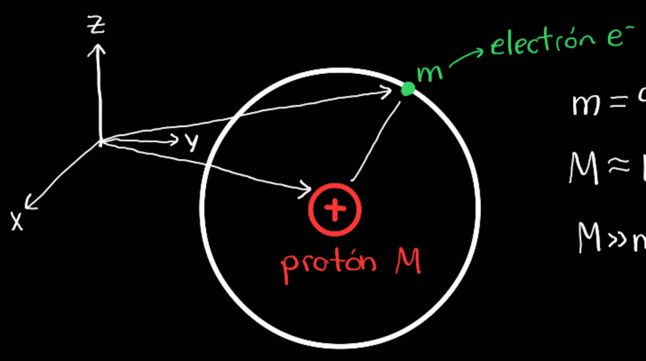
Átomo de Hidrógeno



$m = 9.1\times10^{-28}\,g$

$M \approx 1830 m$

$M \gg m$

carga protón $+e$

carga electrón $-e$

$e = 1.6\times10^{-19}\,C$, $\quad e = 4.8\times10^{-10}\,esu$

6-dimensiones

(6 grados de libertad)

P_{Nx}	P_{Ny}	P_{Nz}	P_{ex}	P_{ey}	P_{ez}
X_N	Y_N	Z_N	Xe	Ye	Ze

$$H = \frac{\vec{P}_N^2}{2M} + \frac{\vec{P}_e^2}{2m} + \frac{(e)(-e)}{|\vec{r}_N - \vec{r}_e|}$$

$$H = \frac{\vec{P}_N^2}{2M} + \frac{\vec{P}_e^2}{2m} - \underbrace{\frac{(e)(-e)}{|\vec{r}_N - \vec{r}_e|}}_{\text{interacción Coulombiana}} \quad \cdots (1)$$

$$\frac{dX_e}{dt} = \frac{\partial H}{\partial t} = \frac{P_{ex}}{m}, \quad \frac{dp_{ex}}{dt} = -\frac{\partial H}{\partial X_e} = +\frac{\partial}{\partial X_e}\frac{e^2}{|\vec{r}_N - \vec{r}_e|} + \frac{1}{2}\frac{2(x_N - x_e)}{(\quad)^{3/2}}$$

$$|\vec{r}_N - \vec{r}_e| = \sqrt{(x_N - x_e)^2 + (y_N - y_e)^2 + (z_N - z_e)^2}$$

Relativa $\quad \vec{r} = \vec{r}_e - \vec{r}_N$

$$\vec{R} = \frac{m\vec{r}_e + M\vec{r}_N}{m+M}$$

$\frac{1}{\mu} = \frac{1}{m} + \frac{1}{M}$ relativa $\qquad \mu = \frac{mM}{m+M}$

$M_T = m + M$ Total

$\vec{p} = \mu\dot{\vec{r}}$ relativo $\qquad \vec{P} = M_T\dot{\vec{R}}$ total

$$\vec{p} = \mu\dot{\vec{r}}_e - \mu\dot{\vec{r}}_N = \frac{\mu}{m}m\dot{\vec{r}}_e - \frac{\mu}{M}M\dot{\vec{r}}_N$$

$$\Rightarrow \vec{p} = \left(\frac{M}{m+M}\right)\vec{P}_e - \left(\frac{m}{m+M}\right)\vec{P}_N$$

$$\Rightarrow \vec{P} = M_T\left(\frac{m\dot{\vec{r}}_e + M\dot{\vec{r}}_N}{M_T}\right) = \vec{P}_e + \vec{P}_N$$

$$\boxed{\vec{P}_e = \frac{m}{M_T}\vec{P} + \vec{p}}$$

$$\boxed{\vec{P}_N = \frac{M}{M_T}\vec{P} - \vec{p}}$$

$$\therefore\ H = \frac{1}{2M}\left(\frac{M}{M_T}\vec{P} - \vec{p}\right)^2 + \frac{1}{2m}\left(\frac{m}{M_T}\vec{P} + \vec{p}\right)^2 - \frac{e^2}{|\vec{r}|}$$

$$= \frac{1}{2M}\left(\frac{M^2}{M_T^2}\vec{P}^2 - 2\frac{M}{M_T}\vec{P}\cdot\vec{p} + \vec{p}^2\right) + \frac{1}{2m}\left(\frac{m^2}{M_T^2}\vec{P}^2 + 2\frac{m}{M_T}\vec{P}\cdot\vec{p} + \vec{p}^2\right) - \frac{e^2}{|\vec{r}|}$$

$$\Rightarrow H = \underbrace{\frac{\vec{P}^2}{2M}}_{CM} + \underbrace{\frac{\vec{p}^2}{2\mu} - \frac{e^2}{|\vec{r}|}}_{\text{relativo}}$$

¡No depende de $\vec{R}$!

Note que,

$$\dot{\vec{P}} = -\frac{\partial H}{\partial \vec{R}} = -\nabla_R H = -\cancelto{0}{\left(\frac{\partial}{\partial x}H, \frac{\partial}{\partial y}H, \frac{\partial}{\partial z}H\right)}$$

i.e. el momento se conserva

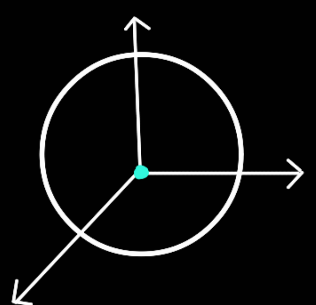
$$\dot{\vec{R}} = \frac{\partial H}{\partial \vec{p}} = \nabla_{\vec{p}} H = \frac{\vec{P}}{M}, \quad \vec{R} = (x, y, z)$$

$\therefore\ \vec{P} = M\dot{\vec{R}}$. Es decir, como $\dot{\vec{P}} = \frac{d\vec{P}}{dt} = 0$ con $\vec{P} = \vec{P}_0$ cte y $\dot{\vec{R}} = \frac{\vec{P}_0}{M}$ tiene vel constante el sistema no depende de la posición del centro de masa.

Más aún, como $M \gg m$ entonces $M \approx M_T$, $\frac{m}{M} \ll 1 \Rightarrow \frac{m}{M_T} \ll 1$. Consecuentemente, $\mu = \frac{mM}{M+m} \approx \frac{mM}{M} = m$. Además, $\frac{m}{M} \ll 1 \Rightarrow \vec{R} \approx \vec{r}_N$. Esto último reduce nuestro hamiltoniano a

$$\boxed{H = \frac{\vec{p}^2}{2m} - \frac{e^2}{r}}, \quad \vec{r} = (x, y, z)$$

similar al problema del Sol y Tierra

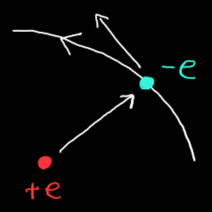


dados cond. inicial $\vec{p}_0$, $\vec{r}_0$.

$$\Rightarrow E = \frac{\vec{p}_0^2}{2m} - \frac{e^2}{r_0} = \frac{\vec{p}^2}{2m} - \frac{e^2}{r}, \quad \frac{dE}{dt} = 0$$

Momento angular

$\vec{L} = \vec{r} \times \vec{p}$



$$\frac{d\vec{r}}{dt} = \frac{\partial H}{\partial \vec{p}} = \frac{\vec{p}}{m}$$

$$\frac{d\vec{p}}{dt} = -\frac{\partial H}{\partial \vec{r}} = -\frac{e^2}{r^2}\hat{r} \,, \quad \hat{r} = \frac{\vec{r}}{r}$$

$L = \vec{r}_0 \times \vec{p}_0 = L_0$, ¡constante!

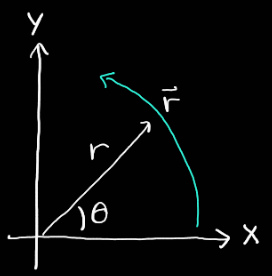
$$\frac{dL}{dt} = 0$$

Es decir de forma clásica $\boxed{\vec{p} = m\frac{d\vec{r}}{dt}}$, $\vec{r} = r\hat{r}$

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$

Demuestre que $\boxed{\frac{\vec{p}^{\,2}}{2m} = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{\vec{L}^2}{2mr^2}}$

$L = mr^2\frac{d\theta}{dt}$ velocidad angular $\omega = \frac{d\theta}{dt}$

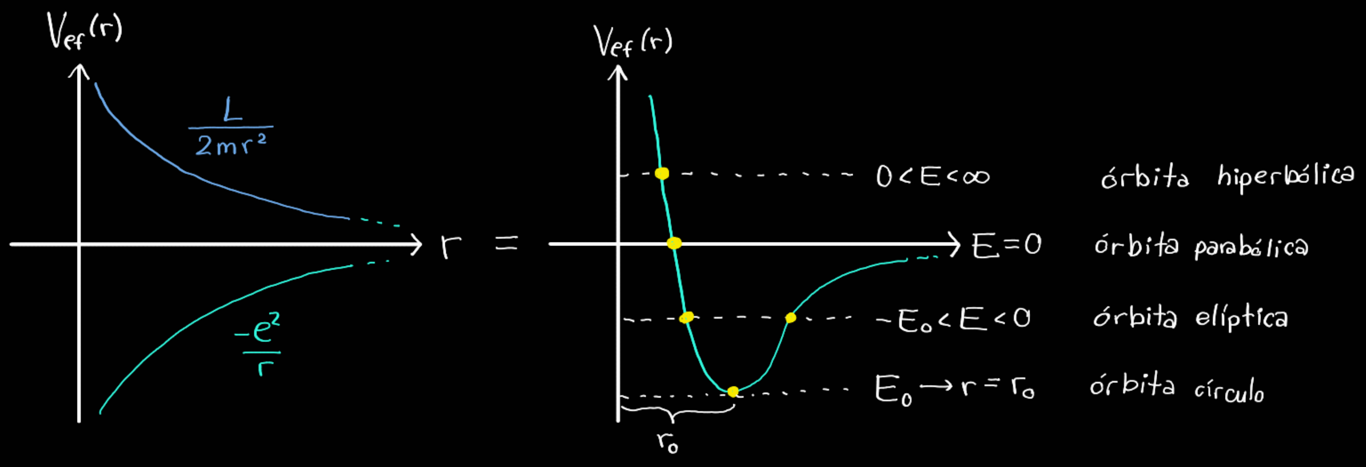


$\propto \boxed{\frac{d\theta}{dt} = \frac{L}{mr^2}}$

Por lo tanto, para el Hamiltoniano del sistema será dado por

$$H = \frac{\vec{p}_r^{\,2}}{2m} + \left[\frac{L^2}{2mr^2} + \frac{e^2}{r}\right] = T + V_{ef}(r) \rightsquigarrow$$ 1 partícula en 1D, $r: 0 \to \infty$

donde $p_r = m\frac{dr}{dt}$ es la proyección en r de $\vec{p}$.



Pero hay un caso más, aquel en que las órbitas precesan