15-150 Fall 2020

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Lecture 5

Efficiency analysis

Today

- **Work and span**
  - sequential and parallel runtime
- **Recurrences**
  - *exact* and *asymptotic* solutions
- **Improving efficiency**
  - careful program design

program $\longrightarrow$ recurrence

what matters

- **Correctness**

  TYPE $f : t_1 \to t_2$

  REQUIRES (value) x $(:t_1)$ such that …

  ENSURES $f\ x \Longrightarrow^{*} v$ $(:t_2)$ such that …

- **Efficiency**

  Information about *evaluation time* of $f\ x$

  $$f\ x \Longrightarrow^{h(x)\ \text{steps}} v$$

  - exact number of steps $h(x)$ depends on x and definition of f

  An ***asymptotic*** estimate is good enough!

  - $h(x)$ is $\mathbf{O}(g(\text{size } x))$ for some notion of size

asymptotic

- We want to estimate the *runtime* $W_f(n)$ for evaluating $f(n)$, for ***large*** n

  assuming basic operations take ***constant time***

- We will give a **big-O** classification

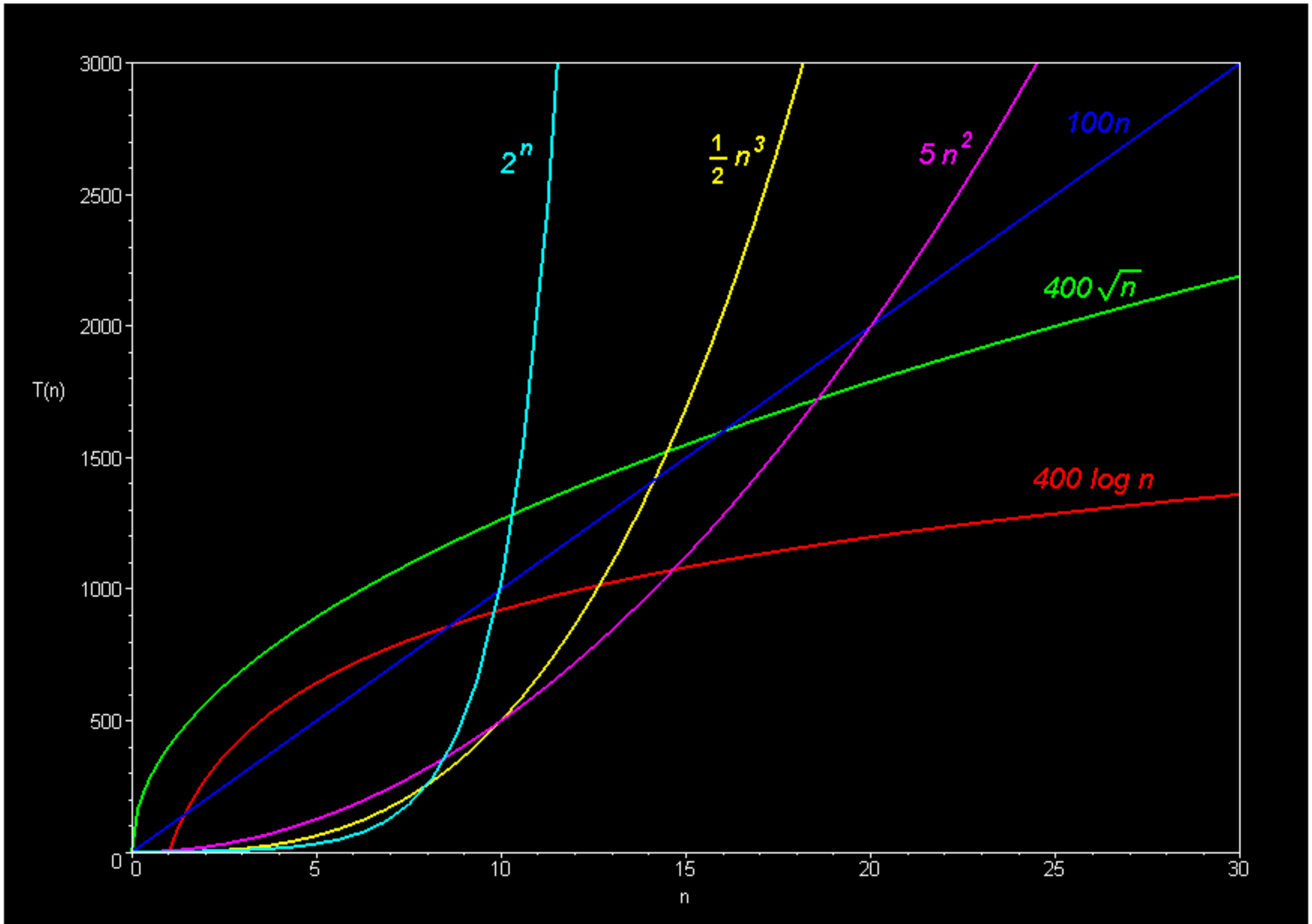
$W_f(n)$ is $O(g(n))$

if there are N and c such that

$\forall n \geq N,\ W_f(n) \leq c\, g(n)$

The graph below compares the running times of various algorithms.

- Linear -- O(n)
- Quadratic -- O(n^2)
- Cubic -- O(n^3)
- Logarithmic -- O(log n)
- Exponential -- O(2^n)
- Square root -- O(sqrt n)



isqrt_1

isqrt_0

isqrt_2

motivation

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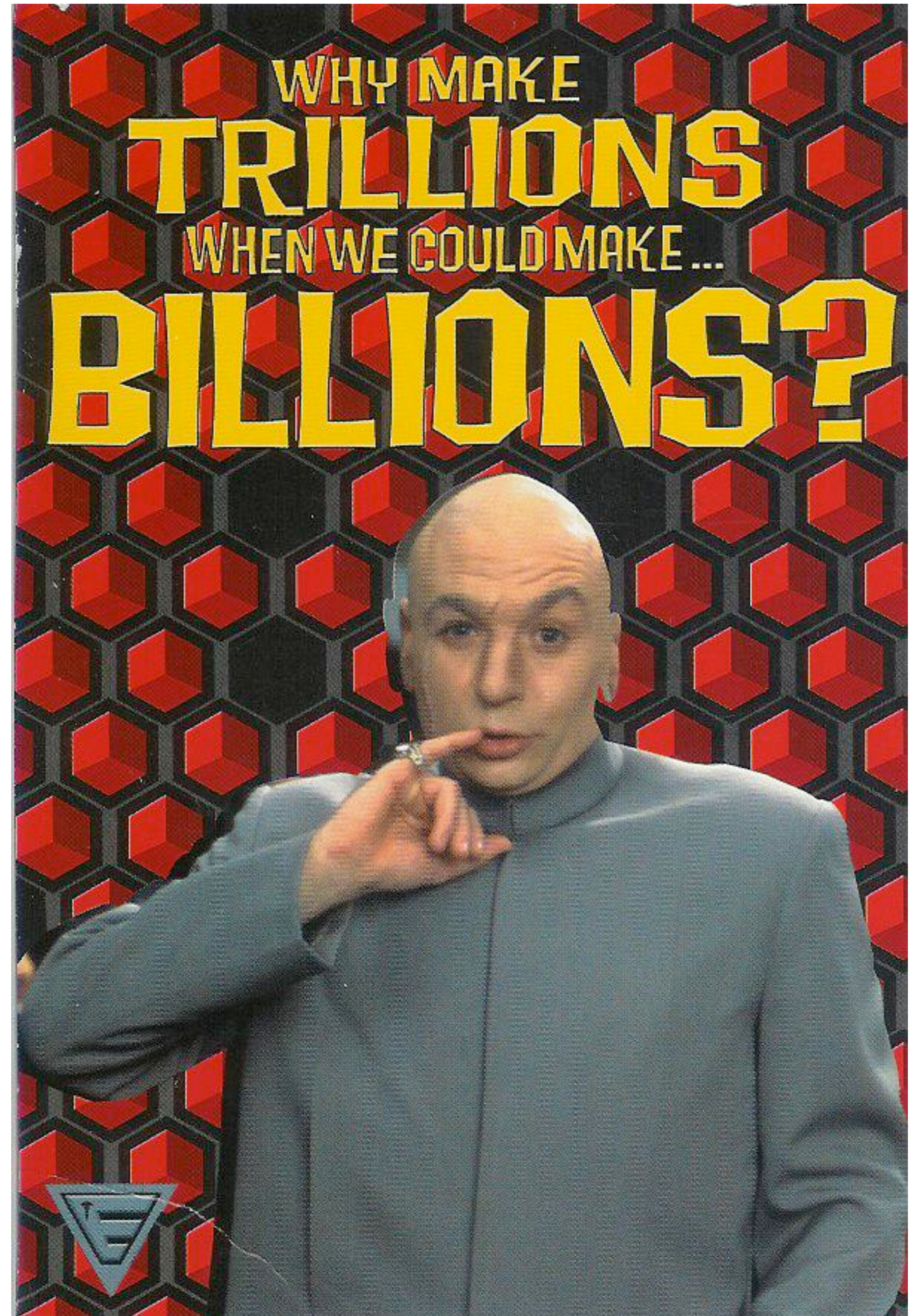

motivation

Why take
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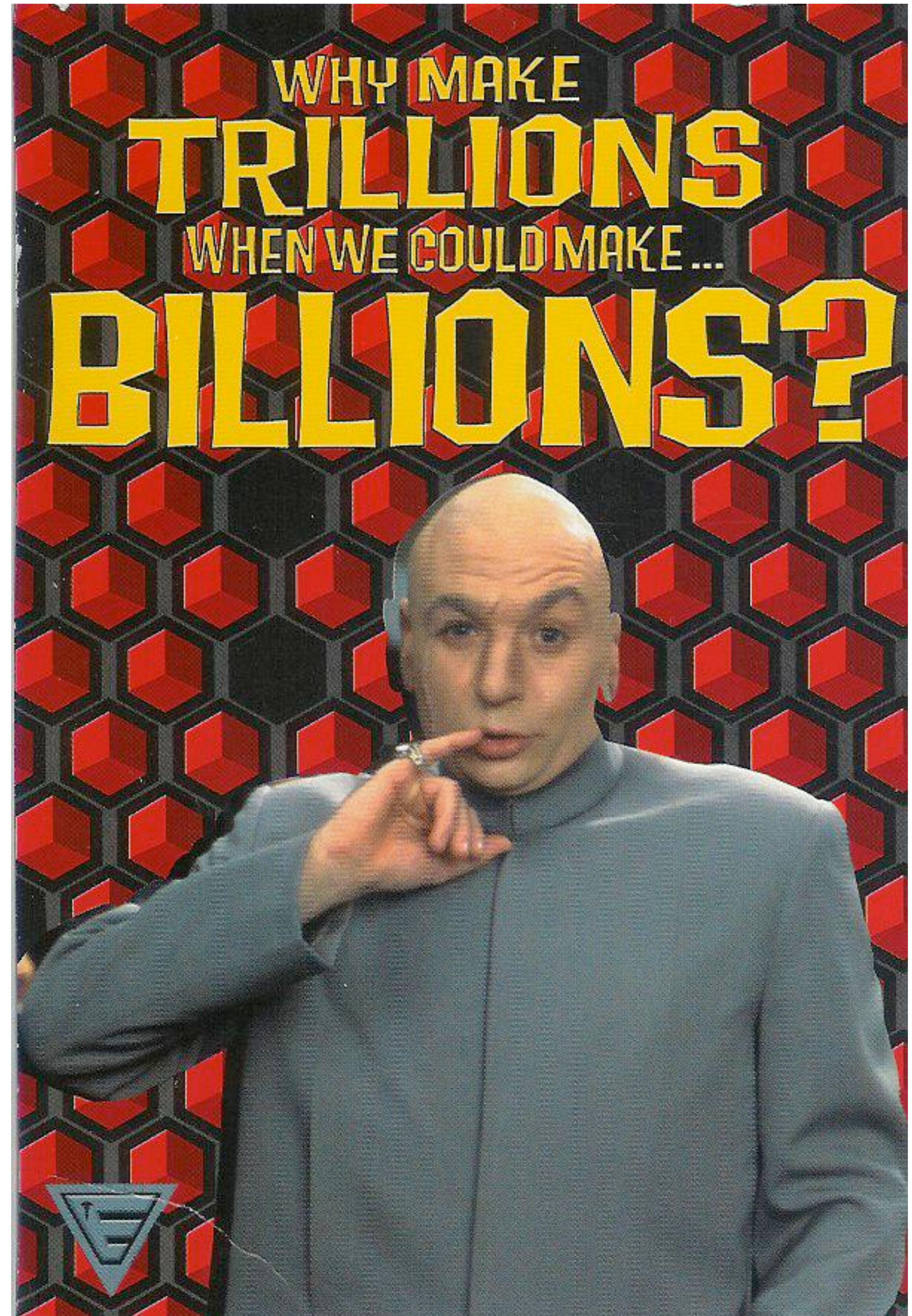

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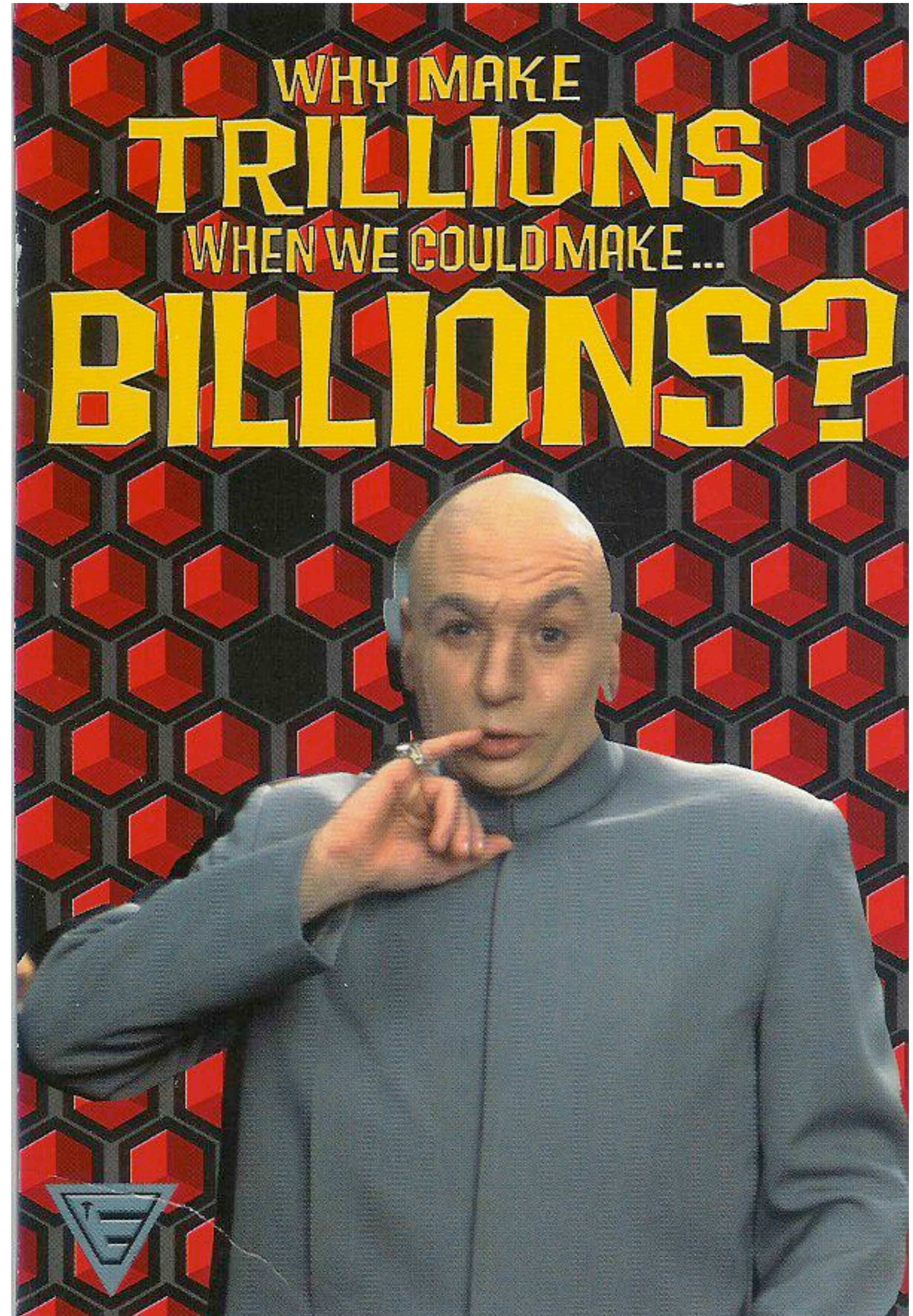
isqrt_0 123456789



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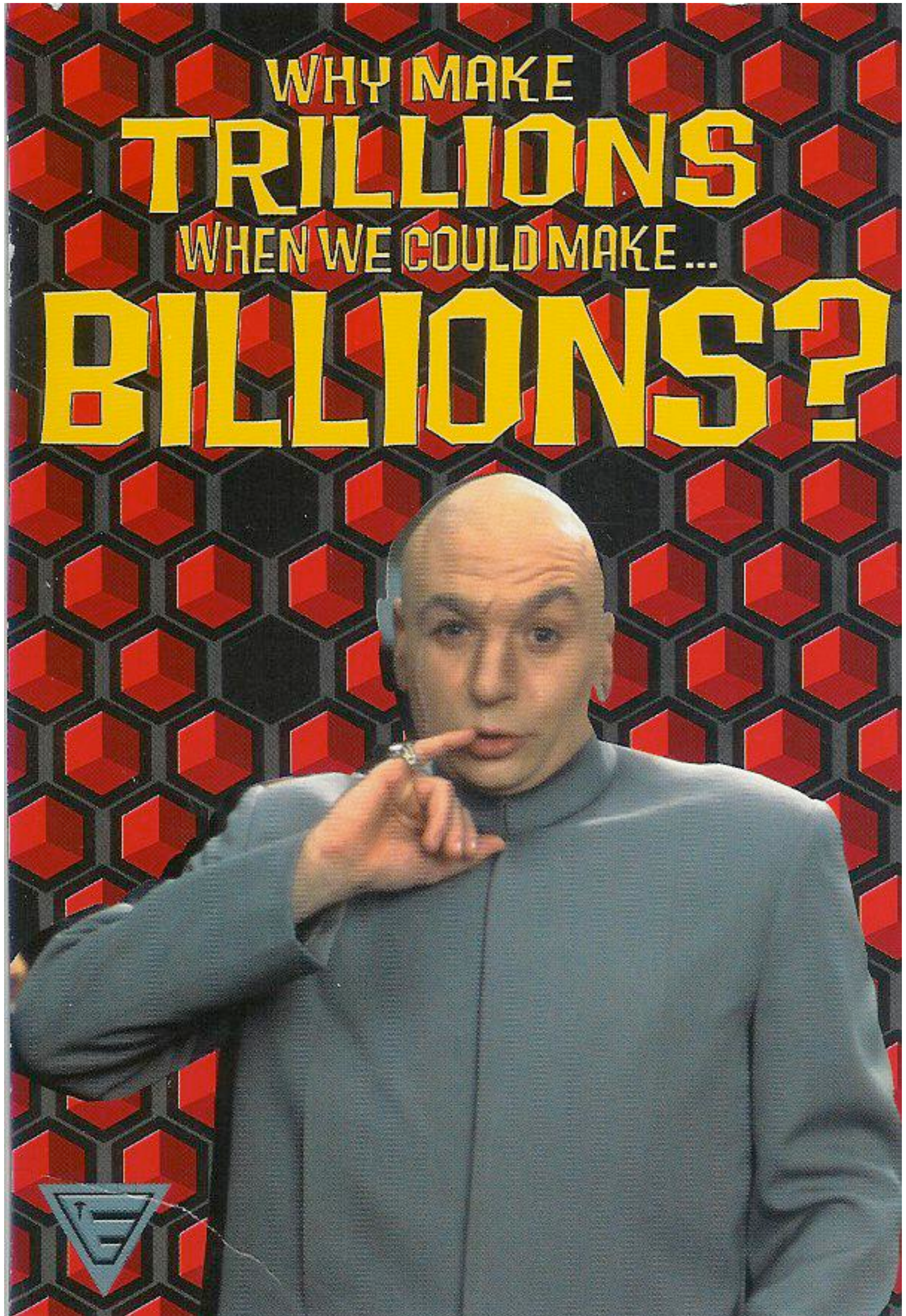

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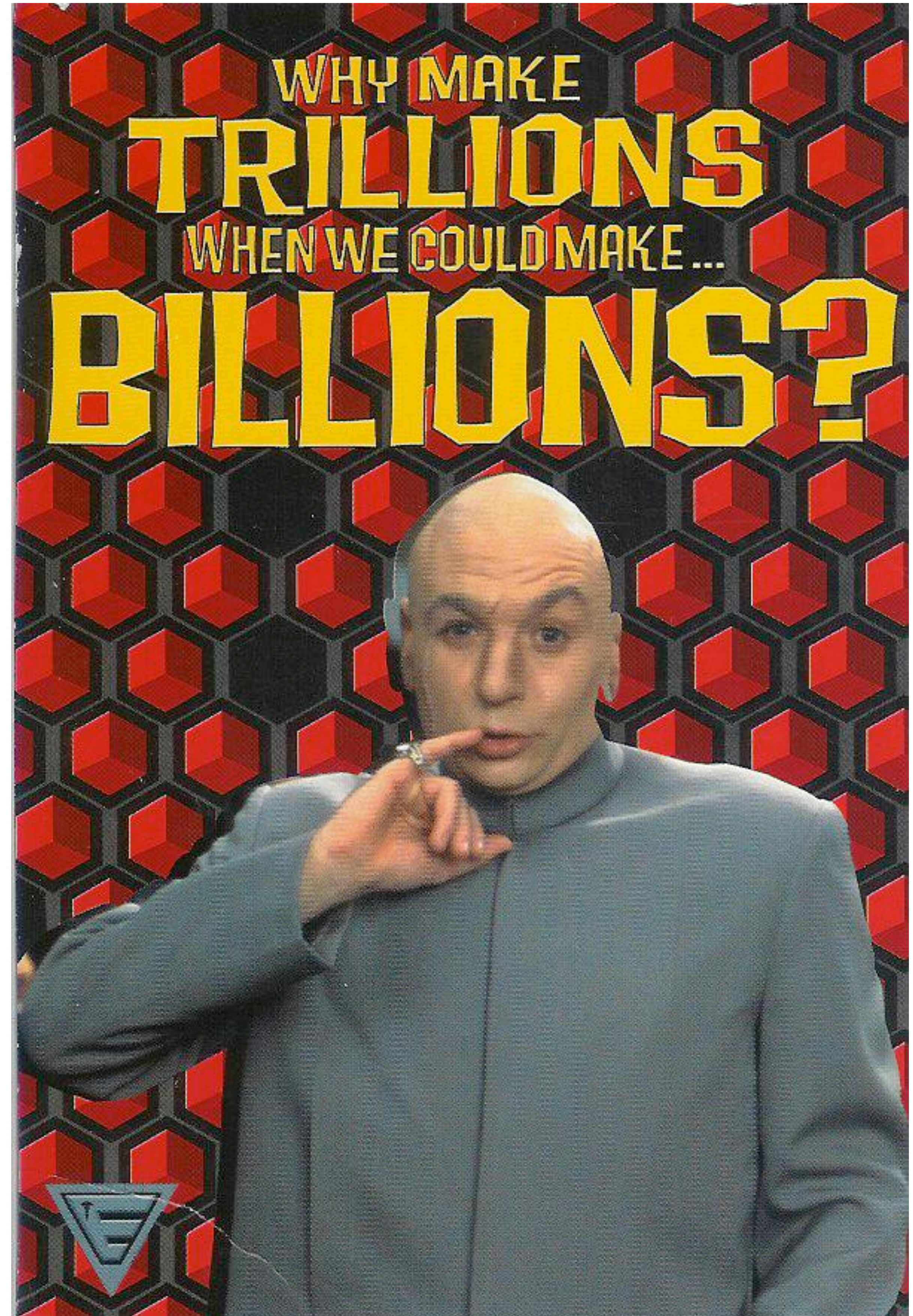

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asymptotically

- ***Ignore*** additive constants

  $n^5+1000000$ is $O(n^5)$

- ***Absorb*** multiplicative constants

  $1000000n^5$ is $O(n^5)$

- Be as accurate as you can

  $O(n^2) \subset O(n^3) \subset O(n^4)$

- Use common terminology

  logarithmic, linear, quadratic, polynomial, exponential

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- Use common terminology

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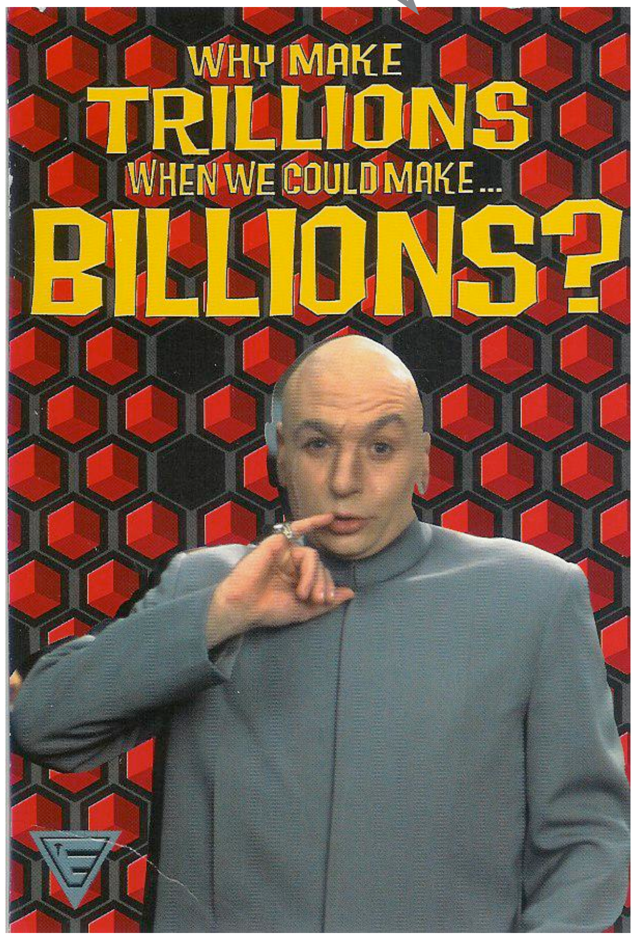
$$O(10^9) = O(10^6)$$



rules of thumb

To calculate work, span for *recursive functions* we can use *recurrence relations*, e.g.

$$W_f(n) = k * W_f(n-1) + c \qquad \text{for } n>0$$

where k, c are constants

- Additive constants ***don't*** matter

  WLOG let c = 1

- Multiplicative constants ***do*** matter

  $W_f(n)$ is $O(k^n)$

  $O(2^n)$ is not the same as $O(3^n)$

work

- $W(e)$, the *work* of e, is the time to evaluate e *sequentially*, on a single processor

  work = total number of operations

- Often we have a function f and a notion of *size* for *argument values*, and want $W_f(n)$, the work of $f(v)$ when v has size n

span

- $S(e)$, the *span* of e, is the time to evaluate e, using parallel evaluation for *independent* code

- Often we have a function f and a notion of *size* for *argument values*, and want $S_f(n)$, the span of $f(v)$ when v has size n

rules of thumb

- Most primitive ops are constant-time
    - but not @ on lists (it does a bunch of :: operations)
- To calculate ***work***,
  \- ***add*** the work for sub-expressions
- To calculate ***span***,
  \- ***max*** the span for *independent* sub-expressions
  \- ***add*** the span for *dependent* sub-expressions

dependence

- **if** b **then** e_1 **else** e_2 — b *before* e_1 *or* e_2
- (**fn** x => e_2) e_1 — e_1 *before* $[\![x{:}v_1]\!]e_2$
- **let val** x = e_1 **in** e_2 **end** — e_1 *before* $[\![x{:}v_1]\!]e_2$

independence

- $(e_1, \ldots, e_n)$ — *tuple components*
- $e_1 + e_2$ — *summands*

work rules

$$W\,(\mathbf{n}) = 0$$

$$W\,(e_1 + e_2) = W\,e_1 + W\,e_2 + 1$$

$$W\,(e_1, e_2) = W\,e_1 + W\,e_2$$

$$W\,(e_1 @ e_2) = W\,e_1 + W\,e_2 + \text{length}\ e_1 + 1$$

$$W\,(\textbf{if}\ b\ \textbf{then}\ e_1\ \textbf{else}\ e_2) \le W\,b + \max(W\,e_1\,,\ W\,e_2\,) + 1$$

span rules

$S\ (\mathbf{n}) = 0$

$S\ (e_1 + e_2) = \max(S\ e_1, S\ e_2) + 1$

$S\ (e_1, e_2) = \max(S\ e_1, S\ e_2)$

$S\ (e_1 @ e_2) = \max(S\ e_1, S\ e_2) + \text{length } e_1 + 1$

$$S\ (\textbf{if } b \textbf{ then } e_1 \textbf{ else } e_2) \leq S\ b + \max(S\ e_1,\ S\ e_2) + 1$$

work and evaluation

- An evaluation step $e \Longrightarrow e'$ represents a basic op, so the exact work for e is the number of steps

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$$
\begin{aligned}
(2+2)+(2+2) &\Longrightarrow 4+(2+2) \\
&\Longrightarrow 4+4 \\
&\Longrightarrow 8
\end{aligned}
$$

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$$W((2+2) + (2+2)) = 3$$

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$$W((2+2) + (2+2)) = 3$$

$$\boxed{W(e_1+e_2) = W(e_1) + W(e_2) + 1}$$

work and application

If $e_1 \Longrightarrow^* (\text{fn } x \texttt{ => } e)$ and $e_2 \Longrightarrow^* v$,
then $W(e_1\ e_2) = W(e_1) + W(e_2) + W([\![x{:}v]\!]e) + 1$

$(\textbf{fn } x \texttt{ => } x+x)\ (2+2)$
$\Longrightarrow (\textbf{fn } x \texttt{ => } x+x)\ 4$
$\Longrightarrow 4+4$
$\Longrightarrow 8$ (3 steps)

$W\ ((\textbf{fn } x \texttt{ => } x+x)\ (2+2))$

work and application

If $e_1 \Longrightarrow^* (\text{fn } x \Rightarrow e)$ and $e_2 \Longrightarrow^* v$,
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$\Longrightarrow 4{+}4$
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$= 0 + 1 + W(4{+}4) + 1$

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exp

```
fun exp (n:int):int =
  if n=0 then 1 else 2 * exp (n-1)
```

Let M be (**fn** n => **if** n=0 **then** 1 **else** 2 * exp(n-1))

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exp 4 $\Longrightarrow^{(1)}$ M 4

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M 4 $\Longrightarrow$ if 4=0 then …
$\Longrightarrow$ if false then …
$\Longrightarrow$ 2 * exp (4-1)
$\Longrightarrow$ 2 * M (4-1)
$\Longrightarrow$ 2 * (M 3)

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$\Longrightarrow^{(5)}$ 2 * (M 3)

$\Longrightarrow^{(5)}$ 2 * (2 * (M 2))

$\Longrightarrow^{(5)}$ 2 * (2 * (2 * (M 1)))

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$$\text{exp } 4 \Longrightarrow^{(28)} 16$$

exp

It's not hard to prove that for all $n \geq 0$,

$$\text{exp } n \Longrightarrow^{(6n+4)} k,$$

where k is the numeral for 2^n

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Does 6n+4 really tell us about actual *runtime* in milliseconds?

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$\text{exp } n \Longrightarrow^{(6n+4)} k$,
where k is the numeral for 2^n

But it's tedious, and why be so accurate?

Does $6n+4$ really tell us about actual *runtime* in milliseconds?

No! But it does tell us runtime is ***linear***.

big-O is big-OK

- It's best to classify runtimes *asymptotically*
- This ignores irrelevant constants…
  (which may be machine-dependent, so not very significant)
- … and ignores runtime on small inputs
  (which may have been special-cased in the code)

$$\text{exp n} \Longrightarrow^{O(n)} 2^n$$

If we *double* n,
the runtime… *doubles*

recurrences

- Given a *recursive definition for function* f and a non-negative ***size*** function that *decreases* in every recursive call
- Extract a ***recurrence relation*** for the ***applicative work*** of f

worst-case work, over all values of size n

$W_f(n)$ = work of f v on values v of size n

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Idea: express $W_f(n)$ in terms of $W_f(m)$, $0 \leq m < n$

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Q: When can this method succeed?

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worst-case work, over all values of size n

$W_f(n)$ = work of f v on values v of size n

Idea: express $W_f(n)$ in terms of $W_f(m)$, $0 \leq m < n$

Q: When can this method succeed?

A: *If the work of f v depends only on the size of v* (!)

example

```
fun Fib(0) = 1
  | Fib(1) = 1
  | Fib(n) = Fib(n-1) + Fib(n-2)
```

size is *value of n*

$$W_{Fib}(0) = c_0$$
$$W_{Fib}(1) = c_0$$
$$W_{Fib}(n) = W_{Fib}(n-1) + W_{Fib}(n-2) + c_1$$

for some constants c_0, c_1

solving a recurrence

WLOG let additive constants be 1

Try to find a *closed form* solution for W(n)
(usually, by guessing and *induction*)

OR Code the recurrence in ML, test for small n,
look for a common pattern

OR Find solution to a *simplified* recurrence
with the same asymptotic properties

OR Appeal to table of standard recurrences

exp

```
fun exp (n:int):int =
  if n=0 then 1 else 2 * exp (n-1)
```

For $n \geq 0$, $\exp n \Longrightarrow^* 2^n$

Let $W_{exp}(n)$ be the runtime for exp(n)

$$W_{exp}(0) = c_0$$

$$W_{exp}(n) = W_{exp}(n-1) + c_1 \quad \text{for } n>0$$

for some constants c_0 and c_1

c_0 : cost for test n=0

c_1 : cost for test n=0, multiply by 2

solution

- Easy to prove by induction on n that

$$W_{exp}(n) = c_0 + n\, c_1 \quad \text{for } n \geq 0$$

$W_{exp}(n)$ is $O(n)$

The work for exp(n) is ***linear***

WLOG let additive constants be 1

comment

If we'd simplified by letting constants be 1,

$W_{exp}(0) = 1$

$W_{exp}(n) = W_{exp}(n-1) + 1$ for $n>0$

we'd have gotten $W_{exp}(n) = 1 + n$

$W_{exp}(n)$ is $O(n)$

The simpler recurrence has the same solution, asymptotically

summary

- We've shown that for $n \geq 0$,
  exp n computes the value of 2^n
  in $O(n)$ steps
- This fact is *independent* of machine details
  (assuming that basic operations are constant time)
- Can we do better?

use parallelism?

(with the same exp function)

```
fun exp (n:int):int =
  if n=0 then 1 else 2 * exp (n-1)
```

- Give a recurrence for the *span* of exp n

It will be *identical* to the recurrence we gave for work, with the same asymptotic solution… why?

There is no advantage to be gained by parallel evaluation here!

a **faster** method?

- The definition of exp relies on the fact that

$$2^n = 2\,(2^{n-1}) \qquad \text{when } n > 0$$

- Everybody knows that

$$2^n = (2^{n \text{ div } 2})^2 \qquad \text{when } n \text{ is even}$$

Let's define

fastexp : int -> int

based on this idea…

fastexp

```
fun square(x:int):int = x * x

fun fastexp (n:int):int =
   if n=0 then 1 else
   if n mod 2 = 0 then square(fastexp (n div 2))
                  else  2 * fastexp(n-1)
```

fastexp

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```

```
fastexp 4 = square(fastexp 2)
          = square(square (fastexp 1))
          = square(square (2 * fastexp 0))
          = square(square (2 * 1))
          = square 4 =16
```

is it faster?

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Code design leads to recurrence…

is it faster?

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Code design leads to recurrence…

$$W_{fastexp}(0) = k_0$$

$$W_{fastexp}(n) = W_{fastexp}(n \text{ div } 2) + k_1 \quad \text{for } n>0, \text{ even}$$

$$W_{fastexp}(n) = W_{fastexp}(n-1) + k_2 \quad \text{for } n>0, \text{ odd}$$

for some constants k_0, k_1, k_2

is it faster?

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Code design leads to recurrence…

$W_{fastexp}(0) = k_0$

$W_{fastexp}(n) = W_{fastexp}(n \text{ div } 2) + k_1$ for $n>0$, even

$W_{fastexp}(n) = W_{fastexp}(n-1) + k_2$ for $n>0$, odd

for some constants k_0, k_1, k_2

k_0: cost for test n=0

k_1: cost for tests n=0, n mod 2 = 0, squaring

k_2: cost for tests n=0, n mod 2 = 0, multiplication by 2

is it faster?

```
fun fastexp (n:int):int =
  if n=0 then 1 else
  if n mod 2 = 0 then square(fastexp (n div 2))
                  else  2 * fastexp(n-1)
```

Expand, then set constants to 1

(asymptotically same as original recurrence)

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Expand, then set constants to 1

$W_{fastexp}(0) = 1$

$W_{fastexp}(1) = 1$

$W_{fastexp}(n) = W_{fastexp}(n \text{ div } 2) + 1$ for $n>1$, even

$W_{fastexp}(n) = W_{fastexp}(n \text{ div } 2) + 1$ for $n>1$, odd

(asymptotically same as original recurrence)

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$$W_{fastexp}(n) = W_{fastexp}(n \text{ div } 2) + 1 \qquad \text{for } n>1$$

(asymptotically same as original recurrence)

approx solution

- $W_{fastexp}(n)$ *is defined like* $\log_2(n)$

  $\log_2 n =$
  if $n=1$ **then** 0 **else** $\log_2 (n \text{ div } 2) + 1$

  $W_{fastexp}(n) =$
  if $n<2$ **then** 1 **else** $W_{fastexp}(n \text{ div } 2) + 1$

- It follows that $W_{fastexp}(n)$ is $O(\log n)$

exercise

- Using ML, discover the relationship between the functions

fun log n = **if** n=1 **then** 0 **else** 1 + log(n **div** 2)

fun W n = **if** n<2 **then** 1 **else** 1 + W(n **div** 2)

(see previous slide)

it's really faster

- Work of exp(n) is O(n)
- Work of fastexp(n) is O(log n)
- O(log n) is a proper subset of O(n)
- fastexp is *asymptotically faster* than exp

list reversal

```
fun rev [ ] = [ ]
  |    rev (x::L) = (rev L) @ [x]
```

For list values A and B,

$W_{@}$(A, B) is *linear* in the *length of* A

Runtime of rev(L)
depends on *length* of L
but not the contents of L

length(rev L)= length(L)

work of rev

```
fun rev [ ] = [ ]
  |   rev (x::L) = (rev L) @ [x]
```

Let $W_{rev}(n)$ be work of rev L when length L = n

$$W_{rev}(0) = 1$$

$$W_{rev}(n) = W_{rev}(n-1) + (n-1) + 1 \qquad \text{for } n>0$$

work of rev

```
fun rev [ ] = [ ]
 |   rev (x::L) = (rev L) @ [x]
```

Let $W_{rev}(n)$ be work of rev L when length L = n

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$W_{rev}(n) =$ for n>0

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$$= W_{rev}(n-2) + (n-1) + n$$

work of rev

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Let $W_{rev}(n)$ be work of rev L when length L = n

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$$W_{rev}(n) = W_{rev}(n-1) + n \qquad \text{for } n>0$$

$$= W_{rev}(n-2) + (n-1) + n$$

$$= 1 + 2 + \ldots + (n-1) + n$$

work of rev

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$W_{rev}(n)$ is $O(n^2)$

work of rev

```
fun rev [ ] = [ ]
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Let $W_{rev}(n)$ be work of rev L when length L = n

$$W_{rev}(0) = 1$$

$$W_{rev}(n) = W_{rev}(n-1) + n \qquad \text{for } n>0$$

$$= W_{rev}(n-2) + (n-1) + n$$

$$= 1 + 2 + \ldots + (n-1) + n$$

$W_{rev}(n)$ is $O(n^2)$ **SLOW!**

faster rev

Surely O(n) should be feasible…

- Use an extra argument to *accumulate* the reversed list

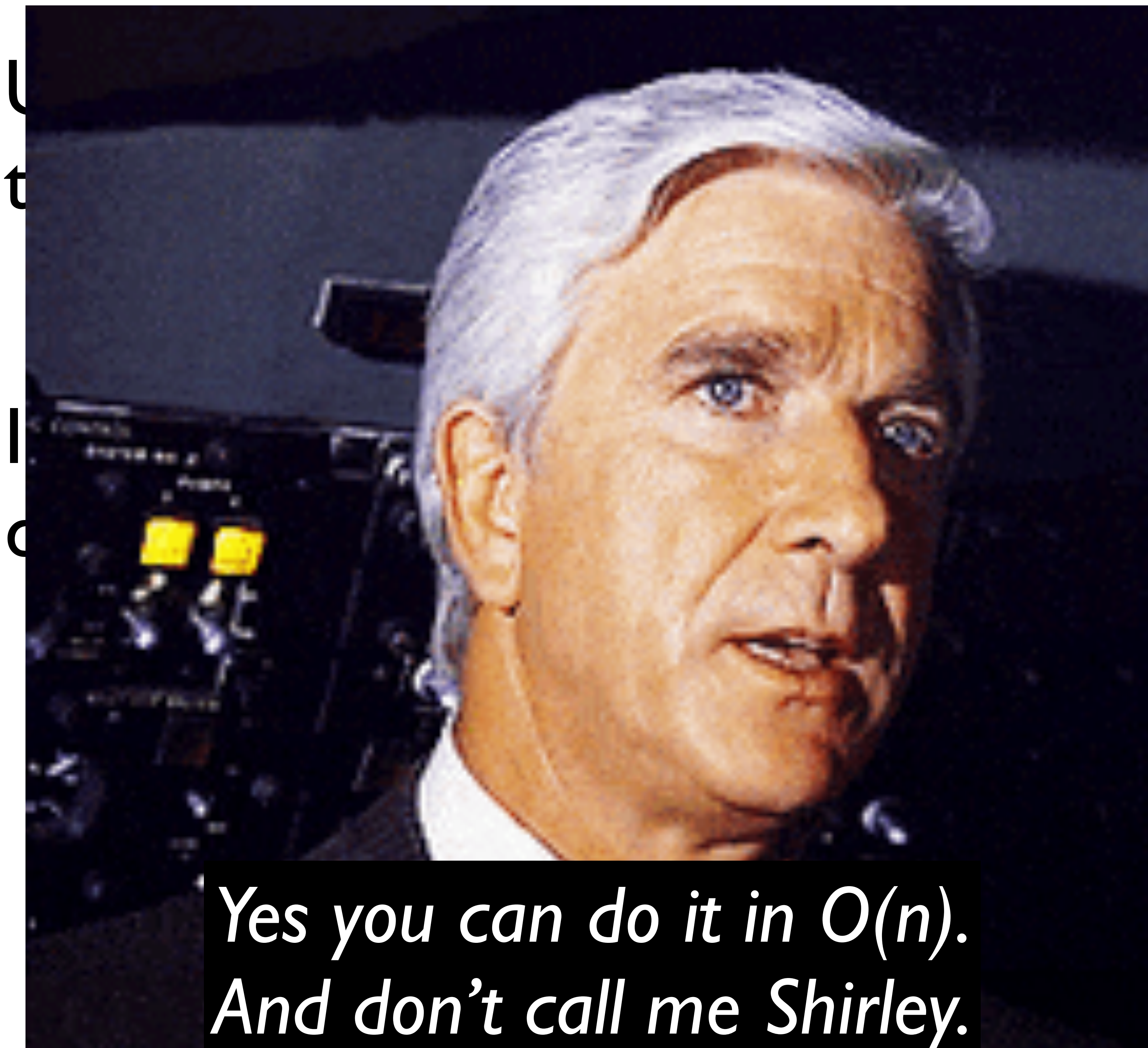
  revver : int list * int list -> int list

- Instead of *append* after the recursive call, do a *cons* before the recursive call

```
fun revver([ ], A) = A
  | revver(x::L, A) = revver(L, x::A)
```

faster rev

Surely O(n) should be feasible…



faster rev

```
fun revver([ ], A) = A
 |  revver(x::L, A) = revver(L, x::A)

fun Rev L = revver(L, [ ])
```

For all L, A, revver(L, A) = (rev L) @ A

For all L, Rev L = rev L

Explain why $W_{Rev}(n)$ is $O(n)$

Hint: analyze W(revver (L, A))

even more faster?

- The definition of fastexp relies on

$$2^n = (2^{n \text{ div } 2})^2 \qquad \text{if } n \text{ is even}$$

$$2^n = 2\,(2^{n-1}) \qquad \text{if } n \text{ is odd}$$

- A moment's thought tells us that

$$2^n = 2\,(2^{(n \text{ div } 2)})^2 \qquad \text{if } n \text{ is odd}$$

Let's define

pow : int -> int

based on this idea…

pow

```
fun pow (n:int):int =
  case n of
     0 => 1
   | 1 => 2
   | _ => let
            val k = pow(n div 2)
          in
            if n mod 2 = 0 then k*k else 2*k*k
          end
```

work of pow(n)

$W_{pow}(0) = 1$
$W_{pow}(1) = 1$
$W_{pow}(n) = 1 + W_{pow}(n \text{ div } 2)$ for $n>1$

Same recurrence as $W_{fastexp}$

Same asymptotic behavior

pow(n) is $O(\log n)$

badpow

```
fun badpow (n:int):int =
  case n of
    0 => 1
  | 1 => 2
  | _ => let
           val k2 = badpow(n div 2)*badpow(n div 2)
         in
           if n mod 2 = 0 then k2 else 2*k2
         end
```

bad idea: does same recursive call twice

work of badpow(n)

$W_{badpow}(0) = 1$
$W_{badpow}(1) = 1$
$W_{badpow}(n) = 1 + 2\,W_{badpow}(n \text{ div } 2)$
for $n>1$

- This implies that $W_{badpow}(n)$ is $O(n)$

But $W_{pow}(n)$ is $O(\log n)$ (faster!)

Bad code design leads to poor performance

summary

Use recurrences for work/span

- recurrence form *mimics* function syntax
- OK to be sloppy with *additive* constants
  - let c = 1, or add/subtract 1

Asymptotic estimates are robust

- independent of architecture
- give information about *scaling*

exercise

- Recall the functions

  isqrt_0 : int -> int
  isqrt_1 : int -> int
  isqrt_2 : int -> int

- Figure out the asymptotic work for

  isqrt_0 n
  isqrt_1 n
  isqrt_2 n

  using recurrences

Try them out on large values of n
and see the differences!

isqrt_0

```
fun isqrt_0 (n : int) : int =
   if n=0 then 0 else
     let
       fun loop i = if n < i*i then i-1 else loop(i+1)
     in
       loop 1
     end
```

- $W_{isqrt\_0}(0) = 1$
- $W_{isqrt\_0}(n) = W_{loop}(1)$ for $n>0$

  How can this be? RHS doesn't seem to use n

isqrt_0

- The loop function used by isqrt_0(n) *does* use the value of n

```
fun loop i = if n < i*i then i-1 else loop(i+1)
```

- Let k be the integer square root of n, so
  $1^2 \leq 2^2 \leq \ldots \leq k^2 \leq n < (k+1)^2$

$$W_{loop}(i) = 1 + W_{loop}(i+1) \quad \text{for } i=1, \ldots, k$$

$$W_{loop}(k+1) = 1$$

Hence $W_{loop}(1)$ is $O(k)$

isqrt_0

- The loop function used by isqrt_0(n) *does* use the value of n

```
fun loop i = if n < i*i then i-1 else loop(i+1)
```

- Let k be the integer square root of n, so $1^2 \leq 2^2 \leq \ldots \leq k^2 \leq n < (k+1)^2$

$$W_{loop}(i) = 1 + W_{loop}(i+1) \quad \text{for } i=1, \ldots, k$$

$$W_{loop}(k+1) = 1$$

Hence $W_{loop}(1)$ is $O(k)$

$$\boxed{\text{So } \; W_{isqrt\_0}(n) \text{ is } O(\sqrt{n})}$$

isqrt_1

```
fun isqrt_1(n) =
  if n=0 then 0 else
   let
     val r = isqrt_1(n - 1) + 1
   in
     if n<r*r then r-1 else r
  end
```

- $W_{isqrt\_1}(0) = 1$
- $W_{isqrt\_1}(n) = 1 + W_{isqrt\_1}(n - 1)$ for $n > 0$

$W_{isqrt\_1}(n)$ is $O(n)$

isqrt_1

```
fun isqrt_1(n) =
  if n=0 then 0 else
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     if n<r*r then r-1 else r
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```

- $W_{isqrt\_1}(0) = 1$
- $W_{isqrt\_1}(n) = 1 + W_{isqrt\_1}(n - 1)$ for $n > 0$

$$\boxed{W_{isqrt\_1}(n) \quad \text{is } O(n)}$$

isqrt_2

```
fun isqrt_2(n) =
   if n=0 then 0 else
    let
      val r = 2 * isqrt_2(n div 4) + 1
    in
      if n<r*r then r-1 else r
   end
```

- $W_{isqrt\_2}(0) = 1$
- $W_{isqrt\_2}(n) = 1 + W_{isqrt\_2}(n \textbf{ div } 4)$ for $n > 0$

$W_{isqrt\_2}(n)$ is $O(\log n)$

isqrt_2

```
fun isqrt_2(n) =
   if n=0 then 0 else
    let
      val r = 2 * isqrt_2(n div 4) + 1
    in
      if n<r*r then r-1 else r
   end
```

- $W_{isqrt\_2}(0) = 1$
- $W_{isqrt\_2}(n) = 1 + W_{isqrt\_2}(n \textbf{ div } 4)$ for $n > 0$

$$\boxed{W_{isqrt\_2}(n) \quad \text{is } O(\log n)}$$

summary

- Asymptotic work analysis "explains" runtime experience

isqrt_0 123456789 *fast*

isqrt_1 123456789 *slowest*

isqrt_2 123456789 *fastest*

$$O(\log n) \subset O(\sqrt{n}) \subset O(n)$$

isqrt_2 isqrt_0 isqrt_1