7/16/18 Lecture Notes: Laplacian on Rectangles + Poisson's formula

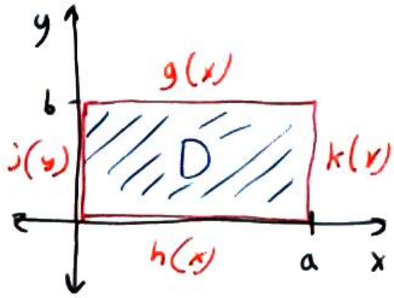
Last time: $\Delta u = u_{xx} + u_{yy}$ $(+u_{zz})$

Properties: Translational invariance
Rotational invariance
Maximum Principle
$\Rightarrow$ uniqueness of solutions to Dirichlet problem
Radial solutions important (next time)

Goal: solve $\Delta u = 0$ on $D = [0,a] \times [0,b]$

may also specify u_x, u_y (or even $u + u_x$)

$$\begin{cases} u(x,b) = g(x) \\ u(x,0) = h(x) \\ u(0,y) = j(y) \\ u(a,y) = k(y) \end{cases}$$



Remarks: 1) No time variable $\rightarrow$ No initial conditions, just boundary

2) If $u \equiv 0$ on ∂D, then $u \equiv 0$ in D. Must use inhomogeneous BC, BUT we'll put this off until the last step.

3) By linearity, can solve w/ 4 different BC, each with just 1 inhomogeneous BC:

E.g. $\left.\begin{array}{l} u(x,b) = g(x) \\ u(x,0) = u(0,y) = u(a,y) = 0 \end{array}\right\}$ We'll solve this one

The lesson of being John Wayne at the first Thanksgiving
(using the same good idea everywhere)

Step 1:

Separation of Variables: $u(x,y) = X(x)\,Y(y)$

$$\Delta u = X''Y + XY'' = 0$$

$$\frac{X''}{X} = \frac{-Y''}{Y} = -\lambda \rightarrow \begin{array}{l} X'' + \lambda X = 0 \\ Y'' - \lambda Y = 0 \end{array}$$

may flip which is $\pm$ based on next step

Step 2: Solve the ODE with <u>Homogeneous</u> BC's

$$X'' + \lambda X = 0$$

$$u(0,y) = u(a,y) = 0 \longrightarrow X(a) = X(0) = 0$$

⋮ lots of work, but we've done it before

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2 \quad X_n(x) = \sin\frac{n\pi}{a}x \quad n = 1, 2, 3, \ldots$$

Step 3: Solve other ODE, using <u>just</u> the <u>one</u> homogeneous <u>BC</u>

NEW

$$Y'' - \lambda Y = 0$$

$$Y(y) = Ae^{\frac{n\pi}{a}y} + Be^{-\frac{n\pi}{a}y}$$

$$\sinh t = \frac{e^t - e^{-t}}{2} \quad \cosh t = \frac{e^t + e^{-t}}{2}$$

Hyperbolic Trig Functions

or alternatively,

$$Y(y) = A\cosh\frac{n\pi}{a}y + B\sinh\frac{n\pi}{a}y$$

from $u(x,0) = 0$

$$0 = Y(0) = A + 0$$

$$Y_n(y) = \sinh\frac{n\pi}{a}y$$

Notable facts: $\sinh 0 = 0$

$\frac{d}{dt}\cosh t = \sinh t$, so

$\frac{d}{dt}\cosh t\Big|_{t=0} = 0$

Step 4: Sum the series and match up with inhomogeneous BC

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{a}x \cdot \sinh\frac{n\pi}{a}y$$

To find A_n:

$$u(x,b) = g(x) = \sum_{n=1}^{\infty} \underbrace{A_n \sinh\frac{n\pi}{a}b}_{\text{coefficients in expansion}} \cdot \sin\frac{n\pi}{a}x$$

$$A_n \underbrace{\sin\frac{n\pi}{a}b} = \frac{2}{a}\int_0^a g(x)\sin\frac{n\pi}{a}x\,dx$$

divide to find A_n

Fun Q: Why never divide by 0?

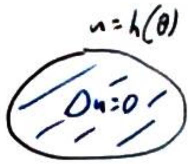
Closing Remarks: 1) Remember to piece back together 4 solutions

2) Watch out for slight differences on different sides of D.

Poisson's Formula: Solving Δu on circular domain

Goal: Solve Dirichlet problem for the circle:

$$\Delta u = 0 \quad \text{in} \quad x^2+y^2<a^2$$
$$u = h(\theta) \quad \text{on} \quad x^2+y^2=a^2$$



Separation of Variables:

$$u(r,\theta) = R(r)\,T(\theta)$$

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

$$R''T + \frac{1}{r}R'T + \frac{1}{r^2}T''R = 0$$

$$\frac{1}{r^2}T''R = -\left(R'' + \frac{1}{r}R'\right)T$$

$$\frac{-T''}{T} = \frac{r^2R''+rR'}{R} = \lambda$$

$$T''+\lambda T = 0$$

$$\left.\begin{aligned} T(0)&=T(2\pi)\\ T'(0)&=T'(2\pi)\end{aligned}\right\} \text{ since } T(\theta) \text{ } 2\pi\text{-periodic}$$

$$T(\theta) = A\cos n\theta + B\sin n\theta$$

$$\lambda = n^2$$

$$R''r^2 + rR' - \lambda R = 0$$

linear, variable coefficient

Guess $R(r) = r^\alpha$

$$\alpha(\alpha-1)r^\alpha + \alpha r^\alpha - n^2r^\alpha = 0$$

$$r^\alpha\left(\alpha(\alpha-1)+\alpha-n^2\right) = 0$$

$$\alpha^2 = n^2$$

$$R(r) = Cr^n + \cancel{Dr^{-n}}$$

blows up at $r=0$

$$\boxed{u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n\left(A_n\cos n\theta + B_n\sin n\theta\right)} \quad (*)$$

On $x^2+y^2=a^2 \rightarrow$ $$\underset{\substack{\parallel\\ h(\theta)}}{u(a,\theta)} = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} a^nA_n\cos n\theta + a^nB_n\sin n\theta$$

$$\boxed{A_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\cos n\phi\, d\phi \qquad B_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\sin n\phi\, d\phi}$$

Can we simplify (*)?

Imitate Dirichlet kernel derivation:

$$u(r,\theta) = \int_0^{2\pi} h(\phi)\frac{\partial\phi}{2\pi} + \sum_{n=1}^{\infty}\frac{r^n}{\pi a^n}\int_0^{2\pi} h(\phi)\Big[\underbrace{\cos n\phi\cos n\theta + \sin n\phi \sin n\theta}_{\cos(n\theta - n\phi)}\Big]\partial\phi$$

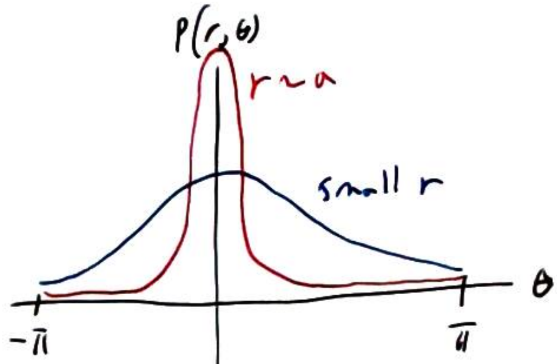
$$u(r,\theta) = \int_0^{2\pi} h(\phi)\underbrace{\Big[1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n(\theta-\phi)\Big]}_{P(r,\theta)\text{ — Poisson kernel}}\frac{\partial\phi}{2\pi} \qquad \text{"convolution"}$$

$$P(r,\theta) = 1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n\theta$$

$$= 1 + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in\theta} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{-in\theta}$$

$$= 1 + \frac{re^{i\theta}}{a - re^{i\theta}} + \frac{re^{-i\theta}}{a - re^{-i\theta}}$$

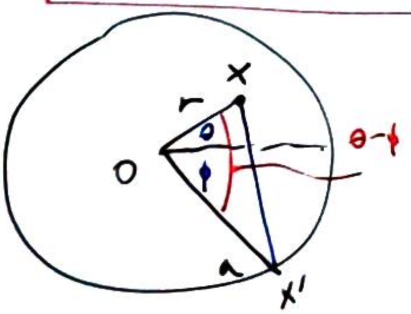
$$= \frac{a^2 - r^2}{a^2 - 2ar\cos\theta + r^2}$$



Is this spiking behavior familiar?

Plugging back in → Poisson's formula

$$\boxed{u(r,\theta) = (a^2 - r^2)\int_0^{2\pi}\frac{h(\phi)}{a^2 - 2ar\cos(\theta-\phi) + r^2}\frac{\partial\phi}{2\pi}}$$



Law of cosines → $|x - x'|^2 = a^2 + r^2 - 2ar\cos(\theta - \phi)$

$$u(x) = \frac{a^2 - |x|^2}{2\pi a}\int_{|x'|=a}\frac{u(x')}{|x-x'|^2}\,ds' \quad (ds = a\,\partial\phi)$$

<u>Mean Value Property</u>: If $\Delta u = 0$ on disc D, u continuous on $\overline{D} = D \cup \partial D$, then the value of u at the center of D is the average of u on ∂D

<u>Pf</u> Take $r=0$ in Poisson's formula

$$u(0) = a^2\int_0^{2\pi}\frac{u(a,\phi)}{a^2}\frac{\partial\phi}{2\pi}$$

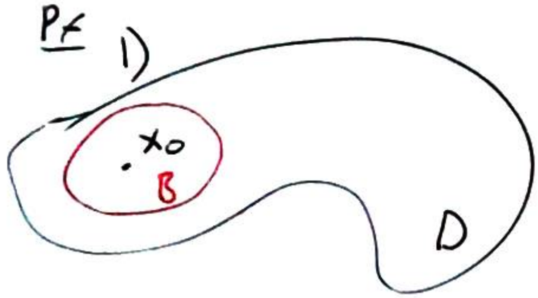
∎

Let $\Delta u = 0$ in D, u continuous on $\bar{D}$, D open, bounded (connected)

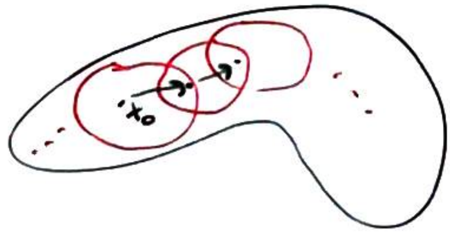
Weak Maximum Principle: $\max_{\bar{D}} u$ attained on ∂D

Strong Maximum Principle: If $\max_{\bar{D}} u$ also attained in D (the inside), then $u \equiv$ constant

Pf 1)



2) Fill D with balls B (where connectedness matters)



If $u(x_0) = \max_{\bar{D}} u$, then $u = \max u$ on all of B by mean-value property

Theorem Let $h(\theta)$ be continuous on ∂D. Let $u(r,\theta)$ be solution to $\Delta u = 0$ in D, $u = h(\theta)$ on ∂D. Then, $\lim_{r\to a^-} u(r,\theta) = h(\theta)$

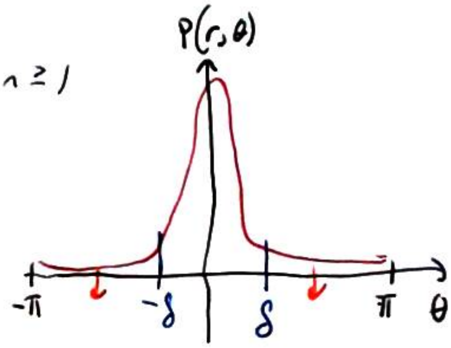
Pf Properties of Poisson kernel $P(r,\theta)$

i) $P(r,\theta) > 0$ for $r < a$

denominator - $a^2 - 2ar\cos\theta + r^2 \geq a^2 - 2ar + r^2 = (a-r)^2 \geq 0$

numerator - $a^2 - r^2 > 0$

ii) $\int_0^{2\pi} P(r,\theta)\frac{d\theta}{2\pi} = 1$ b/c $\int_0^{2\pi} \cos n\theta \, d\theta = 0$ for $n \geq 1$



iii) (Spike Property) For all $\delta > 0$,

$$\max_{\delta \leq \theta \leq 2\pi - \delta} |P(r,\theta)| \to 0 \text{ as } r \to a$$

Why? $P(r,\theta) = \dfrac{a^2 - r^2}{(a-r)^2 + 4ar\sin\theta/2}$

$\to 0$

not $\to 0$ for $\theta \neq 0$

Fix θ_0. $r \to a^-$

$$|u(r,\theta_0) - h(\theta_0)| = \left| \int_0^{2\pi} P(r,\theta_0-\phi)\, h(\phi) \frac{d\phi}{2\pi} - \int_0^{2\pi} P(r,\theta_0-\phi)\, h(\theta_0) \frac{d\phi}{2\pi} \right| \quad \text{by ii)}$$

$$\leq \int^{2\pi} P(r,\theta_0-\phi)\, |h(\phi) - h(\theta_0)| \frac{d\phi}{2\pi} \quad \text{by i)}$$

$$= \int_{\theta_0-\delta}^{\theta_0+\delta} P(r,\theta_0-\phi) \underbrace{|h(\phi)-h(\theta_0)|}_{\substack{\text{small} \\ \text{by continuity}}} \frac{d\phi}{2\pi} + \int_{|\phi-\theta_0|\geq\delta} |h(\phi)-h(\theta_0)| \cdot \underbrace{P(r,\theta_0-\phi)}_{\substack{\text{small by} \\ \text{iii)}}} \frac{d\phi}{2\pi}$$

$\to 0$ by taking $\delta \to 0$ $\qquad\qquad$ $\to 0$ by taking $r \to a^-$

Pointwise convergence is proved!

Last note: Passing derivatives onto $P(r,\theta)$ in Poisson's formula shows every harmonic function is infinitely differentiable.

Midterm Discussion + Pass Back