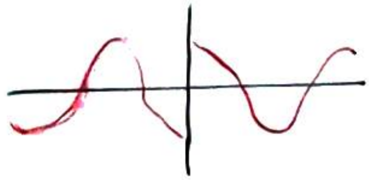
7/2/18 Lecture Notes: Separation of Variables

Last week: Solve $v_{tt} - c^2 v_{xx} = 0 \quad 0 < x < \infty, t > 0$
$v(x,0) = \phi(x) \quad v_t(x,0) = \psi(x) \longrightarrow$ Solve $u_{tt} - c^2 u_{xx} = 0 \quad -\infty < x < \infty, t > 0$
$v(0,t) = 0 \qquad u(x,0) = \phi_{odd}(x) \quad u_t(x,0) = \psi_{odd}(x)$



$$(*) \left[\begin{array}{l} \text{Solve } u_{tt} - c^2 u_{xx} = 0 \quad 0 < x < \ell, \ t > 0 \\ u(x,0) = \phi(x) \quad u_t(x,0) = \psi(x) \\ u(0,t) = u(\ell,t) = 0 \end{array} \right.$$

Reflection method $\rightarrow$ Complicated Formula

<u>Separation of Variables</u>

Find separated solutions $u(x,t) = X(x)T(t)$

Q: What is an example of a function NOT of this form?

Reasons: 1) Can't solve hard problem $\rightarrow$ Solve easier version
2) Will be able to obtain more solutions via linearity
3) Reduces to ODE

2nd-order ODE review:

Solve $ay'' + by' + cy = 0$ If $y = e^{rt}$, $ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$

$$e^{rt} \underbrace{[ar^2 + br + c]}_{\text{find roots } r_1, r_2} = 0$$

$y = a_1 e^{r_1 t} + a_2 e^{r_2 t}$

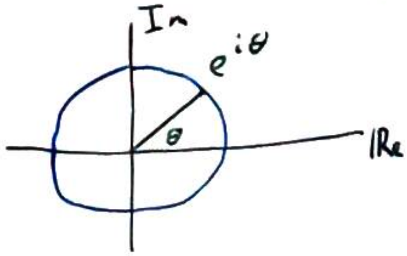
If $r_1, r_2 \in \mathbb{R}$ (real), answer is simple.

Only complication: $r_1 = r_2 = r \longrightarrow y = a_1 e^{rt} + a_2 t e^{rt}$

If $r_1, r_2 \in \mathbb{C}$ (complex-valued),

$r = a \pm bi$ (complex roots to real-coefficient polynomials come in conjugate pairs)

Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$



$$\begin{cases} e^{(a+bi)t} = e^{at}\cos bt + ie^{at}\sin bt \\ e^{(a-bi)t} = e^{at}\cos bt - ie^{at}\sin bt \end{cases}$$

"Change of Basis"

$$y = c_1 e^{at}\cos bt + c_2 e^{at}\sin bt$$

$$u_{tt} - c^2 u_{xx} = 0 \longrightarrow X(x)T''(t) = c^2 X''(x)T(t)$$

$$-\frac{T''}{c^2 T} = \frac{-X''}{X} = \lambda \text{ constant}$$

(Check $\lambda_t = \lambda_x = 0$ by swapping which expression you look at)

$X'' + \lambda X = 0$ $\quad u(0,t) = 0 \quad u(\ell,t) = 0$

$X(0)T(t) = 0 \quad \downarrow \quad$ $T(t) = 0$ is trivial solution

$X(0) = 0 \quad X(\ell) = 0$

$T'' + \lambda c^2 T = 0$

Solutions depend on sign of λ.

$\lambda < 0$: $X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$

$0 = X(0) = A + B$

$0 = X(\ell) = Ae^{\ell\sqrt{-\lambda}} + Be^{-\ell\sqrt{-\lambda}} \Rightarrow A = B = 0$

No nontrivial solutions

$\lambda = 0$: $X(x) = Cx + D$

$X(0) = D = 0$

$X(\ell) = C\ell + D = 0 \Rightarrow C = D = 0$

λ complex: Let $\gamma^2 = -\lambda$, so $X(x) = Ae^{\gamma x} + Be^{-\gamma x}$

$X(0) = A + B$

$X(\ell) = Ae^{\gamma\ell} + Be^{-\gamma\ell}$ $\xrightarrow[\text{determinant}]{\text{check}}$ $e^{2\gamma\ell} = 1$

$2\gamma\ell = 2\pi n i$

$\gamma = \frac{n\pi i}{\ell}$

Nontrivial solutions precisely when $\boxed{-\gamma^2 = \left(\frac{n\pi}{\ell}\right)^2 = \lambda > 0}$

What are solutions when $\lambda = \left(\frac{n\pi}{\ell}\right)^2$?

$$X'' + \left(\frac{n\pi}{\ell}\right)^2 X = 0$$

$$X = C\cos\frac{n\pi}{\ell}x + D\sin\frac{n\pi}{\ell}x$$

$X(0) = C = 0$

$X(\ell) = D\underbrace{\sin\frac{n\pi}{\ell}\cdot\ell}_{0} = 0$

No restriction on D

$X_n(x) = \sin\frac{n\pi}{\ell}x$

Back to $T'' + c^2\lambda T = 0$

$$T_n(t) = A\cos\frac{n\pi c}{\ell}t + B\sin\frac{n\pi c}{\ell}t$$

By linearity, solutions to $u_{tt} - c^2 u_{xx} = 0 \quad 0 < x < \ell \quad t > 0$

$u(0,t) = 0 = u(\ell,t)$

include

$$\boxed{u(x,t) = \sum_n \left(A_n\cos\frac{n\pi c t}{\ell} + B_n\sin\frac{n\pi c t}{\ell}\right)\sin\frac{n\pi x}{\ell}}$$

What about initial conditions?

Check $u(x,0) = \sum_n A_n \sin\frac{n\pi x}{\ell} \overset{?}{=} \phi(x)$

$$u_t(x,t) = \sum_n\left(-\frac{n\pi c}{\ell}A_n\sin\frac{n\pi c t}{\ell} + \frac{n\pi c}{\ell}B_n\cos\frac{n\pi c t}{\ell}\right)\sin\frac{n\pi x}{\ell}$$

$$u_t(x,0) = \sum_n \frac{n\pi c}{\ell}B_n\sin\frac{n\pi x}{\ell} \overset{?}{=} \psi(x)$$

Q: Can we write ϕ, ψ as finite sums of sin and cos?

What about infinite sums?

Your turn:

(1) Solve $u_{tt} - c^2 u_{xx} = 0 \quad 0 < x < \ell,\ t > 0$

$u(x,0) = \phi(x) \quad u_t(x,0) = \psi(x)$

$u_x(0,t) = u_x(\ell,t) = 0$

Neumann Problem

(2) Solve $u_t - k u_{xx} = 0 \quad 0 < x < \ell,\ t > 0$

$u(x,0) = \phi(x)$

$u(0,t) = u(\ell,t) = 0$

Heat Equation