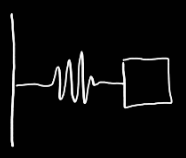
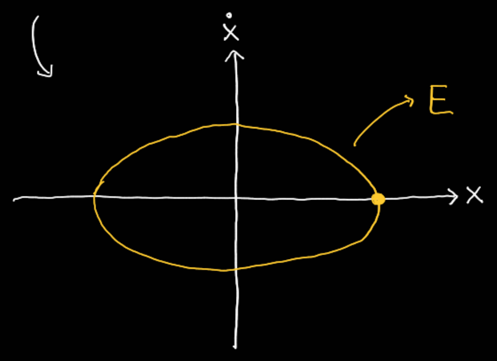
# El oscilador armónico

parte 420



$$L=\frac{1}{2}m\dot{x}^2-\frac{1}{2}kx^2 \cdots (1); \qquad E=\frac{1}{2}m\dot{x}^2+\frac{1}{2}kx^2 \cdots (2)$$

Espacio fase de la energía




Si consideramos $x(0)=x_0$, $\dot{x}(0)=0$ de modo tal que

$$E_0=\frac{1}{2}kx_0^2$$

Más aún, nótese que de (2) tenemos

$$\dot{x}=\sqrt{\frac{2E}{m}-\frac{k}{m}x^2}$$

Si $\dot{x}_0=0$, entonces

$$\frac{2E}{m}=\frac{k}{m}x_0^2$$

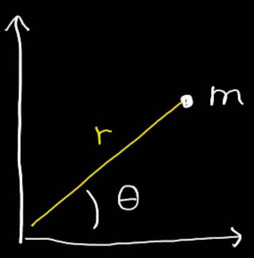
Por otra parte, $\frac{d\dot{x}}{dt}=\sqrt{\frac{k}{m}}\sqrt{x_0^2-x^2}$. Así pues,

$$\int_{x_0}^{x}\frac{dx}{\sqrt{x_0^2-x^2}}=\int_0^t \omega dt \Rightarrow \operatorname{arsin}\left(\frac{x'}{x_0}\right)\Big|_{x_0}^{x}=\omega t$$

$$\Rightarrow \operatorname{arsin}\left(\frac{x}{x_0}\right)-\frac{\pi}{2}=\omega t$$

$$\Rightarrow x(t)=x_0\sin\left(\omega t+\frac{\pi}{2}\right)$$

Dada una partícula de masa puntual $m$ y un campo central $U=U(r)$ se tiene




$$\vec{F}=-\frac{\alpha}{r^2}\hat{r}$$

pero $\vec{r}=r\hat{r}$, $\vec{v}=\dot{r}\hat{r}+r\dot{\theta}\hat{\theta}$, $\vec{r}\times\vec{p}=mr^2\dot{\theta}\,\hat{k}$, de lo cual

$$L=\frac{1}{2}m(\dot{r}^2+r^2\theta^2)-U(r)$$

nótese además $\frac{d}{dt}(mr^2\dot{\theta})=0 \Rightarrow p_\theta=mr^2\theta=\text{cte} \Rightarrow \theta=p_\theta/mr^2 \cdots (*)$

$$E=\frac{1}{2}m\dot{r}^2+\frac{p_\theta^2}{2mr^2}+U(r)$$

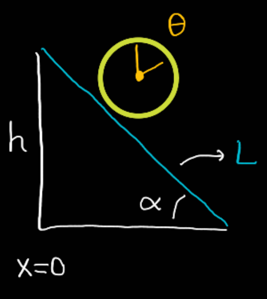
Si $\dot{r}=0$, tenemos puntos de retorno

$$\frac{p_\theta^2}{2mr^2}+U(r)=E$$

entonces, $\frac{dr}{dt}=\sqrt{\frac{2}{m}(E-U(r))-\frac{P}{2mr^2}} \Rightarrow \int dt=\int\frac{dr}{\sqrt{\frac{2}{m}(E-U(r))-\frac{P}{2mr^2}}} \Rightarrow t=t(r)$

Si consideramos por (*) se tendría $d\theta=\frac{p_\theta dt}{mr^2} \Rightarrow \Delta\theta=\int_{r_{min}}^{r_{max}}\frac{p_\theta\, dr/r^2}{\sqrt{2m(E-U(r))-p_\theta^2/r^2}}$

con $U(r)=\frac{p_\theta^2}{2mr^2}-\frac{\alpha}{r}$.




$$-\dot{x}=a\dot{\theta}$$

$$T=\frac{1}{2}m\dot{x}^2+\frac{1}{2}I\dot{\theta}^2$$

$$U=mg(L-x)\sin\alpha$$

$$\Rightarrow \mathcal{L}=\frac{1}{2}m\dot{x}^2+\frac{I}{2a^2}\dot{x}^2-mg(L-x)\sin\alpha$$

$$\Rightarrow \left(m+\frac{I}{2a^2}\right)\ddot{x}=mg\sin\alpha$$

$$\begin{cases} m\ddot{x}=mg\sin\alpha+\lambda \\ I\ddot{\theta}=a\lambda \\ \dot{x}=-a\dot{\theta} \end{cases}$$

Dadas las condiciones y la información que buscamos saber del sistema, podemos descartar a los multiplicadores de Lagrange $\lambda$.

Tenemos que $\mathcal{L}(q,\dot{q})$,

$$\frac{d\mathcal{L}}{dt} = \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \qquad \Rightarrow \frac{d\mathcal{L}}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$$

$$\hookrightarrow \frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \qquad \Rightarrow \frac{d\mathcal{L}}{dt} = \frac{d}{dt}\left[\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i\right]$$

$$\Rightarrow \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i - \mathcal{L}\right] = 0$$

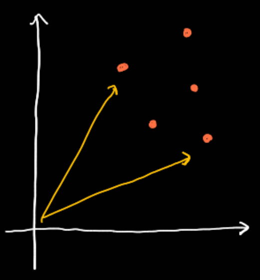
$\therefore \dfrac{\partial L}{\partial \dot{q}_i}\dot{q}_i - \mathcal{L} = \text{cte}$. No obstante, $\mathcal{L} = T(q,\dot{q}) - U(q)$ entonces

$$T = \frac{1}{2}\frac{\partial T}{\partial \dot{q}}\dot{q} \Rightarrow 2T = \frac{\partial L}{\partial \dot{q}}\dot{q}$$

consecuentemente,

$$\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - \mathcal{L} = E \Rightarrow 2T - T + U = E$$

Veamos ahora que ocurre con el momento. Considere entonces un sistema de partículas tales que serán trasladas uniformemente por $\delta\vec{r}_\alpha$,



$$\vec{r}_\alpha \to \vec{r}_\alpha + \delta\vec{r}_\alpha, \quad \delta\vec{r}_\alpha = \vec{\varepsilon}$$

$$\delta\mathcal{L} = \sum_\alpha \frac{\partial L}{\partial \vec{r}_\alpha}\cdot\delta\vec{r}_\alpha \ ; \quad \delta\mathcal{L} = 0 \Rightarrow \sum_\alpha \frac{\partial L}{\partial \vec{r}_\alpha} = 0$$

$$\frac{d}{dt}\left(\sum_\alpha \frac{\partial L}{\partial \dot{\vec{r}}_\alpha}\right) = \frac{\partial L}{\partial \vec{r}_\alpha} = 0 \Rightarrow \vec{p} = \sum_\alpha \frac{\partial L}{\partial \dot{\vec{r}}_\alpha} = \text{cte}$$

Si tuviésemos que $\mathcal{L} = \frac{1}{2}m\vec{v}_\alpha^{\,2} - U(\vec{r}_1, \dots, \vec{r}_f)$, f: grados de libertad

$$\vec{p} = \sum \vec{p}_\alpha = \sum \frac{\partial L}{\partial \dot{\vec{r}}_\alpha} = \sum m\vec{v}_\alpha \Rightarrow \frac{\partial L}{\partial \vec{r}_\alpha} = \frac{\partial U}{\partial \vec{r}_\alpha} = -\vec{F}_\alpha$$

$\Rightarrow \vec{F}_{12} = -\vec{F}_{21}$ (¡recuperamos la 2da Ley!)