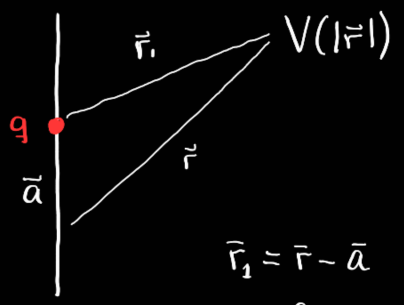
13 abril 2013

Aplicaciones de Polinomios de Legendre



$\bar{r}_1 = \bar{r} - \bar{a}$

$t = \frac{a}{r}$

$$V = \frac{q}{r_1} = \frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} \; ; \quad x = \cos\theta$$

$$= \frac{q}{r\sqrt{1 + \left(\frac{a}{r}\right)^2 - \frac{2ax}{r}}}$$

$$= \frac{q}{r\sqrt{1+t^2-2tx}} \, t^n$$

¿será posible definir un operador tal que factorize los $\frac{a^n}{r^n}$ términos?

$$= \frac{q}{r}\left[1 + \frac{a}{r}\cos\theta + \frac{1}{2}\frac{a^2}{r^2}(3\cos^2\theta - 1) + \cdots\right]$$

$P_0(x) = 1$

$P_1(x) = x = \cos\theta$

$P_2(x) = \frac{1}{2}(3x^2 - 1) = \frac{1}{2}(3\cos^2\theta - 1)$

$P_3(x) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$

Como $V(r) = \frac{q}{r_1} - \frac{q}{r_2}$, si tomamos $\bar{r}_2 = \bar{r} + \bar{a}$

$$|r_2| = \sqrt{(\bar{r}+\bar{a})\cdot(\bar{r}+\bar{a})}$$

$$= \sqrt{r^2 + a^2 + 2ra\cos\theta} \; ; \quad x = \cos\theta$$

$$= r\sqrt{1+t^2-2tx}$$

Nótese que

$$\frac{q}{r^2} = \frac{q}{r} g(x,-t) = \sum_{n=0}^{\infty} P_n(x)(-t)^n \Rightarrow g(x,t) = \frac{1}{(1+t^2-2tx)^{1/2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

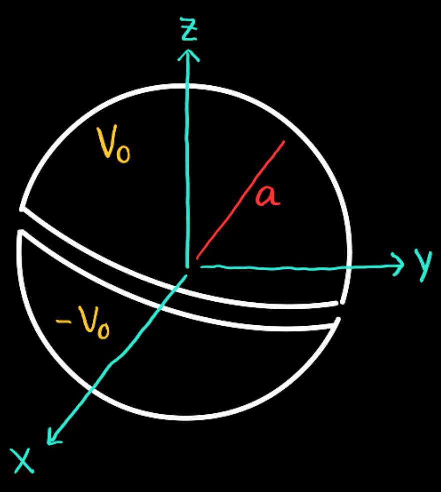
$$\therefore \; V(r,\theta) = \frac{q}{r}\Big[1 + \frac{a}{r}\cos\theta + \frac{a^2}{r^2}\cdot\frac{1}{2}(3\cos^2\theta - 1) + \left(\frac{a}{r}\right)^3\cdot\frac{1}{2}(5\cos^3\theta - 3\cos\theta) + \cdots$$

$$-1 + \frac{a}{r}\cos\theta - \left(\frac{a}{r}\right)^2\cdot\frac{1}{2}(3\cos^2\theta - 1) + \left(\frac{a}{r}\right)^3\cdot\frac{1}{2}(5\cos^3\theta - 3\cos\theta) + \cdots\Big]$$

$$= \frac{2q}{r}\left[\frac{a}{r}\cos\theta + \left(\frac{a}{r}\right)^3\cdot\frac{1}{2}(5\cos^2\theta - 3\cos\theta) + \cdots\right]$$

$$= \frac{2q}{r}\sum_{n=0}^{\infty} P_{2n+1}(\cos\theta)\left(\frac{a}{r}\right)^{2n+1}$$

Ejemplo.



$$\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t} \; ; \qquad \nabla\cdot\bar{E} = 4\pi\rho$$

$$\nabla\times\bar{E} = 0 \qquad \nabla\cdot\bar{E} = 0$$

$$\bar{E} = -\nabla V \qquad -\nabla\cdot(\nabla V) = 0$$

$$\nabla^2 V = 0$$

$$V = V(r,\theta,\varphi) \Rightarrow V = R(r)\,\Theta(\theta)\,\Phi(\varphi)$$

No obstante sabemos que

$$\frac{d^2\Phi}{d\varphi^2} + m^2\Phi = 0 \Rightarrow \Phi = e^{\pm im\varphi} \Rightarrow \Phi = 1$$

por simetría del problema (no hay dependencia del ángulo azimutal).

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{n(n+1)}{r^2}R = 0$$

$$\Rightarrow \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0$$

$$\Rightarrow \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + n(n+1)\Theta = 0$$

$\therefore V(r,\theta) = R(r)\Theta(\theta)$. Así pues,

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{n(n+1)}{r^2}R = 0$$

por Frobenius tendremos

$$R(r)=\sum_{s=0}^{\infty} a_s r^{s+k};\quad R'(r)=\sum_{s=0}^{\infty}(s+k)a_s r^{s+k-1};\quad R''(r)=\sum_{s=0}^{\infty}(s+k)(s+k-1)a_s r^{s+k-2}$$

sustituyendo,

$$\sum_{s=0}^{\infty}(s+k)(s+k-1)a_s r^{s+k-2}+\frac{2}{r}\sum_{s=0}^{\infty}(s+k)a_s r^{s+k-1}-\frac{n(n+1)}{2}\sum_{s=0}^{\infty}a_s r^{s+k}=0$$

$$\sum_{s=0}^{\infty}\left[(s+k)(s+k-1)a_s r^{s+k-2}+\frac{2}{r}\right.$$ completar

$$\Rightarrow k(k-1)+2k-n(n+1)=0$$

Ahora bien, como $\Theta(\theta)=P_n(\cos\theta)$ la solución será

$$V(r,\theta)=\sum_{n=0}^{\infty}\left[a_n r^n P_n(\cos\theta)+\frac{b_n}{r^{n+1}}P_n(\cos\theta)\right]$$

para el potencial pediremos $a_n\equiv 0$ pues en $r\to\infty$ r^n diverge.

$$V(r,\theta)=\sum_{n=0}^{\infty}\frac{b_n}{r^{n+1}}P_n(\cos\theta)$$

$$V(a,\cos\theta)=\begin{cases}+V_0 & \text{si } 0\le\theta\le\frac{\pi}{2}\\ -V_0 & \text{si } \frac{\pi}{2}<\theta\le\pi\end{cases}\qquad V(a,x)=\begin{cases}+V_0 & \text{si } 0\le\theta\le 1\\ -V_0 & \text{si } -1\le\theta<0\end{cases}$$

$$\Rightarrow V(r,\cos\theta)=\sum_{n=0}^{\infty}\frac{b_{2n+1}}{r^{2n+2}}P_{2n+1}(\cos\theta)$$

$$\Rightarrow V(a,\cos\theta)=\sum_{n=0}^{\infty}\frac{b_{2n+1}}{a^{2n+2}}P_{2n+1}(\cos\theta)$$

Tarea

$$\int_0^1 P_{2n+1}(x)=\frac{(-1)^s(2s-1)!!}{(2s+2)!!}$$

Sin embargo, $\int_{-1}^{1}V(a,x)P_{2n+1}(x)dx=\sum_{n=0}^{\infty}\int_{-1}^{1}\frac{b_{2n+1}}{a^{2n+2}}P_{2n+1}(x)P_{2m+1}(x)dx$

$$\Rightarrow\int_{-1}^{1}V(a,x)P_{2m+1}(x)\,dx=\int_{-1}^{0}-V_0P_{2m+1}(x)dx+\int_0^1 V_0P_{2m+1}(x)dx$$

$$=2V_0\int_0^1 P_{2m+1}(x)\,dx$$

$$=\frac{2V_0(-1)^m(2m-1)!!}{(2m+2)!!}$$

Mas aún, $\frac{b_{2n+1}}{a^{2n+2}}\int_{-1}^{1}P_{2n+1}(x)P_{2m+1}(x)dx=\frac{b_{2n+1}}{a^{2n+2}}\delta_{mn}\frac{2}{2(2m+1)+1}$

$$\therefore\ b_{2m+1}=V_0(-1)^m\frac{(2m-1)!!}{(2m+2)!!}a^{m+2}(4m+3)$$

Finalmente,

$$\boxed{V(r,\theta)=V_0\sum_{n=0}^{\infty}(-1)^m(4m+3)\frac{(2m-1)!!}{(2m+2)!!}\left(\frac{a}{r}\right)^{2m+2}P_{2m+1}(\cos\theta),\quad r>a}$$

Fórmula de Rodrigues

$$P_n(x)=\sum_{n=0}^{[\frac{n}{2}]}(-1)^j\frac{(2n-2j)!}{2^n j!(n-j)!(n-2j)!}x^{n-2j};\qquad \left[\frac{n}{2}\right]=\begin{cases}\frac{n}{2} & \text{si } n \text{ es par}\\ \frac{n-1}{2} & \text{si } n \text{ es par}\end{cases}$$

Por un lado, note que

$$\frac{d^n}{dx^n}x^{2n-2j}=\frac{d^{n-1}}{dx^{n-1}}(2n-2j)x^{2n-2j-1}$$

$$= \frac{d^{n-2}}{dx^{n-2}} (2n-2j)(2n-2j-1) x^{2n-2j-2}$$

$$= \frac{d^{n-3}}{dx^{n-3}} (2n-2j)(2n-j-1)(2n-2j-2) x^{2n-2j-3}$$

$$= (2n-2j)(2n-2j-1)\cdots(n-2j+1) \frac{(n-2j)!}{(n-2j)!} x^{2n-j}$$

$$= \frac{(2n-2j)!}{(n-2j)!} x^{2n-j}$$

Por lo tanto,

$$P_n(x) = \sum_{n=0}^{[\frac{n}{2}]} (-1)^j \frac{1}{2^n j! (n-j)!} \frac{d^n}{dx^n} x^{2n-2j}$$

$$= \sum_{n=0}^{[\frac{n}{2}]} (-1)^j \frac{n!}{n!} \frac{1}{2^n j! (n-j)!} \frac{d^n}{dx^n} x^{2n-2j}$$

$$= \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{n=0}^{[\frac{n}{2}]} (-1)^j \frac{n!}{j!(n-j)!} x^{2n-2j}$$

$$= \frac{1}{2^n n!} \frac{d^n}{dx^n} \sum_{n=0}^{[\frac{n}{2}]} (-1)^j \binom{n}{j} x^{2n-2j}$$

$$= \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

Por demostrar $\sum^{[n/2]} \leftrightarrow \sum^{n}$.

Dado $\frac{d^n}{dx^n} x^{2n-2j} = 0$, si $j = [\frac{n}{2}]+1, [\frac{n}{2}]+2, \ldots, [\frac{n}{2}]+n$, $[\frac{n}{2}] = \begin{cases} \frac{n}{2}, & n \text{ par} \\ \frac{n-1}{2}, & n \text{ impar} \end{cases}$

Por ejemplo con $j = [\frac{n}{2}]+1 \Rightarrow \frac{d^n}{dx^n} x^{2n-2(\frac{n}{2}+1)} = \frac{d^n}{dx^n} x^{n-2} = 0$

$$\therefore \boxed{P_n = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n}$$