21 abril 23

Ejercicios

(i) $\ddot{x} = -\omega^2 x \Rightarrow \begin{cases} \dot{x} = v \\ \dot{v} = -\omega^2 x \end{cases}$ $\qquad \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}$

cuyas soluciones son $x = Ae^{\lambda t}$, $v = Be^{\lambda t} \Rightarrow \dot{x} = \lambda x$, $\dot{v} = \lambda v$

(ii) Oscilador de Duffing

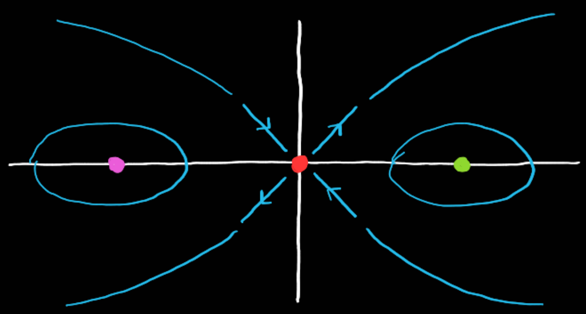
$$H = \frac{1}{2}p^2 \underbrace{- \frac{1}{2}x^2 + \frac{1}{4}x^4}_{-v} \qquad \begin{aligned} \dot{x} &= v \\ \dot{v} &= x - x^3 \end{aligned}$$

Ptos fijos $\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} v = 0 \\ x(1 - x^2) = 0 \end{cases}$ $\qquad$ Ptos críticos $(x, v) = \begin{cases} (0,0) \\ (1,0) \\ (-1,0) \end{cases}$

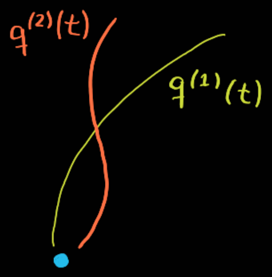
$(x,v) \mapsto (0,0)$ $\qquad \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}$

$(x,v) \mapsto (1,0)$ $\qquad \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$(x,v) \mapsto (-1,0)$ $\qquad \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \end{pmatrix}$



La acción $S = S(q)$



$$S = \int_{t_1}^{t_2} L(q, \dot{q})\, dt$$

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \delta q \right) dt$$

$$= \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \frac{\partial L}{\partial q_i} \delta q_i$$

$$\left[\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} \right) \delta q_i \, dt = \delta S$$

$\Rightarrow \delta S = \frac{\partial L}{\partial \dot{q}_i} \delta q_i(t_2)$, como $p_i = \frac{\partial L}{\partial \dot{q}_i}$ de lo cual $\delta S = p_i \delta q_i$

Mas aún, note que $\frac{dS}{dt} = L$, pero $\frac{dS}{dt} = p_i \dot{q}_i + \frac{\partial S}{\partial t} = L$ de donde

$$\frac{\partial S}{\partial t} = L - p_i \dot{q}_i = -H$$

por lo tanto, $dS = p_i dq_i - H dt$.

$\hookrightarrow$ (Por ejemplo $dS = p_i^{(2)} dq_i^{(2)} - H^{(2)} dt^{(2)} + p_i^{(1)} dt^{(1)} - H^{(1)} dt^{(1)}$)

Reescribiendo a la acción

$$S = \int_{t_1}^{t_2} (p_i dq_i - H dt) \Rightarrow \delta S = \int_{t_1}^{t_2} (\delta p_i dq_i + p_i d(\delta q_i) - \delta H dt)$$

$$dt\, \delta H = \left(\frac{\partial H}{\partial q_i} \delta q_i + \frac{\partial H}{\partial p_i} \delta p_i \right) dt$$

$$\delta S = \overset{0}{\cancel{(p_i \delta q_i)}} \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left(dp_i - \frac{\partial H}{\partial q_i} dt \right) \delta q_i \Rightarrow \begin{cases} \dot{p}_i = -\frac{\partial H}{\partial q_i} \\ \dot{q}_i = \frac{\partial H}{\partial p_i} \end{cases} \Rightarrow p_i = \frac{\partial S}{\partial q}$$

Trayectoria (Conservan Energía)

Dado $H=E$, y $\delta S = -H\delta t$ entonces $\delta S + E\delta t = 0$. Por lo cual,

$$\delta S = \int_{t_0}^{t} p_i \delta q_i - E(t-t_0) = S_0 - E(t-t_0)$$

Buscando $\delta S = 0$ debemos pedir que $\delta S_0 = 0$ (es decir, tiene un mínimo).

Recordando $p_i = \frac{\partial L}{\partial q_i}$ y $E = E\left(q, \frac{dq}{dt_i}\right)$ entonces $L = L\left(q_i, \frac{dq_i}{dt}\right)$,

$$L = \frac{1}{2} a_{ik} \dot{q}_i \dot{q}_k - U(q)$$

$$\frac{\partial L}{\partial \dot{q}_i} = a_{ik} \dot{q}_k = p_i$$

$$\Rightarrow E = \frac{1}{2} a_{ik} \dot{q}_i \dot{q}_k + U(t)$$

$$\Rightarrow \boxed{\int dt = \int \left(\frac{a_{ik}\, dq_i\, dq_k}{2(E-U)}\right)^{1/2}}$$

1- partícula

$$T = \frac{1}{2} m \left(\frac{dl}{dt}\right)^2 \qquad p_i\, dq_i = a_{ik} \frac{dq_k}{dt} dq_i \qquad S_0 = \int p_i\, dq_i$$

donde $S_0 = \int \sqrt{2(E-U)\, a_{ik}\, dq_i\, dq_k}$ y entonces

$$\delta S_0 = \delta \int \sqrt{2m(E-U)}\, dl$$