18 mayo 2023

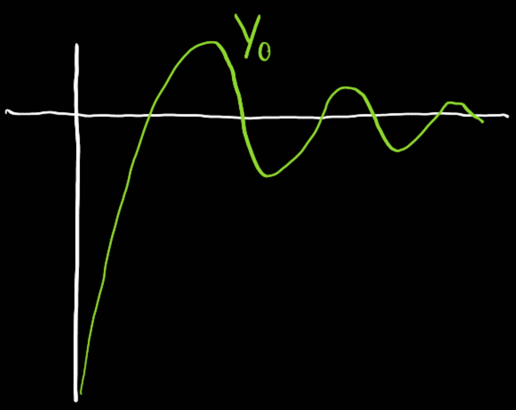
Funciones de Bessel

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\nu^2)y=0 \leftrightarrow x^2Z_\nu''+xZ_\nu'+(x^2-\nu^2)Z_\nu=0$$

$$Z_\nu(x)=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\frac{1}{(j+\nu)!}\left(\frac{x}{2}\right)^{2j+\nu};\quad Z_\nu(x)\leftrightarrow J_\nu(x)\leftrightarrow J_n(x)$$

$J_\nu(x)$: Función de Bessel de 1ª especie

$$y=AJ_\nu(x)+BY_\nu(x) \qquad J_{-n}(x)=(-1)^nJ_n(x)$$



$$Y_\nu(x)=\frac{J_\nu(x)\cos(\nu x)-J_{-\nu}(x)}{\sin(\nu x)}$$

$Y_\nu(x)$: Función de Bessel de 2ª especie (Neumann)

Funciones de Hankel

$$H_\nu^{(1)}(x)=J_\nu(x)+iY_\nu(x)$$
$$H_\nu^{(2)}(x)=J_\nu(x)-iY_\nu(x)$$

$$\psi(x)=A_1H_\nu^{(1)}(x)+B_1H_\nu^{(2)}(x)$$

Funciones de Bessel modificadas

$$x^2y''+xy'+(x^2-\nu^2)y=0 \longrightarrow x^2y''+xy'+(x^2+\nu^2)y$$

Helmholtz: $\nabla^2\psi+k^2\psi=0$

Difusión: $\nabla^2\psi-k^2\psi=0$

$$\psi=\psi(\rho,\varphi,z)=R(\rho)\Phi(\varphi)Z(z)$$

$$\rho^2\frac{d^2R}{d\rho^2}+\rho\frac{dR}{d\rho}-(x^2+\nu^2)R=0$$

Vibración de una membrana circular o tambor

$$\frac{\partial^2u}{\partial t^2}=v^2\nabla^2u$$

$$\nabla^2u=\left(\frac{\partial^2u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\cancelto{0}{\frac{\partial^2u}{\partial\varphi^2}}\right)$$

$$u=u(r,t)=R(r)T(t)$$
$$u=u(r,\varphi,t)\left\{\frac{1}{r^2}\frac{\partial^2u}{\partial t^2}\right\}=R(r)\Phi(\varphi)T(t)$$

Condiciones iniciales y condiciones de contorno

$$t=0\begin{cases}u(r,0)=u_0(r)\\ \left.\dfrac{\partial u(r,t)}{\partial t}\right|_{t=0}=\dot{u}_0(r)\end{cases}\qquad u(a,0)=0$$

Dado $u=R(r)T(t)$ tenemos que

$$R\frac{d^2T}{dt^2}=Tv^2\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)$$

$$\frac{1}{T}\frac{d^2T}{dt^2}=\frac{1}{R}v^2\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)=-\omega^2$$

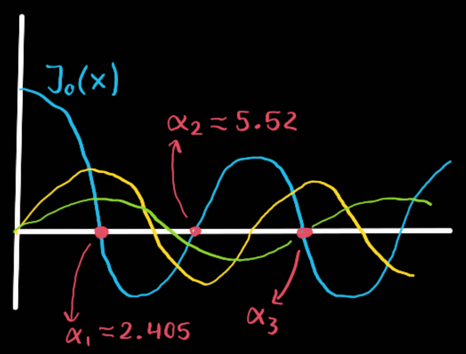
$$\begin{cases}\dfrac{d^2T}{dt^2}+\omega^2T=0\\ \dfrac{d^2R}{dr^2}+\dfrac{1}{r}\dfrac{dR}{dr}+\dfrac{\omega^2}{v^2}R=0\end{cases}$$

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+r^2\frac{\omega^2}{v^2}R=0\ ;\quad x=\frac{\omega}{v}r\ ;\quad \frac{d}{dr}=\frac{dx}{dr}\frac{d}{dx}=\frac{\omega}{v}\frac{d}{dx}$$

$$\Rightarrow \frac{v^2}{\omega^2}x^2\frac{\omega^2}{v^2}\frac{d^2R}{dr^2}+\frac{v}{\omega}x\frac{\omega}{v}\frac{dR}{dr}+x^2R=0$$

$$\Rightarrow x^2\frac{d^2R}{dr^2}+x\frac{dR}{dr}+x^2R=0$$

donde $R(x) = C J_0(x) + B Y_0(x)$, pero $B \equiv 0$. $R(r) = J_0\left(\frac{w}{v} r\right)$



$$R(r) = J_0\left(\frac{w}{v} r\right) \longrightarrow J_0\left(\frac{\omega_n}{v} r\right) \rightarrow \alpha = \frac{\omega a}{v}$$

$\alpha \leftrightarrow \alpha_n$

$\omega \leftrightarrow \omega_n$

$$\frac{d^2 T}{dt^2} + \omega_n^2 T = 0$$

$$T(t) \rightarrow T_n(t) = A_n \cos(\omega_n t) + B_n \sin(\omega_n t)$$

$$u_n(r,t) = R_n(r) T_n(t) = J_0\left(\frac{\alpha_n r}{a}\right)\left(A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right)$$

$$u(r,t) = \sum_{n=1}^{\infty} J_0\left(\frac{\alpha_n r}{a}\right)\left(A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right)$$

Ortogonalidad de las funciones de Bessel

$$\int_0^1 x J_\nu(\alpha_{\nu m} x) J_\nu(\alpha_{\nu \ell} x)\, dx = \begin{cases} 0 & \text{si } m = \ell \\ \frac{1}{2} J_{\nu+1}^2(\alpha_m) & \text{si } m = \ell \end{cases}$$

con $\alpha_{\nu m}$ y $\alpha_{\nu \ell}$ los ceros de m y ℓ de la función de Bessel de orden ν.

Note que $u(r,0) = \sum_{n=1}^{\infty} J\left(\frac{\alpha_n r}{a}\right) A_n$, así pues

$$\int_0^a r\, u(r,0) J_0\left(\frac{\alpha_m r}{a}\right) dr = \sum_{n=1}^{\infty} A_n \int_0^a r\, J_0\left(\frac{\alpha_n r}{a}\right) J_0\left(\frac{\alpha_m r}{a}\right) dr$$

sea $v = r/a \Rightarrow r = va \Rightarrow dr = a\,dv$.

$$a^2 \int_0^1 r\, J_0\left(\frac{\alpha_n r}{a}\right) J_0\left(\frac{\alpha_m r}{a}\right) dr = a^2 \int_0^1 v\, J_0(\alpha_n v) J_0(\alpha_m v) = \frac{a^2}{2} J_1(\alpha_m)$$

$$\therefore A_n = \frac{2}{a^2 J_1^2(\alpha_n)} \int_0^a r\, u_0(r) J_0\left(\frac{\alpha_n}{a} r\right) dr \; ; \quad n = 1, 2, \ldots$$

Para obtener B_n el procedimiento es análogo a partir de

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} J_0\left(\frac{\alpha_n r}{a}\right) \omega_n \left[-A_n \sin(\omega_n t) + B_n \cos(\omega_n t)\right]\Big|_{t=0}$$

$$\ldots \therefore B_n = \frac{2}{a^2 J_1^2(\alpha_n)\, \omega_n} \int_0^a r\, u_0(r) J_0\left(\frac{\alpha_n}{a} r\right) dr \; ; \quad n = 1, 2, \ldots$$