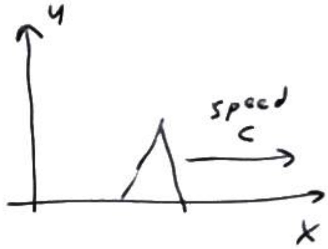
6/19/18: The Wave Equation

Before Class: HW 0, Piazza, Scanning HW, Office Hours

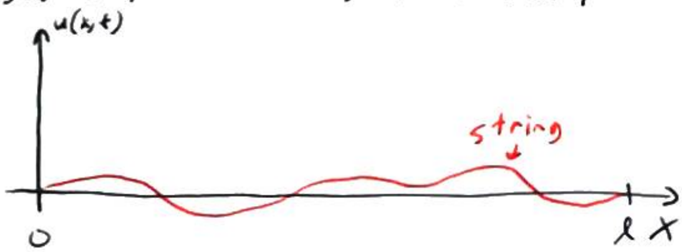
Last Time: Transport Equation $u_t + c u_x = 0$

- derived from physical law
- Initial conditions $u(x,0) = f(x)$
- Solution $u(x,t) = f(x-ct)$



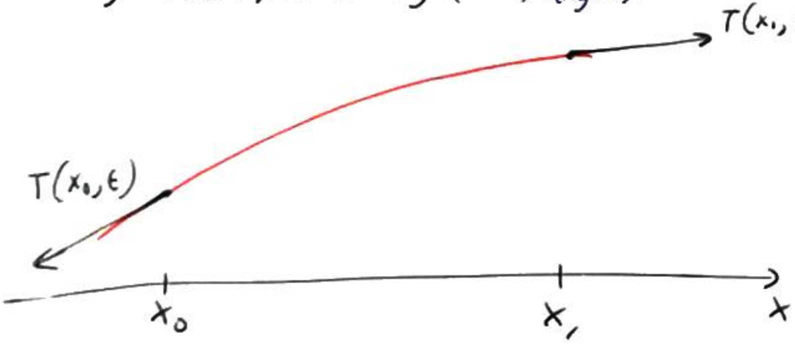
Wave Equation in 1D

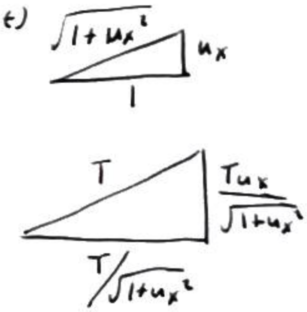
$u(x,t)$ = displacement from equilibrium at position x and time t.



Let $T(x,t)$ = magnitude of tension in string (tangential force)

ρ = density of string (mass/length)





Newton's 3rd Law: $\vec{F} = m\vec{a}$

Simplification: $\sqrt{1+u_x^2} = 1 + \frac{1}{2}\cancel{u_x^2} + \dots$

Horizontal forces: $\left.\frac{T}{\sqrt{1+u_x^2}}\right|_{x_0}^{x_1} = 0 \longrightarrow T$ constant

Vertical forces: $\left.\frac{Tu_x}{\sqrt{1+u_x^2}}\right|_{x_0}^{x_1} = \int_{x_0}^{x_1} \rho u_{tt}\, dx \xrightarrow{\frac{d}{dx_1}}$ $\boxed{\begin{array}{l} Tu_{xx} = \rho u_{tt} \\ u_{tt} = c^2 u_{xx} \quad \left(c = \sqrt{\frac{T}{\rho}}\right) \end{array}}$

(Ignore gravity)

Solving the Wave Equation

$u_{tt} = c^2 u_{xx} \quad -\infty < x < \infty$ (Always specify domain. This can drastically change the solution.)

Idea: $u_{tt} - c^2 u_{xx} = (\partial_t^2 - c^2\partial_x^2)u$
$= (\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0$ $\quad (\partial_t = \partial/\partial t,\ \partial_x = \partial/\partial x)$

Solutions to wave equation include solutions to

$(\partial_t - c\partial_x)u = 0$	$(\partial_t + c\partial_x)u = 0$
$u_t - cu_x = 0$	$u_t + cu_x = 0$
$f(x+ct)$	$g(x-ct)$

Should have $u(x,t) = f(x+ct) + g(x-ct)$ Is this all solutions? Have we addressed this?

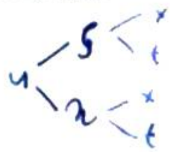
More Rigorous Solution (Motivated by Change of Coordinates method)

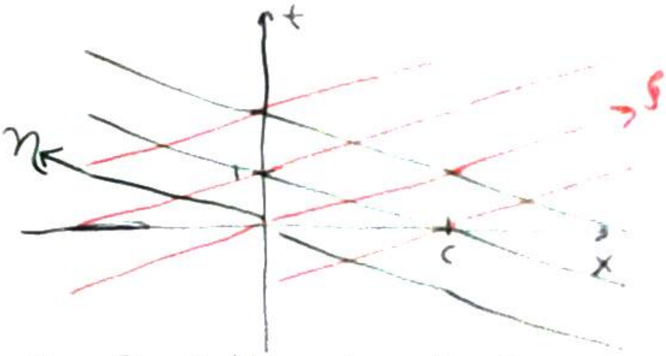
Let $\xi = x + ct,$

$\eta = x - ct$

$\partial_x = \partial_\xi + \partial_\eta$

$\partial_t = c\partial_\xi - c\partial_\eta$





$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = (c\partial_\xi - c\partial_\eta - c\partial_\xi - c\partial_\eta)(c\partial_\xi - c\partial_\eta + c\partial_\xi + c\partial_\eta)u$
$= (-2c\partial_\eta)(2c\partial_\xi)u = 0$

$u_{\xi\eta} = 0$

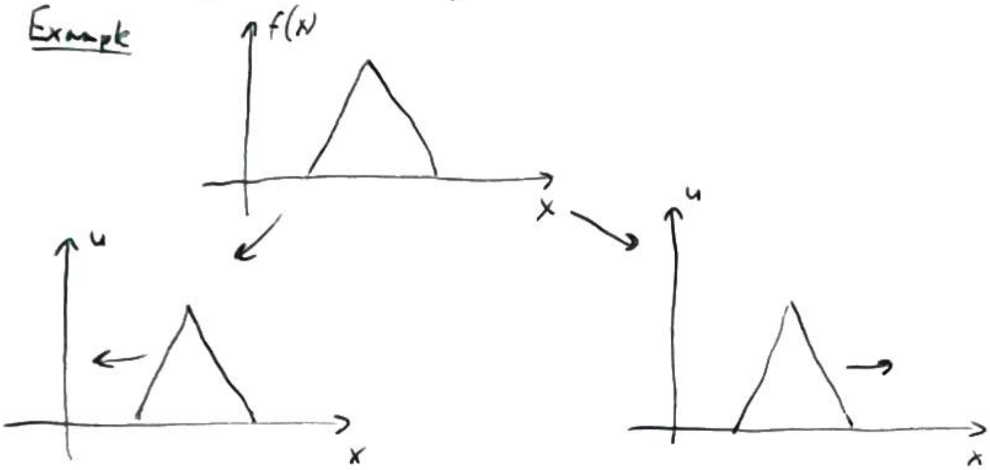
$u = f(\xi) + g(\eta)$

$$\boxed{u(x,t) = f(x+ct) + g(x-ct)}$$

Initial Value Problem

Is $u(x,0) = f(x)$ a good initial condition?

Example



$u(x,t) = f(x+ct)$ $\qquad$ $u(x,t) = f(x-ct)$

Solve: $u_{tt} = c^2 u_{xx} \quad -\infty < x < \infty$

$u(x,0) = \phi(x) \quad u_t(x,0) = \psi(x)$ (Think mass-spring system at each point x. Kind of.)

$t=0$ in solution $\longrightarrow \phi(x) = f(x) + g(x)$, $\boxed{\phi' = f' + g'}$

$u_t = c f'(x+ct) - c g'(x-ct)$

$u_t(x,0) = c f'(x) - c g'(x) \longrightarrow \boxed{\frac{1}{c}\psi = f' - g'}$

Linear Algebra $\longrightarrow$ $f' = \frac{1}{2}\left(\phi' + \psi/c\right) \longrightarrow f(s) = \frac{1}{2}\phi(s) + \frac{1}{2c}\int_0^s \psi + A$

$g' = \frac{1}{2}\left(\phi' - \psi/c\right) \longrightarrow g(s) = \frac{1}{2}\phi(s) - \frac{1}{2c}\int_0^s \psi + B$

[Note: $f(s) + g(s) = \phi(s)$, so $A + B = 0$]

$$u(x,t) = f(x+ct) + g(x-ct)$$
$$= \frac{1}{2}\phi(x+ct) + \frac{1}{2c}\int_0^{x+ct}\psi + \frac{1}{2}\phi(x-ct) - \frac{1}{2c}\int_0^{x-ct}\psi$$

$$\boxed{u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi}$$ d'Alembert's formula

Verify that d'Alembert's formula gives a solution by differentiating it and plugging into the wave equation