7/10/18 Lecture Notes: More Fourier Series Convergence

Last week: Does $\underbrace{\frac{A_0}{2} + \sum_{n\geq 1}^{N} A_n \cos \frac{n\pi}{\ell}x + \sum_{n\geq 1}^{N} B_n \sin \frac{n\pi}{\ell}x}_{S_N(x)} \to f(x)$?

Mean-square convergence: Yes, if $\|f\|_2 < \infty$

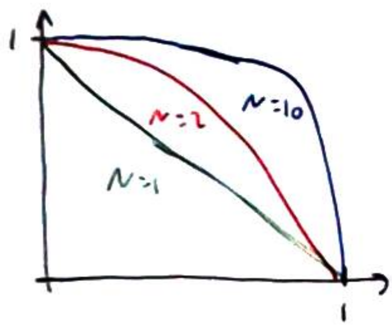
Uniform convergence: Yes, if $f'(x)$ exists and is continuous. (Also true in some other cases)

Today: Pointwise convergence.

$\sum_{n\geq 1}^{N} f_n(x)$ <u>converges pointwise</u> to $f(x)$ if

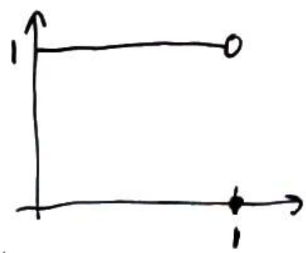
$$\left| f(x) - \sum_{n\geq 1}^{N} f_n(x) \right| \to 0 \text{ as } N \to \infty \text{ for } \underline{\text{each}}\ x$$ (not necessarily at uniform rate)

Example: $f_n(x) = x^{n-1} - x^n$, so $\sum_{n\geq 1}^{N} f_n(x) = 1 - x^N$ on $0 < x < 1$



$\sum_{n\geq 1}^{N} f_n(x) \to f(x)$ pointwise since

for any $0<x<1$ $x^N \to 0$
$1 - x^N \to 1$



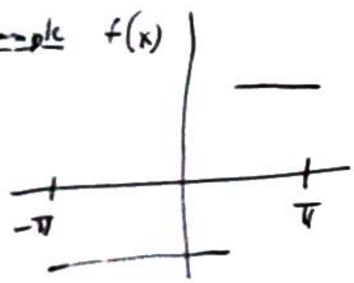
It does NOT converge uniformly.

For each N, take $x_0 = \left(\frac{1}{2}\right)^{1/N}$, $1 - x_0^N = \frac{1}{2}$

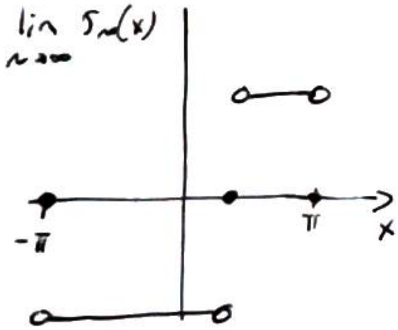
<u>Theorem</u> 1) If $f(x)$ continuous, $f'(x)$ piecewise continuous, then $S_N(x) \to f(x)$ pointwise.

2) If $f(x), f'(x)$ both piecewise continuous, then $S_N(x) \to \frac{1}{2}\left[f(x^+) + f(x^-)\right]$

<u>Example</u> $f(x)$



$\lim_{N\to\infty} S_N(x)$



Pf(#1) Write $S_N(x) = \frac{A_0}{2} + \sum_{n=1}^{N} A_n \cos nx + \sum_{n=1}^{N} B_n \sin nx$

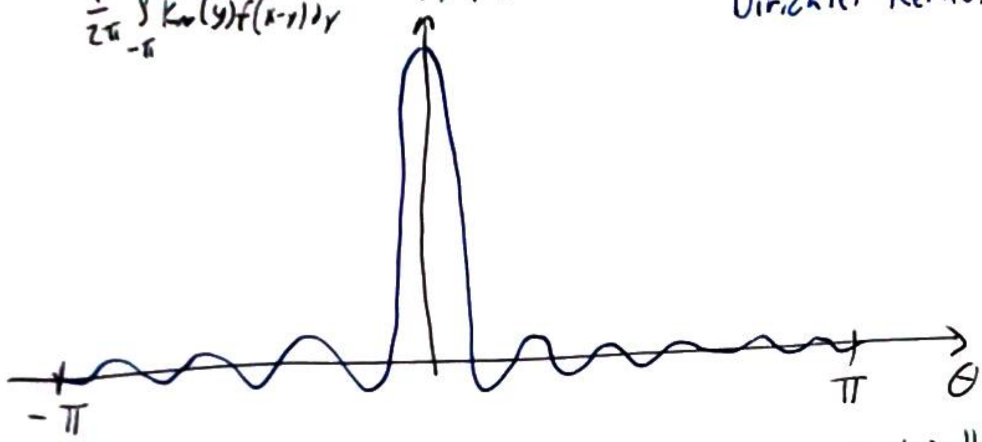
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,dy + \sum_{n=1}^{N} \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\cos ny\,dy \cos nx + \sum_{n=1}^{N} \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\sin ny\,dy\, \sin nx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\Big[1 + 2\sum_{n=1}^{N} \underbrace{\cos ny \cos nx + \sin ny \sin nx}_{\cos(nx-ny)}\Big]\,dy$$

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x-y) f(y)\,dy, \text{ where } \underbrace{K_N(\theta) = 1 + 2\sum_{n=1}^{N} \cos n\theta}_{\text{Dirichlet kernel}}$$

$$\| \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(y) f(x-y)\,dy$$

$K_N(\theta)$



Recall: Solution to $u_t = k u_{xx}$, $u(x,0) = \phi(x)$ is $\int_{-\infty}^{\infty} S(x-y,t)\,\phi(y)\,dy \xrightarrow{*} \phi(x)$ as $t \to 0$

* technically haven't proven yet

Since $\int_{-\pi}^{\pi} K_N(\theta) \frac{d\theta}{2\pi} = 1$, we expect $S_N(x) \to f(x)$

Not as nice as $S(x,t)$, but close enough

Useful formula: $K_N(\theta) = 1 + \sum_{n=1}^{N} e^{in\theta} + e^{-in\theta} = e^{-iN\theta} + \dots + e^{iN\theta}$

$$= \frac{e^{-iN\theta} - e^{+i(N+1)\theta}}{1 - e^{i\theta}} \quad \text{(geometric series formula)}$$

$$= \frac{e^{-i(N+\frac{1}{2})\theta} - e^{i(N+\frac{1}{2})\theta}}{e^{-i\frac{1}{2}\theta} - e^{i\frac{1}{2}\theta}}$$

$$= \frac{\sin\left[(N+\frac{1}{2})\theta\right]}{\sin \frac{1}{2}\theta}$$

WTS $\to 0$

$$S_n(x)-f(x)=\int_{-\pi}^{\pi}K_n(\theta)f(x+\theta)\frac{d\theta}{2\pi}-\int_{-\pi}^{\pi}K_n(\theta)f(x)\frac{d\theta}{2\pi} \quad \text{since } \int_{-\pi}^{\pi}K_n(\theta)\frac{d\theta}{2\pi}=1$$

$$=\int_{-\pi}^{\pi}K_n(\theta)[f(x+\theta)-f(x)]\frac{d\theta}{2\pi} \quad \text{(cancellation!)}$$

$$=\int_{-\pi}^{\pi}\underbrace{\frac{f(x+\theta)-f(x)}{\sin\frac{1}{2}\theta}}_{g(\theta)}\cdot\sin(n+\tfrac{1}{2})\theta\,\frac{d\theta}{2\pi} \quad (*)$$

Note: $\phi_n(\theta):=\sin(n+\frac{1}{2})\theta$ are orthogonal on $[-\pi,\pi]$

and $(*)$ is the formula for a projection coefficient

Bessel's inequality: $\displaystyle\sum_{n=1}^{\infty}\frac{|(g,\phi_n)|^2}{|(\phi_n,\phi_n)|}\le\|g\|_2^2$

Terms must go to 0 if $\|g\|_2<\infty$. g is continuous at $\theta\neq 0$ trivially, at $\theta=0$ b/c $\sin(\theta/2)\approx\theta/2$ and f differentiable. ∎

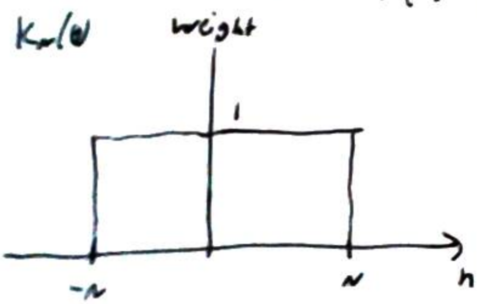
For 2), - same proof but split $\int_{-\pi}^{\pi}$ into $\int_{-\pi}^{0}+\int_{0}^{\pi}$

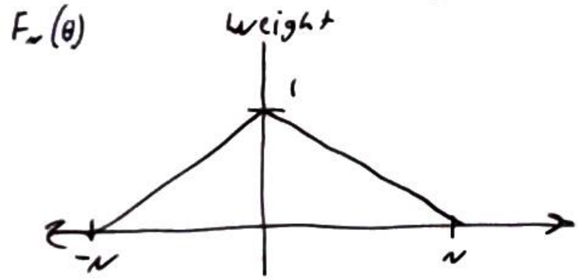
- take one-sided limits to bound $g_+(\theta)$, $g_-(\theta)$

$\boxed{\text{Not on Midterm}}$ Alternate way of summing sines and cosines

- Fejér kernel

Let $F_n(\theta)=\frac{1}{n+1}[K_0(\theta)+K_1(\theta)+\dots+K_n(\theta)]$





$F_n(\theta)$ nicer than $K_n(\theta)$, more like $\delta(x,t)$

1) "Spike" property

not true for K_n

2) $\int_{-\pi}^{\pi}F_n(\theta)\frac{d\theta}{2\pi}=1$

3) $\int_{-\pi}^{\pi}|F_n(\theta)|\frac{d\theta}{2\pi}\le M<\infty$ for all n

$\to \|F_N*f-f\|_2\to 0$ as $N\to\infty$ for $\|f\|_2<\infty$

This proves L^2 Convergence for $K_n(\theta)$ summation by Least-Square Approximation!