

Lecture Notes for  
A Geometrical Introduction to  
Robotics and Manipulation

Richard Murray and Zexiang Li and Shankar S. Sastry  
CRC Press

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# Chapter 3 Manipulator Kinematics

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# 3.1 Forward kinematics



Play/Pause Stop

(a) Adept Cobra i600 (SCARA)

Figure 3.1

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# 3.1 Forward kinematics



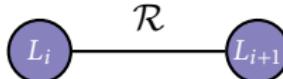
Play/Pause Stop

(a) Adept Cobra i600 (SCARA)

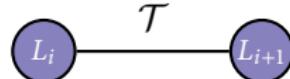
Figure 3.1

◊ Lower Pair Joints:

revolute joint  $S^1 \mapsto SO(2)$



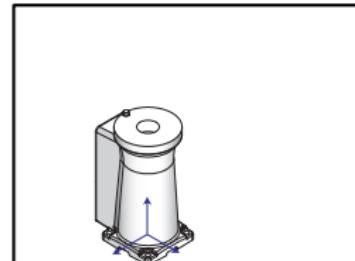
prismatic joint  $\mathbb{R} \mapsto T(1)$



# 3.1 Forward kinematics



(a) Adept Cobra i600 (SCARA)

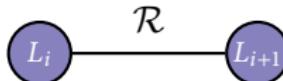


(b) Forward kinematics of SCARA

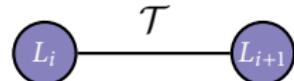
Figure 3.1

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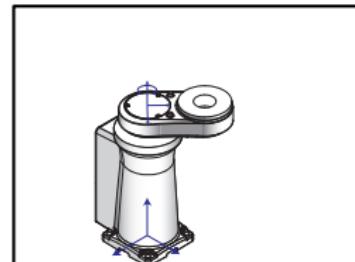


◊ Forward kinematics:



base: stationary

# 3.1 Forward kinematics

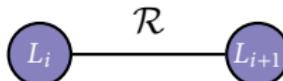


(a) Adept Cobra i600 (SCARA)      (b) Forward kinematics of SCARA

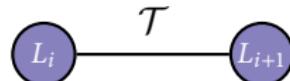
Figure 3.1

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prismatic joint  $\mathbb{R} \mapsto T(1)$



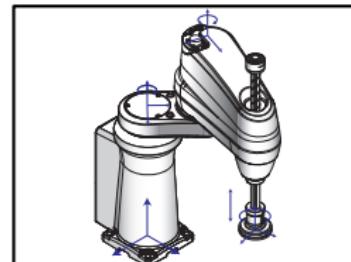
◊ Forward kinematics:



base: stationary

Link 1: first movable link

# 3.1 Forward kinematics

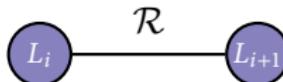


(a) Adept Cobra i600 (SCARA)      (b) Forward kinematics of SCARA

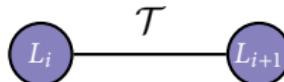
Figure 3.1

◊ Lower Pair Joints:

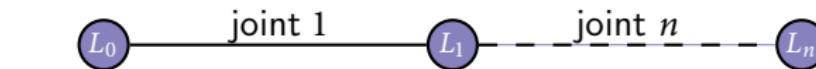
revolute joint  $S^1 \mapsto SO(2)$



prismatic joint  $\mathbb{R} \mapsto T(1)$



◊ Forward kinematics:



base: stationary

Link 1: first movable link

Link n: end-effector

# 3.1 Forward kinematics

## □ Joint space:

Revolute joint:  $S^1, \theta_i \in S^1$  or  $\theta_i \in [-\pi, \pi]$

Prismatic joint:  $\mathbb{R}$

Joint space:  $Q : \underbrace{S^1 \times \cdots \times S^1}_{\text{no. of } R \text{ joint}} \times \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{\text{no. of } P \text{ joint}}$

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# 3.1 Forward kinematics

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Adept  $Q : S^1 \times S^1 \times S^1 \times \mathbb{R}$

Elbow  $Q = \Gamma^6 : \underbrace{S^1 \times \cdots \times S^1}_6$

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# 3.1 Forward kinematics

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Adept  $Q : S^1 \times S^1 \times S^1 \times \mathbb{R}$

Elbow  $Q = \Gamma^6 : \underbrace{S^1 \times \cdots \times S^1}_6$

Reference (nominal) joint config:

$$\theta = (0, 0, \dots, 0) \in Q$$

Reference (nominal) end-effector config:

$$g_{st}(0) \in SE(3)$$

# 3.1 Forward kinematics

## □ Joint space:

Revolute joint:  $S^1, \theta_i \in S^1$  or  $\theta_i \in [-\pi, \pi]$

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Adept  $Q : S^1 \times S^1 \times S^1 \times \mathbb{R}$

Elbow  $Q = \Gamma^6 : \underbrace{S^1 \times \cdots \times S^1}_6$

Reference (nominal) joint config:

$\theta = (0, 0, \dots, 0) \in Q$

Reference (nominal) end-effector config:  $g_{st}(0) \in SE(3)$

Arbitrary configuration  $g_{st}(\theta)$ :

$$g_{st} : \theta \in Q \mapsto g_{st}(\theta) \in SE(3)$$

# 3.1 Forward kinematics

## □ Classical Approach:

$$g_{st}(\theta_1, \theta_2) = g_{st}(\theta_1) \cdot g_{l_1 l_2} \cdot g_{l_2 t}$$

**Disadvantage:** A coordinate frame needed for each link

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# 3.1 Forward kinematics

## □ Classical Approach:

$$g_{st}(\theta_1, \theta_2) = g_{st}(\theta_1) \cdot g_{l_1 l_2} \cdot g_{l_2 t}$$

**Disadvantage:** A coordinate frame needed for each link

## □ The product of exponentials formula:

Consider Fig 3.2.

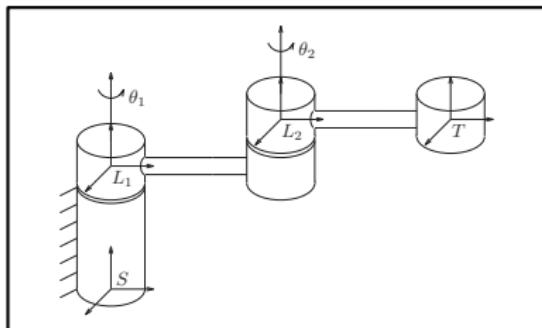


Figure 3.2: A two degree of freedom manipulator

# 3.1 Forward kinematics

Step 1: Rotating about  $\omega_2$  by  $\theta_2$

$$\xi_2 = \begin{bmatrix} -\omega_2 \times q_2 \\ \omega_2 \end{bmatrix}$$

$$g_{st}(\theta_2) = e^{\xi_2 \theta_2} \cdot g_{st}(0)$$

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# 3.1 Forward kinematics

Step 1: Rotating about  $\omega_2$  by  $\theta_2$

$$\xi_2 = \begin{bmatrix} -\omega_2 \times q_2 \\ \omega_2 \end{bmatrix}$$

$$g_{st}(\theta_2) = e^{\hat{\xi}_2 \theta_2} \cdot g_{st}(0)$$

Step 2: Rotating about  $\omega_1$  by  $\theta_1$

$$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix}$$

$$g_{st}(\theta_1, \theta_2) = \underbrace{e^{\hat{\xi}_1 \theta_1} \cdot e^{\hat{\xi}_2 \theta_2}}_{\text{offset}} \cdot g_{st}(0)$$

$$\theta : (0, 0) \mapsto (0, \theta_2) \mapsto (\theta_1, \theta_2)$$

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# 3.1 Forward kinematics

What if another route is taken?

$$\theta : (0, 0) \mapsto (\theta_1, 0) \mapsto (\theta_1, \theta_2)$$

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## 3.1 Forward kinematics

What if another route is taken?

$$\theta : (0, 0) \mapsto (\theta_1, 0) \mapsto (\theta_1, \theta_2)$$

Step 1: Rotating about  $\omega_1$  by  $\theta_1$

$$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix}$$

$$g_{st}(\theta_1) = e^{\hat{\xi}_1 \theta_1} \cdot g_{st}(0)$$

Step 2: Rotating about  $\omega'_2$  by  $\theta_2$

Let  $e^{\hat{\xi}_1 \theta_1} = \begin{bmatrix} R_1 & p_1 \\ 0 & 1 \end{bmatrix}$

$$\omega'_2 = R_1 \cdot \omega_2$$

$$q'_2 = p_1 + R_1 \cdot q_2$$

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# 3.1 Forward kinematics

$$\begin{aligned}\xi'_2 &= \begin{bmatrix} -\omega'_2 \times q'_2 \\ \omega'_2 \end{bmatrix} = \begin{bmatrix} -R_1 \hat{\omega}_2 R_1^T (p_1 + R_1 q_2) \\ R_1 \omega_2 \end{bmatrix} \\ &= \begin{bmatrix} R_1 & \hat{p}_1 R_1 \\ 0 & \bar{R}_1 \end{bmatrix} \begin{bmatrix} -\omega_2 \times q_2 \\ \omega_2 \end{bmatrix} = Ad_{e^{\hat{\xi}_1 \theta_1}} \cdot \xi_2 \Rightarrow \hat{\xi}'_2 = e^{\hat{\xi}_1 \theta_1} \cdot \hat{\xi}_2 \cdot e^{-\hat{\xi}_1 \theta_1}\end{aligned}$$

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# 3.1 Forward kinematics

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$$\begin{aligned}g_{st}(\theta_1, \theta_2) &= e^{\hat{\xi}'_2 \theta_2} \cdot e^{\hat{\xi}_1 \theta_1} \cdot g_{st}(0) \\ &= e^{e^{\hat{\xi}_1 \theta_1} \cdot \hat{\xi}_2 \theta_2 \cdot e^{-\hat{\xi}_1 \theta_1}} \cdot e^{\hat{\xi}_1 \theta_1} \cdot g_{st}(0) \\ &= e^{\hat{\xi}_1 \theta_1} \cdot e^{\hat{\xi}_2 \theta_2} \cdot e^{-\hat{\xi}_1 \theta_1} \cdot e^{\hat{\xi}_1 \theta_1} \cdot g_{st}(0) \\ &= \underbrace{e^{\hat{\xi}_1 \theta_1} \cdot e^{\hat{\xi}_2 \theta_2} \cdot g_{st}(0)}_{\text{Independent of the route taken}}\end{aligned}$$

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## 3.1 Forward kinematics

### □ Procedure for forward kinematic map:

Identify a nominal configuration:

$$\Theta = (\theta_{10}, \dots, \theta_{n0}) = 0, g_{st}(0) \triangleq g_{st}(\theta_{10}, \dots, \theta_{n0})$$

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## 3.1 Forward kinematics

### □ Procedure for forward kinematic map:

Identify a nominal configuration:

$$\Theta = (\theta_{10}, \dots, \theta_{n0}) = 0, g_{st}(0) \triangleq g_{st}(\theta_{10}, \dots, \theta_{n0})$$

Simplification of forward kinematics mapping:

Revolute joint:  $\xi_i = \begin{bmatrix} -\omega \times q_i \\ \omega_i \end{bmatrix}$

Choose  $q_i$  s.t.  $\xi_i$  is simple.

Prismatic joint:  $\xi_i = \begin{bmatrix} v_i \\ 0 \end{bmatrix}$

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# 3.1 Forward kinematics

## □ Procedure for forward kinematic map:

Identify a nominal configuration:

$$\Theta = (\theta_{10}, \dots, \theta_{n0}) = 0, g_{st}(0) \triangleq g_{st}(\theta_{10}, \dots, \theta_{n0})$$

Simplification of forward kinematics mapping:

Revolute joint:  $\xi_i = \begin{bmatrix} -\omega \times q_i \\ \omega_i \end{bmatrix}$

Choose  $q_i$  s.t.  $\xi_i$  is simple.

Prismatic joint:  $\xi_i = \begin{bmatrix} v_i \\ 0 \end{bmatrix}$

Write  $g_{st}(\theta) = e^{\hat{\xi}_1 \theta_1} \dots e^{\hat{\xi}_n \theta_n} \cdot g_{st}(0)$  (product of exponential mapping)

# 3.1 Forward kinematics

## ◇ Example: SCARA manipulator

$$g_{st}(0) = \begin{bmatrix} I & 0 \\ 0 & l_1 + l_2 \\ 0 & l_0 \end{bmatrix}$$

$$\omega_1 = \omega_2 = \omega_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

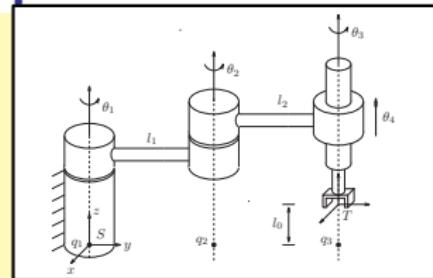


Figure 3.3

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# 3.1 Forward kinematics

## ◇ Example: SCARA manipulator

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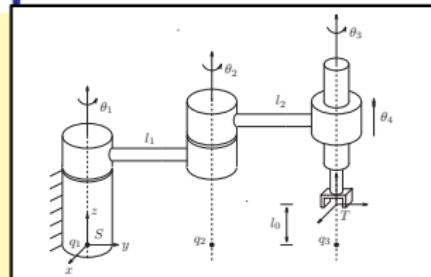


Figure 3.3

$$q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix}, q_3 = \begin{bmatrix} 0 \\ l_1 + l_2 \\ 0 \end{bmatrix}$$

# 3.1 Forward kinematics

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## Example: SCARA manipulator

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$$g_{st}(0) = \begin{bmatrix} I & 0 \\ 0 & l_0 + l_2 \end{bmatrix}$$

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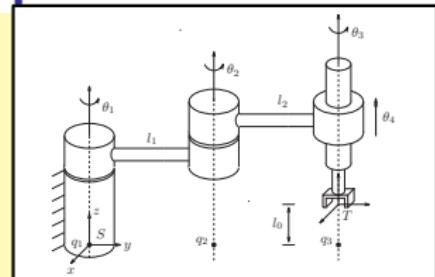


Figure 3.3

$$q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix}, q_3 = \begin{bmatrix} 0 \\ l_1 + l_2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \xi_2 = \begin{bmatrix} l_1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \xi_3 = \begin{bmatrix} l_1 + l_2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \xi_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

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$$g_{st}(\theta) = e^{\hat{\xi}_1 \theta_1} \cdot e^{\hat{\xi}_2 \theta_2} \cdot e^{\hat{\xi}_3 \theta_3} \cdot e^{\hat{\xi}_4 \theta_4} \cdot g_{st}(0) = \begin{bmatrix} R(\theta) & p(\theta) \\ 0 & 1 \end{bmatrix}$$

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# 3.1 Forward kinematics

$$g_{st}(\theta) = e^{\hat{\xi}_1 \theta_1} \cdot e^{\hat{\xi}_2 \theta_2} \cdot e^{\hat{\xi}_3 \theta_3} \cdot e^{\hat{\xi}_4 \theta_4} \cdot g_{st}(0) = \begin{bmatrix} R(\theta) & p(\theta) \\ 0 & 1 \end{bmatrix}$$

$$e^{\hat{\xi}_1 \theta_1} = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad e^{\hat{\xi}_2 \theta_2} = \begin{bmatrix} c_2 & -s_2 & 0 & l_1 s_2 \\ s_2 & c_2 & 0 & l_1 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$e^{\hat{\xi}_3 \theta_3} = \begin{bmatrix} c_3 & -s_3 & 0 & (l_1 s_2) s_3 \\ s_3 & c_3 & 0 & (l_1 s_2) v_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad e^{\hat{\xi}_4 \theta_4} = \begin{bmatrix} I & \begin{bmatrix} 0 \\ 0 \\ \theta_4 \\ 1 \end{bmatrix} \\ 0 & \end{bmatrix}$$

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# 3.1 Forward kinematics

$$g_{st}(\theta) = e^{\hat{\xi}_1 \theta_1} \cdot e^{\hat{\xi}_2 \theta_2} \cdot e^{\hat{\xi}_3 \theta_3} \cdot e^{\hat{\xi}_4 \theta_4} \cdot g_{st}(0) = \begin{bmatrix} R(\theta) & p(\theta) \\ 0 & 1 \end{bmatrix}$$

$$e^{\hat{\xi}_1 \theta_1} = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad e^{\hat{\xi}_2 \theta_2} = \begin{bmatrix} c_2 & -s_2 & 0 & l_1 s_2 \\ s_2 & c_2 & 0 & l_1 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$e^{\hat{\xi}_3 \theta_3} = \begin{bmatrix} c_3 & -s_3 & 0 & (l_1 s_2) s_3 \\ s_3 & c_3 & 0 & (l_1 s_2) v_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad e^{\hat{\xi}_4 \theta_4} = \begin{bmatrix} I & \begin{bmatrix} 0 \\ 0 \\ \theta_4 \\ 1 \end{bmatrix} \\ 0 & \end{bmatrix}$$

$$g_{st}(\theta) = \begin{bmatrix} c_{123} & -s_{123} & 0 & -l_1 s_1 - l_2 s_{12} \\ s_{123} & c_{123} & 0 & l_1 c_1 + l_2 c_{12} \\ 0 & 0 & 1 & l_0 + \theta_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

in which,  $c_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$  and  $c_{12} = \cos(\theta_1 + \theta_2)$ . ◇

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## ◇ Example: Elbow manipulator

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$$g_{st}(0) = \begin{bmatrix} I & \begin{bmatrix} 0 \\ l_1 + l_2 \\ l_0 \\ 1 \end{bmatrix} \\ 0 & \end{bmatrix}$$

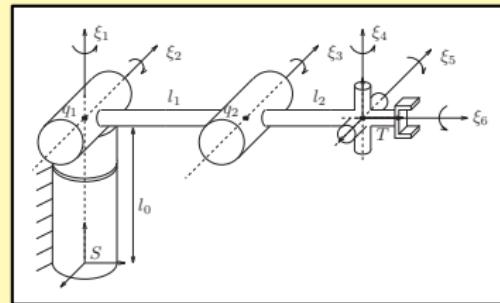


Figure 3.4

$$\xi_1 = \left[ - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

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$$\xi_2 = \left[ - \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} \right] = \begin{bmatrix} 0 \\ -l_0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \xi_3 = \begin{bmatrix} 0 \\ -l_0 \\ l_1 \\ -1 \\ 0 \\ 0 \end{bmatrix},$$

$$\xi_4 = \begin{bmatrix} l_1 + l_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \xi_5 = \begin{bmatrix} 0 \\ -l_0 \\ l_1 + l_2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \xi_6 = \begin{bmatrix} -l_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\rightarrow g_{st}(\theta_1, \dots, \theta_6) = e^{\dot{\xi}_1 \theta_1} \dots e^{\dot{\xi}_6 \theta_6} \cdot g_{st}(0) = \begin{bmatrix} R(\theta) & p(\theta) \\ 0 & 1 \end{bmatrix}$$

$$p(\theta) = \begin{bmatrix} -s_1(l_2 c_2 + l_2 c_{23}) \\ c_1(l_1 c_2 + l_2 c_{23}) \\ l_0 - l_1 s_2 - l_2 s_{23} \end{bmatrix}, \quad R(\theta) = [r_{ij}]$$

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# 3.1 Forward kinematics

in which,  $r_{11} = c_6(c_1c_4 - s_1c_{23}s_4) + s_6(s_1s_{23}c_5 + s_1c_{23}c_4s_5 + c_1s_4s_5)$

$$r_{12} = -c_5(s_1c_{23}c_4 + c_1s_4) + s_1s_{23}s_5$$

$$r_{13} = c_6(-c_5s_1s_{23} - (c_{23}c_4s_1 + c_1s_4)s_5) + (c_1c_4 - c_{23}s_1s_4)s_6$$

$$r_{21} = c_6(c_4s_1 + c_1c_{23}s_4) - (c_1c_5s_{23} + (c_1c_{23}c_4 - s_1s_4)s_5)s_6$$

$$r_{22} = c_5(c_1c_{23}c_4 - s_1s_4) - c_1s_{23}s_5$$

$$r_{23} = c_6(c_1c_5s_{23} + (c_1c_{23}c_4 - s_1s_4)s_5) + (c_4s_1 + c_1c_{23}s_4)s_6$$

$$r_{31} = -(c_6s_{23}s_4) - (c_{23}c_5 - c_4s_{23}s_5)s_6$$

$$r_{32} = -(c_4c_5s_{23}) - c_{23}s_5$$

$$r_{33} = c_6(c_{23}c_5 - c_4s_{23}s_5) - s_{23}s_4s_6$$

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# 3.1 Forward kinematics

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in which,  $r_{11} = c_6(c_1c_4 - s_1c_{23}s_4) + s_6(s_1s_{23}c_5 + s_1c_{23}c_4s_5 + c_1s_4s_5)$

$$r_{12} = -c_5(s_1c_{23}c_4 + c_1s_4) + s_1s_{23}s_5$$

$$r_{13} = c_6(-c_5s_1s_{23} - (c_{23}c_4s_1 + c_1s_4)s_5) + (c_1c_4 - c_{23}s_1s_4)s_6$$

$$r_{21} = c_6(c_4s_1 + c_1c_{23}s_4) - (c_1c_5s_{23} + (c_1c_{23}c_4 - s_1s_4)s_5)s_6$$

$$r_{22} = c_5(c_1c_{23}c_4 - s_1s_4) - c_1s_{23}s_5$$

$$r_{23} = c_6(c_1c_5s_{23} + (c_1c_{23}c_4 - s_1s_4)s_5) + (c_4s_1 + c_1c_{23}s_4)s_6$$

$$r_{31} = -(c_6s_{23}s_4) - (c_{23}c_5 - c_4s_{23}s_5)s_6$$

$$r_{32} = -(c_4c_5s_{23}) - c_{23}s_5$$

$$r_{33} = c_6(c_{23}c_5 - c_4s_{23}s_5) - s_{23}s_4s_6$$

**Simplify forward Kinematics Map:**

Choose base frame or ref. Config. s.t.  $g_{st}(0) = I$



# 3.1 Forward kinematics

## □ Manipulator Workspace:

$$W = \{g_{st}(\theta) | \forall \theta \in Q\} \subset SE(3)$$

### ■ Reachable Workspace:

$$W_R = \{p(\theta) | \forall \theta \in Q\} \subset \mathbb{R}^3$$

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# 3.1 Forward kinematics

## □ Manipulator Workspace:

$$W = \{g_{st}(\theta) | \forall \theta \in Q\} \subset SE(3)$$

### ■ Reachable Workspace:

$$W_R = \{p(\theta) | \forall \theta \in Q\} \subset \mathbb{R}^3$$

### ■ Dextrous Workspace:

$$W_D = \{p \in \mathbb{R}^2 | \forall R \in SO(3), \exists \theta, g_{st}(\theta) = (p, R)\}$$

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## 3.1 Forward kinematics

### ◊ Example: A planar serial 3-bar linkage

### (a) Workspace calculation:

$$g = (x, y, \phi)$$

$$x = l_1 c_1 + l_2 c_{12} + l_3 c_{123}$$

$$y = l_1 s_1 + l_2 s_{12} + l_3 s_{123}$$

$$\phi = \theta_1 + \theta_2 + \theta_3$$

#### (b) Construction of Workspace:

### (c) Reachable Workspace:

(d) Dextrous Workspace:

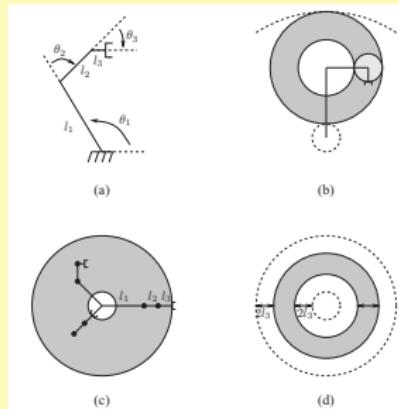


Figure 3.5

# 3.1 Forward kinematics

## ◇ Example: A planar serial 3-bar linkage

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(a) Workspace calculation:

$$g = (x, y, \phi)$$

$$x = l_1 c_1 + l_2 c_{12} + l_3 c_{123}$$

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(b) Construction of Workspace:

(c) Reachable Workspace:

(d) Dextrous Workspace:

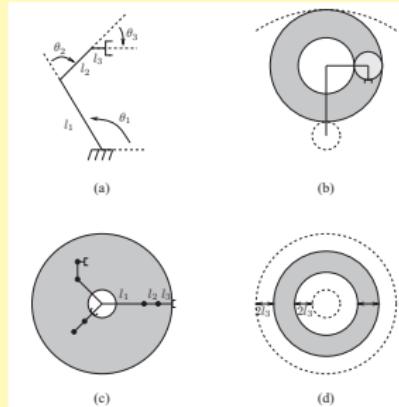


Figure 3.5

□ 6R manipulator with maximum workspace (Paden):

Elbow manipulator and its kinematics inverse.

## 3.2 Inverse Kinematics

### Definition: Inverse kinematics

Given  $g \in SE(3)$ , find  $\theta \in Q$  s.t.

$$gst(\theta) = g, \text{ where } gst : Q \mapsto SE(3)$$

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## 3.2 Inverse Kinematics

### Definition: Inverse kinematics

Given  $g \in SE(3)$ , find  $\theta \in Q$  s.t.

$$gst(\theta) = g, \text{ where } gst : Q \mapsto SE(3)$$

### ◇ Example: A planar example

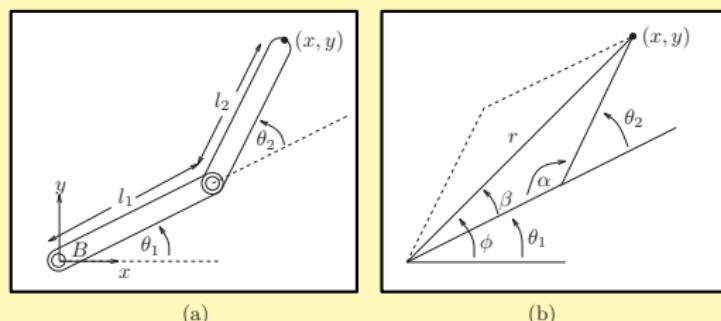


Figure 3.6

$$x = l_1 \cos \theta_1 + l_2 \cos (\theta_1 + \theta_2)$$

$$y = l_1 \sin \theta_1 + l_2 \sin (\theta_1 + \theta_2)$$

Given  $(x, y)$ , solve for  $(\theta_1, \theta_2)$ .

## 3.2 Inverse Kinematics

### ◊ Review:

Polar Coordinates:

$$(r, \phi), r = \sqrt{x^2 + y^2}$$

Law of cosines:

$$\theta_2 = \pi \pm \alpha, \alpha = \cos^{-1} \frac{l_1^2 + l_2^2 - r^2}{2l_1l_2}$$

Flip solution:  $\pi + \alpha$

$$\theta_1 = \text{atan2}(y, x) \pm \beta, \beta = \cos^{-1} \frac{r^2 + l_1^2 - l_2^2}{2l_1r}$$

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## 3.2 Inverse Kinematics

### ◊ Review:

Polar Coordinates:

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Flip solution:  $\pi + \alpha$

$$\theta_1 = \text{atan2}(y, x) \pm \beta, \beta = \cos^{-1} \frac{r^2 + l_1^2 - l_2^2}{2l_1r}$$

### Hight Lights:

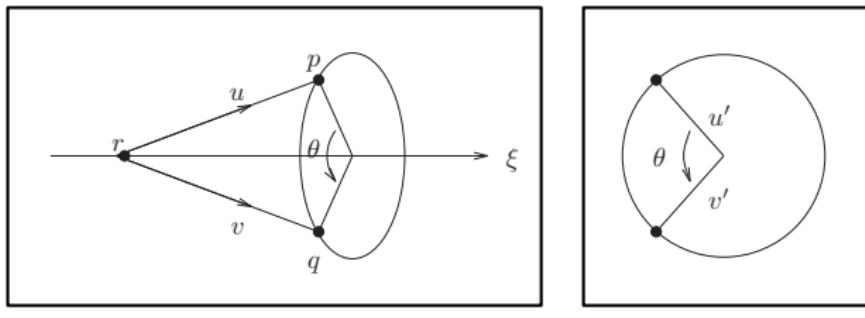
- Subproblems
- Each has zero, one or two solutions!

## 3.2 Inverse Kinematics

### □ Paden-Kahan Subproblems:

#### Subproblem 1: Rotation about a single axis

Let  $\xi$  be a zero-pitch twist, with unit magnitude and two points  $p, q \in \mathbb{R}^3$ . Find  $\theta$  s.t.  $e^{\xi\theta}p = q$



(a)

Figure 3.6

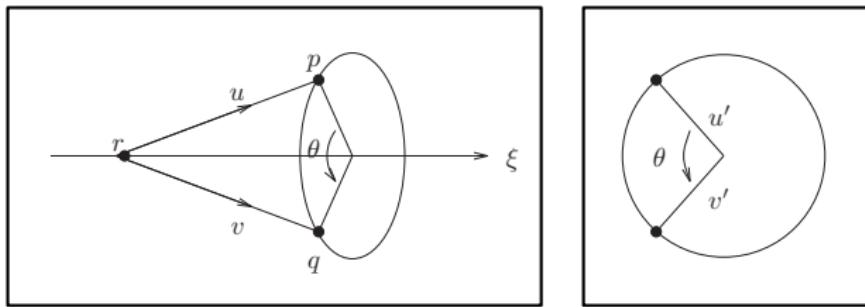
(b)

## 3.2 Inverse Kinematics

### □ Paden-Kahan Subproblems:

#### Subproblem 1: Rotation about a single axis

Let  $\xi$  be a zero-pitch twist, with unit magnitude and two points  $p, q \in \mathbb{R}^3$ . Find  $\theta$  s.t.  $e^{\hat{\xi}\theta} p = q$



(a)

(b)

Figure 3.6

Solution: Let  $r \in l_\xi$ , define  $u = p - r, v = q - r, e^{\hat{\xi}\theta} r = r$

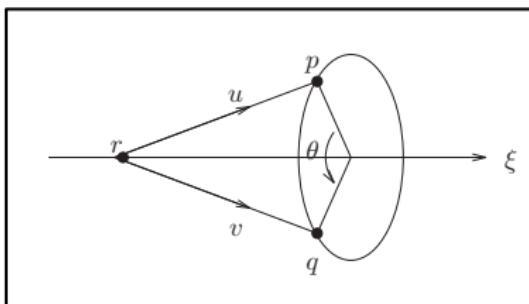
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## 3.2 Inverse Kinematics

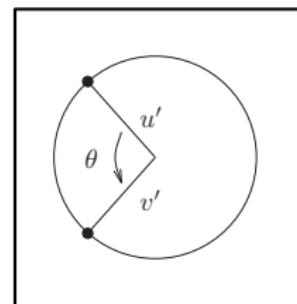
Moreover,

$$\Rightarrow e^{\hat{\xi}\theta} p = q \Rightarrow e^{\hat{\xi}\theta} \underbrace{(p - r)}_u = \underbrace{q - r}_v \Rightarrow \begin{bmatrix} e^{\hat{\omega}\theta} & * \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ 0 \end{bmatrix}$$

$$\Rightarrow e^{\hat{\omega}\theta} u = v \quad \left\{ \begin{array}{l} w^T u = w^T v \\ \|u\|^2 = \|v\|^2 \end{array} \right.$$



(a)



(b)

Figure 3.6

(Continues next slide)

## 3.2 Inverse Kinematics

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$$u' = (I - \omega\omega^T)u, v' = (I - \omega\omega^T)v$$

The solution exists only if

$$\begin{cases} \|u'\|^2 = \|v'\|^2 \\ \omega^T u = \omega^T v \end{cases}$$

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## 3.2 Inverse Kinematics

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$$u' = (I - \omega\omega^T)u, v' = (I - \omega\omega^T)v$$

The solution exists only if

$$\begin{cases} \|u'\|^2 = \|v'\|^2 \\ \omega^T u = \omega^T v \end{cases}$$

- If  $u' \neq 0$ , then

$$u' \times v' = \omega \sin \theta \|u'\| \|v'\|$$

$$u' \cdot v' = \cos \theta \|u'\| \|v'\|$$

$$\Rightarrow \theta = \text{atan2}(\omega^T(u' \times v'), u'^T v')$$

## 3.2 Inverse Kinematics

21

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$$u' \cdot v' = \cos \theta \|u'\| \|v'\|$$

$$\Rightarrow \theta = \text{atan2}(\omega^T(u' \times v'), u'^T v')$$

- If  $u' = 0$ ,  $\Rightarrow$  Infinite number of solutions!



## 3.2 Inverse Kinematics

### Subproblem 2: Rotation about two subsequent axes

Let  $\xi_1$  and  $\xi_2$  be two zero-pitch, unit magnitude twists, with intersecting axes, and  $p, q \in R^3$ . find  $\theta_1$  and  $\theta_2$  s.t.

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p = q.$$

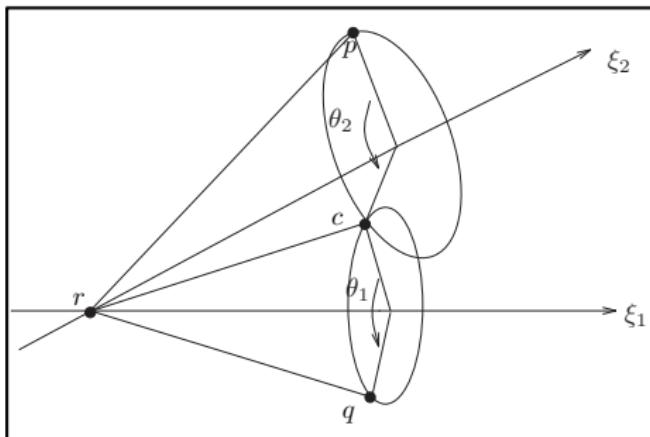


Figure 3.6

(Continues next slide)

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## 3.2 Inverse Kinematics

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**Solution:** If two axes of  $\xi_1$  and  $\xi_2$  coincide, then we get:

**Subproblem 1:**  $\theta_1 + \theta_2 = \theta$

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## 3.2 Inverse Kinematics

**Solution:** If two axes of  $\xi_1$  and  $\xi_2$  coincide, then we get:

**Subproblem 1:**  $\theta_1 + \theta_2 = \theta$

If the two axes are not parallel,  $\omega_1 \times \omega_2 \neq 0$ , then, let  $c$  satisfy:

$$e^{\hat{\xi}_2 \theta_2} p = c = e^{-\hat{\xi}_1 \theta_1} q$$

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## 3.2 Inverse Kinematics

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If the two axes are not parallel,  $\omega_1 \times \omega_2 \neq 0$ , then, let  $c$  satisfy:

$$e^{\hat{\xi}_2 \theta_2} p = c = e^{-\hat{\xi}_1 \theta_1} q$$

Set  $r \in l_{\xi_1} \cap l_{\xi_2}$

$$e^{\hat{\xi}_2 \theta_2} \underbrace{p - r}_{u} = \underbrace{c - r}_{z} = e^{-\hat{\xi}_1 \theta_1} \underbrace{(q - r)}_{v}, \Rightarrow e^{\hat{\omega}_2 \theta_2} u = z = e^{-\hat{\omega}_1 \theta_1} v$$

$$\Rightarrow \begin{cases} \omega_2^T u = \omega_2^T z \\ \omega_1^T v = \omega_1^T z \end{cases}, \|u\|^2 = \|z\|^2 = \|v\|^2$$

## 3.2 Inverse Kinematics

**Solution:** If two axes of  $\xi_1$  and  $\xi_2$  coincide, then we get:

$$\text{Subproblem 1: } \theta_1 + \theta_2 = \theta$$

If the two axes are not parallel,  $\omega_1 \times \omega_2 \neq 0$ , then, let  $c$  satisfy:

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Set  $r \in l_{\xi_1} \cap l_{\xi_2}$

$$e^{\hat{\xi}_2 \theta_2} \underbrace{p - r}_{u} = \underbrace{c - r}_{z} = e^{-\hat{\xi}_1 \theta_1} \underbrace{(q - r)}_{v}, \Rightarrow e^{\hat{\omega}_2 \theta_2} u = z = e^{-\hat{\omega}_1 \theta_1} v$$

$$\Rightarrow \begin{cases} \omega_2^T u = \omega_2^T z \\ \omega_1^T v = \omega_1^T z \end{cases}, \|u\|^2 = \|z\|^2 = \|v\|^2$$

As  $\omega_1, \omega_2$  and  $\omega_1 \times \omega_2$  are linearly independent,

$$z = \alpha \omega_1 + \beta \omega_2 + \gamma (\omega_1 \times \omega_2)$$

$$\Rightarrow \|z\|^2 = \alpha^2 + \beta^2 + 2\alpha\beta \omega_1^T \omega_2 + \gamma^2 \|\omega_1 \times \omega_2\|^2$$

(Continues next slide)

## 3.2 Inverse Kinematics

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$$\begin{aligned}\omega_2^T u &= \alpha \omega_2^T \omega_1 + \beta \\ \omega_1^T v &= \alpha + \beta \omega_1^T \omega_2\end{aligned} \Rightarrow \begin{cases} \alpha &= \frac{(\omega_1^T \omega_2) \omega_2^T u - \omega_1^T v}{(\omega_1^T \omega_2)^2 - 1} \\ \beta &= \frac{(\omega_1^T \omega_2) \omega_1^T v - \omega_2^T u}{(\omega_1^T \omega_2)^2 - 1} \end{cases}$$

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## 3.2 Inverse Kinematics

$$\begin{aligned}\omega_2^T u &= \alpha \omega_2^T \omega_1 + \beta \\ \omega_1^T v &= \alpha + \beta \omega_1^T \omega_2\end{aligned} \Rightarrow \begin{cases} \alpha &= \frac{(\omega_1^T \omega_2) \omega_2^T u - \omega_1^T v}{(\omega_1^T \omega_2)^2 - 1} \\ \beta &= \frac{(\omega_1^T \omega_2) \omega_1^T v - \omega_2^T u}{(\omega_1^T \omega_2)^2 - 1} \end{cases}$$

$$\|z\|^2 = \|u\|^2 \Rightarrow \gamma^2 = \frac{\|u\|^2 - \alpha^2 - \beta^2 - 2\alpha\beta\omega_1^T \omega_2}{\|\omega_1 \times \omega_2\|^2} \quad (*)$$

(\*) has zero, one or two solution(s):

$$\text{Given } z \Rightarrow c \Rightarrow \begin{cases} e^{\hat{\xi}_2 \theta_2} p = c \\ e^{\hat{\xi}_1 \theta_1} p = c \end{cases}$$

for  $\theta_1$  and  $\theta_2$

- 1 Two solutions when the two circles intersect.

## 3.2 Inverse Kinematics

$$\begin{aligned}\omega_2^T u &= \alpha \omega_2^T \omega_1 + \beta \\ \omega_1^T v &= \alpha + \beta \omega_1^T \omega_2\end{aligned} \Rightarrow \begin{cases} \alpha &= \frac{(\omega_1^T \omega_2) \omega_2^T u - \omega_1^T v}{(\omega_1^T \omega_2)^2 - 1} \\ \beta &= \frac{(\omega_1^T \omega_2) \omega_1^T v - \omega_2^T u}{(\omega_1^T \omega_2)^2 - 1} \end{cases}$$

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for  $\theta_1$  and  $\theta_2$

- 1** Two solutions when the two circles intersect.
- 2** One solution when they are tangent

## 3.2 Inverse Kinematics

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$$\begin{aligned}\omega_2^T u &= \alpha \omega_2^T \omega_1 + \beta \\ \omega_1^T v &= \alpha + \beta \omega_1^T \omega_2\end{aligned} \Rightarrow \begin{cases} \alpha &= \frac{(\omega_1^T \omega_2) \omega_2^T u - \omega_1^T v}{(\omega_1^T \omega_2)^2 - 1} \\ \beta &= \frac{(\omega_1^T \omega_2) \omega_1^T v - \omega_2^T u}{(\omega_1^T \omega_2)^2 - 1} \end{cases}$$

$$\|z\|^2 = \|u\|^2 \Rightarrow \gamma^2 = \frac{\|u\|^2 - \alpha^2 - \beta^2 - 2\alpha\beta\omega_1^T \omega_2}{\|\omega_1 \times \omega_2\|^2} \quad (*)$$

(\*) has zero, one or two solution(s):

$$\text{Given } z \Rightarrow c \Rightarrow \begin{cases} e^{\hat{\xi}_2 \theta_2} p = c \\ e^{\hat{\xi}_1 \theta_1} p = c \end{cases}$$

for  $\theta_1$  and  $\theta_2$

- 1** Two solutions when the two circles intersect.
- 2** One solution when they are tangent
- 3** Zero solution when they do not intersect

## 3.2 Inverse Kinematics

### Subproblem 3: Rotation to a given point

Given a zero-pitch twist  $\xi$ , with unit magnitude and  $p, q \in \mathbb{R}^3$ , find  $\theta$  s.t.  $\|q - e^{\hat{\xi}\theta}p\| = \delta$

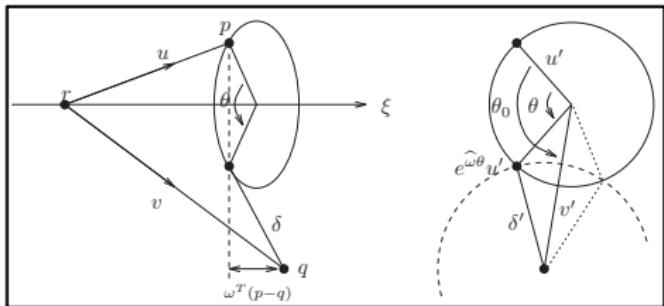


Figure 3.7

## 3.2 Inverse Kinematics

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### Subproblem 3: Rotation to a given point

Given a zero-pitch twist  $\xi$ , with unit magnitude and  $p, q \in \mathbb{R}^3$ , find  $\theta$  s.t.  $\|q - e^{\hat{\xi}\theta} p\| = \delta$

Define:  $u = p - r, v = q - r, \|v - e^{\hat{\omega}\theta} u\|^2 = \delta^2$

$$u' = u - \omega \omega^T u$$

$$v' = v - \omega \omega^T v$$

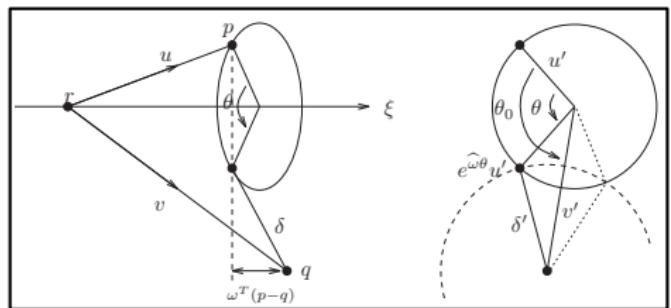


Figure 3.7

$$\Rightarrow u = u' + \omega \omega^T u, v = v' + \omega \omega^T v, \delta'^2 = \delta^2 - |\omega^T(p - q)|^2$$

## 3.2 Inverse Kinematics

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$$\|(\nu' + \omega\omega^T\nu) - e^{\hat{\omega}\theta}(u' + \omega\omega^Tu)\|^2 = \delta^2 \Rightarrow$$
$$\|\nu' - e^{\hat{\omega}\theta}u' + \underbrace{\omega\omega^T(\nu - u)}_{\omega\omega^T(q-p)}\|^2 = \delta^2$$

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$$\|(\nu' + \omega\omega^T\nu) - e^{\hat{\omega}\theta}(u' + \omega\omega^Tu)\|^2 = \delta^2 \Rightarrow \\ \| \nu' - e^{\hat{\omega}\theta}u' + \underbrace{\omega\omega^T(\nu - u)}_{\omega\omega^T(q-p)} \| ^2 = \delta^2$$

$$\|\nu' - e^{\hat{\omega}\theta}u'\|^2 = \delta^2 - \|\omega^T(p-q)\|^2 = \delta'^2, \\ \theta_0 = \text{atan2}(\omega^T(u' \times \nu'), u'^T\nu'), \\ \phi = \theta_0 - \theta \Rightarrow \|u'\|^2 + \|\nu'\|^2 - 2\|u'\| \cdot \|\nu'\| \cos \phi = \delta'^2, \\ \theta = \theta_0 \pm \cos^{-1} \frac{\|u'\| + \|\nu'\| - \delta'^2}{2\|u'\| \cdot \|\nu'\|} \quad (*)$$

Zero, one or two solutions!

## 3.2 Inverse Kinematics

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### □ Solving inverse kinematics using subproblems:

#### Technique 1: Eliminate the dependence on a joint

$e^{\hat{\xi}\theta} p = p$ , if  $p \in l_{\xi}$ . Given  $e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} = g$ , select  $p \in l_{\xi_3}$ ,  $p \notin l_{\xi_1}$  or  $l_{\xi_2}$ , then:

$$gp = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p$$

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## 3.2 Inverse Kinematics

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### □ Solving inverse kinematics using subproblems:

#### Technique 1: Eliminate the dependence on a joint

$e^{\hat{\xi}\theta} p = p$ , if  $p \in l_{\xi}$ . Given  $e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} = g$ , select  $p \in l_{\xi_3}$ ,  $p \notin l_{\xi_1}$  or  $l_{\xi_2}$ , then:

$$gp = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} p$$

#### Technique 2: subtract a common point

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} = g, q \in l_{\hat{\xi}_1} \cap l_{\hat{\xi}_2} \Rightarrow e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} p - q = gp - q \Rightarrow$$

$$\|e^{\hat{\xi}_3\theta_3} p - q\| = \|gp - q\|$$

## 3.2 Inverse Kinematics

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### ◇ Example: Elbow manipulator

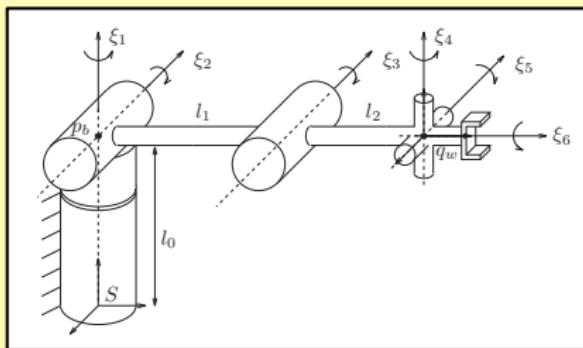


Figure 3.7

$$\begin{aligned}
 g_{st}(\theta) &= e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} g_{st}(0) = g_d \\
 \Rightarrow e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} &= g_d \cdot g_{st}^{-1}(0) = g_1
 \end{aligned}$$

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## 3.2 Inverse Kinematics

### Step 1: Solve for $\theta_3$

Let  $e^{\hat{\xi}_1\theta_1} \dots e^{\hat{\xi}_6\theta_6} q_\omega = g_1 \cdot q_\omega$

$$\Rightarrow e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} q_\omega = g_1 \cdot q_\omega$$

Subtract  $p_b$  from  $g_1 q_\omega$ :

$$\|e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} (e^{\hat{\xi}_3\theta_3} q_\omega - p_b)\| = \|g_1 q_\omega - p_b\|$$

$$\Rightarrow \|e^{\hat{\xi}_3\theta_3} q_\omega - p_b\| \triangleq \delta \leftarrow \text{Subproblem 3}$$

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## 3.2 Inverse Kinematics

### Step 1: Solve for $\theta_3$

Let  $e^{\hat{\xi}_1\theta_1} \dots e^{\hat{\xi}_6\theta_6} q_\omega = g_1 \cdot q_\omega$

$$\Rightarrow e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} q_\omega = g_1 \cdot q_\omega$$

Subtract  $p_b$  from  $g_1 q_\omega$ :

$$\|e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} (e^{\hat{\xi}_3\theta_3} q_\omega - p_b)\| = \|g_1 q_\omega - p_b\|$$

$$\Rightarrow \|e^{\hat{\xi}_3\theta_3} q_\omega - p_b\| \triangleq \delta \leftarrow \text{Subproblem 3}$$

### Step 2: Given $\theta_3$ , solve for $\theta_1, \theta_2$

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} (e^{\hat{\xi}_3\theta_3} q_\omega) = g_1 q_\omega, \text{ Subproblem 2} \Rightarrow \theta_1, \theta_2$$

(Continues next slide)

## 3.2 Inverse Kinematics

**Step 3: Given  $\theta_1, \theta_2, \theta_3$ , solve  $\theta_4, \theta_5$**

$$e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} e^{\hat{\xi}_6\theta_6} = \underbrace{e^{-\hat{\xi}_3\theta_3} e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1} g_1}_{g_2}$$

let  $p \in l_{\xi_6}, p \notin l_{\xi_4}$  or  $l_{\xi_5}, e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5\theta_5} p = g_2 p$ ,  
Subproblem 2  $\Rightarrow \theta_4$  and  $\theta_5$ .

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## 3.2 Inverse Kinematics

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**Step 3: Given  $\theta_1, \theta_2, \theta_3$ , solve  $\theta_4, \theta_5$**

$$e^{\hat{\xi}_4 \theta_4} e^{\hat{\xi}_5 \theta_5} e^{\hat{\xi}_6 \theta_6} = \underbrace{e^{-\hat{\xi}_3 \theta_3} e^{-\hat{\xi}_2 \theta_2} e^{-\hat{\xi}_1 \theta_1} g_1}_{g_2}$$

let  $p \in l_{\xi_6}, p \notin l_{\xi_4}$  or  $l_{\xi_5}, e^{\hat{\xi}_4 \theta_4} e^{\hat{\xi}_5 \theta_5} p = g_2 p$ ,  
Subproblem 2  $\Rightarrow \theta_4$  and  $\theta_5$ .

**Step 4: Given  $(\theta_1, \dots, \theta_5)$ , solve for  $\theta_6$**

$$e^{\hat{\xi}_6 \theta_6} = (e^{\hat{\xi}_1 \theta_1} \dots e^{\hat{\xi}_5 \theta_5})^{-1} \cdot g_1 \triangleq g_3$$

Let  $p \notin l_{\xi_6} \Rightarrow e^{\hat{\xi}_6 \theta_6} p = g_3 \cdot p = q \Leftarrow$  Subproblem 1  
Maximum of solutions: 8



## 3.2 Inverse Kinematics

### ◇ Example: Inverse Kinematics of SCARA

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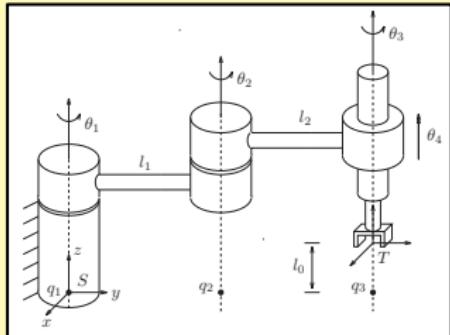


Figure 3.8

$$g_{st}(0) = \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_1 + l_2 \\ 0 & 0 & 1 & l_0 \\ \hline 0 & 0 & 0 & 1 \end{array} \right]$$

$$g_{st}(\theta) = e^{\hat{\xi}_1 \theta_1} \dots e^{\hat{\xi}_4 \theta_4} g_{st}(0)$$

$$= \left[ \begin{array}{ccc|c} c_\phi & -s_\phi & 0 & x \\ s_\phi & c_\phi & 0 & y \\ 0 & 0 & 1 & z \\ \hline 0 & 0 & 0 & 1 \end{array} \right] \triangleq g_d$$

$$p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow p(\theta) = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} \\ l_0 + \theta_4 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \Rightarrow \theta_4 = z - l_0$$

$$e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} = g_d g_{st}^{-1}(0) e^{-\hat{\xi}_4 \theta_4} \triangleq g_1$$

(Continues next slide)

## 3.2 Inverse Kinematics

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Let  $p \in l_{\xi_3}, q \in l_{\xi_1} \Rightarrow e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} p = g_1 p,$

$$\|e^{\hat{\xi}_1 \theta_1} (e^{\hat{\xi}_2 \theta_2} p - q)\| = \|g_1 p - q\|,$$

$$\|e^{\hat{\xi}_2 \theta_2} p - q\| = \delta \leftarrow \text{Subproblem 3 to get } \theta_2$$

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## 3.2 Inverse Kinematics

Let  $p \in l_{\xi_3}, q \in l_{\xi_1} \Rightarrow e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} p = g_1 p,$

$$\|e^{\hat{\xi}_1 \theta_1} (e^{\hat{\xi}_2 \theta_2} p - q)\| = \|g_1 p - q\|,$$

$$\|e^{\hat{\xi}_2 \theta_2} p - q\| = \delta \leftarrow \text{Subproblem 3 to get } \theta_2$$

$$\Rightarrow e^{\hat{\xi}_1 \theta_1} (e^{\hat{\xi}_2 \theta_2} p) = g_1 p \Rightarrow \theta_1 \leftarrow \text{Subproblem 1 to get } \theta_1$$

$$\Rightarrow e^{\hat{\xi}_3 \theta_3} = e^{-\hat{\xi}_2 \theta_2} e^{-\hat{\xi}_1 \theta_1} g_d g_{st}^{-1}(0) e^{-\hat{\xi}_4 \theta_4} \triangleq g_2$$

$$e^{\hat{\xi}_3 \theta_3} p = g_2 p, p \notin l_{\xi_3} \leftarrow \text{Subproblem 1 to get } \theta_3$$

There are a maximum of two solutions!



† End of Section †

## 3.2 Inverse Kinematics

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### ◇ Example: ABB IRB4400

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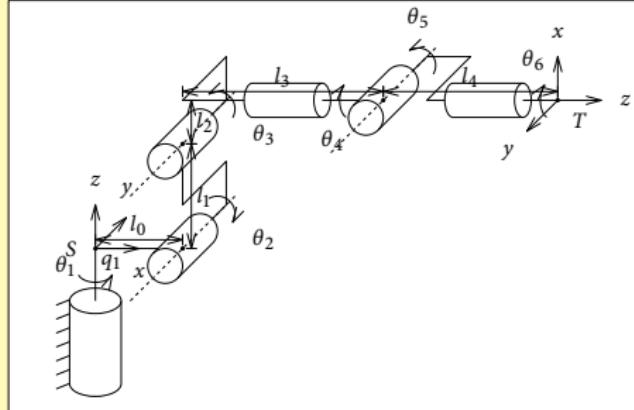
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$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \omega_2 = -\omega_3 = -\omega_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \omega_4 = \omega_6 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, q_2 = \begin{bmatrix} l_0 \\ 0 \\ 0 \end{bmatrix}, q_3 = \begin{bmatrix} l_0 \\ 0 \\ l_1 \end{bmatrix}, p_w := q_4 = q_5 = q_6 = \begin{bmatrix} l_0 + l_3 \\ 0 \\ l_1 + l_2 \end{bmatrix}$$

(Continues next slide)

## 3.2 Inverse Kinematics

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$$g_{st}(0) = \left[ \begin{array}{ccc|c} 0 & 0 & 1 & l_0 + l_3 + l_4 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & l_1 + l_2 \\ \hline 0 & 0 & 0 & 1 \end{array} \right], \xi_i = \left[ \begin{array}{c} q_i \times \omega_i \\ \omega_i \end{array} \right]$$

$$g_{st}(\theta) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} e^{\hat{\xi}_4 \theta_4} e^{\hat{\xi}_5 \theta_5} e^{\hat{\xi}_6 \theta_6} g_{st}(0) := g_d$$

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## 3.2 Inverse Kinematics

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$$g_{st}(0) = \left[ \begin{array}{ccc|c} 0 & 0 & 1 & l_0 + l_3 + l_4 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & l_1 + l_2 \\ \hline 0 & 0 & 0 & 1 \end{array} \right], \xi_i = \left[ \begin{array}{c} q_i \\ \omega_i \end{array} \right]$$

$$g_{st}(\theta) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} e^{\hat{\xi}_4 \theta_4} e^{\hat{\xi}_5 \theta_5} e^{\hat{\xi}_6 \theta_6} g_{st}(0) := g_d$$

$$e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} p_w = g_d p_w =: q \Rightarrow e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} p_w = e^{-\hat{\xi}_1 \theta_1} q$$

$$\Rightarrow 0 = [0 \ 1 \ 0] \cdot e^{-\hat{\xi}_1 \theta_1} q = \cos \theta_1 q_y - \sin \theta_1 q_x, q = \left[ \begin{array}{c} q_x \\ q_y \\ q_z \end{array} \right]$$

$$\Rightarrow \theta_1 = \tan^{-1}(q_y/q_x)$$

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## 3.2 Inverse Kinematics

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$$g_{st}(0) = \left[ \begin{array}{ccc|c} 0 & 0 & 1 & l_0 + l_3 + l_4 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & l_1 + l_2 \\ \hline 0 & 0 & 0 & 1 \end{array} \right], \xi_i = \left[ \begin{array}{c} q_i \\ \omega_i \end{array} \right]$$

$$g_{st}(\theta) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} e^{\hat{\xi}_4 \theta_4} e^{\hat{\xi}_5 \theta_5} e^{\hat{\xi}_6 \theta_6} g_{st}(0) := g_d$$

$$e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} p_w = g_d p_w =: q \Rightarrow e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} p_w = e^{-\hat{\xi}_1 \theta_1} q$$

$$\Rightarrow 0 = [0 \ 1 \ 0] \cdot e^{-\hat{\xi}_1 \theta_1} q = \cos \theta_1 q_y - \sin \theta_1 q_x, q = \left[ \begin{array}{c} q_x \\ q_y \\ q_z \end{array} \right]$$

$$\Rightarrow \theta_1 = \tan^{-1}(q_y/q_x)$$

$$\|e^{\hat{\xi}_3 \theta_3} p_w - q_2\| = \|e^{-\hat{\xi}_1 \theta_1} q - q_2\| =: \delta \leftarrow \text{Subproblem 3 to get } \theta_3$$

$$e^{\hat{\xi}_2 \theta_2} (e^{\hat{\xi}_3 \theta_3} p_w) = e^{-\hat{\xi}_1 \theta_1} q \leftarrow \text{Subproblem 1 to get } \theta_2$$

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## 3.2 Inverse Kinematics

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$$g_{st}(0) = \left[ \begin{array}{ccc|c} 0 & 0 & 1 & l_0 + l_3 + l_4 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & l_1 + l_2 \\ \hline 0 & 0 & 0 & 1 \end{array} \right], \xi_i = \left[ \begin{array}{c} q_i \times \omega_i \\ \omega_i \end{array} \right]$$

$$g_{st}(\theta) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} e^{\hat{\xi}_4 \theta_4} e^{\hat{\xi}_5 \theta_5} e^{\hat{\xi}_6 \theta_6} g_{st}(0) := g_d$$

$$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}p_w = g_d p_w =: q \Rightarrow e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}p_w = e^{-\hat{\xi}_1\theta_1}q$$

$$\Rightarrow 0 = [0 \ 1 \ 0] \cdot e^{-\xi_1 \theta_1} q = \cos \theta_1 q_y - \sin \theta_1 q_x, q = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}$$

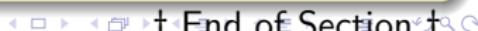
$$\Rightarrow \theta_1 = \tan^{-1}(q_y/q_x)$$

$$\|e^{\hat{\xi}_3\theta_3}p_w - q_2\| = \|e^{-\hat{\xi}_1\theta_1}q - q_2\| =: \delta \leftarrow \text{Subproblem 3 to get } \theta_3$$

$e^{\hat{\xi}_2 \theta_2} (e^{\hat{\xi}_3 \theta_3} p_w) = e^{-\hat{\xi}_1 \theta_1} q \leftarrow \text{Subproblem 1 to get } \theta_2$

$$e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}=e^{-\hat{\xi}_3\theta_3}e^{-\hat{\xi}_2\theta_2}e^{-\hat{\xi}_1\theta_1}g_dg_{st}^{-1}(0)=:g_2$$

Use subproblem 1,2 to solve for  $\theta_4, \theta_5, \theta_6$



End of Section

### 3.3 Manipulator Jacobian

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Given  $g_{st} : Q \rightarrow SE(3)$ ,

$$\theta(t) = (\theta_1(t) \dots \theta_n(t))^T \rightarrow e^{\hat{\xi}_1 \theta_1} \dots e^{\hat{\xi}_n \theta_n} g_{st}(0)$$

and  $\dot{\theta}(t) = (\dot{\theta}_1(t) \dots \dot{\theta}_n(t))^T$ ,

**What is the velocity of the tool frame?**

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### 3.3 Manipulator Jacobian

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Given  $g_{st} : Q \rightarrow SE(3)$ ,

$$\theta(t) = (\theta_1(t) \dots \theta_n(t))^T \rightarrow e^{\hat{\xi}_1 \theta_1} \dots e^{\hat{\xi}_n \theta_n} g_{st}(0)$$

and  $\dot{\theta}(t) = (\dot{\theta}_1(t) \dots \dot{\theta}_n(t))^T$ ,

**What is the velocity of the tool frame?**

$$\begin{aligned}\hat{V}_{st}^s &= \dot{g}_{st}(\theta)g_{st}^{-1}(\theta) = \sum_{i=1}^n \left( \frac{\partial g_{st}}{\partial \theta_i} \dot{\theta}_i \right) g_{st}^{-1}(\theta) \\ &= \sum_{i=1}^n \left( \frac{\partial g_{st}}{\partial \theta_i} g_{st}^{-1}(\theta) \right) \dot{\theta}_i \Rightarrow V_{st}^s = \sum_{i=1}^n \left( \frac{\partial g_{st}}{\partial \theta_i} g_{st}^{-1}(\theta) \right)^\vee \dot{\theta}_i \\ &= \underbrace{\left[ \left( \frac{\partial g_{st}}{\partial \theta_1} g_{st}^{-1}(\theta) \right)^\vee, \dots, \left( \frac{\partial g_{st}}{\partial \theta_n} g_{st}^{-1}(\theta) \right)^\vee \right]}_{J_{st}^s(\theta) \in \mathbb{R}^{6 \times n}} \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}\end{aligned}$$

### 3.3 Manipulator Jacobian

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$$g_{st}(\theta) = e^{\hat{\xi}_1 \theta_1} \dots e^{\hat{\xi}_n \theta_n} g_{st}(0)$$

$$\frac{\partial g_{st}}{\partial \theta_1} g_{st}^{-1}(\theta) = (\hat{\xi}_1 e^{\hat{\xi}_1 \theta_1} \dots e^{\hat{\xi}_n \theta_n} g_{st}(0)) (g_{st}(\theta))^{-1} = \hat{\xi}_1 \Rightarrow$$

$$(\frac{\partial g_{st}}{\partial \theta_1} g_{st}^{-1}(\theta))^{\vee} = \xi_1$$

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### 3.3 Manipulator Jacobian

$$g_{st}(\theta) = e^{\hat{\xi}_1 \theta_1} \dots e^{\hat{\xi}_n \theta_n} g_{st}(0)$$

$$\frac{\partial g_{st}}{\partial \theta_1} g_{st}^{-1}(\theta) = (\hat{\xi}_1 e^{\hat{\xi}_1 \theta_1} \dots e^{\hat{\xi}_n \theta_n} g_{st}(0)) (g_{st}(\theta))^{-1} = \hat{\xi}_1 \Rightarrow$$

$$(\frac{\partial g_{st}}{\partial \theta_1} g_{st}^{-1}(\theta))^{\vee} = \hat{\xi}_1$$

$$\begin{aligned} \frac{\partial g_{st}}{\partial \theta_2} g_{st}^{-1}(\theta) &= (e^{\hat{\xi}_1 \theta_1} \hat{\xi}_2 e^{\hat{\xi}_2 \theta_2} \dots e^{\hat{\xi}_n \theta_n} g_{st}(0)) (g_{st}(\theta))^{-1} \\ &= e^{\hat{\xi}_1 \theta_1} \hat{\xi}_2 e^{\hat{\xi}_2 \theta_2} \dots e^{\hat{\xi}_n \theta_n} g_{st}(0) g_{st}^{-1}(\theta) = e^{\hat{\xi}_1 \theta_1} \hat{\xi}_2 e^{-\hat{\xi}_1 \theta_1} \triangleq \hat{\xi}'_2 \end{aligned}$$

$$(\frac{\partial g_{st}}{\partial \theta_2} g_{st}^{-1}(\theta))^{\vee} = \text{Ad}_{e^{\hat{\xi}_1 \theta_1}} \hat{\xi}_2 = \hat{\xi}'_2 \dots \dots$$

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### 3.3 Manipulator Jacobian

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$$g_{st}(\theta) = e^{\hat{\xi}_1 \theta_1} \dots e^{\hat{\xi}_n \theta_n} g_{st}(0)$$

$$\frac{\partial g_{st}}{\partial \theta_1} g_{st}^{-1}(\theta) = (\hat{\xi}_1 e^{\hat{\xi}_1 \theta_1} \dots e^{\hat{\xi}_n \theta_n} g_{st}(0)) (g_{st}(\theta))^{-1} = \hat{\xi}_1 \Rightarrow$$

$$(\frac{\partial g_{st}}{\partial \theta_1} g_{st}^{-1}(\theta))^{\vee} = \xi_1$$

$$\begin{aligned} \frac{\partial g_{st}}{\partial \theta_2} g_{st}^{-1}(\theta) &= (e^{\hat{\xi}_1 \theta_1} \hat{\xi}_2 e^{\hat{\xi}_2 \theta_2} \dots e^{\hat{\xi}_n \theta_n} g_{st}(0)) (g_{st}(\theta))^{-1} \\ &= e^{\hat{\xi}_1 \theta_1} \hat{\xi}_2 e^{\hat{\xi}_2 \theta_2} \dots e^{\hat{\xi}_n \theta_n} g_{st}(0) g_{st}^{-1}(\theta) = e^{\hat{\xi}_1 \theta_1} \hat{\xi}_2 e^{-\hat{\xi}_1 \theta_1} \triangleq \hat{\xi}'_2 \end{aligned}$$

$$(\frac{\partial g_{st}}{\partial \theta_2} g_{st}^{-1}(\theta))^{\vee} = \text{Ad}_{e^{\hat{\xi}_1 \theta_1}} \xi_2 = \xi'_2 \dots \dots$$

$$(\frac{\partial g_{st}}{\partial \theta_i} g_{st}^{-1}(\theta))^{\vee} = \text{Ad}_{e^{\hat{\xi}_1 \theta_1} \dots e^{\hat{\xi}_{i-1} \theta_{i-1}}} \xi_i \triangleq \xi'_i$$

$$\Rightarrow J_{st}^s(\theta) = [\xi_1, \xi'_2, \dots, \xi'_n]$$

## 3.3 Manipulator Jacobian

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### □ Interpretation of $\xi'_i$ :

- $\xi'_i$  is only affected by  $\theta_1 \dots \theta_{i-1}$
- The twist associated with joint i, at the present configuration.

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# 3.3 Manipulator Jacobian

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## □ Interpretation of $\xi'_i$ :

- $\xi'_i$  is only affected by  $\theta_1 \dots \theta_{i-1}$
- The twist associated with joint i, at the present configuration.

## □ Body jacobian:

$$V_{st}^b = J_{st}^b(\theta) \cdot \dot{\theta}$$

$$J_{st}^b(\theta) = [\xi_1^\dagger \dots \xi_{n-1}^\dagger, \xi_n^\dagger]$$

$$\xi_i^\dagger = \text{Ad}_{e^{\hat{\xi}_{i+1}\theta_{i+1}} \dots e^{\hat{\xi}_n\theta_n} g_{st}(0)} \xi_i$$

Joint twist written with respect to the body frame at the current configuration!

$$J_{st}^s(\theta) = \text{Ad}_{g_{st}(\theta)} \cdot J_{st}^b(\theta)$$

If  $J_{st}^s$  is invertible,  $\dot{\theta}(t) = (J_{st}^s(\theta))^{-1} \cdot V_{st}^s(t)$

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# 3.3 Manipulator Jacobian

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Given  $g(t)$ , how to find  $\theta(t)$ ?

$$\left. \begin{array}{l} 1) \quad \dot{\hat{V}}_{st}^s = \dot{g}(t)g^{-1}(t) \\ 2) \quad \left\{ \begin{array}{l} \dot{\theta}(t) = (J_{st}^s(\theta))^{-1}\hat{V}_{st}^s(t) \\ \theta(0) = \theta_0 \end{array} \right. \end{array} \right\} \Rightarrow \theta(t)$$

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### 3.3 Manipulator Jacobian

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Given  $g(t)$ , how to find  $\theta(t)$ ?

$$\left. \begin{array}{l} 1) \quad \dot{\hat{V}}_{st}^s = \dot{g}(t)g^{-1}(t) \\ 2) \quad \left\{ \begin{array}{l} \dot{\theta}(t) = (J_{st}^s(\theta))^{-1}V_{st}^s(t) \\ \theta(0) = \theta_0 \end{array} \right. \end{array} \right\} \Rightarrow \theta(t)$$

◇ Example: Jacobian for a SCARA manipulator

$$q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, q'_2 = \begin{bmatrix} -l_1 s_1 \\ l_1 c_1 \\ 0 \end{bmatrix},$$

$$q'_3 = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} \\ 0 \end{bmatrix},$$

$$\omega_1 = \omega'_2 = \omega'_3 = [0 \ 0 \ 1]^T$$

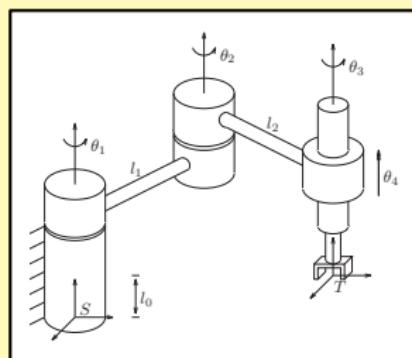


Figure 3.9  
(Continues next slide)

# 3.3 Manipulator Jacobian

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$$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix} = [ \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ ]^T$$

$$\xi'_2 = \begin{bmatrix} -\omega'_2 \times q'_2 \\ \omega'_2 \end{bmatrix} = [ \ l_1 c_1 \ l_1 s_1 \ 0 \ 0 \ 0 \ 0 \ ]^T$$

$$\xi'_3 = \begin{bmatrix} -\omega'_3 \times q'_3 \\ \omega'_3 \end{bmatrix} = [ \ l_1 c_1 + l_2 c_{12} \ l_1 s_1 + l_2 s_{12} \ 0 \ 0 \ 0 \ 1 \ ]^T$$

$$\xi'_4 = \begin{bmatrix} v'_4 \\ 0 \end{bmatrix} = [ \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ ]^T$$

$$J_{st}^s(\theta) = \begin{bmatrix} 0 & l_1 c_1 & l_1 c_1 + l_1 c_{12} & 0 \\ 0 & l_1 s_1 & l_1 s_1 + l_1 s_{12} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$



# 3.3 Manipulator Jacobian

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## ◇ Example: Jacobian of Stanford arm

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$$q_1 = q_2 = \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix},$$

$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\omega'_2 = \begin{bmatrix} -c_1 \\ -s_1 \\ 0 \end{bmatrix}$$

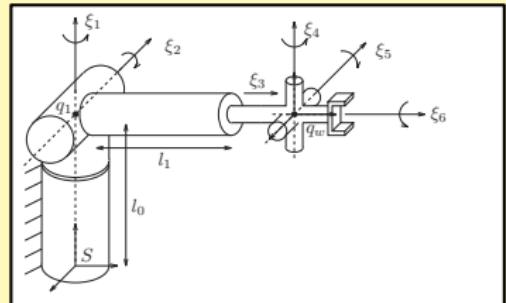


Figure 3.10

$$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xi'_2 = \begin{bmatrix} -\omega'_2 \times q_2 \\ \omega'_2 \end{bmatrix} = \begin{bmatrix} l_0 s_1 & l_0 c_1 & 0 & -c_1 & -s_1 & 0 \end{bmatrix}$$

$$\xi'_3 = \begin{bmatrix} e^{\hat{z}\theta_1} \cdot e^{-\hat{x}\theta_2} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ 0 \end{bmatrix} = \begin{bmatrix} -s_1 c_2 & c_1 c_2 & -s_2 & 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} v_3 \\ 0 \end{bmatrix}$$

(Continues next slide)

# 3.3 Manipulator Jacobian

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$$q'_\omega = \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} + e^{\hat{z}\theta_1} \cdot e^{-\hat{x}\theta_2} \cdot \begin{bmatrix} 0 \\ l_1 + \theta_3 \\ 0 \end{bmatrix} = \begin{bmatrix} -(l_1 + \theta_3)s_1c_2 \\ (l_1 + \theta_3)c_1c_2 \\ l_0 - (l_1 + \theta_3)s_2 \end{bmatrix}$$

$$\omega'_4 = e^{\hat{z}\theta_1} \cdot e^{-\hat{x}\theta_2} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -s_1s_2 \\ c_1s_2 \\ c_2 \end{bmatrix}$$

$$\omega'_5 = e^{\hat{z}\theta_1} \cdot e^{-\hat{x}\theta_2} \cdot e^{\hat{z}\theta_4} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_1c_4 + s_1c_2s_4 \\ -s_1c_4 - c_1c_2s_4 \\ s_2s_4 \end{bmatrix}$$

$$\omega'_6 = e^{\hat{z}\theta_1} \cdot e^{-\hat{x}\theta_2} \cdot e^{\hat{z}\theta_4} \cdot e^{-\hat{x}\theta_5} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -c_5(s_1c_2c_4) + s_1s_2s_5 \\ c_5(c_1c_2c_4 - s_1s_4) - c_1s_2s_5 \\ -s_2c_4c_5 - c_2s_5 \end{bmatrix}$$

$$J_{st}^s = \begin{bmatrix} 0 & -\omega'_2 \times q_1 & \nu'_3 & -\omega'_5 \times q'_\omega & -\omega'_5 \times q'_\omega & -\omega'_6 \times q'_\omega \\ \omega_1 & \omega'_2 & 0 & \omega'_4 & \omega'_5 & \omega'_6 \end{bmatrix}$$



## 3.3 Manipulator Jacobian

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### □ End-effector force:

$$F_t = \begin{bmatrix} \text{force} \\ \text{torque} \end{bmatrix}$$

$$W = \int_{t_1}^{t_2} V_{st}^b \cdot F_t dt = \int_{t_1}^{t_2} \dot{\theta} \cdot \tau dt = \int_{t_1}^{t_2} \dot{\theta}^T (J_{st}^b(\theta))^T \cdot F_t dt$$

$$\Rightarrow \tau = (J_{st}^b)^T F_t = (J_{st}^s)^T F_s$$

- Given  $F_t$ , what  $\tau$  is required to balance that force?
- If we apply a set of joint torques, what is the resulting end-effector wrench?

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### □ Structural force:

**Structural force:** that produces no work on admissible velocity space  $V^b$

$$F^b \cdot V^b = 0, \forall V^b \in \text{Im}J_{st}^b(\theta) \Rightarrow F^b \in (\text{Im}J_{st}^b)^{\perp}$$

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## □ Structural force:

**Structural force:** that produces no work on admissible velocity space  $V^b$

$$F^b \cdot V^b = 0, \forall V^b \in \text{Im}J_{st}^b(\theta) \Rightarrow F^b \in (\text{Im}J_{st}^b)^{\perp}$$

## ◊ Review:

$$\forall A \in \mathbb{R}^{m \times n}, \begin{cases} (\text{Im}A)^{\perp} = \ker A^T \\ (\ker A)^{\perp} = \text{Im}A^T \end{cases}$$

$$(\text{Im}J_{st}^b)^{\perp} = \ker(J_{st}^b)^T, \tau = (J_{st}^b)^T F^b \equiv 0, \forall F^b \in \ker(J_{st}^b)^T$$

# 3.3 Manipulator Jacobian

## ◇ Example: SCARA manipulator

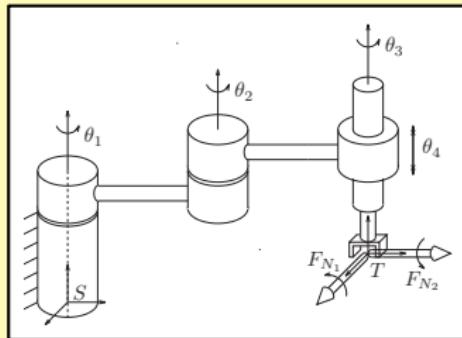


Figure 3.11

$$(J_{st}^s(\theta))^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ l_1 c_1 & l_1 s_1 & 0 & 0 & 0 & 1 \\ l_1 c_1 + l_1 c_{12} & l_1 s_1 + l_1 s_{12} & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$\ker((J_{st}^s(\theta))^T)$ : spanned by

$$F_{N_1} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$F_{N_2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



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# 3.3 Manipulator Jacobian

## □ Singularities:

### Definition:

$\theta$  is called a singular configuration if there  $\exists \dot{\theta} \neq 0$  s.t.

$$V_{st}^s = J_{st}^s(\theta)\dot{\theta} = 0$$

or, a singularity config. is a point  $\theta$  at which  $J_{st}^s$  drops rank.

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## 3.3 Manipulator Jacobian

### □ Singularities:

#### Definition:

$\theta$  is called a singular configuration if there  $\exists \dot{\theta} \neq 0$  s.t.

$$V_{st}^s = J_{st}^s(\theta)\dot{\theta} = 0$$

or, a singularity config. is a point  $\theta$  at which  $J_{st}^s$  drops rank.

Consequence: ( $n = 6$ )

- 1 Can't move in certain directions.
- 2 Large joint motion is required.
- 3 Large structural force.
- 4 Can't apply end-effector force in certain direction force!

# 3.3 Manipulator Jacobian

## □ Singularities for 6R-manipulators:

### Case 1: Two collinear revolute joints

$J(\theta)$  is singular if there exists two joints

$$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix}, \xi_2 = \begin{bmatrix} -\omega_2 \times q_2 \\ \omega_2 \end{bmatrix}$$

s.t.

- 1 The axes are parallel,  $\omega_1 = \pm \omega_2$
- 2 The axes are collinear,  $\omega_i \times (q_1 - q_2) = 0, i = 1, 2$

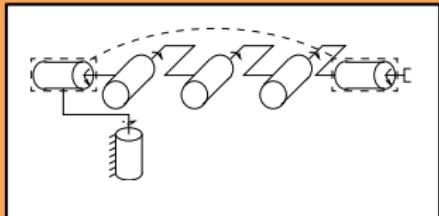


Figure 3.12

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## □ Singularities for 6R-manipulators:

### Case 1: Two collinear revolute joints

$J(\theta)$  is singular if there exists two joints

$$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix}, \xi_2 = \begin{bmatrix} -\omega_2 \times q_2 \\ \omega_2 \end{bmatrix}$$

s.t.

- 1 The axes are parallel,  $\omega_1 = \pm \omega_2$
- 2 The axes are collinear,  $\omega_i \times (q_1 - q_2) = 0, i = 1, 2$

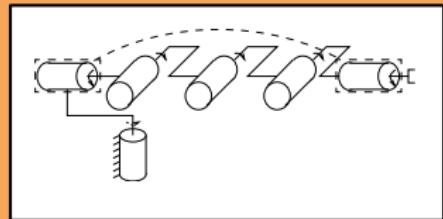


Figure 3.12

### Proof :

Elementary row or column operation do not change rank of  $J(\theta)$ :

$$\begin{aligned} J(\theta) &= \begin{bmatrix} -\omega_1 \times q_1 & -\omega_2 \times q_2 & \cdots \\ \omega_1 & \omega_2 & \cdots \end{bmatrix} \in \mathbb{R}^{6 \times n} \xrightarrow{\omega_1 = \omega_2} \\ J(\theta) &\sim \begin{bmatrix} -\omega_1 \times q_1 & -\omega_2 \times (q_2 - q_1) & \cdots \\ \omega_1 & 0 & \cdots \end{bmatrix} \\ &= \begin{bmatrix} -\omega_1 \times q_1 & 0 & \cdots \\ \omega_1 & 0 & \cdots \end{bmatrix} \end{aligned}$$



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# 3.3 Manipulator Jacobian

**Case 2: Three parallel coplanar revolute joint axes**

$J(\theta)$  is singular if there exists three joints s.t.

- 1 The axes are parallel,  $\omega_i = \pm\omega_j, i, j = 1, 2, 3$
- 2 The axes are coplanar, i.e. there exists a plane with normal  $n$  s.t.

$$n^T \omega_i = 0, n^T (q_i - q_j) = 0, i, j = 1, 2, 3$$

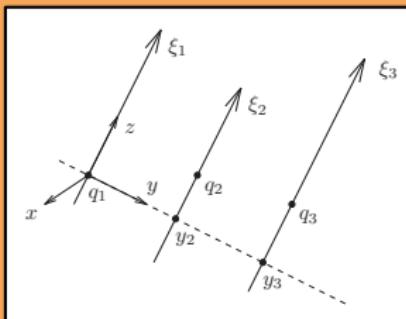


Figure 3.13

(Continues next slide)

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### Proof :

Using change of frame  $J \sim \text{Ad}_g J$  and assume

$$J(\theta) = \begin{bmatrix} -\omega_1 \times q_1 & -\omega_2 \times (q_2 - q_1) & \dots \\ \omega_1 & \omega_2 & \dots \end{bmatrix},$$

$$\text{Ad}_g J(\theta) = \begin{bmatrix} 0 & \pm y_2 & \pm y_3 & \dots \\ 0 & 0 & 0 & \dots \\ 1 & \pm 1 & \pm 1 & \dots \end{bmatrix}$$

Linearly dependent

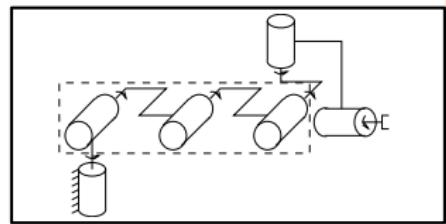


Figure 3.14

Examples are such as the Elbow manipulator in its reference configuration.

(Continues next slide)

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## Case 3: Four intersecting revolute joints axes

$J(\theta)$  is singular if there exists four concurrent revolute joints with intersection point  $q$  s.t.:

$$\omega_i \times (q_i - q) = 0, i = 1, \dots, 4$$

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### Case 3: Four intersecting revolute joints axes

$J(\theta)$  is singular if there exists four concurrent revolute joints with intersection point  $q$  s.t.:

$$\omega_i \times (q_i - q) = 0, i = 1, \dots, 4$$

### Proof :

Choose the frame origin at  $q$ ,

$$p = q_i, i = 1, \dots, 4$$

$$J(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \end{bmatrix}$$

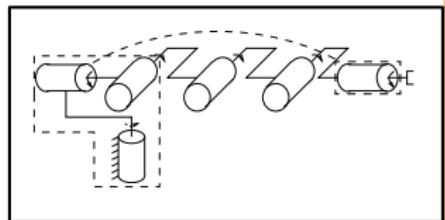


Figure 3.15

# 3.3 Manipulator Jacobian

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## □ Manipulability:

- **Jacobian relation of  $g : \theta \in Q \mapsto g(\theta) \in SE(3)$**

$$V = J(\theta)\dot{\theta} \quad (*)$$

- **Inverse Jacobian:**

Given  $v \in \mathbb{R}^n$ , solve for  $\dot{\theta} \in \mathbb{R}^n$  from  $(*)$



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# 3.3 Manipulator Jacobian

## □ Manipulability:

- **Jacobian relation of  $g : \theta \in Q \mapsto g(\theta) \in SE(3)$**

$$V = J(\theta)\dot{\theta} \quad (*)$$

- **Inverse Jacobian:**

Given  $v \in \mathbb{R}^n$ , solve for  $\dot{\theta} \in \mathbb{R}^n$  from  $(*)$



- **Application: Kinematic control by Inverse Jacobian**

- Input: A desired  $g_d(t) \in SE(3), t \in [0, T]$

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# 3.3 Manipulator Jacobian

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## □ Manipulability:

- **Jacobian relation of  $g : \theta \in Q \mapsto g(\theta) \in SE(3)$**

$$V = J(\theta)\dot{\theta} \quad (*)$$

- **Inverse Jacobian:**

Given  $v \in \mathbb{R}^n$ , solve for  $\dot{\theta} \in \mathbb{R}^n$  from  $(*)$



- **Application: Kinematic control by Inverse Jacobian**

- Input: A desired  $g_d(t) \in SE(3), t \in [0, T]$
- Output:  $\theta(k) = \theta(k\Delta T), \Delta T$  : Sampling period,  $k = 1, \dots, N = [T/\Delta T]$

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# 3.3 Manipulator Jacobian

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## □ Manipulability:

- **Jacobian relation of  $g : \theta \in Q \mapsto g(\theta) \in SE(3)$**

$$V = J(\theta)\dot{\theta} \quad (*)$$

- **Inverse Jacobian:**

Given  $v \in \mathbb{R}^n$ , solve for  $\dot{\theta} \in \mathbb{R}^n$  from  $(*)$



- **Application: Kinematic control by Inverse Jacobian**

- Input: A desired  $g_d(t) \in SE(3), t \in [0, T]$
- Output:  $\theta(k) = \theta(k\Delta T), \Delta T$ : Sampling period,  $k = 1, \dots, N = [T/\Delta T]$
- Step 1: Let  $g_d(k+1) = g(k)e^{\hat{V}\Delta T} = g(\theta(k))e^{\hat{V}\Delta T}$ , solve for

$$\hat{V}\Delta T = \log(g^{-1}(k) \cdot g_d(k+1))$$

# 3.3 Manipulator Jacobian

## □ Manipulability:

- **Jacobian relation of  $g : \theta \in Q \mapsto g(\theta) \in SE(3)$**

$$V = J(\theta)\dot{\theta} \quad (*)$$

- **Inverse Jacobian:**

Given  $v \in \mathbb{R}^n$ , solve for  $\dot{\theta} \in \mathbb{R}^n$  from  $(*)$



- **Application: Kinematic control by Inverse Jacobian**

- Input: A desired  $g_d(t) \in SE(3), t \in [0, T]$
- Output:  $\theta(k) = \theta(k\Delta T), \Delta T$ : Sampling period,  $k = 1, \dots, N = [T/\Delta T]$
- Step 1: Let  $g_d(k+1) = g(k)e^{\hat{V}\Delta T} = g(\theta(k))e^{\hat{V}\Delta T}$ , solve for

$$\hat{V}\Delta T = \log(g^{-1}(k) \cdot g_d(k+1))$$

- Step 2: Solve for  $\dot{\theta}(k)$  from  $V = J(\theta(k)) \cdot \dot{\theta}(k)$  and update

$$\theta(k+1) = \theta(k) + \dot{\theta}(k)\Delta T$$

# 3.3 Manipulator Jacobian

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## ◊ Review: Singular Value Decomposition

Given  $A : \mathbb{R}^n \mapsto \mathbb{R}^m$ , let  $r = \text{rank}(A)$ , then:

$$\dim(R(A)) = \dim(R(A^T)) = r$$

$$\dim(\eta(A)) = n - r, \dim(\eta(A^T)) = m - r$$

$$\mathbb{R}^n = R(A^T) \oplus \eta(A)$$

$$\mathbb{R}^m = R(A) \oplus \eta(A^T)$$

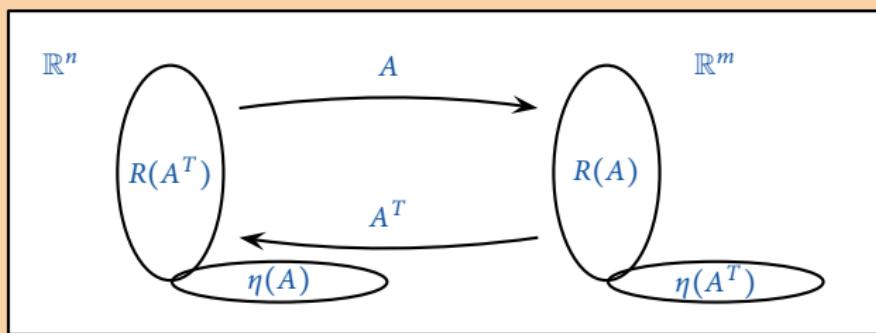


Figure 3.16

# 3.3 Manipulator Jacobian

SVD of  $A$ :

$$A = U\Sigma V^T$$

where:

$$U = [u_1 \cdots u_r, u_{r+1} \cdots u_m] \triangleq [U_1 | U_2] \in \mathbb{R}^{m \times m}$$

$$V = [v_1 \cdots v_r, v_{r+1} \cdots v_n] \triangleq [V_1 | V_2] \in \mathbb{R}^{n \times n}$$

are orthogonal, i.e.  $U \in O(m)$ ,  $V \in O(n)$ , or  $U^T U = I_m$ ,  $V^T V = I_n$ , and  $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_r, \dots, \sigma_p)$ ,  $\sigma_p = \min(m, n)$ , where  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > \sigma_{r+1} = \dots = \sigma_p = 0$

- $\sigma_i$ : Singular value of  $A$ ,  $\sigma_{\max}(A) = \sigma_1$
- $u_i, v_i$ :  $i^{\text{th}}$  left (right) singular vector of  $A$ :  $A v_i = \sigma_i u_i$   
 $A^T u_i = \sigma_i v_i$

## 3.3 Manipulator Jacobian

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### ◊ Review: Properties of SVD

- $A = \sum_{i=1}^r \sigma_i u_i v_i^T \Rightarrow (A^T A) v_k = \sigma_k^2 v_k$  or  $\lambda(A^T A) = \{ \sigma_1^2, \sigma_2^2, \dots, \sigma_r^2, 0, \dots, 0 \}$

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## 3.3 Manipulator Jacobian

### ◊ Review: Properties of SVD

- $A = \sum_{i=1}^r \sigma_i u_i v_i^T \Rightarrow (A^T A) v_k = \sigma_k^2 v_k$  or  $\lambda(A^T A) = \{ \sigma_1^2, \sigma_2^2, \dots, \sigma_r^2, 0, \dots, 0 \}$
- $\text{span}(V_1) = R(A^T), \text{span}(V_2) = \eta(A)$   
 $\text{span}(U_1) = R(A), \text{span}(U_2) = \eta(A^T)$

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## 3.3 Manipulator Jacobian

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### ◊ Review: Properties of SVD

- $A = \sum_{i=1}^r \sigma_i u_i v_i^T \Rightarrow (A^T A) v_k = \sigma_k^2 v_k$  or  $\lambda(A^T A) = \{ \sigma_1^2, \sigma_2^2, \dots, \sigma_r^2, 0, \dots, 0 \}$

- $\text{span}(V_1) = R(A^T)$ ,  $\text{span}(V_2) = \eta(A)$   
 $\text{span}(U_1) = R(A)$ ,  $\text{span}(U_2) = \eta(A^T)$
- Let  $n = m = r$ , then  $A$  maps the unit sphere  $S^{n-1} = \{x \in \mathbb{R}^n | \|x\|_2 = 1\}$  to an ellipsoid with semi-axes  $\sigma_i u_i$ .

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# 3.3 Manipulator Jacobian

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## ◊ Review: Properties of SVD

- $A = \sum_{i=1}^r \sigma_i u_i v_i^T \Rightarrow (A^T A) v_k = \sigma_k^2 v_k$  or  $\lambda(A^T A) = \{ \sigma_1^2, \sigma_2^2, \dots, \sigma_r^2, 0, \dots, 0 \}$

- $\text{span}(V_1) = R(A^T), \text{span}(V_2) = \eta(A)$   
 $\text{span}(U_1) = R(A), \text{span}(U_2) = \eta(A^T)$
- Let  $n = m = r$ , then  $A$  maps the unit sphere  $S^{n-1} = \{x \in \mathbb{R}^n | \|x\|_2 = 1\}$  to an ellipsoid with semi-axes  $\sigma_i u_i$ .

- $\|A\|_F^2 = \sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 = \sum_{i=1}^r \sigma_i^2, \|A\|_2 = \sigma_1$

# 3.3 Manipulator Jacobian

## ◊ Review: Properties of SVD

- $A = \sum_{i=1}^r \sigma_i u_i v_i^T \Rightarrow (A^T A) v_k = \sigma_k^2 v_k$  or  $\lambda(A^T A) = \{ \sigma_1^2, \sigma_2^2, \dots, \sigma_r^2, 0, \dots, 0 \}$

- $\text{span}(V_1) = R(A^T)$ ,  $\text{span}(V_2) = \eta(A)$   
 $\text{span}(U_1) = R(A)$ ,  $\text{span}(U_2) = \eta(A^T)$
- Let  $n = m = r$ , then  $A$  maps the unit sphere  $S^{n-1} = \{x \in \mathbb{R}^n | \|x\|_2 = 1\}$  to an ellipsoid with semi-axes  $\sigma_i u_i$ .

- $\|A\|_F^2 = \sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 = \sum_{i=1}^r \sigma_i^2$ ,  $\|A\|_2 = \sigma_1$

- Sensitivity Analysis for  $Ax = b$ ,  $m = n = r$   

$$A(x + \delta x) = b + \delta b \Rightarrow A\delta x = \delta b$$

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### 3.3 Manipulator Jacobian

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$$\begin{aligned}\frac{\|\delta x\|}{\|x\|} / \frac{\|\delta b\|}{\|b\|} &= \frac{\|b\|}{\|x\|} \frac{\|\delta x\|}{\|\delta b\|} = \frac{\|Ax\|}{\|x\|} \frac{\|A^{-1}\delta b\|}{\|\delta b\|} \\ &\leq \|A\| \|A^{-1}\| \\ &\triangleq k(A) := \frac{\sigma_1(A)}{\sigma_r(A)}, \text{ condition number, } k(A) \geq 1\end{aligned}$$

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### 3.3 Manipulator Jacobian

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$$\triangleq k(A) := \frac{\sigma_1(A)}{\sigma_r(A)}, \text{ condition number, } k(A) \geq 1$$

- Frobenius condition number:  $k_F(A) = \frac{1}{n} \sqrt{\text{tr}(AA^T)\text{tr}(AA^T)^{-1}}$

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### 3.3 Manipulator Jacobian

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$$\frac{\|\delta x\|}{\|x\|} / \frac{\|\delta b\|}{\|b\|} = \frac{\|b\|}{\|x\|} \frac{\|\delta x\|}{\|\delta b\|} = \frac{\|Ax\|}{\|x\|} \frac{\|A^{-1}\delta b\|}{\|\delta b\|} \\ \leq \|A\| \|A^{-1}\|$$

$$\triangleq k(A) := \frac{\sigma_1(A)}{\sigma_r(A)}, \text{ condition number, } k(A) \geq 1$$

- Frobenius condition number:  $k_F(A) = \frac{1}{n} \sqrt{\text{tr}(AA^T)\text{tr}(AA^T)^{-1}}$
- Manipulability Measures:

$$\mu_1(\theta) = \sigma_{\min}(J(\theta))$$

$$\mu_2(\theta) = \frac{\sigma_{\min}(J(\theta))}{\sigma_{\max}(J(\theta))} \triangleq k^{-1}(J(\theta))$$

$$\mu_3(\theta) = \det(J(\theta)) = \prod_{i=1}^n \sigma_i(J(\theta))$$

$$\mu_4(\theta) = k_F^{-1}(J(\theta))$$



## 3.4 Redundant Manipulators

### □ Redundant manipulator:

#### **Definition:**

A manipulator is kinematically redundant if the number of independently controllable joints is greater than the dimension of the task space.

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## 3.4 Redundant Manipulators

### □ Redundant manipulator:

#### Definition:

A manipulator is kinematically redundant if the number of independently controllable joints is greater than the dimension of the task space.

### ◊ Example:

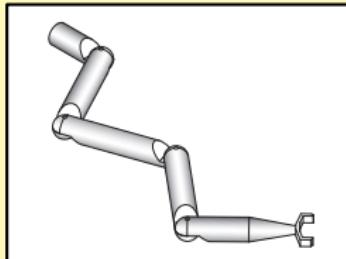
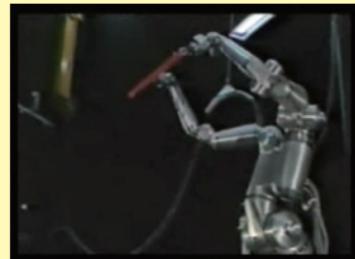


Figure 3.17

$$T = SE(2)$$

$$Q = S^1 \times S^1 \times S^1 \times S^1$$



Play/Pause Stop

Figure 3.18

$$T = SE(3), Q = \Gamma^n$$

## 3.4 Redundant Manipulators

### □ Redundant manipulator:



Figure 3.19: 17-DoF manipulator



Play/Pause  
Stop

Figure 3.20: DLR hand of 4 identical fingers with 4 joints and 3 degrees of freedom each.



Play/Pause  
Stop

Figure 3.21: Honda's Asimo with 34 DoF (3 in the head, 7 in each arm, 2 in each hand, 1 in the torso, 6 in each leg.)



Play/Pause  
Stop

Figure 3.22: OCTARM, a hyper-redundant (continuum) manipulator with 27 DoF

# 3.4 Redundant Manipulators

## • Main use of Redundancy

- Avoid singularities, joint limits and workspace obstacles;
- Optimize certain cost such as joint torque and energy

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# 3.4 Redundant Manipulators

- Main use of Redundancy

- Avoid singularities, joint limits and workspace obstacles;
- Optimize certain cost such as joint torque and energy

- Self-Motion Manifold and Internal Motion:

- Forward Kinematic Map

$$g_{st}(\theta) = e^{\hat{\xi}_1 \theta_1} \cdots e^{\hat{\xi}_n \theta_n} g_{st}(0), n > p, \text{ task space dimension.}$$

$r = n - p (\geq 1)$  : Degree of redundancy

$r \gg 1$  : Hyperredundant

- Jacobian

$$J(\theta)\dot{\theta} = V, V \in \mathbb{R}^p, \dot{\theta} \in \mathbb{R}^n$$

# 3.4 Redundant Manipulators

- **Self-motion manifold**

$$Q_s = \{\theta \in Q | g_{st}(\theta) = g_d\}$$

- **Internal motion space**

$$T_\theta Q_s = \{\dot{\theta} \in T_\theta Q | J(\theta)\dot{\theta} = 0\} \subset T_\theta Q$$

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# 3.4 Redundant Manipulators

- **Self-motion manifold**

$$Q_s = \{\theta \in Q | g_{st}(\theta) = g_d\}$$

- **Internal motion space**

$$T_\theta Q_s = \{\dot{\theta} \in T_\theta Q | J(\theta)\dot{\theta} = 0\} \subset T_\theta Q$$

◊ **Example:**

$$\begin{cases} l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ + l_3 \cos(\theta_1 + \theta_2 + \theta_3) = x \\ l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2) \\ + l_3 \sin(\theta_1 + \theta_2 + \theta_3) = y \end{cases}$$

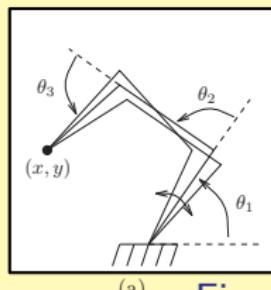
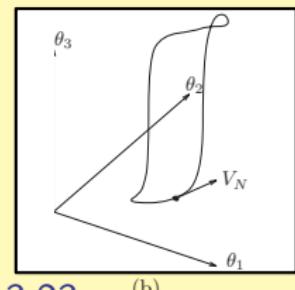


Figure 3.23



$$\frac{\partial p}{\partial \theta} = \left[ \begin{array}{c|c|c} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ \hline l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \end{array} \right],$$

$$v_N = \left[ \begin{array}{c} l_2 l_3 s_3 \\ -l_2 l_3 s_3 - l_1 l_3 s_{23} \\ l_1 l_2 s_3 + l_1 l_3 s_{23} \end{array} \right]$$



## 3.4 Redundant Manipulators

### ◊ Example:

A representation

Fix your palm on the table, and then move your shoulder and elbow joints. This gives the self-motion manifold and the internal motion of the 7-DoF redundant robot shown in Fig. 1.

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## 3.4 Redundant Manipulators

### ◊ Example:

A representation

Fix your palm on the table, and then move your shoulder and elbow joints. This gives the self-motion manifold and the internal motion of the 7-DoF redundant robot shown in Fig. 1.

### □ Redundancy Resolution:

$$V = J\dot{\theta}, J \in \mathbb{R}^{p \times n}, n > p, \text{rank}(J) = k \leq p < n$$

Strategy for  $\dot{\theta} \in \mathbb{R}^n$  given  $V \in \mathbb{R}^p$ .

Case 1:  $k = p$

Case 2:  $k < p$

# 3.4 Redundant Manipulators

## ◊ Review: Least Square Problems

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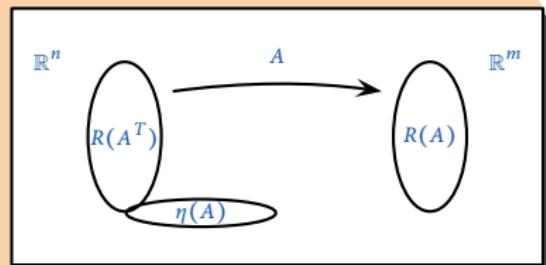
$$Ax = b, A \in \mathbb{R}^{m \times n}, n > m, \text{rank}(A) = k.$$

**Case 1:**

$$k = m \Rightarrow \dim(\eta(A)) = n - k > 0, \\ R(A) = \mathbb{R}^m$$

**P1:**  $\min_{x \in \mathbb{R}^n} \frac{1}{2} \|x\|^2$

s.t.  $Ax - b = 0$



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## 3.4 Redundant Manipulators

**Solution:**  $\varphi(x, \lambda) = \frac{1}{2} \|x\|^2 - \lambda(Ax - b)$ ,  $\lambda \in \mathbb{R}^m$

$$\left( \frac{\partial \varphi}{\partial x} \right)^T = x - A^T \lambda = 0 \Rightarrow x = A^T \lambda$$

$$\left( \frac{\partial \varphi}{\partial \lambda} \right)^T = Ax - b = 0 \Rightarrow AA^T \lambda = b$$

$$\Rightarrow x = A^T (AA^T)^{-1} b \\ \triangleq A^+ b$$

$A^+ = A^T (AA^T)^{-1} \in \mathbb{R}^{m \times m}$ : Moore-Penrose Inverse

In terms of SVD:  $A = U\Sigma V^T = \sum_{i=1}^m \sigma_i u_i v_i^T \Rightarrow A^+ = \sum_{i=1}^m \frac{1}{\sigma_i} v_i u_i^T$

$$x = \left( \sum_{i=1}^m \frac{1}{\sigma_i} v_i u_i^T \right) b = \sum_{i=1}^m \frac{u_i^T b}{\sigma_i} v_i$$

# 3.4 Redundant Manipulators

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## Case 2:

$$k < m \Rightarrow \dim(\eta(A^T)) = m - k$$

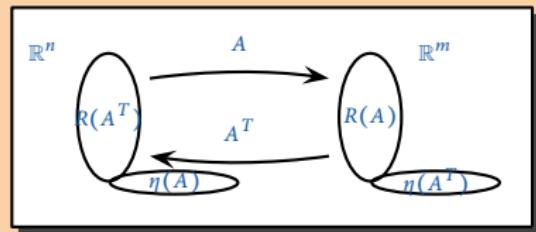
$$\text{P2: } \min_{x \in \mathbb{R}^n} f(x)$$

$$f(x) = \|Ax - b\|^2 + \lambda^2 \|x\|^2$$

$$\text{Solution: } \left( \frac{\partial f}{\partial x} \right)^T = (AA^T + \lambda^2 I)x - A^T b = 0$$

$$\Rightarrow x = (A^T A + \lambda^2 I)A^T b$$

$$= \sum_{i=1}^k \frac{\sigma_i}{\sigma_i^2 + \lambda^2} v_i u_i^T b$$



## 3.4 Redundant Manipulators

- **Redundancy Resolution:**

$$J\dot{\theta} = V, J \in \mathbb{R}^{p \times n}$$

**Case 1:** If  $\text{rank}(J) = p$ , the minimum-norm solution is given by

$$\boxed{\dot{\theta} = J^+ V = \sum_{i=1}^p \frac{u_i^T V}{\sigma_i} v_i} \quad (*)$$

- If  $\sigma_i \ll 1$ , then for  $V = u_i$ ,  $\|V\| = 1$ ,  $\dot{\theta} = \frac{1}{\sigma_i} v_i \Rightarrow \|\dot{\theta}\| \frac{1}{\sigma_i} \gg 1$ , large joint rate needed.
- For cyclic trajectory in task space, (\*) does not give cyclic trajectory in joint space (see [13] Chapter 2)

# 3.4 Redundant Manipulators

## General Solution:

$$\dot{\theta} = \underbrace{J^+ V}_{\in R(J^T)} + \underbrace{(I - J^+ J)\dot{\theta}_0}_{\in \eta(J)}, \dot{\theta}_0 \in \mathbb{R}^n$$

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# 3.4 Redundant Manipulators

## General Solution:

$$\dot{\theta} = \underbrace{J^+ V}_{\in R(J^T)} + \underbrace{(I - J^+ J)\dot{\theta}_0}_{\in \eta(J)}, \dot{\theta}_0 \in \mathbb{R}^n$$

- How to select  $\dot{\theta}_0 \in \mathbb{R}^n$  so as to stay away from singularity, joint limits or workspace obstacles?

Let

$$\varphi(\theta) = \begin{cases} \mu_1^{-1}(\theta) = \sigma_{\min}^{-1}(J) \\ \mu_2^{-1}(\theta) = \frac{\sigma_{\max}(J)}{\sigma_{\min}(J)} \\ \mu_3^{-1}(\theta) = \frac{1}{\det^{1/2}(JJ^{-1})} \end{cases}$$

for singularity avoidance.

(Continues next slide)

## 3.4 Redundant Manipulators

or

$$\varphi(\theta) = \frac{1}{2} \sum_{i=1}^n \left( \frac{\theta_i - \theta_{i,\text{mid}}}{\theta_{i,\text{max}} - \theta_{i,\text{min}}} \right)^2$$

for avoiding joint limits. Then,

$$\dot{\theta} = J^+ V - \lambda_\varphi \nabla \varphi(\theta) \quad (\Delta)$$

where  $\nabla \varphi(\theta) \in \mathbb{R}^n$  : gradient of  $\varphi$ ,  
 $\lambda_\varphi \in \mathbb{R}$  : step size (see [13] on selection of  $\lambda_\varphi$ )

Note  $(\Delta)$  minimizes

$$L(\theta, \dot{\theta}) = \frac{1}{2} \dot{\theta}^T \dot{\theta} + K_\varphi \dot{\theta}^T \nabla \varphi(\theta)$$

## 3.4 Redundant Manipulators

- Damped Least-Square:

$$J\dot{\theta} = V, J \in \mathbb{R}^{p \times n}, p < n$$

$$\begin{aligned}\dot{\theta} &= (J^T J + \lambda^2 I)^{-1} J^T V \\ &= J^T (J^T J + \lambda^2 I)^{-1} V\end{aligned}$$

$\lambda$  : Dampening coefficient. See [10] on selection of  $\lambda$ .

In terms of SVD:

$$\dot{\theta} = \sum_{i=1}^k \frac{\sigma_i}{\sigma_i^2 + \lambda^2} v_i u_i^T V$$

† End of Section †

# 3.5 References

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