Name:
Math 126 Midterm
60 points
7/14/2016

Formula Sheet

$$\Delta_u = \nabla \cdot (\nabla u) = u_{xx} + u_{yy} + u_{zz} \qquad \nabla u = (u_x, u_y, u_z) \qquad \nabla \cdot F = F_x + F_y + F_z$$

$$\iiint_D \nabla \cdot F dx = \iint_{\partial D} F \cdot n dS \qquad \int_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$E = \frac{1}{2} \int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2) dx \qquad c = \sqrt{\frac{T}{\rho}}$$

$$S(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-x^2/(4kt)} \qquad Erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} dp$$

$$\iint_\Delta f = \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s) dy ds$$

$$u(x,t) = \sum_n \left(A_n \cos \frac{n\pi c t}{l} + B_n \sin \frac{n\pi c t}{l} \right) \sin \frac{n\pi x}{l} \qquad \phi(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{l} + B_n \sin \frac{n\pi x}{l} \right)$$

$$A_n = \frac{1}{l} \int_{-l}^{l} \phi(x) \cos \frac{n\pi x}{l} dx \qquad B_n = \frac{1}{l} \int_{-l}^{l} \phi(x) \sin \frac{n\pi x}{l} dx \qquad c_n = \frac{1}{2l} \int_{-l}^{l} \phi(x) e^{-in\pi x/l} dx$$

$$K_N(\theta) = 1 + 2 \sum_{n=1}^{N} \cos n\theta = \frac{\sin \left(N + \frac{1}{2} \right) \theta}{\sin \frac{1}{2} \theta}$$

$$u(r,\theta) = (a^2 - r^2) \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2} \frac{d\phi}{2\pi}$$

$$\iint_{\partial D} v \frac{\partial u}{\partial n} dS = \iiint_D \nabla v \cdot \nabla u dx + \iiint_D v \Delta u dx \qquad \iiint_D (u \Delta v - v \Delta u) dx = \iint_{\partial D} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS$$

$$u(x_0) = \iint_{\partial D} \left[-u(x) \frac{\partial}{\partial n} \left(\frac{1}{|x - x_0|} \right) + \frac{1}{|x - x_0|} \frac{\partial u}{\partial n} \right] \frac{dS}{4\pi}$$

$$u(x_0) = \frac{1}{2\pi} \int_{\partial D} \left[u(x) \frac{\partial}{\partial n} (\log |x - x_0|) - \frac{\partial u}{\partial n} \log |x - x_0| \right] ds$$

Score: 1) ___/5 2) ___/5 3) ___/10 4) ___/15 5) ___/10 6) ___/15 Total:___/60

1)(5 points) Use separation of variables to reduce

$$0 = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

to 2 ODEs. (Do NOT solve the ODEs.)

2)(5 points) Find the solution to $\Delta u = (x^2 + y^2)$ on the disc $x^2 + y^2 < 1$ with boundary conditions $u = 0$ on $x^2 + y^2 = 1$.

3)(10 points) a) State the (weak) Maximum Principle for the heat equation on the interval $0 \leqslant x \leqslant l$. (4 points)
b) Using actual words, prove the uniqueness of solutions to the Dirichlet problem for the heat equation (6 points):

$$u_t - ku_{xx} = f(x,t) \qquad -\infty < x < \infty, t > 0$$
$$u(0,t) = g(t) \qquad u(l,t) = h(t) \qquad t > 0$$
$$u(x,0) = \phi(x) \qquad 0 < x < l$$

4)(15 points) a) Find the general solution to the PDE $2xu_x + u_y = 0$. (5 points)
b) Solve it with the auxiliary condition $u(x,0) = x^2$. Comment on the well-posedness of this problem. (5 points)
c) Solve it with the auxiliary condition $u(0,y) = 5$. Comment on the well-posedness of this problem. (5 points)

5)(10 points) a) State (without proof) the solution to the initial-value problem (3 points):

$$u_{tt} = c^2 u_{xx} \qquad -\infty < x < \infty, -\infty < t < \infty$$

$$u(x,0) = \phi(x) \qquad u_t(x,0) = \psi(x) \qquad -\infty < x < \infty$$

b) Show that your solution satisfies both the initial conditions. (7 points)

6)(15 points) Take for granted that the Fourier cosine series for $f(x) = x$ on the interval $[0, \pi]$ is

$$x = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + ...\right).$$

a) Find the solution to the heat equation $u_t = ku_{xx}$ on $0 < x < \pi$ with initial conditions $u(x, 0) = x$ and boundary conditions $u_x(0, t) = u_x(\pi, t) = 0$. (5 points)
b) Use the theorems from class to discuss the types of convergence this Fourier series has (L^2, pointwise, uniform). Explain why the hypotheses of these theorems are or aren't met. (5 points)
c) Compute $1 + 1/9 + 1/25 + ...$ and justify your answer. (5 points)

Extra Page for Scratch Work or Finishing Answers if Clearly Marked