6/27/18 Lecture Notes: Diffusion with a Source

Background: Derivatives of Integrals

1) Fundamental Theorem of Calculus:

Chain Rule

$$\frac{d}{dx}\int_0^x f(t)\,dt = f(x) \qquad \frac{d}{dx}\int_0^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x)$$

2) Pass Derivative Inside Integral:

$$\frac{d}{dt}\int_a^b f(x,t)\,dx = \lim_{h\to 0}\int_a^b \underbrace{\frac{f(x,t+h)-f(x,t)}{h}}\,dx = \int_a^b f_t(x,t)\,dx$$

$\rightarrow f_t(x,t)$ for all x

$\left|\ \text{"}\ \right| \le \max_{[a,b]} f_t(x,t)$

- Used assumption that f ctsly. differentiable in t

2') Works on $(-\infty,\infty)$ if $\int_{-\infty}^{\infty} |f_t(x,t)|\,dx \le M < \infty$ for all t

3) $\displaystyle\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = ?$ Let $g(t,a,b) = \int_a^b f(x,t)\,dt$

Chain Rule: $\frac{d}{dt} g(t, a(t), b(t))$

$$= g_t + g_a\cdot a'(t) + g_b\, b'(t)$$

$$= \int_{a(t)}^{b(t)} f_t(x,t)\,dx + f(b(t),t)\,b'(t) - f(a(t),t)\,a'(t)$$

Example $f(x,t) = 3x^2t^2 \quad a=0,\ b=1$

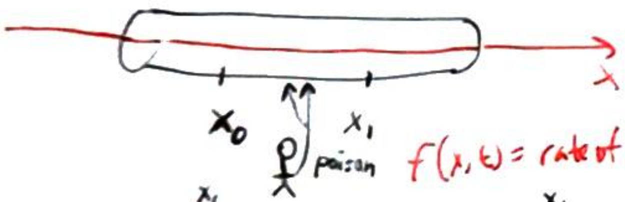
$$\frac{d}{dt}\left[\int_0^1 3x^2t^2\,dx\right] \qquad \int_0^1 \frac{d}{dt}3x^2t^2\,dx = \int_0^1 6x^2t\,dx$$

$$= \frac{d}{dt}\left[\,t^2x^3\Big|_0^1\right] \qquad = t\cdot 2x^3\Big|_0^1 = 2t$$

$$= d/dt\,(t^2) = 2t$$

Diffusion with a Source

$u(x,t)$ = concentration of poison at (x,t) in a pipe



$f(x,t)$ = rate of adding poison at (x,t)

$$M(t) = \int_{x_0}^{x_1} u(x,t)\,dx \qquad \frac{dM}{dt} = \int_{x_0}^{x_1} u_t(x,t)\,dx = ku_x(x_1,t) - ku_x(x_0,t) + \int_{x_0}^{x_1} f(x,t)\,dx$$

$\downarrow \partial/\partial x_1$

$$u_t = ku_{xx} + f(x,t)$$

$$\boxed{u_t - ku_{xx} = f(x,t)}$$

<u>ODE Analogy</u>

Solve $\quad y' + Ay = f(t) \quad y(0) = y_0$

Integrating factor $e^{tA} \rightarrow \quad e^{tA}y' + Ae^{tA}y = e^{tA}f(t)$

$$(ye^{tA})' = e^{tA}f(t)$$

$$ye^{tA} = \int_0^t e^{sA}f(s)\,ds + C$$

Duhamel's Principle $\quad \boxed{y = \int_0^t e^{(t-s)(-A)}f(s)\,ds + y_0 e^{-tA}}$

Write as $\quad S(t)\phi + \int_0^t S(t-s)f(s)\,ds$

$$S(t) = e^{-tA}$$

Solution $\quad u_t - k u_{xx} = f(x,t) \quad -\infty < x < \infty,\ t > 0 \quad u(x,0) = \phi(x)$

$$u(x,t) = \underbrace{\int_{-\infty}^{\infty} S(x-y,t)\,\phi(y)\,dy}_{\text{homogeneous solution}} + \underbrace{\int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)\,f(y,s)\,dy\,ds}_{\text{inhomogeneous solution}}$$

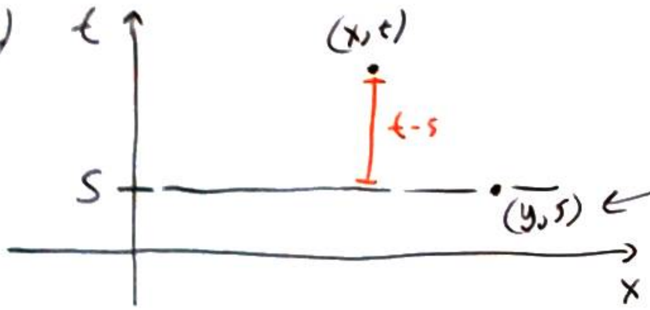
Explanation: 1) Analogy with ODE

Start with $\int_{-\infty}^{\infty} S(x-y,t)\,\phi(y)\,dy$

Replace $\quad t \to t-s \qquad \phi(y) \to f(y,s)$

Integrate $\int_0^t \qquad\qquad ds$

2)



new input of concentration/heat at time s

How much is added? Think of $f(y,s)$ as initial conditions at time s

$$\int_{-\infty}^{\infty} S(x-y,\underbrace{t-s}_{\text{treat time } s \text{ as new time } 0})\,f(y,s)\,dy.$$ This is a rate, so integrate over t.

3) Proof

- Can assume $\phi \equiv 0$ by linearity

First, check $u_t - ku_{xx} = f(x,t)$

$$u_t = \frac{\partial}{\partial_t} \int_0^t \int_{-\infty}^{\infty} S(x-y, t-s) f(y,s)\, dy\, ds$$

$$= \int_0^t \int_{-\infty}^{\infty} \frac{\partial S}{\partial t}(x-y, t-s) f(y,s)\, dy\, ds + \lim_{s \to t} \int_{-\infty}^{\infty} S(x-y, t-s)\, f(y,s)\, dy$$

$\downarrow$ S is solution $\qquad \downarrow$ sub $\epsilon = t-s$, $f(y,s) \to f(y,t)$

$$= \int_0^t \int_{-\infty}^{\infty} k \frac{\partial^2 S}{\partial x^2}(x-y, t-s) f(y,s)\, dy\, ds \quad \underbrace{\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} S(x-y, \epsilon) f(y,t)\, dy}$$

$v(x,t)$ Solution to

$= ku_{xx}$

$v_t = kv_{xx}$, $v(x,0) = f(x,t)$

$$= ku_{xx} + f(x,t)$$

Next, check IC $u(x,0) = \phi(x)$, so

$$\int_0^{t=0} \int_{-\infty}^{\infty} S(x-y, \overset{0}{t-s}) f(y,s)\, dy\, ds = 0 \qquad \int_0^0 \text{anything} = 0 \qquad \blacksquare$$

Lie corrections: 1) Can't pass derivative through integral if $\partial S/\partial t$ unbounded, according to our rule

Fix: Come up with new rule (just need $\int |\partial S/\partial t| < \infty$)

2) Is $\lim_{t \to 0} \int_0^t \text{anything} = 0$?

Fix: Yes, provided integral finite.

Source on a Half-Line

Solve $v_t - kv_{xx} = f(x,t) \quad 0 < x < \infty \quad t > 0$

$v(0,t) = h(t)$

$v(x,0) = \phi(x)$

Dealing with nonhomogeneous BC:

$$V(x,t) = v(x,t) - h(t)$$

$$V_t - kV_{xx} = f(x,t) - h'(t) \quad 0 < x < \infty \quad t > 0$$

$$V(0,t) = 0$$

$$V(x,0) = \phi(x) - h(0)$$

Deal with $f(x,t) - h'(t)$? See homework

Hint: Half-line uses reflection method

How to extend $f(x,t)$?

Movie time!

Next time: Waves with a source

$$u_{tt} - c^2 u_{xx} = f(x,t) \quad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = \phi(x)$$

$$u_t(x,0) = \psi(x)$$

Thinking of $f(x,t)$ as new/added initial conditions at time t,

$u(x,t)$ influenced by f in $\triangle$.