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Math 3012 Combinatorics

1. Closed form solution is $r(n) = 2^n - 1$, induction is:

   (a) Base case: $r(0) = 0$ and $2^0 - 1 = 0$

   (b) Assuming $n = k$ holds, then $k + 1$ can:
   $r(n + 1) = 2r(n) + 1$
   $r(n + 1) = 2(2^n - 1) + 1$
   $r(n + 1) = 2^{n+1} - 2 + 1 = 2^{n+1} - 1$ QED

2. (a) Base case ($n = 1$): 11 - 7 = 4, which is divisible by 4

   (b) Assuming $n = k$ holds, then $k + 1$ can:
   $11^{k+1} - 7^{k+1} = (7+4)*11^k - 7*7^k = 7*11^k + 4*11^k - 7*7^k = 4*11^k + 7(11^k - 7^k)$.
   If it is assumed that $11^n - 7^n$ is a multiple of four, then it follows that $4 * 11^n + 7(11^n - 7^n)$, the case for (n+1) is also a multiple of four (due to the coefficient on 11^n), and thus all positive n results in a multiple of four. QED

3. Euclidean algorithm can: (formula I'm using: $a \% b = a - b * floor(\frac{a}{b})$)

   - $97461 \% 5390 = 441$
   - $5390 \% 441 = 98$
   - $441 \% 98 = 49$
   - $98 \% 49 = 0$

   Thus, gcd(5390,97461) = 49. and by back-tracking:

   - $49 = 441 - 4 * 98$ and $98 = 5390 - 12 * 441$
   - $49 = 441 - 4(5390 - 12 * 441) = -4 * 5390 + 49 * 441$ and $441 = 97461 - 18 * 5390$
   - $-4 * 5390 + 49(97461 - 18 * 5390) = 49 * 97461 - 886 * 5390$

   Thus, a = -886 and b = 49.

4. $g(n) = 3g(n - 1) + 4g(n - 2)$, base cases $g(1) = 4, g(0) = 1$. This is because: in order for all shapes to have a chance of being inserted into the checkerboard, n must be two. Thus two base cases are required; when n is 1 and when n is 0. For the case $n = 0$ there is only 1 way to arrange nothing, and for the case $n = 1$ only two of the pieces (the 1x1 pieces) can fit inside, and there are $2^2 = 4$ ways to arrange that. For the rest of the cases, there are four possible shapes that could fit into the square, and those consider the possibilities of the previous cases, up to $n = 0$. The 3 coefficient ensures that if a black piece is chosen for one n, then it is not chosen for the next n, and so a choice between the remaining three is made. Thus, $g(n) = 3g(n - 1) + 4g(n - 2)$.

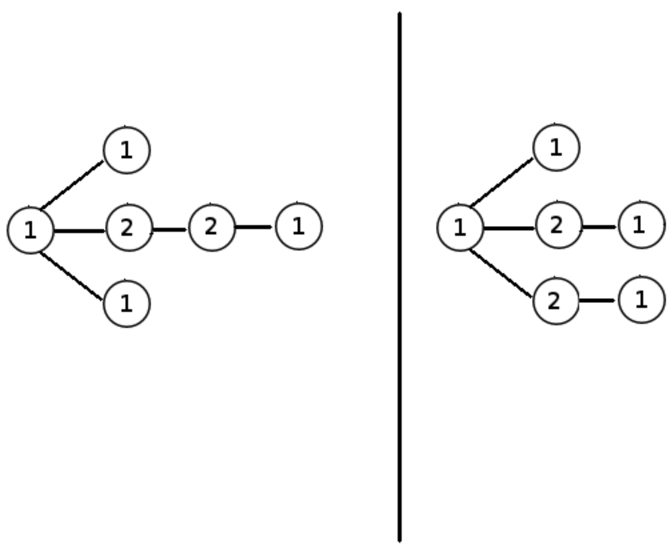
5. The multinomial theorem states that: $(x_1+x_2+x_3)^n = \sum_{k_1+k_2+k_3=n} \binom{n}{k_1, k_2, k_3} x_1^{k_1} x_2^{k_2} x_3^{k_3}$, and so $(2x + 3y^2 + z)^{100}$ can be plugged in. For the term $x^{15}y^{120}z^{25}$, the exponents should sum to 100, which is n, and because y is squared, then if y^2 were to be replaced by another character, say α, then the exponent would only be 60 due to the square multiplying the exponent by 2. Thus, the answer is $\binom{100}{15,60,25}2^{15}3^{60}1^{25}$ which, if multiplied through, becomes $\frac{100!}{15!60!25!}2^{15}3^{60} \approx 7.6808403 \times 10^{71}$.

6. Using the pigeonhole principle, if the 10x10 square was to be divided into 100 smaller, equal-area squares, at least four dots should "go into" one square; that is, occupy the four boundaries of one square. This is guaranteed because 301 is the least number required to guarantee 4 points in a square (3/100 is 3 per square, plus one for a random square, in which the points are at most $\sqrt{2}$ of each other).

7. 2^{4950}; every set with p elements has 2^p different subsets. The number of vertices a graph has is determined by its edges, and because every edge requires two vertices, this becomes the subset $\binom{100}{2}$. Because the number of graphs is a subset of the total number of possibilites with that vertex set, then the answer is $2^{\binom{100}{2}}$.

8. 34; this is because this many unlabelled graphs (graphs that are unlabelled besides how they are connected) could be drawn from five vertices.

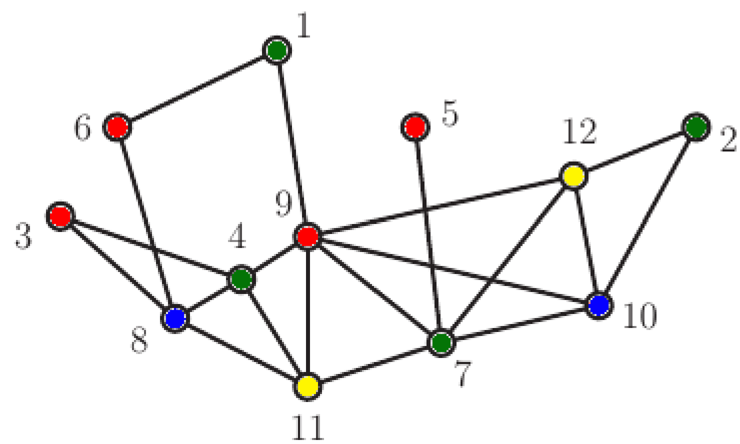
9. There are 47. This includes the null graph (1), the number of graphs that could be formed from vertices without edges (31), the number of graphs that could be formed from a single edge (4), the number of graphs that could be formed from 2 edges (6), and the number of graphs that could be formed from 3 edges (4), and the original graph itself (1), as every graph is its own subset.

10. 

11. Starting from one and proceeding to the lowest level vertex, then:
1, 5, 6, 8, 3, 4, 8, 11, 4, 9, 7, 11, 9, 10, 2, 12, 7, 10, 12, 9, 6, 1.

12. Yes, the graph is Hamiltonian. This is because although many edges join at vertices throughout the graph, a Hamiltonian cycle can be formed between each of the points that only visits each vertex once.

13. (a) A single edge suffices, such as the edge from point 11 to 7.
    (b) Where three vertices exist adjacent to each other, such as the triangle formed by points 11, 7 and 9.
    (c) The triangle containing the vertices 8, 4, 9 and 11.
    (d) The path created by directly linking the vertices 3, 8, 11, 7 and 10.
    (e) If 1 is green, 2 is blue, 3 is red and 4 is yellow, then:



14. Euler's formula states: *Let G be a connected planar graph, and let n, m and f denote, respectively, the numbers of vertices, edges, and faces in a plane drawing of G. Then n - m + f = 2*[1]. In that graph, there are a total of 10 vertices, 13 edges connecting them, one infinite face and 4 bounded faces. According to Euler's formula, 10 - 13 + 1 + 4 should equal 2, and in this case it does.

Bibliography

1. http://www.personal.kent.edu/ rmuhamma/GraphTheory/MyGraphTheory/planarity.htm