EE 127AT/227AT 9/19/2018
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Discussion #4

Exercise 1 (Minimizing a quadratic function) Consider the *unconstrained* optimization problem

$$p^* = \min_x \frac{1}{2} x^\top Q x - c^\top x$$

where $Q = Q^\top \in \mathbb{R}^{n,n}$, $Q \succeq 0$, and $c \in \mathbb{R}^n$ are given. The goal of this exercise is to determine the optimal value p^* and the set of optimal solutions, $\mathcal{X}^{\text{opt}}$, in terms of c and the eigenvalues and eigenvectors of the (symmetric) matrix Q.

1. Assume that $Q \succ 0$. Show that the optimal set is a singleton, and that p^* is finite. Determine both in terms of Q, c.
2. Assume from now on that Q is not invertible. Assume further that Q is diagonal: $Q = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, with $\lambda_1 \geq \ldots \geq \lambda_r > \lambda_{r+1} = \ldots = \lambda_n = 0$, where r is the rank of Q ($1 \leq r < n$). Solve the problem in that case.
3. Now we do not assume that Q is diagonal anymore. Under what conditions (on Q, c) is the optimal value finite? Make sure to express your result in terms of Q and c, as explicitly as possible.

Exercise 2 (Schur complement) Let $A \in \mathbb{R}^{p \times p}$, $C = C^\top \in \mathbb{R}^{q \times q}$, C invertible, $B \in \mathbb{R}^{p \times q}$ and $p + q = n$. Let

$$M = \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$$

1. Prove

$$M = \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix} = \begin{bmatrix} I & BC^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A - BC^{-1}B^\top & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} I & BC^{-1} \\ 0 & I \end{bmatrix}^\top$$

2. Prove that $C \succ 0$ and $A - BC^{-1}B^\top \succ 0 \rightarrow \ M \succ 0$