Verification Logics for Quantum Programs
WPE-II

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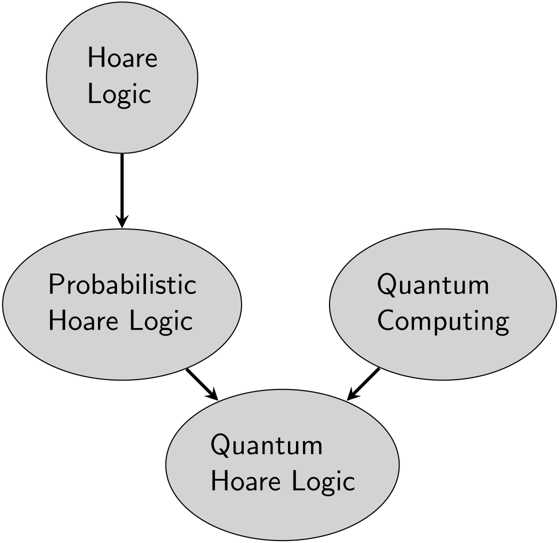
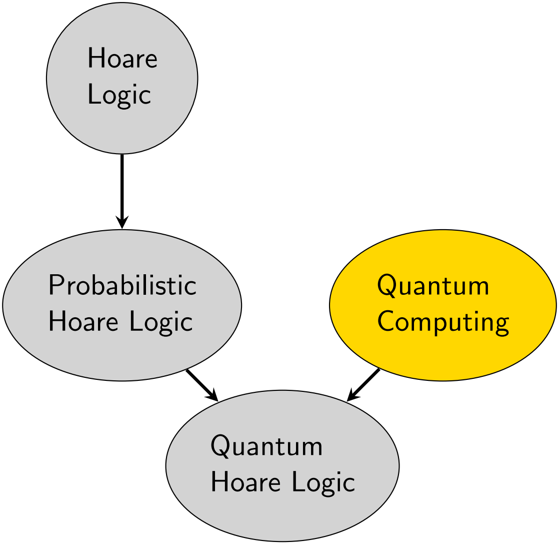
A Rough Outline



Preliminaries: Quantum Computing



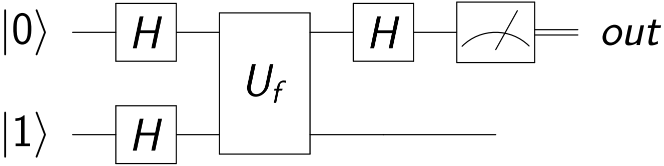
We have a function

$$f : \{0, 1\} \rightarrow \{0, 1\}$$

Is f constant?

Let $f'(x, y) = (x, f(x) \oplus y)$

This allows us to represent f' as a *unitary* matrix U_f, such that $U_f^\dagger U_f = I$.



$$out = \begin{cases} 1 & \text{if } f \text{ is constant} \\ 0 & \text{otherwise} \end{cases}$$

qubits

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

qubits

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Unitaries

$$|0\rangle \;\; \text{—}\boxed{H}\text{—}$$

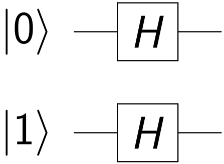
$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Unitaries

$$|0\rangle \; —\boxed{H}—$$

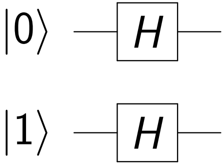
$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Multiqubit States

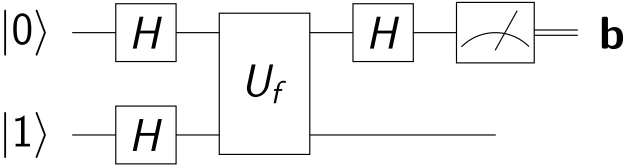


$$(H \otimes H)(|0\rangle \otimes |1\rangle)$$

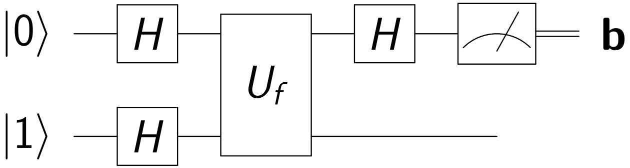
Multiqubit States



$$(H \otimes H)(|0\rangle \otimes |1\rangle) = H|0\rangle \otimes H|1\rangle$$

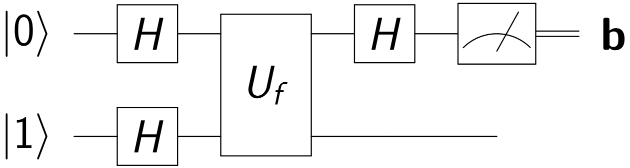


$$\mathbf{q}_0, \mathbf{q}_1 \ := \ |0\rangle \, , |1\rangle \, ;$$



$$\mathbf{q}_0, \mathbf{q}_1 \;:=\; |0\rangle\,,|1\rangle\,;$$
$$\mathbf{q}_0, \mathbf{q}_1 \mathrel{*}= H \otimes H;$$

Deutsch's Algorithm



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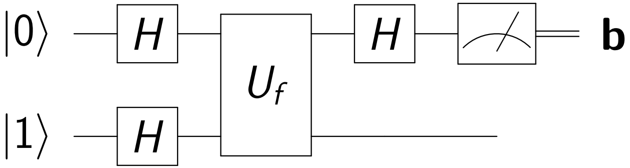

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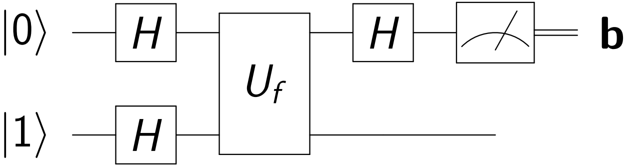


$$\mathbf{q}_0, \mathbf{q}_1 := |0\rangle, |1\rangle;$$

$$\mathbf{q}_0, \mathbf{q}_1 \mathrel{*}= H \otimes H;$$

$$\mathbf{q}_0, \mathbf{q}_1 \mathrel{*}= U_f;$$

$$\mathbf{q}_0 \mathrel{*}= H;$$

$$\mathbf{b} := \mathtt{measure}(\mathbf{q}_0)$$

Our Goal

Where $f(0) = f(1) = 1$ derive the following:

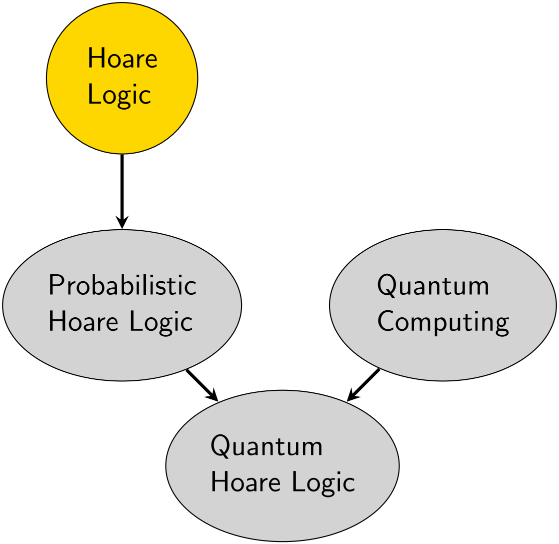
$$\{\text{True}\}$$
$$\mathbf{q}_0, \mathbf{q}_1 \;:=\; |0\rangle\,, |1\rangle\,;$$
$$\mathbf{q}_0, \mathbf{q}_1 \mathrel{*}= H \otimes H;$$
$$\mathbf{q}_0, \mathbf{q}_1 \mathrel{*}= U_f;$$
$$\mathbf{q}_0 \mathrel{*}= H;$$
$$\mathbf{b} \;:=\; \mathtt{measure}(\mathbf{q}_0)$$
$$\{b = 0\}$$

Our Goal

Where $f(0) = f(1) = 1$ derive the following:

$$\{Pr(\text{True}) = 1\}$$
$$\mathbf{q}_0, \mathbf{q}_1 \ := \ |0\rangle\,, |1\rangle\,;$$
$$\mathbf{q}_0, \mathbf{q}_1 \mathrel{*}= H \otimes H;$$
$$\mathbf{q}_0, \mathbf{q}_1 \mathrel{*}= U_f;$$
$$\mathbf{q}_0 \mathrel{*}= H;$$
$$\mathbf{b} := \mathtt{measure}(\mathbf{q}_0)$$
$$\{Pr(\mathbf{b} = 0) = 1\}$$

Hoare Logic


Hoare
Logic
Probabilistic
Hoare Logic
Quantum
Computing
Quantum
Hoare Logic

Hoare Logic

$$\{P\}\ c\ \{Q\}$$

Hoare Logic

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$$\frac{P(\sigma) \qquad c \;/\; \sigma \Downarrow \sigma'}{}$$

Hoare Logic

$$\{P\}\ c\ \{Q\}$$

$$\frac{P(\sigma) \qquad c \,/\, \sigma \Downarrow \sigma'}{Q(\sigma')}$$

$$\{even(x)\}\ x\ := x + 1\ \{odd(x)\}$$

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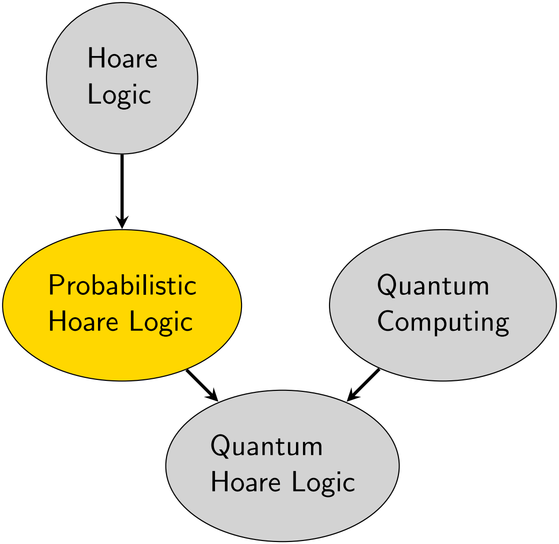
$$\frac{even(\sigma(x)) \qquad x := x + 1\ /\ \sigma \Downarrow \sigma'}{}$$

$$\{\mathit{even}(x)\}\ x\ := x + 1\ \{\mathit{odd}(x)\}$$

$$\frac{\mathit{even}(\sigma(x)) \qquad x\ := x + 1\ /\ \sigma \Downarrow \sigma'}{\mathit{odd}(\sigma'(x))}$$

Probabilistic Hoare Logic



$$\{P\}\ c\ \{Q\}$$

Probabilistic Hoare Logic

$$\{P\}\ c\ \{Q\}$$

$$\textbf{?} \qquad\qquad c \,/\, \textbf{?} \Downarrow \textbf{?}$$

$$\{P\}\ c\ \{Q\}$$

? $\qquad c \,/\, $ **?** $\ \Downarrow\ $ **?**

proposition?

$$\{P\}\ c\ \{Q\}$$

$$\textbf{?} \qquad\qquad c\ /\ \ \textbf{?}\ \Downarrow\ \textbf{?}$$

proposition?

state?

termination?

$$\{P\}\ c\ \{Q\}$$

$$\textbf{?} \qquad\qquad c \;/\; \textbf{?} \Downarrow \textbf{?}$$

proposition?

state?

Probabilistic Hoare Logic

termination?

$$\{P\}\ c\ \{Q\}$$

?

$$c \ / \ \mathcal{D}(\sigma) \Downarrow \mathcal{D}(\sigma)'$$

proposition?

distribution

termination?

$$\{P\}\ c\ \{Q\}$$

$$Pr(X) = p \qquad c \ / \ \mathcal{D}(\sigma) \Downarrow \mathcal{D}(\sigma)'$$

probabilistic
propositions

almost-sure termination

$$\{P\}\ c\ \{Q\}$$

$$Pr(X) = p \qquad c \,/\, \mathcal{D}(\sigma) \Downarrow \mathcal{D}(\sigma)'$$

probabilistic propositions

$$\{\ Pr(even(x)) = \tfrac{1}{3}\ \}$$
$$x := x + 1$$
$$\{\ Pr(odd(x)) = \tfrac{1}{3}\ \}$$

termination?

$$\{P\}\ c\ \{Q\}$$

$$|even(x)| \qquad\qquad c \ / \ \mathcal{D}(\sigma) \Downarrow \mathcal{D}(\sigma)'$$

proposition?

termination?

$$\{P\}\ c\ \{Q\}$$

$$|even(x)| \qquad c \ / \ \mathcal{D}(\sigma) \Downarrow \mathcal{D}(\sigma)'$$

measurable function

partial termination

$$\{P\}\ c\ \{Q\}$$

$$|even(x)| \qquad\qquad c\ /\ \mathcal{D}(\sigma) \Downarrow \mathcal{D}(\sigma)'$$

measurable function

$$\{\ \mathit{even}(x)\ \}$$
$$x\ := x + 1$$
$$\{\ \mathit{odd}(x)\ \}$$

Arithmetic Probabilistic Logic

$$\{\ even(x)\ \}$$
$$x := x + 1$$
$$\{\ odd(x)\ \}$$

$$\forall \mathcal{D}, |even(x)|(\mathcal{D}) \leq |odd(x)|(\llbracket x := x + 1 \rrbracket \mathcal{D})$$

Arithmetic Probabilistic Logic

$$\{ \; even(x) \; \}$$
$$x := x + 1$$
$$\{ \; odd(x) \; \}$$

$$\forall \mathcal{D}, |even(x)|(\mathcal{D}) \leq |odd(x)|(\llbracket x := x + 1 \rrbracket \mathcal{D})$$
$$+ \textit{ probability of non-termination}$$

$$\mathbf{b} := \mathtt{toss}(p)$$

Chadha et. al's EPPL

$$\frac{}{\{P_b^p\}\ \mathbf{b} \ := \mathtt{toss}(p)\ \{P\}}$$

Chadha et. al's EPPL

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$$P_b^p = P[Pr(X) \mapsto$$
$$p * Pr(X[\mathbf{b} \mapsto \mathtt{t}]) + (1 - p) * Pr(X[\mathbf{b} \mapsto \mathtt{f}])]$$

Chadha et. al's EPPL

$$\{\frac{2}{3} * Pr(\texttt{t}) + \frac{1}{3} * Pr(\texttt{f}) = \frac{2}{3}\}$$

$$\mathbf{b} \ \texttt{:=}\ \texttt{toss}(\frac{2}{3})$$

$$\{Pr(\mathbf{b}) = \frac{2}{3}\}$$

Comparison

	Chadha	

[1] Complete version in Chadha et al. 2007

Comparison

	Chadha	
Probability	$b := \texttt{toss}(p)$	

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While Loops	None	

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While Loops	None	
WP-form	Yes	

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Comparison

	Chadha	
Probability	b := toss(p)	
While Loops	None	
WP-form	Yes	
Complete	Almost[1]	

[1] Complete version in Chadha et al. 2007

$$c_1 \quad \bigoplus_{\frac{1}{3}} \quad c_2$$

Den Hartog and De Vink's pH

$$\frac{\{P\}\ c_1\ \{Q_1\} \qquad \{P\}\ c_2\ \{Q_2\}}{\{P\}\ c_1\ \oplus_p\ c_2\ \{Q_1\ \oplus_p\ Q_2\}}$$

Den Hartog and De Vink's pH

$$\frac{\{P\}\ c_1\ \{Q_1\} \qquad \{P\}\ c_2\ \{Q_2\}}{\{P\}\ c_1 \oplus_p c_2\ \{p * Q_1 + (1-p) * Q_2\}}$$

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$$\mathcal{D} \models p * Q_1 + (1-p) * Q_2$$

$$\equiv$$

$$\mathcal{D} = p * \mathcal{D}_1 + (1-p) * \mathcal{D}_2 \text{ s.t.}$$

$$\mathcal{D}_1 \models Q_1 \text{ and } \mathcal{D}_2 \models Q_2$$

$$\frac{\{b?P\}\ c_1\ \{Q_1\} \qquad \{\neg b?P\}\ c_2\ \{Q_2\}}{\{P\}\ \texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2\ \{Q_1 + Q_2\}}$$

Den Hartog and De Vink's pH

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$$\mathcal{D} \models b?P$$

$$\equiv$$

$$\mathcal{D} = b?\mathcal{D}^+ \text{ s.t. } \mathcal{D}^+ \models P$$

Den Hartog and De Vink's pH

$$\frac{P \text{ invariant for } \langle b, c \rangle}{\{P\}\ \texttt{while}\ b\ \texttt{do}\ c\ \ \{P \wedge Pr(b) = 0\}}$$

Comparison

	Chadha	Den Hartog
Probability	$b := \texttt{toss}(p)$	
While Loops	None	
WP-form	Yes	
Complete	Almost[2]	

[2] Complete version in Chadha et al. 2007

Comparison

	Chadha	Den Hartog
Probability	$b := \texttt{toss}(p)$	$c_1 \oplus_p c_2$
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WP-form	Yes	No
Complete	Almost[2]	No

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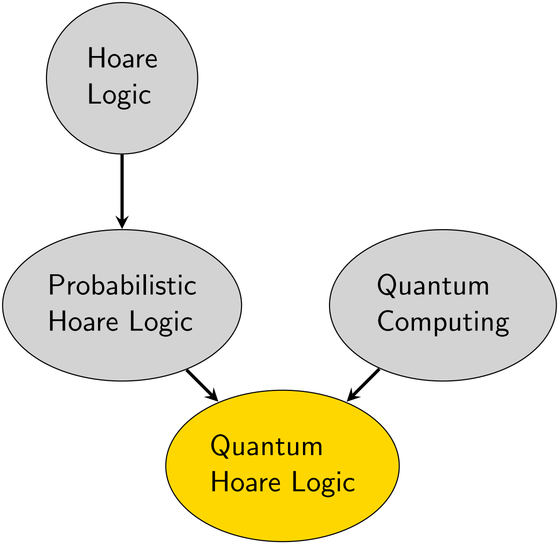
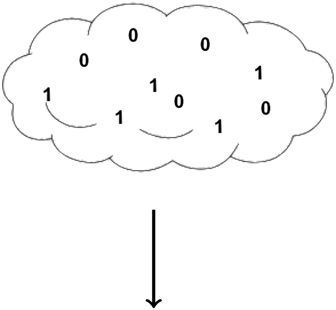
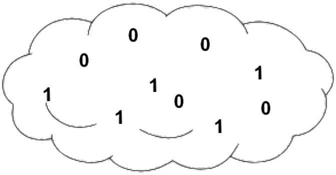
Quantum Hoare Logic



Problems of Measurement



$$\mathbf{b} := \texttt{toss}(p)$$

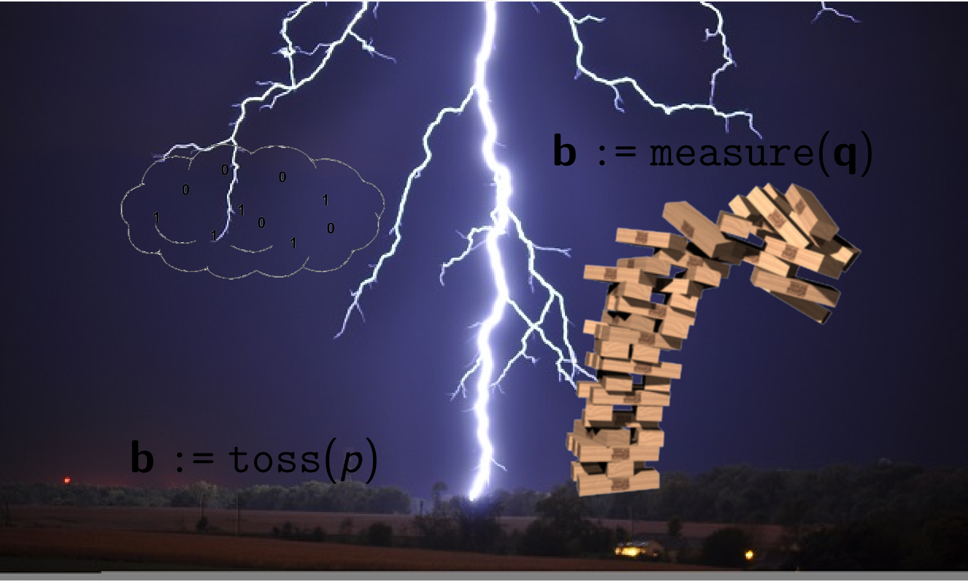


$$\downarrow$$

$$\mathbf{b} := \mathtt{toss}(p)$$

$$\mathbf{b} := \mathtt{measure}(\mathbf{q})$$

Problems of Measurement



Problems of Representation

$$Pr(\mathbf{q}) = \frac{1}{2}$$

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- Neglects entanglement

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- Could correspond to the amplitude $\frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$
- Could also correspond to a *distribution* over $\mathbf{q}$
- Neglects entanglement
- Really want to describe *vectors* or *matrices*

Solution 1: Ensembles!

$$\mathcal{D}(\sigma, |\psi\rangle)$$

σ : map from identifiers to $\mathbb{N}$

$|\psi\rangle$: pure state

Quantum Commands

$$U ::= I \mid H : \mathbf{q} \mid H : \mathbf{qn} \mid \sigma_x : \mathbf{q} \mid \sigma_x : \mathbf{qn}(e, e) \mid \\ \texttt{qif}\ \mathbf{q}\ \texttt{then}\ U\ \texttt{else}\ U \mid UU$$

Classical Commands

$$c ::= U \mid \texttt{skip} \mid \mathbf{b} := b \mid \mathbf{n} := e \mid c\ ;\ c \mid \\ \texttt{if}\ \mathbf{b}\ \texttt{then}\ c\ \texttt{else}\ c \mid \mathbf{b} := \texttt{measure}(\mathbf{qn})$$

EEQPL Assertions

$$Pr(\mathbf{n} = 4) = \frac{1}{2}$$

EEQPL Assertions

$$Pr(\mathbf{n} = 4) = \frac{1}{2}$$

$$Pr(\langle \mathbf{q}_1 : 0, \mathbf{q}_2 : 1 | \, \mathtt{t} \rangle = \frac{1}{\sqrt{2}}) = \frac{1}{3}$$

EEQPL Rules

$$\frac{}{\{P[P(x) \mapsto m_1^{\mathbf{b},\mathbf{q}}(x) + m_0^{\mathbf{b},\mathbf{q}}(x)]\}\ \mathbf{b} := \mathtt{measure}(\mathbf{q})\ \{P\}}$$

Where $m_1^{\mathbf{b},\mathbf{q}}(x)$:

EEQPL Rules

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Where $m_1^{\mathbf{b},\mathbf{q}}(x)$:

- Scales by the probability of the outcome 1

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Where $m_1^{\mathbf{b},\mathbf{q}}(x)$:

- Scales by the probability of the outcome 1
- Sets the amplitudes of all $\mathbf{q} : 0$ valuations to 0

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$$\frac{}{\{P[P(x) \mapsto m_1^{\mathbf{b},\mathbf{q}}(x) + m_0^{\mathbf{b},\mathbf{q}}(x)]\}\ \mathbf{b}\ := \mathtt{measure}(\mathbf{q})\ \{P\}}$$

Where $m_1^{\mathbf{b},\mathbf{q}}(x)$:

- Scales by the probability of the outcome 1
- Sets the amplitudes of all $\mathbf{q} : 0$ valuations to 0
- Scales up the amplitudes of all $\mathbf{q} : 0$ valuations

EEQPL Rules

$$\frac{}{\{P[P(x) \mapsto m_1^{\mathbf{b},\mathbf{q}}(x) + m_0^{\mathbf{b},\mathbf{q}}(x)]\}\ \mathbf{b} := \mathtt{measure}(\mathbf{q})\ \{P\}}$$

Where $m_1^{\mathbf{b},\mathbf{q}}(x)$:

- Scales by the probability of the outcome 1
- Sets the amplitudes of all $\mathbf{q} : 0$ valuations to 0
- Scales up the amplitudes of all $\mathbf{q} : 0$ valuations
- Replaces all instances of $\mathbf{b}$ with $\mathtt{t}$

EEQPL Rules

$$\frac{}{\{P[\langle \omega | t \rangle \mapsto \langle U\omega | t \rangle]\}\ U\ \{P\}}$$

EEQPL Rules

$$\frac{}{\{P[\langle\omega|t\rangle \mapsto \langle U\omega|t\rangle]\}\ U\ \{P\}}$$

$$\{\Box(\tfrac{1}{\sqrt{2}}\langle 0|\,\mathtt{t}\rangle + \tfrac{1}{\sqrt{2}}\langle 1|\,\mathtt{t}\rangle = 1)\}$$
$$H : q$$
$$\{\Box(\langle 0|\,\mathtt{t}\rangle = 1)\}$$

EEQPL Rules

$$\{\Box(\langle 0|\,\mathtt{t}\rangle = 1) \rightarrow\}$$

$$\{\Box(\tfrac{1}{2}\langle 0|\,\mathtt{t}\rangle + \tfrac{1}{2}\langle 0|\,\mathtt{t}\rangle = 1)\} \rightarrow$$

$$\{\Box(\tfrac{1}{2}\langle 0|\,\mathtt{t}\rangle + \tfrac{1}{2}\langle 1|\,\mathtt{t}\rangle + \tfrac{1}{2}\langle 0|\,\mathtt{t}\rangle - \tfrac{1}{2}\langle 1|\,\mathtt{t}\rangle = 1)\}$$

$$H : q$$

$$\{\Box(\tfrac{1}{\sqrt{2}}\langle 0|\,\mathtt{t}\rangle + \tfrac{1}{\sqrt{2}}\langle 1|\,\mathtt{t}\rangle = 1)\}$$

$$H : q$$

$$\{\Box(\langle 0|\,\mathtt{t}\rangle = 1)\}$$

Verifying Deutsch in EEQPL

$$\{\Box(\langle 01|\, t\rangle = 1)\}$$

$$H_2 : (\mathbf{q}_0, \mathbf{q}_1);$$

$$\{\Box(\frac{1}{2}(|\langle 01|\,\mathtt{t}\rangle + \langle 11|\,\mathtt{t}\rangle|^2 + |\langle 00|\,\mathtt{t}\rangle + \langle 10|\,\mathtt{t}\rangle|^2 = 1))\}$$

$$U_f : (\mathbf{q}_0, \mathbf{q}_1);$$

$$\{\Box(\frac{1}{2}(|\langle 00|\,\mathtt{t}\rangle + \langle 10|\,\mathtt{t}\rangle|^2 + |\langle 01|\,\mathtt{t}\rangle + \langle 11|\,\mathtt{t}\rangle|^2 = 1))\}$$

$$H : \mathbf{q}_0;$$

$$\{\Box(|\langle 00|\,\mathtt{t}\rangle|^2 + |\langle 01|\,\mathtt{t}\rangle|^2 = 1)\} \rightarrow$$

$$\{1 * Pr((0 = 0) \wedge 1 * |\langle 00|\,\mathtt{t}\rangle|^2 + 1 * |\langle 01|\,\mathtt{t}\rangle|^2 = 1) + Pr(\dots) = 1\}$$

$$\mathbf{b} := \mathtt{measure}(\mathbf{q}_0);$$

$$\{Pr((\mathbf{b} = 0) \wedge |\langle 00|\,\mathtt{t}\rangle|^2 + |\langle 01|\,\mathtt{t}\rangle|^2 = 1) = 1\} \rightarrow$$

$$\{\Box(\mathbf{b} = 0)\}$$

Verifying Deutsch in EEQPL

$$\{\Box(\langle 01|\, t\rangle = 1)\}$$
$$H_2 : (\mathbf{q}_0, \mathbf{q}_1);$$
$$\{\Box(\frac{1}{2}(|\langle 01|\, \mathtt{t}\rangle + \langle 11|\, \mathtt{t}\rangle|^2 + |\langle 00|\, \mathtt{t}\rangle + \langle 10|\, \mathtt{t}\rangle|^2 = 1))\}$$
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$$\{1 * Pr((0 = 0) \wedge 1 * |\langle 00|\, \mathtt{t}\rangle|^2 + 1 * |\langle 01|\, \mathtt{t}\rangle|^2 = 1) + Pr(\dots) = 1\}$$
$$\mathbf{b} := \mathtt{measure}(\mathbf{q}_0);$$
$$\{Pr((\mathbf{b} = 0) \wedge |\langle 00|\, \mathtt{t}\rangle|^2 + |\langle 01|\, \mathtt{t}\rangle|^2 = 1) = 1\} \rightarrow$$
$$\{\Box(\mathbf{b} = 0)\}$$

Verifying Deutsch in EEQPL

$$\{\Box(\langle 01|\, t\rangle = 1)\}$$

$$H_2 : (\mathbf{q}_0, \mathbf{q}_1);$$

$$\{\Box(\frac{1}{2}(|\langle 01|\, \mathtt{t}\rangle + \langle 11|\, \mathtt{t}\rangle|^2 + |\langle 00|\, \mathtt{t}\rangle + \langle 10|\, \mathtt{t}\rangle|^2 = 1))\}$$

$$U_f : (\mathbf{q}_0, \mathbf{q}_1);$$

$$\{\Box(\frac{1}{2}(\color{red}{|\langle 00|\, \mathtt{t}\rangle + \langle 10|\, \mathtt{t}\rangle|^2} + |\langle 01|\, \mathtt{t}\rangle + \langle 11|\, \mathtt{t}\rangle|^2 = 1))\}$$

$$H : \mathbf{q}_0;$$

$$\{\Box(\color{red}{|\langle 00|\, \mathtt{t}\rangle|^2} + |\langle 01|\, \mathtt{t}\rangle|^2 = 1)\} \rightarrow$$

$$\{1 * Pr((0 = 0) \wedge 1 * |\langle 00|\, \mathtt{t}\rangle|^2 + 1 * |\langle 01|\, \mathtt{t}\rangle|^2 = 1) + Pr(\dots) = 1\}$$

$$\mathbf{b} := \mathtt{measure}(\mathbf{q}_0);$$

$$\{Pr((\mathbf{b} = 0) \wedge |\langle 00|\, \mathtt{t}\rangle|^2 + |\langle 01|\, \mathtt{t}\rangle|^2 = 1) = 1\} \rightarrow$$

$$\{\Box(\mathbf{b} = 0)\}$$

Verifying Deutsch in EEQPL

$$\{\Box(\langle 01|\,t\rangle = 1)\}$$
$$H_2 : (\mathbf{q}_0, \mathbf{q}_1);$$
$$\{\Box(\frac{1}{2}(|\langle 01|\,\mathtt{t}\rangle + \langle 11|\,\mathtt{t}\rangle|^2 + |\langle 00|\,\mathtt{t}\rangle + \langle 10|\,\mathtt{t}\rangle|^2 = 1))\}$$
$$U_f : (\mathbf{q}_0, \mathbf{q}_1);$$
$$\{\Box(\frac{1}{2}(|\langle 00|\,\mathtt{t}\rangle + \langle 10|\,\mathtt{t}\rangle|^2 + |\langle 01|\,\mathtt{t}\rangle + \langle 11|\,\mathtt{t}\rangle|^2 = 1))\}$$
$$H : \mathbf{q}_0;$$
$$\{\Box(|\langle 00|\,\mathtt{t}\rangle|^2 + |\langle 01|\,\mathtt{t}\rangle|^2 = 1)\} \rightarrow$$
$$\{1 * Pr((0 = 0) \wedge 1 * |\langle 00|\,\mathtt{t}\rangle|^2 + 1 * |\langle 01|\,\mathtt{t}\rangle|^2 = 1) + Pr(\dots) = 1\}$$
$$\mathbf{b} := \mathtt{measure}(\mathbf{q}_0);$$
$$\{Pr((\mathbf{b} = 0) \wedge |\langle 00|\,\mathtt{t}\rangle|^2 + |\langle 01|\,\mathtt{t}\rangle|^2 = 1) = 1\} \rightarrow$$
$$\{\Box(\mathbf{b} = 0)\}$$

Comparison

	EEQPL		
Language			
Assertions			
Objects			
While Rule			
WP-form			
Complete			

Comparison

	EEQPL		
Language	Classical		
Assertions			
Objects			
While Rule			
WP-form			
Complete			

Comparison

	EEQPL		
Language	Classical		
Assertions	Truth Functional		
Objects			
While Rule			
WP-form			
Complete			

Comparison

	EEQPL		
Language	Classical		
Assertions	Truth Functional		
Objects	Ensembles		
While Rule			
WP-form			
Complete			

Comparison

	EEQPL		
Language	Classical		
Assertions	Truth Functional		
Objects	Ensembles		
While Rule	None		
WP-form			
Complete			

Comparison

	EEQPL		
Language	Classical		
Assertions	Truth Functional		
Objects	Ensembles		
While Rule	None		
WP-form	Yes		
Complete			

Comparison

	EEQPL		
Language	Classical		
Assertions	Truth Functional		
Objects	Ensembles		
While Rule	None		
WP-form	Yes		
Complete	No		

$$
\begin{aligned}
c ::= {} & \texttt{skip} \mid c \,;\, c \mid \texttt{bit}\ \ \mathbf{b} \mid \texttt{qbit}\ \ \mathbf{q} \mid \texttt{discard}\ \ \mathbf{q} \\
& \mid \mathbf{b} := 0 \mid \mathbf{b} := 1 \mid \vec{\mathbf{q}} \mathrel{*}= U \mid \texttt{while}\ \mathbf{b}\ \texttt{do}\ c \\
& \texttt{if}\ \mathbf{b}\ \texttt{then}\ c\ \texttt{else}\ c \mid \texttt{measure}\ \mathbf{q}\ \texttt{then}\ c\ \texttt{else}\ c
\end{aligned}
$$

$$
\begin{aligned}
c ::= {}& \texttt{skip} \mid c \mathbin{;} c \mid \texttt{bit}\ \ \mathbf{b} \mid \texttt{qbit}\ \ \mathbf{q} \mid \texttt{discard}\ \ \mathbf{q} \\
& \mid \mathbf{b} := 0 \mid \mathbf{b} := 1 \mid \vec{\mathbf{q}} \mathrel{*=} U \mid \texttt{while}\ \mathbf{b}\ \texttt{do}\ c \\
& \texttt{if}\ \mathbf{b}\ \texttt{then}\ c\ \texttt{else}\ c \mid \texttt{measure}\ \mathbf{q}\ \texttt{then}\ c\ \texttt{else}\ c
\end{aligned}
$$

$$
[\![c]\!] : A \to A
$$

Density Matrices

$$|\psi\rangle\langle\psi| = |\psi\rangle \times |\psi\rangle^\dagger$$

Density Matrices

$$|\psi\rangle\langle\psi| = |\psi\rangle \times |\psi\rangle^\dagger$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Density Matrices

$$|\psi\rangle\langle\psi| = |\psi\rangle \times |\psi\rangle^\dagger$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \Rightarrow \begin{pmatrix} \color{red}{\frac{1}{2}} & -\frac{1}{2} \\ -\frac{1}{2} & \color{red}{\frac{1}{2}} \end{pmatrix}$$

Mixed States

$$\sum_i p_i |\psi_i\rangle\langle\psi_i|$$

where

$$Pr(|\psi_i\rangle) = p_i \ \& \ \sum_i p_i = 1$$

Partial Mixed States

$$\sum_i p_i |\psi_i\rangle\langle\psi_i|$$

where

$$Pr(|\psi_i\rangle) = p_i \;\&\; \sum_i p_i \leq 1$$

QHL Assertions

$$Pr(\mathbf{q}) = p$$

QHL Rules: Unitary

$$\frac{}{\{^{\vec{\mathbf{q}}}U^{\dagger}P\}\ \vec{\mathbf{q}} \mathrel{*}= U\ \{P\}}$$

QHL Rules: Unitary

$$\frac{}{\{^{\vec{\mathbf{q}}}U^{\dagger}P\}\ \vec{\mathbf{q}} \mathrel{*}= U\ \{P\}}$$

$$A \models UP$$

$$\equiv$$

$$A = (U \otimes I)A'(U^{\dagger} \otimes I) \text{ s.t. } A' \models P$$

QHL Rules: Measure

$$\frac{\{{}^{q}|1\rangle\langle 0|P\}\ c_1\ \{Q_1\} \qquad \{{}^{q}|1\rangle\langle 1|P\}\ c_2\ \{Q_2\}}{\{P\}\ \texttt{measure}\ q\ \texttt{then}\ c_1\ \texttt{else}\ c_2\ \{Q_1 + Q_2\}}$$

QHL Rules: Measure

$$\frac{\{{}^{q}|1\rangle\langle 0| P\}\ c_1\ \{Q_1\} \qquad \{{}^{q}|1\rangle\langle 1| P\}\ c_2\ \{Q_2\}}{\{P\}\ \texttt{if}\ q\ \texttt{then}\ c_1\ \texttt{else}\ c_2\ \{Q_1 + Q_2\}}$$

QHL Rules: While

$$\frac{\{P \wedge Pr(b = 1) = 1\}\ c\ \{P\}}{\{P \wedge Pr(\mathtt{t}) = 1\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \ \{P \wedge Pr(b = 0) = 1\}}$$

QHL Rules: While

$$\frac{\{P \wedge Pr(b = 1) = 1\}\ c\ \{P\}}{\{P \wedge Pr(\mathtt{t}) = 1\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \ \{P \wedge Pr(b = 0) = 1\}}$$

Subject to the following conditions:

QHL Rules: While

$$\frac{\{P \wedge Pr(b = 1) = 1\}\ c\ \{P\}}{\{P \wedge Pr(\mathtt{t}) = 1\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \ \{P \wedge Pr(b = 0) = 1\}}$$

Subject to the following conditions:

1. The invariant P has no negation, disjunction or existentials.

QHL Rules: While

$$\frac{\{P \wedge Pr(b=1)=1\}\ c\ \{P\}}{\{P \wedge Pr(\mathtt{t})=1\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \ \{P \wedge Pr(b=0)=1\}}$$

Subject to the following conditions:

1. The invariant P has no negation, disjunction or existentials.
2. The program always terminates.

QHL Rules: While

$$\frac{\{P \wedge Pr(b=1)=1\}\ c\ \{P\}}{\{P \wedge Pr(\mathtt{t})=1\}\ \mathtt{while}\ b\ \mathtt{do}\ c\ \ \{P \wedge Pr(b=0)=1\}}$$

Subject to the following conditions:

1. The invariant P has no negation, disjunction or existentials.
2. The program always terminates.
3. The guard is independent of all other variables.

Verifying Deutsch in QHL

$$\{Pr(\text{True}) = 1\}$$
$$\texttt{qbit } \mathbf{q}_0, \mathbf{q}_1;$$
$$\{Pr(q_0 = 0 \wedge q_1 = 0) = 1\} \rightarrow$$
$$\{(H \otimes I) U_f H_2 (I \otimes N) Pr(\mathbf{q}_0 = 0) = 1\}$$
$$\mathbf{q}_1 \mathrel{*}= N;$$
$$\mathbf{q}_0, \mathbf{q}_1 \mathrel{*}= H_2;$$
$$\mathbf{q}_0, \mathbf{q}_1 \mathrel{*}= U_f;$$
$$\mathbf{q}_0 \mathrel{*}= H;$$
$$\{Pr(\mathbf{q}_0 = 0) = 1\}$$
$$\texttt{measure } \mathbf{q}_0 \texttt{ then } \mathbf{b} := 1 \texttt{ else } \mathbf{b} := 0$$
$$\{Pr(\mathbf{b} = 0) = 1\}$$

QHL: Properties

	EEQPL	QHL	
Language	Classical + Quantum		
Objects	Ensembles		
Assertions	Truth Functional		
While Rule	None		
WP-form	Yes		
Complete	No		

QHL: Properties

	EEQPL	QHL	
Language	Classical + Quantum	"Classical" + Quantum	
Objects	Ensembles		
Assertions	Truth Functional		
While Rule	None		
WP-form	Yes		
Complete	No		

QHL: Properties

	EEQPL	QHL	
Language	Classical + Quantum	"Classical" + Quantum	
Objects	Ensembles	Density Matrices	
Assertions	Truth Functional		
While Rule	None		
WP-form	Yes		
Complete	No		

QHL: Properties

	EEQPL	QHL	
Language	Classical + Quantum	"Classical" + Quantum	
Objects	Ensembles	Density Matrices	
Assertions	Truth Functional	Truth Functional	
While Rule	None		
WP-form	Yes		
Complete	No		

QHL: Properties

	EEQPL	QHL	
Language	Classical + Quantum	"Classical" + Quantum	
Objects	Ensembles	Density Matrices	
Assertions	Truth Functional	Truth Functional	
While Rule	None	Limited	
WP-form	Yes		
Complete	No		

QHL: Properties

	EEQPL	QHL	
Language	Classical + Quantum	"Classical" + Quantum	
Objects	Ensembles	Density Matrices	
Assertions	Truth Functional	Truth Functional	
While Rule	None	Limited	
WP-form	Yes	No	
Complete	No		

QHL: Properties

	EEQPL	QHL	
Language	Classical + Quantum	"Classical" + Quantum	
Objects	Ensembles	Density Matrices	
Assertions	Truth Functional	Truth Functional	
While Rule	None	Limited	
WP-form	Yes	No	
Complete	No	No	

Beyond Truth Functional

Truth Functional

- Propositions over states.

Arithmetic

- Propositions over states.

Beyond Truth Functional

Truth Functional

- Propositions over states.
- Propositions about probabilities over states.

Arithmetic

- Propositions over states.
- Measurable functions over distributions.

Beyond Truth Functional

Truth Functional

- Propositions over states.
- Propositions about probabilities over states.
- Propositions about probabilities over quantum (and classical) states.

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- Propositions over states.
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Beyond Truth Functional

Truth Functional

- Propositions over states.
- Propositions about probabilities over states.
- Propositions about probabilities over quantum (and classical) states.

Arithmetic

- Propositions over states.
- Measurable functions over distributions.

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Beyond Truth Functional

Truth Functional

- Propositions over states.
- Propositions about probabilities over states.
- Propositions about probabilities over quantum (and classical) states.

Arithmetic

- Propositions over states.
- Measurable functions over distributions.
- Bounded positive operators over density matrices[3]

[3] D'Hondt and Panangaden, 2006

Bounded Positive Operators

Defined via the Löwner partial order:

$$
\begin{array}{ccccc}
0_{H_x} & \sqsubseteq & M & \sqsubseteq & I_{H_x} \\
& & \equiv & & \\
tr(0\rho) & \leq & tr(M\rho) & \leq & tr(I\rho)
\end{array}
$$

for all density matrices ρ over H_x.

Bounded Positive Operators

Defined via the Löwner partial order:

$$\begin{array}{ccccc} 0_{H_x} & \sqsubseteq & M & \sqsubseteq & I_{H_x} \\ & & \equiv & & \\ 0 & \leq & tr(M\rho) & \leq & tr(\rho) \end{array}$$

for all density matrices ρ over H_x.

Arithmetic Triples

Propositional $\{P\}\ c\ \{Q\}$:

$$\forall \sigma, P(\sigma) \to Q(\llbracket c \rrbracket \sigma)$$

Probabilistic $\{P\}\ c\ \{Q\}$:

$$\forall \mathcal{D}, P(\mathcal{D}) \leq Q(\llbracket c \rrbracket \mathcal{D})$$

Quantum $\{P\}\ c\ \{Q\}$:

$$\forall \rho, tr(P\rho) \leq tr(Q\llbracket c \rrbracket \rho)$$

Arithmetic Triples

Propositional $\{P\}\ c\ \{Q\}$:

$$\forall \sigma, P(\sigma) \to Q(\llbracket c \rrbracket \sigma) \vee \uparrow$$

Probabilistic $\{P\}\ c\ \{Q\}$:

$$\forall \mathcal{D}, P(\mathcal{D}) \leq Q(\llbracket c \rrbracket \mathcal{D}) + Pr(\uparrow)$$

Quantum $\{P\}\ c\ \{Q\}$:

$$\forall \rho, tr(P\rho) \leq tr(Q\llbracket c \rrbracket \rho) + tr(\uparrow)$$

qPD Language

$$c ::= \texttt{skip} \mid c\ ;c \mid \mathbf{q} \ \texttt{:=}\ 0 \mid \vec{\mathbf{q}} \mathrel{*}= U \mid$$
$$\texttt{measure}\ M[\vec{\mathbf{q}}]\ :\ \vec{c} \mid \texttt{while}\ M[\vec{\mathbf{q}}]\ \texttt{do}\ c$$

qPD Unitary Rule

$$\overline{\{U^\dagger P U\}\ \vec{\mathbf{q}}\ *= U\ \{P\}}$$

qPD Measure Rule

$$\frac{\forall m,\ \{P_m\}\ c_m\ \{Q\}}{\{\sum_m M_m^\dagger P_m M_m\}\ \texttt{measure}\ M[\mathbf{q}]\ :\ \vec{c}\ \{Q\}}$$

qPD While Rule

$$\frac{\{Q\}\ c\ \{M_0^\dagger P M_0 + M_1^\dagger Q M_1\}}{\{M_0^\dagger P M_0 + M_1^\dagger Q M_1\}\ \texttt{while}\ M[\vec{\mathbf{q}}]\ \texttt{do}\ c\ \{P\}}$$

qPD While Semantics

$$\text{While End: } \langle \texttt{while } M[\vec{\mathbf{q}}] \texttt{ do } \vec{c}, \rho\rangle \rightarrow \langle \texttt{skip}, M_0 \rho M_0^\dagger \rangle$$

$$\text{While Loop: } \langle \texttt{while } M[\vec{\mathbf{q}}] \texttt{ do } \vec{c}, \rho\rangle \rightarrow$$

$$\langle c \,; \texttt{while } M[\vec{\mathbf{q}}] \texttt{ do } \vec{c}, M_1 \rho M_1^\dagger \rangle$$

We want to show that:

$$\forall \rho, [\![\mathit{deutsch}]\!] \rho = |0\rangle\langle 0| \otimes \rho'$$

for some 2×2 matrix ρ', hence:

$$\forall \rho, tr(I_4 \rho) \leq tr((|0\rangle\langle 0| \otimes I_2)[\![c]\!] \rho)$$

Verifying Deutsch in qPD

We want to show that:

$$\forall \rho, [\![deutsch]\!]\rho = |0\rangle\langle 0| \otimes \rho'$$

for some 2×2 matrix ρ', hence:

$$\forall \rho, tr(I_4 \rho) \leq tr((|0\rangle\langle 0| \otimes I_2)[\![c]\!]\rho)$$

$$P = I_4 \qquad Q = |0\rangle\langle 0| \otimes I_2$$

Verifying Deutsch in qPD

$$\{I_4\} \rightarrow$$
$$\{|0\rangle_1 \langle 0| \, |0\rangle_2 \langle 0| \, (|0\rangle\langle 0| \otimes I_2) \, |0\rangle_2 \langle 0| \, |0\rangle_1 + \ldots\}$$
$$\mathbf{q}_0 := 0; \mathbf{q}_1 := 0$$
$$\{(|0\rangle\langle 0| \otimes I_2)\} \rightarrow \{(I_4 \otimes N)|0\rangle\langle 0| \otimes I_2(I_4 \otimes N)\}$$
$$\mathbf{q}_1 *= N$$
$$\{(|0\rangle\langle 0| \otimes I_2)\} \rightarrow$$
$$\{H_2(I_2 \otimes N)(H \otimes I_2)(|0\rangle\langle 0| \otimes I_2)(H \otimes I_2)(I_2 \otimes N)H_2\}$$
$$\mathbf{q}_0, \mathbf{q}_1 *= H_2;$$
$$\{(I_2 \otimes N)(H \otimes I_2)(|0\rangle\langle 0| \otimes I_2)(H \otimes I_2)(I_2 \otimes N)\}$$
$$\mathbf{q}_0, \mathbf{q}_1 *= U_f;$$
$$\{(H \otimes I_2)(|0\rangle\langle 0| \otimes I_2)(H \otimes I_2)\}$$
$$\mathbf{q}_0 *= H$$
$$\{|0\rangle\langle 0| \otimes I_2\}$$

Verifying Deutsch in qPD

$$\{I_4\} \rightarrow$$

$$\{|0\rangle_1 \langle 0| \, |0\rangle_2 \langle 0| \, (|0\rangle\langle 0| \otimes I_2) \, |0\rangle_2 \langle 0| \, |0\rangle_1 + \dots\}$$

$$\mathbf{q}_0 := 0; \mathbf{q}_1 := 0$$

$$\{(|0\rangle\langle 0| \otimes I_2)\} \rightarrow \{(I_4 \otimes N)|0\rangle\langle 0| \otimes I_2(I_4 \otimes N)\}$$

$$\mathbf{q}_1 \mathrel{*}= N$$

$$\{(|0\rangle\langle 0| \otimes I_2)\} \rightarrow$$

$$\{H_2(I_2 \otimes N)(H \otimes I_2)(|0\rangle\langle 0| \otimes I_2)(H \otimes I_2)(I_2 \otimes N)H_2\}$$

$$\mathbf{q}_0, \mathbf{q}_1 \mathrel{*}= H_2;$$

$$\{(I_2 \otimes N)(H \otimes I_2)(|0\rangle\langle 0| \otimes I_2)(H \otimes I_2)(I_2 \otimes N)\}$$

$$\mathbf{q}_0, \mathbf{q}_1 \mathrel{*}= U_f;$$

$$\{(H \otimes I_2)(|0\rangle\langle 0| \otimes I_2)(H \otimes I_2)\}$$

$$\mathbf{q}_0 \mathrel{*}= H$$

$$\{|0\rangle\langle 0| \otimes I_2\}$$

Comparison

	EEQPL	QHL	qPD
Language	Classical + Quantum	$\sim$Classical +Quantum	
Objects	Ensembles	Density Matrices	
Assertions	Truth Functional	Truth Functional	
While Rule	None	Limited	
WP-form	Yes	No	
Complete	No	No	

Comparison

	EEQPL	QHL	qPD
Language	Classical + Quantum	$\sim$Classical +Quantum	Quantum
Objects	Ensembles	Density Matrices	
Assertions	Truth Functional	Truth Functional	
While Rule	None	Limited	
WP-form	Yes	No	
Complete	No	No	

Comparison

	EEQPL	QHL	qPD
Language	Classical + Quantum	$\sim$Classical +Quantum	Quantum
Objects	Ensembles	Density Matrices	Density Matrices
Assertions	Truth Functional	Truth Functional	
While Rule	None	Limited	
WP-form	Yes	No	
Complete	No	No	

Comparison

	EEQPL	QHL	qPD
Language	Classical + Quantum	$\sim$Classical $+$Quantum	Quantum
Objects	Ensembles	Density Matrices	Density Matrices
Assertions	Truth Functional	Truth Functional	Arithmetic
While Rule	None	Limited	
WP-form	Yes	No	
Complete	No	No	

Comparison

	EEQPL	QHL	qPD
Language	Classical + Quantum	$\sim$Classical +Quantum	Quantum
Objects	Ensembles	Density Matrices	Density Matrices
Assertions	Truth Functional	Truth Functional	Arithmetic
While Rule	None	Limited	Yes
WP-form	Yes	No	
Complete	No	No	

Comparison

	EEQPL	QHL	qPD
Language	Classical + Quantum	$\sim$Classical +Quantum	Quantum
Objects	Ensembles	Density Matrices	Density Matrices
Assertions	Truth Functional	Truth Functional	Arithmetic
While Rule	None	Limited	Yes
WP-form	Yes	No	Yes
Complete	No	No	

Comparison

	EEQPL	QHL	qPD
Language	Classical + Quantum	$\sim$Classical +Quantum	Quantum
Objects	Ensembles	Density Matrices	Density Matrices
Assertions	Truth Functional	Truth Functional	Arithmetic
While Rule	None	Limited	Yes
WP-form	Yes	No	Yes
Complete	No	No	Yes

What's Next?

	EEQPL	QHL	qPD	qPD+
Language	Classical + Quantum	$\sim$Classical +Quantum	Quantum	
Objects	Ensembles	Density Matrices	Density Matrices	
Assertions	Truth Functional	Truth Functional	Arithmetic	
While Rule	None	Limited	Yes	
WP-form	Yes	No	Yes	
Complete	No	No	Yes	

What's Next?

	EEQPL	QHL	qPD	qPD+
Language	Classical + Quantum	∼Classical +Quantum	Quantum	Classical + Quantum
Objects	Ensembles	Density Matrices	Density Matrices	
Assertions	Truth Functional	Truth Functional	Arithmetic	
While Rule	None	Limited	Yes	
WP-form	Yes	No	Yes	
Complete	No	No	Yes	

What's Next?

	EEQPL	QHL	qPD	qPD+
Language	Classical + Quantum	$\sim$Classical +Quantum	Quantum	Classical + Quantum
Objects	Ensembles	Density Matrices	Density Matrices	Density Matrices + ?
Assertions	Truth Functional	Truth Functional	Arithmetic	
While Rule	None	Limited	Yes	
WP-form	Yes	No	Yes	
Complete	No	No	Yes	

What's Next?

	EEQPL	QHL	qPD	qPD+
Language	Classical + Quantum	$\sim$Classical +Quantum	Quantum	Classical + Quantum
Objects	Ensembles	Density Matrices	Density Matrices	Density Matrices + ?
Assertions	Truth Functional	Truth Functional	Arithmetic	Truth Functional?
While Rule	None	Limited	Yes	
WP-form	Yes	No	Yes	
Complete	No	No	Yes	

What's Next?

	EEQPL	QHL	qPD	qPD+
Language	Classical + Quantum	∼Classical +Quantum	Quantum	Classical + Quantum
Objects	Ensembles	Density Matrices	Density Matrices	Density Matrices + ?
Assertions	Truth Functional	Truth Functional	Arithmetic	Truth Functional?
While Rule	None	Limited	Yes	Yes
WP-form	Yes	No	Yes	
Complete	No	No	Yes	

What's Next?

	EEQPL	QHL	qPD	qPD+
Language	Classical + Quantum	$\sim$Classical +Quantum	Quantum	Classical + Quantum
Objects	Ensembles	Density Matrices	Density Matrices	Density Matrices + ?
Assertions	Truth Functional	Truth Functional	Arithmetic	Truth Functional?
While Rule	None	Limited	Yes	Yes
WP-form	Yes	No	Yes	Yes
Complete	No	No	Yes	

What's Next?

	EEQPL	QHL	qPD	qPD+
Language	Classical + Quantum	$\sim$Classical +Quantum	Quantum	Classical + Quantum
Objects	Ensembles	Density Matrices	Density Matrices	Density Matrices + ?
Assertions	Truth Functional	Truth Functional	Arithmetic	Truth Functional?
While Rule	None	Limited	Yes	Yes
WP-form	Yes	No	Yes	Yes
Complete	No	No	Yes	Yes

What's Next?

	EEQPL	QHL	qPD	QLNext
Language	Classical + Quantum	$\sim$Classical +Quantum	Quantum	
Objects	Ensembles	Density Matrices	Density Matrices	
Assertions	Truth Functional	Truth Functional	Arithmetic	
While Rule	None	Limited	Yes	
WP-form	Yes	No	Yes	
Complete	No	No	Yes	

What's Next?

	EEQPL	QHL	qPD	QLNext
Language	Classical + Quantum	$\sim$Classical +Quantum	Quantum	Classical + Quantum
Objects	Ensembles	Density Matrices	Density Matrices	
Assertions	Truth Functional	Truth Functional	Arithmetic	
While Rule	None	Limited	Yes	
WP-form	Yes	No	Yes	
Complete	No	No	Yes	

What's Next?

	EEQPL	QHL	qPD	QLNext
Language	Classical + Quantum	$\sim$Classical +Quantum	Quantum	Classical + Quantum
Objects	Ensembles	Density Matrices	Density Matrices	Density Matrices + ?
Assertions	Truth Functional	Truth Functional	Arithmetic	
While Rule	None	Limited	Yes	
WP-form	Yes	No	Yes	
Complete	No	No	Yes	

What's Next?

	EEQPL	QHL	qPD	QLNext
Language	Classical + Quantum	$\sim$Classical +Quantum	Quantum	Classical + Quantum
Objects	Ensembles	Density Matrices	Density Matrices	Density Matrices + ?
Assertions	Truth Functional	Truth Functional	Arithmetic	Truth Functional?
While Rule	None	Limited	Yes	
WP-form	Yes	No	Yes	
Complete	No	No	Yes	

What's Next?

	EEQPL	QHL	qPD	QLNext
Language	Classical + Quantum	∼Classical +Quantum	Quantum	Classical + Quantum
Objects	Ensembles	Density Matrices	Density Matrices	Density Matrices + ?
Assertions	Truth Functional	Truth Functional	Arithmetic	Truth Functional?
While Rule	None	Limited	Yes	Yes
WP-form	Yes	No	Yes	
Complete	No	No	Yes	

What's Next?

	EEQPL	QHL	qPD	QLNext
Language	Classical + Quantum	∼Classical +Quantum	Quantum	Classical + Quantum
Objects	Ensembles	Density Matrices	Density Matrices	Density Matrices + ?
Assertions	Truth Functional	Truth Functional	Arithmetic	Truth Functional?
While Rule	None	Limited	Yes	Yes
WP-form	Yes	No	Yes	Yes
Complete	No	No	Yes	

What's Next?

	EEQPL	QHL	qPD	QLNext
Language	Classical + Quantum	$\sim$Classical +Quantum	Quantum	Classical + Quantum
Objects	Ensembles	Density Matrices	Density Matrices	Density Matrices + ?
Assertions	Truth Functional	Truth Functional	Arithmetic	Truth Functional?
While Rule	None	Limited	Yes	Yes
WP-form	Yes	No	Yes	Yes
Complete	No	No	Yes	Yes

Thank You