Turán's Theorem Formalization - 3rd Proof

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0.1 Overvieww

Theorem 1 (Turán's Theorem)**.** *If a graph $G = (V, E)$ on n vertices has no p-clique ($p \geq 2$), then*

$$|E| \;\leq\; \Big(1 - \frac{1}{p-1}\Big)\,\frac{n^2}{2}.$$

Set–up. Fix a probability distribution $w = (w_1, \ldots, w_n)$ on the vertices: $w_i \geq 0$ and $\sum_i w_i = 1$. Define the objective

$$f(w) \;=\; \sum_{v_i v_j \in E} w_i w_j,$$

the total "edge weight" of G under w. Our goal is to find the maximal possible value of $f(w)$ subject to $\sum_i w_i = 1$ and $w_i \geq 0$.

1. Concentrate positive weight on a clique. Take any distribution w and two *nonadjacent* vertices v_i, v_j with positive weights. Let s_i be the sum of the weights of the neighbors of v_i, and define s_j similarly. Assume $s_i \geq s_j$ and move the whole weight of v_j onto v_i, obtaining w' with $w'_i = w_i + w_j$ and $w'_j = 0$. A direct calculation gives

$$f(w') \;=\; f(w) + w_j(s_i - s_j) \;\geq\; f(w),$$

so this operation never decreases f. Repeating the move eliminates one positive entry each time until no two positive–weight vertices are nonadjacent; i.e. in an *optimal* distribution the support of w is a clique.

2. Equalize weights inside the clique. Suppose the support is a k–clique and the positive weights are not all equal; say $w_1 > w_2 > 0$. Choose $0 < \varepsilon < w_1 - w_2$ and shift ε from v_1 to v_2: the new distribution w' satisfies

$$f(w') \;=\; f(w) + \varepsilon(w_1 - w_2) - \varepsilon^2 \;>\; f(w).$$

Thus f is strictly improved until all positive weights on the k–clique are equal, $w_i = \frac{1}{k}$ there and 0 elsewhere. For that distribution,

$$f(w) \;=\; \frac{\binom{k}{2}}{k^2} \;=\; \frac{1}{2}\Big(1 - \frac{1}{k}\Big),$$

which is increasing in k. Since G has no p–clique, we must have $k \leq p - 1$, so the best possible value is $\frac{1}{2}(1 - \frac{1}{p-1})$.

3. Conclude the edge bound. For the uniform distribution $w_i = \frac{1}{n}$, we have $f(w) = |E|/n^2$. Hence

$$\frac{|E|}{n^2} \;\leq\; \frac{1}{2}\Big(1 - \frac{1}{p-1}\Big),$$

which is exactly the desired inequality.

Definition 2. A weight function $w : V \to \mathbb{R}_{\geq 0}$ with $\sum_{v \in V} w(v) = 1$.

Theorem 3. *Main goal statement goes here.*

Proof. Sketch the argument here. When fully formal: □

0.2 Preliminaries and notation

Throughout, fix a finite simple graph $G = (V, E)$ with $|V| = n$. We write $\operatorname{supp}(w) := \{v \in V \mid w(v) > 0\}$.

Definition 4 (Weight distributions)**.** A *weight distribution* on G is a function

$$w : V \to \mathbb{R}_{\geq 0} \qquad \text{such that} \qquad \sum_{v \in V} w(v) = 1.$$

In Lean this is the structure `FunToMax` G with fields $w : \alpha \to \mathbb{R}_{\geq 0}$ and $\sum_{v \in V} w(v) = 1$.

Definition 5 (Edge contribution)**.** For a distribution w, the contribution of an (unordered) edge $\{u, v\} \in E$ is

$$\operatorname{vp}(\{u, v\}) := w(u)\, w(v).$$

In Lean this is defined on $\operatorname{Sym}^2(\alpha)$ via the quotient lift.

Definition 6 (Total edge weight)**.** Given a distribution w, the total edge weight is

$$f(w) \;:=\; \sum_{e \in E} \operatorname{vp}(e) \;=\; \sum_{\{u,v\} \in E} w(u)\, w(v).$$

0.3 Minimal–support maximizers (what the code proves)

Theorem 7 (Existence of a non-decreasing distribution with controlled support)**.** *For every distribution w there exist a natural number $m \in \mathbb{N}$ and a distribution w^* such that*

1. $\forall i \in V,\ w(i) = 0 \Rightarrow w^*(i) = 0$ *(zeros are preserved);*
2. $|\{i \in V \mid w^*(i) > 0\}| = m$ *(the support has size m);*
3. $f(w) \leq f(w^*)$ *(the total edge weight does not decrease).*

Definition 8 (Minimal support size m)**.** Let m be the least natural number m for which a distribution w^* as in Theorem 7 exists with $|\operatorname{supp}(w^*)| = m$ and $f(w) \leq f(w^*)$. (Implemented as `Nat.find` of the existence statement.)

Definition 9 (A chosen minimizer Better(W))**.** Fix one such distribution w^* with $|\operatorname{supp}(w^*)| = m$ and $f(W) \leq f(\text{Better}(w^*))$.

Lemma 10 (Properties of Betterw^*)**.**

1. ***Zeros preserved:*** *If $w(i) = 0$ then $w^*(i) = 0$.* *(Lean: `Better_support_included`)*
2. ***Same support size:*** $|\operatorname{supp}(w^*)| = m$. *(Lean: `Better_support_size`)*
3. ***Non-decreasing:*** $f(w) \leq f(w^*$. *(Lean: `Better_non_decr`)*

0.4 Inducing a clique: the $w^{'}$ move

In this section we formalize the first weight–moving step from the proof: given two non–adjacent support vertices, we move all weight from one to the other. This does not decrease the objective and strictly reduces the support size, which will eventually force the support to be a clique.

Definition 11. Let W be a weight function (Definition 4), and let v_j (*loose*) and v_i (*gain*) be distinct vertices. Define the transferred weight function w' by

$$w'(x) \;:=\; \begin{cases} 0, & x = v_j, \\ w(v_i) + w(v_j), & x = v_i, \\ w(x), & \text{otherwise.} \end{cases}$$

We write $w' = \texttt{Improve}$ (*Lean name* `Improve`, with parameters *loose*=v_j and *gain*=v_i).

Lemma 12 (v_i lies in each incident edge)**.** *If* $e \in \mathrm{Inc}(v_i)$ *then* $v_i \in e$*.*

Lemma 13 (Edge value factorization at a fixed endpoint)**.** *For* $e \in \mathrm{Inc}(v_i)$*,*

$$\mathrm{vp}(e) \;=\; w(v_i) \cdot w(v_j)).$$

Lemma 14 (Summing over the incidence set of v_i)**.**

$$\sum_{e \in \mathrm{Inc}(v_i)} \mathrm{vp}(e) \;=\; w(v_i) \cdot \sum_{e \in \mathrm{Inc}(v_i)} w\bigl(\mathrm{other}_e(v_i)\bigr).$$

Lemma 15. *The analogous identity holds with* loose *in place of* gain*.*

Lemma 16 (Adjacency vs. edge membership)**.** *For any* $v, w \in V$*,*

$$v \sim w \quad\Longleftrightarrow\quad \exists e \in E,\; e = \{v, w\}.$$

Lemma 17 (Incidence is contained in the edge set)**.** $\mathrm{Inc}(v) \subseteq E$ *for all* $v \in V$*.*

Lemma 18 (Effect at v_j)**.** *After the transfer, the new weight at* v_j *is* $w'(v_j) = 0$*.*

Lemma 19 (Disjoint incidences when nonadjacent)**.** *If* $v_i \neq v_j$ *and* $v_i \nsim v_j$*, then* $\mathrm{Inc}(v_i)$ *and* $\mathrm{Inc}(v_j)$ *are disjoint.*

Lemma 20 (Partition of edges by changed vs. unchanged)**.** *Let*

$$\mathrm{changed} \;:=\; \mathrm{Inc}(v_i) \;\uplus\; \mathrm{Inc}(v_j)$$

(the disjoint union, by Lemma 19). Then

$$E \;=\; \mathrm{changed} \;\uplus\; (E \setminus \mathrm{changed}).$$

Lemma 21 (Sum split across the partition)**.**

$$\sum_{e\in E} \mathrm{vp}(W,e) \;=\; \sum_{e\in \mathrm{Inc}(v_i)} \mathrm{vp}(e) \;+\; \sum_{e\in \mathrm{Inc}(v_j)} \mathrm{vp}(e) \;+\; \sum_{e\in E\setminus \mathrm{changed}} \mathrm{vp}(e).$$

Lemma 22 (Gain's contribution increases)**.**

$$\sum_{e\in \mathrm{Inc}(v_i)} \mathrm{vp}(w') \;=\; \sum_{e\in \mathrm{Inc}(v_i)} \mathrm{vp}(w) \;+\; w(v_j)\cdot \sum_{e\in \mathrm{Inc}(v_i)} w\big(\mathrm{other}_e(v_i)\big).$$

Lemma 23 (Loose's contribution vanishes)**.**

$$\sum_{e\in \mathrm{Inc}(loose)} \mathrm{vp}(w',e) \;=\; 0.$$

Lemma 24 (Unchanged outside the changed part)**.**

$$\sum_{e\in E\setminus \mathrm{changed}} \mathrm{vp}(w') \;=\; \sum_{e\in E\setminus \mathrm{changed}} \mathrm{vp}(w).$$

Lemma 25 (Monotonicity under one transfer)**.** *Assume*

$$\sum_{e\in \mathrm{Inc}(v_i)} w\big(\mathrm{other}_e(v_i)\big) \;\geq\; \sum_{e\in \mathrm{Inc}(v_j)} w\big(\mathrm{other}_e(v_j)\big),$$

and $v_i \nsim v_j$. *Then*

$$\sum_{e\in E} \mathrm{vp}(w') \;\geq\; \sum_{e\in E} \mathrm{vp}(w).$$

Equivalently, $f(w') \geq f(w)$.

Lemma 26 (Zeros stay zero)**.** *If* $w(x) = 0$ *then* $w'(x) = 0$.

$$\{\, v \in V \mid w'(v) > 0 \,\} \;\subseteq\; \{\, v \in V \mid w(v) > 0 \,\}.$$

Lemma 27 (Support strictly shrinks)**.** *If* $w(v_i) > 0$ *and* $w(v_j) > 0$*, then*

$$\#\{\, i \mid w'(i) > 0 \,\} \;<\; \#\{\, i \mid w(i) > 0 \,\}.$$

Lemma 28 (Support of w^* is a clique)**.** *Let* w *be any weight function. Then the* $\mathrm{supp}(w^*)$ *induces a clique:*

Proof. Assume the contrary. Then two support vertices v_i, v_j are non–adjacent. Without loss of generality assume that the neighbor–sum of v_i is at least as large as that of v_j. Applying the w' move:

- By Lemma 25, the total weight $f(w')$ does not decrease. - By Lemma 27, the support size strictly decreases.

Thus we obtain an "even better" distribution contradicting the minimality of w^*'s support size (Lemma **??**). Therefore no such non–adjacent pair can exist, and the support of w^* is a clique. $\square$

0.5 Second weight moving argument: `Enhance` w^+

Our goal here is to show that unless the weights are already uniform, we can improve the distribution further by transferring a small amount of weight.

Lemma 29 (Support nonempty). *For any weight distribution w, its support is nonempty:*

$$\{v \in V \mid w(v) > 0\} \neq \emptyset.$$

Definition 30 (Uniform clique improvement). For any weight distribution w, we say $\texttt{exists\_uniform\_clique}(w)(m)$ if there exists $w' \in \mathcal{W}$ such that: - $w(v) = 0 \iff w'(v) = 0$, - $\mathrm{supp}(w')$ is a clique, - exactly m vertices have weight $1/m$, - and $f(w) \leq f(w')$.

Definition 31 (Maximal uniform support). For any w:

$$\texttt{max\_uniform\_support}(w) := \max\{m \mid \texttt{exists\_uniform\_clique}(w)(m)\}.$$

Lemma 32 (Existence of best uniform clique). *If the support of w is a clique, then*

$$\texttt{exists\_uniform\_clique}(w)(\texttt{max\_uniform\_support}(w))$$

holds. That is, there exists w' with the same support (a clique), all weights uniform, and $f(w) \leq f(w')$.

Definition 33 (UniformBetter). Given w with clique support, define

$$\texttt{UniformBetter}$$

as the uniformizing distribution $w^{mathrmunif}$ guaranteed by Lemma 32.

Lemma 34 (Support preserved). *For every vertex v,*

$$w(v) = 0 \iff w^{mathrmunif}(v) = 0.$$

Lemma 35 (Support size uniform). *The number of vertices of weight $1/m$ in $\texttt{UniformBetter}(w)$ is exactly*

$$m = \texttt{max\_uniform\_support}(w).$$

Lemma 36 (Edge weight monotonicity). *We have*

$$f(w) \leq f(w^{mathrmunif}).$$

Lemma 37 (Support clique). *The support of $w^{mathrmunif}$ forms a clique.*

Definition 38 (`Enhance`: small ε–transfer). Given distinct vertices v_j (*loose*) and v_i (*gain*) with $w(v_i) < w(v_j)$ and a parameter ε satisfying $0 < \varepsilon < w(v_j) - w(v_i)$, define w^+ by

$$w^+(x) = \begin{cases} w(v_j) - \varepsilon, & x = v_j, \\ w(v_i) + \varepsilon, & x = v_i, \\ w(x), & \text{otherwise.} \end{cases}$$

Then $\sum_x w^+(x) = 1$.

Lemma 39 (Different weights $\Rightarrow$ different vertices). *If $w(v_i) < w(v_j)$ then $v_i \neq v_j$.*

Lemma 40 (Nonpositive ⇒ zero in $\mathbb{R}_{\geq 0}$)**.** *For* $x \in \mathbb{R}_{\geq 0}$*, if* $\neg(x > 0)$ *then* $x = 0$*.*

Lemma 41 (Zeros are preserved by `Enhance`)**.** *For every vertex* u*,*

$$w(u) = 0 \quad \Longleftrightarrow \quad w^+(u) = 0,$$

where w^+ *is the distribution from Definition 38.*

Lemma 42 (Positivity is preserved by `Enhance`)**.** *For every vertex* u*,*

$$w(u) > 0 \quad \Longleftrightarrow \quad w^+(u) > 0.$$

Lemma 43 (`Enhance` keeps the support a clique)**.** *If* $\mathrm{supp}(w)$ *induces a clique, then* $\mathrm{supp}(w^+)$ *also induces a clique.*

0.5.1 Sym2, SimpleGraph technicalities

Definition 44 (Supported edge)**.** An edge $e = \{x, y\} \in$ $operatornameSym^2(\alpha)$ is in the support of w if and only if both endpoints have positive weight:

$$e \in \mathrm{supp}(W) \ :\Longleftrightarrow \ W(x) > 0 \ \wedge \ W(y) > 0.$$

Lemma 45 (Explicit form of supported edges)**.** *For vertices* x, y*,*

$$(x, y).inSupport(W) \quad \Longleftrightarrow \quad W(x) > 0 \wedge W(y) > 0.$$

Lemma 46 (Edges outside support vanish)**.** *If* $e \notin \mathrm{supp}(W)$*, then its contribution is zero:*

$$\mathrm{vp}(W, e) = 0.$$

Lemma 47 (Membership ⇒ positivity)**.** *If* $e \in \mathrm{supp}(W)$ *and* $x \in e$*, then* $W(x) > 0$*.*

Lemma 48 (Other endpoint positive)**.** *If* $e \in \mathrm{supp}(W)$ *and* $x \in e$*, then the opposite endpoint in* e *also has positive weight.*

Lemma 49 (Positivity ⇒ in-support)**.** *If* $W(x) > 0$ *for all* $x \in e$*, then* $e \in \mathrm{supp}(W)$*.*

Definition 50 (Supported incidence set)**.** For a vertex v, the supported incidence set is

$$\texttt{supIncidenceFinset}(W, v) := \{e \in \texttt{incidenceFinset}(v) : e \in \mathrm{supp}(W)\}.$$

Definition 51 (Supported edge set)**.** The supported edge set is

$$(W) := \{e \in \texttt{edgeFinset} : e \in \mathrm{supp}(W)\}.$$

Lemma 52 (Characterization of supported incidence)**.**

$$e \in \textit{supIncidenceFinset}(W, v) \quad \Longleftrightarrow \quad e \in G.\textit{incidenceFinset}(v) \ \wedge \ e \in \mathrm{supp}(W).$$

Lemma 53 (Characterization of supported edges)**.**

$$e \in \textit{supEdgeFinset}(W) \quad \Longleftrightarrow \quad e \in \textit{supEdgeFinset} \ \wedge \ e \in \mathrm{supp}(W).$$

Lemma 54 (Supported incidence ⊆ incidence)**.**

$$\textit{supIncidenceFinset}(W, v) \subseteq \textit{incidenceFinset}(v).$$

Lemma 55 (Element of $s \setminus t$ is in s)**.** *If $a \in s \setminus t$, then $a \in s$.*

Lemma 56 (Supported incidences of v_i and v_j are disjoint (ignoring $\{v_i, v_j\}$))**.** *If $v_i \neq v_j$, then*

$\mathtt{supIncidenceFinset}(w, v_i)\setminus\{\{v_i, v_j\}\}$ *and* $\mathtt{supIncidenceFinset}(w, v_j)\setminus\{\{v_i, v_j\}\}$ *are disjoint.*

Definition 57 (Disjoint union of the two supported incidence sets)**.** Given $v_i \neq v_j$ define

$$\mathtt{incidence\_loose\_gain}(w; v_j, v_i) := \big(\mathtt{supIncidenceFinset}(w, v_i)\setminus\{\{v_i, v_j\}\}\big) \uplus \big(\mathtt{supIncidenceFinset}(w, v_j)\setminus\{\{v_i, v_j$$

Lemma 58 (Disjoint from the single edge $\{v_i, v_j\}$)**.** *We have $\mathtt{incidence\_loose\_gain}(w; v_j, v_i) \cap \{\{v_i, v_j\}\} = \emptyset$.*

Definition 59 (Full incidence block)**.** Let

$$\mathtt{inci\_loose\_gain\_full}(w; v_j, v_i) := \mathtt{incidence\_loose\_gain}(w; v_j, v_i) \uplus \{\{v_i, v_j\}\}.$$

Lemma 60 (Partition of supported edges by the (v_i, v_j) block)**.** *If $v_i \sim v_j$ and $w(v_i) > 0$, $w(v_j) > 0$, then*

$$\mathtt{supEdgeFinset}(w) = \mathtt{inci\_loose\_gain\_full}(w; v_j, v_i) \uplus \Big(\mathtt{supEdgeFinset}(w) \setminus \mathtt{inci\_loose\_gain\_full}(w; v_j, v_i)\Big).$$

Lemma 61 (Sum split over the partition)**.** *With the hypotheses of Lemma 60,*

$$\sum_{e \in \mathtt{supEdgeFinset}(w)} \mathrm{vp}(e) = \Big(\sum_{e \in \mathtt{supIncidenceFinset}(w, v_i)\setminus\{\{v_i, v_j\}\}} \mathrm{vp}(e) + \sum_{e \in \mathtt{supIncidenceFinset}(w, v_j)\setminus\{\{v_i, v_j\}\}} \mathrm{vp}(e) \Big) + \sum_{e \in \{\{v_i, v_j\}\}} \mathrm{vp}($$

$$+ \sum_{e \in \mathtt{supEdgeFinset}(w) \setminus \mathtt{inci\_loose\_gain\_full}(w; v_j, v_i)} \mathrm{vp}(e).$$

Lemma 62 (Sum over all edges equals sum over supported edges)**.**

$$\sum_{e \in E} \mathrm{vp}(e) = \sum_{e \in \mathtt{supEdgeFinset}(w)} \mathrm{vp}(e).$$

Lemma 63 (Factorization on the v_i–incidence)**.** *If $e \in \mathtt{supIncidenceFinset}(w, v_i) \setminus \{dummy\}$, then $\mathrm{vp}(e) = w(v_i) \cdot w(\mathrm{other}_e(v_i))$.*

Lemma 64 (Aggregated factorization at v_i)**.**

$$\sum_{e \in \mathtt{supIncidenceFinset}(w, v_i)\setminus\{dummy\}} \mathrm{vp}(e) = w(v_i) \sum_{e \in (\cdots)^{\#}} w(\mathrm{other}_e(v_i)).$$

Lemma 65 (Other endpoint unchanged under $w \mapsto w^+$ at v_i)**.** *For every $e \in \mathtt{supIncidenceFinset}(w, v_i)\setminus \{\{v_i, v_j\}\}$, $w^+(\mathrm{other}_e(v_i)) = w(\mathrm{other}_e(v_i))$.*

Lemma 66 (Other endpoint unchanged under $w \mapsto w^+$ at v_j)**.** *For every $e \in \mathtt{supIncidenceFinset}(w, v_j)\setminus \{\{v_i, v_j\}\}$, $w^+(\mathrm{other}_e(v_j)) = w(\mathrm{other}_e(v_j))$.*

Lemma 67 (Gain–side sum after $w \mapsto w^+$)**.**

$$\sum_{e \in \mathrm{supIncidenceFinset}(w, v_i)\setminus\{\{v_i, v_j\}\}} \mathrm{vp}(w^+, e) = \sum_{e \in \mathrm{supIncidenceFinset}(w, v_i)\setminus\{\{v_i, v_j\}\}} \mathrm{vp}(w, e) + \varepsilon \sum_{e \in (\cdots)^{\#}} w(\mathrm{other}_e(v_i)).$$

Lemma 68 (Bound for the loose–side "other" weights)**.** *For every $e \in \mathtt{supIncidenceFinset}(w, v_j)\setminus \{\{v_i, v_j\}\}$, $\varepsilon\, w(\mathrm{other}_e(v_j)) \le \mathrm{vp}(w, e)$.*

Lemma 69 (Loose–side sum after $w \mapsto w^+$)**.**

$$\sum_{e \in \mathtt{supIncidenceFinset}(w, v_j)\setminus\{\{v_i, v_j\}\}} \mathrm{vp}(w^+, e) = \sum_{e \in \mathtt{supIncidenceFinset}(w, v_j)\setminus\{\{v_i, v_j\}\}} \mathrm{vp}(w, e) - \varepsilon \sum_{e \in (\cdots)^{\#}} w(\mathrm{other}_e(v_j)).$$

0.5.2 A bijection between the v_j– and v_i–incidence sides

We now formalize the combinatorial bijection between the supported incidence edges at the *loose* vertex v_j and those at the *gain* vertex v_i (excluding the edge $\{v_i, v_j\}$). Intuitively, a supported edge $\{v_j, x\}$ is mapped to $\{v_i, x\}$; the clique hypothesis guarantees the latter is indeed an edge of G, and the support positivity passes to the "other" endpoint x.

Definition 70 (The map the_bij)**.** Given a clique support and $w(v_j) > 0$, $w(v_i) > 0$, define

$$\text{the\_bij} : \big(\texttt{supIncidenceFinset}(w, v_j)\setminus\{\{v_i, v_j\}\}\big)^{\#} \longrightarrow \big(\texttt{supIncidenceFinset}(w, v_i)\setminus\{\{v_i, v_j\}\}\big)^{\#}$$

by sending an edge $\{v_j, x\}$ to $\{v_i, x\}$ (formally, using $\text{other}_e(v_j)$ to refer to x).

Lemma 71 (Image lands in the right set)**.** *For every e in the domain,* $\text{the\_bij}(e)$ *lies in* $\big(\texttt{supIncidenceFinset}(w, v_i) \setminus \{\{v_i, v_j\}\}\big)^{\#}$.

Lemma 72 (Injectivity)**.** *The map* the_bij *is injective.*

Lemma 73 (Surjectivity)**.** *The map* the_bij *is surjective.*

Lemma 74 (Preservation of the "other" weight)**.** *For every e in the domain,*

$$w(\text{other}_e(v_j)) \;=\; w\Big(\text{other}_{\text{the\_bij}(e)}(v_i)\Big).$$

Lemma 75 (Equality of v_j– and v_i–side sums)**.** *We have the identity*

$$\sum_{e\in(\texttt{supIncidenceFinset}(w,v_j)\setminus\{\{v_i,v_j\}\})^{\#}} w(\text{other}_e(v_j)) \;=\; \sum_{e\in(\texttt{supIncidenceFinset}(w,v_i)\setminus\{\{v_i,v_j\}\})^{\#}} w(\text{other}_e(v_i)),$$

obtained by summation along the bijection.

0.5.3 Conclusion to this section

Lemma 76 (Edge values unchanged off the (v_i, v_j) block)**.** *For every $e \in \texttt{supEdgeFinset}(w) \setminus \texttt{inci\_loose\_gain\_full}(w; v_j, v_i)$,*

$$\text{vp}(w^+, e) = \text{vp}(w, e).$$

Lemma 77 (Unchanged sum off the (v_i, v_j) block)**.**

$$\sum_{e\in\texttt{supEdgeFinset}(w)\setminus\texttt{inci\_loose\_gain\_full}(w;v_j,v_i)} \text{vp}(w^+, e) \;=\; \sum_{e\in\texttt{supEdgeFinset}(w)\setminus\texttt{inci\_loose\_gain\_full}(w;v_j,v_i)} \text{vp}(w, e).$$

Lemma 78 (Strict increase on the edge $\{v_i, v_j\}$)**.** *If $v_i \sim v_j$ and $w(v_i) > 0,\ w(v_j) > 0$, then*

$$\text{vp}(w^+, \{v_i, v_j\}) \;>\; \text{vp}(w, \{v_i, v_j\}).$$

Lemma 79 (Supported edge set is preserved)**.**

$$\texttt{supEdgeFinset}(w) \;=\; \texttt{supEdgeFinset}(w^+).$$

Lemma 80 (Total edge weight is nondecreasing under $w \mapsto w^+$)**.**

$$f(w) \;\le\; f(w^+).$$

0.6 Using `Enhance` to force uniform weights

0.7 Equalizing weights on a clique

0.7.1 Extrema on the support and averages

In this subsection we work with a weight function W whose support induces a clique. We: (i) define the extremal weights $W.\max\_weight$ and $W.\min\_weight$ and choose witnesses $operatorname{argmax}, \text{argmin}$; (ii) relate these to the average weight $1/|\operatorname{supp}(W)|$ via avg $\leq$ max, min $\leq$ avg, and the strict versions when min $<$ max; (iii) collect basic sum bounds on the support that will be used to drive a uniformization step later.

Lemma 81 (Support nonempty)**.** *For any weight function W, the support is nonempty:*

$$\{v \in V : W.w(v) > 0\} \neq \emptyset.$$

Definition 82 (Maximum and minimum support weights)**.** Let

$$W.\max\_weight := \max\{ W.w(v) : W.w(v) > 0 \}, \qquad W.\min\_weight := \min\{ W.w(v) : W.w(v) > 0 \}.$$

Lemma 83 (Witnessing vertices)**.** *There exist vertices* $\text{argmax}, \text{argmin} \in \operatorname{supp}(W)$ *such that*

$$w(\text{argmax}) = \max\_weight, \qquad w(\text{argmin}) = \min\_weight.$$

Lemma 84 (Pointwise bounds)**.** *For all $v \in V$, $w(v) \leq W.\max\_weight$. For $v \in \operatorname{supp}(W)$, $\min\_weight \leq W.w(v)$. In particular $W.\min\_weight \leq W.\max\_weight$.*

Lemma 85 (Sum over support)**.**

$$\sum_{v \in V} W.w(v) \;=\; \sum_{v \in \operatorname{supp}(W)} W.w(v) \;=\; 1.$$

Lemma 86 (Support–sum bounds via extrema)**.** *Let $S := \operatorname{supp}(W)$ and $|S| = \#S$. Then*

$$\sum_{v \in S} W.w(v) \;\leq\; |S|\, W.\max\_weight, \qquad \sum_{v \in S} W.w(v) \;\geq\; |S|\, W.\min\_weight.$$

Lemma 87 (Average vs. extrema)**.** *With $S = \operatorname{supp}(W)$,*

$$\frac{1}{|S|} \;\leq\; W.\max\_weight, \qquad W.\min\_weight \;\leq\; \frac{1}{|S|}.$$

Lemma 88 (Strict versions when min $<$ max)**.** *If $W.\min\_weight < W.\max\_weight$, then*

$$\frac{1}{|S|} \;<\; W.\max\_weight \quad \textit{and} \quad W.\min\_weight \;<\; \frac{1}{|S|}.$$

Equivalently, $\sum_{v \in S} W.w(v) < |S|\, W.\max\_weight$ and $\sum_{v \in S} W.w(v) > |S|\, W.\min\_weight$.

Lemma 89 (Flat support iff min=max)**.** *If $W.\min\_weight = W.\max\_weight$, then every $v \in \operatorname{supp}(W)$ has*

$$W.w(v) \;=\; \frac{1}{|S|}.$$

0.7.2 The last weight transfer

In this final step we adjust weights on a clique support until uniform. We define the transfer amount ε as the excess of the maximum weight over the average $1/|S|$, and use the `Enhance` operator (w^+) from Section **??** to move precisely this amount from the maximum vertex to the minimum vertex. This produces the distribution `Enhanced`, which has strictly larger uniform support.

Notation. In this subsection we set $v_j :=$ argmax (the *loose* vertex) and $v_i :=$ argmin (the *gain* vertex). Thereafter we write v_i, v_j instead of argmin, argmax.

Definition 90 (The transfer amount ε)**.** For a weight function W with support S,

$$\varepsilon(W) := W.\max\_weight - \tfrac{1}{|S|}.$$

Lemma 91 (Positivity of ε)**.** *If $W.\min\_weight < W.\max\_weight$, then $\varepsilon(W) > 0$.*

Lemma 92 (ε bounded by gap)**.** *If $W.\min\_weight < W.\max\_weight$, then*

$$\varepsilon(W) < W.w(v_j) - W.w(v_i).$$

Lemma 93 (Argmax–argmin gap implies distinct extrema)**.** *If $W.w(v_i) < W.w(v_j)$ (i.e. $v_i =$ argmin, $v_j =$ argmax), then $W.\min\_weight < W.\max\_weight$.*

Definition 94 (Enhanced distribution)**.** Given W with $W.w(v_i) < W.w(v_j)$,

$$W^+ := \texttt{Enhance}(W,\ v_j,\ v_i,\ \varepsilon(W)).$$

Lemma 95 (Vertices already uniform remain unaffected)**.** *If $W.w(v) = 1/|S|$ then also $(W^+).w(v) = 1/|S|$.*

Lemma 96 (Effect on argmax)**.** *The argmax vertex in W^+ attains exactly the uniform value:*

$$(W^+).w(v_j) = \tfrac{1}{|S|}.$$

Lemma 97 (Uniform count increases)**.** *The number of vertices at uniform weight strictly increases:*

$$\#\{v : (W^+).w(v) = 1/|S|\} \ > \ \#\{v : W.w(v) = 1/|S|\}.$$

Lemma 98 (UniformBetter preserves support)**.** *If $\mathrm{supp}(W)$ is a clique, then*

$$W.w(i) > 0 \iff (w^{mathrmunif})(i) > 0.$$

Lemma 99 (Support is a clique)**.** *If $\mathrm{supp}(W)$ is a clique, then any two distinct vertices $x, y \in \mathrm{supp}(W)$ are adjacent.*

Lemma 100 (UniformBetter has constant support)**.** *If $\mathrm{supp}(W)$ is a clique, then*

$$\forall v \in \mathrm{supp}(W), \quad w^{mathrmunif}(v) = \tfrac{1}{|S|}.$$

Lemma 101 (Uniform edge values)**.** *If $\mathrm{supp}(W)$ is a clique, then for each supported edge e, under the distribution $w^{mathrmunif}$*

$$vp(e) = \left(\tfrac{1}{|S|}\right)^2.$$

Lemma 102 (Clique size from edges)**.** *If $\mathrm{supp}(W)$ has size k, then*

$$|\, G.\texttt{supEdgeFinset}\,| = \tbinom{k}{2}.$$

0.7.3 Final bound

We finish by converting the information we gatehered (uniform weights on a clique support and a precise count of supported edges) into the final bound.

Lemma 103 (Ccomputation)**.** *If $k \geq 1$, then*

$$\Big(\frac{k(k-1)}{2}\Big)\Big(\frac{1}{k}\Big)^2 \;=\; \frac{1}{2}\Big(1 - \frac{1}{k}\Big).$$

Lemma 104 (Monotonicity in the parameter)**.** *For integers $k, q \geq 1$ with $k \leq q$,*

$$\frac{1}{2}\Big(1 - \frac{1}{k}\Big) \;\leq\; \frac{1}{2}\Big(1 - \frac{1}{q}\Big).$$

Lemma 105 (Real–cast version)**.** *For natural numbers $k \leq p$ with $k \geq 1$,*

$$\frac{1}{2}\Big(1 - \frac{1}{k}\Big) \;\leq\; \frac{1}{2}\Big(1 - \frac{1}{p}\Big) \qquad \textit{(as an inequality in } \mathbb{R}\textit{)}.$$

Lemma 106 (Auxiliary casting identity)**.**

Lemma 107 (Final bound for any distribution)**.** *Let $p \geq 2$ and suppose G is p–clique–free. Then for every weight distribution w,*

$$f(w) \;\leq\; \Big(\frac{(p-1)((p-1)-1)}{2}\Big)\Big(\frac{1}{p-1}\Big)^2 \;=\; \frac{1}{2}\Big(1 - \frac{1}{p-1}\Big).$$

Definition 108 (Uniform-on-vertices distribution)**.** The *uniform vertex distribution* is $u : V \to \mathbb{R}_{\geq 0}$ with $u(v) = 1/n$ for all v.

Lemma 109 (Total edge weight under the uniform distribution)**.** *If u is the distribution of Definition 108, then*

$$f(u) = \#E \cdot \Big(\frac{1}{n}\Big)^2.$$

Theorem 110 (Turán's theorem, weighted proof)**.** *Let $p \geq 2$ and suppose G has no p–clique. Then*

$$|E| \;\leq\; \frac{1}{2}\Big(1 - \frac{1}{p-1}\Big)\, n^2.$$