6/18/18 Lecture Notes
What is a PDE?, The Transport Equation, and some ODE review

ODE Review, Part I

Let $f(x)$ be a continuously differentiable function. For which f does

$$f' = f ?$$

Answer: $f(x) = Ae^x$. $A \in \mathbb{R}$

How many degrees of freedom are there for our solutions? 1

What are appropriate conditions to guarantee exactly one solution?

$f(0) = A$, or $f(1) = B$, etc.

If $f'' + bf' + cf = 0$,

How many degrees of freedom? 2

Appropriate conditions?

$f(0) = A \quad f'(0) = B$ (initial conditions)

$f(0) = A \quad f(1) = B$ (boundary conditions) *

What is a PDE?

Let $u = u(x,y)$ or $u = u(x,t)$. Let u_x denote $\frac{\partial u}{\partial x}$.

Examples

nonconstant (variable) coefficient

constant coefficient → 1) $u_x + u_y = 0$ (transport) 2) $u_x + yu_y = 0$ (transport)

nonlinear → 3) $u_x + uu_y = 0$ (shock wave) 4) $u_{xx} + u_{yy} = 0$ (Laplace's equation) ← 2nd order

5) $u_{tt} - u_{xx} + u^3 = 0$ (nonlinear wave) 6) $u_{xx} + u_{yy} + u_{zz} = 0$ (Laplace's equation)

Linearity: The equation $\mathcal{L}u = 0$ is <u>linear</u> if $\mathcal{L}$ is linear.

E.g. $u_x + u_y = 0 \quad \mathcal{L}u = u_x + u_y$

$$\mathcal{L}(u+v) = (u+v)_x + (u+v)_y = u_x + v_x + u_y + v_y = \mathcal{L}(u) + \mathcal{L}(v)$$

Similarly, $\mathcal{L}(cu) = c\mathcal{L}(u)$, $c \in \mathbb{R}$

$\mathcal{L}u = 0$ homogeneous

$\mathcal{L}u = f(x,y)$ inhomogeneous

Example

$$u_{xx} = 0 \qquad\qquad \text{If } u = u(x,y),$$

$$u_x = C \qquad\qquad u_{xx} = 0$$

$$u = Cx + D \qquad\qquad u_x = f(y)$$

$$u = f(y)\,x + g(y)$$

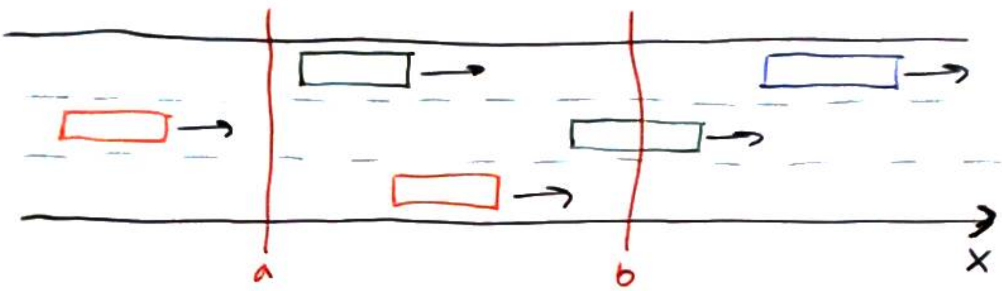
What went wrong?

How many degrees of freedom? Infinite

What are appropriate conditions to guarantee one solution?

See in-class examples

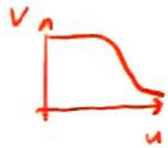
The Transport Equation



Let $u(x,t)$ = density of cars at position x and time t (cars/mile).
(Think fractions of cars, or nanorobot cars so we may consider continuous outputs.)

$$\text{Flow}\left(\tfrac{\text{cars}}{\text{hr}}\right) = \text{Density}\left(\tfrac{\text{cars}}{\text{mi}}\right) \cdot \text{Velocity}\left(\tfrac{\text{mi}}{\text{hr}}\right)$$

$$f(x,t) = u(x,t) \cdot v(u(x,t))$$



Think-Pair-Share: Let $N(t)$ = number of cars between $x=a$ and $x=b$ at time t. Find a formula for dN/dt in terms of f, u, and/or v.

$$\frac{dN}{dt} = \frac{d}{dt}\left[\int_a^b u(x,t)\,dx\right] = f(a,t) - f(b,t)$$

$$= u(a,t)\,v(u(a,t)) - u(b,t)\,v(u(b,t))$$

$$\int_a^b u_t(x,t)\,dx = -\int_a^b \left[u(x,t)\,v(u(x,t))\right]_x dx$$ (Fundamental Theorem of Calculus)

If integrals equal for all a, b, then insides are equal.

$$u_t + [u \cdot v(u)]_x = 0.$$

Simplification: $v(u)$ constant $\longrightarrow$ $u_t + v u_x = 0.$

Moral: PDE come from somewhere!

<u>Method of Characteristics</u>

Goal: Solve $a u_x + b u_y = 0$ (*) Why is this the same as the transport equation?

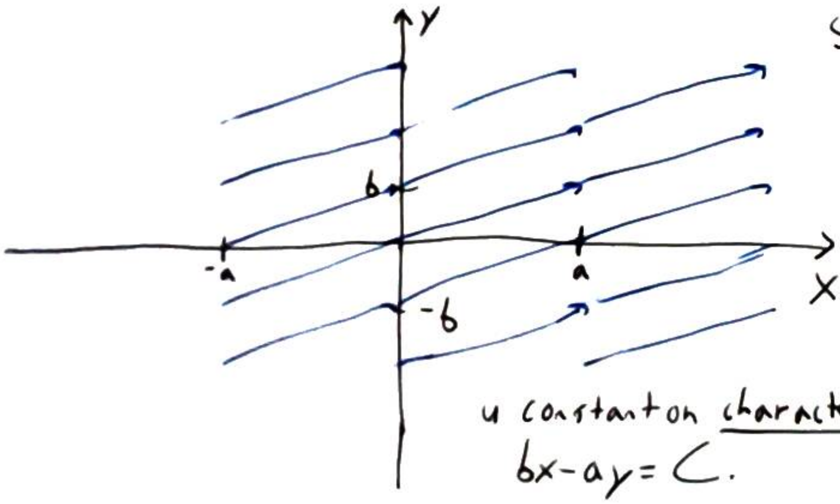
Multivariable Calc Review:

Gradient - $\nabla u(x,y) = \langle u_x, u_y \rangle$

Directional Derivative - $D_{\vec{n}} u = \nabla u \cdot \vec{n}$

Think about relevance

(*) says u doesn't change in direction $\langle a, b \rangle$ since $u \cdot \langle a, b \rangle = 0$



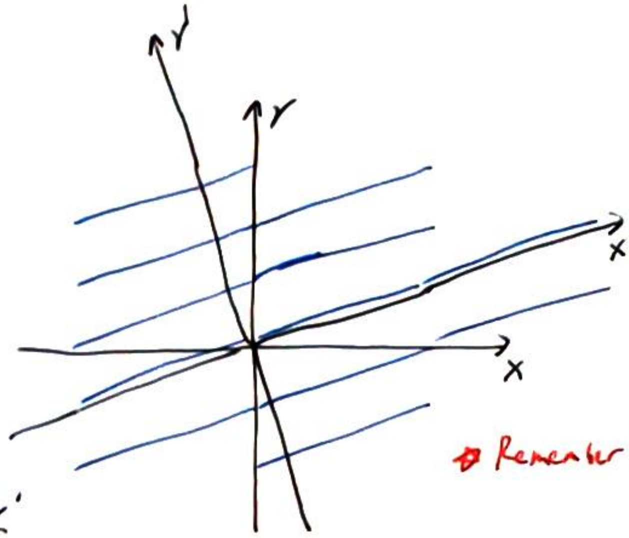
Solution:

$$\boxed{u(x,y) = f(bx - ay)}$$

u constant on <u>characteristic lines</u>

$bx - ay = C.$

Coordinate Method



$x' = ax + by \quad y' = bx - ay$

Sub into $au_x + bu_y = 0$

$$u_x = \frac{\partial u}{\partial x} \overset{*}{=} \frac{\partial u}{\partial x'}\cdot\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = \underline{a u_{x'} + b u_{y'}}$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = \underline{b u_{x'} - a u_{y'}}$$

* Remember the Chain Rule?

$$au_x + bu_y = a(\underline{au_{x'} + bu_{y'}}) + b(\underline{bu_{x'} - au_{y'}})$$
$$= (a^2+b^2)u_{x'} = 0$$
$$u_{x'} = 0$$
$$\boxed{u(x', y') = f(y') = f(bx - ay)}$$

E.g. $2u_x + 5u_y = 0$ with $u(x,0) = \sin x$

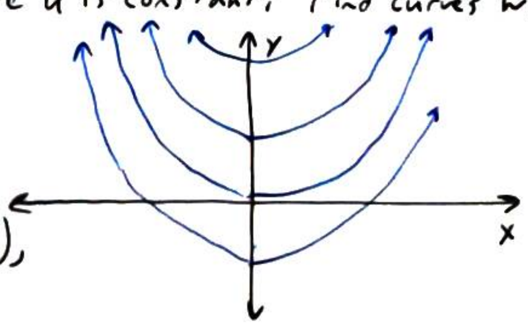
$$u(x,y) = f(5x - 2y)$$
$$u(x,0) = f(5x) = \sin x$$
$$f(w) = \sin w/5$$
$$u(x,y) = f(5x-2y) = \sin\left(\frac{5x-2y}{5}\right)$$

More Characteristics!

Solve $u_x + 2xu_y = 0$ What does the directional derivative say about this?

$u(x,y)$ does not change in the $\langle 1, 2x\rangle$ direction.

What are characteristic curves where u is constant? Find curves with tangent vectors $\langle 1, 2x\rangle$.



$$\frac{dy}{dx} = 2x \longrightarrow y = x^2 + C$$

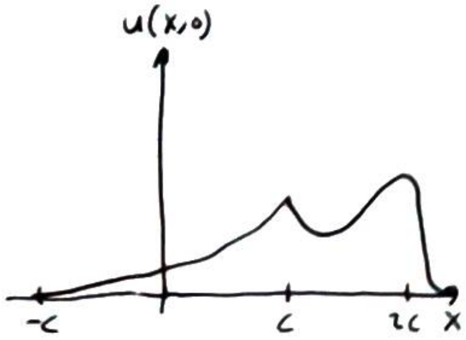
u only depends on which curve you're on (C),

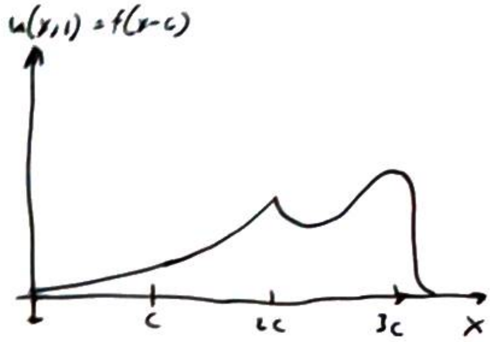
so $u(x,y) = f(C) = f(y - x^2)$

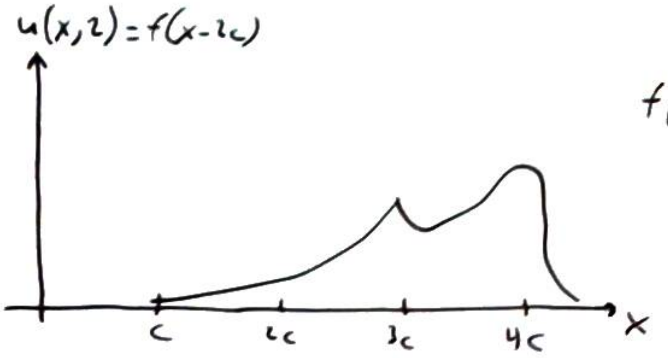
Transport, Revisited

Solution to $u_t + cu_x = 0$ is $u(x,t) = f(x-ct)$, where

$u(x,0) = f(x)$







$f(x)$ moves to the right at speed c

Can one solve $u_x + 2xu_y = 0$, $u(x,0) = f(x)$? What are appropriate conditions for exactly one solution?

Discussion Points: 1) No. Solutions $u(x,y)$ satisfy $u(-1,0) = u(1,0)$, but $f(-1) \neq f(1)$ generally.

2) $u(0,y) = f(y)$ works, but is not the only option.

Summary 1) PDE come in all shapes and sizes

- (non) linear, (in) homogeneous, (non) constant coefficient, nth order, m variables

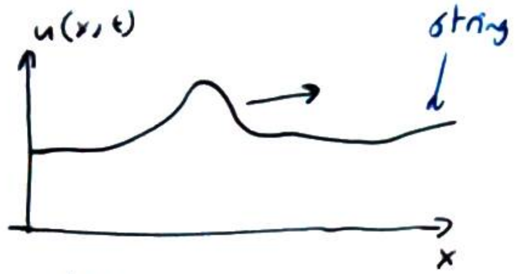
2) Infinite degrees of freedom. When is there exactly one solution? (Well-posedness)

3) PDE come from somewhere! (Usually physical laws)

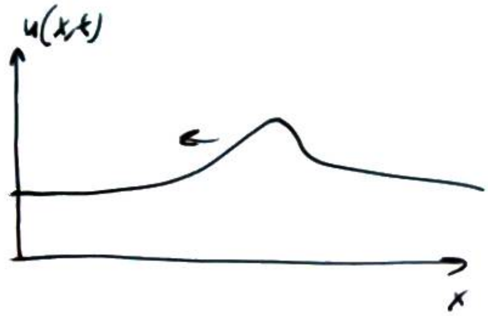
4) Method of characteristics finds curves where $u(x,y)$ is constant, reducing problem to one dimension.

5) Pictures help (Both (x,t) grid and $x, u(x,t)$ snapshots in t

Wave Equation Preview



Solution



Also a solution

How to derive wave equation?

1) Force = mass · acceleration (use physical law)
2) Forces are tension of string, gravity*
3) Find appropriate initial conditions. (Is $u(x,0)$ enough information?)