Homework 2

EECS/BioE C106A/206A
Introduction to Robotics

Due: September 15, 2020

Problem 1. Exponential Coordinates for Rotations

Recall that for any rotation matrix $R \in SO(3)$, there exists a unit axis vector $\omega \in \mathbb{R}^3$, a corresponding skew symmetric matrix $\hat{\omega} \in \mathfrak{so}(3)$, and a scalar θ such that $R = e^{\hat{\omega}\theta}$. Further recall the geometric interpretation of exponential coordinates; to write $R = e^{\hat{\omega}\theta}$ is to state that R implements a rotation about the unit axis ω by θ radians in the positive direction (according to the right hand rule).

(a) Let $\omega = (\omega_1, \omega_2, \omega_3)^T \in \mathbb{R}^3$ be a unit vector and recall that we define the hat operator as

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{1}$$

Note that we denote this operator as either $\hat{\omega}$ or $\omega^\wedge$ interchangeably. Further, we define the "vee" operator $^\vee$ as the inverse of hat, so that $\hat{\omega}^\vee = \omega$. "vee" is defined on $\mathfrak{so}(3)$ and returns a 3-vector.

Let $\theta \in [0, \pi]$ be a scalar. Show that the matrix $\hat{\omega}\theta$ has eigenvalues $\{0, i\theta, -i\theta\}$.

(b) Let R be the rotation matrix for which (ω, θ) is a set of exponential coordinates. i.e. $R = e^{\hat{\omega}\theta}$. Find the eigenvalues of R.

Hint: Recall the properties of the matrix exponential we introduced in Homework 0.

(c) For what values of the rotation angle θ does R have 1 or 2 distinct real eigenvalues? Can it ever have 3 distinct real eigenvalues?

(d) Interpret your answer to part (c) geometrically. When R has exactly 1 real eigenvalue, what is it and what is the corresponding eigenvector? Why does this make sense geometrically given that R is a rotation matrix? What about when R has two distinct real eigenvalues? You should answer this question without ever carrying out a direct eigenvector computation.

Problem 2. Things About Exponential Coordinates

(a) If the rotation matrix $R \in SO(3)$ has exponential coordinates (ω, θ) so that $R = e^{\hat{\omega}\theta}$, what are the exponential coordinates for R^T? Interpret your answer geometrically.

(b) When we write a rotation matrix R_{AB} in exponential coordinates (ω, θ), in what reference frame (A or B) is the axis ω written?

(c) For any $A \in \mathbb{R}^{n \times n}$ and time denoted t, let $Y(t) = e^{At}$. By differentiating the series expansion, show that $\dot{Y}(t) = Ae^{At} = e^{At}A$.

(d) Let $\omega \in \mathbb{R}^3$ be given. Show that $p(t) = e^{\hat{\omega}t}p_0$ solves the vector differential equation

$$\dot{p} = \omega \times p \tag{2}$$

with initial condition $p(0) = p_0$. The symbol $\times$ here stands for the *cross-product* on 3D vectors. Recall that for any $v \in \mathbb{R}^3$: $\hat{\omega}v = \omega \times v$.

Remark: It is a standard result in systems theory that linear ODEs like the one in equation (2) have a unique solution. So in fact, we have shown that $e^{\hat{\omega}t}p_0$ is the unique solution to that differential equation, which is something we claimed in lecture.

Problem 3. Finding Exponential Coordinates

In each of the following subparts, find the exponential coordinates of the rigid body transform requested.

(a) Figure (1) shows a cube undergoing two different rigid body transformations from frame $\{1\}$ to frame $\{2\}$. In both cases, find a set of exponential coordinates for the rigid body transform that maps the cube from its initial to its final configuration, as viewed from frame $\{0\}$. Do this by first finding the equivalent screw motion.

(b) For a point $p_0 \in \mathbb{R}^3$, consider a following rigid body motion in which the velocity of the point is

$$\dot{p}(t) = \omega \times (p(t) - q), \quad p(0) = p_0, \tag{3}$$

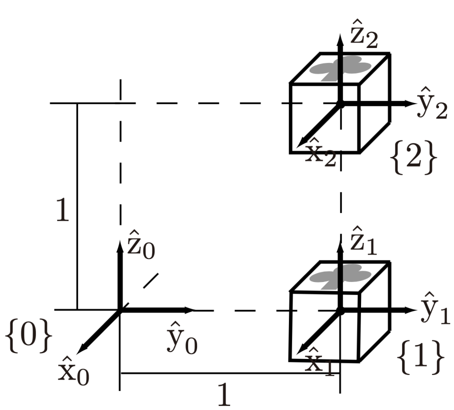
where $\omega = [0, 0, 1]^T \in \mathbb{R}^3$ is the axis of rotation and $q = [1, 1, 1]^T \in \mathbb{R}^3$ is the center of the rotation. $p(t)$ is a coordinate of the point at time t with respect to the frame 0. This is depicted in Figure (2). If $g(t) \in SE(3)$ is a 4×4 matrix such that

$$\begin{bmatrix} p(t) \\ 1 \end{bmatrix} = g \begin{bmatrix} p_0 \\ 1 \end{bmatrix}. \tag{4}$$

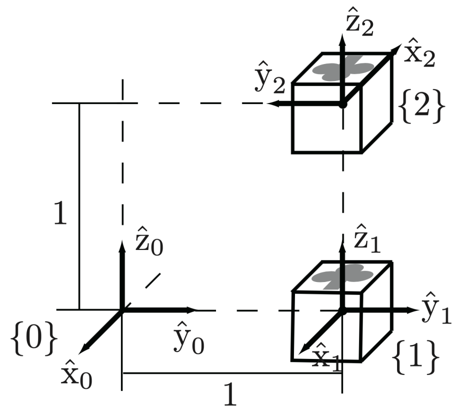
Find a $\xi = (v, \omega) \in \mathfrak{se}(3)$ such that $e^{\hat{\xi}t} = g(t)$.

(c) Let $g = (R, p) \in SE(3)$, with $R = R_x(\pi/2)R_z(\pi)$, and $p = (0, -1/\sqrt{2}, 1/\sqrt{2})$. Find the exponential coordinates of g.

Hint: Try drawing out the transformation between the initial and final frames after applying g, and attempt to find an equivalent screw motion. You shouldn't need to solve this algebraically.



(a) A first screw motion.



(b) A second screw motion.

Figure 1: A cube undergoing two screw motions.

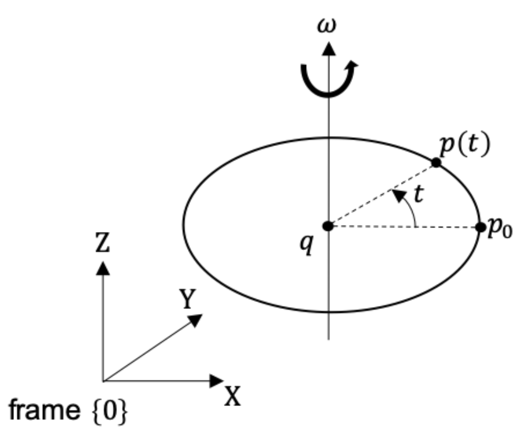


Figure 2: The coordinate of the moving point is $p(t)$. q is $[1, 1, 1]^T$ and ω is $[0, 0, 1]^T$.