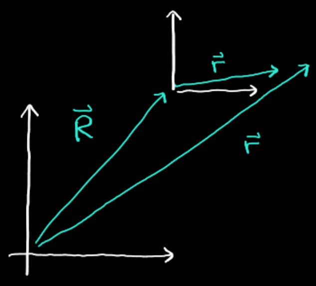
27 feb 23

(*) Recordando que... $T = T_{tras} + T_{rot}$

$$T = \frac{1}{2} M V^2 + \frac{1}{2} \vec{\Omega} \cdot \underset{\sim}{I} \cdot \vec{\Omega}$$

$$\boxed{I_{ij} = \sum_\alpha m_\alpha \left(r_\alpha^2 \delta_{ij} - x_i^\alpha x_j^\alpha \right)}$$ tensor de inercia cuyo origen está en el centro de masas

$$I'_{ij} = \sum_\alpha m_\alpha \left(r'^2_\alpha \delta_{ij} - x_i'^\alpha x_j'^\alpha \right)$$



$\vec{r} = \vec{R} + \vec{r}'$

$$\therefore \; I'_{ij} = \sum_\alpha m_\alpha \left((\vec{r}^2 - \vec{R})^2 \delta_{ij} - (x_i^\alpha - R_i)(x_j^\alpha - R_j) \right)$$

$$= \sum_\alpha m_\alpha \left(r^2 \delta_{ij} - x_i^\alpha x_j^\alpha \right) + \sum_\alpha m_\alpha \left(R^2 \delta_{ij} - R_i R_j \right)$$

$$\boxed{- \cancel{2\vec{R} \cdot \sum_\alpha m_\alpha \vec{r}_\alpha}^{\,0} - \cancel{\sum_\alpha m_\alpha x_i^\alpha R_j}^{\,0} - \cancel{\sum_\alpha m_\alpha x_j^\alpha R_i}^{\,0}}$$

centro de masa visto desde el centro de masa

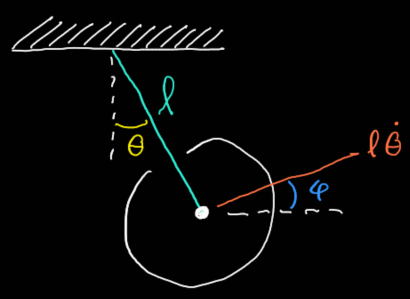
$$\therefore \; I'_{ij} = I_{ij} + M \left(R^2 \delta_{ij} - R_i R_j \right)$$

de donde rescatamos el teorema de ejes principales con $i = j = 1$,

$$I'_1 = I_1 + M R^2$$

denota al eje principal 11

(*)



$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I \dot{\ell}^2$$

$$x = \ell \cos\theta$$

$$y = \ell \sin\theta$$

$$T = \frac{1}{2} m \ell^2 \dot{\theta}^2 + \frac{1}{2} I \dot{\varphi}^2, \quad T = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I \dot{\varphi}^2$$

(*) $\vec{M} = \sum_\alpha m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha$ con $\vec{v}_\alpha = \vec{\Omega} \times \vec{r}_\alpha$, entonces

$$\vec{M} = \sum m_\alpha \vec{r}_\alpha \times (\vec{\Omega} \times \vec{r}_\alpha)$$

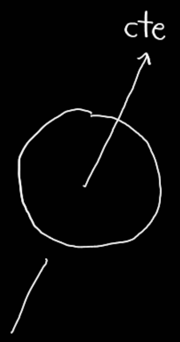
$$= \sum m_\alpha \left(\vec{r}_\alpha^{\,2} \vec{\Omega} - (\vec{r}_\alpha \cdot \vec{\Omega}) \vec{r} \right)$$

$$M_i = \sum_\alpha m_\alpha \left(r_\alpha^2 \Omega_i - x_j^\alpha \Omega_j x_i^\alpha \right)$$

$$= \sum_\alpha m_\alpha \Omega_i \left(r_\alpha^2 \delta_{ji} - x_i^\alpha x_j^\alpha \right)$$

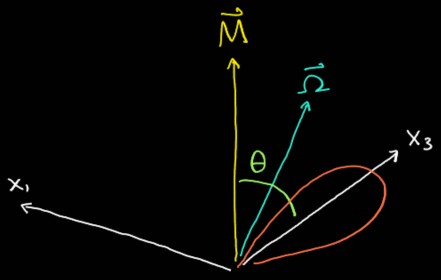
$$= I_{ij} \Omega_j$$

$$\therefore \; \boxed{\vec{M} = \underset{\sim}{I} \cdot \vec{\Omega}.}$$



Si $\dot{\vec{M}} = 0$ (espacio isotrópico) $\Rightarrow$ $\vec{M} = cte$

$$\underset{\sim}{I} = I \mathbb{1} \; ; \quad I_{ij} = I \delta_{ij}$$



x_2 sale del plano

$\vec{M} = (M_1, 0, M_3)$ donde

$$M_1 = I_1 \Omega_1$$

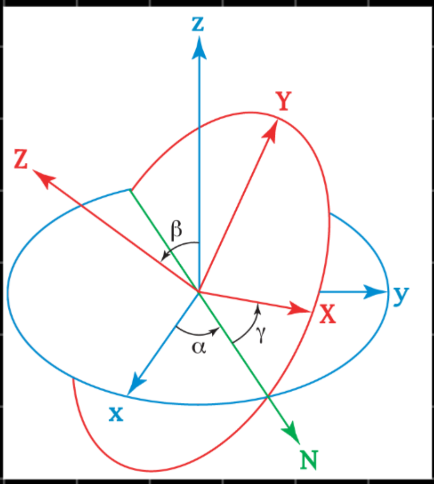
$$M_2 = I_2 \Omega_2 \quad \Rightarrow \quad \Omega_2 = 0$$

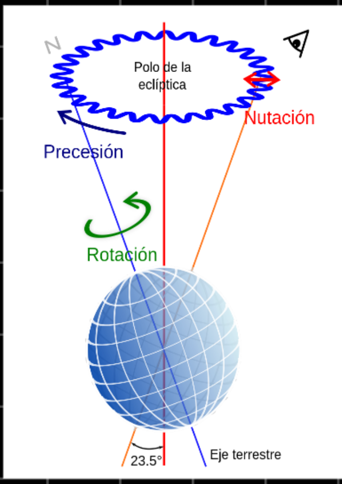
$$M_3 = I_3 \Omega_3$$

$$\vec{v} = \vec{\Omega} \times \vec{r}$$

$$\Omega_3 = \frac{M_3}{I_3} = \frac{M \cos\theta_3}{I_3}, \quad \Omega_2 = \frac{M_1}{I_1} = \frac{M \sin\theta_1}{I_1} \quad \Rightarrow \quad \Omega_{pr} = \frac{M}{I_1}$$

Ángulos de Euler





También se considera la notación $\alpha = \phi, \gamma = \psi$ y $\beta = \theta$.

- $\phi \sphericalangle (x, ON)$
- $\psi \sphericalangle (ON, X)$
- $\theta \sphericalangle (Z, z)$

- $\phi, \psi \in [0, 2\pi)$
- $\theta \in [0, \pi]$

Consideraremos a $(X, Y, Z) = (x_1, x_2, x_3)$.

Si $\dot{\vec{\Theta}} = (\dot{\Theta}_1, \dot{\Theta}_2, \dot{\Theta}_3)$, entonces

$$\dot\theta_1 = \dot\theta \cos\psi \qquad \dot\phi_1 = \dot\phi \sin\theta \sin\psi$$

$$\dot\theta_2 = -\dot\theta \sin\psi \qquad \dot\phi_2 = \dot\phi \sin\theta \cos\psi$$

$$\dot\theta_3 = 0 \qquad \dot\phi_3 = \dot\phi \cos\theta \qquad \dot\psi_3 = \dot\psi$$

$$\Omega_1 = \dot\phi_1 + \dot\theta_1 = \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi$$

$$\Omega_2 = \dot\phi_2 + \dot\theta_2 = \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi \qquad L = T(q, \dot q) - U$$

$$\Omega_3 = \dot\phi_3 + \dot\psi_3 = \dot\phi \cos\theta + \dot\psi$$

$$T_{rot} = \frac{1}{2} I_1 \Omega_1^2 + \frac{1}{2} I_2 \Omega_2^2 + \frac{1}{2} I_3 \Omega_3^2$$

Suponga una peonza simétrica tal que $I_1 = I_2 \neq I_3$,

$$T = \frac{1}{2} I_1 (\dot\phi^2 \sin^2\theta + \dot\theta^2) + \frac{1}{2} I_3 (\dot\phi \cos\theta + \dot\psi)^2$$

Si $\psi = 0$, se tiene que

$$\Omega_1 = \dot\theta; \qquad \Omega_2 = \dot\phi \sin\theta; \qquad \Omega_3 = \dot\phi \cos\theta + \dot\psi$$

$\vec{M} \parallel z$. Rotación intrínseca respecto al eje x_3.

$$M_1 = I_1 \Omega_1 = I_1 \dot\theta$$

$$M_2 = I_2 \Omega_2 = I_2 \dot\phi \sin\theta$$

$$M_3 = I_3 \Omega_3 = I_3 (\dot\phi \cos\theta + \dot\psi) = \frac{\partial L}{\partial \dot\psi}$$

$$M_z = (I_1' \sin^2\theta + I_3 \cos^2\theta) \dot\phi + I_3 \dot\psi \cos\theta = \frac{\partial L}{\partial \dot\phi}$$

$$\left\{ \begin{array}{l} M_1 = 0 \\ M_2 = M \sin\theta \Rightarrow I_2 \dot\phi = M \\ M_3 = M \cos\theta \end{array} \right.$$

$$E = \frac{1}{2}(I_1 + \mu l^2)(\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \frac{1}{2} I_3 (\dot\psi + \dot\phi \cos\theta)^2 + \mu g l \cos\theta$$

$$E' = \frac{1}{2} I_1' \dot\theta^2 + U_{eff}(\theta)$$

$$\dot\phi = \frac{M_z - M_3 \cos\theta}{I_1' \sin\theta}, \qquad \dot\psi = \frac{M_3}{I_3} - \cos(\theta) \frac{M_z - M_3 \cos\theta}{I_1' \sin^2\theta}$$