22 feb 23

Teoremas de conservación

(i) Energía

$$\mathcal{L} = T(q,\dot{q}) - U(q)$$

$$\frac{d}{dt}\Big\{\underbrace{-\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i}_{2T} - \mathcal{L}\Big\} = 0 \Rightarrow \frac{d}{dt}(T+U) = 0 \Rightarrow E = T+U \text{ es cte}$$

$$\mathcal{L} = \sum m_\alpha v_\alpha^2 - U(\vec{r}_\alpha)$$

(ii) Espacio es homogéneo

$$\delta L = 0 \quad \text{si} \quad \vec{r}_\alpha = \vec{r}_\alpha + \vec{\varepsilon}$$

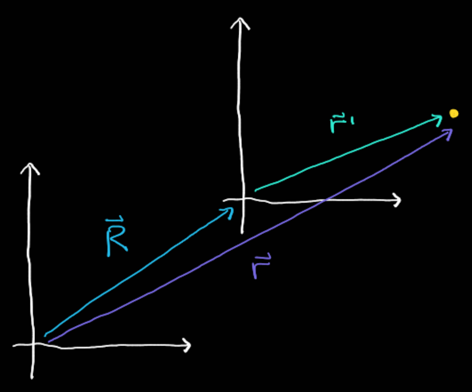
$$\delta L = \sum_\alpha \frac{\partial L}{\partial \vec{r}_\alpha}\cdot \underbrace{\delta\vec{r}_\alpha}_{\vec{\varepsilon}} = \vec{\varepsilon}\cdot\sum\frac{\partial L}{\partial \vec{r}_\alpha} = 0$$

$$\therefore \sum\frac{\partial L}{\partial \vec{r}_\alpha} = 0, \quad U \equiv 0$$

* Ecuación de Euler Lagrange

Como $\frac{\partial L}{\partial \vec{r}_\alpha} \overset{*}{=} \frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}_\alpha}\right) = \dot{\vec{p}}_\alpha$, entonces $\sum \dot{\vec{p}}_\alpha = 0$ y $\frac{d}{dt}\vec{p} = 0$

Consideremos ahora dos sistemas O y O'



$$\vec{r} = \vec{R} + \vec{r}', \quad \vec{v}_\alpha = \vec{V} + \vec{v}_\alpha'$$

$$\vec{p} = \sum m_\alpha \vec{V} + \sum m_\alpha \vec{v}_\alpha'$$

$$\vec{p} = \vec{p}_{CM} + \vec{p}'$$

Siempre que $\exists$ el sistema O' tendremos

$$\vec{p}' = 0$$

$$\vec{p} = \sum m_\alpha \vec{v}_\alpha$$

$$M\vec{V} = \sum m_\alpha \vec{v}_\alpha \Rightarrow \vec{V} = \frac{1}{M}\sum m_\alpha \vec{v}_\alpha$$

Energía interna

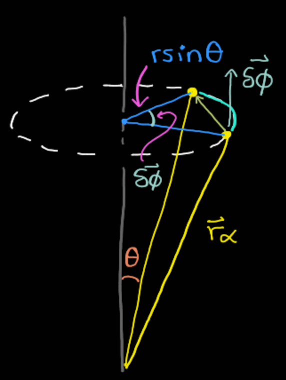
$E = \sum_\alpha \frac{1}{2} m_\alpha \vec{v}_\alpha + U$, donde $\vec{v}_\alpha = \vec{V} + \vec{v}_\alpha'$. Por lo tanto,

$$E = \frac{1}{2}\sum_\alpha m_\alpha \vec{V}^2 + \sum_\alpha m_\alpha \vec{V}\vec{v}_\alpha + \frac{1}{2}\sum_\alpha m_\alpha \vec{v}_\alpha'^2$$

$$= \frac{1}{2}\sum_\alpha m_\alpha \vec{V}^2 + \vec{V}\sum_\alpha m_\alpha \vec{v}_\alpha + \sum_\alpha m_\alpha \vec{v}_\alpha'^2 \Big\} E_i$$

$$\Rightarrow E = \frac{1}{2}M\dot{\vec{R}}^2 + E_i$$

(iii) Isotropía del espacio



$$|\delta\vec{s}| = r\sin\theta\,\delta\phi, \quad \delta\vec{s} = -\vec{r}_\alpha \times \delta\vec{\phi}, \quad \delta\vec{v}_\alpha = -\vec{v}_\alpha\times\delta\vec{\phi}$$

$$\mathcal{L} = \sum p_\alpha$$

$$\Rightarrow \delta L = \sum_\alpha\left[\frac{\partial L}{\partial \vec{r}_\alpha}\cdot\delta\vec{s}_\alpha + \frac{\partial L}{\partial \vec{v}_\alpha}\cdot\delta\vec{v}_\alpha\right]$$

$$= \sum_\alpha\left[\frac{\partial L}{\partial \vec{r}_\alpha}\cdot(\delta\vec{\phi}\times\vec{r}_\alpha) + \frac{\partial L}{\partial \vec{v}_\alpha}\cdot(\delta\vec{\phi}\times\vec{v}_\alpha)\right]$$

$$= \delta\vec{\phi}\cdot\sum_\alpha\left\{\vec{r}_\alpha\times\dot{\vec{p}}_\alpha + \dot{\vec{r}}_\alpha\times\vec{p}_\alpha\right\}$$

$$= \delta\vec{\phi}\cdot\frac{d}{dt}\left[\sum_\alpha \vec{r}_\alpha\times\vec{p}_\alpha\right] = 0$$

el momento ang. depende del sistema de coordenadas

$\therefore \sum_\alpha \vec{r}_\alpha\times\vec{p}_\alpha = \text{cte} = \vec{M}$. Consecuentemente $\vec{M} = \sum_\alpha \vec{r}_\alpha'\times\vec{p}_\alpha + \vec{r}\times\vec{p} = \vec{M}' + \vec{a}\times\vec{p}$.

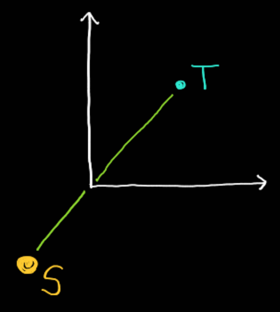
Por otra parte, como $\vec{v}_\alpha = \vec{V} + \vec{v}\,'_\alpha$ tenemos que

$$\vec{M} = \sum_\alpha \vec{r}\,'_\alpha \times (m_\alpha \vec{v}\,'_\alpha) + \sum_\alpha \vec{r}_\alpha \times (m_\alpha \vec{V}) = \sum \vec{r}_\alpha \times \vec{p}\,'_\alpha + \Big(\overbrace{\sum_\alpha m_\alpha \vec{r}_\alpha}^{M\vec{R}}\Big) \times \vec{V}$$

$\therefore\ M = \vec{M} + \vec{R} \times \vec{p}_{CM}.$

Ejemplo. Volvamos al problema de Tierra-Sol tal que ambos generan un campo gravitacional

$$\mathcal{L} = \frac{1}{2} m \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m \dot{\vec{r}}_2^{\,2} - U(|\vec{r}_1 - \vec{r}_2|).$$



$$\text{Sea}\ \left.\begin{array}{l} \vec{r} = \vec{r}_1 - \vec{r}_2 \\ m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0 \end{array}\right\} \text{coord. relativa}$$

$$\vec{r}_1 = \frac{M\vec{r}}{M+m} \qquad \vec{r}_2 = \frac{m\vec{r}}{M+m}$$

$$\boxed{\mathcal{L} = \frac{1}{2}\mu \dot{\vec{r}}^{\,2} - U(r)}, \qquad r(\theta) \to \text{elipse}$$

donde $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$.