Active Fault-Tolerant Control for Nonlinear LPV delayed System

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***Abstract*—This paper proposes a fault estimation and control strategy for nonlinear LPV systems with variable delays. A sliding-mode observer is designed to robustly estimate system states and actuator faults, while an adaptive sliding-mode controller dynamically counteracts faults and uncertainties. The proposed fault-tolerant control architecture emphasizes a modular structure separating estimation and compensation tasks to ensure flexibility and scalability. The method ensures improved fault detection, robust performance, and stability despite time-varying delays , nonlinearities and uncertainties . Simulation results confirm the effectiveness and efficiency of the proposed approach compared to existing techniques.**

Time delay system, LPV Nonlinear system, Active fault tolerant control (aFTC), Sliding Mode control, Actuator faults, LMI optimisation

I. INTRODUCTION

Delayed (LPV) systems have been less studied, but they present many challenges and opportunities for future research. These systems introduce complexities such as actuator degradation, sensor failures and output variations that disrupt system operations and lead to performance degradation. Fault-tolerant control (FTC) has thus proved essential for ensuring stability and reliability, especially in systems affected by delays.Fault-Tolerant Control (FTC) strategies are crucial for maintaining system stability and robustness in the presence of disturbances, uncertainties, and faults. While passive (FTC) ensures baseline robustness by design, Active Fault-Tolerant Control (AFTC) significantly enhances system supervision, adaptability, and reliability by dynamically reconfiguring control actions in real-time after fault detection.

The efficiency of (AFTC) heavily depends on the accuracy of Fault Detection and Isolation (FDI). Robust FDI methods, such as parity space approaches and Extended Kalman Filters [1], are essential to precisely detect and estimate faults, thereby optimizing the performance of the overall FTC system. Recent works [2], [3], [4], [5] have proposed various dynamic and robust strategies, focusing on LPV systems with uncertainties and delays. However, common challenges remain: Many fault estimation techniques, like Sliding Mode Observers (SMOs), are effective but limited to specific nonlinear system classes. Fault estimation (FE) methods can quantify faults but often do not guide fault compensation, highlighting the critical role of an active approach. In nonlinear LPV systems with variable time delays where delays can destabilize control performance the importance of (AFTC) becomes even more pronounced [6], [7].

This paper addresses these challenges by proposing: A sliding-mode observer for robust state and fault estimation under delay variations. An adaptive sliding-mode controller (SMC) capable of real-time adjustment to uncertainties and nonlinearities. Then,a separated estimation-compensation strategy, reinforcing robustness while maintaining computational efficiency.

Compared to previous work [8], [9] relying on SMOs for descriptor systems, the proposed approach offers enhanced adaptability and robustness for systems with local nonlinearities, enabling more accurate fault detection and improved state estimation through dynamic parameter adjustment.

The paper is structured as follows: Section 2 introduces the nonlinear (LPV) system with delays; Section 3 details the adaptive observer design; Section 4 presents the sliding-mode controller and the separate FTC scheme; Section 5illustrates the effectiveness of the proposed methods through simulations; and Section 6 concludes the study.

II. PROBLEM FORMULATION

Let us consider the delayed LPV system (1) written in the following polytopic representation:

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{g}\mu_i(\theta(t))\{A_i(t)+A_{di}x(t-d(t)) \\ &+ B_iu(t)+D_i\xi(t,x)+\Gamma i(x,u)+M_if_a(t)\} \end{aligned} \quad (1)$$

$$y(t) = \sum_{i=1}^{g}\mu_i(\theta(t))\{C_ix(t)\} \quad (2)$$

where $x(t) \in \Re^n$ represents the state vector, $y(t) \in \Re^p$ represents the measurement, $u(t) \in \Re^m$ represents the input, $f_a(t) \in \Re^q$ represents the actuator failure, and $\xi(t) \in \Re^l$ encompasses the uncertainty. This uncertainty represents the unknown and bounded uncertainties belonging to $L2 \in \begin{bmatrix} 0 & \infty \end{bmatrix}$. The functions $varphi_{xi}(x(t))$, $\varphi_2(x(t))$, $\varphi_{yl}(x(t))$, and $\Gamma(x,t)$ are nonlinear and continuous, with $i = 1, 2, 3, 4, 5$.

$A_i \in \Re^{n\times n}$, $A_{di} \in \Re^{n\times n}$, $B_i \in \Re^{n\times m}$, $M_i \in \Re^{n\times q}$, $D_i \in \Re^{n\times l}$, $C_i \in \Re^{p\times n}$ are of appropriate dimensions.

The weighting functions $\mu_i(\theta(t))$ satisfy:

$$\mu_i(\theta(t)) > 0, \quad \sum_{i=1}^{g}\mu_i(\theta(t)) = 1 \quad (3)$$

The delay $d(t)$ satisfies:

$$0 \le d(t) \le d_m, 0 \le \dot{d}(t) \le \tau < 1 \quad (4)$$

a) Assumption H3.1: [10] The uncertainties and faults are unknown but bounded:

$$\|f_a(t)\| \leq \rho_a \ , \|\xi(x,t)\| \leq \beta \tag{5}$$

b) Assumption H3.2: [10]
The matrix $\mathcal{M}$ satisfies:

$$rank(C_i M_i) = rank(M_i) = q \tag{6}$$

c) Assumption H3.3: [6]
The system $\left(A_i + A_{di}e^{-ds}, M_i, C_i\right)$ is minimum phase with relative degree :

$$rank\left(\begin{bmatrix} sI_n - A_i - A_{di}e^{-ds} & M_i \\ C_i & 0_{p\times q} \end{bmatrix}\right) = n + q \tag{7}$$

d) Assumption H3.4 [10]: The function $\Gamma_i(x(t),t)$ is locally Lipschitz on a set $\mathbb{M} \subset \Re^n$:

$$\|\Gamma_i(x_1(t),t) - \Gamma_i(x_2(t),t)\| \leq \gamma_1 \|x_1(t) - x_2(t)\| \tag{8}$$

$\forall x_1(t), x_2(t) \in \mathbb{M}$ where γ_1 and γ_2 are unknown positive Lipschitz constants.

e) Definition: For all matrices X and Y of appropriate dimensions:

$$X^T Y + Y^T X \leq \delta^{-1} X^T X + \delta Y^T Y \tag{9}$$

III. Robust fault estimation design in sliding mode

A. Coordinate system transformation

Using Assumption H3.2, the following transformations are possible for system (1)–(2) [11]:

$$\begin{aligned} T_i A_i T_i^{-1} &= \begin{bmatrix} \bar{A}_{1i} & \bar{A}_{2i} \\ \bar{A}_{31i} & \bar{A}_{4i} \\ \bar{A}_{32i} & \end{bmatrix}, \bar{D}_i = \begin{bmatrix} \bar{D}_{1i} \\ \bar{D}_{2i} \end{bmatrix} \\ T_i \bar{A}_{di} T_i^{-1} &= \begin{bmatrix} A_{d1i} & \bar{A}_{d2i} \\ \bar{A}_{d31i} & \bar{A}_{d4i} \\ \bar{A}_{d32i} & \end{bmatrix}, \bar{M}_{2i} = \begin{bmatrix} 0 \\ \bar{M}_{qi} \end{bmatrix} \\ \bar{M}_i &= \begin{bmatrix} 0 \\ \bar{M}_{2i} \end{bmatrix}, \bar{C}_i = \begin{bmatrix} 0_{p\times(n_1-p)} & \bar{C}_{2i} \end{bmatrix} \end{aligned} \tag{10}$$

where $\bar{C}_{2i} \in \Re^{p\times p}$is non-singular, with $p \geq q$.
Next, to identify the sliding surface:

$$T_{Li} = \begin{bmatrix} I_{n-p} & L_i \\ 0 & \bar{C}_{2i} \end{bmatrix}, L_i = \begin{bmatrix} L_{1i} & 0_{(n-p)q} \end{bmatrix} \tag{11}$$

As a result, the matrices (10) are transformed to take the following forms:

$$\begin{aligned} T_{Li}\bar{A}_i T_{Li}^{-1} &= \begin{bmatrix} \tilde{A}_{1i} & \tilde{A}_{2i} \\ \tilde{A}_{31i} & \tilde{A}_{4i} \\ \tilde{A}_{32i} & \end{bmatrix}, \tilde{D}_i = \begin{bmatrix} \tilde{D}_{1i} \\ \tilde{D}_{2i} \end{bmatrix} \\ T_{Li}\bar{A}_{di} T_{Li}^{-1} &= \begin{bmatrix} \tilde{A}_{d1i} & \tilde{A}_{d2i} \\ \tilde{A}_{d31i} & \tilde{A}_{d4i} \\ \tilde{A}_{d32i} & \end{bmatrix}, \tilde{M}_i = \begin{bmatrix} 0 \\ \tilde{M}_{2i} \end{bmatrix} \\ \tilde{M}_{2i} &= \begin{bmatrix} 0 \\ \tilde{M}_{qi} \end{bmatrix}, \tilde{C}_i = \begin{bmatrix} 0_{p\times(n-p)} & I_p \end{bmatrix} \end{aligned} \tag{12}$$

with $\tilde{M}_{2i} \in \Re^{p\times q}$, $\tilde{C}_2 i \in \Re^{p\times p}$ is a singular matrix, and $p \geq q$.

B. Actuator fault reconstruction based on sliding mode observers

C. Sliding mode observer structure

An adaptive observer for fault estimation is designed to address the challenges associated with the following:

$$\begin{aligned} \dot{\hat{x}}(t) &= \sum_{j=1}^{g} \mu_i(\theta(t)) \Big\{ \tilde{A}_i \hat{x}(t) + \tilde{A}_{di}\hat{x}(t - d(t)) + \Gamma_i(\tilde{x}, t) \\ &+ \tilde{B}_i u(t) + \tilde{G}_{li} e_y(t) + \tilde{G}_{ni} v(t) \Big\} \\ \hat{\tilde{y}}(t) &= \sum_{j=1}^{g} \mu_i(\theta(t)) \Big\{ \tilde{C}_i \hat{\tilde{x}}(t) \Big\} \end{aligned} \tag{13}$$

with $\tilde{\Gamma}_i(x,t) = T_{Li}\begin{bmatrix} \Gamma_{1i}^T(x,t) & \Gamma_{2i}^T(x,t) \end{bmatrix}^T$, $\tilde{e}_y(t) = \hat{\tilde{y}}(t) - \tilde{y}(t)$ represents the estimated output error. The SMO gains $tildeG_{ni}$ and $tildeG_{li}$ are appropriately chosen gain matrices.

$$\tilde{G}_{ni} = \begin{bmatrix} -L_i \bar{C}_{2i}{}^T \\ \bar{C}_{2i}{}^T \end{bmatrix}, \tilde{G}_{li} = \begin{bmatrix} \tilde{G}_{l1i} \\ \tilde{G}_{l2i} \end{bmatrix} \tag{14}$$

with $L_i \in \Re^{(n-p)\times p} = \begin{bmatrix} L_{1,i} & 0_{(n-p)\times q} \end{bmatrix}$ defined in [12]. The robust adaptive sliding mode signal is expressed as follows:

$$v(t) = \begin{cases} -\rho(t) \frac{\bar{P}_2\ \bar{e}_y(t)}{\left\|\bar{P}_2\ \bar{e}_y(t)\right\|} & si\ \bar{e}_y(t) \neq 0 \\ 0 \quad sinon \end{cases} \tag{15}$$

where $\bar{P}_2 \in \Re^{p\times p}$ is a symmetric positive definite matrix that will be determined later and $\rho(t) = \hat{\rho} + \iota_0$ is a known positive scalar that limits the amplitude of the uncertainty and the fault signal. $\hat{\rho}$ is the adaptation term defined by :

$$\dot{\hat{\rho}} = \vartheta \left\| \bar{P}_2 \bar{e}_y(t) \right\|, \hat{\rho}(0) \geq 0 \tag{16}$$

Subsequently ,the state estimation error is now defined as $\dot{\tilde{e}}(t) = \dot{\hat{\tilde{x}}}(t) - \dot{\tilde{x}}(t)$ where we obtain the following:

$$\begin{aligned} \dot{\tilde{e}}(t) &= \sum_{j=1}^{g} \mu_i(\theta(t)) \Big\{ \left(\tilde{A}_i - \tilde{G}_{li}\tilde{C}_i\right) \tilde{e}(t) \\ &+ \tilde{\mathrm{A}}_{di}\tilde{e}(t - d(t)) - \tilde{D}\xi(x,t) + \tilde{\Gamma}_e(x,t) \\ &+ \tilde{G}_{ni} v(t) - \tilde{M}_i f_a(t) \Big\} \end{aligned} \tag{17}$$

It is necessary to demonstrate the following:

$$\begin{aligned} \dot{\tilde{e}}_1(t) &= \sum_{j=1}^{g} \mu_i(\theta(t)) \Big\{ \tilde{A}_{1i}\tilde{e}_1(t) + \tilde{\Gamma}_{e1i}(x,t) + \tilde{G}_{n1i} v(t) \\ &+ \tilde{\mathrm{A}}_{d2i}\tilde{e}_y(t - d(t)) + \tilde{\mathrm{A}}_{d1i}\tilde{e}_1(t - d(t)) \\ &+ \tilde{D}_{1i}\xi(x,t) + \left(\tilde{A}_{2i} - \tilde{G}_{l1i}\right)\tilde{e}_1(t) \Big\} \end{aligned} \tag{18}$$

$$\begin{aligned} \dot{\tilde{e}}_y(t) &= \sum_{j=1}^{g} \mu_i(\theta(t)) \Big\{ \tilde{A}_{3i}\tilde{e}_1(t) + \tilde{\mathrm{A}}_{d4i}\tilde{e}_y(t - d(t)) \\ &+ \left(\tilde{A}_{4i} - \tilde{G}_{l2i}\right)\tilde{e}_y(t)\, \tilde{\mathrm{A}}_{d3i}\tilde{e}_1(t - d(t)) + \tilde{G}_{n2} v(t) \\ &- \tilde{M}_2 f_a(t) - \tilde{D}_2 \xi(x,t) + \tilde{\Gamma}_{e2i}(x,t) \Big\} \end{aligned} \tag{19}$$

where $T_i\Gamma_e(x,t) = \begin{bmatrix} \bar{\Gamma}_{e1i}(x,t) & \bar{\Gamma}_{e2i}(x,t) \end{bmatrix}$. The controlled estimation error $r(t)$ as a function of system input and output.

$$r(t) = H \begin{bmatrix} \dot{\tilde{e}}_1(t) \\ \dot{\tilde{e}}_2(t) \\ \dot{\tilde{e}}_1(t-d(t)) \\ \dot{\tilde{e}}_y(t-d(t)) \end{bmatrix} \tag{20}$$

Let's assume that H is a predefined weight matrix of full rank, i.e. $H = \text{diag}(H_1, H_2, H_3, H_4)$. We consider performing the robust performance measure in its worst-case scenario, as follows:

$$\|H_\infty\| = \sup_{\|\xi\|_2 \neq 0} \frac{\|r(t)\|_2^2}{\|\xi(x,t)\|_2^2} \tag{21}$$

Now, we'll introduce Theorem 1, which describes the conditions necessary for the existence of the suggested observer with a specified performance of $\|H_\infty\| < \sigma$

Theorem 1

Let the delayed (LPV) system (1)–(2) be considered under the verification of Assumptions 1, 2, and 3. Provided that these assumptions are satisfied, the state estimation errors (18)–(19) exhibit asymptotic stability while achieving two objectives: maximizing the admissible Lipschitz constant γ for the nonlinear function $\Gamma(x(t),t)$, and minimizing the gain σ for the system uncertainties $\xi(x,t)$. We note that $0 \leq \lambda \leq 1$), positive constants ε, α, and σ, as well as specific matrices $P_1 > 0$, $P_2 > 0$, $W_{i,}$. The fulfillment of these conditions relies on solving a convex multi-objective optimization problem in the form of an LMI.

minimize $[\lambda(\varepsilon+\alpha) + (1-\lambda)\sigma]$ under the following constraints :

$$\begin{bmatrix} \Omega_i & P\tilde{A}_{di} & P\tilde{D} & P & I_{n-p} & 0_{(n-p)p} \\ * & Z & 0 & 0 & 0 & 0 \\ * & * & -\sigma I_l & 0 & 0 & 0 \\ * & * & * & -\varepsilon I & 0 & 0 \\ * & * & * & * & -\alpha I_{n-p} & 0 \\ * & * & * & * & * & -\alpha I_p \end{bmatrix} < 0 \tag{22}$$

with $\Omega_i = \begin{bmatrix} \Pi_{1,i} & \left(P_1\tilde{A}_3\right)^T \\ (*) & \Pi_{2,i} \end{bmatrix}$ $Z = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix}$ $P\tilde{A}_{di} = \begin{bmatrix} P_1\tilde{A}_{d1} & P_1\tilde{A}_{d2} \\ P_2\tilde{A}_{d3} & P_2\tilde{A}_{d4} \end{bmatrix}$,$P\tilde{D} = \begin{bmatrix} P_1\tilde{D}_1 \\ P_2\tilde{D}_2 \end{bmatrix}$,$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$
and $\Pi_{1,i} = \bar{A}_{1i}^T P_1 + P_1\bar{A}_{1i} + W_i\bar{A}_{3i} + \bar{A}_{3i}^T W_i + S_1 + H_1^T H_1$,
$\Pi_{2,i} = \bar{A}_{4i}^{sT} P_2 + P_2\bar{A}_{4i}^s + S_2 + H_2^T H_2$
$Z_1 = -(1-\tau)S_1 + H_3^T H_3$, $Z_2 = -(1-\tau)S_2 + H_4^T H_4$
After solving the convex multi-objective optimization problem, we demonstrate the following:

$$\begin{aligned} L_i &= {P_1}^{-1}W_i, \sigma^* = \min(\sigma), \alpha^* = \min(\alpha) \\ \varepsilon^* &= \min(\varepsilon), \gamma^* = \min(\gamma) \end{aligned} \tag{23}$$

Proof:The proof of this theorem is represented in [13]

IV. Actuator fault estimation

In this section, it is assumed that the adaptive sliding-mode observer defined by equations (14) has been constructed, and that the range condition $S_g(t) = \{(\tilde{e}_1(t), \tilde{e}_y(t)) \,|\, \tilde{e}_y(t) = 0\}$ is satisfied.
Consequently, we have the following:

$$\begin{aligned} \dot{\tilde{e}}_y &= 0 = \sum_{j=1}^{g} \mu_i(\theta(t))\tilde{A}_{3i}\tilde{e}_1(t) + \tilde{A}_{d3i}\tilde{e}_1(t-d(t)) \\ &+ \tilde{\Gamma}_{e2i}(x,t) + \tilde{G}_{n2}v_{eq}(t) - \tilde{D}_2\xi(x,t) - \tilde{M}_2 f_a(t) \end{aligned} \tag{24}$$

Here, $v_{eq}(t)$ represents the equivalent output error injection signal required to maintain sliding motion. This signal can be approximated at any desired level of accuracy by replacing (17) by the following:

$$v_{eq}(t) = \rho(t)\frac{P_2 e_y}{\|P_2 e_y\| + \delta} \tag{25}$$

where δ is a small positive scalar to reduce the effect of chattering.
We can deduce the following:

$$0 = \sum_{i=1}^{g} \mu_i(\theta(t))\left(\tilde{M}_{2i} f_a(t) - v_{eq}(t)\right) \tag{26}$$

The actuator fault estimate will be obtained as a function of the equivalent error injection signal, and is expressed by the following equations:

$$\hat{f}_a(t) = \sum_{i=1}^{g} \mu_i(\theta(t))\left(\tilde{M}_{2i}\right)^+ \left\{\frac{\bar{P}_0\ \bar{e}_y(t)}{\left\|\bar{P}_0\ \bar{e}_y(t)\right\| + \delta}\right\} \tag{27}$$

V. Fault-tolerant control

A. Sliding Mode Controller Design

This work aims to design a resilient sliding-mode fault-tolerant control (FTC) strategy based on fault estimation (FE) for uncertain systems with local nonlinear models and variable delays.
Let's define a sliding surface S_g as follows:

$$S_g := \{y(t) \in \Re^p : S_c(t) = 0\} \tag{28}$$

Where $S_c(t) \in \Re^m$ is a linear switching function based on state feedback information for nonlinear systems, which can be described as follows:

$$S_c(t) = \sum_{i=1}^{g} \mu_i(\theta(t))\{N_{c,i}y(t)\} \tag{29}$$

with $N_{c,i} = (C_iB_i)^+ - h\left(I_p - C_iB_i(C_iB_i)^+\right)$ and $h \in \Re^{m\times p}$ is an arbitrary matrix .
We note that $(C_iB_i)^+ = \left((C_iB_i)^+(C_iB_i)\right)^1(C_iB_i)^T$.
Before starting the fault-tolerant control (FTC) design, given the assumption that the pair (A_i, B_i) is controllable, and using actuator fault estimation and system state, we suggest formulating robust control as follows:

$$u(t) = u_l(t) + u_n(t) \tag{30}$$

The control input $u_l(t)$ is designed to incorporate the linear component, which is influenced by system states and actuator fault estimation, as specified by the following:

$$u_l(t) = \sum_{j=1}^{g} \mu_i(\theta(t)) \left\{ -k_j \hat{x}(t) - q_i \hat{f}_a(t) \right\} \tag{31}$$

where $q_i hat f_a(t)$ aims to mitigate the impact of actuator faults. It is assumed that $k_j \in \Re^{m \times n}$ and $q_i = B_i M_i$. The nonlinear component $u_n(t)$ responsible for triggering the sliding motion on the surface, S_g, is introduced with an adaptive law as follows:

$$u_n(t) := \begin{cases} \eta(t) \frac{S_c(t)}{\|S_c(t)\|} & if\ S_c(t) = 0 \\ 0 & else \end{cases} \tag{32}$$

$\eta(t) = \hat{\rho}_c + \iota_c + \varpi_c$ where $\iota_c > 0$ and $\varpi_c > 0$ are small positive constants. We use $\hat{\rho}_c$ to determine the term $\eta(t)$.
We note the following:

$$\dot{\hat{\rho}}_c S_c = \varpi_c \|S_c(t)\|, \hat{\rho}_c(0) \geq 0 \tag{33}$$

where ϖ_c is a positive gain. From the above information, using the distinct adaptive nonlinear structure of $u_n(t)$, it is necessary to demonstrate that the system will inevitably converge to the associated sliding mode surface S_g and hold there in finite time. Consequently, when we examine the sliding motion corresponding to the sliding surface S_g described in (45), we consider the Lyapunov–Krasovskii function as our analytical tool.

$$V_c(t) = \frac{1}{2} S_c^T(t) S_c(t) + \frac{1}{2\varpi_c} \tilde{\rho}_c^2 \tag{34}$$

with $\tilde{\rho}_c = \rho_c - \hat{\rho}_c$ is the estimated error of ρ_c. Referring to the dynamic equations of the open-loop system in (2) and (46). The time derivative of (**??**) gives the following:

$$\begin{aligned} \dot{V}_c(t) &= \sum_{i=1}^{g} \sum_{j=1}^{g} \mu_i \mu_j(\theta(t)) \{(N_{c,i} C_i A_i - k_j) \|x(t)\| \\ &+ N_{c,i} C_i A_{di} \|x(t - d(t))\| - \iota_c - \varpi_c\} \|S_c(t)\|\} \end{aligned} \tag{35}$$

We define the connection subsystem with the following expression:

$$\Omega_c = \{x \|x(t)\| \leq \kappa_c\} \tag{36}$$

The reactivity condition, ensuring that the system reaches the sliding surface S, is met when the scalar $iota_c$ is chosen to satisfy $\iota_c > (N_{c,i} C_i A_i - k_j) \kappa_c$ so that we have the following:

$$S^T{}_c(t) S_c(t) \leq -\varpi_c \|S_c(t)\| \tag{37}$$

The suggested sliding mode controller with adaptive law guarantees the presence of optimal sliding motion within a finite time; specifically, $\dot{S}_c(t) S_c(t) = 0, \forall t \geq t_c$. Once the sliding mode has been reached, we turn our attention to evaluating the stability of the closed-lag (LPV) system using local nonlinear models. Let's introduce the equivalent control u_{eq} :

$$\begin{aligned} u_{eq} &= \sum_{i=1}^{g} \mu_j(\theta(t)) \{N_{c,i} C_i \{[A_i x(t) + A_{di} x(t - d(t)) \\ &+ \Gamma(x,t) + D_i(x,t)]\} + u_l(t) \end{aligned} \tag{38}$$

The dynamics of the closed-loop system with the equivalent control law (36) are represented as follows:

$$\begin{aligned} \dot{X}(t) &= \sum_{i=1}^{g} \sum_{j=1}^{g} \mu_i \mu_j(\theta(t)) \{\theta_i (A_i - k_{1j} B_i) x(t) \\ &+ \theta_i A_{di} x(t - d(t)) + \theta_i \Gamma(x,t) + B_{ij} E(t)\} \\ y_c(t) &= \sum_{i=1}^{g} \mu_i(\theta(t)) \{C_i x(t)\} \end{aligned} \tag{39}$$

with $B_{ij} = \begin{bmatrix} Bk_{1j} & M_i & \theta_i D_i \end{bmatrix}$, $E_i(t) = \begin{bmatrix} e_x^T(t) & e_{f_a}^T(t) & \xi^T(x,t) \end{bmatrix}^T$ et $\theta_i = I_n - B_i N_{c,i} C_i$.

B. Separate FTC design with variable delay

. In recent years, considerable research has focused on solving fault-tolerant control (FTC) problems via fault estimation (FE), primarily through a two-step approach:
First ,designing an observer to estimate control effectiveness, and then developing a controller to stabilize the closed-loop system.
The following theorem provides sufficient conditions for the existence of such an FTC control, with prescribed H_∞ performance, using Lyapunov–Krasovskii stability and the multi-objective LMI technique.
Theorem 2
The closed-loop system for a nonlinear system with delay is robustly stable with the admissible Lipschitz constant λ_x maximized simultaneously and the gain ς_x minimized. The H_∞ criteria are guaranteed if there exists a symmetric positive definite matrix $P_x = P_x^T$, Q, and constants $0 \leq \lambda_x \leq 1$, $\alpha_x > 0$ and $\varsigma_x > 0$, such that the following LMI optimization is satisfied:

$$\begin{aligned} & min\left[\lambda_x(\varepsilon_x + \alpha_x) + (1 - \lambda_x)\varsigma_x\right] \\ & \begin{bmatrix} \Psi_{i,j} + C_i^T C_i & \theta_i A_{di} & B_i Q_j & E_i \\ * & (1 - \tau) S_x & 0 & 0 \\ * & * & -2\vartheta + \varsigma_x I_n & 0 \\ * & * & * & \mathrm{N} \end{bmatrix} < 0 \end{aligned} \tag{40}$$

with $\Psi_{i,j} = P_x A_i^T \theta_i{}^T + \theta_i A_i P_x - B_i Q_j - B_i^T Q_j^T + S_{xi}$, $\vartheta = \mu_c P_x$ $K_j = Q_j P_x{}^{-1}$,
$\bar{E}_i = \begin{bmatrix} M_i & \theta_i D_i & P_x C_i^T & P_x & I_n \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$,
$\mathrm{N} = diag\{-\varsigma_x I_n, -\varsigma_x I_q, -\varsigma_x I_l, -I, -\varepsilon_x I, -\alpha_x I\}$.
Solving the multi-objective optimization problem yields the following results:

$$\begin{aligned} \varsigma_x^* &= \min(\varsigma_x), \alpha_x^* = \min(\alpha_x) \\ \varepsilon_x^* &= \min(\varepsilon_x), \lambda_x^* = \min(\lambda_x) \end{aligned} \tag{41}$$

Proof3.2 Let's introduce the Lyapunov function for the closed-loop system as follows:

$$V_x(t) = x^T(t) P_x x(t) + \int_{t-d(t)}^{t} x^T(t) S_x x(t) dx \qquad (42)$$

where $P_x \in \Re^{n \times n}$ is a symmetrical positive-definite matrix. Considering the closed-loop delayed (LPV) system (**??**) and (**??**), the time derivative $\dot{V}_x(t)$ can be expressed as follows:

$$\begin{aligned} \dot{V}_x(t) &= \sum_{i=1}^{g}\sum_{i=1}^{g} \mu_i\mu_j(\theta(t))\left\{x^T(t)\Pi_{xi}x(t)\right. \\ &+ 2x^T(t)P_x\theta_i A_{di}x(t) + 2x^T(t)P_x\theta_i\Gamma(x,t) \\ &- x^T(t-d(t))(1-\tau)S_x x(t-d(t)) \\ &- \left.2x^T(t)P_x B_i E_i(t)\right\} \end{aligned} \qquad (43)$$

with $\Pi_{xi} = P_x{\theta_i}^T{A_i}^T + \theta_i A_i P_x - {Q_j}^T B_i j^T - B_i j Q_j + S_{xi}$. Based on the definition and assumption 4, the following inequality is obtained:

$$\begin{array}{l} 2x^T(t)P_x\theta_i\Gamma(x,t) \le \frac{1}{\varepsilon_c}x^T(t)P_x x(t) \\ +\varepsilon_c\Gamma^T(x,t)\theta^T{}_i\theta_i\Gamma(x,t) \\ \le x^T(t)\left[\frac{1}{\varepsilon_c}P_x^2 + \varepsilon_c\tilde{\gamma}_c^2\right]x(t) \end{array} \qquad (44)$$

We note the following: $\tilde{\gamma}_c = \gamma_c \|\theta_i\|$

To guarantee the robustness of the (LPV) closed-loop delayed system (37)) to perturbations noted $\phi(t)$, we introduce the variable $J(t)$ defined as follows:

$$J(t) = \sum_{i=1}^{g}\sum_{i=1}^{g}\mu_i\mu_j(\theta(t))\left\{\begin{array}{l}\dot{V}_x(t) + Y_L^T(t)Y_L(t)\\ -\varsigma_x\phi^T(t)\phi(t)\end{array}\right\} \qquad (45)$$

We introduce the new variable as follows:

$$\alpha_c := \frac{1}{\varepsilon_c\tilde{\gamma}_c^2} \to \tilde{\gamma}_c = \frac{1}{\sqrt{\varepsilon_c\tilde{\gamma}_c}} \qquad (46)$$

It has been determined that maximization of $\tilde{\gamma}_c$, associated with the nonlinear Lipschitz function $\Gamma(x,t)$, can be achieved by simultaneously minimizing ε_c and α_c, thus implying the need to minimize the linear function $\alpha_c + \varepsilon_c$. Consequently, it remains to demonstrate the following:

$$\Sigma_{sep} = \sum_{i=1}^{g}\sum_{j=1}^{g}\mu_i\mu_j(\theta(t))\left[\begin{array}{cc}\Lambda_{1,ij} & \Lambda_{2ij}\\ (*) & \Lambda_{3ij}\end{array}\right] < 0 \qquad (47)$$

with $\Lambda_{1,ij} = \left[\begin{array}{cc}\Pi_{xi,j} & \theta_i A_{di}\\ * & (1-\tau)S_x\end{array}\right]$, $\Lambda_{2,ij} = \left[\begin{array}{ccccc}P_x M_i & P_x\theta_i D_i & P_x C_i^T & P_x & I_n\\ 0 & 0 & 0 & 0 & 0\end{array}\right]$, $\Lambda_{3ij} = diag\{-\varsigma_x I_n, -\varsigma_x I_q, -\varsigma_x I_l, -I, -\varepsilon_x I, -\alpha_x I\}$. Using Schur's complement, the relation (47) is equivalent to the following:

Next, we construct the matrix X with a single diagonal structure as follows: $X = diag\left\{P_x^{-1}, P_x^{-1}, P_x^{-1}, I_q, I_l, I_p, I_n, I_n\right\}$. By multiplying both sides of Equation (**??**) by X and its transpose, we arrive at Equation (38). Using the definition, we can then establish the following relationship:

$${P_x}^{-1} + {P_x}^{-1} \le \varsigma_x {P_x}^{-1}{P_x}^{-1} + {\varsigma_x}^{-1}I_n \qquad (48)$$

It's equivalent to :

$${P_x}^{-1}\varsigma_x{P_x}^{-1} \le -2{P_x}^{-1} + {\varsigma_x}^{-1}I_n \qquad (49)$$

This completes the Proof.

VI. Application: LPV Diesel Engine Model

In this section, the design of separate and integrated sliding-mode FTC strategies is carried out, using data provided by the sliding-mode observer. The diesel engine model (LPV), obtained from [14], is considered for this purpose (see figure **??**). Diesel engines are designed to withstand severe operating conditions, making them suitable for environments where uncertainty can affect the system. The integration of diesel engines into variable-delay (LPV) systems enables better management of the delays inherent in engine operation. The matrices of the local nonlinear models in (1) can be expressed as follows:

$$\begin{aligned} A_1 &= A_4 = \left[\begin{array}{ccc}-12.6 & 4.6 & 0\\ 0.41 & -2.08 & 0\\ 0.486 & -0.0004 & -1.33\end{array}\right] \\ A_2 &= A_3\left[\begin{array}{ccc}-12.6 & 4.6 & 0\\ 0.41 & -2.08 & 0\\ 0.486 & -0.4404 & -1.33\end{array}\right] \\ A_{d1} &= A_{d2} = A_{d3} = A_{d4} = \left[\begin{array}{ccc}0 & 3.6 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{array}\right] \\ B_1 &= B_2 = B_3 = B_4 = \left[\begin{array}{cc}0 & 0\\ -25.65 & 40.32\\ -18.27 & 1\end{array}\right] \\ C_1 &= C_2 = C_3 = C_4 = \left[\begin{array}{ccc}1 & 0 & 0\\ 1 & 0 & 1\end{array}\right] \\ D_1 &= D_2 = D_3 = D_4 = 0.1 \times \left[\begin{array}{c}1\\ 0.91\\ 0.91\end{array}\right] \\ M_1 &= M_2 = M_3 = M_4 = \left[\begin{array}{c}0.7\\ 0\\ 1\end{array}\right] \end{aligned} \qquad (50)$$

The weighting functions $\mu_i(\theta(t))$ are defined as follows:

$$\begin{aligned} \mu_1(\theta(t)) &= \frac{(\theta_1(t)+0.6)(\theta_2(t)+2)}{4} \\ \mu_2(\theta(t)) &= \frac{(\theta_1(t)+0.6)(2-\theta_2(t))}{4} \\ \mu_3(\theta(t)) &= \frac{(0.6-\theta_1(t))(\theta_2(t)+2)}{4} \\ \mu_4(\theta(t)) &= \frac{(0.6-\theta_1(t))(2-\theta_2(t))}{4} \end{aligned} \qquad (51)$$

with $\theta_1(t) = 0.25\sin(0.1t)$, $\theta_2(t) = 0.1\sin(0.25t)$. The proposed approach encapsulates the non-linearities present in the diesel engine system, enabling efficient management of non-linear dynamics:

$$\Gamma(x,t) = \begin{bmatrix} -3.33 \sin(x_3(t)) \\ 0 \\ 0 \end{bmatrix} \tag{52}$$

The delay is modeled as a variable time delay: $d(t) = 0.1 \sin(0.3t)$. Suppose an actuator fault occurs in the input channel of the diesel engine system. This fault will be added to the equation of state, which can be described as follows:

$$f_a(t) = \begin{cases} 0 & 0 \leq t < 4s \\ \sin(t) & 4 \leq t < 15s \end{cases} \tag{53}$$

We can see that the assumptions 2 and 3 are verified on the basis of the matrices of the nonlinear models, i.e. $rank(C_i M_i) = rank(M_i) = 1$, $\left(A_i + A_{di}e^{-ds}, M, C\right)$ is a minimal phase $n \geq p = 2$ and $q = 1$. The adaptive sliding-mode observer method presented below can be used for the nonlinear diesel engine system to obtain a robust estimate of actuator faults. Based on the information provided by the observer, an adaptive sliding-mode controller will be designed to stabilize the diesel engine system. The design parameters have been chosen as follows $\lambda = 0.1, H_1 = H_3 = I_{n-p}, H_2 = 2I_p, H_4 = I_p$ and $\tilde{A}_{22}^s = diag(-10, -7)$. Using MATLAB's LMI toolbox, we can solve Theorem 1 and design an adaptive observer (14)–(15).

$$\begin{aligned}
G_{l1} &= \begin{bmatrix} 2.420 & 1.388 \\ 9.054 & 1.484 \\ -9.156 & 5.669 \end{bmatrix} G_{n1} = \begin{bmatrix} 1 & 0 \\ 1.09 & 0.30 \\ -1 & 1 \end{bmatrix} \\
G_{l2} &= \begin{bmatrix} 2.864 & 1.032 \\ 9.819 & 1.104 \\ -9.679 & 5.571 \end{bmatrix} G_{n2} = \begin{bmatrix} 1 & 0 \\ 1.18 & 0.22 \\ -1 & 1 \end{bmatrix} \\
G_{l3} &= \begin{bmatrix} 2.839 & 1.045 \\ 10.97 & 1.118 \\ -8464 & 5.569 \end{bmatrix} G_{n3} = \begin{bmatrix} 0.437 & 0.009 \\ 1.182 & 0.22 \\ -1 & 1 \end{bmatrix} \\
G_{l4} &= \begin{bmatrix} 2.227 & 1.494 \\ 9.92 & 1.598 \\ -7.944 & 5.669 \end{bmatrix} G_{n4} = \begin{bmatrix} 1 & 0 \\ 1.049 & 0.32 \\ -1 & 1 \end{bmatrix}
\end{aligned} \tag{54}$$

and we find then :

$$P_e = \begin{bmatrix} 1.912 & 0 & 0 \\ 0 & 1.456 & -0.324 \\ 0 & -0.324 & 1.027 \end{bmatrix} \tag{55}$$

Based on the theorem 2, the gains of the sliding mode controller can be described as follows:

$$\begin{aligned}
k_1 &= \begin{bmatrix} -13.09 & -4.98 & -2.98 \\ 9.54 & 7.17 & 2.11 \end{bmatrix} \\
k_2 &= \begin{bmatrix} -12.94 & -4.92 & -2.97 \\ 9.64 & 7.15 & 2.10 \end{bmatrix} \\
k_3 &= \begin{bmatrix} -12.92 & -5.02 & -2.98 \\ 9.75 & 7.10 & 2.10 \end{bmatrix} \\
k_4 &= \begin{bmatrix} -12.76 & -5.06 & -2.97 \\ 10.10 & 7.19 & 2.15 \end{bmatrix}
\end{aligned} \tag{56}$$

and

$$P_x = \begin{bmatrix} 7.61 & 1.05 & 1.20 \\ 1.05 & 3.86 & 0.71 \\ 1.20 & 0.71 & 0.41 \end{bmatrix} \tag{57}$$

To validate the effectiveness of the proposed scheme, we present a comparative study with the control law used in [9]. The authors employ a state-feedback control law, which can be less effective when the system is affected by complex local nonlinearities such as Lipschitz nonlinearity. In contrast, our integrated FTC approach based on Sliding Mode Control (SMC) forces the system to remain on a sliding surface, thereby ensuring stable performance even in the presence of unexpected parameter variations. Figure 1 illustrates the performance of our integrated FTC based on SMC on a sliding surface for the second actuator fault reconstruction scenario, where the fault changes its behavior. The results show that the separated (AFTC) based on (SMC) can adjust the control law according to the system parameter variations, thus ensuring a fast response (first response at t = 0.14 s). This contrasts with the state-feedback control, which is less effective in facing these challenges (first response at t = 0.19 s). It is clear that our adaptive controller is more robust against chattering effects compared to the state-feedback-based controller. The

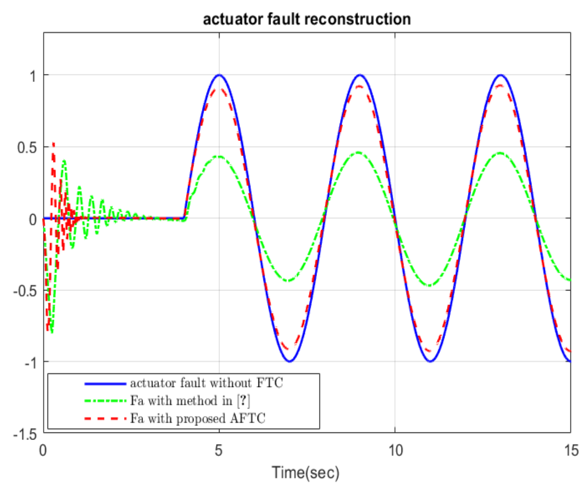

Fig. 1. Actuator fault estimation without FTC (blue line), actuator fault estimation with state feedback from [9] (green line), and actuator fault estimation with the separated AFTC (red line).

comparative study of the time-domain response of the closed-loop system outputs between the separated sliding mode (AFTC) and the method proposed in [9] is shown in Figures 2 and 3. We can conclude that our proposed controllers stabilize the closed-loop system when it is affected by nonlinearities, actuator faults, and variable time delays.

VII. Conclusion

This paper deals with the development of a separated sliding-mode fault-tolerant strategy (FTC) based on , in contrast to classical control approaches. An adaptive sliding-mode observer is first designed to estimate actuator faults

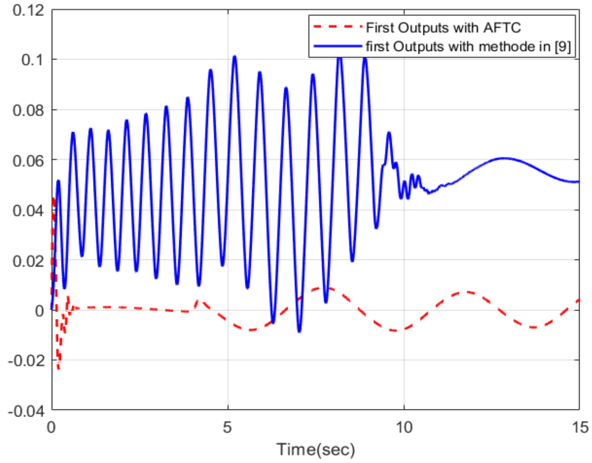

Fig. 2. C

losed-loop system output response: first output response with the separated AFTC (red line) and output response with the method in [9] (blue line)

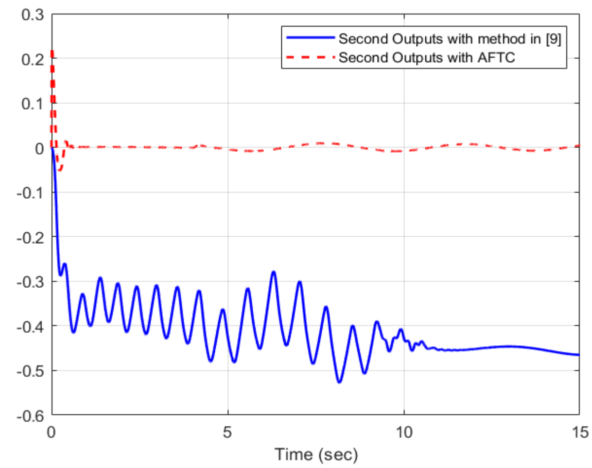

Fig. 3. Closed-loop system output response: second output response with the separated AFTC (red line) and output response with the method in [9] (blue line).

in LPV systems with variable uncertainties and delays. Its stability is guaranteed by a criterion formulated as an LMI. Then, a fault-tolerant sliding-mode control, based solely on the estimated faults, is developed from the compensated fault. The synthesis conditions, obtained via a Lyapunov-Krasovskii approach, are also expressed in LMI form. Multi-objective optimization is used to maximize the Lipschitz constant and uncertainty mitigation. The simulations on a delayed diesel engine confirm the robustness of this separate approach, which facilitates modularity, design and analysis. Finally, prospects are envisaged for integrating advanced compensation schemes, suitable for aerospace systems with long delays and complex non-linearities.

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