August 25, 2015

Math 3012 – Applied Combinatorics Lecture 3

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Binomial Coefficients Everywhere

Foundational Enumeration Problem

Given a set of m identical objects and n distinct cells, the number of ways they can be distributed with the requirement that no cell is empty is

$$\binom{m-1}{n-1}$$

Explanation

A A A A A A | A A | A A A A | A A A A A A A | A | A A A

m objects, $m - 1$ gaps. Choose $n - 1$ of them. In this example, there are 23 objects and 6 cells. We have illustrated the distribution (6, 2, 4, 7, 1, 3).

Equivalent Problem

Restatement

How many solutions in positive integers to the equation:

$$x_1 + x_2 + x_3 + \dots x_n = m$$

Given a set of m identical objects and n distinct cells, the number of ways they can be distributed with the requirement that no cell is empty is

$$\binom{m-1}{n-1}$$

Building on What We Know

Restatement

How many solutions in non-negative integers to the equation:

$$x_1 + x_2 + x_3 + \dots x_n = m$$

Answer

$$\binom{m+n-1}{n-1}$$

Explanation Add n artificial elements, one for each variable.

Mixed Problems

Problem How many integer solutions in non-negative integers to the equation:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 142$$

Subject to the constraints:

$$x_1, x_2, x_5, x_7 \geq 0; \quad x_3 \geq 8; \quad x_4 > 0; \quad x_6 > 19$$

Answer

$$\binom{\mathbf{119}}{\mathbf{6}}$$

Good = All - Bad

Problem How many integer solutions in non-negative integers to the equation:

$$x_1 + x_2 + x_3 + x_4 = 63$$

Subject to the constraints:

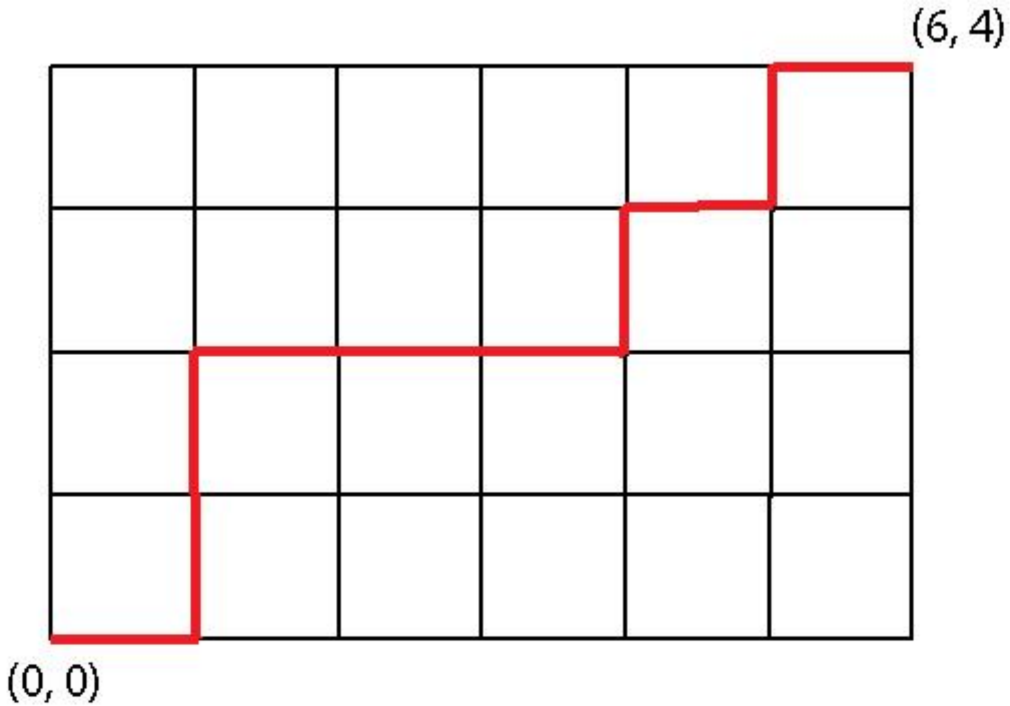
$$x_1, x_2 \geq 0; \quad 2 \leq x_3 \leq 5; \quad x_4 > 0$$

Answer

$$\binom{63}{3} - \binom{59}{3}$$

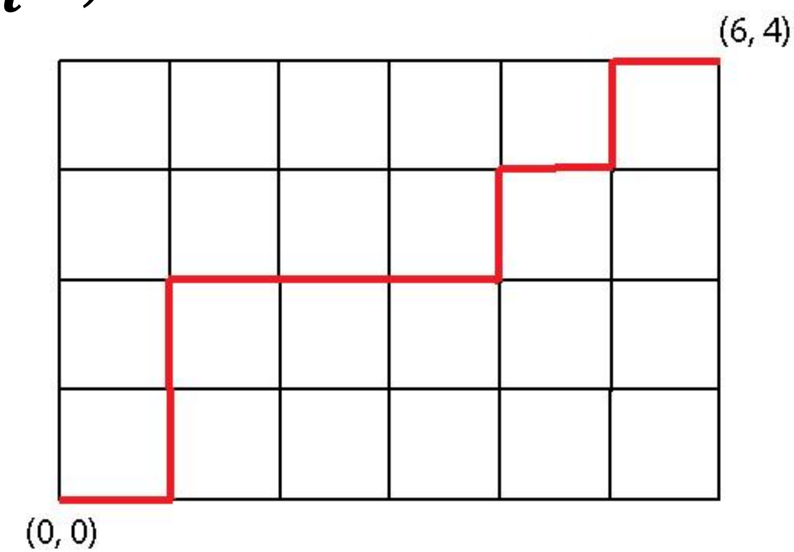
Lattice Paths (1)

Restriction Walk on edges of a grid. Only allowable moves are R (right) and U (up), i.e., no L (left) and no D (down) moves are allowed.



Lattice Paths (2)

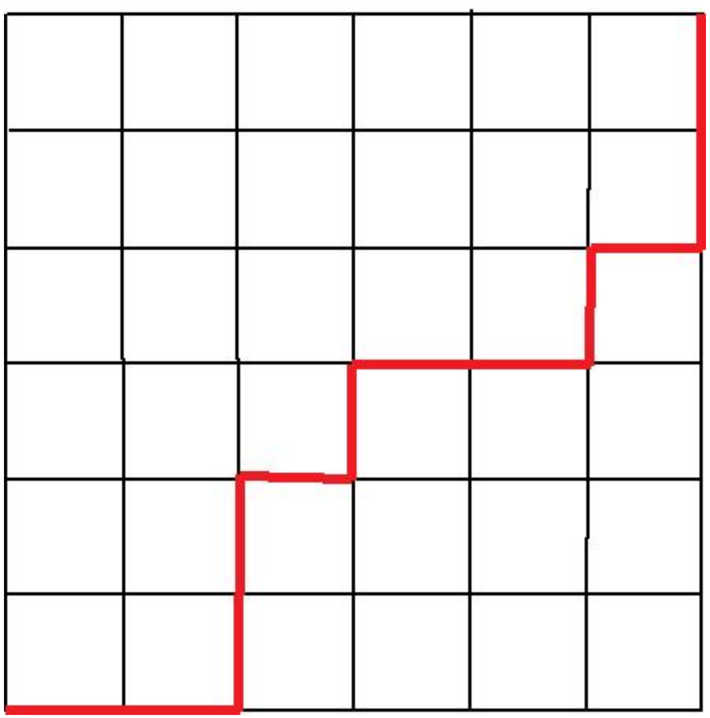
Observation The number of lattice paths from (0, 0) to (m, n) is $\binom{m+n}{m}$.



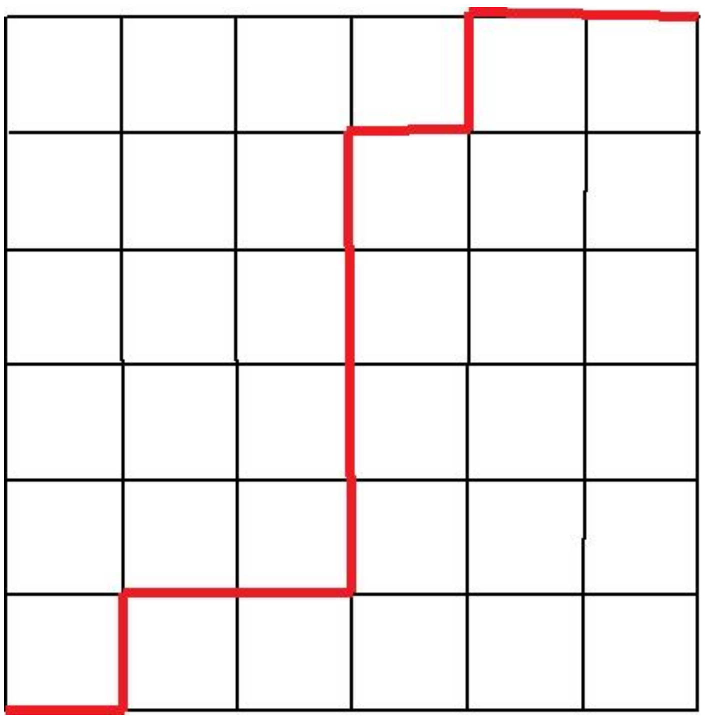
Explanation A lattice path corresponds to a choice of m horizontal moves in a sequence of m + n moves. Here the choices are: RUURRRURUR

Lattice Paths – Not Above Diagonal

Question How many lattice paths from (0, 0) to (n, n) never go above the diagonal?



Good



Bad

Lattice Paths – Not Above Diagonal

Solution The number of lattice paths from (0, 0) to (n, n) which never go above the diagonal is the Catalan Number:

$$\frac{\binom{2n}{n}}{n+1}$$

Observation The first few Catalan numbers are:

1, 1, 2, 5, 14. What is the next one?

Parentheses and Catalan Numbers

Basic Problem How many ways to parenthesize an expression like:

$$x_1 * x_2 * x_3 * x_4 * \ldots * x_n$$

For example, when $n = 4$, we have 5 ways:

$$x_1 * (x_2 * (x_3 * x_4))$$
$$x_1 * ((x_2 * x_3) * x_4))$$
$$(x_1 * x_2) * (x_3 * x_4)$$
$$((x_1 * x_2) * x_3) * x_4$$
$$(x_1 * (x_2 * x_3)) * x_4$$

Can you verify that there are 14 ways when $n = 5$?

Using Recurrence Equations (1)

Basic Problem How many regions are determined by n lines that intersect in general position?

Answer

$d_1 = 2$

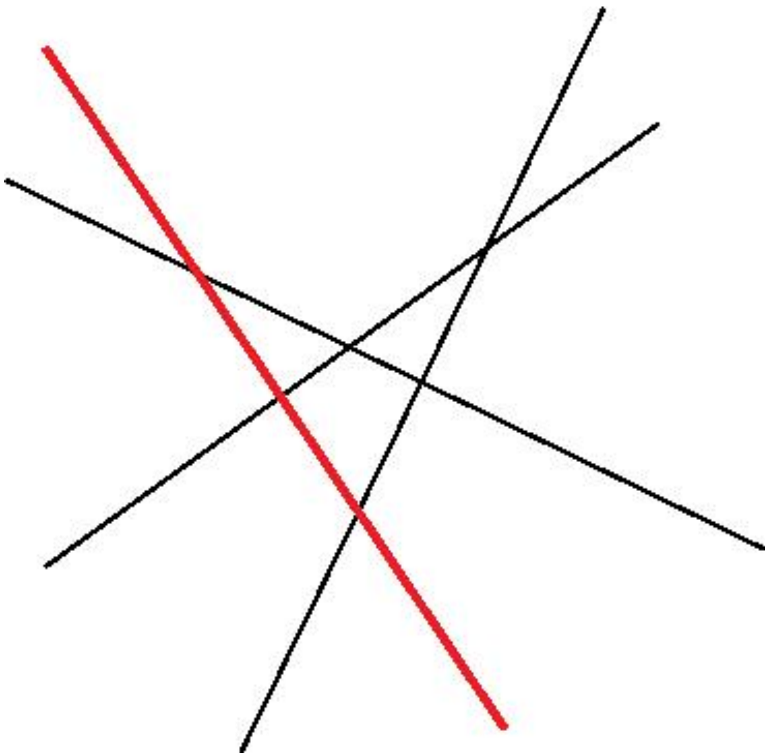
$d_{n+1} = d_n + n+1$ when $n \geq 0$.

So $d_2 = 2 + (1+1) = 4$

$d_3 = 4 + (2+1) = 7$

$d_4 = 7 + (3+1) = 11$

What are d_5 and d_6?



Using Recurrence Equations (2)

Basic Problem How many regions are determined by n circles that intersect in general position?

Answer

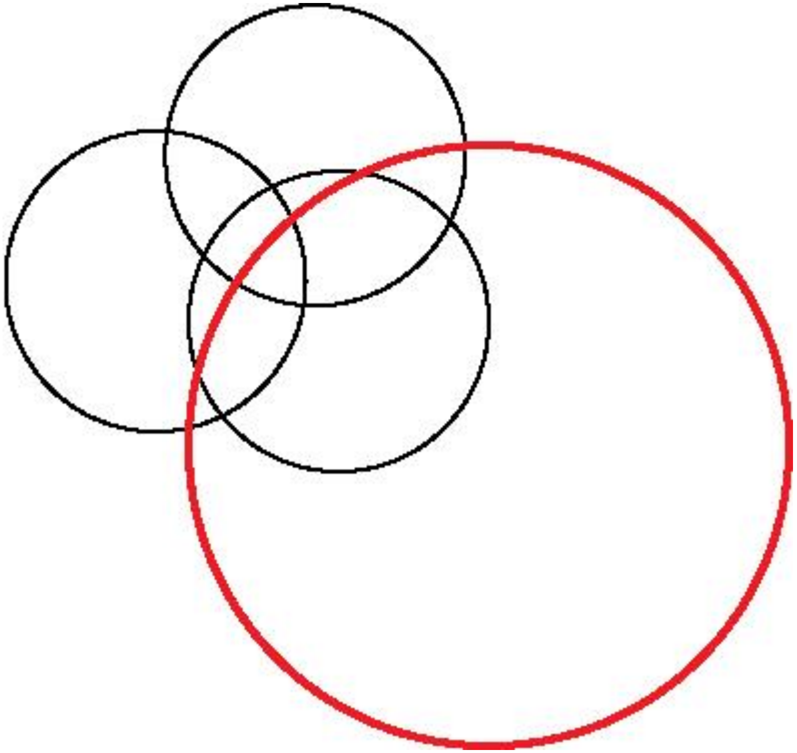
$d_1 = 2$

$d_{n+1} = d_n + 2n$ when $n \geq 0$.

So $d_2 = 2 + 2*1 = 4$

$d_3 = 4 + 2*2 = 8$

$d_4 = 8 + 2*3 = 14$



What are d_5 and d_6?

Using Recurrence Equations (3)

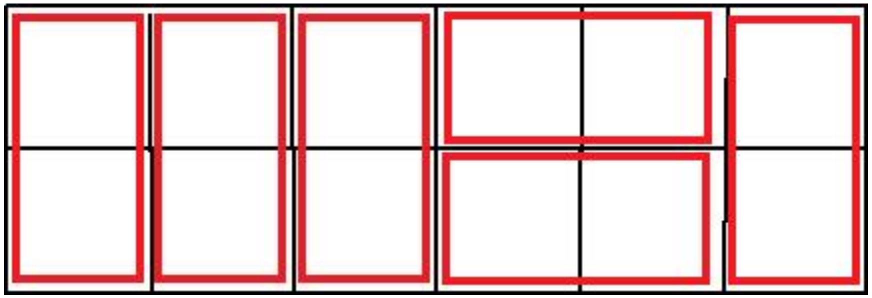
Basic Problem How many ways to tile a $2 \times n$ grid with dominoes of size 1×2 and 2×1?

Answer

$d_1 = 1$

$d_2 = 2$

$d_{n+2} = d_{n+1} + d_n$ when $n \geq 0$.



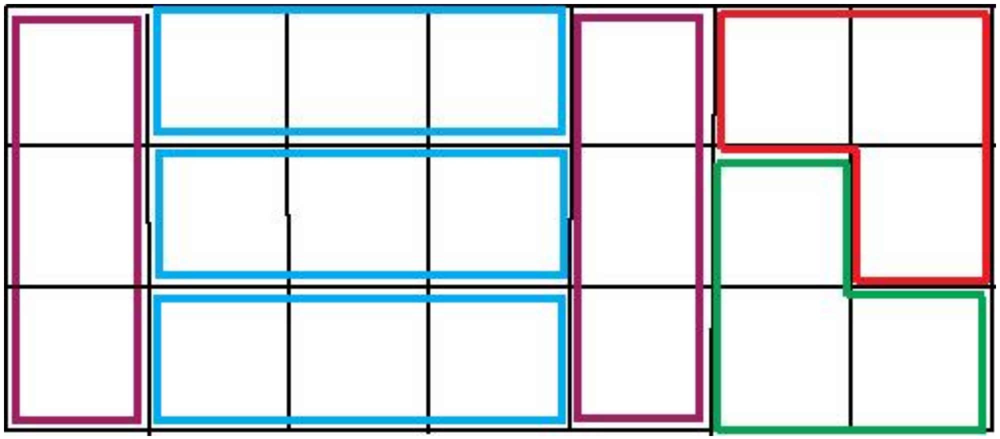
So $d_3 = 2 + 1 = 3$

$d_4 = 3 + 2 = 5$

What are d_5 and d_6?

Challenge Problem (4)

Basic Problem How many ways to tile a 3 x n grid with tiles of the four shapes illustrated here?



Partial Answer
$d_1 = 1$
$d_2 = 2$
$d_3 = 4$

What are d_5 and d_6?

Cash Prize One dollar to first person who can correctly evaluate d_{20}.

Using Recurrence Equations (5)

Basic Problem How ternary sequences do not contain 01 in consecutive positions?

Answer

$t_1 = 3$

$t_2 = 8$

$t_n = 3t_{n-1} - t_{n-2}$ when $n \geq 2$.

So $t_3 = 3 \times 8 - 3 = 21$

$t_4 = 3 \times 21 - 8 = 55$

What is t_5?

Critical Question

Question If you know that:

$a_1 = 14$
$a_2 = 23$
$a_3 = -96$
$a_4 = 52$ and

$a_{n+4} = 9\,a_{n+3} - 7\,a_{n+2} + 8\,a_{n+1} + 13\,a_n$ when $n \geq 1$, then you can calculate a_n for any positive integer n. Is this good enough, or would you like to know even more about a_n ?

Basis for Long Division

Theorem If m and n are positive integers, there are unique integers q and r with $q \geq 0$ and $0 \leq r < m$ so that

$$n = q\,m + r$$

Question Is this obvious or does it require an explanation/proof?