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Unit:- 1
Laplace transformation

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Q i) why Laplace transformation.
→ It is widely used by electronic engineers to solve differen eqⁿ occurring in the analysis of electronic circuit. It is heart of digital signal processing. It is used in process control & nuclear physics.

Defⁿ:- let ~~F(u)~~ F(t) be the function of t define for t greater than 0 the Laplace transform define by

$$L[F(t)] = \int_0^\infty e^{-st} \cdot F(t) \cdot dt$$

$$L[F(t)] = \int_0^\infty e^{-st} F(t) \cdot dt = \cancel{F(s)}\, F(s)$$

where "s" is parameter which may be real imaginary

* Laplace transformation of some elementary function.

$F(t) = 1$

$$L[F(t)] = \int_0^\infty e^{-st} \cdot F(t)\, dt$$

$$= \left[\frac{-e^{-st}}{s}\right]_0^\infty = -\frac{1}{s}\left[e^{-\infty} - e^{0}\right]$$

$$\boxed{L[F(t)] = \frac{1}{s}} \qquad s > 0$$

(2) $F(t) = \sin(at)$

$$L[F(t)] = \int_0^\infty e^{-st} F(t)\, dt = F(s)$$

$$= \int_0^\infty e^{-st} \cdot \sin(at) \cdot dt$$

$$= \frac{e^{-st}}{s^2 + a^2}\left[-s\sin(at) - a\cos at\right]_0^\infty$$

$$= \frac{a}{s^2 + a^2} \qquad s > a$$

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$$\int e^{at}\sin(bt) = \frac{e^{at}}{a^2+b^2}\left[a\sin bt - b\cos bt\right]$$

$$\int e^{at}\cos(bt) = \frac{e^{at}}{a^2+b^2}\left[a\cos bt + b\sin(bt)\right]$$

(3). $F(t) = \cos(bt)$

$$L[F(t)] = \int_0^\infty e^{-st}\,F(t)\,dt = F(s)$$

$$= \int_0^\infty e^{-st}\cos(bt)$$

$$= \frac{e^{-st}}{s^2+b^2}\left[s\cos bt + b\,\cancel{\cos}\,\sin bt\right]_0^\infty$$

$$= 0 - \frac{1}{s^2+b^2}\left[-1\,s\right]$$

$$\boxed{= F(t) = \frac{s}{s^2+b^2} \qquad s>b}$$

$F(t) = \sin(ht)$

$F(s) = \dfrac{a}{s^2-a^2} \qquad s>a$

$F(t) = \cos(ht)$

$F(s) = \dfrac{s}{s^2-a^2} \qquad s>a$

$F(t) = e^{at}$

$F(s) = \dfrac{1}{s-a}$

$F(t) = t^n$

$F(s) = \dfrac{\Gamma(n+1)}{s^{n+1}}$

For $n > -1$

$= \dfrac{n!}{s^{n+1}}$ For n is +ve integer

e.g. $f(t) = t^{5/2}$

$$L[F(t)] = \frac{\Gamma\left(\frac{3}{2}+1\right)}{s^{5/2}}$$

$$\Gamma(n+1) = n\,\Gamma(n)$$

$$= \frac{3}{2}\,\Gamma\left(\frac{3}{2}\right)$$

$$= \frac{3}{2}\times\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$$

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1. e^{at+b}
2. $(2t+3)^3$
3. $\sin(2t).\cos t$
4. $\sin^2 4t$

$\Rightarrow$ e^{ax+b}

$$L[F(t)] = \int_0^\infty e^{-st} F(t)\,dt = F(s)$$

$$= \int_0^\infty e^{-st}.e^{at+b}\,dt$$

$$= \int_0^\infty e^{(s-a)t+b}\,dt$$

$$= \int_0^\infty \left[\frac{e^{-[(s-a)t+b]}}{(s-a)}\right]_0^\infty dt$$

$$= \frac{e^b}{s-a}$$

$e^{at} = \frac{1}{s-a}$

$= e^{at}.e^{b}$

$= \frac{1}{s-a}e^b$

$\boxed{= \frac{e^b}{s-a}}$

2. $(2t+3)^3$ $\qquad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$\Rightarrow$ $F(t) = (2t+3)^3$

$= (2t)^3 + [3(2t)^2(3)] + 3[(2a)(3)^2] + (3)^3$

$= 8t^3 + 3(4t^2)(3) + 3(2a)(9) + 27$

$= 8t^3 + 36t^2 + 54t + 27$

$= 8.\frac{3!}{s^4} + 36\times\frac{2!}{s^3} + 54\times\frac{1!}{s^2} + 27\times\frac{1}{s}$

$= \frac{48}{s^4} + \frac{72}{s^3} + \frac{54}{s^2} + \frac{27}{s}$

$c^x = e^{x \log c}$

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(3). $\sin 2t \cdot \cos t$

$\Rightarrow$ $\sin A \cdot \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

$= \frac{1}{2} \sin(2t+t) + \sin(2t-t)$

$= \frac{1}{2} \sin(3t) + \sin t$

$= \frac{1}{2}\left[\frac{3}{s^2+3^2} + \frac{1}{s^2+1^2}\right]$

Ans $= \frac{1}{2}\left[\frac{3}{s^2+9} + \frac{1}{s^2+1}\right]$

4). $f(t) = \sin^2(4t)$

$\boxed{\sin^2\theta = \frac{1-\cos 2\theta}{2}}$

$\Rightarrow$ $\frac{1-\cos 8t}{2}$

$= \frac{1}{2} - \frac{\cos 8t}{2}$

$= \frac{1}{2s} - \frac{1}{2} \frac{s}{s^2+8^2}$

(5) $f(t) = 4^t$

~~$e^x = \log$~~

$\boxed{c^x = e^{x \log c}}$

$\Rightarrow$ ~~$4^{t \log}$~~

$f(t) = 4^t$

$L[f(t)] = L(4^t) = F(s)$

$= L(e^{t \log 4})$

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$= \frac{1}{s - \log 4}$ where $s > (\log 4)$

(6). $3\cos(4t + 7)$

$\Rightarrow$ $F(t) = 3\cos(4t + 7)$

$\cos(A+B) = \cancel{\cos(A+B) \sin(A+B)}$
$\cos A \cos B - \sin A \sin B$

$L[F(t)] = L(3\cos(4t+7))$

$= L[3(\cos 4t \cos 7 - \sin 4t \sin 7)]$

$= L\left[3\left(\cos 7 \cdot \frac{s}{s^2+4^2} - \sin 7 \cdot \frac{4}{s^2+4^2}\right)\right]$

$= L\left[3\left(\cos 7 \cdot \frac{s}{s^2+16} - \sin 7 \frac{4}{s^2+16}\right)\right]$

$$\boxed{= \frac{3}{s^2+16}\left(s \cdot \cos 7 - 4\sin 7\right)}$$

*

Tues

* properties of Laplace transform.

1) First shifting theorem.

if $L[F(t)] = F(s)$

$L[e^{-at} . F(t)] = F(s+a)$

where $\boxed{s \longrightarrow s+a}$

e.g.) $e^{-at} . \sin(bt)$

Let $F(t) = \sin bt$

$= \frac{b}{s^2+b^2}$

soln. replace $s = s+a$

$= \frac{b}{(s+a)^2+b^2}$

2) $e^{at} \cos(bt)$

Let $F(t) = \cos(bt)$

$= \frac{s}{s^2+b^2}$

replace $s = s+a$

$= \frac{s+a}{(s+a)^2+b^2} = \frac{s-a}{(s-a)^2+b^2}$

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(3) $(t+2)^2 e^{4t}$

(4) $e^{-3t} \sin^2 t$

(5) $\cosh(at) \cdot \sin at$

$\Rightarrow$ $F(t) = (t+2)^2 e^{4t}$

$$e^{at} = \frac{1}{s-a} = \frac{1}{s-4}$$

$$= (t+2)^2 \ \cancel{\&}\ e^{4t}$$

$$= \cancel{2}t^2 + 2(2t) + 2^2 \ e^{4t}$$

$$= \cancel{2}t^2 + 4t + 4 \cdot e^{4t}$$

$$= \cancel{2t^2 + 8 \cdot e^{4t}}$$

$$= \cancel{2}t^2 + 4t + 4 \cdot e^{4t}$$

$$= \frac{2!}{s^3} + \frac{4\cancel{s}}{s^2} + 4 \cdot \frac{1}{s} \cdot \cancel{e^{4t}}$$

$$= \frac{\cancel{2}\ 2 + 4s + 4s^2}{s^3} \cdot \cancel{e^{4t}}$$

$$= \frac{2 + 4(s-4) + 4(s-4)^2}{\cancel{s^3}(s-4)^3} \times \cancel{\frac{1}{s-4}}$$

$$= \frac{2 + 4s - 16 + 4s^2 - 32s + 64}{(s-4)^3} \times \cancel{\frac{1}{s-4}}$$

$$= \frac{4s^2 - 28s + 50}{(s-4)^3}$$

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(3) $e^{-3t} \sin^2 t$

$e^{at} = \frac{1}{s+a} = \frac{1}{s+3}$

$\Rightarrow$

$F(t) = \sin^2 t$

$= \left(\frac{1-\cos 2t}{2}\right)$

$= \frac{1}{2} - \cos 2t$

$= \frac{1}{2s} \left[- \frac{1}{2} * \frac{s}{s^2+a^2} \right]$

$= \frac{1}{2} \cdot \left[\frac{1}{s} * \frac{s}{s^2+4} \right]$

$= \frac{1}{8}$

$L[F(t) \cdot e^{-3t}] = \frac{1}{2} \left(\frac{1}{(s+3)} * \frac{(s+3)}{(s+3)^2+4} \right)$

$= \frac{1}{2} \left(\frac{s}{(s+3)^2+4} - \frac{(s+3)}{(s+3)^2+4} \right)$

$= \frac{1}{2} \left[\frac{s-s-3}{(s+3)^2+4} \right]$

$= \frac{1}{2} \left[\frac{2-3}{s^2+6s+9+4} \right] = \frac{1}{2} \left[\frac{2-s}{s^2+6s+13} \right]$

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$$\boxed{\cosh(at) = \frac{e^{at}+e^{-at}}{2}}$$

$$\boxed{\sinh(at) = \frac{e^{at}-e^{-at}}{2}}$$

(4) $\cosh(at) \cdot \sin at$.

$\Rightarrow$ $\cos(at) \quad \sin(at)$

$$= \left[\frac{e^{at}+e^{-at}}{2} \cdot \sin(at)\right]$$

$$= \frac{1}{2}\left[e^{at} \cdot \sin at + e^{-at} \sin at\right]$$

$$= \frac{1}{2} \cdot \frac{a}{s^2-a^2} + \frac{1}{2} \cancel{\sin} \frac{a}{s^2+a^2}$$

$$= \frac{1}{2} \frac{a}{(s-a)^2-a^2} + \frac{1}{2} \frac{a}{(s+a)^2+a^2}$$

(5) $\sinh at . \sin at$

$\Rightarrow$ $f(t) = \sin at . \sin at$

$$= \left[\frac{e^{at}+e^{-at}}{2} \quad \sin at\right]$$

$$= \frac{1}{2}\left[\left(e^{at}-e^{-at}\right) . \sin at\right]$$

$$= \frac{1}{2} e^{at} \sin at - \frac{1}{2} e^{-at} \sin at$$

$$= \frac{1}{2} \cdot \frac{a}{s^2-a^2} - \frac{1}{2} \frac{a}{s^2-a^2}$$

$$= \frac{1}{2} \frac{a}{(s-a)^2-a^2} - \frac{1}{2} \frac{a}{(s^2+a^2)-a^2}$$

$$= \frac{a}{(2s^2-2a^2)-a^2} - \frac{a}{(2s^2+2a^2)-a^2} = \frac{a}{4s^2-4a^2-a^2} - \frac{a}{(4s^2+4a^2)}$$

$$= \frac{a}{-4s^2a^2+4a^4} - \frac{a}{4s^2a^2-4a^4} =$$

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* second shifting property-

if $L[F(t)] = F(s)$

&

$$F(t) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$$

$$L[F(t)] = e^{-as} \cdot F(s)$$

ex:- 1).

$$F(t) = \begin{cases} \cos(t - 2\pi/3) & t > 2\pi/3 \\ 0 & t < 2\pi/3 \end{cases}$$

$F(t) = \cos t$

$F(t-a)$ $\cancel{F(\cos t)} = \cos(t - 2\pi/3)$

$a = 2\pi/3$

$L[F(t)] = \cos t$

$F(s) = \dfrac{s}{s^2+1}$

$L[F(t)] = e^{-as} \cdot F(s)$.

$$\boxed{= e^{-2\pi/3 \cdot s} \cdot \frac{s}{s^2+1}}$$

2).

$$F(t) = \begin{cases} (t-1)^3 & t > 1 \\ 0 & t < 0 \end{cases}$$

let

$F(t) = t^3$

$F(t-a) = (t-1)^3$

$\underline{\underline{a = 1}}$

$L[f(t)] = t^3$

$= \frac{3!}{s^4} = \frac{6}{s^4}$

By second shifting.

$L[f(t)] = e^{-as} \cdot f(s)$

$\boxed{= e^{-s} \cdot \frac{6}{s^4}}$

* Laplace transforme derivatives.

$L[F(t)] = F(s)$

$\cancel{L[F'(t)}$

1. $L[F'(t)] = sF(s) - F(0)$

2. $L[F''(t)] = s^2 F(s) - sF(0) - F'(0)$

3. $L[F'''(t)] = s^3 F(s) - s^2 F(0) - sF'(0) - F''(0)$

1. obtain laplace transform of $\frac{d^2y}{dt^2} - \frac{3dy}{dt} + 3y$

given that $y(0) = 2$ $F(0) = 2$

$y'(0) = -4$. $F'(0) = -4$

$\Rightarrow = s^2 F(s) - sF(0) - F'(0) - 3[sF(s) - F(0)] + 3F(s)$

$= s^2 \cdot F(s) - s(2) + 4 - 3[sF(s) - 2] + 3F(s)$

$= s^2 F(s) - 2s + 4 - 3sF(s) + 6 + 3F(s)$

$= s^2 F(s) - 3sF(s) + 3F(s) - 2s + 10$