Math 126 Final 8/11/2016 (100 points, 110 minutes)

Remember: Write everything you want to be graded in your green/blue book. Show your work, using words when any step may be unclear. And relax, take a deep breath. You've been studying this stuff all semester and now is your chance to show off your knowledge.

Short Answer

1) Give an example of a homogeneous linear PDE, a nonhomogeneous linear PDE, and a nonlinear PDE. Be sure to label which is which. (5 points)

2) Using the Laplacian in polar coordinates, find all radially symmetric solutions to $\Delta u = u_{xx} + u_{yy} = 0$ in two dimensions (that is, those solutions which only depend on r). (5 points)

3) Solve $u_x + xu_y = 0$ with the auxiliary condition $u(0, y) = e^{-y}$. [No need to draw a picture here.] (5 points)

4) State Kirchoff's formula and Huygen's Principle, then how Kirchoff's Formula implies Huygen's Principle. (5 points)

5) State the solution to the heat equation $u_t = u_{xx}$ on the half-line $x > 0$ with initial conditions $u(x, 0) = \phi(x)$ and boundary condition $u_x(0, t) = 0$. Briefly explain how we derived this solution. (5 points)
Not as Short Answer

6) Let $f(x) = a$ for $x < 0$ and $f(x) = b$ for $x > 0$. Using definitions, compute the derivative of $f(x)$ in the sense of a distribution (don't reference derivatives of any known functions). (10 points)

7) Solve $u_t + uu_x = 0$ with initial condition $u(x,0) = -x$ for as long as a continuous solution exists. Sketch some characteristics. (10 points)

8) Find the solution to the wave equation $u_{xx} = u_{tt}$ on $0 < x < \pi, t > 0$ with Dirichlet boundary conditions $u(0,t) = u(\pi,t) = 0$, where $u(x,0) = \phi(x) = 2\sin x + \sin 3x$ and $u_t(x,0) = \psi(x) = \sin x - \sin 2x$. Write your solution without any Σ signs. (10 points)

9) (15 points) a) Write the scheme for the heat equation $u_t = u_{xx}$ that uses the forward difference for the t-derivative and the centered difference for the x-derivative. (3 points)

b) Apply this scheme on the interval $0 < x < 5$ with $\Delta x = \Delta t = 1$ with initial conditions $u(x, 0) = x(5 - x)$ and Neumann boundary conditions $u_x(0, t) = u_x(5, t) = 0$. Do so until $t = 4$. (10 points)

c) Is this scheme stable? [Hint: Your answer should involve the letter s.] (2 points)

10) Take for granted that the Green's function for the Laplacian on the half-space $\{(x, y)|y > 0\} \subset \mathbb{R}^2$ is

$$G(x, y, x_0, y_0) = \frac{1}{2\pi} \log \sqrt{(x - x_0)^2 + (y - y_0)^2} - \frac{1}{2\pi} \log \sqrt{(x - x_0)^2 + (y + y_0)^2}.$$

Use this to find the solution to the Dirchlet problem $u_{xx} + u_{yy} = 0$ for $y > 0$ with $u(x, 0) = h(x)$. (10 points)

11) (15 points) a) Write the solution to the heat equation $u_t = u_{xx}$ for the real-line $-\infty < x < \infty$ with initial conditions $u(x,0) = \phi(x)$. (3 points)

b) Take the Fourier transform (in x) of the PDE and its initial conditions to get an ODE. Then take the Fourier transform of your solution and show it satisfies this ODE. [In class, we used this technique to find the solution, but I just want you to show that it works.] (12 points)

Last Few Points

12) Say something true about calculus of variations. (3 points)

13) How might you expect to use what you learned in this class in your future? (2 points)