DATA 605 : Assignment Week 9

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Problem 11 page 363 : The price of one share of stock in the Pilsdorff Beer Company is given by Y_n on the nth day of the year. Finn observes that the differences $X_n = Y_{n+1} - Y_n$ appear to be independent random variables with a common distribution having mean $\mu = 0$ and variance $\sigma^2 = 1/4$. If $Y_1 = 100$, estimate the probability that Y_{365} is

a. ≥ 100

b. ≥ 110

c. ≥ 120

Answer :

$$\begin{aligned} X_n &= Y_{n+1} - Y_n \\ Y_2 &= X_1 + Y_1 \\ E(Y_2) &= E(X_1) + E(Y_1) \\ &= 0 + 100 \\ &= 100 \end{aligned}$$

$$\begin{aligned} V(Y_2) &= V(X_1) + V(Y_1) \\ &= 1/4 + 0 \\ &= 1/4 \end{aligned}$$

Apply the same formula and we will find the expected value of Y356 and the variance of Y365

$$\begin{aligned} X_n &= Y_{n+1} - Y_n \\ Y_{n+1} &= X_n - Y_n \\ Y_{365} &= X_{364} + \ldots + X_1 + Y_1 \\ E(Y_{365}) &= E(X_{364} + \ldots + X_1 + Y_1) \\ &= 100 \end{aligned}$$

$$\begin{aligned} V(Y_{365}) &= V(X_{364}) + \ldots + V(X_1) + V(Y_1) \\ &= 364(1/4) \\ &= \frac{365}{4} \\ \sigma &= \sqrt{\frac{365}{4}} \end{aligned}$$

a.

$$\geq 100$$

$$P(Y_{365} \geq 100) = P(Y_{365} - 100 \geq 0) = P(\frac{Y_{365} - 100}{\sqrt{\frac{365}{4}}} \geq 0)$$

=0.5 by the central limit theorem

b.

$$\geq 110$$

$$P(Y_{365} \geq 10) = P(Y_{365} - 100 \geq 10) = P(\frac{Y_{365} - 100}{\sqrt{\frac{365}{4}}} \geq \frac{10}{\sqrt{\frac{365}{4}}})$$

=0.147 by the central limit theorem

c.

$$\geq 120$$

$$P(Y_{365} \geq 20) = P(Y_{365} - 100 \geq 20) = P(\frac{Y_{365} - 100}{\sqrt{\frac{365}{4}}} \geq \frac{20}{\sqrt{\frac{365}{4}}})$$

=0.018 by the central limit theorem

Calculate the expected value and variance of the binomial distribution using the moment generating function.

Answer :

For binomial distribution, $P(X = k) = \binom{n}{k} p^k q^{n-k}$, where $q = 1 - p$.

The moment generating function is $M_X(t) = (q + pe^t)^n$.

The first moment is $M'_X(t) = n(q + pe^t)^{n-1} pe^t$.

The expected value is the first moment evaluated at $t = 0$:

$$\begin{aligned} E(X) = M'_X(0) &= n(q + pe^0)^{n-1} pe^0 \\ &= n(q+p)^{n-1} p \\ &= np(1-p+p)^{n-1} \\ &= np1^{n-1} \\ &= np \end{aligned}$$

The second moment is $M''_X(t) = n(n-1)(q + pe^t)^{n-2} p^2 e^{2t} + n(q + pe^t)^{n-1} pe^t$.

Evaluate the second moment at $t = 0$:

$$\begin{aligned} E(X^2) = M''_X(0) &= n(n-1)(q + pe^0)^{n-2} p^2 e^0 + n(q + pe^0)^{n-1} pe^0 \\ &= n(n-1)(1-p+p)^{n-2} p^2 + n(1-p+p)^{n-1} p \\ &= n(n-1)p^2 + np \end{aligned}$$

The variance is $V(X) = E(X^2) - E(X)^2$:

$$
\begin{aligned}
V(X) &= n(n-1)p^2 + np - n^2p^2 \\
&= np((n-1)p + 1 - np) \\
&= np(np - p + 1 - np) \\
&= np(1-p) \\
&= npq
\end{aligned}
$$

We arrived at the known definitions for binomial distribution - $E(X) = np$ and $V(X) = npq$.

Calculate the expected value and variance of the exponential distribution using the moment generating function

Answer :

For exponential distribution, $f(x) = \lambda e^{-\lambda x}$.

The moment generating function is $M_X(t) = \frac{\lambda}{\lambda - t}, t < \lambda$.

Using WolframAlpha, we get $M'_X(t) = \frac{\lambda}{(\lambda-t)^2}$ and $M''_X(t) = \frac{2\lambda}{(\lambda-t)^3}$.

Expected value:

$$
\begin{aligned}
E(X) = M'_X(0) &= \frac{\lambda}{(\lambda - 0)^2} \\
&= \frac{\lambda}{\lambda^2} \\
&= \frac{1}{\lambda}
\end{aligned}
$$

Variance:

$$
\begin{aligned}
V(X) = E(X^2) - E(X)^2 &= M''_X(0) - M'_X(0)^2 \\
&= \frac{2\lambda}{(\lambda - 0)^3} - \frac{1}{\lambda^2} \\
&= \frac{2\lambda}{\lambda^3} - \frac{1}{\lambda^2} \\
&= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \\
&= \frac{1}{\lambda^2}
\end{aligned}
$$

We arrived at the known definitions for binomial distribution - $E(X) = 1/\lambda$ and $V(X) = 1/\lambda^2$.
