F20 PHYSICS 137B: Section notes for Born approximation

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In these notes, I work out a scattering problem in detail, so that you can get a feel for the general technique for solving such problems.

1 A brief review

In lecture, you saw the (first) Born approximation as:

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}|\tilde{V}(\mathbf{k}-\mathbf{k}')|^2, \tag{1.1}$$

where $\tilde{V}$ is the Fourier transform of the scattering potential V. This is the first term in a series approximation – the details require us to develop the basic theory of Green's functions, which we have not covered. If you are interested, take a look at Griffiths 10.4.3.

Some special cases of the Born approximation are:

- Low energy scattering: we take $|\mathbf{k}-\mathbf{k}'| \to 0$, so:

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int V(\mathbf{r})\, d^3\mathbf{r}\right|. \tag{1.2}$$

  Note that this case is best dealt with using partial wave expansion.

- Spherically symmetric scattering:

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{\hbar^4 k^2 \sin^2(\theta/2)}\left|\int rV(r)\sin(\kappa r)\, dr\right|^2, \tag{1.3}$$

  where $\kappa = 2k\sin(\theta/2)$.

I would generally cover the Yukawa potential as well, but it looks like you did it during lecture, so make sure to review that derivation as well!

When is the Born approximation valid? The general form of the scattered wavefunction is

$$\psi_s(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}V(\mathbf{r}_0)\psi_s(\mathbf{r}_0)d^3\mathbf{r}_0. \tag{1.4}$$

We want the scattered part to be smaller than the incident part; consider the case of a scattering potential with characteristic length a, and $ka < 1$. The exponent in the integral is order 1 (it is a phase), while the denominator is order a, so the integral is of order $\frac{ma^2|V|}{\hbar^2}$ and the validity condition is $\frac{ma^2|V|}{\hbar^2} \ll 1$. We see that this means, roughly, the kinetic energy of a particle confined in the potential should be much larger than the potential.

In the limit $ka \gg 1$, then the exponential oscillates rapidly. We would apply something called the "stationary phase approximation" or "saddle point" approximation – the phase mostly cancels everywhere, and the integral is dominated by places where the derivative vanishes. This leads to a condition that $\frac{m|V|a}{\hbar^2 k} = \frac{|V|a}{\hbar v} \ll 1$, where v is the velocity of a particle with momentum $\hbar k$.

2 Griffiths 10.20

We consider the Gaussian potential:

$$V(r) = V_0 e^{-\mu r^2/a^2}. \tag{2.1}$$

This potential is spherically symmetric – it has no dependence on θ or ϕ, so we can use the spherically symmetric formula for the Born approximation:

$$f(\theta) == \frac{2m}{\hbar^2 \kappa} \int_0^\infty r V_0 e^{-\mu r^2/a^2} \sin(\kappa r)\, dr \tag{2.2}$$

This integral can be done by parts (if doing by hand) or using an integral calculator:

$$f(\theta) == -\frac{m V_0 \sqrt{\pi}}{2\hbar^2 \mu^{3/2}} e^{-\kappa^2/4\mu}, \tag{2.3}$$

giving us the cross-section

$$\frac{d\sigma}{d\Omega} = \frac{\pi m^2 V_0^2}{4\hbar^4 \mu^3} e^{-\kappa^2/2\mu}. \tag{2.4}$$

We want the total cross-section, so we should integrate this over all solid angles:

$$\sigma = \int \frac{\pi m^2 V_0^2}{4\hbar^4 \mu^3} e^{-\kappa^2/2\mu} d\Omega \tag{2.5}$$

$$= \frac{\pi^2 m^2 V_0^2}{2\hbar^4 \mu^3} \int e^{-2k^2 \sin^2(\theta/2)/\mu} \sin\theta d\theta \tag{2.6}$$

$$= \frac{\pi^2 m^2 V_0^2}{2\hbar^4 \mu^2 k^2} \left(1 - e^{-2k^2/\mu}\right). \tag{2.7}$$

The integral can be done by substitution $u = \sin(\theta/2)$.

This point is mostly unrelated, but if we wanted to, say, compute the *average* cross-section per solid angle (e.g. if our detector did not subtend the entire solid angle), we would divide the result above by 4π.