F20 PHYSICS 137B: How to average

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In this brief note, I explain how to perform generic averaging of functions.

1 Discrete case

The set up is as follows: one has a discrete random variable X, a function f that takes X as an argument, and a probability distribution $\{p_i\}$ for X corresponding to values $\{X_i\}$. You are likely familiar with the standard notion of the average of X: it is an expectation value:

$$\bar{X} = \langle X \rangle = \frac{\sum_i p_i X_i}{\sum_i p_i}. \tag{1.1}$$

This can be generalized to obtain the average of the function f over X:

$$\langle f(X) \rangle = \frac{\sum_i p_i f(X_i)}{\sum_i p_i}. \tag{1.2}$$

That is, we see that the distribution on X induces an identical distribution on f, where we treat $f(X)$ as the random variable taking values $f(X_i)$.

2 Continuous case

The set up is identical to the discrete case, except that X now takes continuous values, which we label as x. The probability distribution is now continuous as well – the standard terminology is *probability density function* $\rho(x)$. The interpretation of this quantity is that $\rho(x)\, dx$ is the probability of obtaining a value of X that lies in the interval $x \to x + dx$. Note that $\rho(x)$ by itself has no physical meaning in terms of pure probability. Expectation values are computed as in the discrete case, but we now use integrals instead of sums:

$$\langle X \rangle = \frac{\int x\rho(x)\, dx}{\int \rho(x)\, dx}, \tag{2.1}$$

and for functions f of X:

$$\langle f(X) \rangle = \frac{\int f(x)\rho(x)\, dx}{\int \rho(x)\, dx}. \tag{2.2}$$

The integrals are over the domain of $\rho(x)$.

In the specific case of averaging over a sphere, i.e. averaging over all incoming/outgoing vector orientations in 3D, we use the probability density function $d\Omega = \sin\theta\, d\theta\, d\phi$. So averaging a function f over a sphere corresponds to:

$$\langle f \rangle = \frac{\int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) \sin\theta\, d\theta\, d\phi}{\int_0^{2\pi} \int_0^{\pi} \sin\theta\, d\theta\, d\phi} = \frac{1}{4\pi} \int_0^{2\pi} f(\theta, \phi) \sin\theta\, d\theta\, d\phi \tag{2.3}$$