Multiple solutions of AR equations

Consider the AR (1) equation:

$$X_t = \phi X_{t-1} + W_t,$$

with $|\phi| < 1$, (W_t) a zero-mean white noise process with variance σ^2, and $t \in \mathbb{Z}$. For each of the following processes, (i) Show that it solves the AR (1) equation (ii) and determine if it is weakly stationary. (Remember too the definition of the unique stationary solution.)

1. $X_t = \sum_{j=0}^{\infty} \phi^j W_{t-j}$

2. Fix $c \in \mathbb{R}$. $X_t = c\phi^t + \sum_{j=0}^{\infty} \phi^j W_{t-j}$

Invertibility and uniqueness

In this problem, we will develop intuitions on how imposing invertibility resolves the non-uniqueness of MA processes, as well as how the root condition leads to an AR (∞) representation. To keep our discussion simple, consider an arbitrary MA (q) process (X_t) where all the roots of the MA polynomial

$$\theta(z) = 1 + \theta_1 z + ... + \theta_q z^q, \theta_q \neq 0$$

are real-valued and none of the roots have magnitude exactly equal to 1. The underlying white noise process (W_t) is i.i.d. zero mean Gaussian white noise with variance σ^2.

3. Assume the MA process $X_t = W_t - 4W_{t-1}$.
   (a) Show the process is not invertible.

   (b) Show that $\tilde{X}_t = \tilde{W}_t - \frac{1}{4}\tilde{W}_{t-1}$, where $\tilde{W}_t \overset{iid}{\sim} N(0, 16\sigma^2)$, is invertible, and that the distribution of X_t and $\tilde{X}_t$ are the same.

   (c) For the invertible process, find the corresponding $AR(\infty)$ form.

4. Assume the MA process $X_t = W_t - \frac{7}{2}W_{t-1} + \frac{3}{2}W_{t-2}$.

   (a) Show the process is not invertible.

   (b) Show that $\tilde{X}_t = \tilde{W}_t - \frac{5}{6}\tilde{W}_{t-1} + \frac{1}{6}\tilde{W}_{t-2}$, where $\tilde{W}_t \overset{iid}{\sim} N(0, 9\sigma^2)$, is invertible, and that the distribution of X_t and $\tilde{X}_t$ are the same.

   (c) For the invertible version of the MA process, find the corresponding AR (∞) representation.

5. *(Sample ACF vs. Theoretical ACF)* Let W_t be a white noise process with variance 1. Consider the two following processes X_t and Y_t:

$$X_t = \frac{1}{4}X_{t-1} + W_t \qquad Y_t = W_t + 2W_{t-1} - 2W_{t-4}$$

   (a) Derive the ACF of X_t.

   (b) Derive the ACF of Y_t.

   (c) Compare your values with those generated by `ARMAacf` for the first 20 lags. Do they match?

   (d) Simulate both X_t and Y_t using the `arima.sim` function, and plot their sample ACFs. Are they close to the theoretical ACFs?