Lecture Notes 7/30/18: Heat Eqn in 3D and Intro to Distributions

3D Heat Equation: $u_t = k\Delta u$

$$u(\vec{x},0) = \phi(\vec{x})$$

1D solution: $u(x,t) = \int_{-\infty}^{\infty} S(x-x',t)\phi(x')\,dx'$, where

$$S(x,t) = \frac{\partial Q}{\partial x} = \frac{1}{\sqrt{4\pi k t}} e^{-x^2/4kt},$$

and Q is solution to $u_t = k u_{xx}$ with

$$u(x,0) = H(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$$ $H(x)$ is Heaviside function

Back to 3D:

Claim: $\boxed{u(\vec{x},t) = \frac{1}{(4\pi k t)^{3/2}} \iiint e^{-|\vec{x}-\vec{x}'|^2/4kt} \phi(\vec{x}')\,d\vec{x}'}$

$\boxed{= \iiint S_3(\vec{x}-\vec{x}',t)\phi(\vec{x}')\,d\vec{x}'}$, where

$$S_3(x,y,z,t) = S(x,t)S(y,t)S(z,t).$$

Proof: Step 1) Check $u_t = k\Delta u$ Q: Why does it suffice to check S?

$$\frac{\partial S_3}{\partial t} \underset{\text{product rule}}{=} \frac{\partial S}{\partial t}(x,t)S(y,t)S(z,t) + S(x,t)\frac{\partial S}{\partial t}(y,t)S(z,t) + S(x,t)S(y,t)\frac{\partial S}{\partial t}(z,t)$$

$\downarrow S_t = kS_{xx}$

$$= k\frac{\partial^2 S}{\partial x^2}(x,t)S(y,t)S(z,t) + \text{ similar terms}$$

$$= k\Delta S_3 \qquad \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

takes $S(y,t), S(z,t) \to 0$, simplifying product rule

Step 2: Check initial conditions * different from book

$$u(x,y,z,t) = \iiint \frac{\partial G}{\partial x}(x-x',t)\,\frac{\partial G}{\partial y}(y-y',t)\,\frac{\partial G}{\partial z}(z-z',t)\,\phi(x',y',z')\,dz'\,dy'\,dx'$$

$$\int_{-\infty}^{\infty} \frac{\partial G}{\partial z}(z-z',t)\,\phi(x',y',z')\,dz' = -\int_{-\infty}^{\infty} \frac{\partial G}{\partial z'}(z-z',t)\,\phi(x',y',z')\,dz'$$

IBP

$$= \int_{-\infty}^{\infty} \underbrace{G(z-z',t)}_{=\,H(z-z')\text{ at }t=0}\,\phi_{z'}(x',y',z')\,dz' - \underbrace{G(z-z',t)\,\phi(x',y',z')\Big|_{z'=-\infty}^{\infty}}_{\text{decay of }\phi}$$

$$u(x,y,z,0) = \iint \frac{\partial G}{\partial x}(x-x',t)\,\frac{\partial G}{\partial y}(y-y',t)\left[\int_{-\infty}^{\infty} H(z-z')\,\phi_{z'}(x',y',z')\,dz'\right]dy'\,dx'$$

$$\int_{-\infty}^{z} \phi_{z'}(x',y',z')\,dz' = \phi(x',y',z) \xrightarrow[\text{times}]{\text{2 more}} \phi(x,y,z) \checkmark$$

Note: We basically showed

$$\int_{-\infty}^{\infty} H'(x-y)\,\phi(y)\,dy = \phi(x) \quad \text{or} \quad \int_{-\infty}^{\infty} H'(x)\,\phi(x)\,dx = \phi(0) \ \text{(change of variables)}$$

since $G(x,0) = H(x)$

Later today: $H'(x) = \delta(x)$, the Dirac delta function

Alternate method for checking IC: Show $S_3(x,t)$ has same 3 special properties as Poisson kernel

2D: $S_2(x,y,t) = S(x,t)\,S(y,t) = \frac{1}{4\pi k t} e^{-(x^2+y^2)/4kt}$

Small Detour: Schrödinger's Equation

Equation for wave function of an H atom:

potential

$$-i\hbar u_t = \frac{\hbar^2}{2m}\Delta u + \frac{e^2}{r}u$$

no potential

Free Schrödinger equation (with nice units):

$$-iu_t = \tfrac{1}{2}\Delta u$$

On a bounded domain: Solve using separation of variables (like with heat equation)

E.g. on $[0,\pi]$, $u(x,t) = \sum_{n=1}^{\infty} A_n \sin nx \, e^{-in^2t}$.

On unbounded domain: Solve with source function (almost exactly like with heat equation)

E.g. $\mathbb{R}^3$ Plugging in $k = i/2$ into heat solution,

$$\boxed{u(\vec{x},t) = \frac{1}{(2\pi i t)^{3/2}} \iiint e^{-|\vec{x}-\vec{x}'|^2/2it} \phi(\vec{x}')\,d\vec{x}'}$$

Q: Is this correct?

Solution: Correct

Method: Needs work

- If $k = i/2$, then no decay of $e^{-|\vec{x}|^2/4kt}$
- Instead, use $k = \varepsilon + i/2$, taking $\sqrt{\varepsilon + i/2}$ to have positive real part $\rightarrow$ decay
- Take limit as $\varepsilon \to 0$
- Shows $i^{3/2} = \frac{1+i}{\sqrt{2}}$ in solution

Problem 1: $u_{tt} = c^2 u_{xx}$ has solutions $u(x,t) = f(x+ct) + g(x-ct)$

Solution should be C^2, but this makes sense for any f, g.

* (How) can we take derivatives of any functions?

Problem 2: $v(\vec{x}) = \frac{1}{4\pi|\vec{x}|}$ is "fundamental solution" for Δ in $\mathbb{R}^3$.

* Why isn't v defined at $\vec{x} = 0$?

Problem 3: $u(x,t) = \int S(x-y,t)\phi(y) \to \phi(x)$ as $t \to 0$.

* Why can't we just plug in $t = 0$?

Answer: Distributions

Example: Dirac delta "function" $\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$ $\int \delta(x)\,dx = 1$

$\delta(x)$ not a function, but only need it for 2 purposes:

1) $\int_{-\infty}^{\infty} \delta(x)\phi(x)\,dx = \phi(0)$ Rule: $\phi(x) \mapsto \phi(0)$

2) Integration by parts

<u>Defn</u> The set of <u>test functions</u>, C_0^∞, is functions which are infinitely differentiable and zero outside some finite interval.

(Kind of) Example: $\begin{cases} e^{-1/x^2} & x>0 \\ 0 & x \le 0 \end{cases}$

A <u>distribution</u> is a rule* which takes in C_0^∞ functions, and spits out numbers

Write (f, ϕ) for output (like inner product, wink, wink)

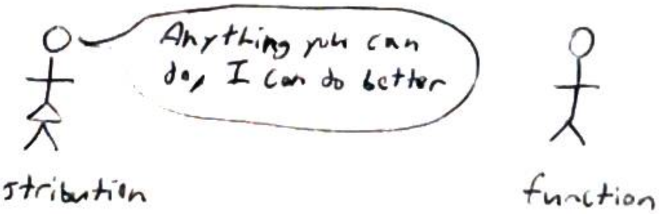
distribution   function $\in C_0^\infty$

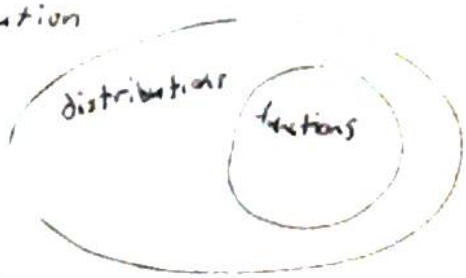
* continuous* linear functional

* If $\phi_n, \phi \in C_0^\infty$ all vanish outside common interval, $\phi_n^{(k)} \to \phi^{(k)}$ uniformly, $\forall k$, then $(f, \phi_n) \to (f, \phi)$

Example: Let $f(x)$ be a function.

$\phi \longmapsto \int_{-\infty}^{\infty} f(x)\phi(x)\,dx$ is a distribution





Distributions include functions and more.

<u>Defn</u> If f_n, f are distributions, we say <u>$f_n \to f$ weakly</u> if $(f_n, \phi) \to (f, \phi)$ for all $\phi \in C_0^\infty$.

Example: $\int_{-\infty}^{\infty} S(x,t)\phi(x)\,dx \longrightarrow \phi(0)$ as $t \to 0$

Answer to Problem 3: $S(x,t) \to \delta(x)$ weakly as $t \to 0$

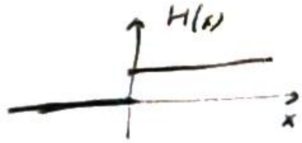
Can't plug in $t=0$ because $\delta(x)$ not a function

Example: Dirichlet kernel $K_N(\theta)$

$$\int_{-\pi}^{\pi} K_N(\theta)\phi(\theta)\,d\theta \to 2\pi\,\phi(0) \text{ means } K_N \to 2\pi\delta \text{ as } N \to \infty$$

weakly

<u>Question</u>: Is $H(x)$ differentiable?



Answer 1: No, see Intro Calc.

Answer 2: Yes, $H(x)$ is a distribution

If $f(x)$ is a differentiable function, $\phi \in C_0^\infty$,

decayed ϕ

$$\int_{-\infty}^{\infty} f'(x)\phi(x)\,dx = \underbrace{-\int_{-\infty}^{\infty} f(x)\phi'(x)\,dx}_{} + \underbrace{\cancel{f(x)\phi(x)\Big|_{-\infty}^{\infty}}}_{0}$$ (use differentiability of ϕ)

makes sense for any distribution f

The derivative of a distribution f is the rule

$$\phi \mapsto -(f, \phi').$$

If f is a differentiable function, this definition coincides with the original

Example
$$\int_{-\infty}^{\infty} H'(x)\phi(x)\,dx = -\int_{-\infty}^{\infty} H(x)\phi'(x)\,dx$$
$$= -\int_0^{\infty} \phi'(x)\,dx = -\phi(x)\Big|_0^{\infty} = \phi(0)$$

H' turns ϕ into $\phi(0)$, so $H' = \delta$

Question: What is δ'? Answer: The rule $\phi \mapsto -\phi'(0)$

— Why we want test functions to be infinitely differentiable

General rule: Anything you can do with functions, you can do with distributions, except multiplying two distributions
(distribution · function okay)

Answer to Problem 1: We can take derivatives of $f(x+ct)+g(x-ct)$ for any f, g by viewing them as distributions

Note: Can do all this in 2D and 3D

More Examples

- $\delta(\vec{x})$ in 3D $\iiint \delta(\vec{x})\phi(\vec{x})\,d\vec{x} = \phi(0)$
- $\phi \longmapsto \iint_S \phi \, dS$, where S is surface
- $\phi \longmapsto \int_C \phi \, ds$, where C is a curve

  (Can view δ as integrating ϕ at a point)

- If X is a random variable, $\phi \longmapsto \mathbb{E}[\phi(X)]$
- $\phi \longmapsto \lim\limits_{\varepsilon \to 0} \int_{|x|>\varepsilon} \frac{\phi(x)}{x}\,dx$