27 feb 23

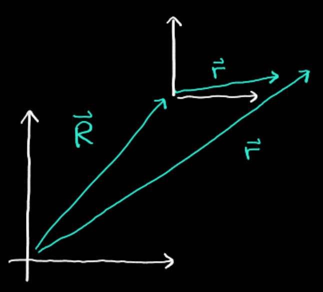
(*) Recordando que... $T = T_{tras} + T_{rot}$

$$T = \frac{1}{2} M V^2 + \frac{1}{2}\vec{\Omega}\cdot\underset{\sim}{I}\cdot\vec{\Omega}$$

$$\boxed{I_{ij} = \sum_\alpha m_\alpha \left(r_\alpha^2 \delta_{ij} - x_i^\alpha x_j^\alpha\right)}$$

tensor de inercia cuyo origen está en el centro de masas

$$I'_{ij} = \sum_\alpha m_\alpha \left(r'^2_\alpha \delta_{ij} - x_i'^\alpha x_j'^\alpha\right)$$



$\vec{r} = \vec{R} + \vec{r}'$

$$\therefore\ I'_{ij} = \sum_\alpha m_\alpha \left((\vec{r}^2 - \vec{R})^2 \delta_{ij} - (x_i^\alpha - R_i)(x_j^\alpha - R_j)\right)$$

$$= \sum_\alpha m_\alpha \left(r^2 \delta_{ij} - x_i^\alpha x_j^\alpha\right) + \sum_\alpha m_\alpha \left(R^2 \delta_{ij} - R_i R_j\right)$$

$$-2\vec{R}\cdot\cancelto{0}{\sum_\alpha m_\alpha \vec{r}_\alpha} - \cancelto{0}{\sum_\alpha m_\alpha x_i^\alpha} R_j - \cancelto{0}{\sum_\alpha m_\alpha x_j^\alpha} R_i$$

centro de masa visto desde el centro de masa

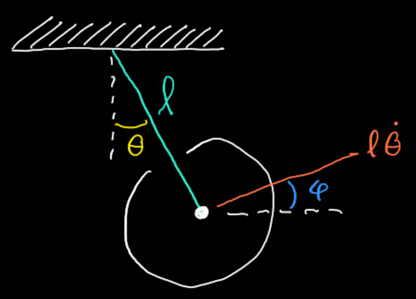
$$\therefore\ I'_{ij} = I_{ij} + M\left(R^2 \delta_{ij} - R_i R_j\right)$$

de donde rescatamos el teorema de ejes principales con $i = j = 1$,

$$I'_1 = I_1 + MR^2$$

denota al eje principal 11

(*)



$$T = \frac{m}{2}(x^2 + y^2) + \frac{1}{2} I \dot{\ell}^2$$

$$x = \ell\cos\theta$$

$$y = \ell\sin\theta$$

$$T = \frac{1}{2} m \ell^2 \dot{\theta}^2 + \frac{1}{2} I \dot{\varphi}^2, \quad T = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I \dot{\varphi}^2$$

(*) $\vec{M} = \sum_\alpha m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha$ con $\vec{v}_\alpha = \vec{\Omega}\times\vec{r}_\alpha$, entonces

$$\vec{M} = \sum m_\alpha \vec{r}_\alpha \times \left(\vec{\Omega}\times\vec{r}_\alpha\right)$$

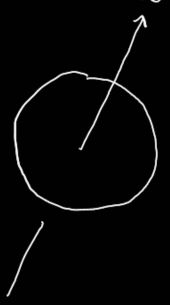
$$= \sum m_\alpha \left(\vec{r}_\alpha^{\,2}\vec{\Omega} - (\vec{r}_\alpha\cdot\vec{\Omega})\vec{r}\right)$$

$$M_i = \sum_\alpha m_\alpha \left(r_\alpha^2 \Omega_i - x_j^\alpha \Omega_j x_i^\alpha\right)$$

$$= \sum_\alpha m_\alpha \Omega_i \left(r_\alpha^2 \delta_{ji} - x_i^\alpha x_j^\alpha\right)$$

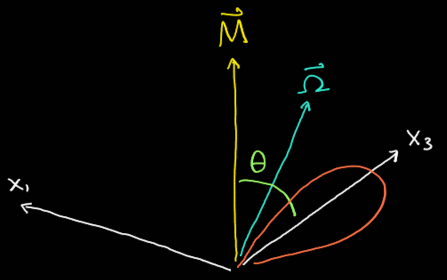
$$= I_{ij}\Omega_j$$

$$\therefore\ \boxed{\vec{M} = \underset{\sim}{I}\cdot\vec{\Omega}.}$$



Si $\dot{\vec{M}} = 0$ (espacio isotrópico) $\Rightarrow \vec{M} = cte$

$$\underset{\sim}{I} = I\mathbb{1}\ ; \quad I_{ij} = I\delta_{ij}$$



x_2 sale del plano

$\vec{M} = (M_1, 0, M_3)$ donde

$$M_1 = I_1\Omega_1$$

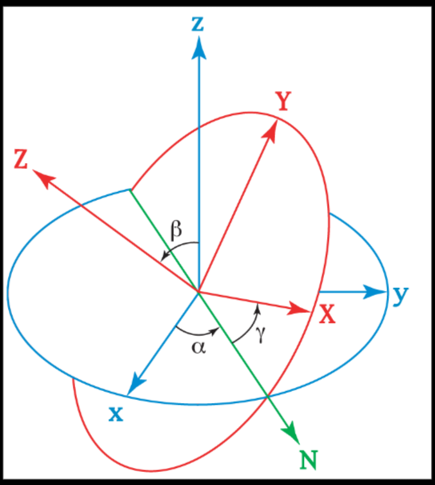
$$M_2 = I_2\Omega_2 \Rightarrow \Omega_2 = 0$$

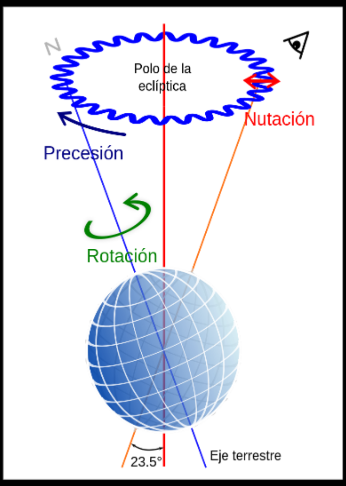
$$M_3 = I_3\Omega_3$$

$$\vec{v} = \vec{\Omega}\times\vec{r}$$

$$\Omega_3 = \frac{M_3}{I_3} = \frac{M\cos\theta_3}{I_3}, \quad \Omega_2 = \frac{M_1}{I_1} = \frac{M\sin\theta_1}{I_1} \Rightarrow \Omega_{pr} = \frac{M}{I_1}$$

Ángulos de Euler





También se considera la notación $\alpha = \phi, \gamma = \psi$ y $\beta = \theta$.

- $\phi \measuredangle (x, ON)$
- $\psi \measuredangle (ON, X)$
- $\theta \measuredangle (Z, z)$

- $\phi, \psi \in [0, 2\pi)$
- $\theta \in [0, \pi]$

Consideraremos a $(X, Y, Z) = (x_1, x_2, x_3)$.

Si $\vec{\dot{\theta}} = (\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)$, entonces

$$\dot{\theta}_1 = \dot{\theta}\cos\psi \qquad \dot{\phi}_1 = \dot{\phi}\sin\theta\sin\psi$$
$$\dot{\theta}_2 = -\dot{\theta}\sin\psi \qquad \dot{\phi}_2 = \dot{\phi}\sin\theta\cos\psi$$
$$\dot{\theta}_3 = 0 \qquad \dot{\phi}_3 = \dot{\phi}\cos\theta \qquad \dot{\psi}_3 = \dot{\psi}$$

$$\Omega_1 = \dot{\phi}_1 + \dot{\theta}_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$\Omega_2 = \dot{\phi}_2 + \dot{\theta}_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \qquad L = T(q, \dot{q}) - U$$
$$\Omega_3 = \dot{\phi}_3 + \dot{\psi}_3 = \dot{\phi}\cos\theta + \dot{\psi}$$

$$T_{rot} = \frac{1}{2}I_1\Omega_1^2 + \frac{1}{2}I_2\Omega_2^2 + \frac{1}{2}I_3\Omega_3^2$$

Suponga una peonza simétrica tal que $I_1 = I_2 \neq I_3$,

$$T = \frac{1}{2}I_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}I_3(\dot{\phi}\cos\theta + \dot{\psi})^2$$

Si $\psi = 0$, se tiene que

$$\Omega_1 = \dot{\theta}; \qquad \Omega_2 = \dot{\phi}\sin\theta; \qquad \Omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$$

$\vec{M} \parallel z$. Rotación intrínseca respecto al eje x_3.

$$M_1 = I_1\Omega_1 = I_1\dot{\theta}$$
$$M_2 = I_2\Omega_2 = I_2\dot{\phi}\sin\theta$$
$$M_3 = I_3\Omega_3 = I_3(\dot{\phi}\cos\theta + \dot{\psi}) = \frac{\partial L}{\partial \dot{\psi}}$$
$$M_z = (I_1'\sin^2\theta + I_3\cos^2\theta)\dot{\phi} + I_3\dot{\psi}\cos\theta = \frac{\partial L}{\partial \dot{\phi}}$$

$$\begin{cases} M_1 = 0 \\ M_2 = M\sin\theta \quad \Rightarrow \quad I_2\dot{\phi} = M \\ M_3 = M\cos\theta \end{cases}$$

$$E = \frac{1}{2}(I_1 + \mu l^2)(\dot{\theta}^2 + \dot{\theta}^2\sin^2\theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 + \mu g l\cos\theta$$

$$E' = \frac{1}{2}I_1'\dot{\theta}^2 + U_{eff}(\theta)$$

$$\dot{\phi} = \frac{M_z - M_3\cos\theta}{I_1'\sin\theta}, \qquad \dot{\psi} = \frac{M_3}{I_3} - \cos(\theta)\frac{M_z - M_3\cos\theta}{I_1'\sin^2\theta}$$