Spectral Density Continued

Jared Fisher

Lecture 11b

Announcements

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- New Grading Policy: Homework drop may be used on a project checkpoint instead.
- New Grading Policy: Contact me if you are concerned with failing/not passing and we'll work out a late homework/checkpoint submission option (maximum score of C- or P).

Schedule

- Tuesday 4/27: Lecture on Spectral density part 2

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- Monday 5/10: Final Project Report and Forecasts due

Disclaimer

Like last time, we're discussing several things about the frequency domain, but not too in depth. The purpose is to

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1. Give you exposure to a set of tools that are available
2. Connect several things we've been talking about this semester

Recap: Intro to spectral density

Definition: Discrete Fourier Transform

For data $x_0, \ldots, x_{n-1} \in C$ the discrete Fourier transform (DFT) is given by $b_0, \ldots, b_{n-1} \in C$, where

$$b_j = \sum_{t=0}^{n-1} x_t \exp\left(-\frac{2\pi ijt}{n}\right) \text{ for } j = 0, \ldots, n-1.$$

(In R, the DFT is calculated by the function *fft()*.)

Definition: Periodogram

For real values data $x_0, \ldots, x_{n-1}$ with DFT $b_0, \ldots, b_{n-1}$ the **periodogram** is defined as

$$I(j/n) = \frac{|b_j|^2}{n} \quad \text{for } j = 1, \ldots, \lfloor n/2 \rfloor$$

Theorem: Connection between periodogram and $\hat{\gamma}$

For some data $x_0, \ldots, x_{n-1}$ let $\hat{\gamma}(h)$ for $h = 0, \ldots, n-1$ be its sample ACVF. Then

$$I(j/n) = \sum_{h=-(n-1)}^{n-1} \hat{\gamma}(h) \exp\left(-\frac{2\pi ijh}{n}\right) \text{ for } j = 1, \ldots, \lfloor n/2 \rfloor.$$

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- We've shown that every dataset can be written in terms of sinusoids.
- The magnitude of the sinusoid component with frequency j/n is given by the respective periodogram $I(j/n)$.
- But this is a discrete representation and leads to leakage!
- Now we extend these definitions to the process $\{X_t\}$ itself.
- Remember that ACVF is related to the periodogram, and that leads to the following natural process-analog of the periodogram.

Definition: Spectral Density

For a stationary process with ACVF $\gamma_X(h)$ with $\sum_{h=-\infty}^{\infty} |\gamma_X(h)| < \infty$ we define the *spectral density* as

$$f(\lambda) := \sum_{h=-\infty}^{\infty} \gamma_X(h) \exp\left(-2\pi i \lambda h\right) \text{ for } -1/2 \leq \lambda \leq 1/2.$$

Notes on the Spectral Density

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- f is always nonnegative: $f(\lambda) \geq 0$
- Like the periodogram, the spectral density gives the strengths of sinusoids at various frequencies contributing to a stationary stochastic process.

Thoerem: ACVF and Spectral Density

For a stationary process with spectral density $f(\lambda)$, $-1/2 \leq \lambda \leq 1/2$, it holds for its ACVF that

$$\gamma_X(h) = \int_{-1/2}^{1/2} e^{2\pi i \lambda h} f(\lambda) d\lambda = \int_{-1/2}^{1/2} \cos(2\pi \lambda h) f(\lambda) d\lambda.$$

Definition: Linear Time Invariant Filter

A linear time-invariant filter with coefficients $\{a_j\}$ for $j = \ldots, -2, -1, 0, 1, 2, 3, \ldots$ transforms an input time series $\{X_t\}$ into an output time series $\{Y_t\}$ via

$$Y_t = \sum_{j=-\infty}^{\infty} a_j X_{t-j}.$$

In the above definition, the coefficients $\{a_j\}$ are often assumed to satisfy $\sum_{j=-\infty}^{\infty} |a_j| < \infty$.

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\begin{aligned}
\gamma_Y(h) =& \operatorname{cov}(Y_t, Y_{t+h}) \\
=& \operatorname{cov}\left(\sum_j a_j X_{t-j}, \sum_k a_k X_{t+h-k}\right) \\
=& \sum_{j,k} a_j a_k \operatorname{cov}(X_{t-j}, X_{t+h-k}) \\
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- Note that the above calculation shows also that $\{Y_t\}$ is stationary (like you did on your homework earlier!).

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- Combining this with the ACVF of $\{Y_t\}$ from the last slide, we get the spectral density f_Y of the output $\{Y_t\}$:
$$\begin{aligned}\gamma_Y(h) &= \sum_j \sum_k a_j a_k \int e^{2\pi i (h-k+j)\lambda} f_X(\lambda) d\lambda \\ &= \int e^{2\pi i h\lambda} f_X(\lambda) \left(\sum_j \sum_k a_j a_k e^{-2\pi i k\lambda} e^{2\pi i j\lambda} \right) d\lambda\end{aligned}$$

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\end{aligned}$$

- We'll simplify this rearranged formula on the last line.

Definition: Transfer Function

For a time invariant linear filter with coefficients $\{a_j\}$, we define the **transfer function**

$$A(\lambda) := \sum_j a_j e^{-2\pi i j\lambda} \text{ for } -1/2 \leq \lambda \leq 1/2. \tag{1}$$

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- Thus

$$\begin{aligned} A(\lambda) &= \sum_j a_j e^{-2\pi i j \lambda} \\ &= \sum_j a_j [\cos(2\pi j \lambda) - i \sin(2\pi j \lambda)] \\ &= \left[\sum_j a_j \cos(2\pi j \lambda) \right] - i \sum_j a_j \sin(2\pi j \lambda) \end{aligned}$$

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- Conjugate: $\overline{A(\lambda)} = \left[\sum_j a_j \cos(2\pi j \lambda)\right] + i \sum_j a_j \sin(2\pi j \lambda) = \sum_j a_j e^{2\pi i j \lambda}$

Using the Transfer Function

- Recall our previous equation for the ACVF of Y:

$$\gamma_Y(h) = \int e^{2\pi i h\lambda} f_X(\lambda) \left(\sum_j \sum_k a_j a_k e^{-2\pi i k\lambda} e^{2\pi i j\lambda} \right) d\lambda$$

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- Applying the definition of the transfer function:

$$\gamma_Y(h) = \int e^{2\pi i \lambda h} f_X(\lambda) A(\lambda) \overline{A(\lambda)} d\lambda,$$

where, of course, $\overline{A(\lambda)}$ denotes the complex conjugate of $A(\lambda)$.

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- As a result, we have

$$\gamma_Y(h) = \int e^{2\pi i\lambda h} f_X(\lambda) \left| A(\lambda) \right|^2 d\lambda.$$

- This is clearly of the form $\gamma_Y(h) = \int e^{2\pi i\lambda h} f_Y(\lambda) d\lambda$.

Definition: Power Transfer Function

The function $\lambda \mapsto |A(\lambda)|^2$ is called the **power transfer function**.

Use of the Power Transfer Function

- We therefore have

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- So what does the filter do to the spectrum? It modifies the spectrum by multiplying it with the power transfer function $|A(\lambda)|^2$.
- Depending on the value of $|A(\lambda)|^2$, some frequencies may be enhanced in the output while other frequencies will be diminished.
- Thus, the spectral density is very useful while studying the properties of a filter.
- While the autocovariance function of the output series γ_Y depends in a complicated way on that of the input series γ_X, the dependence between the two spectral densities is very simple.

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- This corresponds to the weights $a_0 = 1$ and $a_s = -1$ and $a_j = 0$ for all other j.
- Then the transfer function is given by

$$
\begin{aligned}
A(\lambda) &= \sum_j a_j e^{-2\pi i j \lambda} \\
&= a_0 e^{-2\pi i (0) \lambda} + a_s e^{-2\pi i s \lambda} \\
&= (1)e^0 + (-1)e^{-2\pi i s \lambda} \\
&= 1 - e^{-2\pi i s \lambda} \\
&= 1 - \cos(2\pi s \lambda) + i \sin(2\pi s \lambda)
\end{aligned}
$$

Example: Power Transfer Function of the Differencing Filter

- The power transfer function:

$$
\begin{aligned}
|A(\lambda)|^2 &= \sqrt{Re(A(\lambda))^2 + Im(A(\lambda))^2}^2 \\
&= [1 - \cos(2\pi s\lambda)]^2 + \sin^2(2\pi s\lambda) \\
&= 1 - 2\cos(2\pi s\lambda) + \cos^2(2\pi s\lambda) + \sin^2(2\pi s\lambda) \\
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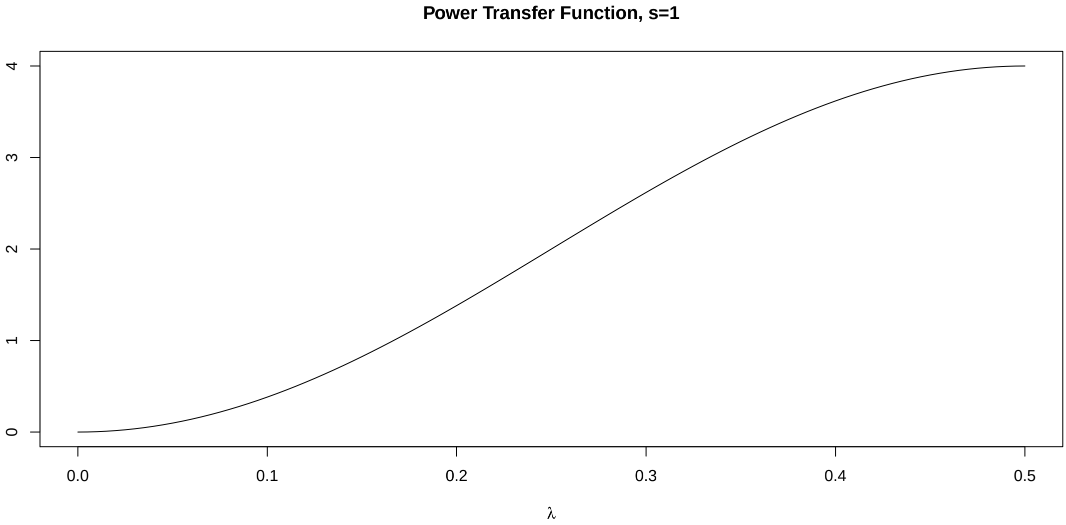
$$

- To understand this function, we only need to consider the interval $[0, 1/2]$ because it is symmetric on $[-1/2, 1/2]$.

$s = 1$



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- When $s = 1$, the function $|A(\lambda)|^2$ is increasing on $[0, 1/2]$.
- This means that first order differencing enhances the higher frequencies in the data and diminishes the lower frequencies.
- Therefore, it will make the data "more wiggly" as it elminates low frequency elements (i.e. trend!).

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- For higher values of s, the function $A(\lambda)$ goes up and down and takes the value zero for $\lambda = 0, 1/s, 2/s, \ldots$.

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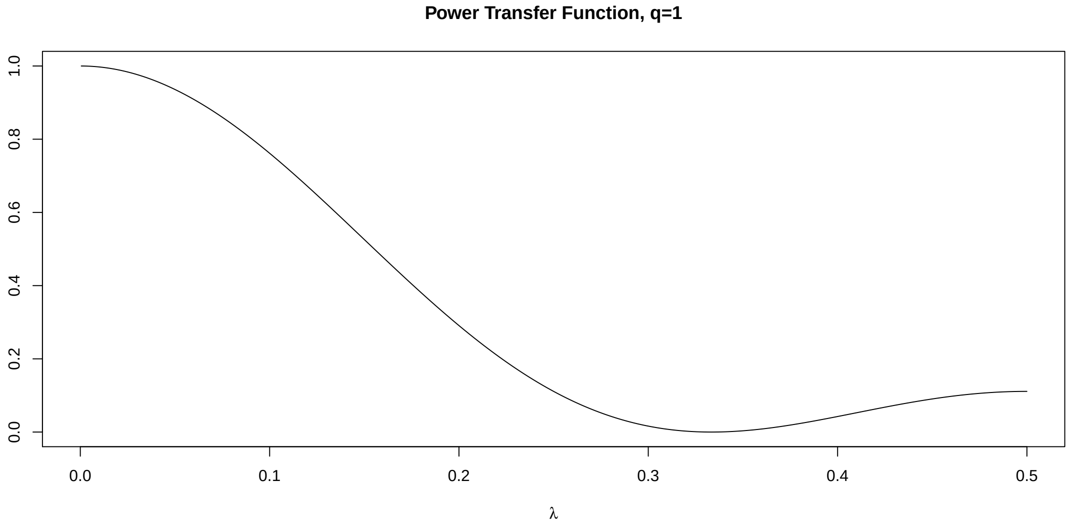
- For higher values of s, the function $A(\lambda)$ goes up and down and takes the value zero for $\lambda = 0, 1/s, 2/s, \ldots$.
- In other words, it eliminates all components of period s.

New Content

Example: Power Transfer Function of Smoothing Filter



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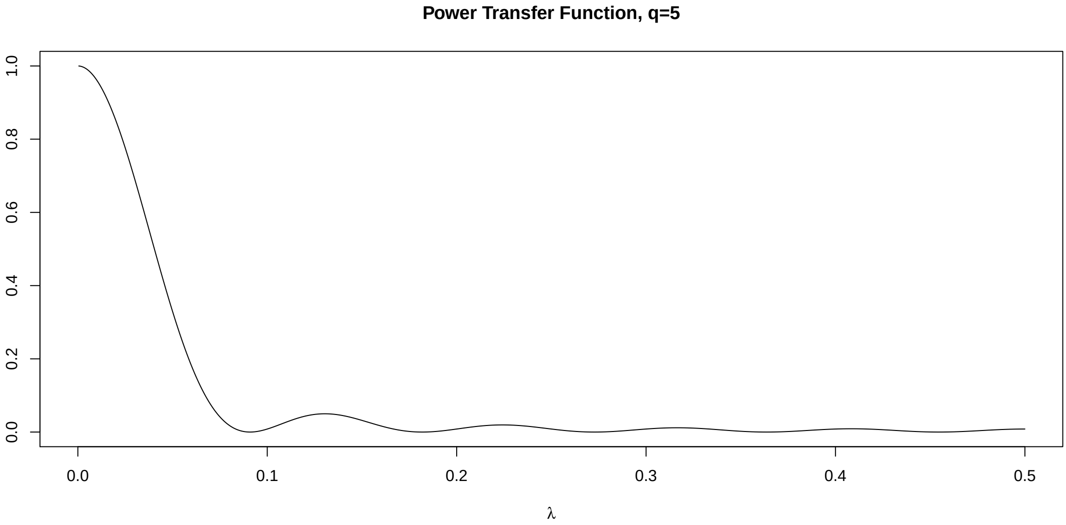

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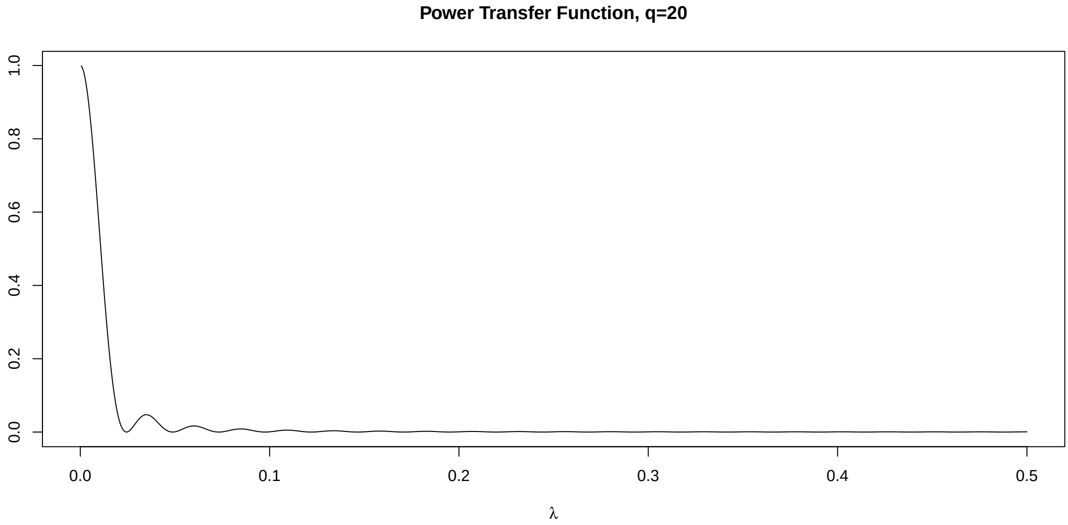

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- For $-1/2 \leq \lambda \leq 1/2$, the transfer function is

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\begin{aligned}
A(\lambda) &= \sum_{j=-q}^{q} \frac{1}{2q+1} e^{-2\pi i j \lambda} \\
&= \frac{\sum_{j=-1}^{-q} e^{-2\pi i j \lambda} + 1 + \sum_{j=1}^{q} e^{-2\pi i j \lambda}}{2q+1} \\
&= \frac{\sum_{j=0}^{-q} e^{-2\pi i j \lambda} - 1 + \sum_{j=0}^{q} e^{-2\pi i j \lambda}}{2q+1}
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- When $\lambda = 0$ it is easy to see that and $A(0) = \frac{q+1-1+q+1}{2q+1} = 1$.
- When $\lambda \neq 0$ then $\exp(2\pi i \lambda) \neq 1$ and this function can be evaluated using the geometric series formula, e.g. $\sum_{j=0}^{q} e^{-2\pi i j \lambda} = \frac{1-e^{2\pi i \lambda (q+1)}}{1-e^{2\pi i \lambda}}$.

Example: Power Transfer Function of Smoothing Filter

- Then, because

$$e^{i\theta} - 1 = \cos\theta + i\sin\theta - 1 = 2e^{i\theta/2}\sin(\theta/2)$$

we get

$$S_q(\lambda) = \frac{\sin \pi q\lambda}{\sin \pi\lambda} e^{i\pi\lambda(q-1)}.$$

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- Thus

$$S_q(\lambda) + S_q(-\lambda) = 2\frac{\sin(\pi q\lambda)}{\sin(\pi\lambda)}\cos(\pi\lambda(q-1)),$$

which implies that the transfer function is given by

$$A(\lambda) = \frac{1}{2q+1}\left(2\frac{\sin(\pi(q+1)\lambda)}{\sin(\pi\lambda)}\cos(\pi q\lambda) - 1\right).$$

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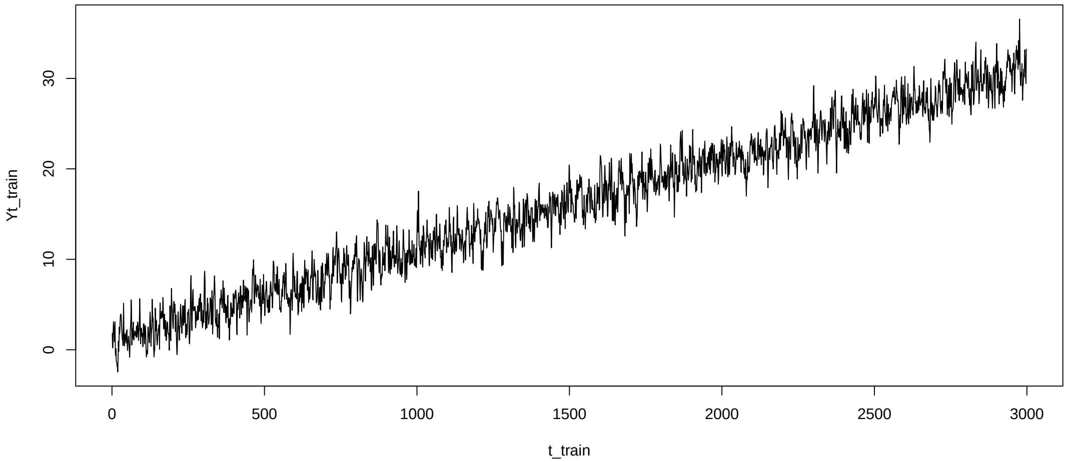
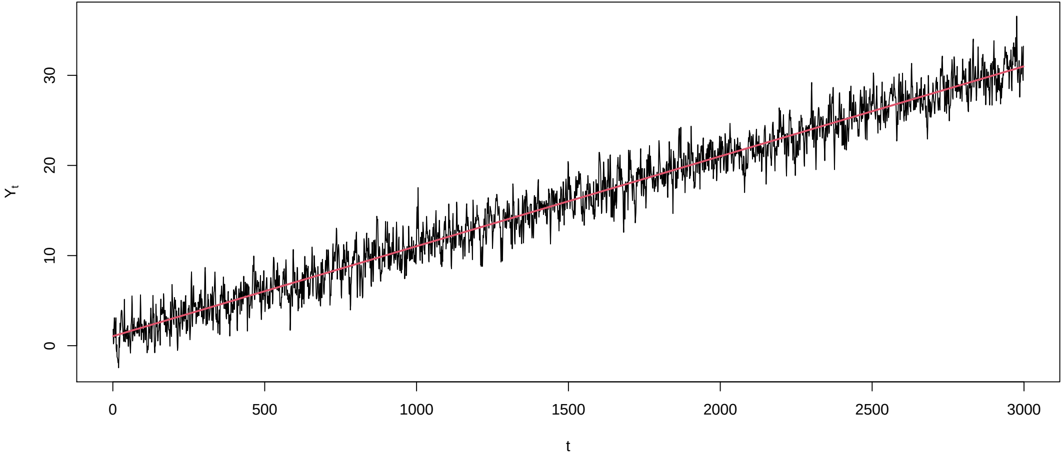
- For q large, it drops to zero very quickly $\Rightarrow$ the filter kills the high frequency components in the input process.

Spectral density of ARMA process

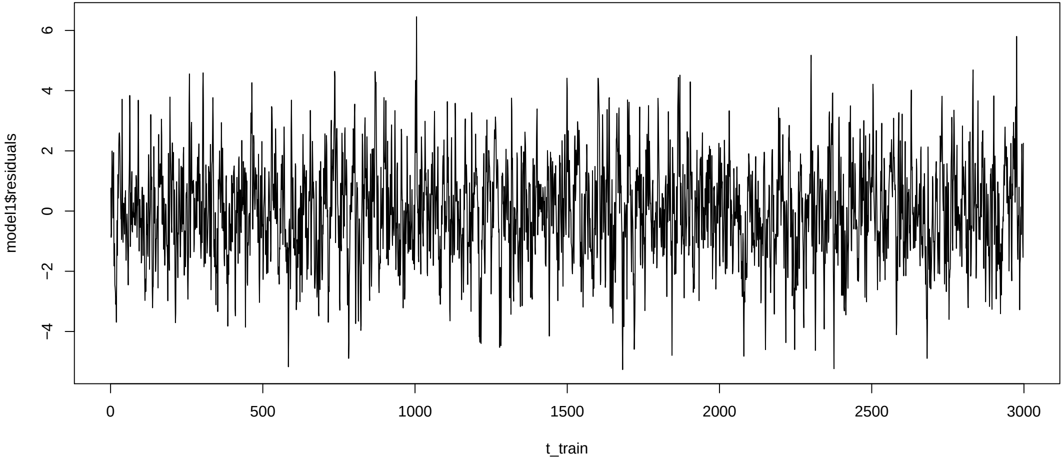
Big Picture from Lecture 6b: modeling and forecasting



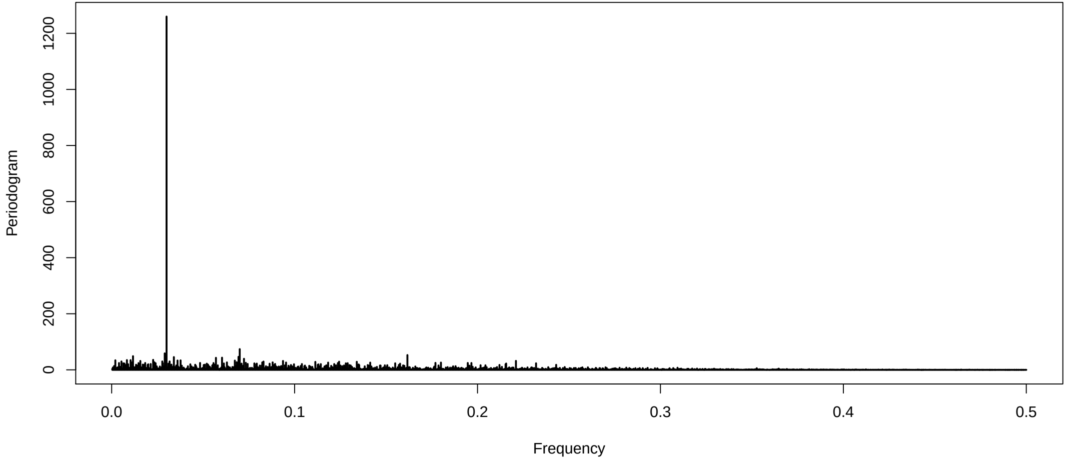
Model the linear trend



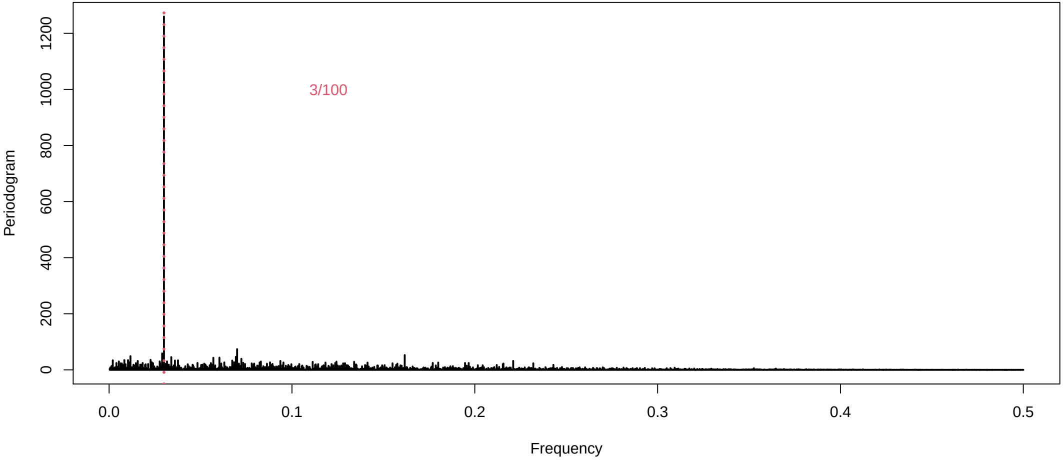
Residuals with Trend removed



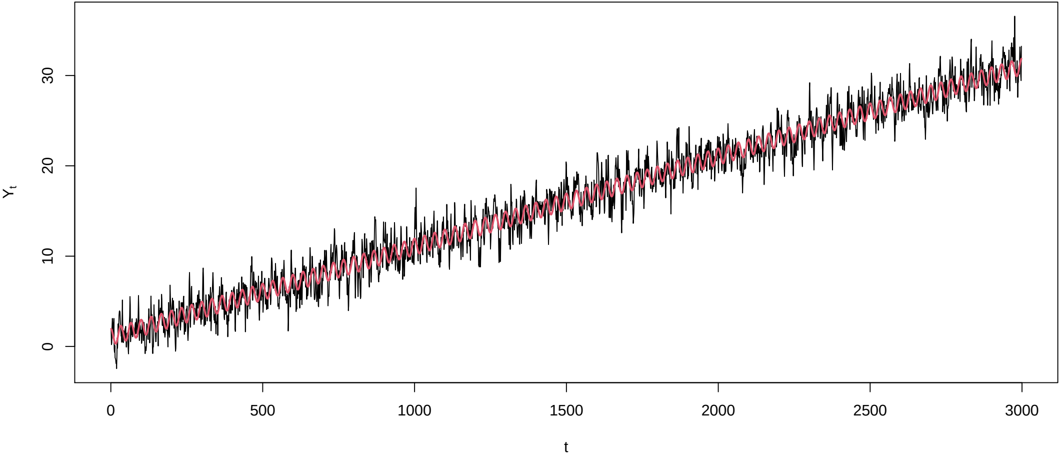
No more trend, check periodogram for seasonality



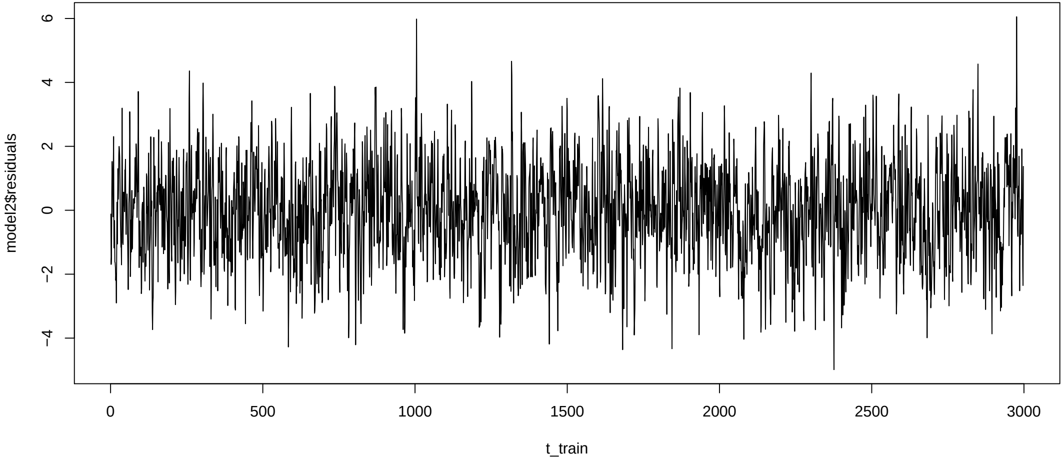
Frequency is clearly 3/100



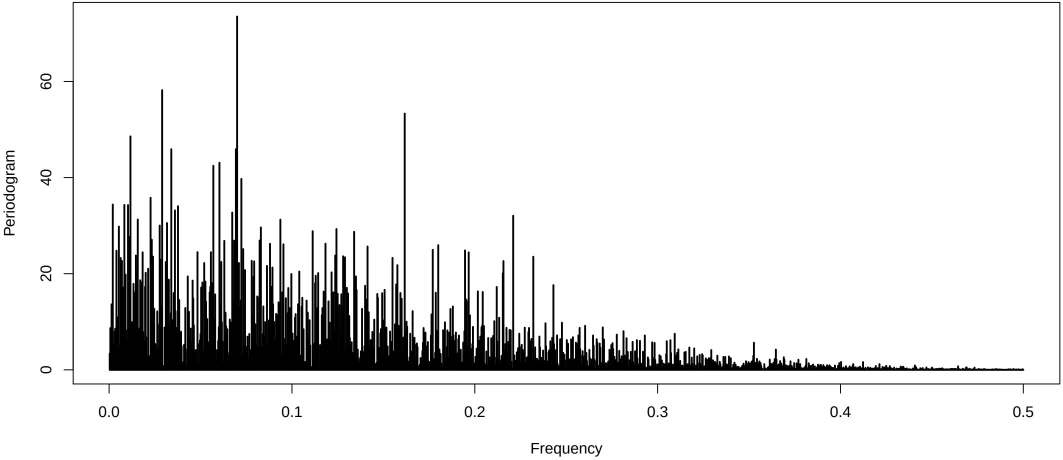
Add Sinusoid to model



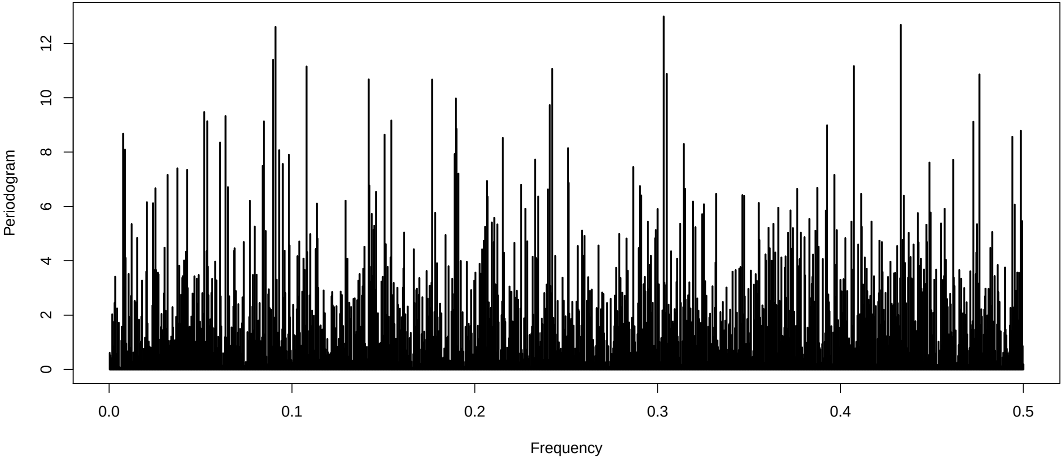
Residuals without Linear Trend and Sinusoid



No more large spikes either



For reference: Periodogram of Gaussian Noise



Comment

Because we know

$$f_Y(\lambda) = f_X(\lambda) \left| A(\lambda) \right|^2 \text{ for } -1/2 \leq \lambda \leq 1/2$$

we can compute the spectral density of the unique stationary solution of a causal ARMA process.

Theorem: Spectral Density of ARMA Process

Let $\{X_t\}$ be a stationary causal ARMA process $\phi(B)X_t = \theta(B)W_t$ with ϕ and θ having no common roots.

Then, for the definition of spectral density f_X of $\{X_t\}$ that uses the ACVF:

$$f(\lambda) := \sum_{h=-\infty}^{\infty} \gamma_X(h) \exp\left(-2\pi i \lambda h\right) \text{ for } -1/2 \leq \lambda \leq 1/2.$$

it holds that

$$f_X(\lambda) = \sigma_W^2 \frac{|\theta(e^{-2\pi ij\lambda})|^2}{|\phi(e^{-2\pi ij\lambda})|^2} \text{ for } -1/2 \leq \lambda \leq 1/2$$

Proof

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- Let $A_\phi(\lambda)$ denote the transfer function of this filter.
- Then we have
$$f_U(\lambda) = |A_\phi(\lambda)|^2 f_X(\lambda).$$

Proof (page 2)

- Similarly, using the fact that $U_t = \theta(B)W_t$ and that the spectral density of white noise is constant, $f_W(\lambda) = \sigma_W^2$, we write

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where $A_\theta(\lambda)$ is the transfer function of the filter with coefficients $a_0 = 1$ and $a_j = \theta_j$ for $1 \leq j \leq q$ and $a_j = 0$ for all other j.

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- Equating the two $f_U(\lambda)$,

$$f_X(\lambda) = \frac{|A_\theta(\lambda)|^2}{|A_\phi(\lambda)|^2}\sigma_W^2 \text{ for } -1/2 \leq \lambda \leq 1/2.$$

Proof (page 3)

- Now

$$\begin{aligned} A_\phi(\lambda) &= 1 - \phi_1 e^{-2\pi i(1)\lambda} - \phi_2 e^{-2\pi i(2)\lambda} - \cdots - \phi_p e^{-2\pi i(p)\lambda} \\ &= \phi(e^{-2\pi ij\lambda}) \end{aligned}$$

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- Note that the denominator $A_\phi(\lambda)$ is non-zero for all λ because of stationarity.
- Similarly $A_\theta(\lambda) = \theta(e^{-2\pi ij\lambda})$, which completes the proof:
$$f_X(\lambda) = \sigma_W^2 \frac{|\theta(e^{-2\pi ij\lambda})|^2}{|\phi(e^{-2\pi ij\lambda})|^2} \text{ for } -1/2 \leq \lambda \leq 1/2$$

Example: MA(1)

For the MA(1) process: $X_t = W_t + \theta W_{t-1}$, we have $\phi(z) = 1$ and $\theta(z) = 1 + \theta z$. Therefore

$$
\begin{aligned}
f_X(\lambda) &= \sigma_W^2 \left|1 + \theta e^{2\pi i \lambda}\right|^2 \\
&= \sigma_W^2 \left|1 + \theta \cos 2\pi\lambda + i\theta \sin 2\pi\lambda\right|^2 \\
&= \sigma_W^2 \left[(1 + \theta \cos 2\pi\lambda)^2 + \theta^2 \sin^2 2\pi\lambda\right] \\
&= \sigma_W^2 \left[1 + \theta^2 + 2\theta \cos 2\pi\lambda\right] \text{ for } -1/2 \le \lambda \le 1/2.
\end{aligned}
$$

Check that for $\theta = -1$, the quantity $1 + \theta^2 + 2\theta\cos(2\pi\lambda)$ equals the power transfer function of the first differencing filter.

Example: MA(1)

Visualize in code!

Example: AR(1)

For AR(1): $X_t - \phi X_{t-1} = W_t$, we have $\phi(z) = 1 - \phi z$ and $\theta(z) = 1$. Thus

$$f_X(\lambda) = \sigma_W^2 \frac{1}{|1 - \phi e^{2\pi i \lambda}|^2} = \frac{\sigma_W^2}{1 + \phi^2 - 2\phi \cos 2\pi\lambda}$$

for $-1/2 \leq \lambda \leq 1/2$.

Example: AR(2)

For the AR(2) model: $X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = W_t$, we have $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$ and $\theta(z) = 1$. Here it can be shown that

$$f_X(\lambda) = \frac{\sigma_W^2}{1 + \phi_1^2 + \phi_2^2 - 2\phi_1(1-\phi_2)\cos 2\pi\lambda - 2\phi_2 \cos 4\pi\lambda}$$

for $-1/2 \leq \lambda \leq 1/2$.

Various other examples

In code and on the whiteboard

Parametric Spectral Density Estimation

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- For convenience, usually a parametric spectral estimator is obtained by fitting an AR(p) model, where the order p is determined by model selection such as AIC or BIC.
- The following theorem shows that any spectral density can be approximated arbitrary close by the spectrum of an AR process, see Property 4.7 in TSA4e.

Theorem: AR Spectal Approximation

Let $g(\lambda)$ be the spectral density of a stationary process. Then, given $\epsilon > 0$, there is a time series with the representation

$$\phi(B)X_t = W_t,$$

for some finite order p polynomial ϕ and some white noise W_t with variance σ^2, such that

$$|f_X(\lambda) - g(\lambda)| < \epsilon \quad \text{for all } \lambda \in [-1/2, 1/2].$$

Moreover, the roots of ϕ outside the unit circle.

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- For further reading on parametric density estimation see TSA4e Chapter 4.5.