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Turán's Theorem Formalization

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September 16, 2025

0.1 Concentrating support on a clique - Improve Operation

Definition 1 (A better distribution). *Better* Given a weight function W , a choice of a weight function **Better** W with $\text{supp}(\text{Better } W) \subseteq \text{supp}(W)$ and $W.\text{fw} \leq (\text{Better } W).\text{fw}$.

Definition 2 (Single transfer). *Improve* Given distinct vertices $\text{loose} \neq \text{gain}$, the weight function **Improve** W $\text{loose} \text{ gain}$ moves a small amount from loose to gain .

Lemma 3 (Sum splitting along the partition). *Improve_{partition}sum_{split}lem : Improve_{edge}Finset_{partition}Sum_{over E} splits as the sum over the gain-incidence, plus the loose-incidence, plus the complement.*

Lemma 4 (Gain-incidence increases). *Improve_{gain}contribution_{increase}The sum on the gain-incidence increases by $W.w \text{ loose}$ times the sum of the other-endpoint weights incident to ‘gain’.*

Lemma 5 (Loose-incidence becomes zero). *Improve_{loose}contribution_{zero}The sum on the loose-incidence is zero after **Improve**.*

Lemma 6 (Unchanged complement). *Improve_{unchanged}edge_{sum}Edges outside the union of gain/loose incidence*

Lemma 7 (Transfer does not decrease fw). *Improve_{total_wweight_nondeclem : Improve_{partition}sum_{split}, lem : Improve_{total_wweight_nondeclem} (Improve $W \text{ loose gain}$).fw.}*

Lemma 8 (Improve strictly reduces support). *Improve_{support_{strictly}, reduceddef : Improve, lem : Improve_{support_{strictly}, reduceddef}}*

Theorem 9 (Support of **Better** is a clique). *Better_{forms_{clique}def : Improve, lem : Improve_{total_wweight_nondeclem} every two distinct vertices of positive weight are adjacent in G .}*

0.2 The Enhance Operation

Definition 10 (Enhance). *Enhance* Defines the operation of transferring weight from one vertex to another, provided the two vertices are non-adjacent. This operation is central to the second phase of the proof, where we later reduce the support size while ensuring the edge weight does not decrease.

Lemma 11 (Sum over support). *sum_{over_ssupport}Expresses the total vertex weight as the sum of weights over the support*

Lemma 12 (Supported edge partition). *supported_{edge}partition.Splits the edgeset into edges incident to the chosen vertex*

Lemma 13 (Enhance gain sum). *Enhance_{gain_s}um*Showsthatunder??, thecontributionofthegainvertexesedges

Lemma 14 (Enhance loose sum). *Enhance_{loose_s}um*Showsthatunder??, thecontributionoftheloosevertexesedges

Definition 15 (Bijection inside the clique). *the_{bij}*Providesabijectionbetweenthesupportedincidenceedgesat'lo

Lemma 16 (Bijection preserves). *the_{bij_s}ame*Showsthatthebijectionpreservesthe"other"weight :
foranyedgefromthesupportedincidencesetofloose, theweighatthe"other"vertexequalsthatinitsimageunder

Lemma 17 (Loose/gain equality). *Enhance_sum_{loose_g}ain_equal*Showsthatthetotalweightmovedfromtheloosev

Lemma 18 (Complement unchanged). *Enhance_sum_complement_unchanged*Showsthatedgesnotincidenttogain

Lemma 19 (Edge contribution increase). *Enhance_edge_gain_{loose_i}ncrease*Provesthatthenetcontributionfromg

Lemma 20 (Support edges unchanged). *Enhance_support_edges_same*Showsthatforverticesoutsideofgainandlo

Theorem 21 (Enhance increases edge weight). *Enhance_iotal_weight_strinclem* : supported_edge_partition, lem

0.3 Equalizing the weights on the clique - Enhanced

Definition 22 (Carefully chosen ε). *the*Define $0 := \max -1/|\text{supp}|$.

Definition 23 (Maximising the number of uniform vertices). *max_uni form_support*Definethemaximalmachiev

Lemma 24 (Best uniform distribution exists). *exists_{best_u}ni form_def* : max_uni form_supportThereexistsadistr
W.fw.

Definition 25 (UniformBetter). *UniformBetter* lem:exists_{best_u}ni formAchoiceofamaximiserfrom??.

Definition 26 (Enhanced). *Enhanced def:Enhance, def:the Defines ‘Enhanced’ weight function : transferring weight from the argmax vertex ‘lose’ to the argmin vertex ‘gain’, using the previous in Section 2 defining*

Lemma 27. *Enhanced_unaffected def : Enhance, def : Enhanced Show that under ‘Enhanced’ every vertex that*

Lemma 28. *Enhanced_effect_{argmax} def : Enhance, def : Enhanced Show that the weight at the argmax vertex*

Lemma 29. *Enhanced_inc_uniform_count def : Enhanced, lem : Enhanced_effect_{argmax}, lem : Enhanced_una*

Lemma 30. *def:UniformBetter The support of W forms a clique if and only if the support at UniformBetter also forms a clique.*

Lemma 31 (Uniform weights on the support). *UniformBetter_constant_support lem : UniformBetter, def : En*

Lemma 32 (Edge values under UniformBetter). *UniformBetter_edges_value lem : UniformBetter_constant_supp*

Lemma 33 (Edge count in a clique). *clique_size lem : UniformBetter_facts If the support has size k , then the number of edges is $k(k-1)/2$.*

Lemma 34 (A light computation). *computation $(k(k-1)/2) \cdot (1/k)^2 = \frac{1}{2} (1 - 1/k)$ for $k > 0$.*

Lemma 35 (Monotonicity of the bound). *bound bound_real The function $k \mapsto \frac{1}{2}(1 - \frac{1}{k})$ is nondecreasing in k (for $k \geq 1$).*

Theorem 36 (Final bound inside a clique). *finale_bound lem : Better_nond_ecr, lem : Better_forms_clique, lem : U*
 $p - 1$, then

$$W.\text{fw} \leq \frac{1}{2} (1 - 1/(p - 1)).$$

Definition 37 (Uniform weights over all vertices). *UnivFun The uniform vertex-weight function assigning $1/|V|$ to each vertex.*

Lemma 38 (Total weight under UnivFun). *UnivFun_weight def : UnivFun(UnivFun G).fw = $\#E \cdot (1/|V|)^2$.*

Theorem 39 (Turán’s Theorem). *turans def:UnivFun, lem:UnivFun_weight, lem : finale_bound, lem : computation Let $p \geq 2$ and let G be a p -clique-free graph. Then*

$$\#E \leq \frac{1}{2} \left(1 - \frac{1}{p-1} \right) (\#V)^2.$$