Hermite Polynomials

The third frequently used orthogonal system is the Hermite system defined by

$$H_0(x) = 1, \quad H_1(x) = 2x$$

and the recursive equation.

$$H_n(x) - 2xH_{n-1}(x) + 2(n-1)H_{n-2}(x) = 0, \quad n \geqslant 0.$$

These functions form a complete orthogonal system over the infinite interval $(-\infty, \infty)$ w.r.t. the weight function e^{-x^2}.

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The 2-D system functions

$\{ 1, \cos(mx_2), \sin(mx_2), \cos(mx_1), \sin(mx_1),$

$\cos(mx_1)\cos(nx_2),$

$\sin(mx_1)\sin(nx_2), \quad \sin(mx_1),$

$\sin(mx_1)\cos(nx_2), \quad \sin(nx_2) \}$ for arbitrary $m, n \geq 1.0$

Multivariate Functions.

- Constructing a complete orthogonal systems of multivariate functions.

Let $\{u_1(x), u_2(x), \dots\}$ be a complete orthogonal system over the interval $[a, b]$ w.r.t. a weight function $w(x)$.

Defn: [a complete system]

A system of functions S defined over a domain D is called Complete, if for any given piecewise continuous function over D, is a sequence $\{u_i(x)\}$ whose elements are finite linear combinations of the elements of S can be found, such that $\{u_i(x)\}$ approximate $f(x)$ arbitrarily closely in the mean.

Example $\{1, \cos(mx), \sin(mx)\}$, $m \geq 1$

The functions are orthogonal over the interval $[0, 2\pi]$ with respect to $w(x) = 1$.

* This system is complete over the interval $[0, 2\pi]$.

* A complete system of functions may NOT be an orthogonal system.

* Orthogonal $\Rightarrow$ Orthonormal

$$u_i(x) \rightarrow \frac{u_i(x)}{\sqrt{A_i}} = u_i^*(x)$$

$$\int_a^b w(x) u_i^*(x) u_j^*(x) dx = \frac{1}{\sqrt{A_i}} \frac{1}{\sqrt{A_i}} A_i = 1, 0.$$

* Defn: Let $f(x)$ be a piecewise continuous function and $\{u_1(x), u_2(x), \cdots, u_m(x)\}$ a system of functions, defined over the same domain.

If $\lim_{i \to \infty} u_i(x) = f(x)$, $f(x)$ is continuous at x

$\lim_{i \to \infty} u_i(x) = \frac{1}{2}[f(x_+) + f(x_-)]$, f has a jump at x

the sequence $\{u_i(x)\}$ is said to approximate $f(x)$ arbitrarily closely in the mean.

Univariate Functions

Let $u(x), v(x)$ be real-valued integrable functions of one variable on $a \leq x \leq b$, $I = [a, b]$.

The function $w(x)$ be a nonnegative integrable function over I for which $\int_a^b w(x)dx > 0$.

Defn [The inner product of $u(x), v(x)$ over I]

$$\langle u, v \rangle = \int_a^b u(x)v(x)dx \quad \text{on} \quad [a, b].$$

Defn [The norm of $u(x)$ over I]

$$\langle u, u \rangle^{\frac{1}{2}} = \left[\int_a^b u^2(x)dx\right]^{\frac{1}{2}}.$$

Defn [The orthogonal function over I w.r.t the weight function $w(x)$]

$$\int_a^b w(x)u(x)v(x)dx = 0$$

Defn [The orthogonal system over I w.r.t. $w(x)$]

For a set of integrable functions $u_1(x), u_2(x), \cdots, u_m(x)$ on I.

$$\int_a^b w(x)u_i(x)u_j(x)dx = A\delta_{ij}, \quad 1 \leq i, j \leq m, \quad A \neq 0, \quad \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j. \end{cases}$$

$\langle A = 1 \Rightarrow$ orthonormal $\rangle$.

Defn: [Linearly independent]

Given a set of functions $u_1(x), u_2(x), \ldots, u_m(x)$ over I, the integrable functions are L.I.;

$$c_1 u_1(x) + c_2 u_2(x) + \cdots + c_m u_m(x) = 0 \quad \text{iff} \quad c_1 = c_2 = \cdots = c_m = 0.$$

Theorem An orthogonal system is L.I.

Let $u_1(x), u_2(x), \ldots, u_m(x)$ are orthogonal w.r.t. $w(x)$ over I.

$$\sum_{i=1}^{m} c_i u_i(x) = 0 \quad , \quad a \leq x \leq b.$$

Multiplying this equation by $w(x) u_i(x)$ and integrating over I provides

$$\int_a^b w(x) u_j(x) \sum_{i=1}^{m} c_i u_i(x)\, dx = 0$$

$$\sum_{j=1}^{m} \int_a^b c_i\, w(x) u_i(x) u_j(x)\, dx = 0$$

$$\sum_{j=1}^{m} c_i A_j = 0 \quad (A_j \neq 0) \Rightarrow c_i = 0$$

$$\int_a^b w(x) u_i(x) u_j(x)\, dx = A_i \delta_{ij} \quad 1 \leq i, j \leq m$$

$$c_j A_j = 0 \quad 1 \leq j \leq m$$

$$A_j \neq 0 \Rightarrow c_j = 0.$$