24 marzo 2023

Ayudantía

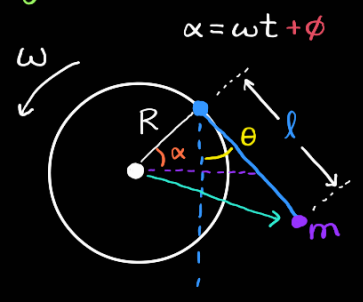
Dada $\mathcal{L}(q, \dot{q}, t)$ el Hamiltoniano es

$$H(q, p; t) = \sum_{i=1}^{n} \dot{q}_i p_i - \mathcal{L}$$

$$\frac{\partial H}{\partial p} = \dot{q}, \quad \frac{\partial H}{\partial q} = -\dot{p}, \quad \frac{\partial H}{\partial t} = \frac{dH}{dt} = -\frac{\partial \mathcal{L}}{\partial t}.$$

De los siguientes ejercicios encuentre el Lagrangiano, el Hamiltoniano y las ecuaciones de movimiento

Ejercicio 1.



(a) Tómese como punto de referencia el centro del disco.

$$\vec{s} = R(\cos(\omega t), \sin(\omega t)) + \ell(\sin\theta, -\cos\theta)$$

$$\vec{s} = (R\cos(\omega t) + \ell\sin\theta, R\sin(\omega t) - \ell\cos\theta)$$

$$\dot{\vec{s}} = (-R\omega\sin(\omega t) + \ell\dot{\theta}\cos\theta, R\omega\cos(\omega t) + \ell\dot{\theta}\sin\theta)$$

$$\begin{aligned}|\dot{\vec{s}}|^2 &= R^2\omega^2(\omega t) + \ell^2\dot{\theta}^2\cos^2\theta - 2R\omega\ell\dot{\theta}\sin(\omega t)\cos\theta \\ &\quad + R^2\omega^2\cos^2(\omega t) + \ell^2\dot{\theta}^2\sin^2\theta + 2R\omega\ell\dot{\theta}\cos(\omega t)\sin\theta \\ &= R^2\omega^2 + \ell^2\dot{\theta}^2 + 2R\omega\ell\dot{\theta}\sin(\theta - \omega t)\end{aligned}$$

$$U = mgy = mg(R\sin(\omega t) - \ell\cos\theta)$$

(i) $$\mathcal{L} = \frac{1}{2}m(R^2\omega^2 + \ell^2\dot{\theta}^2) + mR\omega\ell\dot{\theta}\sin(\theta - \omega t) - mg(R\sin(\omega t) - \ell\cos\theta)$$

(ii) Momento generalizado

$$p = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m\ell^2\dot{\theta} + mR\omega\ell\sin(\theta - \omega t)$$

$$\mathcal{H} = p_\theta\dot{\theta} - \mathcal{L}$$

$$\dot{\theta} = \frac{1}{m\ell^2}(p_\theta - mR\ell\omega\sin(\theta - \omega t))$$

$$\Rightarrow \mathcal{H} = \frac{p_\theta}{m\ell^2}(p_\theta - mR\ell\omega\sin(\theta - \omega t)) - \frac{1}{2}mR^2\omega^2 - \frac{1}{2}m\ell^2\dot{\theta}^2$$

$$- mR\omega\ell\dot{\theta}\sin(\theta - \omega t) + mg(R\sin(\omega t) - \ell\cos\theta)$$

$$\Rightarrow \mathcal{H} = \frac{p_\theta}{m\ell^2}(p_\theta - mR\ell\omega\sin(\theta - \omega t)) - \frac{1}{2}mR^2\omega^2$$

$$- \frac{1}{2}m\ell^2\left(\frac{1}{m\ell^2}(p_\theta - mR\ell\omega\sin(\theta - \omega t))\right)^2$$

$$- mR\omega\ell\frac{1}{m\ell^2}(p_\theta - mR\ell\omega\sin(\theta - \omega t))\sin(\theta - \omega t)$$

$$+ mg(R\sin(\omega t) - \ell\cos\theta)$$

$$\Rightarrow \mathcal{H} = \frac{p_\theta^2}{m\ell^2} - \frac{p_\theta}{m\ell^2}mR\ell\omega\sin(\theta - \omega t) - \frac{1}{2}mR^2\omega^2$$

$$- \frac{1}{2}m\ell^2\left(\frac{1}{m^2\ell^4}\right)(p_\theta - mR\ell\omega\sin(\theta - \omega t))^2$$

$$- mR\omega\ell\frac{1}{m\ell^2}p_\theta\sin(\theta - \omega t) + mR\omega\ell\frac{1}{m\ell^2}mR\ell\omega\sin(\theta - \omega t))\sin(\theta - \omega t)$$

$$+ mg(R\sin(\omega t) - \ell\cos\theta)$$

$$\Rightarrow \mathcal{H} = \frac{P_\theta^2}{ml^2} - \frac{P_\theta}{l} R\omega \sin(\theta - \omega t) - \frac{1}{2} mR^2\omega^2$$

$$- \frac{1}{2ml^2}\left(P_\theta^2 - 2P_\theta mRl\omega \sin(\theta - \omega t) + m^2R^2l^2\omega^2 \sin^2(\theta - \omega t)\right)$$

$$- \frac{R\omega}{l} P_\theta \sin(\theta - \omega t) + \frac{R\omega}{l} mRl\omega \sin(\theta - \omega t))\sin(\theta - \omega t)$$

$$+ mg(R\sin(\omega t) - l\cos\theta)$$

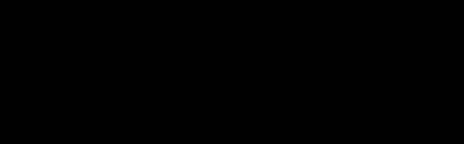
$$\Rightarrow \mathcal{H} = \frac{P_\theta^2}{ml^2} - \frac{P_\theta}{l} R\omega \sin(\theta - \omega t) - \frac{1}{2} mR^2\omega^2$$

$$- \frac{1}{2ml^2}\left(P_\theta^2 - 2P_\theta mRl\omega \sin(\theta - \omega t) + m^2R^2l^2\omega^2 \sin^2(\theta - \omega t)\right)$$

$$- \frac{R\omega}{l} P_\theta \sin(\theta - \omega t) + mR^2\omega^2 \sin(\theta - \omega t)\sin(\theta - \omega t)$$

$$+ mgR\sin(\omega t) - mgl\cos\theta$$

Ejercicio 2.



$$U = \frac{1}{2}k_1x^2 + \frac{1}{2}k_2x^2 = \frac{1}{2}(k_1 + k_2)x^2$$

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}(k_1 + k_2)x^2$$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \Rightarrow \dot{x} = \frac{p}{m}$$

$$\mathcal{H} = \frac{p^2}{m} - \frac{p^2}{2m} + \frac{1}{2}(k_1 + k_2)x^2 \Rightarrow \mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}(k_1 + k_2)x^2$$

Ejercicio 3. $\mathcal{L} = \dot{q}_1^2 + \frac{\dot{q}_2^2}{a + bq_1} + k_1q_1^2 + k_2q_1\dot{q}_2$. Encuentre H y las ecuaciones de movimiento.