Stationarity

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Lecture 1b

Announcements

Assignments

- The Practice Homework (sometimes called HW0) is "due" tomorrow (1/27), but you can submit it anytime for practice.
- Your first graded assignment is the Homework 1, which will be posted today, and is due before 11:59pm Pacific, February 3, on Gradescope.
- The first phase of your project is due Wednesday February 10, on Gradescope. The full assignment instructions will be posted soon, but for now please reply to the poll on what kind of data you would like to analyze.
- Please see the Calendar on bCourses for due dates and to reference the weekly numbering system.

Waitlist and Concurrent Students

- I will add anyone on the waitlist/CE to bCourses so you can access materials and Gradescope
- Being on bCourses does not mean you are enrolled in the course
- I will begin processing applications when waitlist has emptied, and the stat office will give final approval after that
- The waitlist is fuller than I would expect,
- I will accept applications in the order they were received.

Accomodations and Schedule Conflicts

- Please let me know of any conflicts or accomodations, religious or otherwise, as soon as possible.
- If you know that I have or will be receiving an accomodations letter from DSP on your behalf, please reach out to me privately over email as soon as possible to discuss your letter. If I get a letter but don't hear from you, I will be reaching out soon.

Recap

Time Series

- (Univariate time series) we don't have n subjects at 1 point in time, we have 1 subject at n points in time...
- Our model: $Y_t = f(t) + X_t$
- $f(t)$ can be thought of as the signal
- X_t can be thought of as the noise

Definitions (TSA4e Example 1.8)

Random variables $X_1, ..., X_n$ will be denoted as

- White noise: if they have mean zero, variance σ^2, and are uncorrelated
- IID noise: if they are white noise AND are independent and identically distributed (IID).
- Gaussian [white] noise: if they are IID noise AND are normally distributed, $X_i \sim N(0, \sigma^2)$

Definitions:

- Autocovariance (Definition 1.2):

$$\begin{aligned}\gamma_x(s,t) &= cov(X_s, X_t)\\ &= E\left[(X_s - E[X_s])(X_t - E[X_t])\right]\end{aligned}$$

- [mental aside: let $s > t$ and $h = s - t$. h is the number of "lags"]
- Sample autocovariance (Definition 1.14):

$$\hat{\gamma}(h) = n^{-1}\sum_{t=1}^{n-h}(x_{t+h} - \bar{x})(x_t - \bar{x})$$

Definitions:

- Autocorrelation function "ACF" (Definition 1.3):

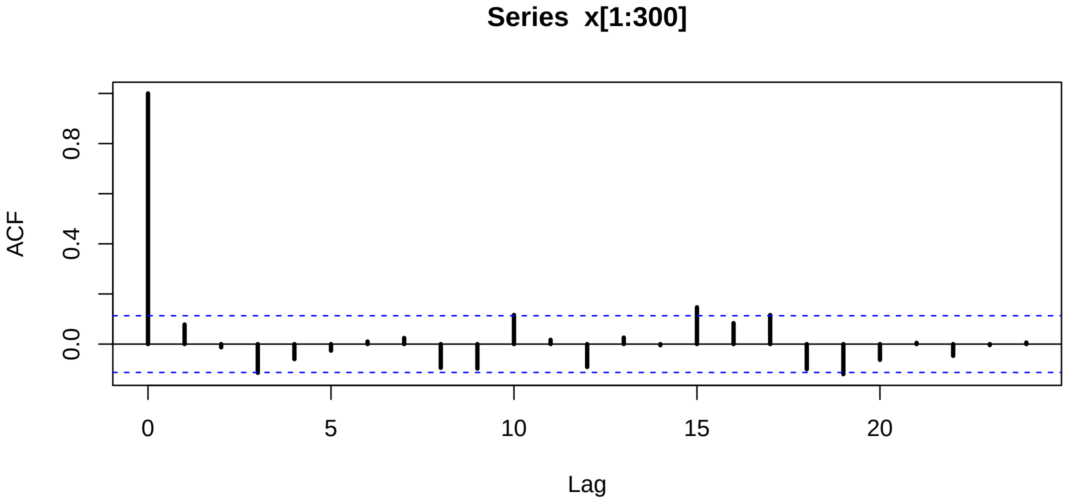
$$\rho(s,t) = \frac{\gamma_x(s,t)}{\sqrt{\gamma_x(s,s)\gamma_x(t,t)}} = \frac{cov(X_s, X_t)}{\sqrt{var(X_s)var(X_t)}}$$

- Sample autocorrelation (Definition 1.15):

$$r_h = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)} = \frac{\sum_{t=1}^{n-h}(x_t - \bar{x})(x_{t+h} - \bar{x})}{\sum_{t=1}^{n}(x_t - \bar{x})^2}$$

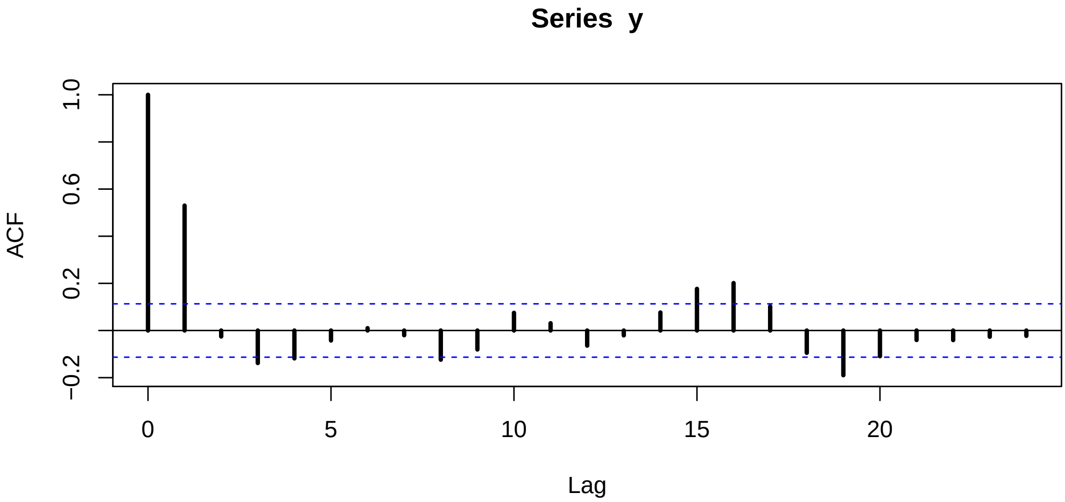
ACF plot

```
x = rnorm(301)
acf(x[1:300],lwd=3)
```



ACF plot - simple Moving Average

```
y = .5*(x[1:300] + x[2:301])
acf(y,lwd=3)
```



CI for Sample Correlations

- Wouldn't it be great if those dashed blue lines were the appropriate confidence interval?

Simplified Theorem A.7 (see Property 1.2)

- Under general conditions, if x_t is white noise, then for n large, and with arbitrary but fixed H, then the sample autocorrelations

$$r_1, ..., r_H \overset{iid}{\sim} N(0, 1/n)$$

- In other words

$$\sqrt{n} \begin{pmatrix} r_1 \\ \vdots \\ r_H \end{pmatrix} \to N(0, I) \quad \text{as } n \to \infty$$

- Key takeaway: $var(r_h) = 1/n$ (Equation 1.38)

Confidence Interval

- For white noise $r_1, ..., r_H \overset{iid}{\sim} N(0, 1/n)$, for

$$P\left(|r_h| > 1.96n^{-\frac{1}{2}}\right) \approx P\left(|N(0,1)| > 1.96\right) = 5\%$$

- So for $n = 100$, $1.96n^{-\frac{1}{2}} = 1.96/\sqrt{100} = .196$

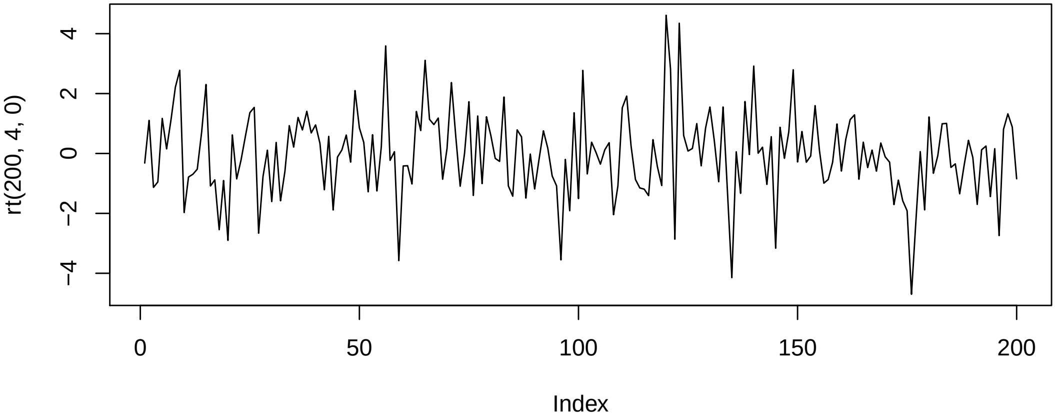
Expand our definition of Noise

Modeling Idea

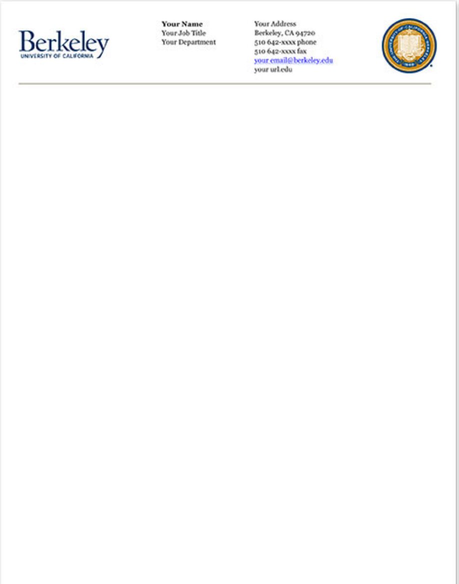
- $Y_t = f(t) + X_t$
- In this class we'll first model f such that $y_t - \hat{f}(t)$ exhibits steady behavior over time
- But what do we mean by "steady" or "stable"?
- White noise is correct, but we're going to look at a broader group of processes as steady
- "Stationary" is the adjective, "Stationarity" is the noun

White Noise

```
plot(rt(200,4,0),type='l')
```



Stationery



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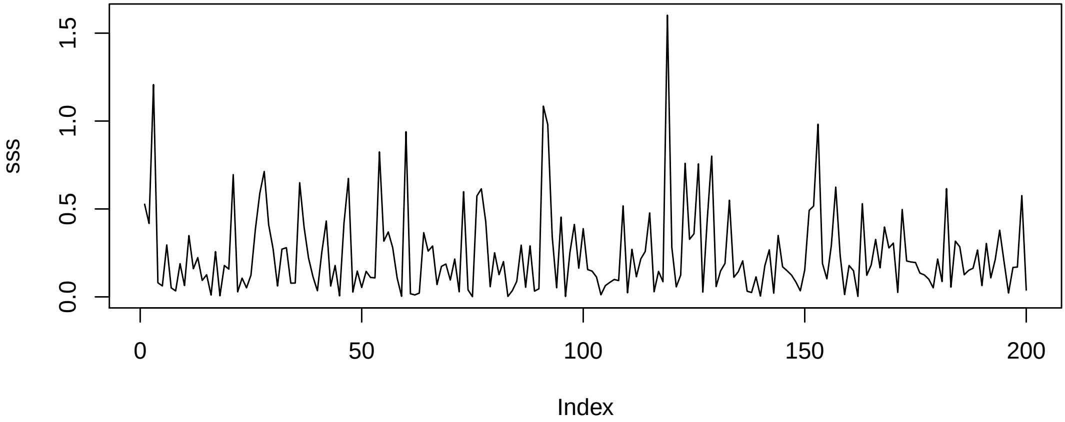
510 642-xxxx fax
your email@berkeley.edu
your url.edu

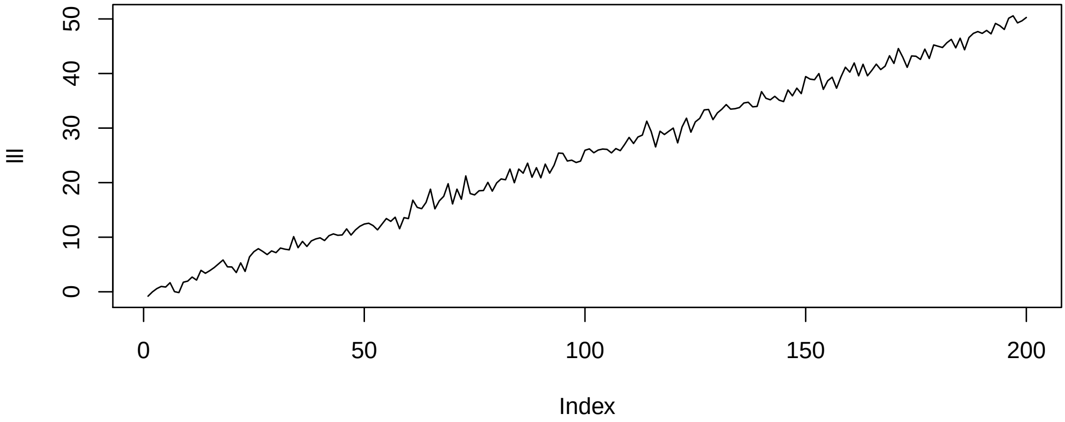
Stationary (but not White Noise, why?)

```
sss = rexp(200,4)
plot(sss,type='l')
```



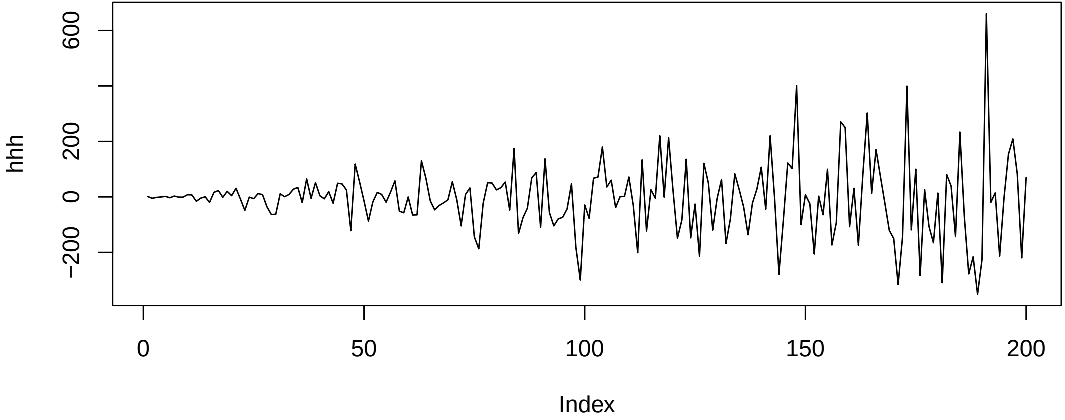
Not Stationary - linear trend

```
lll = (1:200)/4+rt(200,4,0)
plot(lll,type='l')
```

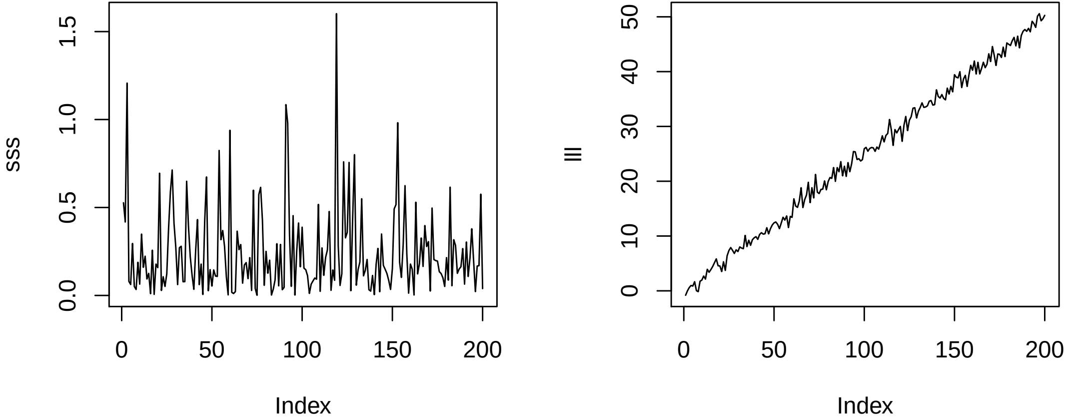


Not Stationary - heteroskedastic

```r
hhh = rnorm(200,0,1:200)
plot(hhh,type='l')
```



Side by Side



sss
0.0
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200
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lll
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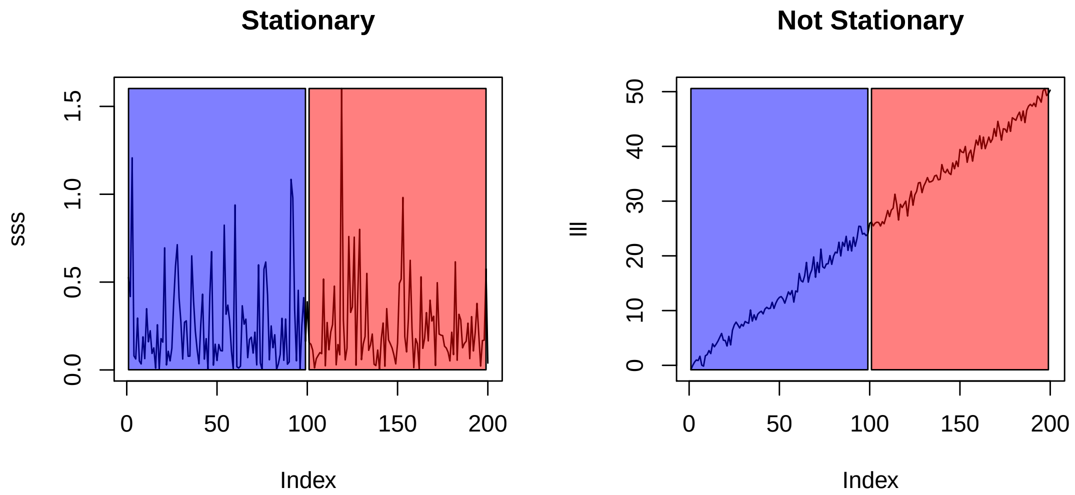
Side by Side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Side by Side

```
## `stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
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```
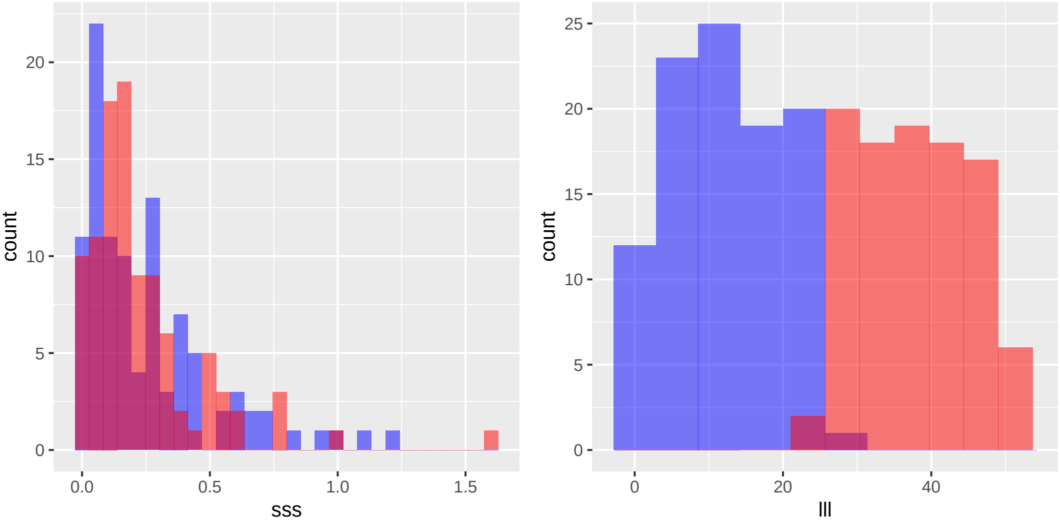
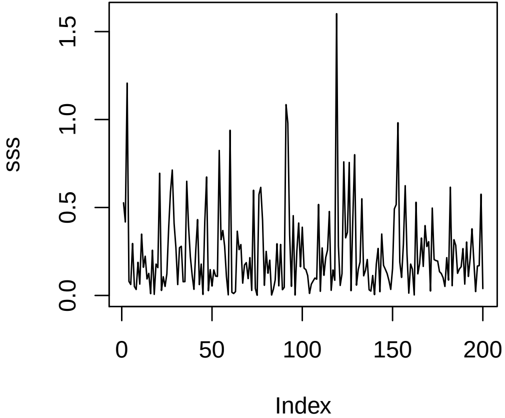



Side by Side



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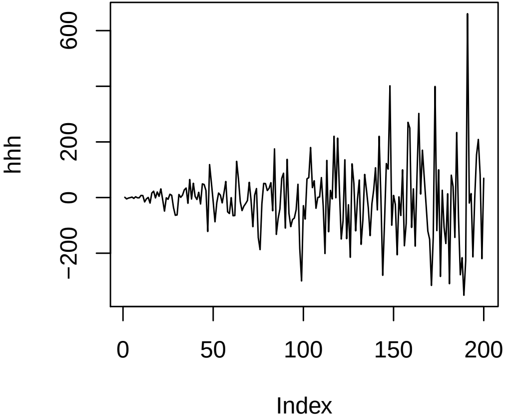

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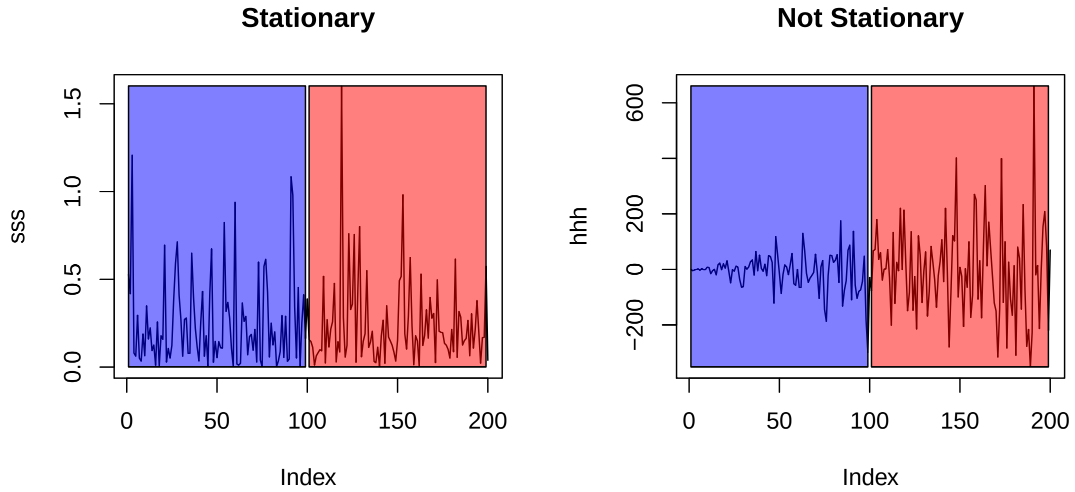
Side by Side



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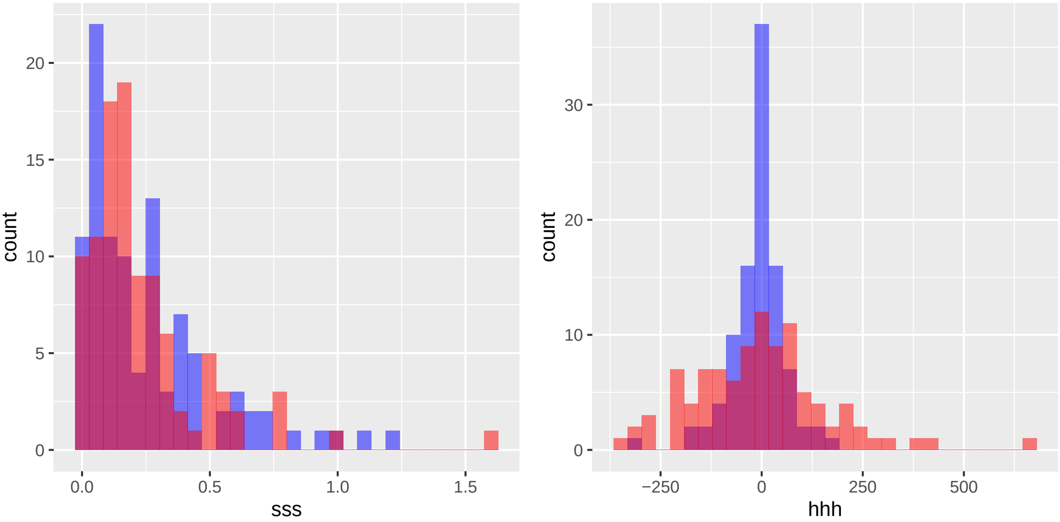
Side by Side



Stationary

Stationary essentially means that the dependence structure of $\{X_t\}$ is invariant over time and hence, we can learn while observing more and more data.

(we'll pause here, then formally define "stationary")

Strictly Stationary (TSA4e Definition 1.6)

A strictly stationary time series is one for which the probabilistic behavior of every collection of values $(X_{t_1}, X_{t_2}, ..., X_{t_k})$ is identical to that of the time shifted set $(X_{t_{1+h}}, X_{t_{2+h}}, ..., X_{t_{k+h}})$. That is,

$Pr((X_{t_1} \leq c1, ..., X_{t_k} \leq c_k) = Pr(X_{t_{1+h}} \leq c_1, ..., X_{t_{k+h}} \leq c_k)$ for

- all $k = 1, 2, ...$
- all time points $t_1, t_2, ..., t_k$
- all numbers $c_1, c_2, ..., c_k$
- all time shifts $h = ..., -2, -1, 0, 1, 2, ...$

Strictly Stationary - Rephrased Definition

A doubly infinite sequence of random variables $\{Xt\}$:

$$..., X_{-2}, X_{-1}, X_0, X_1, X_2, ...$$

is strictly stationary if for every choice of times $t_1, ..., t_k$ and lag h, the joint distribution of

$$(X_{t_1}, X_{t_2}, ..., X_{t_k})$$

is the same as the joint distribution of

$$(X_{t_{1+h}}, X_{t_{2+h}}, ..., X_{t_{k+h}}).$$

Challenge

- Really hard, if not impossible, to show a time series is strictly stationary in applied practice...
- So we'll usually look at something else

Weakly Stationary (TSA4e Definition 1.7)

A weakly stationary time series, X_t, is a finite variance process such that

1. the mean value function, $E(X_t)$, is constant and does not depend on time t, and
2. the autocovariance function, $\gamma(s, t)$ depends on s and t only through their difference $|s - t|$.

Weakly Stationary - Rephrased Definition

A doubly infinite sequence of random variables $\{X_t\}$ is weak stationary if

1. The mean of the random variable X_t, denoted by $E(X_t)$, is the same for all times t
2. The covariance between X_t and X_s is the same as the covariance between X_{t+h} and X_{s+h} for every choice of times t, s, and lag h.

Stationarity

In this course, when we say "stationary" we will always mean "weakly stationary".

Example: White Noise

Let $\{X_t\}$ be white noise. Recall the definition of white noise:

- $E(X_t) = 0, \quad \forall t$
- $Var(X_t) = \sigma^2, \quad \forall t$
- $Cov(X_t, X_s) = \begin{cases} 0 & t \neq s \\ \sigma^2 & t = s \end{cases}$

Is White Noise stationary?

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Is White Noise stationary?

- $E(X_t)$ is the same for all t
- $Cov(X_t, X_s)$ only depends on $|s - t|$ not on s or t individually:
- $|s - t| = 0 \Rightarrow Cov(X_t, X_s) = \sigma^2$
- $|s - t| > 0 \Rightarrow Cov(X_t, X_s) = 0$
- Yes, white noise is stationary, and we similarly see that IID noise and Gaussian noise are stationary too.

Example: IID Noise

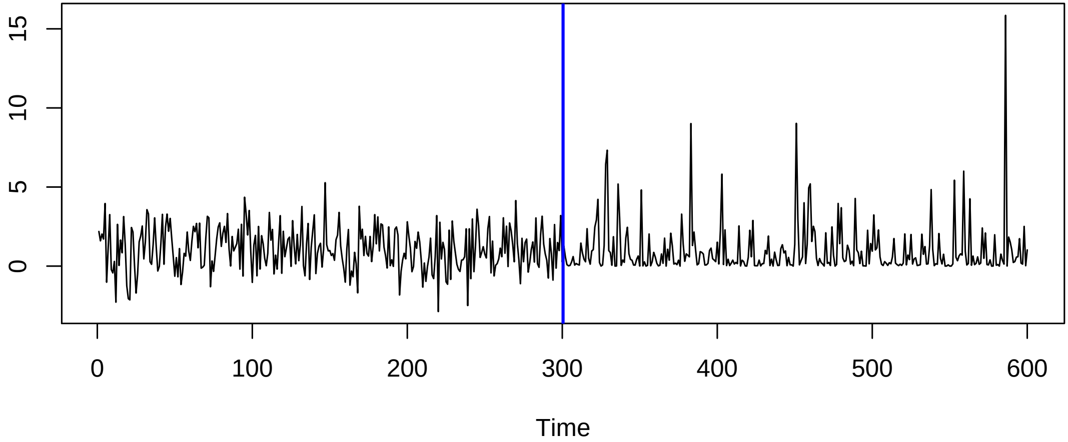
Let $\{X_t\}$ be IID noise. Recall the definition of IID noise:

- $\{X_t\}$ is white noise
- X_t and X_s are independent for all $t \neq s$
- $X_t \sim A$ for all t (identically distributed)

Is IID Noise strongly stationary?

- For $k = 1$ and any h, the joint distribution of $(X_{t_1}, X_{t_2}, ..., X_{t_k})$ is the same as the joint distribution of $(X_{t_{1+h}}, X_{t_{2+h}}, ..., X_{t_{k+h}})$ for $k = 1$ (3rd point above)
- For $k > 1$, because of independence, the multivariate distribution will be the same too.
- So yes, IID noise is strongly stationary, as is Gaussian noise (The case where A is the Normal/Gaussian distribution).

Example: Half Gaussian, Half Chi-Square



Extension: Stationarity and Gaussian Processes

A Few Extra Notes on Stationarity

- Note that the concept of stationarity (both weak and strong), as well as the notion of autocovariance and autocorrelation functions $\gamma(h)$ and $\rho(h)$ applies to the random variables $\{X_t\}$, not to a specific data set (which is a single realization of the random variables $\{X_t\}$).
- Thus, strictly speaking, we cannot say that a particular data set is stationary. We can only say that a particular data set is a realization of stationary time series random variables.
- Any strong stationary time series with existing means and autocovariances is also weakly stationary. The other direction does not hold, in general.

Gaussian Process

- An exception, where weak and strong stationarity are equivalent are Gaussian processes.
- Definition: The sequence $\{X_t\}$ is said to be a **Gaussian process** if for every choice of times $t_1, \ldots, t_k$ the joint distribution of $(X_{t_1}, \ldots, X_{t_k})$ is multivariate normal.
- Recall that $(X_{t_1}, \ldots, X_{t_k})$ is multivariate normal if and only if every *linear combination* of $(X_{t_1}, \ldots, X_{t_k})$ is univariate normal.
- In particular, it is much stronger than saying that each of $X_{t_1}, \ldots, X_{t_k}$ has a univariate normal distribution.
- The multivariate normal distributions are uniquely determined by their means and covariances. Hence, we obtain

$$\text{Weak Stationarity} + \text{Gaussian Process} \implies \text{Strong Stationarity}$$

Gaussian Process

We won't dive into Gaussian processes in this course any further though. But, now try a Google Image search for "Gaussian Process", it's pretty cool