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Turán's Theorem Formalization

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0.1 Concentrating support on a clique - Improve Operartion

Definition 1 (A better distribution)**.** Better Given a weight function W, define $w^* :=$ `Better`(W) such that $\operatorname{supp}(w^*) \subseteq \operatorname{supp}(W)$ and

$$W.\mathrm{fw} \leq w^*.\mathrm{fw}.$$

In words: w^* is a distribution improving or preserving the edge weight and supported within that of W.

Definition 2 (Single transfer)**.** Improve Given distinct vertices $v_j \neq v_i$, define $w' :=$ `Improve`(W, v_j, v_i) by transferring a small amount of weight from v_j (loose) to v_i (gain):

$$w'(v_j) = W(v_j) - \varepsilon, \quad w'(v_i) = W(v_i) + \varepsilon,$$

for some small $\varepsilon > 0$, and $w'(v) = W(v)$ for $v \neq v_i, v_j$. In words: weight is shifted from v_j to v_i preserving total weight.

Lemma 3 (Sum splitting along the partition)**.** *$Improve_{p}artition_{s}um_{s}plitThesumoveredgesE$ of the function* $\boldsymbol{vp}$ *splits as*

$$\sum_{e \in E} \boldsymbol{vp}(w', e) = \sum_{\substack{e \in E \\ e \ni v_i}} \boldsymbol{vp}(w', e) + \sum_{\substack{e \in E \\ e \ni v_j}} \boldsymbol{vp}(w', e) + \sum_{\substack{e \in E \\ e \not\ni v_i, v_j}} \boldsymbol{vp}(w', e).$$

In words: the total sum decomposes into parts incident to v_i, incident to v_j, and the rest.

Lemma 4 (Gain-incidence increases)**.** *$Improve_{g}ain_{c}ontribution_{i}ncreaseTheincrementinthesumoveredgesinc$ is*

$$\sum_{e \ni v_i} \boldsymbol{vp}(w', e) - \sum_{e \ni v_i} \boldsymbol{vp}(W, e) = \varepsilon \sum_{v_k \in N(v_i)} W(v_k),$$

where $N(v_i)$ is the neighborhood of v_i. In words: the gain vertex's contribution increases by ε times the sum of weights of its neighbors.

Lemma 5 (Loose-incidence becomes zero)**.** *$Improve_{l}oose_{c}ontribution_{z}eroThesumoveredgesincidenttov_j$ after the transfer satisfies*

$$\sum_{e \ni v_j} \boldsymbol{vp}(w', e) = 0.$$

In words: the loose vertex's incident edge contributions vanish after the transfer.

Lemma 6 (Unchanged complement)**.** *$Improve_{u}nchanged_{e}dge_{s}umForedgese$ not incident to v_i or v_j,*

$$\boldsymbol{vp}(w', e) = \boldsymbol{vp}(W, e).$$

In words: edges outside the gain and loose neighborhoods remain unchanged.

Lemma 7 (Transfer does not decrease fw)**.** $Improve_totalweight_nondeclem : Improve_partition_sum_split, lem : I$
w'.fw. In words: the total edge weight does not decrease after applying `Improve`.

Lemma 8 (Improve strictly reduces support)**.** $Improve_support_strictly_reduceddef : ImproveIftheneighborhoo$
$|\mathrm{supp}(W)|$. In words: the transfer strictly reduces the number of vertices with positive weight.

Theorem 9 (Support of `Better` is a clique)**.** $Better_forms_cliquedef : Improve, lem : Improve_totalweight_nonde$
`Better`(W) forms a clique:

$$\forall v_a, v_b \in \mathrm{supp}(w^*), v_a \neq v_b \implies \{v_a, v_b\} \in E.$$

In words: every two distinct vertices with positive weight in w^* are adjacent.

0.2 The Enhance Operation

Definition 10 (Enhance)**.** Enhance Given distinct non-adjacent vertices v_j, v_i, define $w' :=$ `Enhance`$(W, v_j, v_i, \varepsilon)$ by transferring $\varepsilon > 0$ weight from v_j to v_i:

$$w'(v_j) = W(v_j) - \varepsilon, \quad w'(v_i) = W(v_i) + \varepsilon, \quad w'(v) = W(v) \text{ for } v \neq v_i, v_j,$$

with the condition $\{v_j, v_i\} \notin E$. In words: Enhance transfers weight between non-adjacent vertices to increase edge weight.

Lemma 11 (Sum over support)**.** $sum_over_supportThetotalvertexweightsatisfies \sum_{v \in \mathrm{supp}(W)} W(v) =$ 1. In words: the weights sum to 1 over the support.

Lemma 12 (Supported edge partition)**.** $supported_edge_partitionTheedgesetE$ *partitions as*

$$E = E_{v_i} \cup E_{v_j} \cup E_{\mathrm{rest}},$$

where

$$E_{v_i} = \{e \in E : v_i \in e\}, \quad E_{v_j} = \{e \in E : v_j \in e\}, \quad E_{\mathrm{rest}} = E \setminus (E_{v_i} \cup E_{v_j}).$$

In words: edges are split into those incident to v_i, to v_j, and the rest.

Lemma 13 (Enhance gain sum)**.** $Enhance_gain_sumUnder$ **??**$, thechangeinthesumoveredgesincidenttov_i$ *satisfies*

$$\sum_{e \in E_{v_i}} \boldsymbol{vp}(w', e) - \sum_{e \in E_{v_i}} \boldsymbol{vp}(W, e) = \varepsilon \sum_{v_k \in N(v_i)} W(v_k).$$

In words: the gain vertex's edge contribution increases by ε times the sum of its neighbors' weights.

Lemma 14 (Enhance loose sum)**.** $Enhance_loose_sumUnder$ **??**$, thesumoveredgesincidenttov_j$ *satisfies*

$$\sum_{e \in E_{v_j}} \boldsymbol{vp}(w', e) = 0.$$

In words: the loose vertex's incident edge contributions become zero after Enhance.

Definition 15 (Bijection inside the clique)**.** the$_{b}ijDefineabijection\phi : \{e \in E_{v_j} \setminus \{s(v_j, v_i)\}\} \to \{e \in E_{v_i} \setminus \{s(v_j, v_i)\}\}$ mapping edges incident to v_j (except $s(v_j, v_i)$) to edges incident to v_i (except $s(v_j, v_i)$). In words: this bijection pairs edges incident to v_j with edges incident to v_i within the clique.

Lemma 16 (Bijection preserves)**.** *$the_{b}ij_{s}ameForanyedgee$ incident to v_j (excluding $s(v_j, v_i)$), the "other" vertex weight satisfies*

$$W(\text{other}(e, v_j)) = W(\text{other}(\phi(e), v_i)).$$

In words: the bijection preserves weights at the other endpoints of edges.

Lemma 17 (Loose/gain equality)**.** $Enhance_{s}um_{l}oose_{g}ain_{e}qualdef : the_{b}ij, lem : the_{b}ij_{s}ameThetotalweighttr$
$\sum_{e \in E_{v_j}} \boldsymbol{vp}(w', e) + \sum_{e \in E_{v_i}} \boldsymbol{vp}(w', e) \geq \sum_{e \in E_{v_j}} \boldsymbol{vp}(W, e) + \sum_{e \in E_{v_i}} \boldsymbol{vp}(W, e)$. In words: the combined edge contributions of loose and gain vertices do not decrease after Enhance.

Lemma 18 (Complement unchanged)**.** *$Enhance_{s}um_{c}omplement_{u}nchangedForedgese \in E_{\text{rest}}$,*

$$\boldsymbol{vp}(w', e) = \boldsymbol{vp}(W, e).$$

In words: edges not incident to v_i or v_j remain unaffected by Enhance.

Lemma 19 (Edge contribution increase)**.** $Enhance_{e}dge_{g}ainloose_{i}ncreaseThenetedgecontributionsatisfies \sum_{e}$
$\sum_{e \in E_{v_i} \cup E_{v_j}} \boldsymbol{vp}(W, e)$. In words: the total contribution from gain and loose vertices does not decrease.

Lemma 20 (Support edges unchanged)**.** *$Enhance_{s}upport_{e}dges_{s}ameForanyvertexv \notin \{v_i, v_j\}$, the edge contributions satisfy*

$$\sum_{e \ni v} \boldsymbol{vp}(w', e) = \sum_{e \ni v} \boldsymbol{vp}(W, e).$$

In words: vertices outside gain and loose retain their edge contributions after Enhance.

Theorem 21 (Enhance increases edge weight)**.** $Enhance_{t}otal_{w}eight_{s}tricinclem : supported_{e}dge_{p}artition, lem$
$w'.\text{fw} > W.\text{fw}$. In words: the Enhance operation strictly improves the total edge weight.

0.3 Equalizing the weights on the clique - EnhanceD

Definition 22 (Carefully chosen ε)**.** the$_{e}psDefine$$\mathsf{the_}\varepsilon := \max\Big\{\varepsilon > 0 \mid w'(v_j) = W(v_j) - \varepsilon \geq 0, \quad w'(v_i) = W$
In words: $\mathsf{the\_}\varepsilon$ is the maximal ε transferring weight from the argmax vertex v_j to the argmin vertex v_i without violating support constraints.

Definition 23 (Maximising the number of uniform vertices)**.** $\max_u niform_s upportDefinem :=$ $\max\{k \mid \exists w$ with $\operatorname{supp}(w) \subseteq \operatorname{supp}(W)$, $w.\mathrm{fw} \geq W.\mathrm{fw}$, and $w(v) = \frac{1}{k}$ for at least k vertices$\}$. In words: m is the maximal number of vertices with uniform weight $1/k$ achievable without decreasing edge weight.

Lemma 24 (Best uniform distribution exists)**.** *$exists_b est_u niformdef : max_u niform_s upportThereexistsw_M$ with $\operatorname{supp}(w_M) \subseteq \operatorname{supp}(W)$, $w_M.\mathrm{fw} \geq W.\mathrm{fw}$, and with at least m vertices having weight $1/m$. In words: a maximiser w_M achieving the maximal uniform vertex count exists.*

Definition 25 (UniformBetter)**.** UniformBetter lem:exists$_b est_u niformDefinew_M :=$ `UniformBetter`(W) as such a maximiser with maximal uniform support.

Definition 26 (Enhanced)**.** Enhanced def:Enhance, def:the$_e psDefinew^+ :=$ `Enhanced`(W) $:=$ `Enhance`$(W, v_j, v_i,$ `the_`$\varepsilon)$, where $v_j = \arg\max_v W(v)$, $v_i = \arg\min_v W(v)$. In words: Enhanced transfers maximal weight from the heaviest to the lightest vertex.

Lemma 27. *$Enhanced_u naffecteddef : Enhance, def : EnhancedForanyvertexv$ with $W(v) = \frac{1}{|\operatorname{supp}(W)|}$,*

$$w^+(v) = W(v).$$

In words: vertices already at uniform weight remain unchanged under Enhanced.

Lemma 28. *$Enhanced_e ffect_a rgmaxdef : Enhance, def : EnhancedTheweightattheargmaxvertexv_j$ after Enhanced satisfies*

$$w^+(v_j) = \frac{1}{|\operatorname{supp}(W)|}.$$

In words: Enhanced reduces the argmax vertex's weight to the uniform level.

Lemma 29. *$Enhanced_i nc_u niform_c ountdef : Enhanced, lem : Enhanced_e ffect_a rgmax, lem : Enhanced_u naf$ increases after Enhanced:*

$$|\{v : w^+(v) = 1/|\operatorname{supp}(W)|\}| > |\{v : W(v) = 1/|\operatorname{supp}(W)|\}|.$$

In words: Enhanced increases the count of uniform weight vertices.

Lemma 30. *def:UniformBetter The support of W forms a clique if and only if the support of $w_M =$ `UniformBetter`(W) forms a clique:*

$$\operatorname{supp}(W) \text{ clique} \iff \operatorname{supp}(w_M) \text{ clique}.$$

In words: the clique property is preserved by UniformBetter.

Lemma 31 (Uniform weights on the support)**.** *$UniformBetter_c onstant_s upportdef : UniformBetter, def : Enh$ $\operatorname{supp}(w_M)$,*

$$w_M(v) = \frac{1}{|\operatorname{supp}(w_M)|}.$$

In words: the weights of all support vertices in UniformBetter are uniform.

Lemma 32 (Edge values under UniformBetter)**.** *$UniformBetter_edges_valuelem : UniformBetter_constant_supp$*
$\{v_a, v_b\}$ with $v_a, v_b \in \mathrm{supp}(w_M)$,

$$\boldsymbol{vp}(w_M, e) = \left(\frac{1}{|\mathrm{supp}(w_M)|}\right)^2.$$

In words: every supported edge has value equal to the square of the uniform vertex weight.

Lemma 33 (Edge count in a clique)**.** *$clique_sizelem : UniformBetter_factsIf |\mathrm{supp}(w_M)| = k$, then*

$$|\{e \in E : e \subseteq \mathrm{supp}(w_M)\}| = \frac{k(k-1)}{2}.$$

In words: the supported edges form a complete graph on k vertices.

Lemma 34 (A light computation)**.** *computation For $k > 0$,*

$$\frac{k(k-1)}{2} \cdot \left(\frac{1}{k}\right)^2 = \frac{1}{2}\left(1 - \frac{1}{k}\right).$$

In words: the total edge weight for a clique with uniform weights simplifies to $\frac{1}{2}(1 - \frac{1}{k})$.

Lemma 35 (Monotonicity of the bound)**.** *bound $bound_realThefunction f(k) :=$* $\frac{1}{2}\left(1 - \frac{1}{k}\right)$ is nondecreasing for $k \geq 1$. In words: the bound increases as the clique size increases.

Theorem 36 (Final bound inside a clique)**.** *$finale_boundlem : Better_forms_clique, lem : Better_non_decr, lem : B$*
is supported on a clique of size $k \leq p - 1$, then

$$W.\mathrm{fw} \leq \frac{1}{2}\left(1 - \frac{1}{p-1}\right).$$

In words: the total edge weight is bounded by the Turán bound for cliques of size less than p.

Definition 37 (Uniform weights over all vertices)**.** UnivFun Define

$$\texttt{UnivFun}(G)(v) := \frac{1}{|V|} \quad \forall v \in V.$$

In words: the uniform vertex weight function assigns equal weight $1/|V|$ to every vertex.

Lemma 38 (Total weight under `UnivFun`)**.** *$UnivFun_weightdef : UnivFunThetotaledgeweightsatisfies(\texttt{UnivF}$*
$|E| \cdot \left(\frac{1}{|V|}\right)^2$. In words: the total edge weight under uniform vertex weights equals the number of edges times the square of the uniform weight.

Theorem 39 (Turán's Theorem)**.** *turans def:UnivFun, lem:UnivFun*$_weight, lem : finale_bound, lem : computation Let p \geq 2$ *and let* G *be a* p*-clique-free graph. Then*

$$|E| \leq \frac{1}{2}\left(1 - \frac{1}{p-1}\right)|V|^2.$$

In words: the number of edges in a p*-clique-free graph is bounded by Turán's theorem.*