14 marzo 23

Funciones Especiales

Función Gamma $\Gamma(z)$

Definición de Euler $\quad \Gamma(z) = \lim\limits_{n\to\infty} \dfrac{1\cdot 2\cdot 3\cdots n\cdot n^z}{z(z+1)(z+2)\cdots(z+n)}$

Nótese que
$$\Gamma(z+1) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n\cdot n^{z+1}}{(z+1)(z+2)\cdots(z+1+n)}$$
$$= \lim_{n\to\infty} \frac{zn}{(z+n+1)} \frac{1\cdot 2\cdot 3\cdots n\cdot n^z}{(z+1)(z+2)\cdots(z+n)}$$
$$= \lim_{n\to\infty} \frac{zn}{z+n+1} \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n\cdot n^z}{(z+1)(z+2)\cdots(z+n)}$$
$$= z\,\Gamma(z)$$

$\therefore$ $\boxed{\Gamma(z+1) = z\,\Gamma(z)}$.

A partir de la definición obtenemos que
$$\begin{cases} \Gamma(1) = 1 \\ \Gamma(2) = 1 \\ \Gamma(3) = 2 \\ \quad\vdots \\ \Gamma(n+1) = n! \end{cases}$$

Representaciones de $\Gamma(z)$

(i) $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$

(ii) $\Gamma(z) = 2\int_0^\infty e^{-t^2} t^{2z-1} dt$

(iii) $\Gamma(z) = \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1} dt$

Sea $u^2 = t \Rightarrow 2u\,du = dt$. Por lo cual, si $e^{u^2} = e^t$, entonces

$\cdots \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1} dt$

Dada $F(z,n) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt$, sea $\omega = \frac{t}{n} \Rightarrow n\,d\omega = dt$. Si $t\to 0 \Rightarrow \omega = 0$, si $t = n \Rightarrow \omega = 1$. Por lo cual, si se toma

$$u_1 = (1-\omega)^n \Rightarrow du_1 = -n(1-\omega)^{n-1}d\omega; \qquad dv_1 = \omega^{z-1}d\omega \Rightarrow v_1 = \frac{\omega^z}{z}$$

Entonces

$$F(z,n) = \int_0^1 (1-\omega)^n \omega^{z-1} d\omega$$
$$= n^z\left[\cancel{(1-\omega)^n \frac{\omega^z}{z}\Big|_0^1}^{\,0} + \frac{n}{z}\int_0^1 (1-\omega)^{n-1}\omega^z d\omega\right]$$
$$= \frac{n^z n}{z}\int_0^1 (1-\omega)^{n-1}\omega^z d\omega$$

$$u_2 = (1-\omega)^{n-1} \Rightarrow du_2 = -(n-1)(1-\omega)^{n-2}d\omega; \qquad dv_2 = \omega^z d\omega \Rightarrow v_2 = \frac{\omega^{z+1}}{z+1}$$

$$\Rightarrow F(z,n) = \frac{n^z n}{z}\int_0^1 (1-\omega)^{n-1}\omega^z d\omega$$
$$= \frac{n^z n}{z}(n-1)\frac{1}{z+1}\int_0^1 (1-\omega)^{n-2}\omega^{z+1} d\omega$$
$$\vdots$$
$$=$$

Dado $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$. **P.D.** $\lim\limits_{n\to\infty}\left(1 - \frac{t}{n}\right)^n = e^{-t}$.

Observe que $(a+b)^n = \sum\limits_{k=0}^{n}\binom{n}{k} a^k b^{n-k}$. Tomando $a = -\frac{t}{n}$, $b = 1$ se sigue

$$\left(1-\frac{t}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\left(-\frac{t}{n}\right)^k \underset{n\to\infty}{\overset{?}{=}} \sum_{k=0}^{n}\binom{n}{k}\left(-\frac{t}{n}\right)^k = e^{-t}$$

Se tiene que

$$\lim_{n\to\infty}\frac{1}{n^k}\binom{n}{k} = \lim_{n\to\infty}\frac{1}{n^k}\frac{n!}{(n-k)!\,k!} = \frac{1}{k!}\lim_{n\to\infty}\frac{n!}{n^k(n-k)!} = \frac{1}{k!}$$

La fórmula de Stirling $\ln N! = N\ln N - N$, por lo cual

$$\ln\left(\frac{n!}{n^k(n-k)!}\right) = n\ln(n) - k\ln(n) - \cancel{n} - (n-k)\ln(n-k) + \cancel{n} - k$$

$$\ln(n-k) = \ln n\left(1-\frac{k}{n}\right) = \ln(n) + \ln\left(1-\frac{k}{n}\right) = \ln(n) - \frac{k}{n}$$

$$\begin{aligned}\therefore \ln\left(\frac{n!}{n^k(n-k)!}\right) &= \ln(n!) - \ln(n^k n(n-k)!) \\ &= n\ln(n) - \ln(n^k) - \ln(n(n-k)!) \\ &= n\ln(n) - k\ln(n) - \ln(n) - \ln[(n-k)!] \\ &= n\ln(n) - k\ln(n) - \ln(n) - (n-k)\ln(n-k) - n + k \\ &\quad - (n-k)\ln(n) + (n-k)\frac{k}{n} \\ &= \cdots = -\frac{k^2}{n}\end{aligned}$$

$$\therefore \lim_{n\to\infty}\ln\left[\frac{n!}{n^k(n-k)!}\right] = \lim_{n\to\infty} -\frac{k^2}{n} = 0$$

$$\therefore \lim_{n\to\infty}\left(1-\frac{t}{n}\right)^n = e^{-t}$$

$$\therefore \lim_{n\to\infty} F(n,z) = \int_0^\infty e^{-t}t^{z-1}dz$$

Extensión de $\Gamma(x)$ a $\mathbb{R}^-$

Considere $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt \Rightarrow \Gamma(1) = \int_0^\infty e^{-t}dt = -e^{-t}\Big|_0^\infty = 1$

Como $\Gamma(x+1) = x\Gamma(x)$, entonces

* $\Gamma(0) = \frac{\Gamma(1)}{0} \to \infty$
* $\Gamma(-1) = -\Gamma(-1+1) = -\Gamma(0) \to -\infty$
* $\Gamma(-2) = \frac{\Gamma(-2+1)}{-2} = -\frac{\Gamma(-1)}{2} \to \infty$

$\vdots$

$$\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}} = \frac{\Gamma\left(\frac{1}{2}\right)}{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)} = \frac{4}{3}\Gamma\left(\frac{1}{2}\right)$$

Note que

si $t=0$ $u=0$

$t\to\infty$ $u\to\infty$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t}t^{\frac{1}{2}-1}dt = \int_0^\infty e^{-t}t^{-1/2}dt; \quad t=u^2 \Rightarrow dt = 2u\,du$$

$$\Rightarrow \int_0^\infty e^{-u^2}\frac{1}{u}2u\,du = 2\int_0^\infty e^{-u^2}du \Rightarrow \int_0^\infty e^{-u^2}du = \boxed{\frac{\sqrt{\pi}}{2}}$$

$$\therefore \Gamma\left(-\frac{5}{2}\right) = \frac{\Gamma\left(-\frac{3}{2}\right)}{\left(-\frac{5}{2}\right)} = -\frac{\sqrt{\pi}}{\left(\frac{15}{8}\right)} = -\frac{8}{15}\sqrt{\pi}$$

Función Digamma $\digamma(z)$

Como $z! = z\Gamma(z) = \lim_{n\to\infty} \dfrac{n!}{(z+1)(z+2)\cdots(z+n)} n^z$

$$\ln z! = \lim_{n\to\infty}\left[\ln n! + z\ln(n) - \ln(z+1) - \ln(z+2) - \ldots - \ln(z+n)\right]$$

$$\frac{d}{dz}\ln z! \equiv \digamma(z) = \lim_{n\to\infty}\left(\ln(n) - \frac{1}{(z+1)} - \frac{1}{(z+2)} - \cdots - \frac{1}{(z+n)}\right)$$

Función Poligamma $\digamma^m(z) = \dfrac{d^{m+1}}{dz^{m+1}}\ln(z!)$

$$\digamma(z) = \lim_{n\to\infty}\left(\ln(n) - \frac{1}{(z+1)} - \frac{1}{(z+2)} - \cdots - \frac{1}{(z+n)}\right)$$

$$\frac{d}{dz}\digamma(z) = \lim_{n\to\infty}\left(\frac{1}{(z+1)^2} - \frac{1}{(z+2)^2} - \cdots - \frac{1}{(z+n)^2}\right) = \boxed{}$$

$$\frac{d^2}{dz^2}\digamma(z) = \lim_{n\to\infty}\left[\frac{-2(z+1)}{(z+1)^4} - \frac{2(z+2)}{(z+2)^4} - \cdots - \frac{-2(z+n)}{(z+n)^4}\right] = -\sum_{n=1}^{\infty}\frac{2}{(z+n)^3}$$

$$\digamma^m(z) = \left(\frac{d}{dz}\right)^m \digamma(z) = \left(\frac{d}{dz}\right)^{m+1}\ln\Gamma(z)$$