7/3/18 Lecture Notes: Fourier Series

Today different from book, what Strauss on midterm

Last time: Solve $u_t - k u_{xx} = 0$ on $0 < x < \ell,\ t > 0$

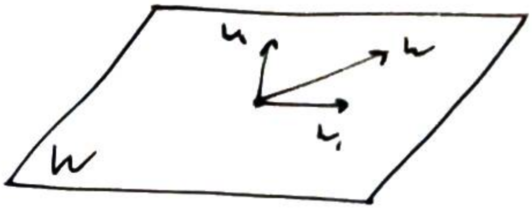
$$u(0,t) = u(\ell,t) = 0$$

$$u(x,0) = \phi(x)$$

Solution is $\boxed{u(x,t) = \sum_n A_n e^{-\left(\frac{n\pi}{\ell}\right)^2 t} \sin \frac{n\pi}{\ell} x,}$

provided $\phi(x) = \sum_n A_n \sin \frac{n\pi}{\ell} x$ Q: When is this true?

Linear algebra review



$x \cdot y = x_1 y_1 + \ldots + x_n y_n$

$x \cdot y = 0$ means x, y orthogonal

$\|x\| = \sqrt{x \cdot x}$ is a norm

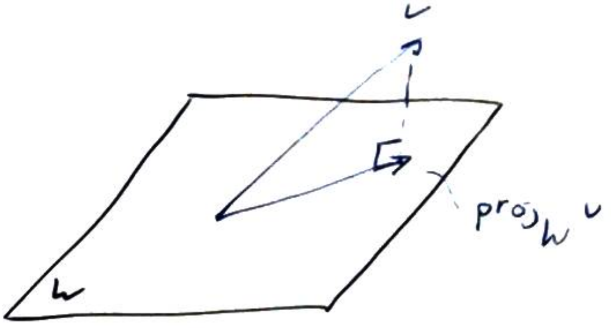
$\{x_1, \ldots, x_k\}$ orthogonal if $x_j \cdot x_\ell = 0,\ j \neq \ell$

orthonormal if $x_j \cdot x_\ell = \delta_{j\ell} = \begin{cases} 1 & j = \ell \\ 0 & j \neq \ell \end{cases}$

If $\{w_1, \ldots, w_k\}$ orthonormal basis for W,

$w = (w \cdot w_1) w_1 + \ldots + (w \cdot w_k) w_k$ for all $w \in W$ ← divide each term by $\|w_i\|^2$ if orthogonal set

If $W \subseteq V$ subspace,



$\text{proj}_W v = (v \cdot w_1) w_1 + \ldots + (v \cdot w_k) w_k$

is the vector $y \in W$ which minimizes $\|v - y\|$

Big Idea: Project $\phi(x)$ onto $\text{Span}\{\sin\frac{n\pi}{\ell}x : n \in \mathbb{N}\}$ (and cosines)

Need Inner Product: $(f,g) := \int_a^b f(x)\overline{g(x)}\,dx$ for $f,g : [a,b] \to \mathbb{C}$

Claim: On $[-\ell,\ell]$, $\{1, \sin\frac{n\pi}{\ell}x, \cos\frac{n\pi}{\ell}x\}$ orthogonal [Note: Want $\|f\|^2 = (f,f) = \int |f|^2 \geq 0$]

Part of Proof: $\int_{-\ell}^{\ell} 1 \cdot \cos\frac{n\pi}{\ell}x\,dx = 0$ b/c n full periods of cosine

$$\int_{-\ell}^{\ell} \sin\frac{m\pi}{\ell}x \cos\frac{n\pi}{\ell}x\,dx = 0 \quad \text{b/c} \quad \int_{-\ell}^{\ell} \text{odd function}(x)\,dx = 0$$

$$\int_{-\ell}^{\ell} \cos\frac{m\pi}{\ell}x \cos\frac{n\pi}{\ell}x\,dx = \int_{-\ell}^{\ell} \frac{1}{2}\left(e^{\frac{imx}{\ell}} + e^{\frac{-imx}{\ell}}\right) \cdot \frac{1}{2}\left(e^{\frac{inx}{\ell}} + e^{\frac{-inx}{\ell}}\right)dx = 0$$

since $\pm m \pm n \neq 0$

If $f(x)$ a function on $[-\ell,\ell]$, its <u>Fourier series</u> is

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell}, \text{ where}$$

$$A_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx, \quad B_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$ (projection coefficients)

normalization factor

Example $f(x) = x$ on $-\pi < x < \pi$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\overset{\text{odd}}{x}\overbrace{\cos nx}^{\text{even}}}_{\text{odd}}\,dx = 0 \qquad B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\,dx = \frac{-1}{n\pi} x\cos nx\Big|_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi}\cos nx\,dx$$

0 b/c n periods of cosine

$$= \frac{-2}{n}\cos(n\pi) = (-1)^{n+1} \cdot \frac{2}{n}$$

$$f(x) = x \overset{*}{=} 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots\right)$$

Sine Series $0<x<\ell$ 0) Start with $\phi:[0,\ell]\to\mathbb{C}$

1) Extend $\phi \to \phi_{odd}$
2) Find Fourier series for ϕ_{odd}
3) Only get sines b/c ϕ_{odd} is odd
4) Compute → formula involving only ϕ

$$A_n = \frac{1}{\ell}\int_{-\ell}^{\ell} \phi_{odd}(x)\sin\frac{n\pi x}{\ell} = \frac{1}{\ell}\int_{-\ell}^{0} -\phi(-x)\sin\frac{n\pi x}{\ell}\,dx + \frac{1}{\ell}\int_{0}^{\ell}\phi(x)\sin\frac{n\pi x}{\ell}\,dx$$

$$= \frac{2}{\ell}\int_{0}^{\ell}\phi(x)\sin\frac{n\pi x}{\ell}\,dx$$

Reminder: $\phi(x) \overset{*}{=} \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{\ell}$

Note: We could also start from scratch again (and show $\sin\frac{n\pi x}{\ell}$ orthogonal), but by this method, convergence results for Fourier series immediately transfer over to sine series

Application: Solve $u_t - ku_{xx} = 0 \quad 0<x<\pi$

$u(0,t) = u(\pi,t) = 0$

$u(x,0) = x$

$$u(x,t) \overset{*}{=} \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2}{n}\sin nx\, e^{-n^2 t}$$

Cosine series: Even extension →

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{\ell}, \text{ where } A_n = \frac{2}{\ell}\int_0^{\ell}\phi(x)\cos\frac{n\pi x}{\ell}\,dx$$

Complex Fourier series: Change of basis!

complex conjugation ↓

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi}{\ell}x}, \text{ where } c_n = \frac{1}{2\ell}\int_{-\ell}^{\ell}\phi(x)e^{-in\pi x/\ell}\,dx$$

Remark: This works for <u>any</u> orthogonal set of functions.

Idea: Instead of proving orthogonality each time, obtain it from boundary conditions.

satisfying BC

Suppose X_1, X_2 solutions to $X''+\lambda X=0$. Then

$$\begin{gathered}\int_a^b \left(-X_1''\bar{X}_2 + X_1\bar{X}_2''\right)dx \overset{*}{=} \left(-X_1'\bar{X}_2 + X_1\bar{X}_2'\right)\Big|_a^b \\ \| \\ (\lambda_1-\bar{\lambda}_2)\int_a^b X_1\bar{X}_2\,dx\end{gathered}$$

* "Green's 2nd identity" via integration by parts

<u>Theorem</u> If $f'(x)g(x)-f(x)g'(x)\Big|_a^b=0$ for all f, g satisfying boundary conditions, then all solutions to $X''+\lambda X$ satisfying BC are orthogonal when λ's different. Furthermore, λ is real-valued.

Such X are called eigenfunctions

Such boundary conditions are called symmetric

<u>Pf</u> Above + take $\lambda=\lambda_1=\lambda_2$ to show real since $\lambda-\bar{\lambda}=0$

<u>Theorem</u> If in addition, $f(x)f'(x)\Big|_a^b \le 0$ for all f satisfying BC, all eigenvalues λ are ≥ 0.