6/20/18 Lecture Notes: Wave Equation Examples + Principles, Some Heat Equation

Last time: Wave equation $u_{tt} - c^2 u_{xx} = 0$

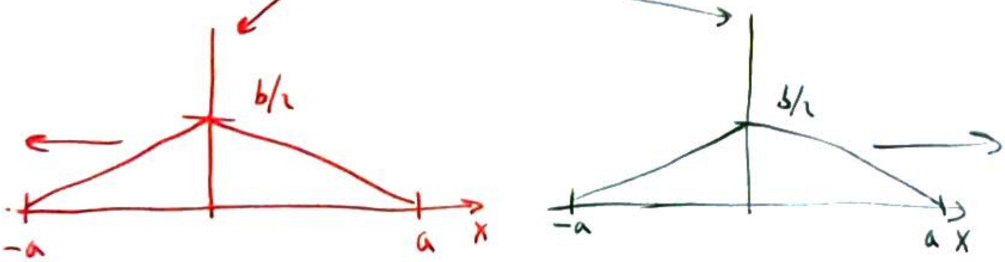
Solution: $u(x,t) = f(x+ct) + g(x-ct)$

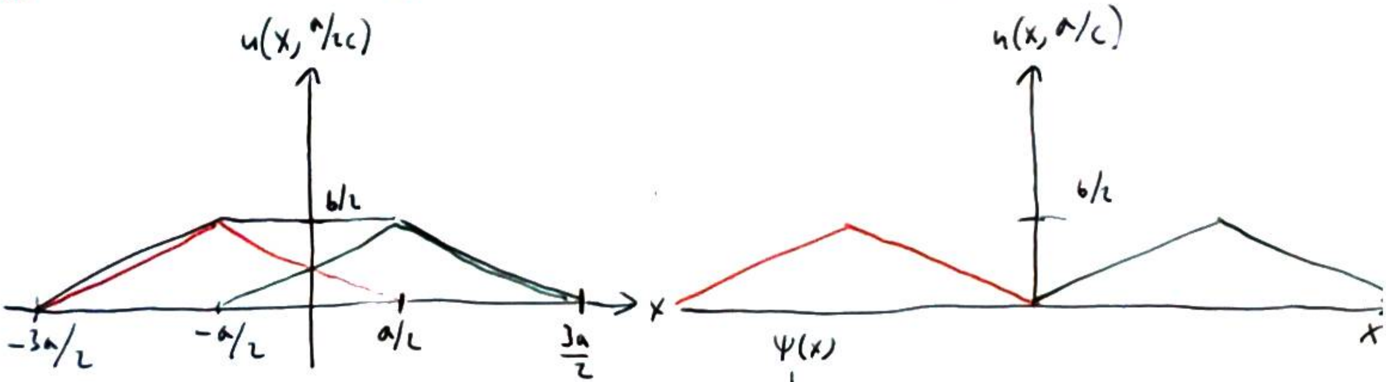
Initial Conditions: $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$

d'Alembert's formula: $u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi$

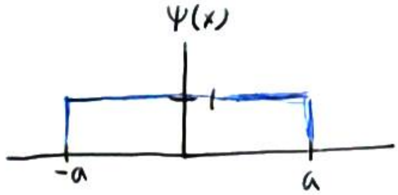
Examples 1) The Plucked String $\phi(x)$ $\qquad\qquad \psi(x) = 0$

$u(x,t) = \underline{\frac{1}{2}\phi(x+ct)} + \underline{\frac{1}{2}\phi(x-ct)}$

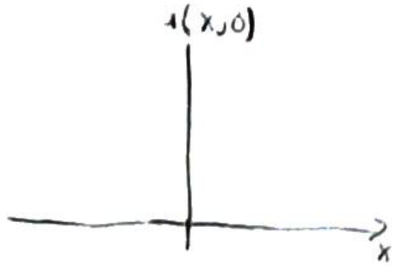




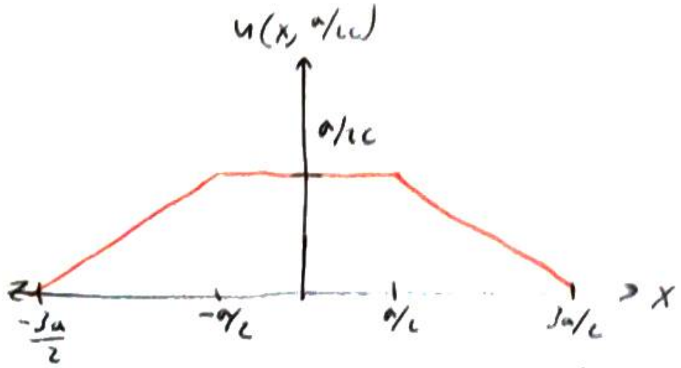
2) The Hammer Blow: $\phi(x) = 0$



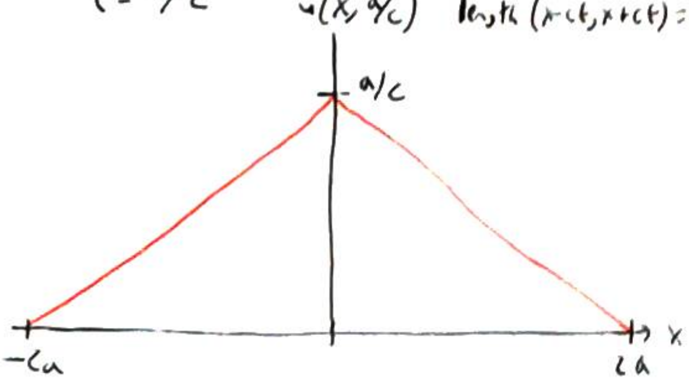
$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds = \frac{1}{2c}\,\text{length}\left[(x-ct, x+ct) \cap (-a, a)\right]$$



$t = a/2c$ length$(x-ct, x+ct) = a$

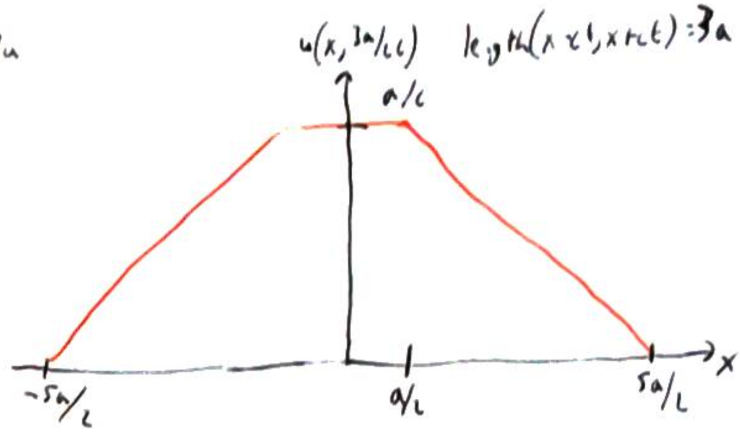


$t = a/c$



length$(x-ct, x+ct) = 2a$

length$(x-ct, x+ct) = 3a$



Why doesn't this happen on a piano?

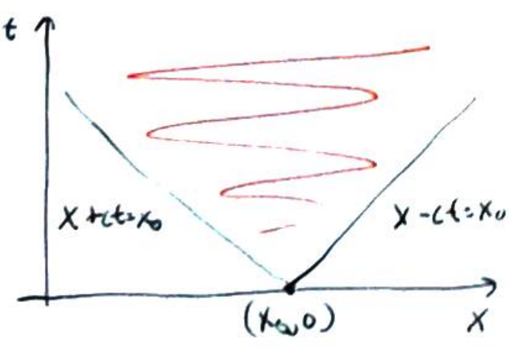
There the domain is bounded, whereas our model has unbounded domain.

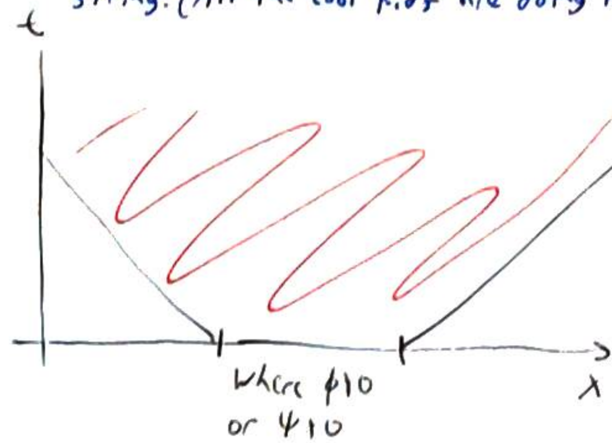
Note: The wave equation is linear, so it suffices to separately consider cases where $\phi = 0$ and $\psi = 0$.

Causality: Waves move at speed $\leq c$

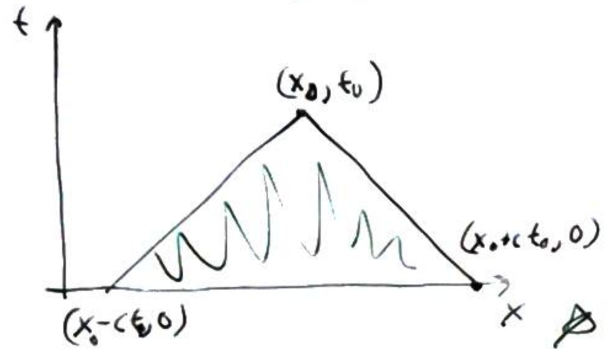
Think about trying to send your friend a message in Morse code by plucking the string. (All the cool kids are doing it.)

View #1: Domain of influence





View #2: Domain of dependence



Note: This is much more interesting in 2 and 3 dimensions.

⊳ Show animations ⊲

Energy Quick IBP review: $(fg)' = fg' + f'g$, so $\int fg' = fg - \int f'g$

Kinetic energy $\sim \frac{1}{2}mv^2 \sim \frac{1}{2}\int \rho u_t^2\, dx$ To think about: Why is integral finite?

Potential energy $\sim \frac{1}{2}\int T u_x^2\, dx$

Define $E(t) := \int_{-\infty}^{\infty} \rho u_t^2 + T u_x^2\, dx$

$$\frac{dE}{dt} = \frac{1}{2}\int_{-\infty}^{\infty} 2\rho u_t u_{tt} + \underbrace{2T u_x u_{xt}}\, dx$$

$$\int_{-\infty}^{\infty} T u_x u_{xt}\, dx \overset{IBP}{=} \underbrace{T u_x u_t \Big|_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} T u_{xx} u_t\, dx$$

Small cheat: Only take u's which decay as $|x| \to \infty$.

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} u_t \underbrace{(\rho u_{tt} - T u_{xx})}_{0} dx = 0$$

Energy is conserved!

To work on with partner: What if $u_{tt} - c^2 u_{xx} = f(x,t)$?

Is energy conserved?

A: Replace the above 0 with $\rho f(x,t)$ to get

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} \rho u_t f(x,t)\, dx$$

>0 if external force in direction of u_t

<0 if " " " opposite direction of u_t

Big concept

Uniqueness Theorem for Wave Equation

If u_1, u_2 both solve

$$u_{tt} - c^2 u_{xx} = f(x,t) \qquad u(x,0) = \phi(x)$$
$$-\infty < x < \infty,\ t \geq 0 \qquad u_t(x,0) = \psi(x)$$

Then $u_1 = u_2$.

Proof Let $v = u_1 - u_2$. We want to show $v \equiv 0$.

$$v_{tt} - c^2 v_{xx} = (u_1 - u_2)_{tt} - c^2 (u_1 - u_2)_{xx}$$
$$= f(x,t) - f(x,t) = 0$$

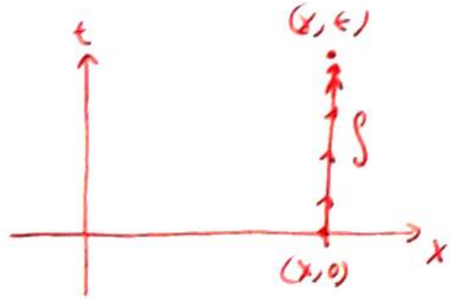
$v(x,0) = v_t(x,0) = 0$, so by energy conservation

$$E(t) = E(0) = \frac{1}{2}\int_{-\infty}^{\infty} \rho \cdot 0^2 + T \cdot 0^2 \, dx = 0$$

$$\frac{1}{2}\int_{-\infty}^{\infty} \rho v_t^2 + T v_x^2 \, dx = 0 \quad \text{for all } t$$

$$\rho v_t^2 + T v_x^2 = 0 \quad \text{for all } t \text{ and all } x.$$

In particular, $v_t \equiv 0$ and



$$v(x,t) = v(x,0) + \int_0^t v_t(x,s)\,ds$$
$$= 0 + 0 = 0$$

Therefore, $u_1 = u_2$.

Wave Equation Lessons

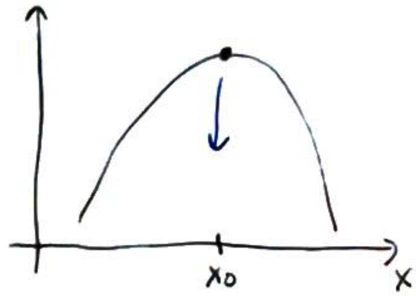
1) Don't just find solutions. Learn to find properties of those solutions. (Sometimes that's all you can do.) Energy, speed of travel

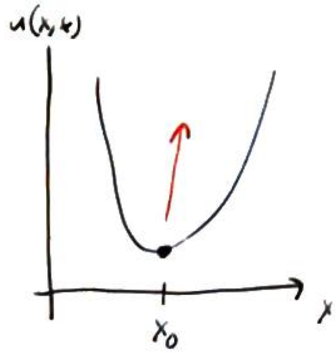
2) Existence and uniqueness of solutions to initial-value problems are important. They can be established in multiple ways.

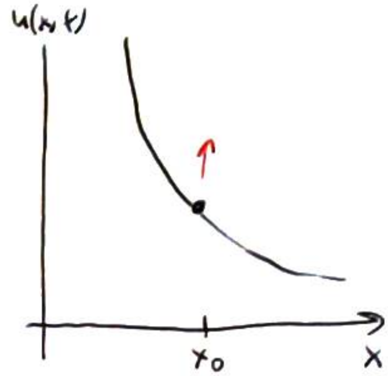
Intro to the Heat Equation

1) Intuition

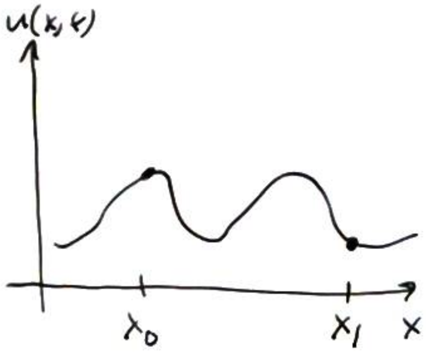
u(x,t): temperature at (x,t)







Does $u(x_0,t)$ increase or decrease in t?



Let $m(t) = \int_{x_0}^{x_1} u(x,t)\,dx \rightarrow \frac{dm}{dt} = \int_{x_0}^{x_1} u_t(x,t)\,dx$

Fink's law of diffusion: Temperature "moves" from high to low, at a rate proportional to the gradient, u_x

$$\frac{dm}{dt} = k u_x(x_1,t) - k u_x(x_0,t) = \int_{x_0}^{x_1} u_t(x,t)\,dx$$

$$\Big\downarrow \partial/\partial x_1$$

$\boxed{u_t = k u_{xx}}$ Check this satisfies intuition

Things to look out for

1) Heat spreads out over time. What can we say about maxima of solutions?
2) Is energy conserved?
3) General Principles: Stability: Does a small error in measuring temperature lead to a small or large error in the solution?