Lecture Notes for
A Geometrical Introduction to Robotics and Manipulation

Richard Murray and Zexiang Li and Shankar S. Sastry
CRC Press

Zexiang Li[1] and Yuanqing Wu[1]

[1]ECE, Hong Kong University of Science & Technology

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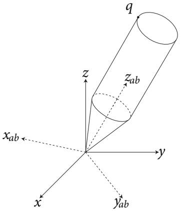
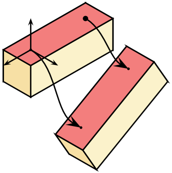
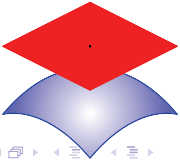
Chapter 2 Rigid Body Motion

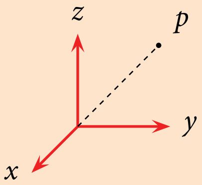






2.1 Rigid Body Transformations

§ Notations:



$$p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \text{ or } p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

For $p \in \mathbb{R}^n, n = 2, 3$(2 for planar, 3 for spatial)

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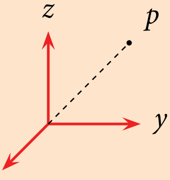

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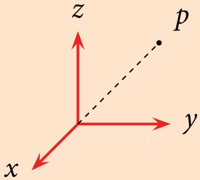

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Matrix: $A \in \mathbb{R}^{n \times m}, A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$

2.1 Rigid Body Transformations

□ Description of point-mass motion:

$$p(0) = \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix}: \text{ initial position}$$

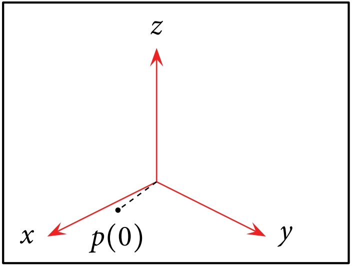

Figure 2.1

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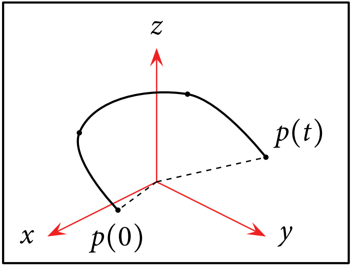

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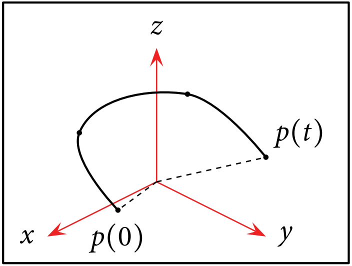

Figure 2.1

Definition: Trajectory

A **trajectory** is a curve $p : (-\varepsilon, \varepsilon) \mapsto \mathbb{R}^3, p(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$

Rigid Body Motion:

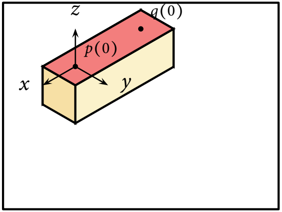

Figure 2.2

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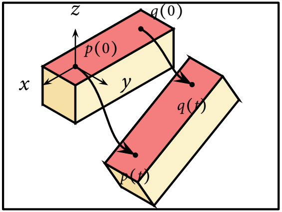

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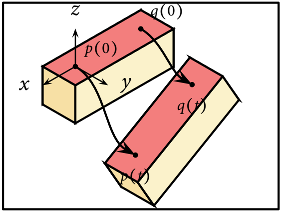

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$$\|p(t) - q(t)\| = \|p(0) - q(0)\| = \text{constant}$$

Definition: Rigid body transformation

$$g : \mathbb{R}^3 \mapsto \mathbb{R}^3$$

s.t.

1. Length preserving: $\|g(p) - g(q)\| = \|p - q\|$
2. Orientation preserving: $g_*(v \times \omega) = g_*(v) \times g_*(\omega)$

2.2 Rotational Motion in $\mathbb{R}^3$

□ Rotational Motion:

1 Choose a reference frame A (spatial frame)

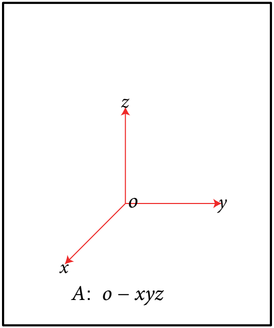

Figure 2.3

□ Rotational Motion:

1 Choose a reference frame A (spatial frame)
2 Attach a frame B to the body (body frame)

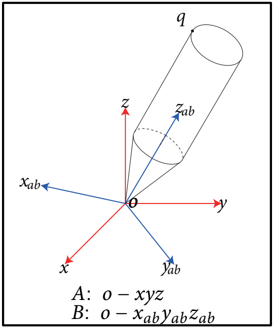

Figure 2.3

$$x_{ab} \in \mathbb{R}^3: \quad \text{coordinates of } x_b \text{ in frame } A$$

$$R_{ab} = [x_{ab}\ y_{ab}\ z_{ab}] \in \mathbb{R}^{3\times 3}: \quad \text{Rotation (or orientation) matrix of } B \text{ w.r.t. } A$$

□ Property of a Rotation Matrix:

Let $R = [r_1 \ r_2 \ r_3]$ be a rotation matrix

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$$\det(R^T R) = \det R^T \cdot \det R = (\det R)^2 = 1, \det R = \pm 1$$

As $\det R = r_1^T (r_2 \times r_3) = 1 \Rightarrow \det R = 1$

2.2 Rotational Motion in $\mathbb{R}^3$

Definition:

$$SO(3) = \left\{ R \in \mathbb{R}^{3\times3} \,\middle|\, R^T R = I, \det R = 1 \right\}$$

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4. $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$

◇ Review: Examples of group

1. $(\mathbb{R}^3, +)$

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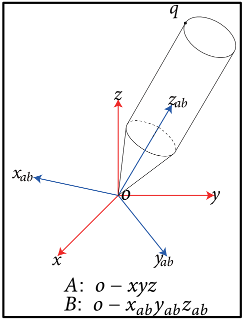
Configuration and rigid transformation:

- $R_{ab} = [x_{ab}\ y_{ab}\ z_{ab}] \in SO(3)$
  Configuration Space



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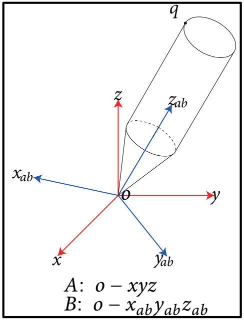
- Let $q_b = \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} \in \mathbb{R}^3$: coordinates of q in B.

$$q_a = x_{ab} \cdot x_b + y_{ab} \cdot y_b + z_{ab} \cdot z_b$$

$$= [x_{ab}\ y_{ab}\ z_{ab}] \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = R_{ab} \cdot q_b$$



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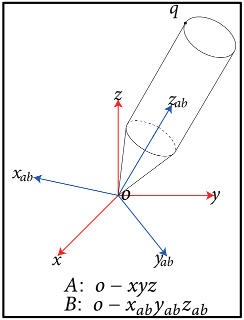

Figure 2.3

- A configuration $R_{ab} \in SO(3)$ is also a transformation:

$$R_{ab} : \mathbb{R}^3 \to \mathbb{R}^3, R_{ab}(q_b) = R_{ab} \cdot q_b = q_a$$

A config. $\Leftrightarrow$ A transformation in $SO(3)$

2.2 Rotational Motion in $\mathbb{R}^3$

Property 2: R_{ab} preserves distance between points and orientation.

1. $\|R_{ab} \cdot (p_b - q_b)\| = \|p_a - q_a\|$
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Proof :

For $a \in \mathbb{R}^3$, let $\hat{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$

Note that $\hat{a} \cdot b = a \times b$

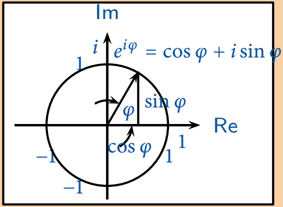
1 follows from
$$\begin{aligned} \|R_{ab}(p_b - p_a)\|^2 &= (R_{ab}(p_b - p_a))^T R_{ab}(p_b - p_a) \\ &= (p_b - p_a)^T R_{ab}^T R_{ab}(p_b - p_a) \\ &= \|p_b - p_a\|^2 \end{aligned}$$

2 follows from $R\hat{v}R^T = (Rv)^\wedge$ (prove it yourself) $\square$

2.2 Rotational Motion in $\mathbb{R}^3$

□ Parametrization of $SO(3)$ (the exponential coordinate):

◇ **Review:** $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$



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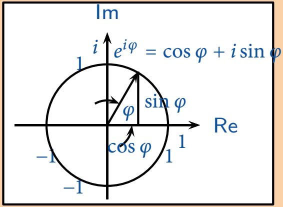

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◇ Review:

$$\begin{cases} \dot{x}(t) = ax(t) \\ x(0) = x_0 \end{cases} \Rightarrow x(t) = e^{at}x_0$$

(Continues next slide)

2.2 Rotational Motion in $\mathbb{R}^3$

$$R \in SO(3), R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$r_i \cdot r_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad \leftarrow \text{6 constraints}$$

$\Rightarrow$ 3 independent parameters!

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Consider motion of a point q on a rotating link

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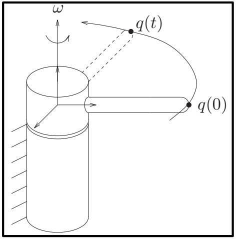

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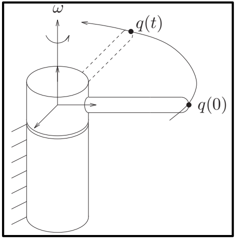
$$\Rightarrow q(t) = e^{\hat{\omega} t} q_0 \text{ where } e^{\hat{\omega} t} = I + \hat{\omega} t + \frac{(\hat{\omega} t)^2}{2!} + \frac{(\hat{\omega} t)^3}{3!} + \cdots$$



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$\Rightarrow$ 3 independent parameters!

Consider motion of a point q on a rotating link

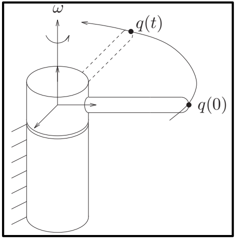

Figure 2.5

$$\begin{cases} \dot{q}(t) = \omega \times q(t) = \hat{\omega} q(t) \\ q(0)\text{: Initial coordinates} \end{cases}$$

$$\Rightarrow q(t) = e^{\hat{\omega} t} q_0 \text{ where } e^{\hat{\omega} t} = I + \hat{\omega} t + \frac{(\hat{\omega} t)^2}{2!} + \frac{(\hat{\omega} t)^3}{3!} + \cdots$$

By the definition of rigid transformation, $R(\omega, \theta) = e^{\hat{\omega}\theta}$. Let $so(3) = \{\hat{\omega} | \omega \in \mathbb{R}^3\}$ or $so(n) = \{S \in \mathbb{R}^{n \times n} | S^T = -S\}$ where $\wedge : \mathbb{R}^3 \mapsto so(3) : \omega \mapsto \hat{\omega}$, we have:

Property 3: $\exp : so(3) \mapsto SO(3), \hat{\omega}\theta \mapsto e^{\hat{\omega}\theta}$

2.2 Rotational Motion in $\mathbb{R}^3$

Rodrigues' formula ($\|\omega\| = 1$):

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1 - \cos\theta)$$

Rodrigues' formula ($\|\omega\| = 1$):

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1 - \cos\theta)$$

Proof :

Let $a \in \mathbb{R}^3$, write

$$a = \omega\theta, \omega = \frac{a}{\|a\|} (\text{or } \|\omega\| = 1), \text{and } \theta = \|a\|$$

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\theta + \frac{(\hat{\omega}\theta)^2}{2!} + \frac{(\hat{\omega}\theta)^3}{3!} + \cdots$$

As

$$\hat{a}^2 = aa^T - \|a\|^2 I, \hat{a}^3 = -\|a\|^2\hat{a}$$

we have:

$$e^{\hat{\omega}\theta} = I + (\theta - \frac{\theta^3}{3!} + \frac{\theta^3}{5!} - \cdots)\hat{\omega} + (\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \cdots)\hat{\omega}^2$$

$$= I + \hat{\omega}\sin\theta + \hat{\omega}^2(1 - \cos\theta)$$

$\square$

2.2 Rotational Motion in $\mathbb{R}^3$

Rodrigues' formula for $\|\omega\| \neq 1$:

$$e^{\hat{\omega}\theta} = I + \frac{\hat{\omega}}{\|\omega\|}\sin\|\omega\|\theta + \frac{\hat{\omega}^2}{\|\omega\|^2}(1 - \cos\|\omega\|\theta)$$

Rodrigues' formula for $\|\omega\| \neq 1$:

$$e^{\hat{\omega}\theta} = I + \frac{\hat{\omega}}{\|\omega\|}\sin\|\omega\|\theta + \frac{\hat{\omega}^2}{\|\omega\|^2}(1 - \cos\|\omega\|\theta)$$

Proof for Property 3:

Let $R \triangleq e^{\hat{\omega}\theta}$, then:

$$(e^{\hat{\omega}\theta})^{-1} = e^{-\hat{\omega}\theta} = e^{\hat{\omega}^T\theta} = (e^{\hat{\omega}\theta})^T$$

$$\Rightarrow R^{-1} = R^T \Rightarrow R^T R = I \Rightarrow \det R = \pm 1$$

From $\det\exp(0) = 1$, and the continuity of det function w.r.t. θ, we have $\det e^{\hat{\omega}\theta} = 1, \forall\theta \in \mathbb{R}$ $\square$

2.2 Rotational Motion in $\mathbb{R}^3$

Property 4: The exponential map is onto.

Property 4: The exponential map is onto.

Proof :

Given $R \in SO(3)$, to show $\exists \omega \in \mathbb{R}^3, \|\omega\| = 1$ and θ s.t. $R = e^{\hat{\omega}\theta}$

Let

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

and

$$v_\theta = 1 - \cos\theta, c_\theta = \cos\theta, s_\theta = \sin\theta$$

By Rodrigues' formula

$$e^{\hat{\omega}\theta} = \begin{bmatrix} \omega_1^2 v_\theta + c_\theta & \omega_1\omega_2 v_\theta - \omega_3 s_\theta & \omega_1\omega_3 v_\theta + \omega_2 s_\theta \\ \omega_1\omega_2 v_\theta + \omega_3 s_\theta & \omega_2^2 v_\theta + c_\theta & \omega_2\omega_3 v_\theta - \omega_1 s_\theta \\ \omega_1\omega_3 v_\theta - \omega_2 s_\theta & \omega_2\omega_3 v_\theta + \omega_1 s_\theta & \omega_3^2 v_\theta + c_\theta \end{bmatrix}$$

(continues next slide)

2.2 Rotational Motion in $\mathbb{R}^3$

Taking the trace of both sides,

$$\mathrm{tr}(R) = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta = \sum_{i=1}^{3} \lambda_i$$

where λ_i is the eigenvalue of $R, i = 1, 2, 3$

Case 1: $\mathrm{tr}(R) = 3$ or $R = I,\ \theta = 0 \Rightarrow \omega\theta = 0$

Taking the trace of both sides,

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Case 1: $\operatorname{tr}(R) = 3$ or $R = I,\ \theta = 0 \Rightarrow \omega\theta = 0$

Case 2: $-1 < \operatorname{tr}(R) < 3,$

$$\theta = \arccos\frac{\operatorname{tr}(R) - 1}{2} \Rightarrow \omega = \frac{1}{2s_\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

2.2 Rotational Motion in $\mathbb{R}^3$

Taking the trace of both sides,

$$\mathrm{tr}(R) = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta = \sum_{i=1}^{3} \lambda_i$$

where λ_i is the eigenvalue of $R, i = 1, 2, 3$

$$\begin{aligned}
\text{Case 1:} \quad & \mathrm{tr}(R) = 3 \text{ or } R = I, \ \theta = 0 \Rightarrow \ \omega\theta = 0 \\
\text{Case 2:} \quad & -1 < \mathrm{tr}(R) < 3, \\
& \theta = \arccos \frac{\mathrm{tr}(R) - 1}{2} \Rightarrow \omega = \frac{1}{2s_\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \\
\text{Case 3:} \quad & \mathrm{tr}(R) = -1 \Rightarrow \cos\theta = -1 \Rightarrow \theta = \pm\pi
\end{aligned}$$

(continues next slide)

2.2 Rotational Motion in $\mathbb{R}^3$

Following are 3 possibilities:

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \omega = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \omega = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Note that if $\omega\theta$ is a solution, then $\omega(\theta \pm n\pi), n = 0, \pm 1, \pm 2, \ldots$ is also a solution. $\square$

2.2 Rotational Motion in $\mathbb{R}^3$

Definition: Exponential coordinate

$\omega\theta \in \mathbb{R}^3$, with $e^{\hat{\omega}\theta} = R$ is called the exponential coordinates of R

Exp :

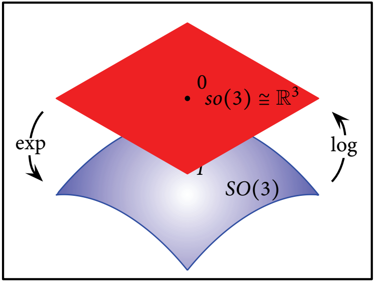

Figure 2.6

2.2 Rotational Motion in $\mathbb{R}^3$

Definition: Exponential coordinate

$\omega\theta \in \mathbb{R}^3$, with $e^{\hat{\omega}\theta} = R$ is called the exponential coordinates of R

Exp :

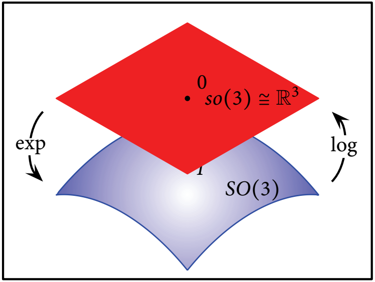

Figure 2.6

Property 5: exp is 1-1 when restricted to an open ball in $\mathbb{R}^3$ of radius π.

2.2 Rotational Motion in $\mathbb{R}^3$

Theorem 1 (Euler):
Any orientation is equivalent to a rotation about a fixed axis $\omega \in \mathbb{R}^3$ through an angle $\theta \in [-\pi, \pi]$.

1707–1783

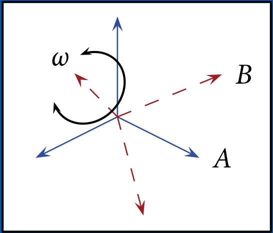

Figure 2.7

$SO(3)$ can be visualized as a solid ball of radius π.

□ Other Parametrizations of $SO(3)$:

- XYZ fixed angles (or Roll-Pitch-Yaw angle)

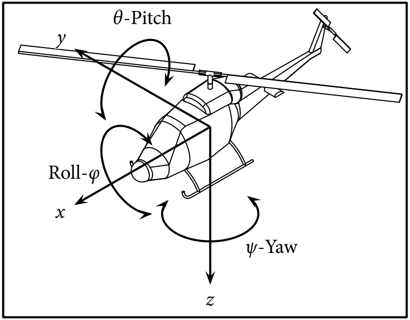

Figure 2.8

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2.2 Rotational Motion in $\mathbb{R}^3$

- XYZ fixed angles (or Roll-Pitch-Yaw angle) Continued

$$
\begin{aligned}
R_x(\varphi) &:= e^{\hat{x}\varphi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix} \\
R_y(\theta) &:= e^{\hat{y}\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \\
R_z(\psi) &:= e^{\hat{z}\psi} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
R_{ab} &= R_x(\varphi)R_y(\theta)R_z(\psi) \\
&= \begin{bmatrix} c_\theta c_\psi & -c_\theta s_\psi & s_\theta \\ s_\varphi s_\theta c_\psi + c_\varphi s_\psi & -s_\varphi s_\theta s_\psi + c_\varphi c_\psi & -s_\varphi c_\theta \\ -c_\varphi s_\theta c_\psi + s_\varphi s_\psi & c_\varphi s_\theta s_\psi + s_\varphi c_\psi & c_\varphi c_\theta \end{bmatrix}
\end{aligned}
$$

2.2 Rotational Motion in $\mathbb{R}^3$

- ZYX Euler angle

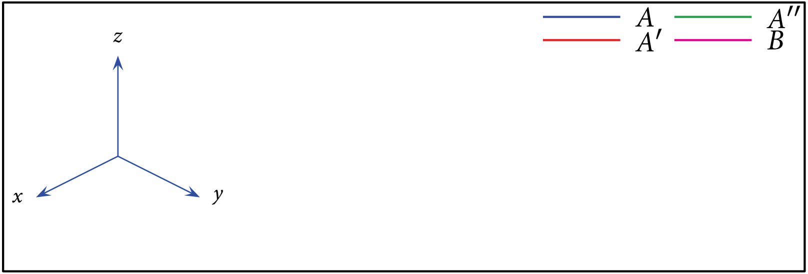

Figure 2.9

2.2 Rotational Motion in $\mathbb{R}^3$

- ZYX Euler angle

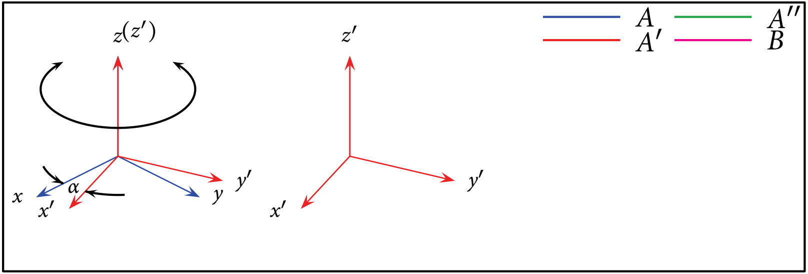

Figure 2.9

$$R_{aa'} = R_z(\alpha)$$

2.2 Rotational Motion in $\mathbb{R}^3$

- ZYX Euler angle

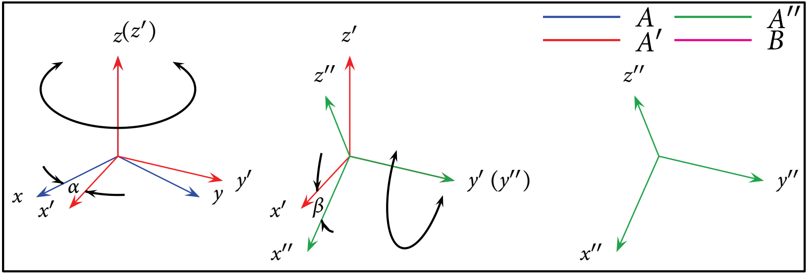

Figure 2.9

$$R_{aa'} = R_z(\alpha) \qquad R_{a'a''} = R_y(\beta)$$

2.2 Rotational Motion in $\mathbb{R}^3$

- ZYX Euler angle

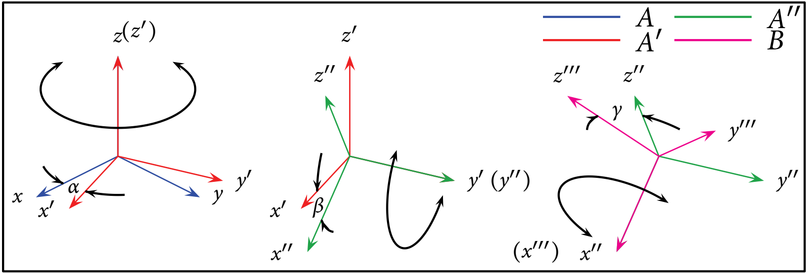

Figure 2.9

$$R_{aa'} = R_z(\alpha) \qquad R_{a'a''} = R_y(\beta) \qquad R_{a''b} = R_x(\gamma)$$

$$R_{ab} = R_z(\alpha)R_y(\beta)R_x(\gamma)$$

(continues next slide)

2.2 Rotational Motion in $\mathbb{R}^3$

- *ZYX* Euler angle (continued)

$$R_{ab}(\alpha,\beta,\gamma) = \begin{bmatrix} c_\alpha c_\beta & -s_\alpha c_\gamma + c_\alpha s_\beta s_\gamma & s_\alpha s_\gamma + c_\alpha s_\beta c_\gamma \\ s_\alpha c_\beta & c_\alpha c_\gamma + s_\alpha s_\beta s_\gamma & -c_\alpha s_\gamma + s_\alpha s_\beta c_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{bmatrix}$$

Note: When $\beta = \frac{\pi}{2}$, $\cos\beta = 0$, $\alpha + \gamma = \text{const} \Rightarrow$ singularity!

$$\beta = \operatorname{atan2}(-r_{31}, \sqrt{r_{32}^2 + r_{33}^2})$$
$$\alpha = \operatorname{atan2}(r_{21}/c_\beta, r_{11}/c_\beta)$$
$$\gamma = \operatorname{atan2}(r_{32}/c_\beta, r_{33}/c_\beta)$$

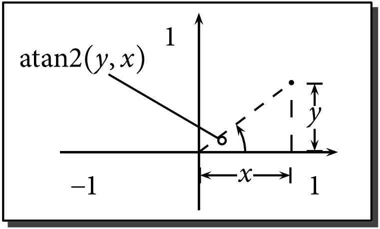

Figure 2.10

§ Quaternions:

$Q = q_0 + q_1 i + q_2 j + q_3 k$
where $i^2 = j^2 = k^2 = -1, i \cdot j = k, j \cdot k = i, k \cdot i = j$

§ Quaternions:

$Q = q_0 + q_1 i + q_2 j + q_3 k$
where $i^2 = j^2 = k^2 = -1, i \cdot j = k, j \cdot k = i, k \cdot i = j$

Property 1: Define $Q^* = (q_0, q)^* = (q_0, -q), q_0 \in \mathbb{R}, q \in \mathbb{R}^3$
$\|Q\|^2 = QQ^* = q_0{}^2 + q_1{}^2 + q_2{}^2 + q_3{}^2$

2.2 Rotational Motion in $\mathbb{R}^3$

§ Quaternions:

$Q = q_0 + q_1 i + q_2 j + q_3 k$
where $i^2 = j^2 = k^2 = -1, i \cdot j = k, j \cdot k = i, k \cdot i = j$

Property 1: Define $Q^* = (q_0, q)^* = (q_0, -q), q_0 \in \mathbb{R}, q \in \mathbb{R}^3$
$\|Q\|^2 = QQ^* = {q_0}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2$

Property 2: $Q = (q_0, q), P = (p_0, p)$
$QP = (q_0 p_0 - q \cdot p, q_0 p + p_0 q + q \times p)$

§ Quaternions:

$Q = q_0 + q_1 i + q_2 j + q_3 k$
where $i^2 = j^2 = k^2 = -1, i \cdot j = k, j \cdot k = i, k \cdot i = j$

Property 1: Define $Q^* = (q_0, q)^* = (q_0, -q), q_0 \in \mathbb{R}, q \in \mathbb{R}^3$
$\|Q\|^2 = QQ^* = {q_0}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2$

Property 2: $Q = (q_0, q), P = (p_0, p)$
$QP = (q_0 p_0 - q \cdot p, q_0 p + p_0 q + q \times p)$

Property 3: (a) The set of unit quaternions forms a group
(b) If $R = e^{\hat{\omega}\theta}$, then $Q = (\cos\frac{\theta}{2}, \omega \sin\frac{\theta}{2})$
(c) Q acts on $x \in \mathbb{R}^3$ by QXQ^*, where $X = (0, x)$

2.2 Rotational Motion in $\mathbb{R}^3$

□ Unit Quaternions:

Given $Q = (q_0, q), q_0 \in \mathbb{R}, q \in \mathbb{R}^3$, the vector part of QXQ^* is given by $R(Q)x$, recall that

$$q_0 = \cos\frac{\theta}{2}, q = \omega \sin\frac{\theta}{2}$$

and the Rodrigues' formula:

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1 - \cos\theta)$$

then

$$\begin{aligned} R(Q) &= I + 2q_0\hat{q} + 2\hat{q}^2 \\ &= \begin{bmatrix} 1 - 2(q_2^2 + q_3^2) & -2q_0q_3 + 2q_1q_2 & 2q_0q_2 + 2q_1q_3 \\ 2q_0q_3 + 2q_1q_2 & 1 - 2(q_1^2 + q_3^2) & -2q_0q_1 + 2q_2q_3 \\ -2q_0q_2 + 2q_1q_3 & 2q_0q_1 + 2q_2q_3 & 1 - 2(q_1^2 + q_2^2) \end{bmatrix} \end{aligned}$$

where $\|Q\| \triangleq q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$

(continues next slide)

□ Quaternions (continued):

Conversion from Roll-Pitch-Yaw angle to unit quaternions:

$$Q = (\cos\frac{\varphi}{2}, x\sin\frac{\varphi}{2})(\cos\frac{\theta}{2}, y\sin\frac{\theta}{2})(\cos\frac{\psi}{2}, z\sin\frac{\psi}{2}) \Rightarrow$$

$$q_0 = \cos\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} - \sin\frac{\varphi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2}$$

$$q = \begin{bmatrix} \cos\frac{\varphi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} + \sin\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} \\ \cos\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} - \sin\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} \\ \cos\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} + \sin\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} \end{bmatrix}$$

Conversion from unit quaternions to roll-pitch-yaw angles (?)

† End of Section †

2.3 Rigid motion in $\mathbb{R}^3$

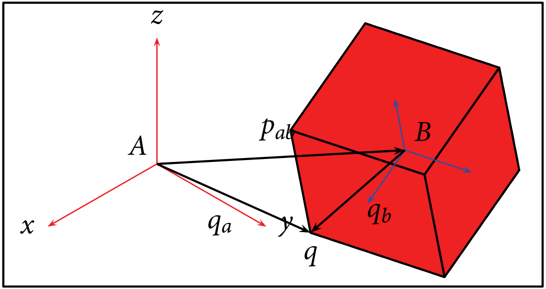

Figure 2.11

$p_{ab} \in \mathbb{R}^3$: Coordinates of the origin of B

$R_{ab} \in SO(3)$: Orientation of B relative to A

$SE(3) : \{(p, R) | p \in \mathbb{R}^3, R \in SO(3)\}$: Configuration Space

2.3 Rigid motion in $\mathbb{R}^3$

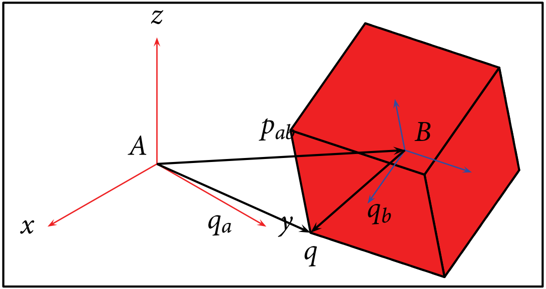

Figure 2.11

$p_{ab} \in \mathbb{R}^3$: Coordinates of the origin of B

$R_{ab} \in SO(3)$: Orientation of B relative to A

$SE(3) : \{(p, R) | p \in \mathbb{R}^3, R \in SO(3)\}$: Configuration Space

Or…as a transformation:

$$g_{ab} = (p_{ab}, R_{ab}) : \mathbb{R}^3 \mapsto \mathbb{R}^3$$

$$q_b \mapsto q_a = p_{ab} + R_{ab} \cdot q_b$$

2.3 Rigid motion in $\mathbb{R}^3$

Homogeneous Representation:

Points:

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \in \mathbb{R}^3 \quad \Longrightarrow \quad \bar{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix} \in \mathbb{R}^4$$

2.3 Rigid motion in $\mathbb{R}^3$

Homogeneous Representation:

Points:

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \in \mathbb{R}^3 \quad \Longrightarrow \quad \bar{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix} \in \mathbb{R}^4$$

Vectors:

$$v = p - q = \begin{bmatrix} p_1 - q_1 \\ p_2 - q_2 \\ p_3 - q_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \Longrightarrow \quad \bar{v} = \bar{p} - \bar{q} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix} - \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}$$

2.3 Rigid motion in $\mathbb{R}^3$

□ Homogeneous Representation:

Points:

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \in \mathbb{R}^3 \Rightarrow \bar{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix} \in \mathbb{R}^4$$

Vectors:

$$v = p - q = \begin{bmatrix} p_1 - q_1 \\ p_2 - q_2 \\ p_3 - q_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \Rightarrow \bar{v} = \bar{p} - \bar{q} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix} - \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}$$

1. Point-Point = Vector
2. Vector+Point = Point
3. Vector+Vector = Vector
4. Point+Point: Meaningless

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2.3 Rigid motion in $\mathbb{R}^3$

$$q_a = p_{ab} + R_{ab} \cdot q_b$$

$$\begin{bmatrix} q_a \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}}_{\bar{g}_{ab}} \begin{bmatrix} q_{ab} \\ 1 \end{bmatrix}$$

$$\bar{q}_a = \bar{g}_{ab} \cdot \bar{q}_b$$

$$\boxed{g_{ab} = (p_{ab}, R_{ab})}$$

$$\Downarrow$$

$$\boxed{\bar{g}_{ab} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}}$$

2.3 Rigid motion in $\mathbb{R}^3$

$$q_a = p_{ab} + R_{ab} \cdot q_b$$

$$\begin{bmatrix} q_a \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}}_{\overline{g}_{ab}} \begin{bmatrix} q_{ab} \\ 1 \end{bmatrix}$$

$$\overline{q}_a = \overline{g}_{ab} \cdot \overline{q}_b$$

$$\boxed{g_{ab} = (p_{ab}, R_{ab})}$$

$$\boxed{\overline{g}_{ab} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}}$$

□ Composition Rule:

$$\overline{q}_b = \overline{g}_{bc} \cdot \overline{q}_c$$

$$\overline{q}_a = \overline{g}_{ab} \cdot \overline{q}_b$$

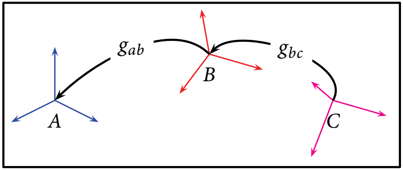

Figure 2.12

2.3 Rigid motion in $\mathbb{R}^3$

$$q_a = p_{ab} + R_{ab} \cdot q_b$$

$$\begin{bmatrix} q_a \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}}_{\bar{g}_{ab}} \begin{bmatrix} q_{ab} \\ 1 \end{bmatrix}$$

$$\bar{q}_a = \bar{g}_{ab} \cdot \bar{q}_b$$

$$\boxed{g_{ab} = (p_{ab}, R_{ab})}$$

$$\Downarrow$$

$$\boxed{\bar{g}_{ab} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}}$$

- **Composition Rule:**

$$\bar{q}_b = \bar{g}_{bc} \cdot \bar{q}_c$$

$$\bar{q}_a = \bar{g}_{ab} \cdot \bar{q}_b = \underbrace{\bar{g}_{ab} \cdot \bar{g}_{bc}}_{\bar{g}_{ac}} \cdot \bar{q}_c$$

$$\bar{g}_{ac} = \bar{g}_{ab} \cdot \bar{g}_{bc} = \begin{bmatrix} R_{ab}R_{bc} & R_{ab}p_{bc} + p_{ab} \\ 0 & 1 \end{bmatrix}$$

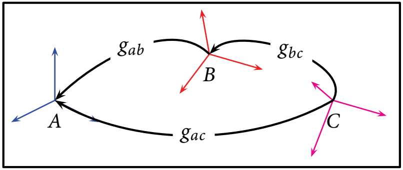

Figure 2.12

2.3 Rigid motion in $\mathbb{R}^3$

□ Special Euclidean Group:

$$SE(3) = \left\{ \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \middle| p \in \mathbb{R}^3, R \in SO(3) \right\}$$

2.3 Rigid motion in $\mathbb{R}^3$

□ Special Euclidean Group:

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Property 4: $SE(3)$ forms a group.

2.3 Rigid motion in $\mathbb{R}^3$

☐ Special Euclidean Group:

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Property 4: $SE(3)$ forms a group.

Proof :

1. $g_1 \cdot g_2 \in SE(3)$
2. $e = I_4$
3. $(\bar{g})^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$
4. Associativity: Follows from property of matrix multiplication

☐

§ Induced transformation on vectors:

$$\bar{v} = s - r = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}, \bar{g}_* \bar{v} = \bar{g}s - \bar{g}r = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix} = \begin{bmatrix} Rv \\ 0 \end{bmatrix}$$

The bar will be dropped to simplify notations

Property 5: An element of $SE(3)$ is a rigid transformation.

2.3 Rigid motion in $\mathbb{R}^3$

Exponential coordinates of $SE(3)$:

For rotational motion:

$$\dot{p}(t)=\omega \times (p(t) - q)$$

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix}=\begin{bmatrix} \hat{\omega} & -\omega \times q \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p \\ 1 \end{bmatrix}$$

$$\text{or } \dot{\bar{p}}=\hat{\xi} \cdot \bar{p} \Rightarrow \bar{p}(t) = e^{\hat{\xi}t}\bar{p}(0)$$

$$\text{where } e^{\hat{\xi}t}=I + \hat{\xi}t + \frac{(\hat{\xi}t)^2}{2!} + \cdots$$

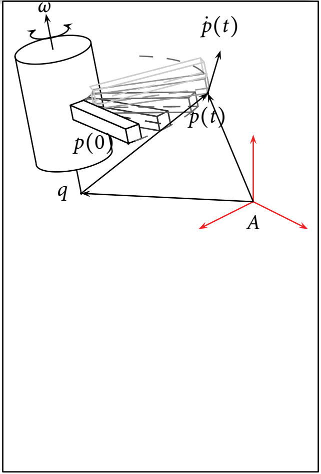

Figure 2.13

2.3 Rigid motion in $\mathbb{R}^3$

Exponential coordinates of $SE(3)$:

For rotational motion:

$$\dot{p}(t)=\omega\times(p(t)-q)$$

$$\begin{bmatrix}\dot{p}\\0\end{bmatrix}=\begin{bmatrix}\hat{\omega} & -\omega\times q\\0 & 0\end{bmatrix}\begin{bmatrix}p\\1\end{bmatrix}$$

$$\text{or } \dot{\bar{p}}=\hat{\xi}\cdot\bar{p}\Rightarrow\bar{p}(t)=e^{\hat{\xi}t}\bar{p}(0)$$

$$\text{where } e^{\hat{\xi}t}=I+\hat{\xi}t+\frac{(\hat{\xi}t)^2}{2!}+\cdots$$

For translational motion:

$$\dot{p}(t)=v$$

$$\begin{bmatrix}\dot{p}(t)\\0\end{bmatrix}=\begin{bmatrix}0 & v\\0 & 0\end{bmatrix}\begin{bmatrix}p\\1\end{bmatrix}$$

$$\dot{\bar{p}}(t)=\hat{\xi}\cdot\bar{p}(t)\Rightarrow\bar{p}(t)=e^{\hat{\xi}t}\bar{p}(0)$$

$$\hat{\xi}=\begin{bmatrix}0 & v\\0 & 0\end{bmatrix}$$

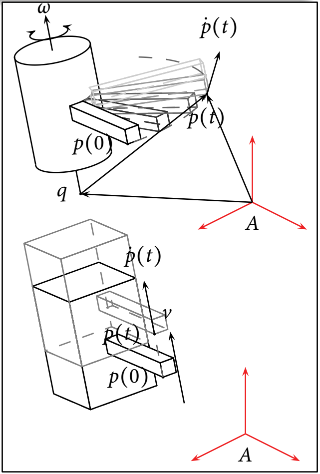

Figure 2.13

2.3 Rigid motion in $\mathbb{R}^3$

Definition:

$$se(3) = \left\{ \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \middle| v, \omega \in \mathbb{R}^3 \right\}$$

is called the twist space. There exists a 1-1 correspondence between $se(3)$ and $\mathbb{R}^6$, defined by $\wedge : \mathbb{R}^6 \mapsto se(3)$

$$\xi := \begin{bmatrix} v \\ \omega \end{bmatrix} \mapsto \hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$$

2.3 Rigid motion in $\mathbb{R}^3$

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Property 6: $\exp : se(3) \mapsto SE(3), \hat{\xi}\theta \mapsto e^{\hat{\xi}\theta}$

2.3 Rigid motion in $\mathbb{R}^3$

Definition:

$$se(3) = \left\{ \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \middle| v, \omega \in \mathbb{R}^3 \right\}$$

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$$\xi := \begin{bmatrix} v \\ \omega \end{bmatrix} \mapsto \hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$$

Property 6: $\exp : se(3) \mapsto SE(3), \hat{\xi}\theta \mapsto e^{\hat{\xi}\theta}$

Proof :

Let $\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$

- If $\omega = 0$, then $\hat{\xi}^2 = \hat{\xi}^3 = \cdots = 0$, $e^{\hat{\xi}\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix} \in SE(3)$

(continues next slide)

2.3 Rigid motion in $\mathbb{R}^3$

- If ω is not 0, assume $\|\omega\| = 1$.

Define:

$$g_0 = \begin{bmatrix} I & \omega \times v \\ 0 & 1 \end{bmatrix}, \hat{\xi}' = g_0^{-1} \cdot \hat{\xi} \cdot g_0 = \begin{bmatrix} \hat{\omega} & h\omega \\ 0 & 0 \end{bmatrix}$$

where $h = \omega^T \cdot v$.

$$e^{\hat{\xi}\theta} = e^{g_0 \cdot \hat{\xi}' \cdot g_0^{-1}} = g_0 \cdot e^{\hat{\xi}'\theta} \cdot g_0^{-1}$$

and as

$$\hat{\xi}'^2 = \begin{bmatrix} \hat{\omega}^2 & 0 \\ 0 & 0 \end{bmatrix}, \hat{\xi}'^3 = \begin{bmatrix} \hat{\omega}^3 & 0 \\ 0 & 0 \end{bmatrix}$$

we have

$$e^{\hat{\xi}'\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & h\omega\theta \\ 0 & 1 \end{bmatrix} \Rightarrow e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})\hat{\omega}v + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix}$$

$\square$

2.3 Rigid motion in $\mathbb{R}^3$

$$p(\theta) = e^{\hat{\xi}\theta} \cdot p(0) \Rightarrow g_{ab}(\theta) = e^{\hat{\xi}\theta}$$

If there is offset,

$$g_{ab}(\theta) = e^{\hat{\xi}\theta} g_{ab}(0)(\text{ Why?})$$

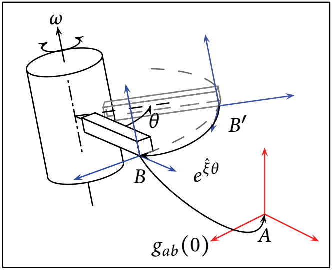

Figure 2.14

2.3 Rigid motion in $\mathbb{R}^3$

Property 7: $exp : se(3) \mapsto SE(3)$ is onto.

2.3 Rigid motion in $\mathbb{R}^3$

Property 7: $exp : se(3) \mapsto SE(3)$ is onto.

Proof :

Let $g = (p, R), R \in SO(3), p \in \mathbb{R}^3$

Case 1: $(R = I)$ Let

$$\hat{\xi} = \begin{bmatrix} 0 & \frac{p}{\|p\|} \\ 0 & 0 \end{bmatrix}, \theta = \|p\| \Rightarrow e^{\hat{\xi}\theta} = g = \begin{bmatrix} I & p \\ 0 & 1 \end{bmatrix}$$

2.3 Rigid motion in $\mathbb{R}^3$

Property 7: $exp : se(3) \mapsto SE(3)$ is onto.

Proof :

Let $g = (p, R), R \in SO(3), p \in \mathbb{R}^3$

Case 1: $(R = I)$ Let

$$\hat{\xi} = \begin{bmatrix} 0 & \frac{p}{\|p\|} \\ 0 & 0 \end{bmatrix}, \theta = \|p\| \Rightarrow e^{\hat{\xi}\theta} = g = \begin{bmatrix} I & p \\ 0 & 1 \end{bmatrix}$$

Case 2: $(R \neq I)$

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})(\omega \times v) + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} e^{\hat{\omega}\theta} = R \\ (I - e^{\hat{\omega}\theta})(\omega \times v) + \omega\omega^T v\theta = p \end{cases}$$

Solve for $\omega\theta$ from previous section. Let $A = (I - e^{\hat{\omega}\theta})\hat{\omega} + ww^T\theta$, $Av = p$. Claim:

$$A = (I - e^{\hat{\omega}\theta})\hat{\omega} + ww^T\theta := A_1 + A_2$$

$$\ker A_1 \cap \ker A_2 = \phi \Rightarrow v = A^{-1}p$$

□

$\xi\theta \in \mathbb{R}^6$: Exponential coordinates of $g \in SE(3)$

2.3 Rigid motion in $\mathbb{R}^3$

□ Screws, twists and screw motion:

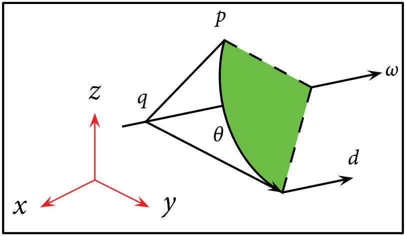

Figure 2.15

Screw attributes

Pitch: $h = \frac{d}{\theta}(\theta = 0, h = \infty), d = h \cdot \theta$

Axis: $l = \{q + \lambda\omega | \lambda \in \mathbb{R}\}$

Magnitude: $M = \theta$

□ Screws, twists and screw motion:

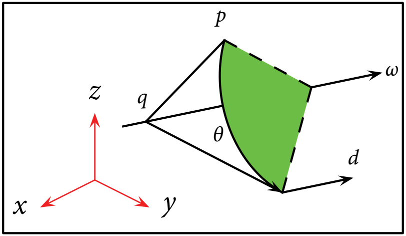

Figure 2.15

Screw attributes

Pitch:	$h = \frac{d}{\theta} (\theta = 0, h = \infty), d = h \cdot \theta$
Axis:	$l = \{q + \lambda\omega \mid \lambda \in \mathbb{R}\}$
Magnitude:	$M = \theta$

Definition:
A **screw** S consists of an axis l, pitch h, and magnitude M. A **screw motion** is a rotation by $\theta = M$ about l, followed by translation by $h\theta$, parallel to l. If $h = \infty$, then, translation about v by $\theta = M$

2.3 Rigid motion in $\mathbb{R}^3$

Corresponding $g \in SE(3)$:

$$g \cdot p = q + e^{\hat{\omega}\theta}(p - q) + h\theta\omega$$

$$g \cdot \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta\omega \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} \Rightarrow$$

$$g = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta\omega \\ 0 & 1 \end{bmatrix}$$

2.3 Rigid motion in $\mathbb{R}^3$

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$$g \cdot p = q + e^{\hat{\omega}\theta}(p - q) + h\theta\omega$$

$$g \cdot \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta\omega \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} \Rightarrow$$

$$g = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta\omega \\ 0 & 1 \end{bmatrix}$$

On the other hand...

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})\omega \times v + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix}$$

If we let $v = -\omega \times q + h\omega$, then

$$(I - e^{\hat{\omega}\theta})(-\hat{\omega}^2 q) = (I - e^{\hat{\omega}\theta})(-\omega\omega^T q + q) = (I - e^{\hat{\omega}\theta})q$$

Thus, $e^{\hat{\xi}\theta} = g$

2.3 Rigid motion in $\mathbb{R}^3$

Corresponding $g \in SE(3)$:

$$g \cdot p = q + e^{\hat{\omega}\theta}(p - q) + h\theta\omega$$

$$g \cdot \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})q + h\theta\omega \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} \Rightarrow$$

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On the other hand...

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (I - e^{\hat{\omega}\theta})\omega \times v + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix}$$

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$$(I - e^{\hat{\omega}\theta})(-\hat{\omega}^2 q) = (I - e^{\hat{\omega}\theta})(-\omega\omega^T q + q) = (I - e^{\hat{\omega}\theta})q$$

Thus, $e^{\hat{\xi}\theta} = g$

For pure rotation $(h = 0)$: $\xi = (-\omega \times q, \omega)$

For pure translation: $g = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$, $\Rightarrow \xi = (v, 0)$, and $e^{\hat{\xi}\theta} = g$

2.3 Rigid motion in $\mathbb{R}^3$

Screw associated with a twist:

$\xi = (v, \omega) \in \mathbb{R}^6$

1. Pitch: $h = \begin{cases} \dfrac{\omega^T v}{\|\omega\|^2}, & \text{if } \omega \neq 0 \\ \infty, & \text{if } \omega = 0 \end{cases}$
2. Axis: $l = \begin{cases} \dfrac{\omega \times v}{\|\omega\|^2} + \lambda\omega, & \lambda \in \mathbb{R}, \text{if } \omega \neq 0 \\ 0 + \lambda v & \lambda \in \mathbb{R}, \text{if } \omega = 0 \end{cases}$
3. Magnitude: $M = \begin{cases} \|\omega\|, & \text{if } \omega \neq 0 \\ \|v\|, & \text{if } \omega = 0 \end{cases}$

Screw associated with a twist:

$\xi = (v, \omega) \in \mathbb{R}^6$

1. Pitch: $h = \begin{cases} \dfrac{\omega^T v}{\|\omega\|^2}, & \text{if } \omega \neq 0 \\ \infty, & \text{if } \omega = 0 \end{cases}$
2. Axis: $l = \begin{cases} \dfrac{\omega \times v}{\|\omega\|^2} + \lambda\omega, & \lambda \in \mathbb{R}, \text{if } \omega \neq 0 \\ 0 + \lambda v & \lambda \in \mathbb{R}, \text{if } \omega = 0 \end{cases}$
3. Magnitude: $M = \begin{cases} \|\omega\|, & \text{if } \omega \neq 0 \\ \|v\|, & \text{if } \omega = 0 \end{cases}$

Special cases:

1. $h = \infty$, Pure translation (prismatic joint)
2. $h = 0$, Pure rotation (revolute joint)

2.3 Rigid motion in $\mathbb{R}^3$

Screw	**Twist: $\hat{\xi}\theta$**
Case 1: Pitch: $h = \infty$ Axis: $l = \{q + \lambda v \mid \|v\| = 1, \lambda \in \mathbb{R}\}$ Magnitude: M	$\theta = M$, $\hat{\xi} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$
Case 2: Pitch: $h \neq \infty$ Axis: $l = \{q + \lambda \omega \mid \|\omega\| = 1, \lambda \in \mathbb{R}\}$ Magnitude: M	$\theta = M$, $\hat{\xi} = \begin{bmatrix} \hat{\omega} & -\hat{\omega}q + h\omega \\ 0 & 0 \end{bmatrix}$

2.3 Rigid motion in $\mathbb{R}^3$

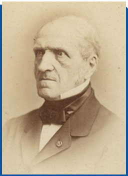
Screw	Twist: $\hat{\xi}\theta$
Case 1: Pitch: $h = \infty$ Axis: $l = \{q + \lambda v \mid \|v\| = 1, \lambda \in \mathbb{R}\}$ Magnitude: M	$\theta = M,$ $\hat{\xi} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$
Case 2: Pitch: $h \neq \infty$ Axis: $l = \{q + \lambda \omega \mid \|\omega\| = 1, \lambda \in \mathbb{R}\}$ Magnitude: M	$\theta = M,$ $\hat{\xi} = \begin{bmatrix} \hat{\omega} & -\hat{\omega}q + h\omega \\ 0 & 0 \end{bmatrix}$

Definition: Screw Motion

Rotation about an axis by $\theta = M$, followed by translation about the same axis by $h\theta$

2.3 Rigid motion in $\mathbb{R}^3$

Theorem 2 (Chasles):
Every rigid body motion can be realized by a rotation about an axis combined with a translation parallel to that axis.



1793–1880

2.3 Rigid motion in $\mathbb{R}^3$

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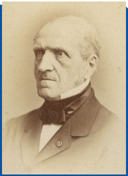

1793–1880

Proof :

For $\hat{\xi} \in se(3)$:

$$\hat{\xi} = \hat{\xi}_1 + \hat{\xi}_2 = \begin{bmatrix} \hat{\omega} & -\omega \times q \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & h\omega \\ 0 & 0 \end{bmatrix}$$

$$[\hat{\xi}_1, \hat{\xi}_2] = 0 \Rightarrow e^{\hat{\xi}\theta} = e^{\hat{\xi}_1\theta} e^{\hat{\xi}_2\theta}$$

$\square$

† End of Section †

2.4 Velocity of a Rigid Body

◇ Review: Point-mass velocity

$$q(t) \in \mathbb{R}^3, t \in (-\varepsilon, \varepsilon), v = \frac{\mathrm{d}}{\mathrm{d}t} q(t) \in \mathbb{R}^3, a = \frac{\mathrm{d}^2}{\mathrm{d}t^2} q(t) = \frac{\mathrm{d}}{\mathrm{d}t} v(t) \in \mathbb{R}^3$$

2.4 Velocity of a Rigid Body

◇ Review: Point-mass velocity

$$q(t) \in \mathbb{R}^3, t \in (-\varepsilon, \varepsilon), v = \frac{\mathrm{d}}{\mathrm{d}t} q(t) \in \mathbb{R}^3, a = \frac{\mathrm{d}^2}{\mathrm{d}t^2} q(t) = \frac{\mathrm{d}}{\mathrm{d}t} v(t) \in \mathbb{R}^3$$

□ Velocity of Rotational Motion:

$$R_{ab}(t) \in SO(3), t \in (-\varepsilon, \varepsilon), \; q_a(t) = R_{ab}(t) q_b$$

$$V^a = \frac{\mathrm{d}}{\mathrm{d}t} q_a(t) = \dot{R}_{ab}(t) q_b = \dot{R}_{ab}(t) R_{ab}^T(t) R_{ab}(t) q_b = \dot{R}_{ab} R_{ab}^T q_a$$

$$R_{ab}(t) R_{ab}^T(t) = I \Rightarrow \dot{R}_{ab} R_{ab}^T + R_{ab} \dot{R}_{ab}^T = 0, \dot{R}_{ab} R_{ab}^T = -(\dot{R}_{ab} R_{ab}^T)^T$$

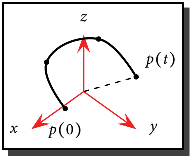

Figure 2.1

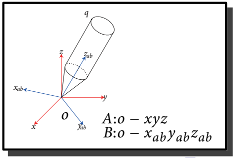

Figure 2.3

2.4 Velocity of a Rigid Body

Denote spatial angular velocity by:

$$\hat{\omega}^s_{ab} = \dot{R}_{ab} R^T_{ab}, \omega_{ab} \in \mathbb{R}^3$$

Then

$$V^a = \hat{\omega}^s_{ab} \cdot q_a = \omega^s_{ab} \times q_a$$

Body angular velocity:

$$\hat{\omega}^b_{ab} = R^T_{ab} \cdot \dot{R}_{ab}, v^b \triangleq R^T_{ab} \cdot v^a = \omega^b_{ab} \times q_b$$

Relation between body and spatial angular velocity:

$$\omega^b_{ab} = R^T_{ab} \cdot \omega^s_{ab} \text{ or } \hat{\omega}^b_{ab} = R^T_{ab} \hat{\omega}^s_{ab} R_{ab}$$

□ Generalized Velocity:

$$g_{ab} = \begin{bmatrix} R_{ab}(t) & p_{ab}(t) \\ 0 & 1 \end{bmatrix}, q_a(t) = g_{ab}(t) q_b$$

$$\frac{\mathrm{d}}{\mathrm{d}t} q_a(t) = \dot{g}_{ab}(t) q_b = \dot{g}_{ab} \cdot g_{ab}^{-1} \cdot g_{ab} \cdot q_b = \hat{V}^s_{ab} \cdot q_a$$

□ Generalized Velocity:

$$g_{ab} = \begin{bmatrix} R_{ab}(t) & p_{ab}(t) \\ 0 & 1 \end{bmatrix}, q_a(t) = g_{ab}(t)q_b$$

$$\frac{\mathrm{d}}{\mathrm{d}t} q_a(t) = \dot{g}_{ab}(t)q_b = \dot{g}_{ab} \cdot g_{ab}^{-1} \cdot g_{ab} \cdot q_b = \hat{V}_{ab}^s \cdot q_a$$

$$\begin{aligned} \hat{V}_{ab}^s = \dot{g}_{ab} \cdot g_{ab}^{-1} &= \begin{bmatrix} \dot{R}_{ab} & \dot{p}_{ab} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_{ab}^T & -R_{ab}^T p_{ab} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \dot{R}_{ab} R_{ab}^T & -\dot{R}_{ab} R_{ab}^T p_{ab} + \dot{p}_{ab} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \hat{\omega}_{ab}^s & -\omega_{ab}^s \times p_{ab} + \dot{p}_{ab} \\ 0 & 0 \end{bmatrix} \triangleq \begin{bmatrix} \hat{\omega}_{ab}^s & v_{ab}^s \\ 0 & 0 \end{bmatrix} \end{aligned}$$

☐ (Generalized) Spatial Velocity:

$$V_{ab}^s = \begin{bmatrix} v_{ab}^s \\ \omega_{ab}^s \end{bmatrix} = \begin{bmatrix} -\omega_{ab}^s \times p_{ab} + \dot{p}_{ab} \\ (\dot{R}_{ab} R_{ab}^T)^\vee \end{bmatrix}$$

$$v_{q_a} = \omega_{ab}^s \times q_a + v_{ab}^s$$

Note: $v_{q_b} = g_{ab}^{-1} \cdot v_{q_a} = g_{ab}^{-1} \cdot \dot{g}_{ab} \cdot q_b = \hat{V}_{ab}^b \cdot q_b$

□ (Generalized) Spatial Velocity:

$$V_{ab}^s = \begin{bmatrix} v_{ab}^s \\ \omega_{ab}^s \end{bmatrix} = \begin{bmatrix} -\omega_{ab}^s \times p_{ab} + \dot{p}_{ab} \\ (\dot{R}_{ab} R_{ab}^T)^\vee \end{bmatrix}$$

$$v_{q_a} = \omega_{ab}^s \times q_a + v_{ab}^s$$

Note: $v_{q_b} = g_{ab}^{-1} \cdot v_{q_a} = g_{ab}^{-1} \cdot \dot{g}_{ab} \cdot q_b = \hat{V}_{ab}^b \cdot q_b$

□ (Generalized) Body Velocity:

$$\hat{V}_{ab}^b = g_{ab}^{-1} \dot{g}_{ab} = \begin{bmatrix} R_{ab}^T \dot{R}_{ab} & R_{ab}^T \dot{p}_{ab} \\ 0 & 0 \end{bmatrix} \triangleq \begin{bmatrix} \hat{\omega}_{ab}^b & v_{ab}^b \\ 0 & 0 \end{bmatrix}$$

$$V_{ab}^b = \begin{bmatrix} v_{ab}^b \\ \omega_{ab}^b \end{bmatrix} = \begin{bmatrix} R_{ab}^T \dot{p}_{ab} \\ (R_{ab}^T \dot{R}_{ab})^\vee \end{bmatrix}$$

2.4 Velocity of a Rigid Body

□ Relation between body and spatial velocity:

$$\begin{aligned}
\hat{V}^s_{ab} &= \dot{g}_{ab} \cdot g^{-1}_{ab} = g_{ab} \cdot g^{-1}_{ab} \cdot \dot{g}_{ab} \cdot g^{-1}_{ab} = g_{ab} \cdot \hat{V}^b_{ab} \cdot g^{-1}_{ab} \\
&= \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega}^b_{ab} & v^b_{ab} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T_{ab} & -R^T_{ab} p_{ab} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega}^b_{ab} R^T_{ab} & -\hat{\omega}^b_{ab} R^T_{ab} p_{ab} + v^b_{ab} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} R_{ab} \hat{\omega}^b_{ab} R^T_{ab} & -R_{ab} \hat{\omega}^b_{ab} R^T_{ab} p_{ab} + R_{ab} v^b_{ab} \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

□ Relation between body and spatial velocity:

$$
\begin{aligned}
\hat{V}^s_{ab} &= \dot{g}_{ab} \cdot g_{ab}^{-1} = g_{ab} \cdot g_{ab}^{-1} \cdot \dot{g}_{ab} \cdot g_{ab}^{-1} = g_{ab} \cdot \hat{V}^b_{ab} \cdot g_{ab}^{-1} \\
&= \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega}^b_{ab} & v^b_{ab} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T_{ab} & -R^T_{ab} p_{ab} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega}^b_{ab} R^T_{ab} & -\hat{\omega}^b_{ab} R^T_{ab} p_{ab} + v^b_{ab} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} R_{ab} \hat{\omega}^b_{ab} R^T_{ab} & -R_{ab} \hat{\omega}^b_{ab} R^T_{ab} p_{ab} + R_{ab} v^b_{ab} \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

$$
V^s_{ab} = \begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix} = \underbrace{\begin{bmatrix} R_{ab} & \hat{p}_{ab} R_{ab} \\ 0 & R_{ab} \end{bmatrix}}_{\mathrm{Ad}_g} V^b_{ab}
$$

$$
\mathrm{Ad}_g = \begin{bmatrix} R & \hat{p} R \\ 0 & R \end{bmatrix} \in \mathbb{R}^{6\times 6}, \text{ for } g = (p, R)
$$

□ Properties of Adjoint mapping:

$$g^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \Rightarrow$$

$$\mathrm{Ad}_{g^{-1}} = \begin{bmatrix} R^T & (-R^T p)^\wedge R^T \\ 0 & R^T \end{bmatrix}$$

$$= \begin{bmatrix} R^T & -R^T \hat{p} \\ 0 & R^T \end{bmatrix} = (\mathrm{Ad}_g)^{-1}$$

and $\mathrm{Ad}_{g_1 \cdot g_2} = \mathrm{Ad}_{g_1} \cdot \mathrm{Ad}_{g_2}$

2.4 Velocity of a Rigid Body

□ Properties of Adjoint mapping:

$$g^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \Rightarrow$$

$$\mathrm{Ad}_{g^{-1}} = \begin{bmatrix} R^T & (-R^T p)^\wedge R^T \\ 0 & R^T \end{bmatrix}$$

$$= \begin{bmatrix} R^T & -R^T \hat{p} \\ 0 & R^T \end{bmatrix} = (\mathrm{Ad}_g)^{-1}$$

and $\mathrm{Ad}_{g_1 \cdot g_2} = \mathrm{Ad}_{g_1} \cdot \mathrm{Ad}_{g_2}$

The map $\mathrm{Ad} : SE(3) \mapsto GL(\mathbb{R}^6), \mathrm{Ad}(g) = \mathrm{Ad}_g$ is a group homomorphism

Matrix Rep	Vector Rep
$\hat{\xi} \in se(3)$	$\xi \in \mathbb{R}^6$
$g \cdot \hat{\xi} \cdot g^{-1} \in se(3)$	$\mathrm{Ad}_g \xi \in \mathbb{R}^6$

2.4 Velocity of a Rigid Body

◇ Example: Velocity of Screw Motion

$$g_{ab}(\theta) = e^{\hat{\xi}\theta(t)} g_{ab}(0), \quad \frac{\mathrm{d}}{\mathrm{d}t} e^{\hat{\xi}\theta(t)} = \hat{\xi}\dot{\theta}(t) e^{\hat{\xi}\theta(t)} = \dot{\theta}(t) e^{\hat{\xi}\theta(t)} \hat{\xi}$$

$$\begin{aligned} \hat{V}^s_{ab} &= \dot{g}_{ab} \cdot g^{-1}_{ab} = (\hat{\xi}\dot{\theta} e^{\hat{\xi}\theta(t)} g_{ab}(0)) \cdot (g^{-1}_{ab}(0) e^{-\hat{\xi}\theta(t)}) \\ &= \hat{\xi}\dot{\theta} \Rightarrow V^s_{ab} = \xi\dot{\theta} \end{aligned}$$

$$\begin{aligned} \hat{V}^b_{ab} &= g^{-1}_{ab} \cdot \dot{g}_{ab} = g^{-1}_{ab}(0) e^{-\hat{\xi}\theta} \cdot e^{\hat{\xi}\theta} \hat{\xi}\dot{\theta} g_{ab}(0) \\ &= g^{-1}_{ab}(0) \hat{\xi}\dot{\theta} g_{ab}(0) = (\mathrm{Ad}_{g^{-1}_{ab}(0)} \xi)^\wedge \dot{\theta} \Rightarrow V^b_{ab} = \mathrm{Ad}_{g^{-1}_{ab}(0)} \xi\dot{\theta} \end{aligned}$$

2.4 Velocity of a Rigid Body

□ Metric Property of $se(3)$:

Let $g_i(t) \in SE(3)$, $i = 1, 2$, be representations of the same motion, obtained using coordinate frame A and B. Then,

$$\boxed{g_2(t) = g_0 \cdot g_1(t) \cdot g_0^{-1} \Rightarrow V_2^s = \mathrm{Ad}_{g_0} \cdot V_1^s}$$

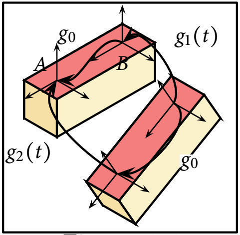

Figure 2.2

(Continues next slide)

2.4 Velocity of a Rigid Body

$$\|V_2^s\|^2 = (\mathrm{Ad}_{g_0} \cdot V_1^s)^T (\mathrm{Ad}_{g_0} \cdot V_1^s) = (V_1^s)^T \mathrm{Ad}_{g_0}^T \cdot \mathrm{Ad}_{g_0} \cdot V_1^s$$

$$\begin{aligned}\mathrm{Ad}_{g_0}^T \cdot \mathrm{Ad}_{g_0} &= \begin{bmatrix} R_0^T & 0 \\ -R_0^T \hat{p}_0 & R_0^T \end{bmatrix} \begin{bmatrix} R_0 & \hat{p}_0 R_0 \\ 0 & R_0 \end{bmatrix} \\ &= \begin{bmatrix} I & R_0^T \hat{p}_0 R_0 \\ -R_0^T \hat{p}_0 R_0 & I - R_0^T \hat{p}_0^2 R_0 \end{bmatrix}\end{aligned}$$

In general, $\|V_2^s\| \neq \|V_1^s\|$, or there exists no bi-invariant metric on $se(3)$.

□ Coordinate Transformation:

$$g_{ac}(t) = g_{ab}(t) \cdot g_{bc}(t)$$

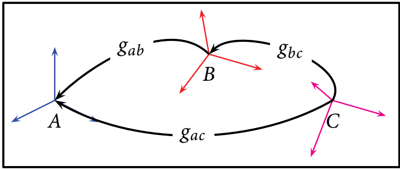

Figure 2.12

$$\begin{aligned}\hat{V}^s_{ac} &= \dot{g}_{ac} \cdot g_{ac}^{-1} \\ &= (\dot{g}_{ab} \cdot g_{bc} + g_{ab} \cdot \dot{g}_{bc})(g_{bc}^{-1} \cdot g_{ab}^{-1}) \\ &= \dot{g}_{ab} \cdot g_{ab}^{-1} + g_{ab} \cdot \dot{g}_{bc} \cdot g_{bc}^{-1} \cdot g_{ab}^{-1} = \hat{V}^s_{ab} + g_{ab} \hat{V}^s_{bc} g_{ab}^{-1}\end{aligned}$$

$$\Rightarrow V^s_{ac} = V^s_{ab} + Ad_{g_{ab}} V^s_{bc}$$

□ Coordinate Transformation:

$$g_{ac}(t) = g_{ab}(t) \cdot g_{bc}(t)$$

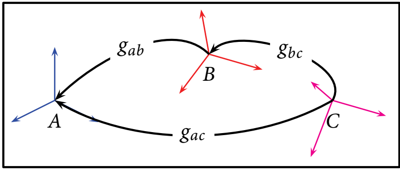

Figure 2.12

$$\begin{aligned}\hat{V}^s_{ac} &= \dot{g}_{ac} \cdot g_{ac}^{-1} \\ &= (\dot{g}_{ab} \cdot g_{bc} + g_{ab} \cdot \dot{g}_{bc})(g_{bc}^{-1} \cdot g_{ab}^{-1}) \\ &= \dot{g}_{ab} \cdot g_{ab}^{-1} + g_{ab} \cdot \dot{g}_{bc} \cdot g_{bc}^{-1} \cdot g_{ab}^{-1} = \hat{V}^s_{ab} + g_{ab} \hat{V}^s_{bc} g_{ab}^{-1}\end{aligned}$$

$$\Rightarrow V^s_{ac} = V^s_{ab} + Ad_{g_{ab}} V^s_{bc}$$

Similarly: $V^b_{ac} = Ad_{g_{bc}^{-1}} V^b_{ab} + V^b_{bc}$

Note: $V^s_{bc} = 0 \Rightarrow V^s_{ac} = V^s_{ab}, \; V^b_{ab} = 0 \Rightarrow V^b_{ac} = V^b_{bc}$

2.4 Velocity of a Rigid Body

◇ Example:

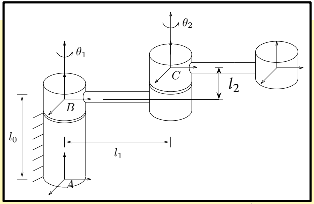

Figure 2.16

$$g_{ab}(\theta_1) = \begin{bmatrix} c_{\theta_1} & -s_{\theta_1} & 0 & 0 \\ s_{\theta_1} & c_{\theta_1} & 0 & 0 \\ 0 & 0 & 1 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$g_{bc}(\theta_2) = \begin{bmatrix} c_{\theta_2} & -s_{\theta_2} & 0 & 0 \\ s_{\theta_2} & c_{\theta_2} & 0 & l_1 \\ 0 & 0 & 1 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ V_{ab}^s = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \dot\theta_1, \ V_{bc}^s = \begin{bmatrix} l_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \dot\theta_2$$

$$V_{ac}^s = V_{ab}^s + Ad_{g_{ab}} \cdot V_{bc}^s = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \dot\theta_1 + \begin{bmatrix} l_1 c_{\theta_1} \\ l_1 s_{\theta_1} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \dot\theta_2$$

† End of Section †

2.5 Wrenches & Reciprocal Screws

□ Wrenches:

Let $F_c = \begin{bmatrix} f_c \\ \tau_c \end{bmatrix} \in \mathbb{R}^6$, $f_c, \tau_c \in \mathbb{R}^3$ be force or moment applied at the origin of C

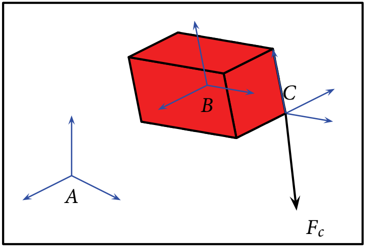

Figure 2.17

2.5 Wrenches & Reciprocal Screws

□ Wrenches:

Let $F_c = \begin{bmatrix} f_c \\ \tau_c \end{bmatrix} \in \mathbb{R}^6$, $f_c, \tau_c \in \mathbb{R}^3$ be force or moment applied at the origin of C

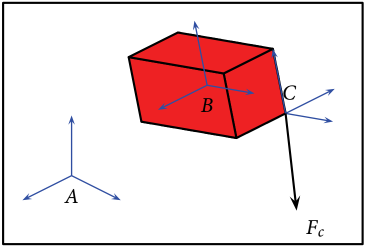

Figure 2.17

Generalized power: $\delta W = F_c \cdot V^b_{ac} = \langle f_c, v^b_{ac} \rangle + \langle \tau_c, \omega^b_{ac} \rangle$

2.5 Wrenches & Reciprocal Screws

Wrenches:

Let $F_c = \begin{bmatrix} f_c \\ \tau_c \end{bmatrix} \in \mathbb{R}^6$, $f_c, \tau_c \in \mathbb{R}^3$ be force or moment applied at the origin of C

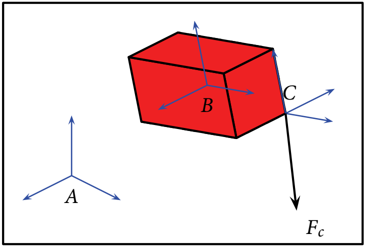

Figure 2.17

Generalized power: $\delta W = F_c \cdot V_{ac}^b = \langle f_c, v_{ac}^b \rangle + \langle \tau_c, \omega_{ac}^b \rangle$

Work: $W = \int_{t_1}^{t_2} V_{ac}^b \cdot F_c \mathrm{d}t$

$$V_{ab}^b \cdot F_b = (\mathrm{Ad}_{g_{bc}} \cdot V_{ac}^b)^T \cdot F_b$$
$$= (V_{ac}^b)^T \mathrm{Ad}_{g_{bc}}^T \cdot F_b = (V_{ac}^b)^T \cdot F_c, \forall V_{ac}^b$$
$$\Rightarrow F_c = \mathrm{Ad}_{g_{bc}}^T \cdot F_b$$

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2.5 Wrenches & Reciprocal Screws

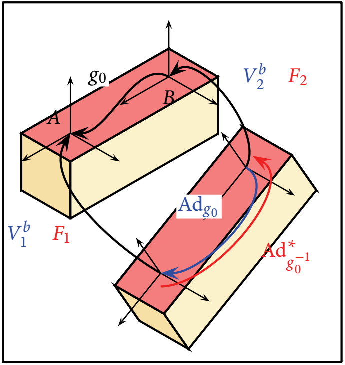

Figure 2.18

$$V_2^s = \mathrm{Ad}_{g_0^{-1}} \cdot V_1^s$$

$$\left(V_2^b = \mathrm{Ad}_{g_0^{-1}} \cdot V_1^b\right)$$

$$\Rightarrow V_1^b = \mathrm{Ad}_{g_0} \cdot V_2^b$$

$$F_2 = \mathrm{Ad}^*_{g_0} F_1$$

□ Screw coordinates for a wrench:

Generate a wrench associated with S:

- ($h \neq \infty$): force of mag. M along l, and torque of mag. hM about l.
- ($h = \infty$): pure torque of mag. M about l

$$F = \begin{cases} M \begin{bmatrix} \omega \\ -\omega \times q + h\omega \end{bmatrix} & h \neq \infty \\ M \begin{bmatrix} 0 \\ \omega \end{bmatrix} & h = \infty \end{cases}$$

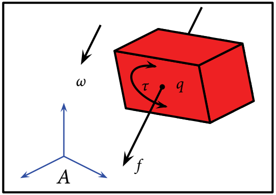

Figure 2.19

F: wrench along the screw S.

(Continues next slide)

2.5 Wrenches & Reciprocal Screws

Screw coordinates for a wrench (Continued):

1. Pitch:

$$h = \begin{cases} \dfrac{f^T\tau}{\|f\|^2} & \text{if } f \neq 0 \\ \infty & \text{if } f = 0 \end{cases}$$

2. Axis:

$$l = \begin{cases} \dfrac{f \times \tau}{\|f\|^2} + \lambda f, \lambda \in \mathbb{R} & \text{if } f \neq 0 \\ 0 + \lambda\tau, \lambda \in \mathbb{R} & \text{if } f = 0 \end{cases}$$

3. Magnitude:

$$M = \begin{cases} \|f\| & \text{if } f \neq 0 \\ \|\tau\| & \text{if } f = 0 \end{cases}$$

2.5 Wrenches & Reciprocal Screws

Theorem 3 (Poinsot):
Every collection of wrenches applied to a rigid body is equivalent to a force applied along a fixed axis plus a torque about the axis.

1777-1859

2.5 Wrenches & Reciprocal Screws

Theorem 3 (Poinsot):
Every collection of wrenches applied to a rigid body is equivalent to a force applied along a fixed axis plus a torque about the axis.

1777-1859

Multi-fingered grasp:

$$F_o = \sum_{i=1}^{k} \mathrm{Ad}^T_{g^{-1}_{oc_i}} \cdot F_{c_i}$$

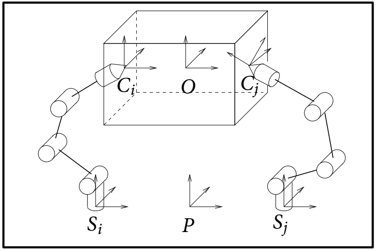

Figure 2.20

2.5 Wrenches & Reciprocal Screws

◻ Reciprocal screws:

$$V = \begin{bmatrix} v \\ \omega \end{bmatrix}, \ F = \begin{bmatrix} f \\ \tau \end{bmatrix}$$

$$F \cdot V = f^T \cdot v + \tau^T \cdot \omega$$

$$\downarrow \quad \downarrow$$

$$S_2 \quad S_1$$

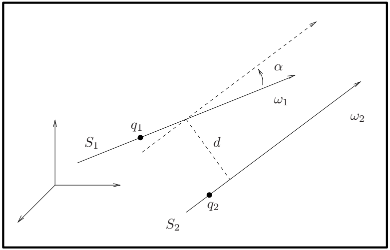

Figure 2.21

$$\alpha = \operatorname{atan2}((\omega_1 \times \omega_2) \cdot n, \omega_1 \cdot \omega_2)$$

$$S_1 \odot S_2 = M_1 M_2 ((h_1 + h_2) \cos\alpha - d \sin\alpha)$$

$$= 0 \text{ if reciprocal}$$

(continues next slide)

2.5 Wrenches & Reciprocal Screws

Given
$V = M_1 \begin{bmatrix} q_1 \times \omega_1 + h_1\omega_1 \\ \omega_1 \end{bmatrix}$, $F = M_2 \begin{bmatrix} \omega_2 \\ q_2 \times \omega_2 + h_2\omega_2 \end{bmatrix}$,
Let $q_2 = q_1 + dn$, then

$$\begin{aligned} V \cdot F &= M_1M_2(\omega_2 \cdot (q_1 \times \omega_1 + h_1\omega_1) + \omega_1 \cdot (q_2 \times \omega_2 + h_2\omega_2)) \\ &= M_1M_2(\omega_2 \cdot (q_1 \times \omega_1) + h_1\omega_1 \cdot \omega_2 \\ &+ \omega_1 \cdot ((q_1 + dn) \times \omega_2) + h_2\omega_1 \cdot \omega_2) \\ &= M_1M_2((h_1 + h_2)\cos\alpha - d\sin\alpha) \end{aligned}$$

◇ Example: Basic joints

- Revolute joint: $\xi = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix}$

$$\xi^{\perp} = \mathrm{span}\left\{ \begin{bmatrix} \omega_i \\ q \times \omega_i \end{bmatrix}, \begin{bmatrix} 0 \\ v_j \end{bmatrix} \middle| \begin{array}{l} \omega_i \in S^2, i = 1, 2, 3 \\ v_j \cdot \omega = 0, j = 1, 2 \end{array} \right\}: \text{5-system}$$

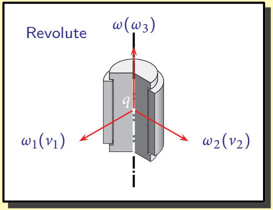

Figure 2.22

◇ Example: Basic joints

- Revolute joint: $\xi = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix}$

$$\xi^\perp = \text{span}\left\{ \begin{bmatrix} \omega_i \\ q \times \omega_i \end{bmatrix}, \begin{bmatrix} 0 \\ v_j \end{bmatrix} \middle| \begin{array}{l} \omega_i \in S^2, i = 1,2,3 \\ v_j \cdot \omega = 0, j = 1,2 \end{array} \right\}: \text{5-system}$$

- Prismatic joint: $\xi = \begin{bmatrix} v \\ 0 \end{bmatrix}$

$$\xi^\perp = \text{span}\left\{ \begin{bmatrix} \omega_i \\ q \times \omega_i \end{bmatrix}, \begin{bmatrix} 0 \\ v_j \end{bmatrix} \middle| \begin{array}{l} \omega_i \cdot v = 0, i = 1,2 \\ v_j \in S^2, j = 1,2,3 \end{array} \right\}: \text{5-system}$$

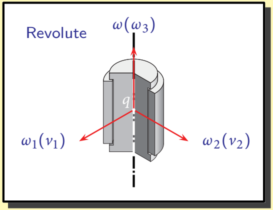

Figure 2.22

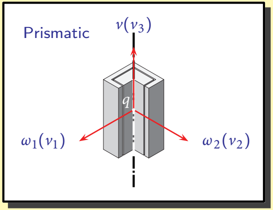

Figure 2.23

◇ Example: Basic joints (continued)

- Spherical joint: $\xi = \operatorname{span}\left\{\begin{bmatrix} -\omega_i \times q \\ \omega_i \end{bmatrix} \middle| \omega_i \in S^2, i = 1,2,3\right\}$

$\xi^\perp = \operatorname{span}\left\{\begin{bmatrix} \omega_i \\ q \times \omega_i \end{bmatrix} \middle| \omega_i \in S^2, i = 1,2,3\right\}$: 3-system

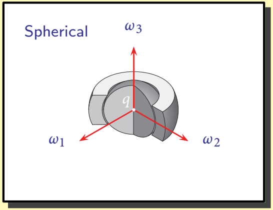

Figure 2.24

⋄ Example: Basic joints (continued)

- Spherical joint: $\xi = \text{span}\left\{\begin{bmatrix} -\omega_i \times q \\ \omega_i \end{bmatrix} \middle| \omega_i \in S^2, i = 1,2,3\right\}$

$\xi^\perp = \text{span}\left\{\begin{bmatrix} \omega_i \\ q \times \omega_i \end{bmatrix} \middle| \omega_i \in S^2, i = 1,2,3\right\}$: 3-system

- Universal joint: $\xi = \text{span}\left\{\begin{bmatrix} q \times x \\ x \end{bmatrix}, \begin{bmatrix} q \times y \\ y \end{bmatrix}\right\}$

$\xi^\perp = \text{span}\left\{\begin{bmatrix} \omega_i \\ q \times \omega_i \end{bmatrix}, \begin{bmatrix} 0 \\ z \end{bmatrix} \middle| \omega_i \in S^2, i = 1,2,3\right\}$: 4-system

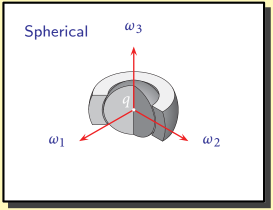

Figure 2.24

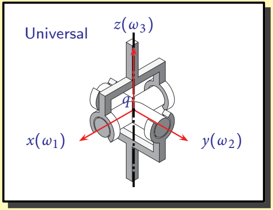

Figure 2.25

◇ Example: Kinematic chains

- Universal-Spherical Dyad:

$$\xi = \text{span}\left\{\begin{bmatrix} q_1 \times x \\ x \end{bmatrix}, \begin{bmatrix} q_1 \times y \\ y \end{bmatrix} \begin{bmatrix} q_2 \times \omega_i \\ \omega_i \end{bmatrix} \middle| \omega_i \in S^2, i = 1, 2, 3\right\}$$

$$\xi^\perp = \text{span}\left\{\begin{bmatrix} v \\ q_1 \times v \end{bmatrix} \middle| v = \frac{q_2 - q_1}{\|q_2 - q_1\|}\right\}$$

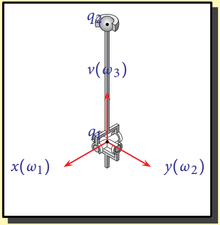

Figure 2.26

◇ Example: Kinematic chains

- Universal-Spherical Dyad:

$$\xi = \text{span}\left\{ \begin{bmatrix} q_1 \times x \\ x \end{bmatrix}, \begin{bmatrix} q_1 \times y \\ y \end{bmatrix} \begin{bmatrix} q_2 \times \omega_i \\ \omega_i \end{bmatrix} \middle| \omega_i \in S^2, i = 1, 2, 3 \right\}$$

$$\xi^{\perp} = \text{span}\left\{ \begin{bmatrix} v \\ q_1 \times v \end{bmatrix} \middle| v = \frac{q_2 - q_1}{\|q_2 - q_1\|} \right\}$$

- Revolute-Spherical Dyad:

zero pitch screws passing through the center of the sphere, lie on a plane containing the axis of the revolute joint: 2-system

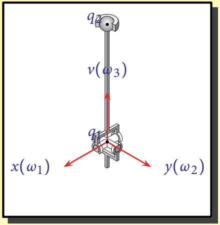

Figure 2.26

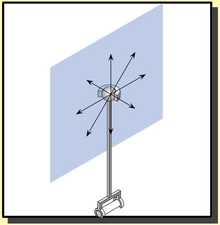

Figure 2.27

† End of Section †

2.6 References

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