7/5/18 Lecture Notes: Convergence of Fourier Series

Last time: If $f:[-\ell,\ell]\to\mathbb{C}$, the <u>Fourier series</u> of f is

$$f(x) \overset{*}{=} \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi}{\ell}x + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{\ell}x,$$

where $A_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi}{\ell}x\,dx \quad B_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi}{\ell}x\,dx$ (projection coefficients)

Question 1: When else can we do this?

Important: Only used orthogonality of $\{1, \sin\frac{n\pi}{\ell}x, \cos\frac{n\pi}{\ell}x\}$.

Suppose X_1, X_2 solutions to $X''+\lambda X=0$ ($-X''=\lambda X$, so X is "eigenfunction") and satisfy given boundary conditions (BC)

Then, $\int_a^b(-X_1''\overline{X_2}+X_1\overline{X_2''})\,dx \overset{*}{=} (-X_1'\overline{X_2}+X_1\overline{X_2}')\Big|_a^b$ — Green's 2nd identity via IBP

$$\begin{array}{ccc} \int_a^b(-X_1''\overline{X_2}+X_1\overline{X_2''})\,dx & = & (-X_1'\overline{X_2}+X_1\overline{X_2}')\Big|_a^b \\ \| & & \| \\ (\lambda_1-\overline{\lambda_2})\int_a^b X_1\overline{X_2}\,dx & & 0 \text{ by (BC)?} \end{array}$$

<u>Definition</u> (BC) are <u>symmetric</u> if f,g satisfying (BC) implies
$$f'(x)g(x)-f(x)g'(x)\Big|_a^b = 0$$

E.g. Dirichlet $X(a)=X(b)=0$
Neumann $X'(a)=X'(b)=0$
Periodic $\begin{cases} X(a)=X(b) \\ X'(a)=X'(b)\end{cases}$

<u>Theorem</u> If (BC) symmetric, then
1) All eigenvalues $\lambda\in\mathbb{R}$
2) Eigenfunctions with different λ are orthogonal

3) Furthermore, if $f(x)f'(x)\Big|_a^b \leq 0$ for all f satisfying (BC), then all eigenvalues λ are ≥ 0. (See HW 6, 5.3:15)

Simplify notation: Think of Fourier series as singly infinite sum
$$f(x) \overset{*}{=} \sum_{n=1}^{\infty} A_n X_n$$

Question 2:

What does it mean for $\sum_{n=1}^{\infty} f_n(x)$ to equal $f(x)$? When does it happen for Fourier series?

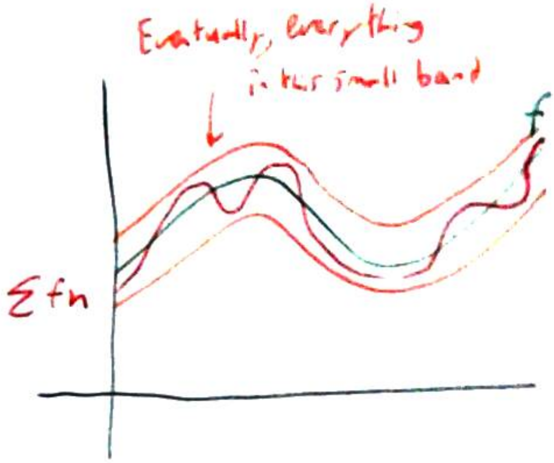
Simpler question: What about $\sum_{n=1}^{\infty} a_n = S$?

A: $\lim_{N\to\infty} \left| \sum_{n=1}^{N} a_n - S \right| = 0$

Do later at (*)

Answer for functions: $\lim_{N\to\infty} \left\| \sum_{n=1}^{N} f_n - f \right\|_\infty = 0$,

where $\|f-g\|_\infty = \max |f(x)-g(x)|$



This is called uniform convergence. (convergence at each point, but at uniform rate)

~~Alternative~~ Answer: Measure distance between f, g by

$$\|f-g\|_2 = \left(\int_a^b |f(x)-g(x)|^2 \, dx \right)^{1/2}, \quad \lim_{N\to\infty} \left\| \sum_{n=1}^{N} f_n - f \right\|_2 = 0 \; *$$

Q: Why? Because it matches with the inner product we used for projection

* This is called mean-square (or L^2) convergence

Speaking of projection, what does it tell us?

1) Least square approximation: If $\|f\|_2 < \infty$, then

$$\left\| f - \sum_{n=1}^{N} c_n X_n \right\|_2$$

is minimized precisely when $c_1 = A_1, \ldots, c_N = A_N$ (A_i are Fourier coefficients)

2) Bessel's Inequality: $\left\| \sum_{n=1}^{N} A_n X_n \right\|_2 \le \|f\|_2$ for all N (since it's a projection)

square both sides, apply orthogonality, $N \to \infty$

$$\sum_{n=1}^{\infty} A_n^2 \|X_n\|_2^2 \le \|f\|_2^2$$

<u>Theorem</u> If $\|f\|_2 < \infty$, then the Fourier series of f converges to f in the mean-square sense

Pf Ask me to explain until my excitedness scares you off.

Cor: Parseval's equality: $\sum_{n=1}^{\infty} |A_n|^2 \|X_n\|_2^2 = \|f\|_2^2$ when $\|f\|_2 < \infty$.

Example: $x =^* 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \dots\right) \quad -\pi < x < \pi$

$$\sum_{n=1}^{\infty} \frac{4}{n^2} \underbrace{\int_{-\pi}^{\pi} \sin^2 nx\, dx}_{\pi} = \int_{-\pi}^{\pi} x^2 dx = \left.\frac{x^3}{3}\right|_{-\pi}^{\pi}$$

$$4\pi \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^3}{3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Open Problem: Find $\sum_{n=1}^{\infty} \frac{1}{n^3}$

(*) Uniform Convergence

Theorem If $f(x)$ is differentiable, and $f'(x)$ is continuous, then the Fourier series of f converges to f uniformly.

(take $l = \pi$ for simplification)

Pf Let $S_N(x) = \frac{A_0}{2} + \sum_{n=1}^{N} A_n \cos nx + \sum_{n=1}^{N} B_n \sin nx$

$$\|f(x) - S_N(x)\|_\infty \le \sum_{n=N+1}^{\infty} \|A_n \cos nx + B_n \sin nx\|_\infty$$

$$\le \sum_{n=N+1}^{\infty} |A_n| + |B_n|$$

Since f' exists,

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \cancelto{0}{\left.\frac{1}{n\pi} f(x)\sin nx\right|_{-\pi}^{\pi}} - \int_{-\pi}^{\pi} \frac{1}{n\pi} f'(x) \sin nx\, dx = \frac{-1}{n} B_n'$$

Similarly, $\frac{1}{n}A_n' = B_n$, where A_n', B_n' are Fourier coefficients of f'.

(Note: This shows smooth f has decaying Fourier coefficients.)

$$\|f - S_N\|_\infty \le \sum_{n=N+1}^{\infty} \frac{1}{n}\left(|A_n'| + |B_n'|\right)$$

Cauchy-Schwarz $|x \cdot y| \le \|x\| \cdot \|y\|$

$$\le \left(\sum_{n=N+1}^{\infty} \frac{1}{n^2}\right)^{1/2} \left(\sum_{n=1}^{\infty} \left(|A_n'| + |B_n'|\right)^2\right)^{1/2}$$

$$\le \left(\sum_{n=N+1}^{\infty} \frac{1}{n^2}\right)^{1/2} \left(\sum_{n=1}^{\infty} 4\left(|A_n'|^2 + |B_n'|^2\right)\right)^{1/2}$$

$\downarrow$ 0 $\qquad < \infty$ by Bessel's inequality for $f'(x)$ (continuous functions are square integrable)