

# F20 PHYSICS 137B: Quick review of 137A

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These notes serve as a brief review of material that you are expected to know for 137B. In particular, we assume a knowledge of spin. Please feel free to contact me if you want to discuss any prerequisites or topics for the course. Below, I assume a familiarity with bracket notation. Note that I also do not discuss the hydrogen atom in these notes; most of the interesting physics in the system is due to the dependencies on angular momentum. I encourage you to review the appropriate material in Griffiths if you would like a refresher on solving the spherical Schrodinger equation.

## 1 Schrodinger Equation

The basic object in the Schrodinger picture of quantum mechanics is the wavefunction of a particle  $\Psi$ . This wavefunction is the solution to the Schrodinger equation (SE):

$$\hat{H} |\Psi\rangle = i\hbar \frac{\partial}{\partial t} |\Psi\rangle, \quad (1.1)$$

where  $\hat{H} = \left(-\frac{\hbar^2}{2m} \nabla^2 + V\right)$  is the Hamiltonian operator. The solution of the SE can be constructed as the sum of eigenfunctions of the Hamiltonian operator (a standard result in Fourier analysis):

$$|\Psi(\mathbf{x}, t)\rangle = \sum_i c_i e^{-iE_i t/\hbar} |\psi(\mathbf{x})\rangle, \quad (1.2)$$

where the eigenvalue  $E_i$  is the energy associated to the state  $|\psi(\mathbf{x})\rangle$ . These are sometimes called “stationary” states, but I will generally refer to them as energy eigenfunctions. The eigenfunctions are determined by the eigenvalue equation:

$$\hat{H} |\psi\rangle = \left(-\frac{\hbar^2}{2m} \nabla^2 + V\right) |\psi\rangle = E |\psi\rangle. \quad (1.3)$$

This is the time-independent SE (TISE). In particular, the general solution must hold true for a given initial condition  $|\Psi(\mathbf{x}_0, 0)\rangle$ :

$$|\Psi(\mathbf{x}, 0)\rangle = \sum_i c_i |\psi_i(x)\rangle, \quad (1.4)$$

and the time evolution then follows by appending the appropriate exponential factors. The coefficients are determined by applying the orthogonality of non-degenerate eigenfunctions:

$$c_i = \langle \psi_i(x) | \Psi(\mathbf{x}, 0) \rangle \quad (1.5)$$

You should recall and be able to derive the solutions to the TISE for several basic potentials, including the infinite square well, delta function potentials, and the harmonic oscillator.

## 2 Observables

Observables  $\mathcal{O}$  in the Schrodinger picture are Hermitian (self-adjoint) operators acting on a Hilbert space of states  $\mathcal{H}$ . The spectral theorem for such operators ensures the existence of a “spectral decomposition”:

$$\mathcal{O} = \sum_i a_i P_i, \quad (2.1)$$

where  $a_i$  are the eigenvalues of  $\mathcal{O}$ , and  $P_i = |\phi_i\rangle\langle\phi_i|$  are projectors onto the associated one-dimensional eigenspace in  $\mathcal{H}$ . That is,  $\mathcal{O}|\phi_i\rangle = a_i|\phi_i\rangle$ . A standard postulate of quantum mechanics stipulates that any measurement of  $\mathcal{O}$  on a state  $\psi$  will yield an eigenvalue  $a_i$  with probability  $|\langle\phi_i|\psi\rangle|^2$ . If one obtains the result  $a_i$ , then the post-measurement state has collapsed to the corresponding eigenfunction  $|\phi_i\rangle$ .

In quantum mechanics, we are usually interested in the result of multiple (asymptotically infinite) measurements rather than a single measurement. The average of the measurement results in such a setting is computed as an expectation value:

$$\langle\mathcal{O}\rangle = \langle\psi|\mathcal{O}|\psi\rangle. \quad (2.2)$$

The Hamiltonian is no different; its associated measured quantity is the energy. However, it is special because it directly appears in the Schrodinger equation (we say that the Hamiltonian *generates time translation*). In particular, an operator is *conserved* if it commutes with the Hamiltonian:

$$[H, \mathcal{O}] = H\mathcal{O} - \mathcal{O}H = 0. \quad (2.3)$$

A couple notes: the above is an *operator* equation, meaning you should stick a  $|\psi\rangle$  to the right of all the terms and a “for all  $|\psi\rangle$ ” at the end, in order to make physical sense of the equation. The idea of conservation in time is precisely what you remember from introductory classical mechanics: the operator does not evolve in time. More specifically, suppose I obtain some set of statistics for  $\mathcal{O}$  in a state  $|\psi\rangle$ . If I then allow the state to evolve according to the Hamiltonian, I will obtain the same set of statistics for  $\mathcal{O}$  in the new, time-evolved state.

Commuting operators are also important, because such operators are *mutually diagonalizable*. That is, there exists some basis in which both operators are diagonal. In the language we have been using, this means that one can choose a set of states  $\{|\psi_i\rangle\}$  such that those states are eigenstates of both operators. Physically, this means it is possible to measure each operator to unlimited precision simultaneously.

## 3 Angular momentum

You are first introduced to quantum angular momentum via the hydrogen atom. Namely, the energy eigenfunctions of the hydrogen atom are determined by 3 quantum numbers  $n, l, m$ , with  $l$  and  $m$  corresponding to the angular momentum of the atom. The quantum-ness arises from the fact that these solutions are quantized: they can only be integers. Moreover, for each component of the angular momentum, we can associate to it a corresponding operator  $L_x, L_y, L_z$ . These are defined by “canonically quantizing” their classical definitions:

$$L_i = \epsilon_{ijk} x_j p_k, \quad (3.1)$$

where  $\epsilon_{ijk}$  is the totally anti-symmetric tensor and repeated indices are summed over. By canonical quantization, we mean that we simply replace the position coordinate  $x_j$  with the position operator, and similarly for the momentum. One can show that these operators obey the fundamental commutation relation:

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k. \quad (3.2)$$

In particular, note that different components of the angular momentum do not commute, and hence cannot be simultaneously known. However, if one may define the total angular momentum operator as  $L^2 = L_i L_i$ , you can verify that  $L^2$  commutes with any  $L_i$ , and hence can be known simultaneously with any single component.

In doing computations, we usually work in the basis of eigenstate of  $L_z$ , and hence it is useful to introduce a formalism for how to deal with the action of  $L_x$  and  $L_y$  on such states. This is done via “ladder operators.” The usual setup is as follows: we simultaneously diagonalize  $L^2$  and  $L_z$  so that we have a basis of states  $|l, m\rangle$ , where  $l$  is the number associated to  $L^2$  and  $m$  to  $L_z$ . One can readily derive using the ladder method:

$$L^2 |l, m\rangle = \hbar^2 l(l+1) |l, m\rangle, \quad (3.3)$$

$$L_z |l, m\rangle = \hbar m |l, m\rangle, \quad (3.4)$$

$$L_{\pm} |l, m\rangle = \hbar \sqrt{l(l+1) - m(m \pm 1)} |l, m \pm 1\rangle \quad (3.5)$$

where  $L_{\pm} = L_x \pm iL_y$ . One can also easily show that  $|m| \leq l$ .

## 4 Spin

The theory of angular momentum becomes more intricate when we introduce spin, largely because spin can be half-integral. The formalism is the same as before: we have a spin operator for each component  $S_i$  and the total spin  $S^2$ . They obey the same commutation relations and the set-up is usually the same:  $|s, m\rangle$  is the basis of simultaneous eigenstates with eigenvalues  $\hbar^2 s(s+1)$  and  $\hbar m$  for  $S^2$  and  $S_z$ , respectively.

The most common case for this course will be  $s = 1/2$ , and  $m = \pm 1/2$ . Here, it is possible to explicitly compute the form of each  $S_i$  and  $S^2$  in the  $S_z$  basis, which gives the famous Pauli matrices.

There will be many situations where we will have to add angular momentum, e.g. a situation involving 2 electrons. The key here is that the individual components  $S_i^1, S_i^2$  do not commute with the total angular momentum  $(S^1 + S^2)^2$ . We must in fact make a choice about which basis to work in: the *product* basis, in which we keep track of the individual spins, or the *coupled* basis, in which we only keep track of the total spin. Notationally:

$$|j, m\rangle = \sum_{m_1+m_2=m} C_{m_1 m_2 m}^{s_1 s_2 j} |s_1 m_1\rangle |s_2 m_2\rangle. \quad (4.1)$$

The derivation of this formula is beyond the scope of this course, but the point is that an eigenstate of the *total* spin  $|j, m\rangle$  is generally a linear combination of the tensor product of eigenstates of the *individual* spins. The coefficients that appear in the sum are called Clebsch-Gordon coefficients – generally these will be provided for you on e.g. a table.