15 feb 23

Recordemos que, $J = \int_{x_1}^{x_2} f(y(x), y'(x); x)\,dx$ y dado un parámetro α

$$J(\alpha) = \int_{x_1}^{x_2} f(y(x,\alpha), y'(x,\alpha); x)\,dx$$

Ejemplos simples:

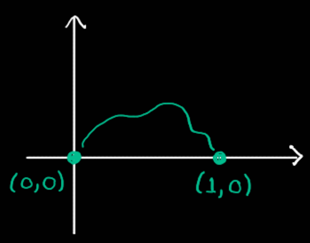
(i) $y = x$

$$y(x,\alpha) = x + \alpha \sin(x) \qquad J(\alpha) = \int_0^{2\pi} (1 + 2\alpha\cos(x) + \alpha^2\cos^2(x))\,dx$$

$$f = \left(\frac{dy}{dx}\right)^2, \quad x \in [0, 2\pi] \qquad = 2\pi + \alpha^2\pi$$

$$\frac{dy}{dx} = 1 + \alpha\cos(x)$$

(ii) Buscamos la distancia mínima entre dos puntos



$$ds = \sqrt{dx^2 + dy^2}$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \Rightarrow S = \int_0^1 \sqrt{1 + (y')^2}\,ds$$

$$y(x) = 0$$

$$y(x,\alpha) = \alpha^2(x^2 - x)$$

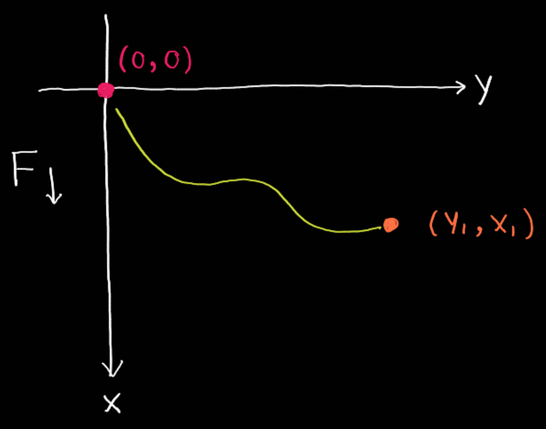
$$\Rightarrow S(\alpha) = \int_0^1 \left[(4\alpha^2)x^2 + (-4\alpha^2)x + (\alpha^2+1)\right]^{1/2} dx$$

$$= \frac{1}{2}\sqrt{\alpha^2+1} + \frac{1}{2\alpha}\sinh^{-1}(\alpha)$$

$$= \frac{1}{2}\left(1 + \frac{\alpha^2}{2} + \cdots\right) + \frac{1}{2\alpha}\left(\alpha - \frac{1}{6}\alpha^3 + \cdots\right)$$

$$= 1 + \frac{\alpha^2}{6} + O(\alpha^3)$$

La Braquistócrona



$$T + U = E$$

$$\frac{1}{2}mv^2 - mgx = \cancel{E}^{\,0} \Rightarrow v = \sqrt{2gx}$$

$$t = \int_0^{x_1} \frac{ds}{v} \; ; \quad ds = \sqrt{1 + (y')^2}$$

$$\Rightarrow t = \int_0^{x_1} \frac{\sqrt{1 + (y')^2}}{\sqrt{2gx}}\,dx$$

Aquí $f = \left[\dfrac{1 + (y')^2}{2gx}\right]^{1/2}$. De modo tal que,

$$\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) - \frac{\partial f}{\partial y} = 0$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \Rightarrow \frac{\partial f}{\partial y'} = cte = \frac{1}{\sqrt{2a}} = \left\{\frac{(y')^2}{x[1 + (y')^2]}\right\}^{1/2}$$

$$\Rightarrow y'(x) = \frac{dy}{dx} = \frac{x}{(2ax - x^2)^{1/2}}$$

$$\Rightarrow y(x) = \int \frac{x}{(2ax - x^2)^{1/2}}\,dx$$

$$\Rightarrow y(\theta) = a(\theta - \sin\theta) + A$$

$$\Rightarrow \begin{cases} x(\theta) = a(1 - \cos\theta) \\ dx = a\sin\theta\, d\theta \end{cases}$$

parametrización de una cicloide

Ahora bien, $f = f(y, y', z, z'; x)\,dx$ y $g = g(y, z; x)$ tales que

restricción geométrica

$$y(x,\alpha) = y(x) + \alpha\, y(x)$$

$$z(x,\alpha) = z(x) + \alpha\, z(x)$$

$$\eta_1(x_1) = \eta_2(x_1) = \eta_1(x_2) = \eta_2(x_2)$$

Como $\dfrac{\partial J(\alpha)}{\partial \alpha} = \displaystyle\int \frac{\partial f}{\partial \alpha}\,dx$, entonces

$$\frac{\partial J}{\partial \alpha} = \int \left[\left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha} \right) + \left(\frac{\partial f}{\partial z}\frac{\partial z}{\partial \alpha} + \frac{\partial f}{\partial z'}\frac{\partial z'}{\partial \alpha} \right) \right] dx$$

$$\Rightarrow \left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = \int \left[\left(\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial y'}\right) \right)\frac{\partial y}{\partial \alpha} + \left(\frac{\partial f}{\partial z} + \frac{d}{dt}\left(\frac{\partial f}{\partial z'}\right) \right)\frac{\partial z}{\partial \alpha} \right] dx = 0 \quad \cdots (i)$$

Ahora bien, nótese que $\dfrac{\partial J(\alpha)}{\partial \alpha} = \displaystyle\int \frac{\partial f}{\partial \alpha}\,dx$, y además

$$dg = \left(\frac{\partial g}{\partial y}\overbrace{\frac{\partial y}{\partial \alpha}}^{\eta_1(x)} + \frac{\partial g}{\partial z}\overbrace{\frac{\partial z}{\partial \alpha}}^{\eta_2(x)} \right) d\alpha = 0 \Rightarrow \frac{\eta_2(x)}{\eta_1(x)} = -\frac{\partial g/\partial y}{\partial g/\partial z} \quad \cdots (ii)$$

Sustituyendo esto último en (i) tenemos que

$$\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = \int \left[\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial y'}\right) + \left(\frac{\partial f}{\partial z} + \frac{d}{dt}\left(\frac{\partial f}{\partial z'}\right) \right)\frac{\eta_2(x)}{\eta_1(x)} \right] \eta_1(x)\,dx = 0$$

Sin embargo,

$$\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial y'}\right) = -\left(\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right) \right)\frac{\eta_2}{\eta_1}$$

$$\Rightarrow \underbrace{\left(\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial y'}\right) \right)\left(\frac{\partial g}{\partial y}\right)^{-1}}_{\lambda(x)} \overset{(ii)}{=} \left(\frac{\partial f}{\partial z} - \frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right) \right)\left(\frac{\partial g}{\partial z}\right)^{-1}$$

$$\Rightarrow \begin{cases} \dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = \lambda(x)\dfrac{\partial g}{\partial y} \\[2ex] \dfrac{\partial f}{\partial z} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial z'}\right) = \lambda(x)\dfrac{\partial g}{\partial z} \end{cases}$$