Turán's Theorem Formalization

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0.1 Concentrating support on a clique - Improve Operartion

Definition 1 (Weight distributions)**.** Given a finite simple graph $G = (V, E)$, a *weight distribution* is a function

$$w : V \to \mathbb{R}_{\geq 0} \qquad \text{with} \qquad \sum_{v \in V} w(v) = 1.$$

A probability distribution on the vertex Set.

Definition 2 (Total edge weight)**.** For $w \in W$, the total edge weight is

$$f(w) \;:=\; \sum_{e \in E} \mathrm{vp}(w, e).$$

In words: sum the edge values over all edges of the graph.

Definition 3 (A better distribution)**.** Given a weight function w, define w^* such that $\mathrm{supp}(w^*) \subseteq \mathrm{supp}(w)$ and

$$w.\mathrm{fw} \leq w^*.\mathrm{fw}.$$

That is w^* is a distribution with non-decreasing total edge weight with the original support of w preserved.

Definition 4 (Improve Operation)**.** Given distinct vertices $v_j \neq v_i$, define $w' := \texttt{Improve}(W, v_j, v_i)$ by moving *all* weight from v_j to v_i:

$$w'(v_j) = 0, \qquad w'(v_i) = w(v_i) + w(v_j), \qquad w'(v) = w(v) \text{ for } v \notin \{v_i, v_j\}.$$

Lemma 5 (Sum split)**.** *The sum over edges E of the function* $\boldsymbol{vp}$ *splits as*

$$\sum_{e \in E} \boldsymbol{vp}(w', e) = \sum_{\substack{e \in E \\ v_i \in e}} \boldsymbol{vp}(w', e) + \sum_{\substack{e \in E \\ v_j \in e}} \boldsymbol{vp}(w', e) + \sum_{\substack{e \in E \\ v_i, v_j \notin e}} \boldsymbol{vp}(w', e).$$

That is the total sum decomposes into parts incident to v_i, incident to v_j, and the rest.

Lemma 6 (Gain-incidence increases)**.** *The increment in the sum over edges incident to v_i is*

$$\sum_{v_i \in e} \boldsymbol{vp}(w', e) - \sum_{v_i \in e} \boldsymbol{vp}(w, e) = \varepsilon \sum_{v_y \in N(v_i)} w(v_y),$$

where $N(v_i)$ is the neighborhood of v_i.

Lemma 7 (Loose-incidence becomes zero)**.** *The sum over edges incident to v_j after the transfer satisfies*

$$\sum_{v_j \in e} \boldsymbol{vp}(w', e) = 0.$$

Lemma 8 (Unchanged complement)**.** *For edges e not incident to v_i or v_j,*

$$\boldsymbol{vp}(w', e) = \boldsymbol{vp}(w, e).$$

Edges outside v_i and v_j neighborhoods remain unchanged under the operation.

Lemma 9 (Improve results in non decreaasing distribution)**.** *Given non adjacent vertices v_i, v_j and assuming that the neightborsum of weights at s_i is equal or greater than that of s_j, we have*

$$f(w) \le f(w').$$

that is the total edge weight does not decrease after applying `Improve`*.*

Lemma 10 (Improve strictly reduces support)**.** *The new distribution w' has a strictly smaller support:*

$$|\mathrm{supp}(w')| < |\mathrm{supp}(w)|.$$

Theorem 11 (Support of `Better` is a clique)**.** *The support of w^* forms a clique:*

$$\forall v_x, v_y \in \mathrm{supp}(w^*), v_y \ne v_x \implies \{v_y, v_x\} \in E.$$

In words: every two distinct vertices with positive weight in w^* are adjacent.

0.2 The Enhance Operation

Definition 12 (Enhance Operation)**.** Given distinct non-adjacent vertices v_j, v_i, define w^+ by transferring $\varepsilon > 0$ weight from v_j to v_i:

$$w^+(v_j) = w(v_j) - \varepsilon, \quad w^+(v_i) = W(v_i) + \varepsilon, \quad w^+(v) = W(v) \text{ for } v \ne v_i, v_j,$$

with the condition $\{v_j, v_i\} \notin E$. The Enhance transfers weight between non-adjacent vertices to increase edge weight.

Lemma 13 (Supported edge partition)**.** *The edge set E partitions as*

$$E = E_{v_i} \cup E_{v_j} \cup E_{\text{rest}},$$

where

$$E_{v_i} = \{e \in E : v_i \in e\}, \quad E_{v_j} = \{e \in E : v_j \in e\}, \quad E_{\text{rest}} = E \setminus (E_{v_i} \cup E_{v_j}).$$

In words: edges are split into the incidence set to v_i, the one to v_j, and the rest.

Lemma 14 (Enhance gain sum)**.** *Under Theorem 12, the change in the sum over edges incident to v_i satisfies*

$$\sum_{e \in E_{v_i}} \boldsymbol{vp}(w^+, e) - \sum_{e \in E_{v_i}} \boldsymbol{vp}(w, e) = \varepsilon \sum_{v_y \in N(v_i)} w(v_y).$$

That is the gain vertex's edge contribution increases by ε times the sum of its neighbors' weights.

Lemma 15 (Enhance loose sum)**.** *Under Theorem 12, the sum over edges incident to v_j satisfies*

$$\sum_{e \in E_{v_j}} \boldsymbol{vp}(w^+, e) = 0.$$

That is the loose vertex's incident edge contributions become zero after Enhance.

Definition 16 (Bijection inside the clique)**.** Define a bijection

$$\phi : \{e \in E_{v_j} \setminus \{s(v_j, v_i)\}\} \to \{e \in E_{v_i} \setminus \{s(v_j, v_i)\}\}$$

mapping edges incident to v_j (except $s(v_j, v_i)$) to edges incident to v_i (except $s(v_j, v_i)$). In words: this bijection pairs edges incident to v_j with edges incident to v_i within the clique.

Lemma 17 (Bijection preserves)**.** *For any edge* e *incident to* v_j *(excluding* $s(v_j, v_i)$*), the "other" vertex weight satisfies*

$$w(\mathrm{other}(e, v_j)) = w(\mathrm{other}(\phi(e), v_i)).$$

In words: the bijection preserves weights at the other endpoints of edges.

Lemma 18 (Loose/gain equality)**.** *The total weight transfer balances the edge contributions:*

$$\sum_{e \in E_{v_j}} \boldsymbol{vp}(w^+, e) + \sum_{e \in E_{v_i}} \boldsymbol{vp}(w', e) \geq \sum_{e \in E_{v_j}} \boldsymbol{vp}(w, e) + \sum_{e \in E_{v_i}} \boldsymbol{vp}(w, e).$$

In words: the combined edge contributions of loose and gain vertices do not decrease after Enhance.

Lemma 19 (Complement unchanged)**.** *For edges* $e \in E_{\mathrm{rest}}$*,*

$$\boldsymbol{vp}(w^+, e) = \boldsymbol{vp}(w, e).$$

In words: edges not incident to v_i *or* v_j *remain unaffected by Enhance.*

Lemma 20 (Edge contribution increase)**.** *The total edge contribution satisfies:*

$$\sum_{e \in E_{v_i} \cup E_{v_j}} \boldsymbol{vp}(w^+, e) \geq \sum_{e \in E_{v_i} \cup E_{v_j}} \boldsymbol{vp}(w, e).$$

That is, the total contribution from gain and loose vertices does not decrease.

Lemma 21 (Support edges unchanged)**.** *For any vertex* $v \notin \{v_i, v_j\}$*, the edge contributions satisfy*

$$\sum_{e \ni v} \boldsymbol{vp}(w^+, e) = \sum_{e \ni v} \boldsymbol{vp}(w, e).$$

In words: vertices outside gain and loose retain their edge contributions after Enhance.

Theorem 22 (Enhance increases edge weight)**.** *For a given distribution* $w \in W$ *Applying* `Enhance` *(*w^+*) strictly increases the total edge weight:*

$$f(w^+) > f(w)$$

That is, the Enhance operation strictly improves the total edge weight contribution.

0.3 Equalizing the weights on the clique - EnhanceD

Definition 23 (Maximising the number of uniform vertices)**.** For a given distribution w, K is the maximal number of uniform vertices achievable without decreasing the total edge weight

$$K := \max\{N_a(w)\}$$

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Lemma 24 (Best uniform distribution exists)**.** *There exists* w_M *with* $\mathrm{supp}(w_M) \subseteq \mathrm{supp}(w)$*,* $w_M.\mathrm{fw} \geq W.\mathrm{fw}$*, and with at least* m *vertices having weight* $1/m$*. In words: a maximiser* w_M *achieving the maximal uniform vertex count exists.*

Definition 25 (UniformBetter)**.** Given $w \in \mathcal{W}$ whose support induces a clique, define

$$w_M := \texttt{UniformBetter}(w)$$

to be the witness provided by Theorem 24: it preserves the zero set of W, its support is a clique, satisfies $f(w_M) \geq f(w)$, and achieves the maximal number $K = \texttt{max\_uniform\_support}(w)$ of vertices with weight $1/k$ (where $k = |\operatorname{supp}(w)|$).

Definition 26 (Carefully chosen ε)**.** Define

$$\texttt{the\_} \ := \ w_{\max} \ - \ \frac{1}{k}.$$

In words: $\texttt{the\_}$ is the difference between the largest vertex weight and the average $1/k$.

Definition 27 (Enhanced Operation)**.** Let $v_{\max}$ and $v_{\min}$ be vertices attaining the maximal and minimal weights of w, respectively. Set $\varepsilon := \texttt{the\_}$ and define

$$w^+ \ := \ \texttt{Enhance}(w, v_{\max}, v_{\min}, \texttt{the\_}).$$

In words: w^+ transfers the carefully chosen ε from the heaviest to the lightest vertex.

Lemma 28. *For any vertex v with $w(v) = \frac{1}{|\operatorname{supp}(w)|}$,*

$$w^+(v) = w(v).$$

In words: vertices already at uniform weight remain unchanged under Enhanced.

Lemma 29. *The weight at the argmax vertex v_j after Enhanced satisfies*

$$w^+(v_j) = \frac{1}{|\operatorname{supp}(w)|}.$$

That is, Enhanced reduces the argmax vertex's weight to the uniform weight.

Lemma 30. *The number of vertices with weight $\frac{1}{|\operatorname{supp}(W)|}$ increases after Enhanced:*

$$|\{v : w^+(v) = 1/|\operatorname{supp}(w)|\}| > |\{v : W(v) = 1/|\operatorname{supp}(w)|\}|.$$

Lemma 31 (Uniform weights on the support)**.** *For every vertex $v \in \operatorname{supp}(w_M)$,*

$$w_M(v) = \frac{1}{|\operatorname{supp}(w_M)|}.$$

That is the weights of all support vertices in UniformBetter are uniform.

Lemma 32 (Edge values under UniformBetter)**.** *For any edge $e = \{v_a, v_b\}$ with $v_a, v_b \in \operatorname{supp}(w_M)$,*

$$\boldsymbol{vp}(w_M, e) = \left(\frac{1}{|\operatorname{supp}(w_M)|}\right)^2.$$

In words: every supported edge has value equal to the square of the uniform vertex weight.

Lemma 33 (Edge count in a clique)**.** *If $|\operatorname{supp}(w_M)| = k$, then*

$$|\{e \in E : e \subseteq \operatorname{supp}(w_M)\}| = \frac{k(k-1)}{2}.$$

Lemma 34 (computation)**.** *For $k > 0$,*

$$\frac{k(k-1)}{2} \cdot \left(\frac{1}{k}\right)^2 = \frac{1}{2}\left(1 - \frac{1}{k}\right).$$

That is the total edge weight for a clique with uniform weights simplifies to $\frac{1}{2}(1 - \frac{1}{k})$.

Lemma 35 (Monotonicity of the bound)**.** *The function*

$$f(k) := \frac{1}{2}\left(1 - \frac{1}{k}\right)$$

is nondecreasing for $k \geq 1$.

Theorem 36 (Final bound inside a clique)**.** *If w is supported on a clique of size $k \leq p-1$, then*

$$f(w) \leq \frac{1}{2}\left(1 - \frac{1}{p-1}\right).$$

In words: the total edge weight is bounded by the Turán bound for cliques of size less than p.

Definition 37 (Uniform weights over all vertices)**.** Define

$$\texttt{UnivFun}(G)(v) := \frac{1}{|V|} \quad \forall v \in V.$$

That is, the uniform vertex weight distribution assigns equal weight $1/|V|$ to every vertex.

Lemma 38 (Total weight under `UnivFun`)**.** *The total edge weight satisfies*

$$(\mathit{UnivFun}(G)).\mathrm{fw} = |E| \cdot \left(\frac{1}{|V|}\right)^2.$$

THat is, the total edge weight under uniform vertex weights equals the number of edges times the square of the uniform weight.

Theorem 39 (Turán's Theorem)**.** *Let $p \geq 2$ and let G be a p-clique-free graph. Then*

$$|E| \leq \frac{1}{2}\left(1 - \frac{1}{p-1}\right)|V|^2.$$