7/9/18 Lecture Notes: Laplace's Equation

Motivation: What are steady-state solutions ($u_t \equiv 0$) to the heat equation, $u_t = k u_{xx}$?

Answer in 1D: $u_{xx} = 0$

$u(x,t) = Ax + B$ (no dependence on t allowed)

What about higher dimensions?

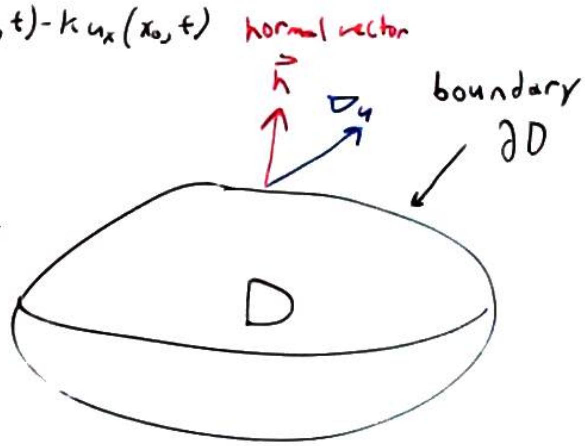
First, we need heat equation in higher dimensions.

Recall: In 1D $\quad x_0 \qquad\qquad x_1$

Define $m(t) = \int_{x_0}^{x_1} u(x,t)\,dt$. Then,

$$\frac{dm}{dt} = \int_{x_0}^{x_1} u_t(x,t)\,dt = k u_x(x_1,t) - k u_x(x_0,t)$$

$$\longrightarrow \quad u_t = k u_{xx}$$



In 3D: Define $m = \iiint_D u(x,y,z,t)\,dx\,dy\,dz$

$$\frac{dm}{dt} = \iiint_D u_t(x,y,z,t)\,dx\,dy\,dz$$

Also, $$\frac{dm}{dt} = \iint_{\partial D} k \overbrace{\nabla u \cdot \vec{n}}^{\text{directional derivative}}\,dS = k\iiint_D \underbrace{\operatorname{div}(\nabla u)}_{\substack{\parallel \\ u_{xx}+u_{yy}+u_{zz} := \Delta u}}\,dx\,dy\,dz$$

Heat equation is $\boxed{u_t = k\Delta u}$

$\Delta u = 0$ Laplace's equation (solutions called harmonic)

$\Delta u = f$ Poisson's equation

Derive this for fun using another vector calculus theorem

Note: $\Delta u = u_{xx} + u_{yy}$ in 2-D

Meaning of Laplace's equation $\Delta u = 0$

1) Steady state solutions to heat (and wave) equations

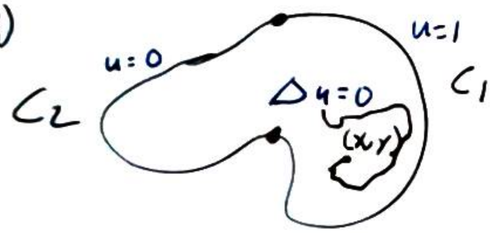
2) If $f(z) = u(z) + iv(z)$ is analytic (complex differentiable), then taking $z = x+iy$, $\Delta u = \Delta v = 0$.

Example: $f(z) = e^z = e^x \cdot e^{iy}$

$$= \underbrace{e^x \cos y}_{u(x,y)} + \underbrace{i e^x \sin y}_{iv(x,y)}$$

$u_{xx} + u_{yy} = u - u = 0$ $\qquad$ $v_{xx} + v_{yy} = v - v = 0$

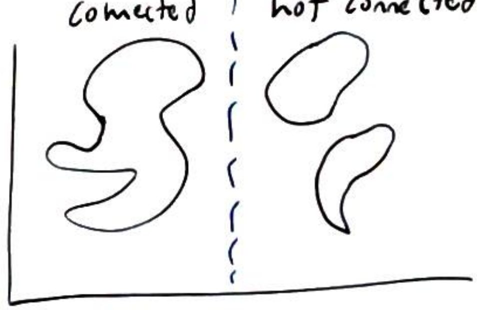
3)



Let $u(x,y)$ be the solution to

$$\Delta u = 0 \quad \text{in} \quad D$$
$$u = 1 \text{ on } C_1$$
$$u = 0 \text{ on } C_2.$$

Then, the probability a Brownian motion beginning at $(x,y) \in D$ exits through C_1 is $u(x,y)$.



(Weak) Maximum Principle Let D be any connected, bounded, open set. Let $u(x,y)$ (or $u(x,y,z)$) be harmonic on D, continuous on $\bar{D} = D \cup \partial D$. Then, $\max_D u$, $\min_D u$ occur on ∂D.

[(Strong) Maximum Principle: If $\max_D u$ occurs inside D, then u is constant.

to be proven later

Pf idea (Weak) At interior max, $u_{xx} < 0$, $u_{yy} < 0$, but

$$\Delta u = u_{xx} + u_{yy} = 0. \Rightarrow \Leftarrow$$

Actual Proof Let $v(x,y) = u(x,y) + \varepsilon(x^2+y^2)$ $\quad 0<\varepsilon<1$

$$\Delta v = \Delta u + \varepsilon \Delta(x^2+y^2) = 4\varepsilon > 0$$

By above reasoning, v has no interior maximum.

Say max at $(x_0,y_0) \in \partial D$. Then, for all $(x,y) \in D$,

$$u(x,y) \le v(x,y) \le v(x_0,y_0) = u(x_0,y_0) + \varepsilon(x_0^2+y_0^2)$$
$$\le \max_{\partial D} u + \varepsilon \ell^2,$$

where ℓ is largest value of x^2+y^2 in D (use boundedness)

Taking $\varepsilon \to 0$, $u(x,y) \le \max_{\partial D} u$ for all $(x,y) \in D$

- Proof for minimum similar, or apply maximum principle to $-u$.

Dirichlet problem $\Delta u = f$ in D

$u = h$ on ∂D

Uniqueness: Suppose u_1, u_2 both solutions and let $w = u_1 - u_2$.

Then, $\Delta w = 0$ in D

$w = 0$ on ∂D

Max/min principle $\Rightarrow w \equiv 0$, so $u_1 \equiv u_2$.

Invariance under certain change of coordinates:

Physical idea: The laws of physics don't change depending on
- where you stand, or
- which direction you face

Neither should the Laplacian

Note: From here on, focus on understanding 2D case proofs the most, 3D work same way

Translational Invariance (in 2D): $\begin{array}{l} x' = x+a \\ y' = y+b \end{array} \rightarrow \begin{array}{l} u_{x'} = u_x \\ u_{y'} = u_y \end{array} \rightarrow \begin{array}{l} u_{x'x'} = u_{xx} \\ u_{y'y'} = u_{yy} \end{array}$

$$u_{xx} + u_{yy} = u_{x'x'} + u_{y'y'}$$

Translational Invariance in 3D: Same

Rotational Invariance (in 2D)

Recall: $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ is a rotation matrix

$x' = x\cos\alpha + y\sin\alpha$

$y' = -x\sin\alpha + y\cos\alpha$

$$u_x = \frac{\partial u}{\partial x'}\cdot\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = u_{x'}\cos\alpha - u_{y'}\sin\alpha$$

Similarly, $u_y = u_{x'}\sin\alpha + u_{y'}\cos\alpha$

$$u_{xx} = (u_{x'}\cos\alpha - u_{y'}\sin\alpha)_{x'}\underbrace{\cos\alpha}_{\partial x'/\partial x} - (u_{x'}\cos\alpha - u_{y'}\sin\alpha)_{y'}\underbrace{\sin\alpha}_{\partial y'/\partial y}$$

$$+\ u_{yy} = (u_{x'}\sin\alpha + u_{y'}\cos\alpha)_{x'}\sin\alpha + (u_{x'}\sin\alpha + u_{y'}\cos\alpha)_{y'}\cos\alpha$$

Blue terms cancel

$$u_{xx} + u_{yy} = (u_{x'x'} + u_{y'y'})(\cos^2\alpha + \sin^2\alpha) = u_{x'x'} + u_{y'y'}$$

Rotational Invariance in 3D: Same proof, but use arbitrary 3×3 orthogonal matrix B and change of coordinates $\vec{x} \mapsto B\vec{x}$

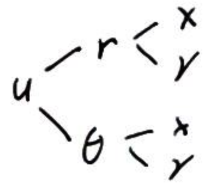
(orthogonal means $B^T B = I$)

Laplacian in Polar Coordinates (2D)

$x = r\cos\theta \quad y = r\sin\theta$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

hard to find from definition



1) Find Jacobian $J = \begin{pmatrix} \partial x/\partial r & \partial y/\partial r \\ \partial x/\partial\theta & \partial y/\partial\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$

corresponds to changing coordinates the other way

Inverse formula for 2×2 matrices

$J^{-1} = \begin{pmatrix} \partial r/\partial x & \partial\theta/\partial x \\ \partial r/\partial y & \partial\theta/\partial y \end{pmatrix} = \begin{pmatrix} \cos\theta & \frac{-\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{pmatrix}$

Therefore,

$$\partial/\partial x = \cos\theta\,\partial/\partial r - \frac{\sin\theta}{r}\,\partial/\partial\theta$$

$$\partial/\partial y = \sin\theta\,\partial/\partial r + \frac{\cos\theta}{r}\,\partial/\partial\theta$$

lots of algebra

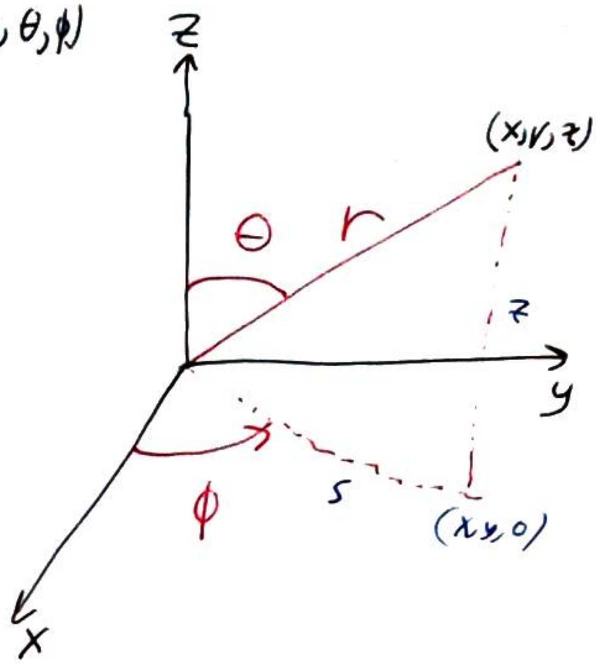
Get single derivative by product rule

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

Note: Rotation, i.e. replacing θ with $\theta + \alpha$, doesn't change Δu. Rotational invariance!

Laplacian in Spherical Coordinates (r,θ,ϕ)

Note: This follows book, which uses "incorrect" physics notation, which swaps θ and ϕ.



$x = s\cos\phi$

$s = r\sin\theta$

$y = s\sin\phi$

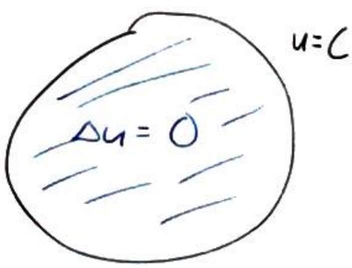
$z = r\cos\theta$

$$\Delta u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left[u_{\theta\theta} + (\cot\theta)u_\theta + \frac{1}{\sin^2\theta}u_{\phi\phi}\right]$$

Pf idea
1) Decompose change of coordinates $(x,y,z) \to (s,\phi,z) \to (r,\theta,\phi)$
2) Apply 2D result at each step, noting that
   - For fixed z, $(x,y) \longleftrightarrow (s,\phi)$ is Cartesian $\leftrightarrow$ Polar
   - For fixed ϕ, $(s,z) \longleftrightarrow (r,\theta)$ is Cartesian $\leftrightarrow$ Polar

Moral: Don't always reinvent the wheel.

Radial Solutions to $\Delta u = 0$
- Recall, Δ is invariant under rotation.
- If $\Delta u = 0$ in ball
  $u = C$ on boundary of ball (sphere),



then u is radial (only depends on r, distance from the origin)

Question: What are radial solutions to $\Delta u = 0$?

Additional motivation: Previously, we solved $\begin{array}{l} u_t - k u_{xx} = 0 \\ u(x,0) = \underbrace{\delta(x)}_{\text{Dirac delta function}} \end{array} \longrightarrow S(x,t)$, fundamental solution

Next, we solve $\Delta u = \delta(x,y,z)$

$\downarrow$

fundamental solution

By similar logic, fundamental solution is radial.

2D: $\Delta u = u_{rr} + \frac{1}{r} u_r + \overset{\text{radial}}{\cancel{\frac{1}{r^2} u_{\theta\theta}}} = 0$) Integrating Factor

$$r u_{rr} + u_r = 0$$
$$(r u_r)_r = 0$$
$$r u_r = c_1$$
$$u = c_1 \log r + c_2$$

3D: $\Delta u = u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2}\underbrace{\left[\cancel{u_{\theta\theta}} + \cancel{(\cot\theta) u_\theta} + \frac{1}{\sin^2\theta} \cancel{u_{\phi\phi}}\right]}_{\text{radial}} = 0$

$$u_{rr} + \frac{2}{r} u_r = 0$$
$$r^2 u_{rr} + 2r u_r = 0$$
$$(r^2 u_r)_r = 0$$
$$r^2 u_r = c_1$$
$$u = -c_1/r + c_2$$

) Integrating Factor