https://ro-gut.github.io/turan3 https://github.com/ro-gut/turan3 https://ro-gut.github.io/turan3/docs

Turán's Theorem Formalization

ro-gut

September 17, 2025

0.1 Concentrating support on a clique - Improve Operartion

Definition 1 (A better distribution). Better Given a weight function W, a choice of a weight function `Better` W with supp(`Better` W) $\subseteq$ supp(W) and W.fw $\leq$ (`Better` W).fw.

Definition 2 (Single transfer). Improve Given distinct vertices `loose` $\neq$ `gain`, the weight function `Improve` W `loose` `gain` moves a small amount from `loose` to `gain`.

Lemma 3 (Sum splitting along the partition). *$Improve_partition_sum_splitSumming$* *vp* *over E splits as the sum over the gain-incidence, plus the loose-incidence, plus the complement.*

Lemma 4 (Gain-incidence increases). *$Improve_gain_contribution_increaseThesumonthegain-incidenceincreasesbyW.w$ `loose` times the sum of the other-endpoint weights incident to ‘gain‘.*

Lemma 5 (Loose-incidence becomes zero). *$Improve_loose_contribution_zeroThesumontheloose-incidenceiszeroafter$* *`Improve`.*

Lemma 6 (Unchanged complement). *$Improve_unchanged_edge_sumEdgesoutsidetheunionofgain/looseincidenc$*

Lemma 7 (Transfer does not decrease fw). *$Improve_total_weight_nondeclem : Improve_partition_sum_split, lem : I$* *(`Improve` W `loose` `gain`).fw.*

Lemma 8 (Improve strictly reduces support). *$Improve_support_strictly_reduceddef : ImproveIftheneighbourho$*

Theorem 9 (Support of `Better` is a clique). *$Better_forms_cliquedef : Improve, lem : Improve_total_weight_nonde$* *$everytwodistinctverticesofpositiveweightareadjacentinG$.*

0.2 The Enhance Operation

Definition 10 (Enhance). Enhance Defines the operation of transferring weight from one vertex to another, provided the two vertices are non-adjacent. This operation is central to the second phase of the proof, where we later reduce the support size while ensuring the edge weight does not decrease.

Lemma 11 (Sum over support). *$sum_over_supportExpressesthetotalvertexweightasthesumofweightsoverthes$*

Lemma 12 (Supported edge partition). *$supported_edge_partitionSplitstheedgesetintoedgesincidenttothechosen$*

Lemma 13 (Enhance gain sum). $Enhance_gain_sumShowsthatunder$ **??**, $thecontributionofthegainvertexsedges$

Lemma 14 (Enhance loose sum). $Enhance_loose_sumShowsthatunder$ **??**, $thecontributionofthelooseverte xsedg$

Definition 15 (Bijection inside the clique). the$_bijProvidesabijectionbetweenthesupportedincidenceedgesat`lo$

Lemma 16 (Bijection preserves). $the_bij_sameShowsthatthebijectionpreservesthe"other"weight :$
$foranyedgefromthesupportedincidencesetofloose, theweighatthe"other"vertexequalsthatinitsimageunedert$

Lemma 17 (Loose/gain equality). $Enhance_sum_loose_gain_equalShowsthatthetotalweightmovedfromtheloosev$

Lemma 18 (Complement unchanged). $Enhance_sum_complement_unchangedShowsthatedgesnotincidenttogain$

Lemma 19 (Edge contribution increase). $Enhance_edge_gainloose_increaseProvesthatthenetcontributionfromg$

Lemma 20 (Support edges unchanged). $Enhance_support_edges_sameShowsthatforverticesoutsideofgainandlo$

Theorem 21 (Enhance increases edge weight). $Enhance_total_weight_stricinclem : supported_edge_partition, lem :$

0.3 Equalizing the weights on the clique - EnhanceD

Definition 22 (Carefully chosen ε). the$_epsDefine$`the_`$\varepsilon := \max - \frac{1}{|\mathrm{supp}|}$.

Definition 23 (Maximising the number of uniform vertices). $\max_uniform_supportDefinethemaximal$ m $achiev$

Lemma 24 (Best uniform distribution exists). $exists_best_uniformdef : max_uniform_supportThereexistsadistr$
W.fw.

Definition 25 (UniformBetter). UniformBetter lem:exists$_best_uniformAchoiceofamaximiserfrom$**??**.

Definition 26 (Enhanced)**.** Enhanced def:Enhance, def:the$_e psDefinesthe$Enhanced$weightfunction:$ $transferringweightfromtheargmaxvertex$loose$totheargminvertex$gain$, using ??withtheamount$the_$\varepsilon$.

Lemma 27. *$Enhanced_unaffecteddef: Enhance, def: EnhancedShowsthatunder$Enhanced$everyvertexthato$*

Lemma 28. *$Enhanced_effect_argmaxdef: Enhance, def: EnhancedShowsthattheweightattheargmaxvertex$*

Lemma 29. *$Enhanced_inc_uniform_countdef: Enhanced, lem: Enhanced_effect_argmax, lem: Enhanced_unaf$*

Lemma 30. *def:UniformBetter The support of W forms a clique if and only if the support at UniformBetter also forms a clique.*

Lemma 31 (Uniform weights on the support)**.** *$UniformBetter_constant_supportdef: UniformBetter, def: Enh$*

Lemma 32 (Edge values under UniformBetter)**.** *$UniformBetter_edges_valuelem: UniformBetter_constant_suppo$*

Lemma 33 (Edge count in a clique)**.** *$clique_sizelem: UniformBetter_factsIfthesupporthassizek, thenthenum$ 1)/2.*

Lemma 34 (A light computation)**.** *computation $\big(k(k-1)/2\big)\cdot(1/k)^2 = \frac{1}{2}\,(1 - 1/k)$ for $k > 0$.*

Lemma 35 (Monotonicity of the bound)**.** *bound $bound_realThefunctionk\mapsto$ $\frac{1}{2}(1-\frac{1}{k})$ is nondecreasing in k (for $k \geq 1$).*

Theorem 36 (Final bound inside a clique)**.** *$finale_boundlem: Better_non_decr, lem: Better_forms_clique, lem: U$ $p-1$, then*

$$W.\mathrm{fw} \;\le\; \tfrac{1}{2}\,(1 - 1/(p-1)).$$

Definition 37 (Uniform weights over all vertices)**.** UnivFun The uniform vertex-weight function assigning $1/|V|$ to each vertex.

Lemma 38 (Total weight under `UnivFun`)**.** *$UnivFun_weightdef: UnivFun$(`UnivFun` G).fw=$\#E\cdot(1/|V|)^2$.*

Theorem 39 (Turán's Theorem)**.** *turans def:UnivFun, lem:$UnivFun_weight, lem:$ $finale_bound, lem: computationLetp\geq 2$ and let G be a p-clique-free graph. Then*

$$\#E \;\le\; \tfrac{1}{2}\Big(1 - \tfrac{1}{p-1}\Big)\,(\#V)^2.$$