8/2/18 Lecture Notes

Correction: $\sigma_k^2 = \int_{-\infty}^{\infty} k^2 |F(k)|^2 \frac{dk}{2\pi}$, since $1 = \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(k)|^2 \frac{dk}{2\pi}$

Last time: Function/distribution $f(x)$ has Fourier transform

$$\hat{f}(k) = F(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx, \text{ and}$$

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{ikx} dk/2\pi$$

Rules: $\frac{\partial f}{\partial x} \longrightarrow ikF(k)$

$xf(x) \longrightarrow i\frac{\partial F}{\partial k}$

Solving PDE: a) $S_t = S_{xx}$ $\quad -\infty < x < \infty$ $\quad$ ($S(x,t)$ is source function)

b) $S(x,0) = \delta(x)$

Take F.T. of above in x (NOT t).

$$\hat{S}(k,t) = \int_{-\infty}^{\infty} S(x,t) e^{-ikx} dx$$

a) $\rightarrow$ $\frac{\partial \hat{S}}{\partial t} = (ik)^2 \hat{S} = -k^2 \hat{S}$

b) $\rightarrow$ $\hat{S}(k,0) = 1$

This is an ODE in t for each k, so

$\hat{S}(k,t) = e^{-k^2 t}$ What has this Fourier transform?

$e^{-x^2/2} \longrightarrow \sqrt{2\pi}\, e^{-k^2/2}$ $\quad$ Want $t = \frac{1}{2a^2} \rightarrow a = \frac{1}{\sqrt{2t}}$

$f(ax) \longrightarrow \frac{1}{|a|} F\left(\frac{k}{a}\right)$

$a > 0$ $\quad e^{-a^2x^2/2} \rightarrow \frac{1}{a}\sqrt{2\pi}\, e^{-k^2/2a^2}$

$$S(x,t) = \frac{a^{-1}}{\sqrt{2\pi}} e^{-a^2x^2/2} = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$$

* Not in book

Now, let $u_t = u_{xx}$ $\quad -\infty < x < \infty$

$u(x,0) = \phi(x)$

Fourier Transform $\rightarrow$ $\begin{cases} \hat{u}_t = -k^2 \hat{u} \\ \hat{u}(k,0) = \hat{\phi}(k) \end{cases}$ ODE in t for each k

$G(k,t) = \hat{\phi}(k)\, e^{-k^2 t}$. Recall $\widehat{f * g}(x) = \hat{f}(k)\, \hat{g}(k)$, so

$$u(x,t) = \phi * \text{Inverse FT}\left(e^{-k^2 t}\right)$$

$$= \phi * S$$

Convolution in x

$$= S * \phi$$

$$= \int_{-\infty}^{\infty} S(x-y,t)\, \phi(y)\, dy$$

Solutions with IC = $\delta(x)$ help find all solutions

Wave Equation: $S_{tt} = c^2 S_{xx}$

$S(x,0) = 0, \quad S_t(x,0) = \delta(x)$

F.T. $\rightarrow$ $\left.\begin{array}{l} \hat{S}_{tt} = -c^2 k^2 \hat{S} \\ \hat{S}(k,0) = 0 \\ \hat{S}_t(k,0) = 1 \end{array}\right\} \xrightarrow[\text{ODE}]{\text{Solve}} \hat{S}(k,t) = \frac{1}{kc} \sin kct$

Solutions are sines and cosines

What has this Fourier transform?

$H(a - |x|) \rightarrow \frac{2}{k} \sin ak \xrightarrow{a = ct} H(ct - |x|) \rightarrow \frac{2}{k} \sin kct$ } linearity

$\frac{1}{2c} H(c^2 t^2 - x^2) \rightarrow \frac{1}{kc} \sin kct$

(Can also do in 3D)

(See book)

$$S(x,t) = \frac{1}{2c} H(c^2 t^2 - x^2)$$

Can we do the entire class in one hour?

Laplace's Equation: $u_{xx} + u_{yy} = 0 \quad y > 0$

$u(x,0) = \delta(x)$

F.T. in x $\rightarrow$ $\hat{u}(k,y) = \int_{-\infty}^{\infty} e^{-ikx} u(x,y)\, dx$, and

$\left.\begin{array}{l} -k^2 \hat{u} + \hat{u}_{yy} = 0 \\ \hat{u}(k,0) = 1 \end{array}\right\} \xrightarrow[\text{in } y]{\text{Solve ODE}} \hat{u}(k,y) = c_1 e^{yk} + c_2 e^{-yk}$

What Fourier transforms are these?

Issue with decay?

Choose solution with decay when $y > 0$. This depends on k.

$$\hat{u}(k,y) = e^{-y|k|} \qquad \text{F.T of what?}$$

$$e^{-y|k|} \xrightarrow{(*)} \frac{2y}{y^2+k^2} \quad (y>0)$$

$$? \longrightarrow e^{-y|k|}$$

We'll compute inverse Fourier transform, though computation is much like (*)

$$u(x,y) = \int_{-\infty}^{\infty} e^{ikx} e^{-y|k|} \frac{dk}{2\pi}$$

$$= \int_0^{\infty} e^{k(ix-y)} \frac{dk}{2\pi} + \int_{-\infty}^{0} e^{k(ix+y)} \frac{dk}{2\pi}$$

$$= \frac{1}{2\pi(ix-y)} e^{ikx-ky}\Big|_{k=0}^{\infty} + \frac{1}{2\pi(ix+y)} e^{ikx+ky}\Big|_{k=-\infty}^{0}$$

$$u(x,y) = \frac{1}{2\pi}\left(\frac{1}{y-ix} + \frac{1}{y+ix}\right) = \frac{y}{\pi(x^2+y^2)}$$

* not in book exactly

Solve $u_{xx} + u_{yy} = 0 \quad y > 0$

$u(x,0) = \phi(x)$

$$\text{F.T} \longrightarrow \begin{array}{l} -k^2\hat{u} + \hat{u}_{yy} = 0 \\ \hat{u}(k,0) = \hat{\phi}(k) \end{array} \xrightarrow[\text{for each } k]{\text{Solve ODE}} \hat{u}(k,y) = \hat{\phi}(k) e^{-y|k|}$$

$$u(x,y) = \frac{y}{\pi(x^2+y^2)} *_x \phi$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-x_0)^2+y^2}\, \phi(x_0)\, dx_0$$

Discuss studying for final