_S u \, dS = u(0)$ [$S = \partial D$, a sphere centered at 0]

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Corollary: Strong Maximum Principle in $\mathbb{R}^3$. (pf as before)

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Uniqueness of Dirichlet Problem (Energy methods)

Suppose u_1, u_2 solve $\Delta u = f$ in D

$u = h$ on ∂D.

Then $w = u_1 - u_2$ solves $\Delta v = 0$ in D

$v = 0$ on ∂D

Plug in $u = v$ in Green's first identity $\rightarrow$

$$\iint \cancel{w} \frac{\partial w}{\partial n} \, dS = \iiint_D |\nabla w|^2 dx + \iiint_D w \cancel{\Delta w} \, dx$$

$\nabla w \equiv 0$, so w constant, $= 0$ by BC.

Dirichlet's Principle: Let u be solution to $\begin{cases} \Delta u = 0 & \text{in } D \\ u = h & \text{on } \partial D \end{cases}$ and w any function with $w = h$ on ∂D

Then, $E[w] \geq E[u]$, where

$$E[w] = \frac{1}{2} \iiint_D |\nabla w|^2 dx$$

Harmonic functions minimize energy!

Pf Expand around u. Let v be any function that is 0 on ∂D.

Then, $u + \varepsilon v = h$ on ∂D. If u minimizes E, then

$$E[u + \varepsilon v] = E[u] + \underbrace{\varepsilon \iiint_D \nabla u \cdot \nabla v}_{= -\varepsilon \iiint_D \Delta u \cdot v \, dx \text{ by Green's 1st Identity}} + \varepsilon^2 E[v]$$

minimized when $\varepsilon = 0$, so coefficient $\iiint_D \Delta u \cdot v \, dx$ on ε is 0.

v is arbitrary, so $\Delta u \equiv 0$.

Next time: Use Green's Identities, special radial harmonic functions to solve Dirichlet problem.