Math 126 Discussion Problems for 7/11/2018

1. Confirm that d'Alembert's formula for solutions to the wave equation solves the initial value problem by differentiating and plugging in initial values. (You may assume that ϕ and ψ have as many derivatives as you need. For fun, try to figure out the minimal number.)

2. In class earlier, we proved uniqueness of solutions to the Dirichlet problem for the heat equation on the interval $0 \leq x \leq l$. Show that there is at most one solution to the heat equation $u_t - ku_{xx} = f(x,t)$ with $u(x,0) = \phi(x)$ on $-\infty < x < \infty$ provided that $\lim_{|x|\to\infty} u(x,t) = 0$ for all $t \geq 0$. Which methods can you use on this problem? Maximum Principle, Energy Methods, both?

3. Use energy methods to prove uniqueness to the wave equation initial value problem $u_{tt} - c^2 u_{xx} = f(x,t)$ with $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$.

4. Let $H(x) = 0$ for $x < 0$ and $H(x) = -1$ for $x > 0$. Let $f(x)$ be a continuously differentiable function that goes to 0 as $x \to \infty$.

a) Show that $\int_{-\infty}^{\infty} H(x)f'(x)dx = f(0)$.

b) Integrate by parts to show that $\int_{-\infty}^{\infty} -H'(x)f(x) = f(0)$. What function is $-H'(x)$? Does this even make sense?

c) Okay, now that you've thought about part b), let $\delta(x)$ be a function which is ∞ at $x = 0$ and 0 everywhere else whose integral is 1. Show that $\int_{-\infty}^{\infty} \delta(x)f(x)dx = f(0)$.

d) Show that for any x, $\int_{-\infty}^{\infty} f(x-y)\delta(y)dy = f(x)$.

e) Solve the heat equation on the real line with initial conditions $u(x,0) = \delta(x)$. Explain why we say the heat equation has infinite speed of propogation.

5. (Exercise 3.4.5) Let $f(x,t)$ be any function and let $u(x,t) = \frac{1}{2c}\int\int_{\Delta} f$, where Δ is the triangle of dependence. Verify directly by differentiation that $u_{tt} = c^2 u_{xx} + f$ and $u(x,0) = u_t(x,0) = 0$. Hint: Begin by writing the formula as the iterated integral

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y,s)dyds$$

and differentiate with care using the rule in the Appendix.

6. Solve the eigenvalue problem $-X''(x) = \lambda X(x)$ with $X'(0) = X'(l) = 0$. Check all possible eigenvalues.

7. Let A be a 2x2 symmetric matrix.

a) For $u, v \in \mathbb{R}^2$, show that $(Au) \cdot v = u \cdot (Av)$.

b) Show that A has only real eigenvalues.

c) Let $f(x)$ and $g(x)$ be functions satisfying symmetric boundary conditions on the interval $[a,b]$. Show that

$$\int_a^b f''(x)g(x)dx = \int_a^b f(x)g''(x)dx.$$

d) Explain how the two scenarios are analogous.

8. Consider the initial-value problem for the wave equation on the interval $-\pi < x < \pi$ with homogeneous Dirichlet boundary conditions. That is, $u_{tt} = c^2 u_{xx}$ with $u(x,0) = \phi(x)$, $u_t(x,o) = \psi(x)$, and $u(-\pi,t) = u(\pi,t) = 0$.

a) Write down the formula for the solution in terms of series expansions we derived previously. The coefficients in your solution should reference the Fourier cofficients of ϕ and ψ.

b) Now that you have the Fourier expansions for ϕ and ψ, plug their 2π-periodic extensions into d'Alembert's formula. What do you get? (Hint: You will want to use the sine and cosine addition formulas at some point. Also, integrating a Fourier series term-by-term is okay, even though differentiating presents problems.)