28 feb 23

# Series de Fourier

Sea $f$ una función de variable real. Sabemos que $f$ se puede descomponer en la suma de una función par y una impar, $f = f_p + f_i$, donde

$$f_p(x) = \frac{f(x)+f(-x)}{2} \qquad y \qquad f_i(x) = \frac{f(x)-f(-x)}{2}$$

La meta es expresar a $f$ de la forma

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx) + \sum_{n=0}^{\infty} b_n \sin(nx)$$

donde la primer suma es $f_p(x)$ y la segunda es $f_i(x)$. Manipulando para hayar los coeficientes,

$$f(x)\cos(0x) = \sum_{n=0}^{\infty} a_n \cos(nx)\cos(0x) + \sum_{n=0}^{\infty} b_n \sin(nx)\cos(0x)$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\cos(0x)\,dx = \int_{-\pi}^{\pi} \sum_{n=0}^{\infty} a_n \cos(nx)\cos(0x) + \sum_{n=0}^{\infty} b_n \sin(nx)\cos(0x)\,dx$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\,dx = \sum_{n=0}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nx)\cos(0x)\,dx + \sum_{n=0}^{\infty} b_n \int_{-\pi}^{\pi} \sin(nx)\cos(0x)\,dx = a_0\pi$$

$$\therefore \boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx}.$$

Por otro lado,

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \sum_{n=0}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx + \sum_{n=0}^{\infty} b_n \int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = a_m\pi$$

de modo que

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx}$$

Análogamente, $\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx}$. A $a_n$ y $b_n$ se les conoce por coeficientes del desarrollo es serie de Fourier.

Suponga que ahora se busca una serie para una función que no necesariamente está definida en $[-\pi,\pi]$ sino en $[-L,L]$. Proponemos un cambio de variable $z = \frac{L}{\pi}x$, de modo que si $x=-\pi \Rightarrow z=-L$ y si $x=\pi \Rightarrow z=L$. De esta forma para una función $f(z)$ se tiene que

$$a_n = \frac{1}{\pi}\int_{-L}^{L} f(z)\cos\left(\frac{\pi n}{L}z\right)\frac{\pi}{L}\,dz = \frac{1}{L}\int_{-L}^{L} f(z)\cos\left(\frac{n\pi}{L}x\right)dz$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(z)\sin\left(\frac{\pi n}{L}z\right)dz$$

y la serie se forma en

$$f(z) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{L}z\right) + \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi}{L}z\right)$$

Ejemplo: Sea $f(x) = \begin{cases} 0, & x\in[-\pi,0) \\ x, & x\in[0,\pi] \end{cases}$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left(\frac{x}{n}\sin(nx)\Big|_0^{\pi} - \int_0^{\pi}\frac{1}{n}\sin(nx)\,dx\right) = \frac{1}{\pi}\cdot\frac{1}{n^2}\cos(nx)\Big|_0^{\pi} = \frac{1}{\pi n^2}\left((-1)^n - 1\right)$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_0^{\pi} x\sin(nx)\,dx = \cdots = \frac{(-1)^{n+1}}{n}$$

En $\mathbb{C}$, Fourier se ve como:

$$f(x) = \sum_{n=0}^{\infty} a_n \left( \frac{e^{inx} + e^{-inx}}{2} \right) + \sum_{n=0}^{\infty} b_n \left( \frac{e^{inx} - e^{-inx}}{2} \right)$$

de donde se obtiene

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2}(a_n - ib_n) e^{inx} + \sum_{n=0}^{\infty} \frac{1}{2}(a_n + ib_n) e^{-inx}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2} z_n e^{inx} + \sum_{n=0}^{\infty} \frac{1}{2} \bar{z}_n e^{inx}$$

$$= z_0 + \sum_{n=0}^{\infty} \frac{1}{2} z_n e^{inx} + \sum_{n=0}^{\infty} \frac{1}{2} \bar{z}_n e^{inx} ; \quad z_0 = \frac{a_0}{2}$$

$\therefore\ f(x) = z_0 + \sum_{n=-1}^{-\infty} z_n e^{inx}$.

# Fórmula de Parseval

Sea una función continua a trozos en el intervalo $[-\pi, \pi]$ y sean $a_0$, $a_n$, $b_n$ coeficientes de Fourier. Entonces:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\, dx = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

Demostración.

$$f^2(x) = \frac{a_0}{2} f(x) + \sum_{n=0}^{\infty} a_n f(x) \cos(nx) + \sum_{n=0}^{\infty} b_n f(x) \sin(nx)$$

$$\Rightarrow \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) dx = \frac{a_0}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx + \sum_{n=0}^{\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} a_n f(x) \cos(nx) + \sum_{n=0}^{\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} b_n f(x) \sin(nx)$$

$$\Rightarrow \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\, dx = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

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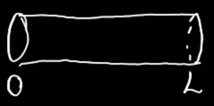
# Ecuación del calor




Condiciones de frontera

$$\frac{\partial u}{\partial t}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$$

$$u(x,0) = f(x), \quad 0 < x < L$$

$$\frac{\partial u}{\partial t}(x,t) = \beta \frac{\partial^2 u}{\partial x^2}(x,t)$$

Sea $u(x,t) = X(x)T(t)$ de modo tal que

$$\frac{\partial}{\partial t} X(x)T(t) = \beta \frac{\partial^2}{\partial x^2} X(x)T(t) \Rightarrow X(x) \frac{\partial T}{\partial t}(t) = \beta T(t) \frac{\partial^2 X(x)}{\partial x^2}$$

$$\Rightarrow X(x) \frac{\partial T}{\partial t}(t) = \beta T(t) \frac{\partial^2 X(x)}{\partial x^2}$$

$$\Rightarrow \frac{1}{T} \frac{\partial T}{\partial t} = \beta \frac{1}{X} \frac{\partial^2 X}{\partial x^2} = -\lambda^2$$

Por lo tanto, $\frac{dT}{dt} = -\lambda^2 T$ y $\frac{d^2 X}{dx^2} = -\frac{\lambda^2}{\beta} X$. Esto tiene entonces como soluciones a

$$T = U^{-\lambda^2 t} \quad \text{y} \quad X = A \cos\left(\frac{\lambda}{\sqrt{\beta}} x\right) + B \sin\left(\frac{\lambda}{\sqrt{\beta}} x\right)$$

$$X'(x) = A \frac{\lambda}{\sqrt{\beta}} \sin\left(\frac{\lambda}{\sqrt{\beta}} x\right) + B \frac{\lambda}{\sqrt{\beta}} \cos\left(\frac{\lambda}{\sqrt{\beta}} x\right)$$

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x}(0,T) &= X'(0)T(t) = 0 \\ \frac{\partial u}{\partial x}(L,T) &= X'(0)T(t) = 0 \end{aligned}}$$

$$X'(0) = \frac{B\lambda}{\sqrt{\beta}}, \qquad X'(L) = -\frac{A\lambda}{\sqrt{\beta}} \sin\left(\frac{\lambda L}{\sqrt{\beta}}\right)$$

$$\Rightarrow \frac{B\lambda}{\sqrt{\beta}} = 0 \Leftrightarrow B = 0, \text{ entonces } A \neq 0 \Rightarrow \frac{\lambda L}{\sqrt{\beta}} = n\pi$$

$$\Rightarrow \lambda = \frac{n\pi\sqrt{\beta}}{L}$$

$$\therefore\ X_n(x) = A_n \cos\left(\frac{\lambda_n x}{\sqrt{\beta}}\right)$$

Por otra parte, $T_n(t) = B_n e^{-\lambda_n^2 t}$ $\Big\}$ $U_n(x,t) = a_n e^{-\lambda_n^2 t} \cos\left(\frac{n\pi}{L} x\right)$, entonces

$$U(x,t) = \sum_n a_n e^{-\lambda_n^2 t} \cos\left(\frac{n\pi}{L} x\right)$$

Falta arreglar $U(x,0) = f(x)$,

$$U(x,0) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{L} x\right) = f(x) \, ; \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{L} x\right) dx$$