SARIMA: Extensions of ARMA

Jared Fisher

Lecture 8b

Announcements

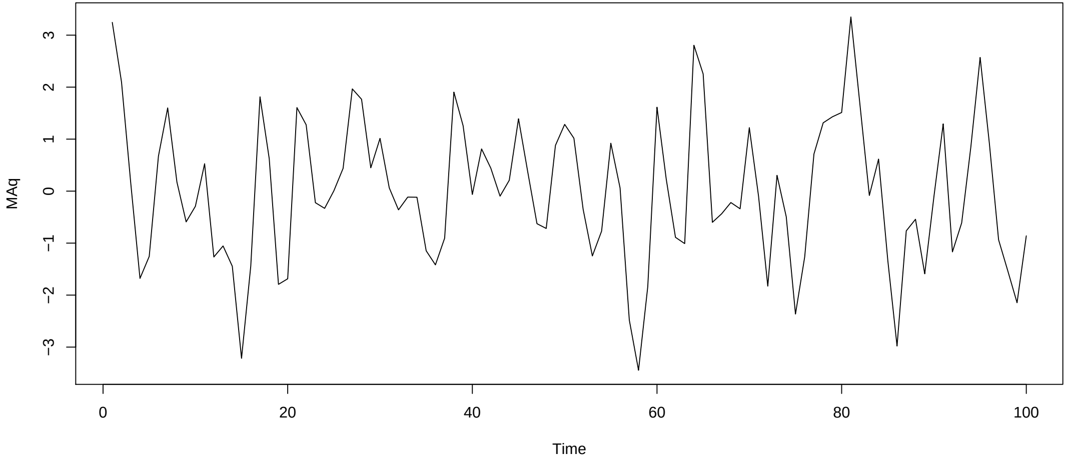
Announcements

- Homework 5 is due next week, Wednesday April 7 by 11:59pm.
- Midterm 2 is the following week, Thursday April 15.

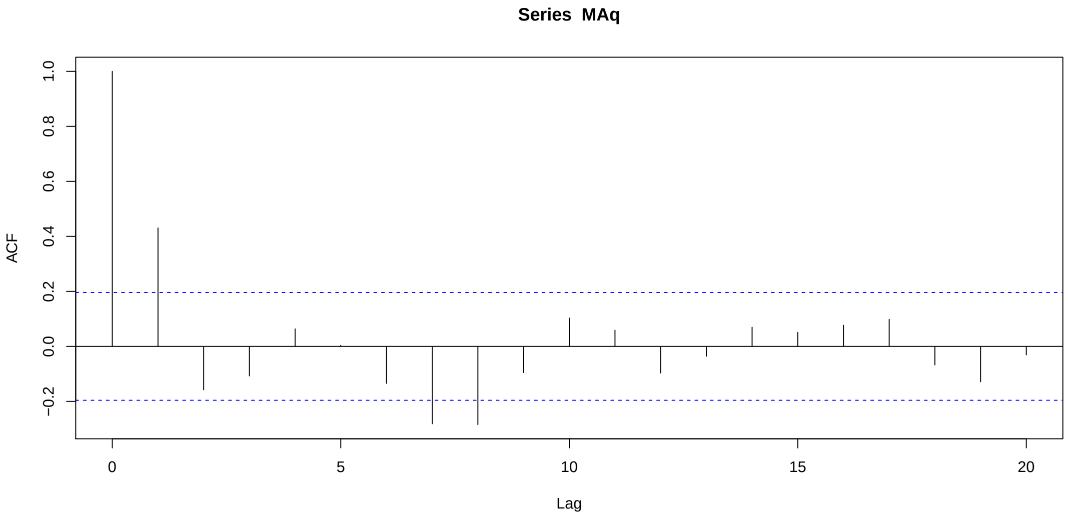
Recap via examples

MA(3)

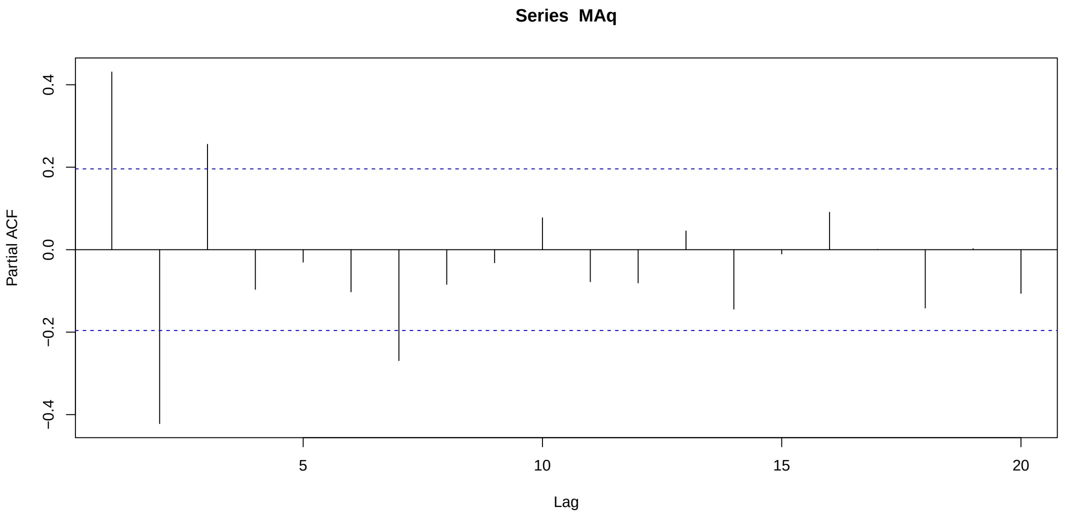
```
MAq = arima.sim(n=100,model=list(ma=c(.9,0,-.2)))
plot.ts(MAq)
```



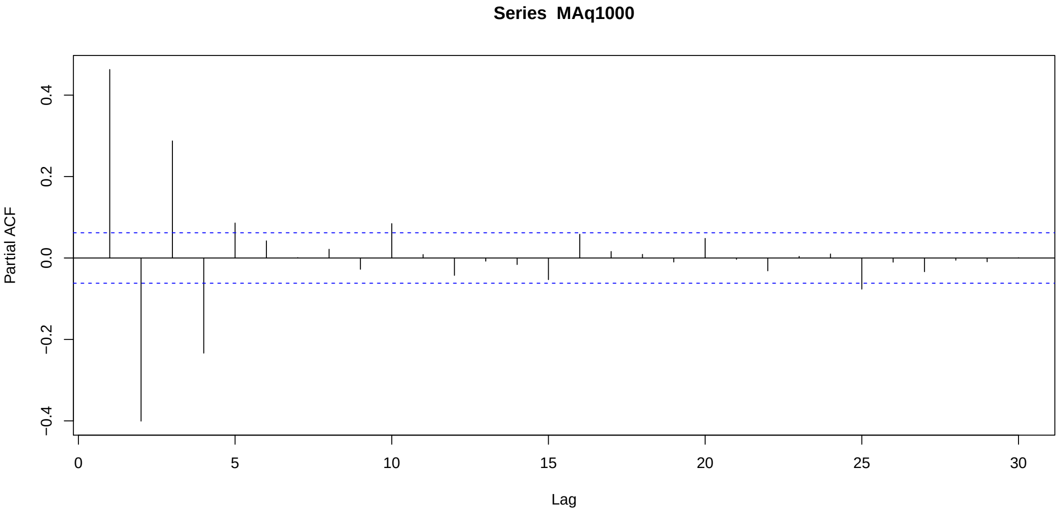
MA(3)



MA(3)

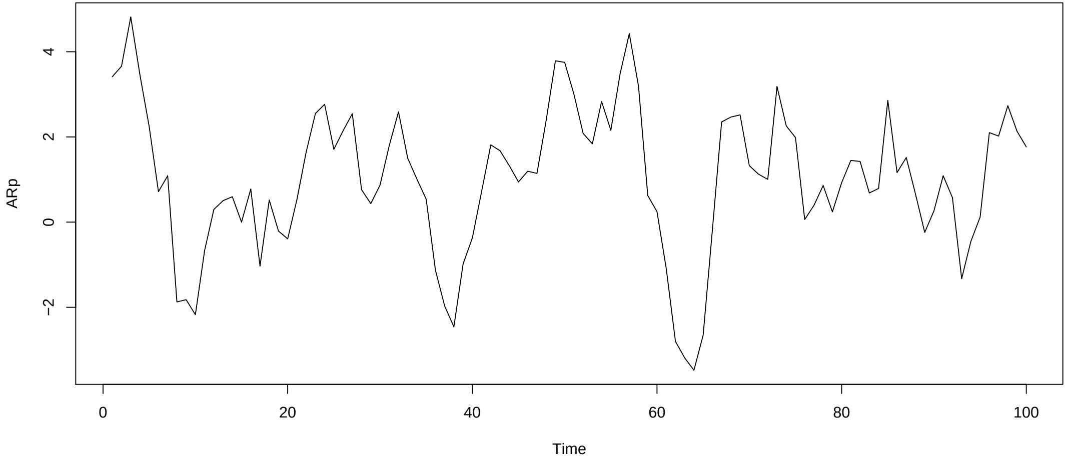


MA(3), n=1000

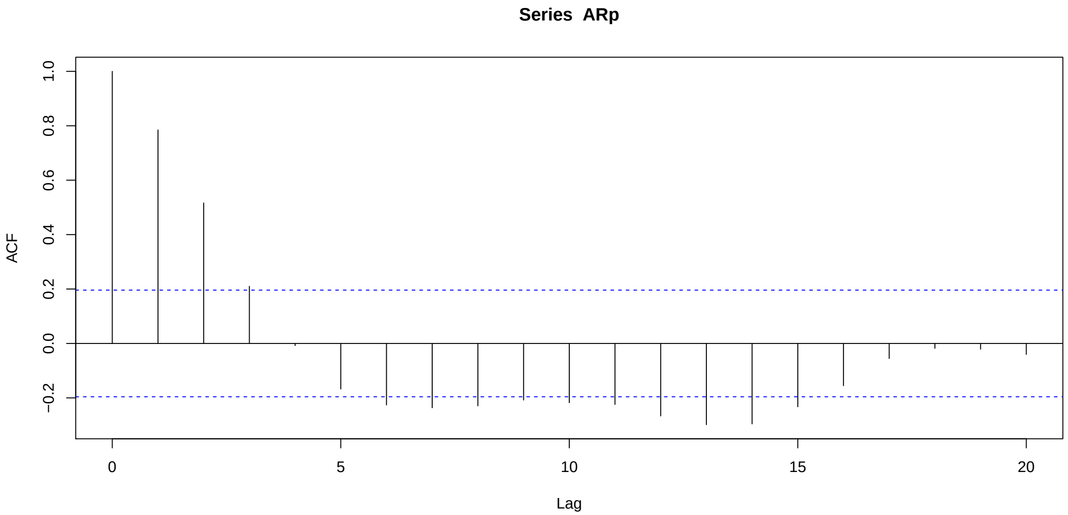


AR(3)

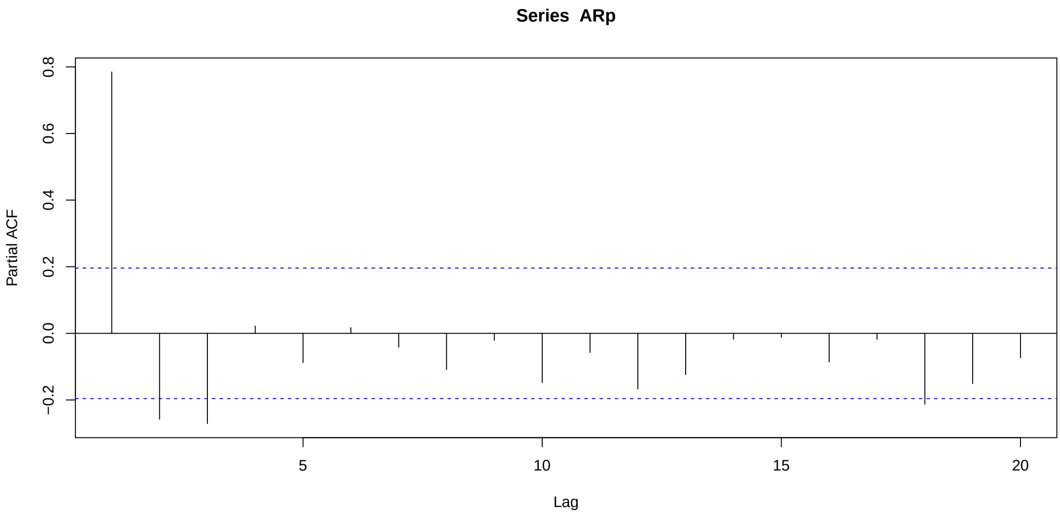
```
ARp = arima.sim(n=100,model=list(ar=c(.9,0,-.2)))
plot.ts(ARp)
```



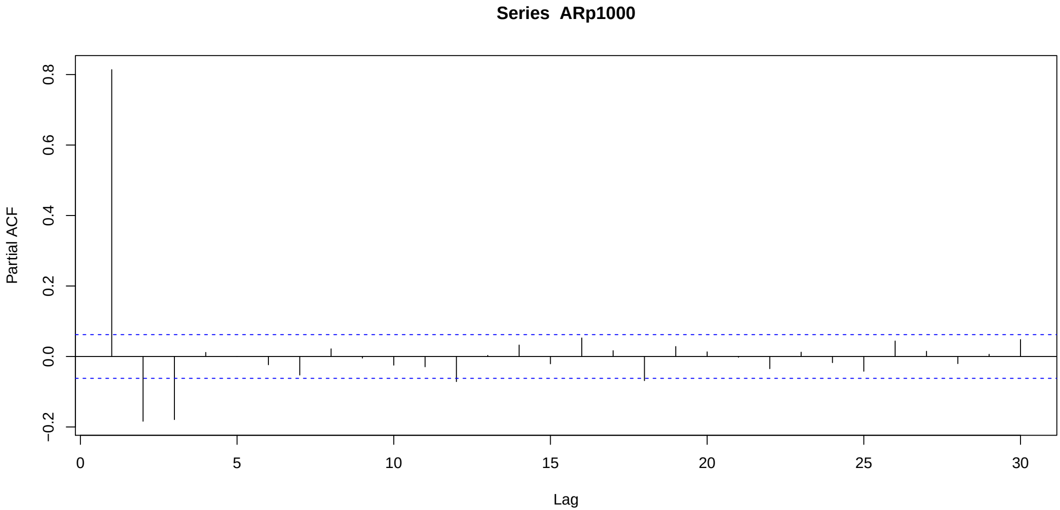
AR(3)



AR(3)

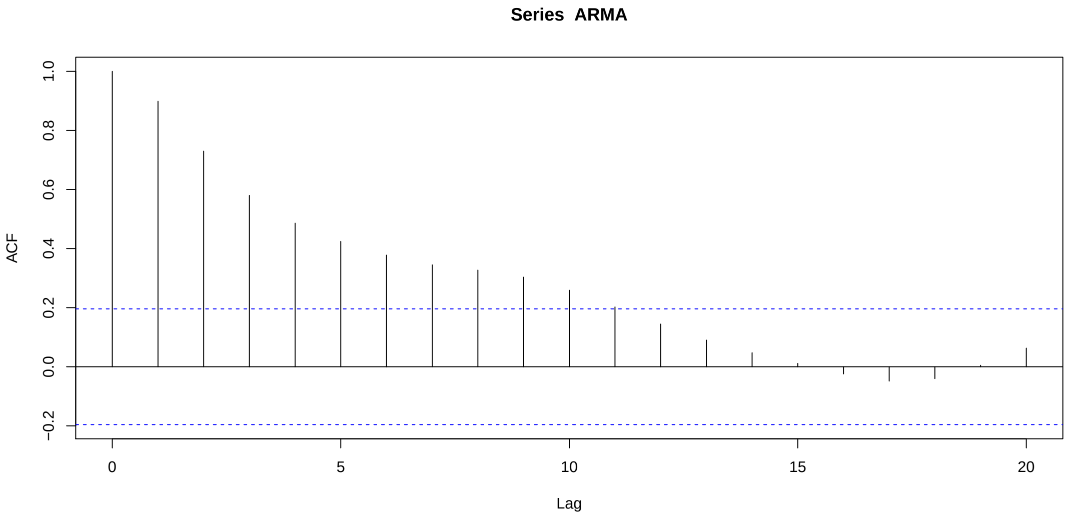


AR(3), n=1000

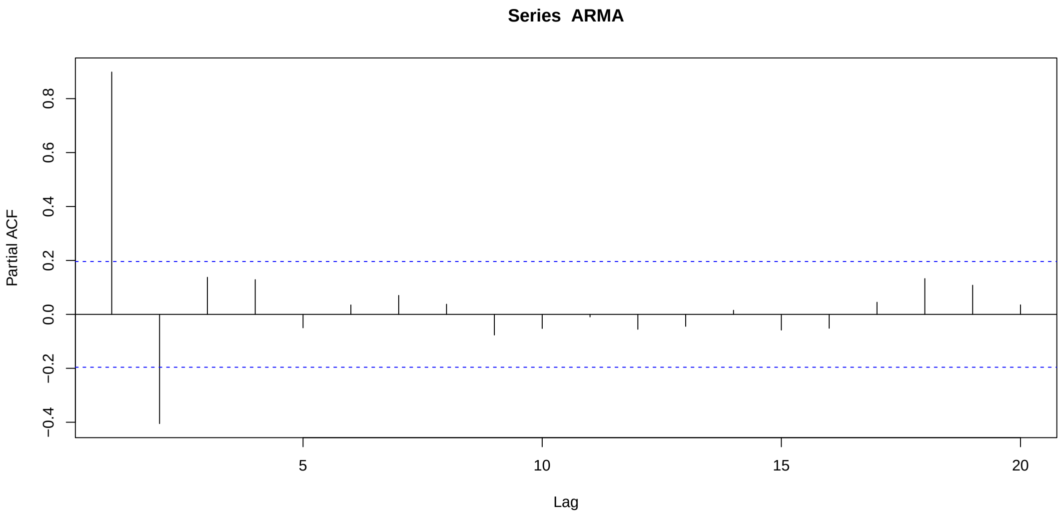

Series ARp1000
Partial ACF
0.8
0.6
0.4
0.2
0.0
−0.2
0
5
10
15
20
25
30
Lag

p=3 (!)

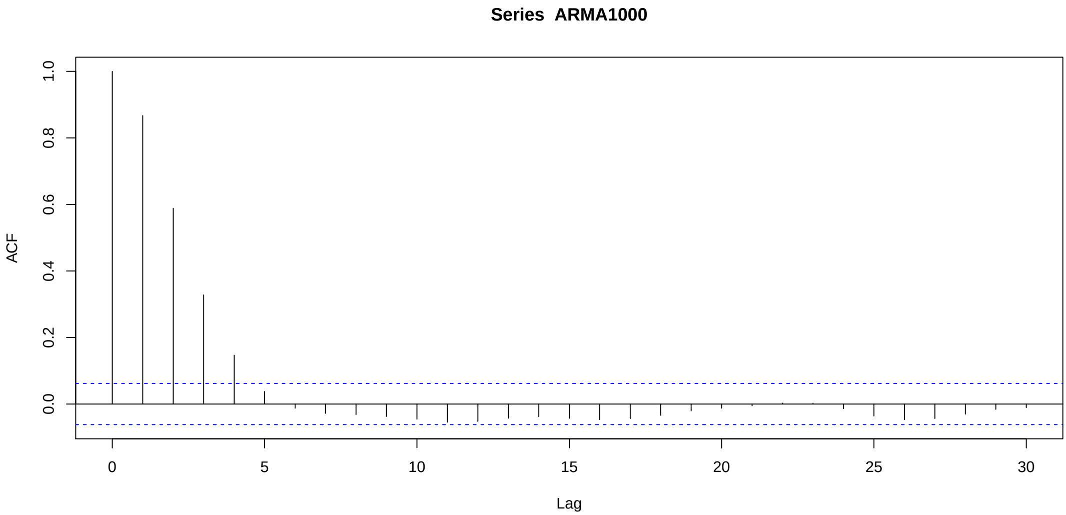
ARMA(p,q)



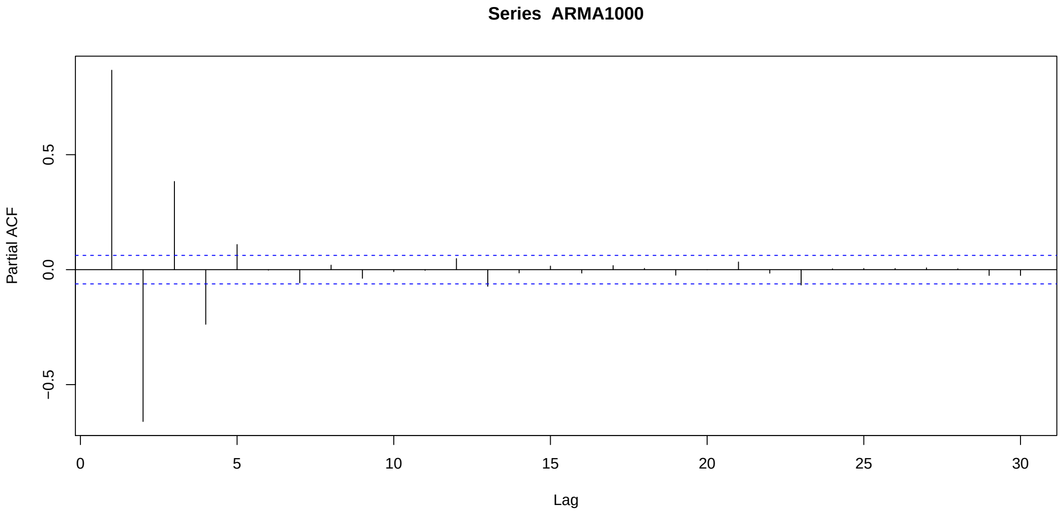
ARMA(p,q)



ARMA(p,q), n=1000



ARMA(p,q), n=1000

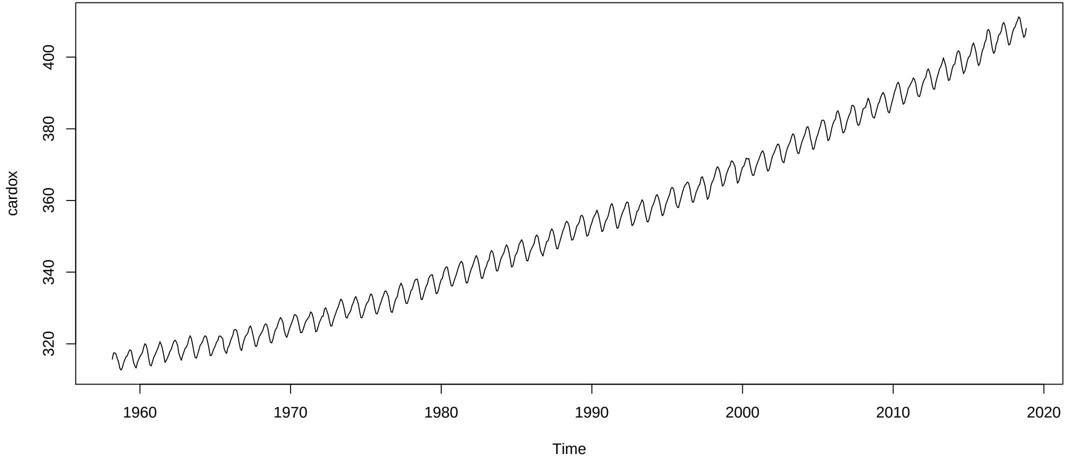


p=???, q=???

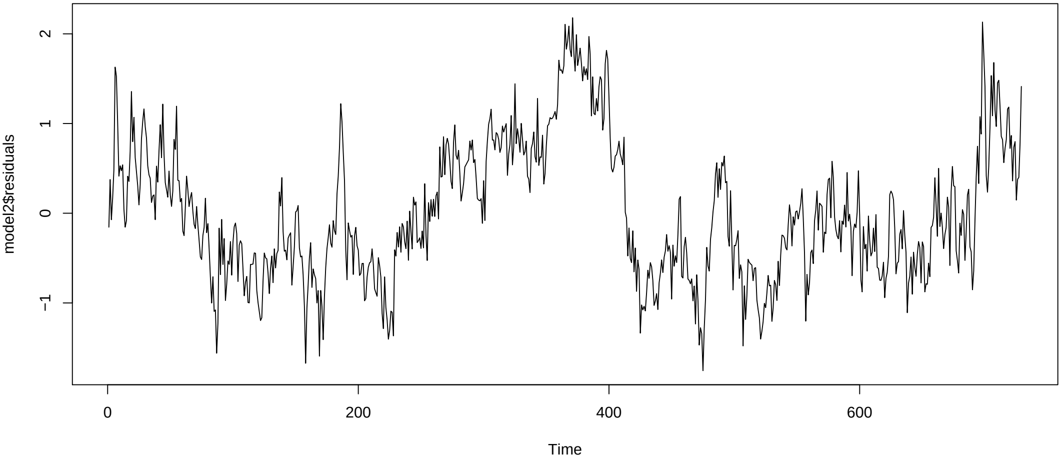
Another example

Hawaii Carbon Dioxide data

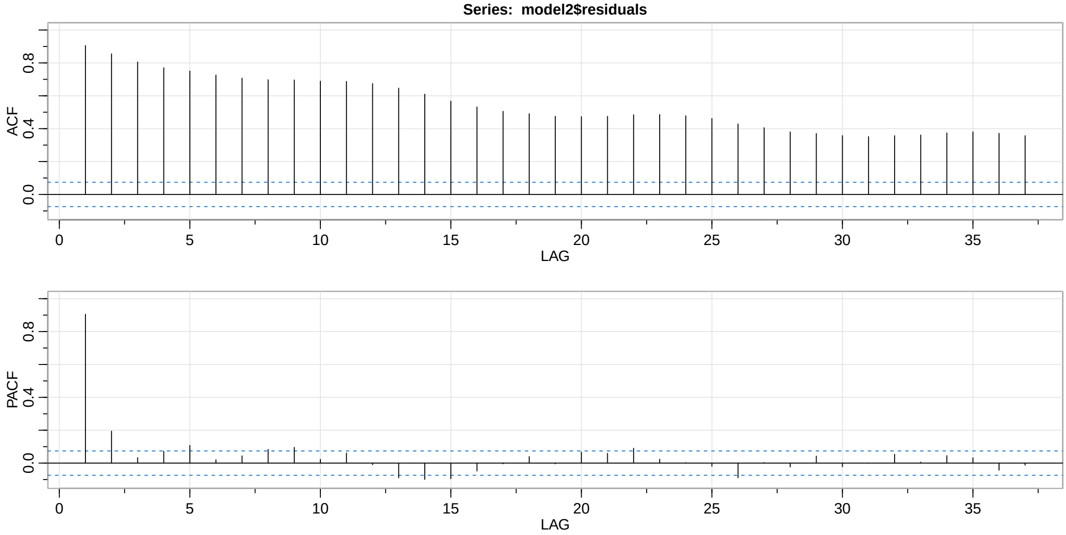
```
library(astsa)
plot.ts(cardox)
```



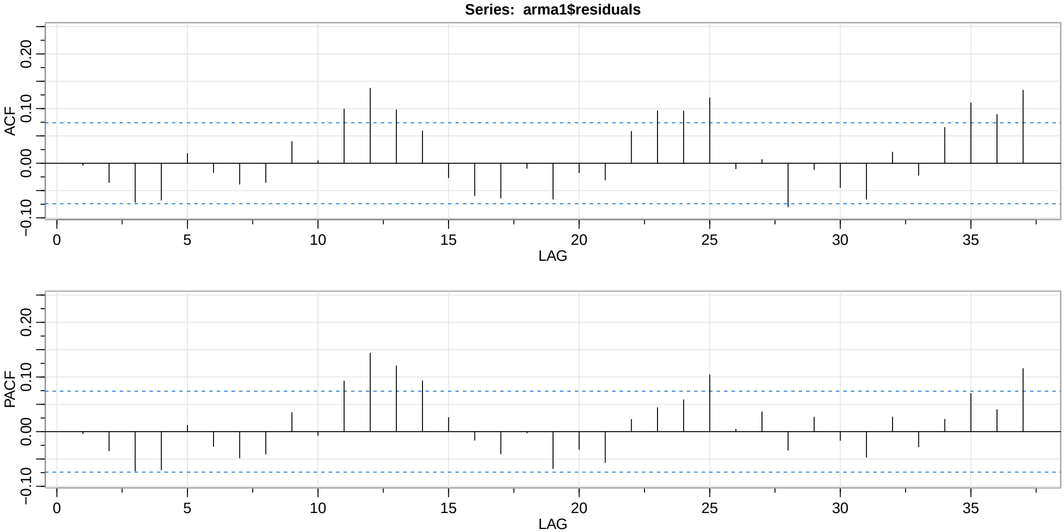
Residuals after Quadratic trend and Monthly Seasonality



Residuals after Quadratic trend and Monthly Seasonality



. . . after AR(2)



Series: arma1$residuals
ACF
−0.10
0.00
0.10
0.20
0
5
10
15
20
25
30
35
LAG
PACF
−0.10
0.00
0.10
0.20
0
5
10
15
20
25
30
35
LAG

ARIMA

ARIMA models

- ARIMA is essentially differencing plus ARMA. The “I” stands for “integrated” .
- Differencing can be used on time series data to remove trends and seasonality. For example,

1. Removing polynomial trends: Suppose $Y_t = \mu_t + X_t$ where μ_t is a polynomial of order d and X_t is stationary, then differencing of order d: $\nabla^d Y_t = (I - B)^d Y_t$ results in stationary data to which an ARMA model can be fit.
2. Removing seasonality (and linear trend): Suppose $Y_t = a + bt + s_t + X_t$, where there is seasonality with period S: $s_{t-S} = s_t$. A lag S difference results in a stationary process, $\nabla_S Y_t = (I - B^S) Y_t$, to which an ARMA model can be fit.
3. Random walk models: $Y_t = Y_{t-1} + X_t$ where X_t is stationary. Then $\nabla Y_t = Y_t - Y_{t-1} = Y_{t-1} + X_t - Y_{t-1} = X_t$ is stationary and an ARMA model can be fit to X_t.

- These and similar models, which after appropriate differencing reduce to ARMA models, are called ARIMA models.

Definition: ARIMA

A process V_t is said to be ARIMA(p, d, q) if $X_t = (I - B)^d V_t$ is ARMA$(p,\ q)$ with mean μ. In other words:

$$\phi(B)(X_t - \mu) = \theta(B)W_t,$$

where $\{W_t\}$ is white noise.

ARIMA in R

- For fitting an ARIMA model in R one can employ the function *arima(dataset,order=*$c(p, d, q)$*)*.
- Gives you estimates of
  - μ (under the name intercept),
  - $\phi_1, \ldots, \phi_p$
  - $\theta_1, \ldots, \theta_q$.
  - Their estimated standard errors.
  - σ^2
- The function *predict* will yield predictions, see *help(predict.Arima)*

Code

R code on ARIMA

Seasonal ARMA models

Definition: Seasonal ARMA

The doubly infinite sequence $\{X_t\}$ is said to be a seasonal ARMA(P, Q) process with period S if it is stationary and if it satisfies the difference equation

$$\Phi(B^S)X_t = \Theta(B^S)W_t$$

where $\{W_t\}$ is white noise and

$$\Phi(B^S) = 1 - \Phi_1 B^S - \Phi_2 B^{2S} - \cdots - \Phi_P B^{PS}$$

and

$$\Theta(B^S) = 1 + \Theta_1 B^S + \Theta_2 B^{2S} + \cdots + \Theta_Q B^{QS}.$$

Sparsity of Seasonal ARMA

- Can also be viewed as ARMA(PS, QS) models.
- However Seasonal ARMA have $P + Q + 1$ parameters (the 1 is for σ^2) while a general ARMA(PS, QS) model will have $PS + QS + 1$ parameters. So these are much sparser models.
- Example: ARMA(12,12), with $\phi_1 = ... = \phi_{11} = \theta_1 = ... = \theta_{11} = 0$ could instead be $\text{ARMA}(1, 1)_{12}$.
- This is sometimes instead written with an S in front ("SARMA") and/or with the subscript in square brackets: ARMA(1,1)[12].

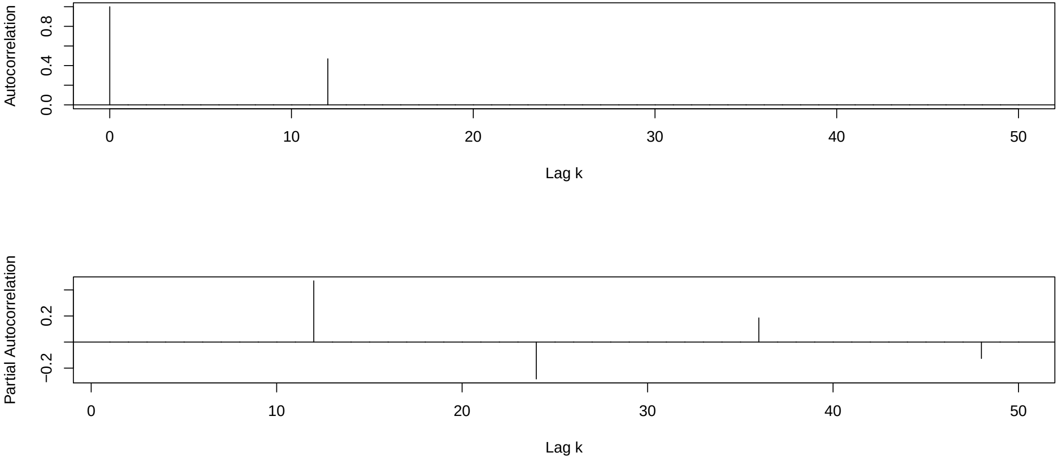
Notes on Seasonal ARMA

- Unique stationary solution exists to $\Phi(B^S)X_t = \Theta(B^S)W_t$ if and only if every root of $\Phi(z^S)$ has magnitude different from one.
- Causal stationary solution exists if and only if every root of $\Phi(z^S)$ has magnitude strictly larger than one.
- Invertible stationary solution exists if and only if every root of $\Theta(z^S)$ has magnitude strictly larger than one.

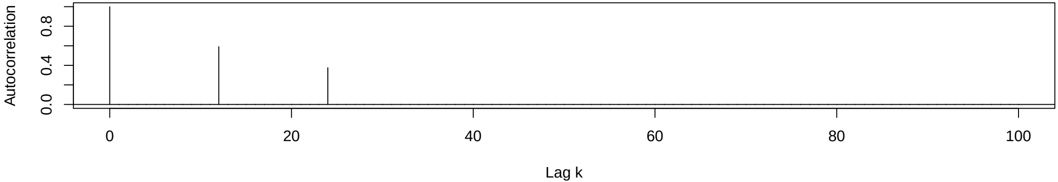
(P)ACF of Seasonal ARMA

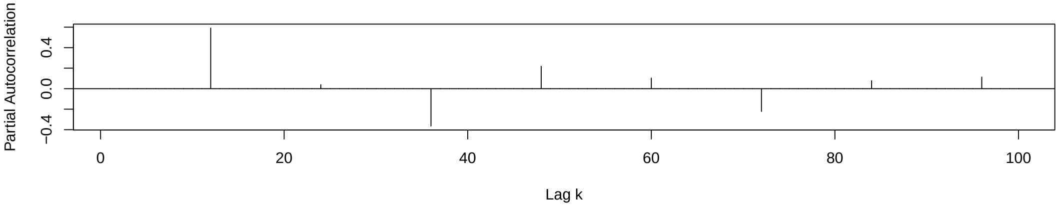
- The ACF and PACF of these models are non-zero only at the seasonal lags $h = 0, S, 2S, \ldots$.
- At these seasonal lags, the ACF and PACF of these models behave just as the case of the non-seasonal/standard ARMA model: $\Phi(B)X_t = \Theta(B)W_t$.

Seasonal MA(1) model

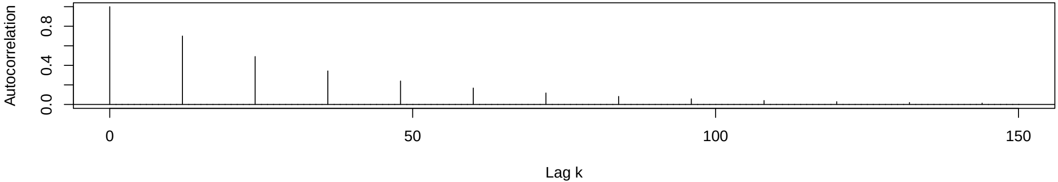


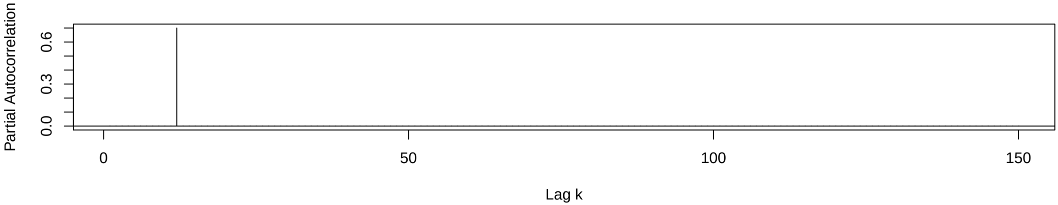
Seasonal MA(2) model



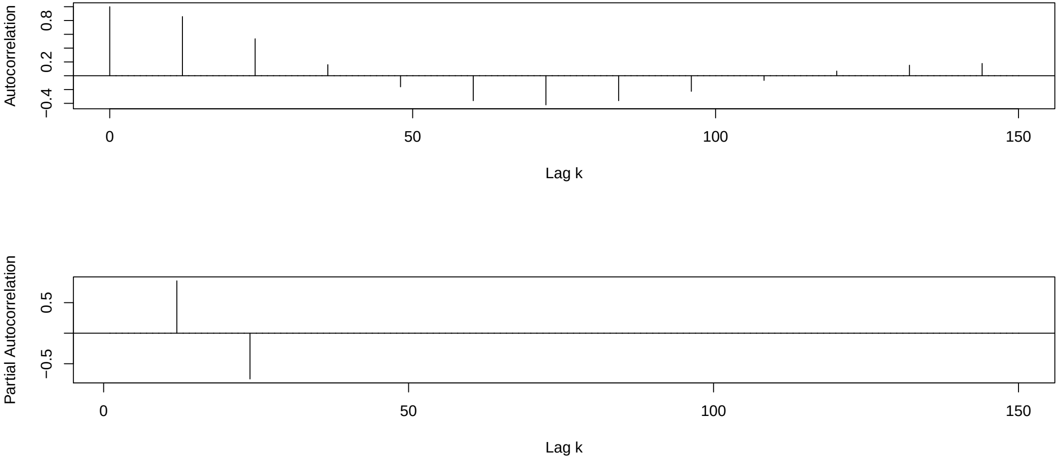


Seasonal AR(1) model

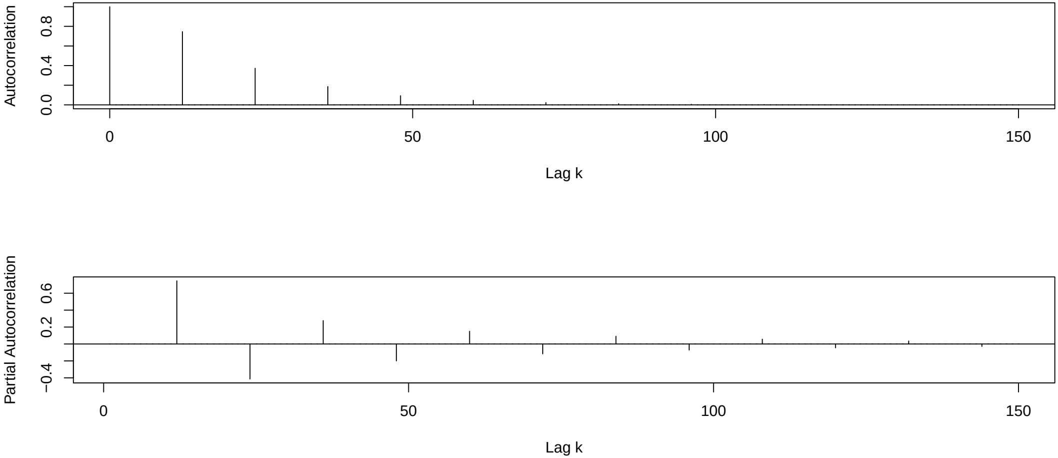




Seasonal AR(2) model



Seasonal ARMA(1,1) model



Multiplicative seasonal ARMA models

Multiplicative seasonal ARMA models

Sometimes it is useful to combine ARMA and seasonal ARMA (by multiplication) to obtain models with desirable properties of their acf functions.

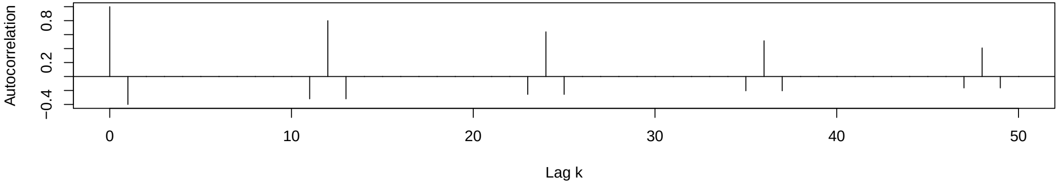
Definition: MSARMA

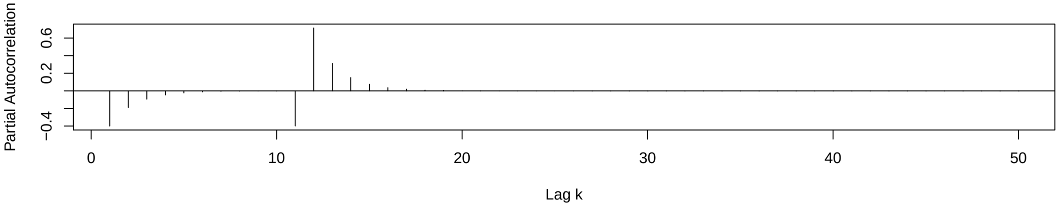
The Multiplicative Seasonal Autoregressive Moving Average Model ARMA(p, q) $\times$ (P, Q)$_S$ is defined as the stationary solution to the difference equation:

$$\Phi(B^S)\phi(B)X_t = \Theta(B^S)\theta(B)W_t,$$

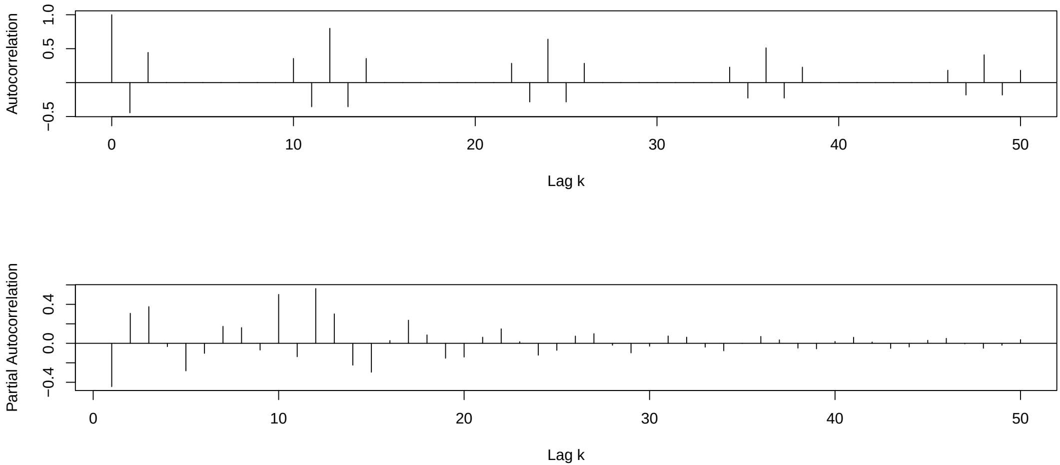
for some white noise process $\{W_t\}$.

ARMA$(0,1)X(1,0)_{12}$ model:

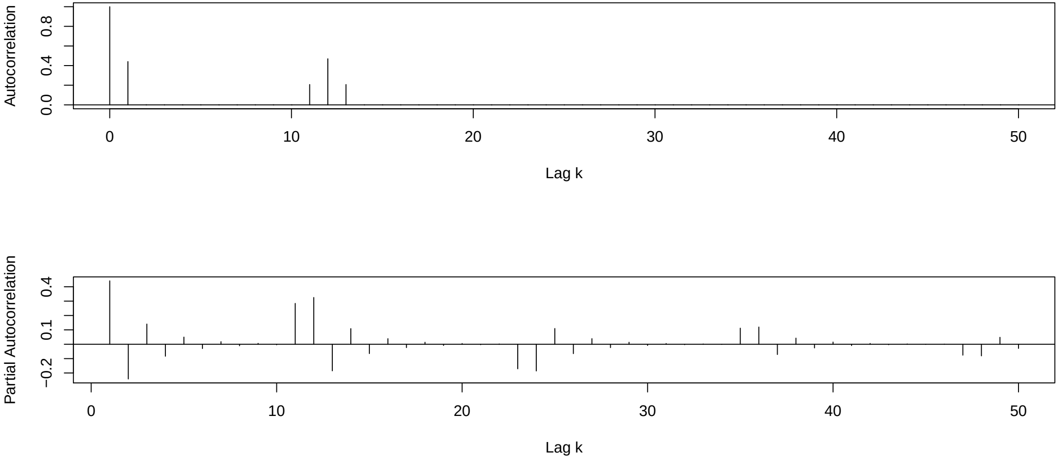




ARMA$(0,2)X(1,0)_{12}$ model:



ARMA$(0,1)X(0,1)_{12}$ model:



Example: different “co2”

- Recall R code for Canadian co2 levels
- In the co2 dataset in R, for the first and seasonal differenced data (with period 12), we observe sample autocorrelations which are significantly non-zero at lags 1, 11, 12 and 13, only.
- We can use a MA(13) model to this data but that will have 14 parameters and therefore will likely overfit the data.
- We can get a much more parsimonious model for this dataset by *combining* the MA(1) model with a seasonal MA(1) model of period 12.

Example: co2

- Specifically, consider the model ARMA(0,1) $\times$ $(0,1)_{12}$ model

$$X_t = (1 + \Theta B^{12})(1 + \theta B)W_t.$$

- This model has the autocorrelation function:

$$\rho_x(1) = \frac{\theta}{1+\theta^2} \quad \text{and} \quad \rho_X(12) = \frac{\Theta}{1+\Theta^2}$$

  and

$$\rho_X(11) = \rho_X(13) = \frac{\theta\Theta}{(1+\theta^2)(1+\Theta^2)}.$$

- At every other lag, the autocorrelation $\rho_X(h)$ equals zero. This is therefore a suitable model for the first and seasonal differenced data in the co2 dataset.

Note

- In general, when you get a stationary dataset whose correlogram shows interesting patterns at seasonal lags, consider using a multiplicative seasonal ARMA model. You may use the R function *ARMAacf* to understand the autocorrelation and partial autocorrelation functions of these models.
- Let's look at many examples in R, then we'll finally combine differencing with multiplicative seasonal ARMA models.

SARIMA

SARIMA models

- Definition: A process V_t is said to be ARIMA$(p, d, q) \times (P, D, Q)_S$, if after differencing d times and seasonal differencing D times, it follows a multiplicative seasonal ARMA model, that is, if it satisfies the difference equation:

$$\Phi(B^S)\phi(B)\nabla_S^D\nabla^d V_t = \Theta(B^S)\theta(B)W_t.$$

- Recall that $\nabla_S^D = (1 - B^S)^D$ and $\nabla^d = (1 - B)^d$ denote the differencing operators.
- Note: $X_t = \nabla_S^D\nabla^d V_t$ is stationary
- In R this model can be fit to the data by using the function *arima*() with the *seasonal* argument or with the *sarima*() function