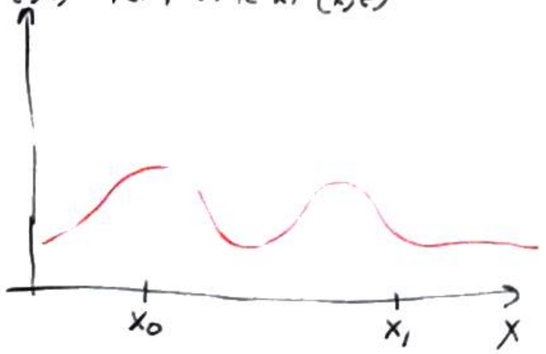
6/21/18: The Diffusion/Heat Equation

Pre-Lecture: Problem 1.3.1: $\vec{F} = m\vec{a}$

add extra force   velocity at point is u_t, should integrate to get total force on segment

- Sections 1.3, 1.4, 1.5 split over many lectures (good midterm review)
- HW1 due Friday at 7pm

$u(x,t)$ = temperature at (x,t)



Let $M(t)$ = "total temperature" in $[x_0, x_1]$ at time t, i.e.

$$\int_{x_0}^{x_1} u(x,t)\,dx$$

$dM/dt =$ 1) $\int_{x_0}^{x_1} u_t(x,t)\,dx$

2) Fick's law: Temperature moves from hot to cold, at a rate proportional to the gradient, u_x

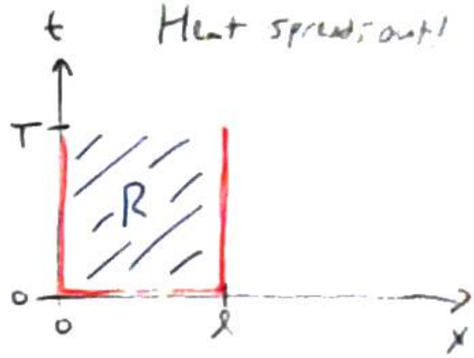
$$k u_x(x_1,t) - k u_x(x_0,t) = \int_{x_0}^{x_1} u_t(x,t)\,dx$$

$$\Big\downarrow \; \partial/\partial x_1$$

$$\boxed{u_t = k u_{xx}}$$

Maximum Principle If $u_t = ku_{xx}$ in R $(0 \le x \le \ell, 0 \le t \le T)$, then $\max\limits_R u$ is obtained at $t=0$ or $x=0$ or $x=\ell$.

Heat spreads out!



Note: Also applies to translated regions.

Minimum Principle Same for $\min\limits_R u$ (since $-u$ also a solution to the heat equation)

Incorrect Proof If u obtains max at interior point (x_0,t_0), then

$u_{xx}(x_0,t_0) < 0$, $u_t(x_0,t_0) = 0$, but $u_t = ku_{xx}$ $\Rightarrow\Leftarrow$.

What goes wrong? u_{xx} not necessarily 0 at maximum.

Correct Proof

Let $M = \max\limits_{\sqcup} u$ Let $v(x,t) = u(x,t) + \varepsilon x^2$ for $0 < \varepsilon < 1$

$$v_t - kv_{xx} = \cancel{u_t - ku_{xx}} - 2\varepsilon k < 0$$

Why $v_t(x_0,t_0) \ge 0$?

Possibility of

Analyze v: If v obtains max at interior point (x_0,t_0), then

$v_{xx}(x_0,t_0) \le 0$ and $v_t(x_0,t_0) \ge 0 \rightarrow$ Contradiction

(just like we wanted for u)

Conclude: $\max\limits_R v$ on $\sqcup$ and is $\le M + \varepsilon \ell^2$

For $(x,t) \in R$, $v(x,t) \le M + \varepsilon \ell^2$

$u(x,t) \le M + \varepsilon(\ell^2 - x^2)$ $\quad \varepsilon \to 0$

$u(x,t) \le M$ ∎

Dirichlet Problem What are appropriate initial conditions?

$u_t - ku_{xx} = f(x,t)$ $\quad 0 < x < \ell,\ t > 0$

$u(x,0) = \phi(x)$

$u(0,t) = g(t)$ $\quad u(\ell,t) = h(t)$

Uniqueness of Solutions Note: We haven't even established existence yet!

Suppose u_1, u_2 solutions to Dirichlet problem. Let $w = u_1 - u_2$, so

$$w_t - k w_{xx} = 0 \quad 0 < x < \ell,\ t > 0$$
$$w(x,0) = 0$$
$$w(0,t) = w(\ell,t) = 0$$

w is 0 on $\sqcup$, so by the maximum principle, $w \equiv 0$ and $u_1 = u_2$.

Energy methods Let $E(t) = \frac{1}{2}\int_0^\ell w^2(x,t)\,dx$ (same w for proof, energy always same)

$$dE/dt = \int_0^\ell w w_t\,dx \overset{w_t = k w_{xx}}{=} k\int_0^\ell w w_{xx}\,dx$$

IBP

$$= \underbrace{k w w_x \Big|_0^\ell}_{0 \text{ by boundary conditions}} - k\int_0^\ell w_x^2\,dx \le 0$$

$0 = E(0) \ge E(t) \ge 0$, so $E(t) = 0$ for all t, $w \equiv 0$.

A PDE (with initial and/or boundary conditions) is well-posed if it has $\underbrace{\text{uniqueness, existence}}_{\text{exactly one solution}}$, and stability

Stability: close initial conditions $\Rightarrow$ close solutions

Example $u_t - k u_{xx} = 0 \quad 0 < x < \ell,\ t > 0 \qquad u_1(x,0) = \phi_1(x)$

$u(0,t) = 0 = u(\ell,t) \qquad u_2(x,0) = \phi_2(x)$

$w = u_1 - u_2$ solves heat equation with $w(x,0) = \phi_1(x) - \phi_2(x)$

Energy: $E(t) \le E(0)$

$$\int_0^\ell |u_1 - u_2|^2 \, dx \le \int_0^\ell |\phi_1 - \phi_2|^2 \, dx \quad \text{for all } t \qquad \|u_1 - u_2\|_{L^\infty_t L^2_x} \le \|\phi_1 - \phi_2\|_{L^2_x}$$

Maximum Principle: $w(x,t) \le \max_x w(x,0)$ [0 on boundary]

$$u_1 - u_2 \le \max_x |\phi_1 - \phi_2| \quad \text{for all } x, t$$

$$\text{Min. Principle} \rightarrow u_1 - u_2 \ge -\max |\phi_1 - \phi_2| \quad \text{for all } x, t$$

Conclude: $\max_{0 \le x \le \ell} |u_1(x,t) - u_2(x,t)| \le \max_{0 \le x \le \ell} |\phi_1(x) - \phi_2(x)|$ for all t

$$\|u_1 - u_2\|_{L^\infty} \le \|\phi_1 - \phi_2\|_{L^\infty}$$