# 7/21/18 Lecture Notes: Finite Difference Method for the Wave Equation

Last time: Discretize $u(x,t) \to u(j\Delta x, n\Delta t) = u_j^n$

"Solve" $u_t = u_{xx}$ by setting

Forward difference $\to$ $\dfrac{u_j^{n+1} - u_j^n}{\Delta t} = \dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}$ $\leftarrow$ centered 2nd difference

Stability guaranteed when $s := \dfrac{\Delta t}{(\Delta x)^2} \leq 1/2$

Today $u_{tt} = c^2 u_{xx} \longrightarrow \dfrac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{(\Delta t)^2} = c^2 \dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}$ both centered 2nd differences

Set $s = \dfrac{c^2(\Delta t)^2}{(\Delta x)^2}$

$$u_j^{n+1} - 2u_j^n + u_j^{n-1} = s(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

$$\boxed{u_j^{n+1} = s(u_{j+1}^n + u_{j-1}^n) + 2(1-s)u_j^n - u_j^{n-1}}$$

Template:

$$\begin{array}{ccc} & * & \\ s & 2-2s & s \\ & -1 & \end{array}$$

1) Makes sense since solution depends on $u(x,0)$ and $u_t(x,0)$.

2) How to deal with IC?

<u>Initial conditions:</u>

Heat equation $u(x,0) = \phi(x) \longrightarrow u_j^0 = \phi(j\Delta x) = \phi_j$

Wave equation $u(x,0) = \phi(x) \longrightarrow u_j^0 = \phi(j\Delta x) = \phi_j$

$u_t(x,0) = \psi(x) \longrightarrow \dfrac{u_j^1 - u_j^{-1}}{2\Delta t} = \psi(j\Delta x) = \psi_j$

use centered difference for $O(\Delta x^2)$ error

In $n=0$, finding $u_j^1$, use $u_j^1 + u_j^{-1} = s(u_{j+1}^0 + u_{j-1}^0) + 2(1-s)u_j^0$

Not given, but sub in $u_j^1 - 2\Delta t \underbrace{\psi(j\Delta x)}_{\psi_j}$

$$u_j^1 = \underbrace{\frac{s}{2}\left(\phi_{j+1}+\phi_{j-1}\right)}_{\text{look familiar?}} + (1-s)\phi_j + \psi_j \Delta t$$

Plan: 1) Let $u_j^0 = \phi_j$

2) Use to get $u_j^1$

3) Use template for all other $u_j^n$ $(n \geq 2)$

Stability

Can check by example that $s = 2 \to$ unstable (makes sense since all

$s = 1 \to$ stable (coefficients nonnegative)

$$\text{Recall } u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \underbrace{\left(A_n \cos\frac{n\pi c t}{\ell} + B_n \sin\frac{n\pi c t}{\ell}\right)}_{C_n e^{i\frac{n\pi c t}{\ell}} + d_n e^{-i\frac{n\pi c t}{\ell}}} \begin{matrix}\sin \\ \text{or} \\ \cos\end{matrix}\left(\frac{n\pi x}{\ell}\right)$$

because complex exponentials are better

Note: $e^{inx} = (e^{ix})^n$, so in

separation of variables, guess*

* Again, justify with discrete Fourier transform

$$u_j^n = (\eta)^j (\xi)^n, \quad \text{with } e^{ik\Delta x} = \eta$$

$$\text{Recall: } u_j^{n+1} = s\left(u_{j+1}^n + u_{j-1}^n\right) + 2(1-s)\,u_j^n - u_j^{n-1}$$

$$\eta^j \xi^{n+1} = s\left(\eta^{j+1}\xi^n + \eta^{j-1}\xi^n\right) + 2(1-s)\eta^j\xi^n - \eta^j \xi^{n-1} \quad \Big) \div \eta^j \xi^n$$

$$\xi = s\eta + s\frac{1}{\eta} + 2(1-s) - \frac{1}{\xi}$$

$$\xi + \frac{1}{\xi} - 2 = s\left(\eta + \frac{1}{\eta} - 2\right)$$

$$\text{Since } \eta = e^{ik\Delta x},\ \eta + \frac{1}{\eta} - 2 = e^{ik\Delta x} + e^{-ik\Delta x} - 2$$

$$= 2\cos k\Delta x - 2$$

$$\xi + \frac{1}{\xi} - 2 = 2\underbrace{s\left[\cos k\Delta x - 1\right]}_{p \leq 0} \qquad \xi^2 - 2(1+p)\xi + 1 = 0$$

$$\xi = 1 + p \pm \sqrt{p^2 + 2p}$$

(Largest at $p = 0$, $p = -2$)

If $p < -2$, one root $< -1$, so $|\xi| > 1 \rightarrow$ unstable

If $p > -2$, $\xi = 1 + p \pm i\sqrt{-p^2 - 2p}$,

$|\xi|^2 = (1+p)^2 - p^2 - 2p = 1$, so $|\xi| = 1 \rightarrow$ stable

If $p = -2$, $\xi = -1 \rightarrow$ stable

Stability when: $p \geq -2$

$$s(\cos k\Delta x - 1) \geq -2 \quad \text{for all } k$$

$$s \leq \frac{2}{1 - \cos k\Delta x} \leq 1$$

Note $s \leq 1 \iff c \leq \frac{\Delta x}{\Delta t}$, so discrete propagation speed has to be $\geq$ actual speed

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<u>Nonlinear example</u>

$u_{tt} - \Delta u + u + u^7 = 0$ (non linear wave in 3D)

Look for radial solutions, so

$$u_{tt} - u_{rr} - \frac{2}{r} u_r + u + u^7 = 0$$

Let $v(r,t) = u(r,t) \cdot r$ $\quad v_{tt} = r u_{tt}$

$v_r = r u_r + u$

$v_{rr} = r u_{rr} + 2u_r$

$\cdot r$ and sub

$$v_{tt} - v_{rr} + v + r^{-6} v^7 = 0$$

$$v(0,t) = 0$$

Scheme: $$\underbrace{\frac{v_j^{n+1} - 2v_j^n + v_j^{n-1}}{(\Delta t)^2}}_{v_{tt}} = \underbrace{\frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{(\Delta r)^2}}_{v_{rr}} - \underbrace{\frac{1}{2}\left(v_j^{n+1} + v_j^{n-1}\right)}_{v} \underbrace{- \frac{1}{8} \underbrace{(j\Delta r)^{-6}}_{r^{-6}} \frac{(v_j^{n+1})^8 - (v_j^{n-1})^8}{v_j^{n+1} - v_j^{n-1}}}_{v^7}$$

Conserves energy