27 marzo 23

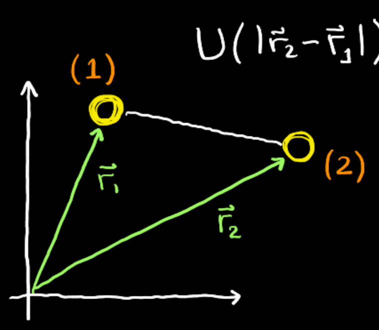
Ayudantía

Fuerza central

$$\vec{F} = f(r)\hat{r}$$

$$\frac{dL}{dt} = \frac{d}{dt}(\vec{r}\times\vec{p}) = \frac{d\vec{r}}{dt}\times\vec{p} + \cancel{\vec{r}\times\frac{d\vec{p}}{dt}}^{\,0} = \cancel{\dot{\vec{r}}\times m\dot{\vec{r}}}^{\,0}$$

$$\therefore \frac{dL}{dt} = 0 \quad \Rightarrow \vec{L} = L_0$$



$U(|\vec{r}_2 - \vec{r}_1|)$ $\qquad (x_1, y_1, z_1) \qquad (x_2, y_2, z_2)$

$$\mathcal{L} = \frac{1}{2}m_1|\dot{\vec{r}}_1|^2 + \frac{1}{2}m_2|\dot{\vec{r}}_2|^2 - U(|\vec{r}_2 - \vec{r}_1|)$$

denotamos $\vec{r} = \vec{r}_2 - \vec{r}_1$. Además,

$$\vec{R}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$$

$$\Rightarrow (m_1+m_2)\vec{R}_{cm} = m_1\vec{r}_1 + m_2\vec{r}_2$$

$$\Rightarrow \begin{cases} (m_1+m_2)\vec{R}_{cm} - m_1\vec{r} = (m_1+m_2)\vec{r}_2 \\ (m_1+m_2)\vec{R}_{cm} - m_2\vec{r} = (m_1+m_2)\vec{r}_1 \end{cases}$$

$$\Rightarrow \vec{r}_2 = \frac{m_1\vec{r}}{m_1+m_2}, \quad \vec{r}_1 = \frac{m_2\vec{r}}{m_1+m_2} \quad \Rightarrow \dot{\vec{r}}_2 = \frac{m_1\dot{\vec{r}}}{m_1+m_2}, \quad \dot{\vec{r}}_1 = \frac{m_2\dot{\vec{r}}}{m_1+m_2}$$

$$\mathcal{L} = \frac{1}{2}m_2\frac{m_1^2}{(m_1+m_2)^2}|\dot{\vec{r}}|^2 + \frac{1}{2}m_1\frac{m_2^2}{(m_1+m_2)^2}|\dot{\vec{r}}|^2 - U(\vec{r})$$

$$= \frac{1}{2}\frac{m_2 m_1}{(m_1+m_2)^{\cancel{2}}}\cancel{(m_1+m_2)}|\dot{\vec{r}}|^2 - U(\vec{r})$$

llamamos por masa reducida a $\mu = \frac{m_1 m_2}{m_1+m_2} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)^{-1}$. Entonces,

$$\mathcal{L} = \frac{1}{2}\mu|\dot{\vec{r}}|^2 - U(\vec{r}) = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(\vec{r})$$

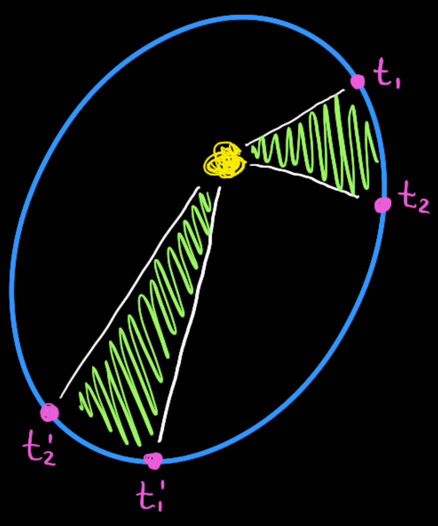
$$p_\theta = \frac{\partial \mathcal{L}}{\partial\dot{\theta}} = \boxed{\mu r^2\dot{\theta} = \ell}, \quad r^2\dot{\theta}^2 = r^2\left(\frac{\ell^2}{\mu^2 r^4}\right) = \frac{\ell^2}{\mu^2 r^2}.$$ Consecuentemente,

$$\mathcal{L} = \underbrace{\frac{1}{2}\mu\left(\dot{r}^2 + \frac{\ell^2}{\mu^2 r^2}\right)}_{K} - \underbrace{U(\vec{r})}_{U}$$

potencial generalizado

$$\Rightarrow E = K + U = \boxed{\frac{1}{2}\mu\dot{r}^2 + \boxed{\frac{\ell^2}{2\mu r^2} + U(r)}}$$

1ª Cuadratura



$$dA = \frac{1}{2}r^2\theta \Rightarrow \frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} \Rightarrow \frac{dA}{dt} = \frac{\ell}{2\mu}$$

$$\therefore E = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell}{2\mu r^2} + U$$

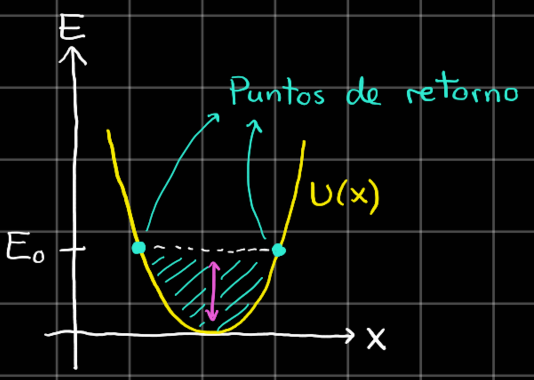
$$\dot{r} = \left(\frac{2}{\mu}\left[(E-U) - \frac{\ell^2}{2\mu r^2}\right]\right)^{1/2}$$

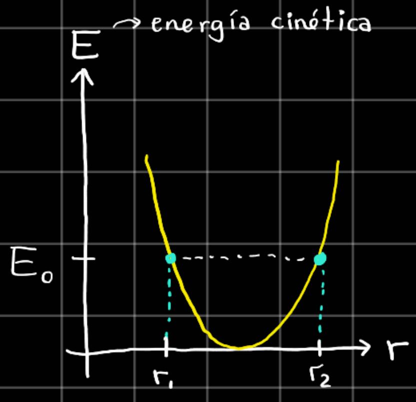
$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{dr}{d\theta}$$

Sea $\dot{\theta}\frac{dr}{d\theta} = F(r) \longrightarrow \frac{\ell}{\mu r^2}\frac{dr}{d\theta} = F(r) \Rightarrow \frac{\ell}{\mu r^2 F(r)} = d\theta$

$$\int_{r_0}^{r} \frac{\ell/\mu r'^2}{\frac{2}{\mu}\left[(E-U) - \frac{\ell^2}{2\mu r'^2}\right]} dr' = \int_{\theta_0}^{\theta} d\theta'$$

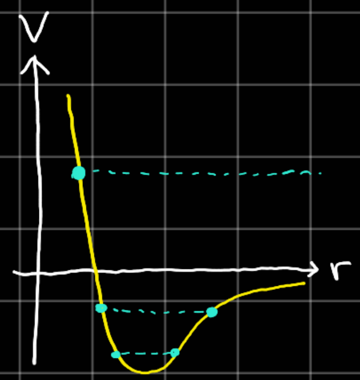
Vector Runge Lenz. $\vec{A} = (\vec{\ell} \times \dot{\vec{r}} + \mu\hat{r})$





$U = -\frac{k}{r}$

$V = -\frac{k}{r} + \frac{\ell}{2\mu r^2}$

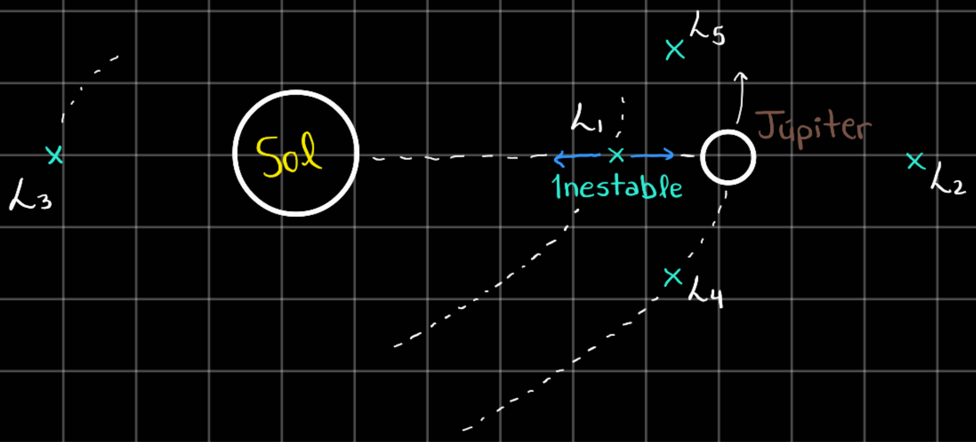


$E < 0$ (Elipses)

$E = 0$ (Parábolas)

$E > 0$ (Hipérbola)

Puntos de Lagrange



$L = \frac{1}{2}\mu\left(\dot{r}^2 + \frac{\ell^2}{\mu^2 r^2}\right) + \frac{k}{r}$; $\frac{\partial L}{\partial \dot{r}} = \mu\dot{r}$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \mu\ddot{r}$

$$\frac{\partial L}{\partial r} = -\frac{\ell^2}{\mu^2 r^3} - \frac{k}{r^2}$$

Entonces, $\mu\ddot{r} + \frac{\ell^2}{\mu^2 r^3} + \frac{k}{r^2} = 0$. Denotando $u = 1/r$ se obtiene

$$\frac{du}{dt} = \frac{d}{dt}\left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{dr}{dt}$$

$$\frac{d^2u}{dt^2} = \frac{d}{dt}\left(\frac{1}{r^2}\dot{r}\right) = -\frac{\ddot{r}}{r^2} + \frac{2}{r^3}\dot{r}^2$$

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt}, \quad \frac{d^2r}{dt^2} = \frac{d}{dt}\left(-\frac{1}{u^2}\frac{du}{dt}\right) = -\frac{\ddot{u}}{u^2} + \frac{2}{u^3}\dot{u}^2$$