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Turán's Theorem Formalization

ro-gut

September 16, 2025

0.1 Concentrating support on a clique - Improve Operartion

Definition 1 (A better distribution)**.** Better Given a weight function W, a choice of a weight function `Better` W with $\mathrm{supp}(\texttt{Better}\ W) \subseteq \mathrm{supp}(W)$ and $W.\mathrm{fw} \leq (\texttt{Better}\ W).\mathrm{fw}$.

Definition 2 (Single transfer)**.** Improve Given distinct vertices `loose` $\neq$ `gain`, the weight function `Improve` W `loose` `gain` moves a small amount from `loose` to `gain`.

Lemma 3 (Sum splitting along the partition)**.** *$Improve_{p}artition_{s}um_{s}plitlem : Improve_{e}dgeFinset_{p}artitionSu$ over E splits as the sum over the gain-incidence, plus the loose-incidence, plus the complement.*

Lemma 4 (Gain-incidence increases)**.** *$Improve_{g}ain_{c}ontribution_{i}ncreaseThesumonthegain-incidenceincreasesbyW.w$ `loose` times the sum of the other-endpoint weights incident to 'gain'.*

Lemma 5 (Loose-incidence becomes zero)**.** *$Improve_{l}oose_{c}ontribution_{z}eroThesumontheloose-incidenceiszeroafter$ `Improve`.*

Lemma 6 (Unchanged complement)**.** *$Improve_{u}nchanged_{e}dge_{s}umEdgesoutsidetheunionofgain/looseincidenc$*

Lemma 7 (Transfer does not decrease fw)**.** *$Improve_{t}otal_{w}eight_{n}ondeclem : Improve_{p}artition_{s}um_{s}plit, lem : I$ (`Improve` W `loose` `gain`).fw.*

Lemma 8 (Improve strictly reduces support)**.** *$Improve_{s}upport_{s}trictly_{r}educeddef : Improve, lem : Improve_{l}oo$*

Theorem 9 (Support of `Better` is a clique)**.** *$Better_{f}orms_{c}liquedef : Improve, lem : Improve_{t}otal_{w}eight_{n}onde$ $everytwodistinctverticesofpositiveweightareadjacentinG$.*

0.2 The Enhance Operation

Definition 10 (Enhance)**.** Enhance Defines the operation of transferring weight from one vertex to another, provided the two vertices are non-adjacent. This operation is central to the second phase of the proof, where we later reduce the support size while ensuring the edge weight does not decrease.

Lemma 11 (Sum over support)**.** *$sum_{o}ver_{s}upportExpressesthetotalvertexweightasthesumofweightsoverthes$*

Lemma 12 (Supported edge partition)**.** *$supported_{e}dge_{p}artitionSplitstheedgesetintoedgesincidenttothechosen$*

Lemma 13 (Enhance gain sum). $Enhance_g ain_s umShowsthatunder$ **??**$, thecontributionofthegainvertexsedges$

Lemma 14 (Enhance loose sum). $Enhance_l oose_s umShowsthatunder$**??**$, thecontributionofthelooseverte xsedg$

Definition 15 (Bijection inside the clique). the$_b ijProvidesabijectionbetweenthesupportedincidenceedgesat`lo$

Lemma 16 (Bijection preserves). $the_b ij_s ameShowsthatthebijectionpreservesthe"other"weight :$
$foranyedgefromthesupportedincidencesetofloose, theweighatthe"other"vertexequalsthatinitsimageunedert$

Lemma 17 (Loose/gain equality). $Enhance_s um_l oose_g ain_e qualShowsthatthetotalweightmovedfromthelooseve$

Lemma 18 (Complement unchanged). $Enhance_s um_c omplement_u nchangedShowsthatedgesnotincidenttogain$

Lemma 19 (Edge contribution increase). $Enhance_e dge_g ainloose_i ncreaseProvesthatthenetcontributionfromg$

Lemma 20 (Support edges unchanged). $Enhance_s upport_e dges_s ameShowsthatforverticesoutsideofgainandlo$

Theorem 21 (Enhance increases edge weight). $Enhance_t otal_w eight_s tricinclem : supported_e dge_p artition, lem :$

0.3 Equalizing the weights on the clique - EnhanceD

Definition 22 (Carefully chosen ε). the$Define0 :=$ max–1/|supp|.

Definition 23 (Maximising the number of uniform vertices). $\max_u niform_s upportDefinethemaximal$machiev

Lemma 24 (Best uniform distribution exists). $exists_b est_u niformdef : max_u niform_s upportThereexistsadistr$
W.fw.

Definition 25 (UniformBetter). UniformBetter lem:exists$_b est_u niformAchoiceofamaximiserfrom$**??**.

Definition 26 (Enhanced)**.** Enhanced def:Enhance, def:the$Defines`Enhanced`weightfunction:$
$transferingweightfromtheargmaxvertex`loose`totheargminvertex`gain`, usingthepreviousinSection2defin$

Lemma 27. $Enhanced_unaffecteddef:Enhance, def:EnhancedShowsthatunder`Enhanced`everyvertextha$

Lemma 28. $Enhanced_effect_argmaxdef:Enhance, def:EnhancedShowsthattheweightattheargmaxverte$

Lemma 29. $Enhanced_inc_uniform_countdef:Enhanced, lem:Enhanced_effect_argmax, lem:Enhanced_unaf$

Lemma 30. *def:UniformBetter The support of W forms a clique if and only if the support at UniformBetter also forms a clique.*

Lemma 31 (Uniform weights on the support)**.** $UniformBetter_constant_supportlem:UniformBetter, def:En$

Lemma 32 (Edge values under UniformBetter)**.** $UniformBetter_edges_valuelem:UniformBetter_constant_suppo$

Lemma 33 (Edge count in a clique)**.** $clique_sizelem:UniformBetter_factsIfthesupporthassizek, thenthenum$
1)/2.

Lemma 34 (A light computation)**.** *computation* $\big(k(k-1)/2\big)\cdot(1/k)^2 = \frac{1}{2}\,(1 - 1/k)$ *for* $k > 0$.

Lemma 35 (Monotonicity of the bound)**.** *bound* $bound_realThefunctionk\mapsto \frac{1}{2}(1-\frac{1}{k})$ *is nondecreasing in* k *(for* $k \geq 1$*).*

Theorem 36 (Final bound inside a clique)**.** $finale_boundlem:Better_non_decr, lem:Better_forms_clique, lem:U$
$p-1$*, then*

$$W.\mathrm{fw} \;\leq\; \tfrac{1}{2}\,(1 - 1/(p-1)).$$

Definition 37 (Uniform weights over all vertices)**.** UnivFun The uniform vertex-weight function assigning $1/|V|$ to each vertex.

Lemma 38 (Total weight under `UnivFun`)**.** $UnivFun_weightdef:UnivFun$*(*`UnivFun` *G).fw=#E·*$(1/|V|)^2$.

Theorem 39 (Turán's Theorem)**.** *turans def:UnivFun, lem:UnivFun*$_weight, lem:$
$finale_bound, lem:computationLetp$$\geq 2$ *and let G be a p-clique-free graph. Then*

$$\#E \;\leq\; \tfrac{1}{2}\Big(1 - \tfrac{1}{p-1}\Big)\,(\#V)^2.$$