Problema de Sturm-Liouville

$\mathcal{L}u + \lambda w u = 0$; $\quad \mathcal{L} = P_0(x)\frac{d^2}{dx^2} + P_1(x)\frac{d}{dx} + P_2(x)$

Ecuación de Laguerre $\quad xy'' + (1-x)y' + ay = 0$

Como $P_0 = x$; $\frac{dP_0}{dx} \neq P_1$. Entonces,

$$\exp\left(\int^x \frac{P_1(x')}{P_0(x')}dx'\right) = \exp\left(\int^x \frac{(1-x')}{x'}dx'\right)$$

$$= \exp\left(\int^x \frac{dx'}{x'} - \int^x dx'\right)$$

$$= e^{\ln(x) - x}$$

$$= xe^{-x}$$

$$\Rightarrow xe^{-x}y'' + e^{-x}(1-x)y' + ae^{-x}y = 0$$

$$P_0 = xe^{-x};\quad \frac{d}{dx}P_0 = -xe^{-x} + e^{-x} = e^{-x}(1-x) = P_1(x)$$

Aquí $W(x) = e^{-x}$. Buscamos a y b tales que

$$V^*(x)\,P_0(x)\,\frac{du}{dx}\Big|_a^b = 0 \qquad u \leftrightarrow y$$

de lo cual $[a,b] \rightarrow [0,\infty)$.

Ecuación de Hermite $\quad y'' - 2xy' + 2\alpha y = 0$

$$\int^x \frac{P_1(x')}{P_0(x')}dx' = \int^x -2x'dx' = -x^2 \Rightarrow \exp\left(\int^x \frac{P_1(x')}{P_0(x')}dx'\right) = e^{-x^2}$$

Por lo cual, $P_0' = P_1$; $P_0 = e^{-x^2}$; $\frac{dP_0}{dx} = -2xe^{-x^2}$. Como buscamos a y b tales que

$$V^*(x)\,P_0(x)\,\frac{du}{dx}\Big|_a^b = 0$$

entonces $[a,b] = (-\infty,\infty)$.

El Oscilador armónico (pt. 420)

$$y'' + \omega^2 y = 0 \longrightarrow \begin{cases} y = A\sin(\omega x) \\ y = B\cos(\omega x) \end{cases}$$

Para $\mathcal{L}u + \lambda\omega u = 0$, con $P_0\frac{d^2u}{dx^2} + P_1\frac{du}{dx} + P_2 u = 0$ entonces

$$P_0 = 1,\quad P_1 = 0;\quad w = 1;\quad \lambda = \omega^2$$

Por lo cual $v^* P_0 u'\Big|_a^b = 0 \Rightarrow m\cos(mx)\cos(mx)\Big|_a^b = 0$

$$\Rightarrow m\cos(mb)\cos(mb) = m\cos(ma)\cos(ma)$$

Si $a = x_0 \Rightarrow b = x_0 + 2\pi$

Ec. de Legendre $(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$

donde $[a,b] \to [-1,1]$; $w(x) = 1$.

Sea el conjunto $u_n(x) = x^n$, $n \in \mathbb{N}_0$.

Desarrollamos los primeros tres términos

Como $\varphi_n(x) = \dfrac{\psi_n(x)}{\left(\int_a^b \psi_n^2(x) w(x) dx\right)^{1/2}}$, donde

$$\psi_i(x) = u_i(x) + a_{i0}\varphi_0(x) + a_{i1}\varphi_1(x) + \cdots + a_{i\,i-1}\varphi_{i-1}(x)$$

$(n=0)$ $u_0 = 1$, $\psi_0 = 1$, $\varphi_0 = 1/\sqrt{2}$

$(n=1)$ $u_1 = x$, $\psi_1(x) = x + \frac{1}{\sqrt{2}} a_{10}$; $a_{10} = \int_{-1}^{1} \frac{x}{\sqrt{2}} dx = 0 \Rightarrow \psi_1(x) = x$

$$\varphi_1(x) = \frac{x}{\left(\int_{-1}^{1} x^2 dx\right)^{1/2}} = \sqrt{\frac{3}{2}}\, x$$

$(n=2)$ $u_2 = x^2$,

$$\psi_2(x) = u_2(x) + a_{20}\varphi_0(x) + a_{21}\varphi_1(x) = x^2 + \frac{1}{\sqrt{2}} a_{20} + a_{21}\sqrt{\frac{3}{2}}\, x$$

$$a_{20} = \int_{-1}^{1} \frac{x^2}{\sqrt{2}} dx = \frac{1}{\sqrt{2}}\left(-\frac{2}{3}\right), \quad a_{21} = \int_{-1}^{1} x^3 \sqrt{\frac{3}{2}}\, dx = 0$$

$$\Rightarrow \psi_2(x) = x^2 - \frac{1}{3}$$

$$\varphi_2(x) = \frac{x^2 - 1/3}{\left(\int_{-1}^{1} (x^2 - 1/3)^2 dx\right)^{1/2}}$$

Función Gamma

$$\Gamma(z) = \int_{-\infty}^{\infty} e^{-t} t^{z-1} dt \qquad \Gamma(z) = \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1} dt \qquad Re(z) > 0$$

$$\Gamma(z) = 2\int_0^{\infty} e^{-t^2} t^{2z-1} dt \qquad \Gamma(z+1) = z\Gamma(z)$$

Ejemplo. $I = \int_0^{\infty} \sqrt{x}\, e^{-x^3} dx$.

Hint: $x^3 = t$; $x = t^{1/3}$; $\sqrt{x} = t^{1/6} \Rightarrow 3x^2 dx = dt \Rightarrow dx = dt/3t^{2/3}$

Ejemplo: $I = \int_0^a x^4 \sqrt{a^2 - x^2}\, dx$

Sea $x^2 = a^2 t$; $x = a t^{1/2} \Rightarrow 2x dx = a^2 dt \Rightarrow dx = \frac{a^2 dt}{2x} = \frac{a^2 dt}{2a t^{1/2}} = \frac{1}{2} a t^{-1/2} dt$.

$$\Rightarrow I = \int_0^1 a^4 t^2 \sqrt{a^2 - a^2 t}\left(\tfrac{1}{2}\right) a t^{-1/2} dt = \frac{a^6}{2}\int_0^1 t^{3/2}\sqrt{1-t}\, dt$$

$$\Rightarrow I = \frac{a^6}{2}\frac{\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}{\Gamma(4)}$$

P.d. $B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \equiv \int_0^1 t^{x-1}(1-t)^{y-1} dt$

Veamos quien es $m!n!$. Por un lado note que

$$(z+1)! = \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

$$z! = \Gamma(z+1) = \int_0^\infty e^{-t} t^{z} dt$$

De modo tal que

$$m!n! = \lim_{a^2\to\infty} \int_0^{a^2} e^{-u} u^m du \int_0^{a^2} e^{-v} v^n dv$$

Sea entonces

$$u \to x^2 \qquad du = 2x\,dx \qquad si\ u = a^2 \Rightarrow x = a$$

$$v \to y^2 \qquad dv = 2y\,dy \qquad si\ v = a^2 \Rightarrow y = a$$

Por lo cual,

$$m!n! = \lim_{a\to\infty} 4 \int_0^a e^{-x^2} x^{2m}\, x\,dx \int_0^a e^{-y^2} y^{2n}\, y\,dy$$

$$= \lim_{a\to\infty} 4 \int_0^a e^{-x^2} x^{2m+1} dx \int_0^a e^{-y^2} y^{2n+1} dy$$

Considere $x = r\cos\theta,\ y = r\sin\theta \Rightarrow dx\,dy = r\,dr\,d\theta$

$$m!n! = \lim_{a\to\infty} 4 \int_0^a e^{-r^2} r^{2m+2n+3} dr \int_0^{\pi/2} \cos^{2m+1}\theta \sin^{2n+1}\theta\, d\theta$$

$\therefore\ \Gamma(z) = 2\int_0^\infty e^{-u^2} u^{2z-1} du$. Si $z = m+n+1$

$$m!n! = 2(m+n+1)! \int_0^{\pi/2} \cos^{2m+1}\theta\ \sin^{2n+1}\theta\, d\theta$$

$$\Rightarrow \frac{m!n!}{(m+n+1)!} = 2\int_0^{\pi/2} \cos^{2m+1}\theta\ \sin^{2n+1}\theta\, d\theta = B(m+1, n+1)$$

Note que $\cos^{2m+1}\theta \sin^{2n+1}\theta = \cos^{2m}\theta\cos\theta \sin^{2n}\theta\sin\theta = (\cos^2\theta)^m (\sin^2\theta)^n \cos\theta\sin\theta$.

Si $t = \sin^2\theta \Rightarrow \cos^2\theta = 1-\sin^2\theta = 1-t \Rightarrow dt = 2\sin\theta\cos\theta\, d\theta$

$$\Rightarrow 2\int_0^{\pi/2} \cos^{2m+1}\theta\ \sin^{2n+1}\theta\, d\theta = 2\int_0^1 (1-t)^m t^n \frac{dt}{2} = \int_0^1 (1-t)^m t^n dt$$