6/25/18 Lecture Notes: The Diffusion Equation on the Real Line and Half-Line

How was HW1?

Last time: Heat/Diffusion Equation. $u_t = ku_{xx}$

Maximum Principle: Maximum over R obtained on $\sqcup$

Energy Methods: $E(t) = \frac{1}{2}\int u(x,t)^2\,dx$ non increasing

Both Imply → Uniqueness of solution to Dirichlet problem:

$$u_t - ku_{xx} = f(x,t) \qquad u(0,t) = g(t)$$
$$u(x,0) = \phi(x) \qquad u(\ell,t) = h(t)$$

Stability - close $\phi \Rightarrow$ close u

Goal: Solve $u_t = ku_{xx}$ $\quad -\infty < x < \infty,\ t > 0$ $\quad (*)$

$$u(x,0) = \phi(x)$$

Important Properties of solutions: If $u(x,t)$, $v(x,t)$ satisfy $(*)$, then so do

1) $u(x-y,t)$ (translation invariance)
2) $u_x(x,t)$, $u_t(x,t)$, $u_{xx}(x,t)$, etc. (derivatives)
3) $(au+bv)(x,t)$ (linearity)
4) $\sum_{j=1}^{n} a_j u(x-y_j,t) \longrightarrow \int_{-\infty}^{\infty} u(x-y,t)g(y)\,dy$ (convolution)
5) $u(ax, a^2t)$ (dilation/scaling invariance)

How to Solve Heat Equation: meme Edition

Brain Activity Level	Idea
low	Solve when $\phi(x)$ nice, like either 1 or 0
medium	Solve when $\phi(x) = \begin{cases} 1 & x=0 \\ 0 & x\neq 0 \end{cases}$ and take linear combinations to get other $\phi(x)$.
high	Solve when $\phi(x) = \delta(x) = \begin{cases} \infty & x=0 \\ 0 & x\neq 0 \end{cases}$ with $\int \delta = 1$ (Dirac delta function)
explosion	Solve when $\phi(x) = \int_{-\infty}^{x} \delta(t)\,dt = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x<0 \end{cases}$ to get actual function, then take derivative

Step 1: Find $Q(x,t)$ such that

$$Q_t = k Q_{xx} \quad \text{and} \quad Q(x,0) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

Guess $Q(x,t) = g(p)$, $p = x/\sqrt{4kt}$

Step 2: Solve for g:

$$Q_t = \frac{\partial g}{\partial p} \cdot \frac{\partial p}{\partial t} = \frac{-1}{2t} \cdot \frac{x}{\sqrt{4kt}} g'(p) = \frac{-1}{2t} p\, g'(p)$$

$$Q_{xx} = \frac{1}{4kt} g''(p)$$

$$Q_t - k Q_{xx} = \frac{1}{t}\left[-\frac{1}{2} p\, g'(p) - \frac{1}{4} g''(p) \right] = 0$$

Integrating Factor e^{p^2}

$$\Rightarrow g''(p) + 2p\, g'(p) = 0 \Rightarrow e^{p^2} g'' + 2p e^{p^2} g' = 0$$

$$(e^{p^2} g')' = 0$$

$$e^{p^2} g' = c_1$$

$$g' = c_1 e^{-p^2}$$

$$g(p) = c_1 \int_0^p e^{-s^2} ds + c_2$$

Step 3: Find c_1, c_2.

Fun Fact: $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$$Q(x,t) = c_1 \int_0^{x/\sqrt{4kt}} e^{-p^2} dp + c_2$$

For $x > 0$, $1 = \lim_{t \searrow 0} Q = c_1 \int_0^{\infty} e^{-p^2} dp + c_2 = c_1 \frac{\sqrt{\pi}}{2} + c_2$

For $x < 0$, $0 = \lim_{t \searrow 0} Q = c_1 \int_0^{-\infty} e^{-p^2} dp + c_2 = -c_1 \frac{\sqrt{\pi}}{2} + c_2$

Linear Algebra $\Rightarrow c_1 = 1/\sqrt{\pi}$, $c_2 = 1/2$

Step 4: Find solution for arbitrary $\phi(x)$ (Back to Brain Meme)

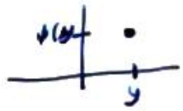
Q solution $\Rightarrow$ $S = \partial Q / \partial x$ solution

$$\Rightarrow \boxed{u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy \text{ solution}} \quad (\phi \to 0 \text{ as } |x| \to \infty)$$

[Remember: $S(x,t)$ solution for $\phi(x) = \delta(x)$

$S(x-y,t)$ solution for $\phi(x) = \delta_{x=y}$

$S(x-y,t)\phi(y)$ solution for IC $\phi(y)\delta_{x=y}$



Sum up (integrate) over all y.]

Claim: $u(x,0) = \phi(x)$ (as defined above)

__Proof__ $u(x,t) = \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y,t)\phi(y)\,dy$

$$= -\int_{-\infty}^{\infty} \frac{\partial Q}{\partial y}(x-y,t)\phi(y)\,dy$$

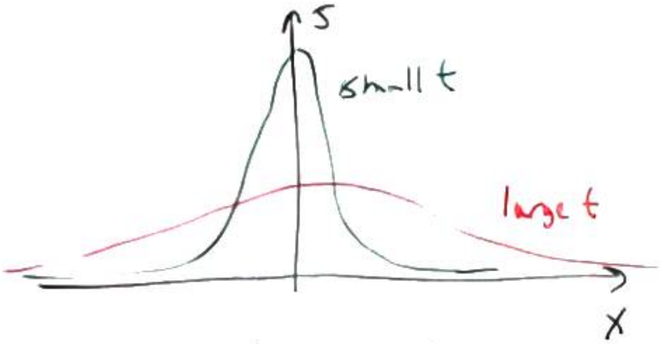
IBP

$$= \int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)\,dy - \underbrace{\cancel{Q(x-y,t)\phi(y)\Big|_{y=-\infty}^{\infty}}}_{0}$$

$$u(x,0) = \int_{-\infty}^{\infty} Q(x-y,0)\phi'(y)\,dy$$

$$= \int_{-\infty}^{x} \phi'(y)\,dy = \phi\Big|_{-\infty}^{x} = \phi(x) \qquad ▩$$

Fundamental Solution: $S(x,t) = \frac{\partial Q}{\partial x} = \frac{1}{2\sqrt{\pi k t}} e^{-x^2/4kt} \quad t > 0$



Useful tool: The <u>error function</u>

$$\mathcal{E}rf(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2}\,dp$$

Note: $S(x,t) \to \delta(x)$ as $t \to 0$

Example: Solve $u_t = ku_{xx}$ $\quad -\infty < x < \infty \quad t > 0$

$u(x,0) = e^x$

Solution: $u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} e^y \, dy$

exponent: $-\frac{x^2 - 2xy + y^2 - 4kty}{4kt} = -\frac{(y+2kt-x)^2 + 4k^2t^2 + 4ktx}{4kt}$

Sub $p = \frac{y+2kt-x}{\sqrt{4kt}}$ $\quad dp = \frac{dy}{\sqrt{4kt}}$ $\quad = -\frac{(y+2kt-x)^2}{4kt} + kt + x$

$u(x,t) = e^{kt+x} \int_{-\infty}^{\infty} e^{-p^2} \frac{dp}{\sqrt{\pi}} = e^{kt+x}$

Diffusion on the Half-Line

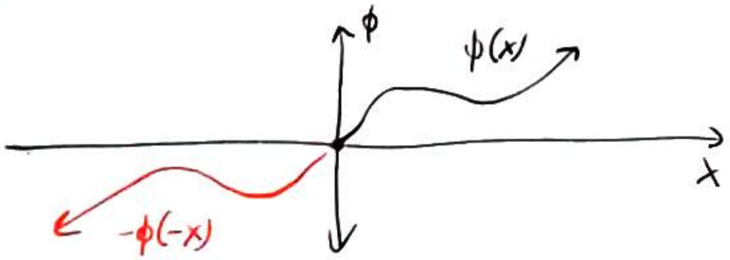
Mathematician, fire, bucket story

$v_t - kv_{xx} = 0 \quad 0 < x < \infty \quad 0 < t < \infty$

$v(x,0) = \phi(x)$ — Initial condition

$v(0,t) = 0$ — Boundary condition

Reflection method: Switch to the (full) real line!



call this ϕ_{odd}, the odd extension of ϕ

New Goal: Solve $u_t - ku_{xx} = 0$, $u(x,0) = \phi_{odd}(x)$ $\xrightarrow{\text{HW 2.4:11}}$ $u(x,t)$ odd as function of x, so $u(0,t) = 0$, restriction to $t \geq 0$ is v.

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\,\phi_{odd}(y)\,dy$$

$$= \int_{0}^{\infty} S(x-y,t)\phi(y)dy - \int_{-\infty}^{0} S(x-y,t)\,\phi(-y)\,dy$$

Change of Variables

$y \to -y$

$dy \to -dy$

$$+ \int_{\infty}^{0} S(x+y,t)\,\phi(y)\,dy$$

$$= \int_{0}^{\infty} \left[S(x-y,t) - S(x+y,t)\right] \phi(y)\,dy$$

For $x>0$, $$\boxed{v(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{0}^{\infty} \left[e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt}\right] \phi(y)\,dy}$$

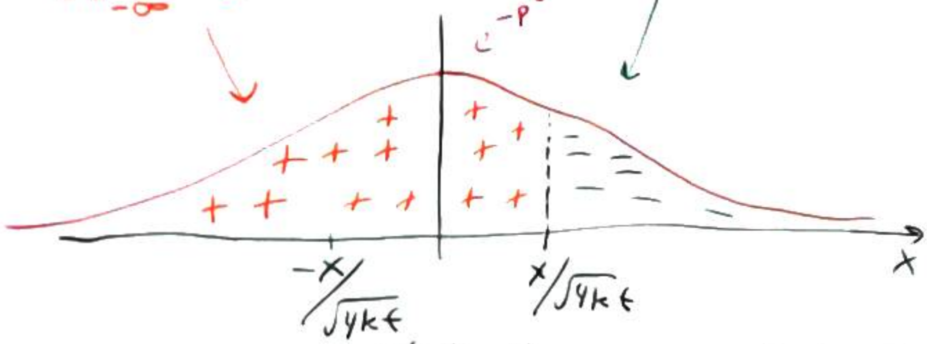
<u>Example</u> Let $\phi(x)=1$ for $x>0$ $\rightarrow$ $\phi_{odd}(x) = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$

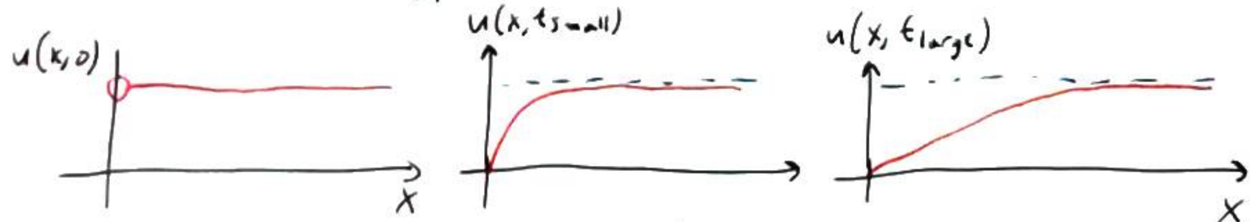
$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-(x-y)^2/4kt}\,dy - \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-(x+y)^2/4kt}\,dy$$

$p = (x-y)/\sqrt{4kt}$ $\qquad$ $p = (x+y)/\sqrt{4kt}$

$dp = -dy/\sqrt{4kt}$ $\qquad$ $dp = dy/\sqrt{4kt}$

$$\underbrace{-\int_{x/\sqrt{4kt}}^{-\infty} e^{-p^2}\,dp/\sqrt{\pi}}_{\frac{1}{\sqrt{\pi}}\int_{-\infty}^{x/\sqrt{4kt}} e^{-p^2}dt} - \underbrace{\int_{x/\sqrt{4kt}}^{\infty} e^{-p^2}\,dp/\sqrt{\pi}} = \mathrm{Erf}\left(x/\sqrt{4kt}\right)$$





Note: Infinite speed of propogation