Turán's Theorem Formalization

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0.1 Concentrating support on a clique - Improve Operartion

Definition 1 (A better distribution)**.** Given a weight function W, a choice of a weight function `Better` W with supp(`Better` W) $\subseteq$ supp(W) and W.fw $\leq$ (`Better` W).fw.

Definition 2 (Single transfer)**.** Given distinct vertices `loose` $\neq$ `gain`, the weight function `Improve` W `loose` `gain` moves a small amount from `loose` to `gain`.

Lemma 3 (Sum splitting along the partition)**.** *Summing `vp` over E splits as the sum over the gain-incidence, plus the loose-incidence, plus the complement.*

Lemma 4 (Gain-incidence increases)**.** *The sum on the gain-incidence increases by $W.w$ `loose` times the sum of the other-endpoint weights incident to 'gain'.*

Lemma 5 (Loose-incidence becomes zero)**.** *The sum on the loose-incidence is zero after `Improve`.*

Lemma 6 (Unchanged complement)**.** *Edges outside the union of gain/loose incidences keep the same value under `Improve`.*

Lemma 7 (Transfer does not decrease fw)**.** *Under the neighbourhood condition used in the code, W.fw $\leq$ (`Improve` W `loose` `gain`).fw.*

Lemma 8 (Improve strictly reduces support)**.** *If the neighbourhood condition fails, the transfer strictly reduces the support of W.*

Theorem 9 (Support of `Better` is a clique)**.** *The support of (`Better`W) forms a clique: every two distinct vertices of positive weight are adjacent in G.*

0.2 The Enhance Operation

Definition 10 (Enhance)**.** Defines the operation of transferring weight from one vertex to another, provided the two vertices are non-adjacent. This operation is central to the second phase of the proof, where we later reduce the support size while ensuring the edge weight does not decrease.

Lemma 11 (Sum over support)**.** *Expresses the total vertex weight as the sum of weights over the support.*

Lemma 12 (Supported edge partition)**.** *Splits the edge set into edges incident to the chosen vertices ("gain" and "loose") and the remaining edges.*

Lemma 13 (Enhance gain sum)**.** *Shows that under* **??***, the contribution of the gain vertex's edges increases by exactly the transferred weight multiplied by the sum of its neighbor weights.*

Lemma 14 (Enhance loose sum)**.** *Shows that under* **??***, the contribution of the loose vertex's edges becomes zero.*

Definition 15 (Bijection inside the clique)**.** Provides a bijection between the supported incidence edges at 'loose' (without s(loose, gain)) and the supported incidence edges at 'gain' (without s(loose, gain))

Lemma 16 (Bijection preserves)**.** *Shows that the bijection preserves the "other" weight: for any edge from the supported incidence set of loose, the weigh at the "other" vertex equals that in its image uneder the bijection*

Lemma 17 (Loose/gain equality). *Shows that the total weight moved from the loose vertex to the gain vertex balances correctly in the edge contributions.*

Lemma 18 (Complement unchanged). *Shows that edges not incident to gain or loose are unaffected by* **??**.

Lemma 19 (Edge contribution increase). *Proves that the net contribution from gain and loose together does not decrease.*

Lemma 20 (Support edges unchanged). *Shows that for vertices outside of gain and loose, the contributions remain identical before and after applying* **??**.

Theorem 21 (Enhance increases edge weight). *Establishes that applying* **??** *strictly increases the total edge weight*

0.3 Equalizing the weights on the clique - EnhanceD

Definition 22 (Carefully chosen ε). Define `the_` $:= \max - \frac{1}{|\mathrm{supp}|}$.

Definition 23 (Maximising the number of uniform vertices). Define the maximal m achievable by any distribution with support contained in that of W and with at least as much edge weight.

Lemma 24 (Best uniform distribution exists). *There exists a distribution realising this maximum with edge weight* $\geq W.\mathrm{fw}$.

Definition 25 (UniformBetter). A choice of a maximiser from **??**.

Definition 26 (Enhanced). Defines ‘Enhanced‘ weight function : transfering weight from the argmax vertex ‘loose‘ to the argmin vertex ‘gain‘, using the previous in Section 2 defined function ‘Enhance‘ by the amount defined ‘the$_\varepsilon^\grave{}$

Lemma 27. *Shows that under ‘Enhanced‘ every vertex that originally had weight 1/|support|, remains with the same weight*

Lemma 28. *Shows that the weight at the argmax vertex becomes the target uniform weight after Enhanced.*

Lemma 29. *Shows that the number of uniform weight vertices increases after Enhanced.*

Lemma 30. *The support of W forms a clique if and only if the support at UniformBetter also forms a clique.*

Lemma 31 (Uniform weights on the support). *Every support vertex of* `UniformBetter`W *has weight* $1/|\mathrm{supp}|$.

Lemma 32 (Edge values under UniformBetter). *Every supported edge has value* $(1/|\mathrm{supp}|)^2$.

Lemma 33 (Edge count in a clique). *If the support has size k, then the number of supported edges is* $k(k-1)/2$.

Lemma 34 (A light computation). $(k(k-1)/2) \cdot (1/k)^2 = \frac{1}{2}(1 - 1/k)$ *for* $k > 0$.

Lemma 35 (Monotonicity of the bound). *The function* $k \mapsto \frac{1}{2}(1 - \frac{1}{k})$ *is nondecreasing in k (for $k \geq 1$).*

Theorem 36 (Final bound inside a clique)**.** *If W is supported on a clique of size $k \le p-1$, then*

$$W.\mathrm{fw} \;\le\; \tfrac{1}{2}\,(1 - 1/(p-1)).$$

Definition 37 (Uniform weights over all vertices)**.** The uniform vertex-weight function assigning $1/|V|$ to each vertex.

Lemma 38 (Total weight under `UnivFun`)**.** $(\mathit{UnivFun}\, G).\mathrm{fw} = \#E \cdot (1/|V|)^2$.

Theorem 39 (Turán's Theorem)**.** *Let $p \ge 2$ and let G be a p-clique-free graph. Then*

$$\#E \;\le\; \tfrac{1}{2}\left(1 - \tfrac{1}{p-1}\right)(\#V)^2.$$