13 marzo 23

(i) $\mathcal{L}=\mathcal{L}(q,\dot{q},t)$

$$d\mathcal{L}=\frac{\partial \mathcal{L}}{\partial q_k}dq_k+\frac{\partial \mathcal{L}}{\partial \dot{q}_k}d\dot{q}_k+\frac{\partial \mathcal{L}}{\partial t}dt$$

Proponemos $\frac{\partial \mathcal{L}}{\partial \dot{q}_k}=p_k$, entonces

$$p_k\, d\dot{q}_k=d(p_k\dot{q}_k)-\dot{q}_k dp_k$$

$$d\mathcal{L}=\frac{\partial \mathcal{L}}{\partial q_k}dq_k+p_k d\dot{q}_k+\frac{\partial \mathcal{L}}{\partial t}dt$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right)=\frac{\partial \mathcal{L}}{\partial q_k}=\dot{p}_k}$$

$$\Rightarrow d(p_k\dot{q}_k-\mathcal{L})=-\frac{\partial \mathcal{L}}{\partial q_k}dq_k+\dot{q}_k dp_k-\frac{\partial \mathcal{L}}{\partial t}dt$$

$$dH=-\dot{p}_k\,dq_k+\dot{q}_k\,dp_k-\frac{\partial \mathcal{L}}{\partial t}dt$$

ecuaciones canónicas de Hamilton

$$\dot{q}_k=\frac{\partial H}{\partial p_k}\qquad \dot{p}_k=-\frac{\partial H}{\partial q_k}$$

$$\boxed{\frac{\partial H}{\partial t}=-\frac{\partial \mathcal{L}}{\partial t}}$$

Así $H=H(p,q,t)$. Por lo cual,

$$\frac{dH}{dt}=\frac{\partial H}{\partial q_k}\dot{q}_k+\frac{\partial H}{\partial p_k}\dot{p}_k+\frac{\partial H}{\partial t}=\frac{\partial H}{\partial t}$$

$\frac{dH}{dt}=0 \Rightarrow H=\text{cte}.$

Habíamos visto que cuando el Lagrangiano no depende del tiempo, entonces

$$E=\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_k-\mathcal{L}=T+V$$

Sabíamos que $\mathcal{L}=\sum\frac{1}{2}m_\alpha v_\alpha^2-U(|\vec{r}_\alpha|)$. Los momentos generalizados serán,

$$\vec{p}_\alpha=m_\alpha\vec{v}_\alpha \Rightarrow \vec{v}_\alpha=\frac{1}{m_\alpha}\vec{p}_\alpha$$

y entonces $H=\sum_\alpha \vec{p}_\alpha\cdot\vec{v}_\alpha-\mathcal{L}=\sum_\alpha\left[\frac{p_\alpha^2}{m_\alpha}-\frac{p_\alpha^2}{2m_\alpha}\right]+U(|\vec{r}_\alpha|)$. Por lo cual,

$$H=\sum_\alpha\frac{p_\alpha^2}{2m}+U(|\vec{r}|)$$

como $\vec{v}_\alpha=\frac{1}{m_\alpha}\vec{p}_\alpha$ entonces $\dot{\vec{p}}_\alpha=-\frac{\partial U}{\partial \vec{r}_\alpha}$.

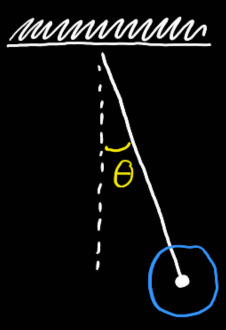
Considere ahora el caso de una partícula en coord. cilíndricas,

$$H=\frac{1}{2m}\left(p_r^2+\frac{1}{r^2}p_\varphi^2+p_z^2\right)+U(\vec{r})$$

Para el caso de las coord. esféricas,

$$H=\frac{1}{2m}\left(p_r^2+\frac{1}{r^2}p_\theta^2+\frac{1}{r^2\sin\theta}p_\phi^2\right)+U(r)$$

Ejemplo:



$$\mathcal{L}=\frac{1}{2}ml^2\dot{\theta}^2-mgl(1-\cos\theta)\qquad p_\theta=ml^2\dot{\theta}\qquad \dot{\theta}=\frac{p_\theta}{I}$$

$$\Rightarrow \mathcal{L}=\frac{1}{2}I\dot{\theta}^2-mgl\cos\theta)$$

$$\Rightarrow H=\frac{2p_\theta^2}{2I}-\frac{p_\theta^2}{2I}+mgl(1-\cos\theta)$$

$$\Rightarrow H=\frac{p_\theta^2}{2I}+mgl(1-\cos\theta)$$

Por lo tanto, $\dot{p}=-mgl\sin\theta \Rightarrow I\ddot{\theta}=-mgl\sin\theta$. Denotando a la velocidad angular $\dot{\theta}$ por ω, i.e. $\dot{\theta}=\omega$, entonces

$$\dot{\omega}=-\frac{mgl}{I}\sin\theta$$

Por otra parte, la energía será dada por

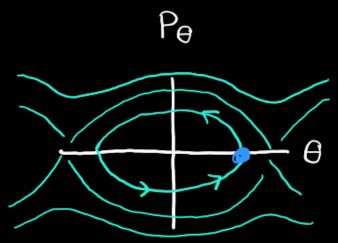
$$E = \frac{P_\theta^2}{2I} + mg\ell(1-\cos\theta)$$

$$\Rightarrow E_0 = mg\ell(1-\cos\theta_0);$$

$\theta(0) = \theta_0,\ \dot{\theta}(0) = 0,$
$\theta_0 \ll 1$

Ahora bien, hacemos θ variar de modo tal que

$$E_0 = \frac{P_\theta^2}{2I} + \frac{1}{2}mg\ell\theta^2$$



Note que $1-\cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$. Consecuentemente

$$\frac{P_\theta^2}{2I} = 2mg\ell\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right) \Rightarrow \frac{P_\theta^2}{2I} = 2mg\ell\cos^2\left(\frac{\theta}{2}\right); \quad \text{si } \theta_0 = \pi$$

de lo cual se obtiene

$$\frac{d\theta}{dt} = \sqrt{\frac{4mg\ell}{I}\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right)}$$

canónicas de Hamilton

Paréntesis de Poisson

Sea $f = f(q, p, t)$. Entonces,

$$\frac{df}{dt} = \frac{\partial f}{\partial q_k}\dot{q}_k + \frac{\partial f}{\partial p_k}\dot{p}_k + \frac{\partial f}{\partial t} = \boxed{\frac{\partial f}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial H}{\partial q_k}} + \frac{\partial f}{\partial t}$$

$$\frac{df}{dt} = \boxed{[H, f]} + \frac{\partial f}{\partial t}$$

Concretamente, se define $[g, h] = \frac{\partial h}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial h}{\partial p_k}\frac{\partial g}{\partial q_k}$.

Propiedades:

(i) $[f, g] = -[g, f]$

(ii) $[f, c] = 0$, si $c \in \mathbb{C}$

(iii) $[f_1 + f_2, g] = [f_1, g] + [f_2, g]$

(iv) $\frac{\partial}{\partial t}[f, g] = \left[\frac{\partial f}{\partial t}, g\right] + \left[f, \frac{\partial g}{\partial t}\right]$

(v) $[f, q_k] = \frac{\partial f}{\partial q_k}$; $[f, p_k] = \frac{\partial f}{\partial p_k}$

(vi) $[q_i, q_j] = \delta_{ij}$

(vii) $[p_i, p_j] = \delta_{ij}$

$\Rightarrow [p_i, q_k] = 0$

$(viii)$ $[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$

Si f y g son ctes de mov. $\Rightarrow [f, g]$ también lo es.