From Compactness To Completeness

or how to make Propositional Logic, First-Order Logic, and other logics fun and easier to learn

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30 June 2020

Contents

Preface

Though my emphases are a little different in these lecture notes, I follow a trend tried by a few others in recent years of introducing and proving the Compactness Theorem before the Completeness Theorem. Doing it this way, Completeness becomes a consequence of Compactness. The other way around, which is standard in many textbooks, invokes Completeness (as well as Soundness) to prove Compactness .[1]

There are good reasons for reversing the traditional approach. Perhaps the chief reason is to avoid getting immersed in the fiddly details of formal-proof systems and, thus, to also avoid the concern of dealing with as many proofs of Completeness as there are proof systems (a welcome avoidance when part of our study involves different logics and their proof systems).

From my own teaching experience, many students of computer science, and some of the very best, are not inspired by the relatively large amount of syntactic details they have to absorb in the traditional approach before reaching applications related to their own interests. Formal proofs are far less permissive of syntactic imprecision than semi-formal or informal proofs, and it takes more than one semester to learn and appreciate the benefits of the former over the latter, especially when the latter can be very partially spelled out with no loss of rigor (often the case). Put differently, the traditional approach would force students to learn a good deal of *proof theory* and its unsparing precision on syntax earlier than later, before reaching more interesting applications based on as much of *model theory* as time permits towards the end of the semester. Starting with Compactness, we reverse this order, emphasize more model theory and semantic notions early on, and can present interesting (though still small) applications much earlier in the semester, with a distinctly algebraic or semantic flavor and a smaller amount of definitions related to syntax and formal derivations.

But there are other advantages to starting with Compactness. It makes it easier to grasp topological aspects of the notion (and the origin of its name). We can go deep in a topological direction, by explaining Compactness purely in terms of notions such as the *finite-intersection property*, *ultrafilters*, *ultraproducts*, and others, but that would take us further afield from the focus on formal logic. I choose a watered-down approach. In the proof of Compactness in Section 1, we construct a maximal satisfiable set of propositional wff's without any explicit reference to topological notions, although these are lurking right under the surface. I delay making explicit connections to topology until Appendix F.

The simplest logic we consider in these notes is *propositional logic*, and the most expressive is *first-order logic*. Also unusual is the gradual transition from the former to the latter, as five intermediate logics are introduced: the *logic of quantified Boolean formulas* (in Section 2), *equality logic* (in Section 3), what I call *zeroth-order logic* (in Section 4), and *equational logic* and *quasi-equational logic* (in Section 5). This is not a linear progression in expressive power from *propositional logic* to *first-order logic*, as some of our intermediate logics are extended by unrelated features (*i.e.*, incomparable in their standard interpretation); it is more a progression in the difficulty of studying them. Throughout the main body of these notes, I strive to minimally invoke proof systems and formal proofs, although these are covered in Appendix C in some details for the interested student.

We reduce Compactness for *first-order logic*, and for every intermediate logic, to Compactness for *propositional logic*. Thus, there is only one proof of Compactness in these notes, that of Compactness for *propositional logic*, from which Compactness for all the other logics follow – this is the common thread holding them all together.[2]

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[1] A typical example is the proof of the Compactness Theorem in the textbook *A Mathematical Introduction to Logic* by H.B. Enderton [4]; the proof at the end of Section 2.5 invokes the Completeness Theorem, as well as the Soundness Theorem, to prove Compactness. Another textbook, common in computer science departments, is *Logic in Computer Science* by M. Huth and M. Ryan [5], which omits altogether proofs for Soundness and Completeness (on page 96), and then invokes Completeness to prove Compactness as a consequence (on page 137).

[2] I do not claim originality for this approach. You will find it in some books from the 1960's and perhaps earlier, though it did not

Which is again different from the traditional approach. Some textbooks include proofs of Completeness twice, and then Compactness as a corollary via Soundness twice too: first for *propositional logic*, and again for *first-order logic*, on the grounds that the proof of Completeness for the latter is markedly different from that for the former (and it is, at least in the details). Other textbooks do the proof of Completeness for only one of the two logics, referring to some other textbook for a proof for the other logic. But in either case, connections related to Compactness between the two logics are lost or glossed over.[3]

The preceding are all among the many benefits of starting with Compactness and proving it only once for all the logics. But there is no free lunch and there is work to do. Most of the hard work is about setting up the means for the transition from Compactness for *propositional logic* to Compactness for the other logics, *i.e.*, the means to reduce the latter to the former. The reductions for the *logic of quantified Boolean formulas* and for *equality logic* are relatively straightforward based on two simple ideas (quantifier-elimination and a method of substitution) that are greatly amplified in what is called *Herbrand theory* in later sections. *Herbrand theory* is what we need for the reductions of the more complex logics: *zeroth-order logic*, *equational logic*, *quasi-equational logic*, and *first-order logic*.

An additional benefit of an excursion through *Herbrand theory* is that it has other important uses outside these notes. It plays the role of a *transfer principle* by reducing many questions of first-order logic to questions of propositional logic, all separate from Compactness. It can thus provide a unifying background for the study of other topics beyond the scope of these notes, such as the *tableaux* and *resolution* methods and *unification theory* (all good material for a follow-up course stressing *algorithmic* and *proof-theoretic* methods).

Finally, the obvious question: Why do we stop our presentation of "*From Compactness To Completeness*" at first-order? Can't we extend it to *second-order logic*, if only to fragments of the latter? The short answer: It takes a tiny amount of second-order quantification to collapse the entire edifice built on Compactness.

How to read these lecture notes

The Preface (this section) and the last Section 7 are for context and can be read separately from everything else.

The material in Section 1 on *propositional logic* gradually builds up, through successive sections on more complex logics, and ends in Section 6 on *first-order logic*. These sections, one for every logic, are intended to be read sequentially. Throughout, I insert small exercises (those that are untitled) right after incomplete or outlined arguments, which typically ask for supplying missing details; I consider them an integral part of the material and doing them should give a better grip on separating what is essential from the chaff in logical arguments. In the last subsection of every section, I include a few small applications and related exercises (those are titled) of the kind students in computer science encounter elsewhere in their studies.

I rarely cover all these sections in their entirety in a 14-week semester, where approximately half of the course work is devoted to separate material on the theory and pragmatics of using *SAT/SMT solvers* and *automated*

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seem to gain wide acceptance. For example, this approach is implicit in R.M. Smullyan's book *First-Order Logic* [8] (see the proofs of Theorem 6 at the end of Chapter VI and Theorem 2 in Chapter VII). And it is explicit in G. Kreisel's and J.L. Krivine's book *Elements of Mathematical Logic* [6] (see their *Finiteness Theorem*, Theorem 12, in Chapter 2). However, it takes some doing to decode the notation in these two books, somewhat different from that in more recent publications.

[3] Here are three examples. In the book *Models and Ultraproducts: An Introduction* by J.L. Bell and A.B. Slomson [1], Completeness is proved twice, once for *propositional logic* in Section 2.3 and once for *first-order logic* in Section 3.5, the latter not given as a consequence of the former, and Compactness in both cases given after Completeness in Sections 2.4 and 5.4, respectively. In the book *Mathematical Logic* by J.D. Monk [7], Completeness is proved for *propositional logic* in Theorems 8.28 and 8.29, and again separately not as a corollary for *first-order logic* in Theorems 11.19 and 11.20; Compactness is then given as a consequence of the latter only, in Theorem 11.22. In the book *Logic and Structure* by D. van Dalen [10], Completeness is proved twice differently, once for *propositional logic* in Section 1.5, once for *first-order logic* in Section 3.1, with Compactness for the latter given as a consequence in Section 3.2.

theorem provers.[4] Material in these notes is therefore the basis of about one-half of the lectures, typically interspersed with the other half. I try to choose a pace that suits most of the students, who are typically first-year graduate students with a background in computer science. At a slower pace, I may reach Section 4 or Section 5; at a faster pace with a smaller or more advanced group of students, I may reach the end of Section 6.

All the appendices should be read as needed, or as much as students desire to read on their own. Except possibly for Appendices D and E, their material is not part of what I present in lecture, though I refer to them in homework assignments and in case there are questions in lecture. Appendices A, B, and C, are mostly a review of terminology and notation used in earlier parts of the notes. My preference is to leave Appendices D and E to students to read and learn by themselves, giving more time for applications in lectures; those two appendices present concrete examples (the familiar *de Morgan's laws*, *prenex forms*, and *skolemization*) of how we can deal with the same formulas in two different ways, semantically and proof-theoretically. I include Appendix F for those interested in knowing the topological connections of Compactness.

[4]Typically, I have used the SAT/SMT solver **Z3** with a **Python** or **OCaml** interface for friendlier interaction. The automated theorem prover **Prover9**, and its companion counter-example searcher **Mace4**, have been particularly easy to set up and use.

1 Propositional Logic (PL)

Let $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$ be the set of well-formed formulas of *propositional logic* over the set $\mathcal{P}$ of *propositional variables*. We say a set $\Gamma \subseteq \mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$ is *finitely satisfiable* iff every finite subset of Γ is satisfiable. If Γ is a finite set, then "finitely satisfiable" coincides with "satisfiable".

We write $\mathsf{models}(\Gamma)$ to denote the set of models of Γ. In the propositional case, $\mathsf{models}(\Gamma)$ is the set of all truth assignments to the propositional variables that satisfy every $\varphi \in \Gamma$. The next lemma is a preliminary result for the Compactness Theorem.

Lemma 1. *Let $\Gamma \subseteq \mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$ and $\varphi \in \mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$. If Γ is finitely satisfiable, then $\Gamma \cup \{\varphi\}$ or $\Gamma \cup \{\neg\varphi\}$ (or possibly both) is finitely satisfiable.*

Proof. Suppose the conclusion of the lemma does not hold: Both $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are not finitely satisfiable. Hence, there are finite subsets $\Gamma_1 \subseteq \Gamma$ and $\Gamma_2 \subseteq \Gamma$ such that both $\Gamma_1 \cup \{\varphi\}$ and $\Gamma_2 \cup \{\neg\varphi\}$ are not satisfiable. Hence, both:

$$\mathsf{models}(\Gamma_1) \cap \mathsf{models}(\varphi) \;=\; \varnothing \qquad \text{and} \qquad \mathsf{models}(\Gamma_2) \cap \mathsf{models}(\neg\varphi) \;=\; \varnothing.$$

Hence, both $\mathsf{models}(\Gamma_1) \subseteq \mathsf{models}(\neg\varphi)$ and $\mathsf{models}(\Gamma_2) \subseteq \mathsf{models}(\varphi)$. Hence,

$$\mathsf{models}(\Gamma_1 \cup \Gamma_2) \;=\; \mathsf{models}(\Gamma_1) \cap \mathsf{models}(\Gamma_2) \;\subseteq\; \mathsf{models}(\neg\varphi) \cap \mathsf{models}(\varphi) \;=\; \varnothing.$$

Hence, the finite subset $\Gamma_1 \cup \Gamma_2$ does not have models, *i.e.*, is not satisfiable. Hence, Γ is not finitely satisfiable, and the hypothesis of the lemma does not hold either. $\square$

1.1 Compactness in PL

Theorem 2 (Compactness for Propositional Logic, Version I)**.** *Let $\Gamma \subseteq \mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$. It then follows that: Γ is satisfiable $\Leftrightarrow$ Γ is finitely satisfiable.*

Proof. The implication "$\Rightarrow$" is immediate. The non-trivial implication is "$\Leftarrow$": If Γ is finitely satisfiable, then Γ is satisfiable.

The set of propositional variables is $\mathcal{P} = \{p_0, p_1, p_2, \ldots\}$. Let $\varphi_1, \varphi_2, \varphi_3, \ldots$ be a fixed, countably infinite, enumeration of all the formulas in $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$. We define a nested sequence of supersets of Γ as follows:

$$\Delta_0 \quad = \Gamma,$$

$$\Delta_{i+1} = \begin{cases} \Delta_i \cup \{\varphi_i\} & \text{if } \Delta_i \cup \{\varphi_i\} \text{ is finitely satisfiable,} \\ \Delta_i \cup \{\neg\varphi_i\} & \text{otherwise.} \end{cases}$$

Clearly, $\Gamma = \Delta_0 \subseteq \Delta_1 \subseteq \Delta_2 \subseteq \Delta_3 \subseteq \cdots$. By induction on $i \geqslant 0$, using Lemma 1, every Δ_i is a finitely satisfiable set of propositional wff's. We now define:

$$\Delta \;=\; \bigcup_i \Delta_i \qquad \text{(the limit of the } \Delta_i\text{'s)}$$

Two facts about Δ follow from its definition:

1. For every propositional wff φ, either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$, but not both.
   This is why Δ is said *maximal finitely satisfiable*, soon to be shown just *maximal satisfiable*.

2. Since every propositional variable p_i is a wff itself, either $p_i \in \Delta$ or $\neg p_i \in \Delta$, but not both.

We next define a truth assignment σ as follows:

$$\sigma(p_i) \;=\; \begin{cases} \textit{true} & \text{if } p_i \in \Delta, \\ \textit{false} & \text{if } \neg p_i \in \Delta. \end{cases}$$

Claim: σ satisfies a propositional wff φ iff $\varphi \in \Delta$. We leave the proof of this claim as an (easy) exercise.

Hence, σ is a valuation satisfying every wff in Δ, *i.e.*, $\sigma \in \mathsf{models}(\Delta)$. Hence, because $\Gamma \subseteq \Delta$, it is also the case that σ satisfies every wff in Γ. Hence, Γ is satisfiable. □

Exercise 3. Provide the details in the preceding proof showing that there is "a fixed, countably infinite, enumeration of all the formulas in $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$". Although not needed in the proof, we can state a stronger assertion: The fixed enumeration of all the formulas in $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$ is *computable*, *i.e.*, can be generated by an infinitely-running computer program. □

Exercise 4. In the definition of the nested sequence of Δ_i's in the preceding proof, we did *not* write:

$$\Delta_{i+1} \;=\; \begin{cases} \Delta_i \cup \{\varphi_i\} & \text{if } \Delta_i \cup \{\varphi_i\} \text{ is finitely satisfiable,} \\ \Delta_i \cup \{\neg\varphi_i\} & \text{if } \Delta_i \cup \{\neg\varphi_i\} \text{ is finitely satisfiable.} \end{cases}$$

Explain why. *Hint*: Exhibit a set Γ of wff's and a single wff φ such that both $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are satisfiable. □

Exercise 5. Prove the **claim** in the penultimate paragraph of the proof of Theorem 2. There is no harm in simplifying the syntax a little, by restricting the logical connectives to two, say, $\{\neg, \vee\}$ or $\{\neg, \wedge\}$. *Hint*: Use structural induction on propositional wff's. □

Lemma 6. *Let $\Gamma \subseteq \mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$ and $\varphi \in \mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$, both arbitrary. We then have:*
$\Gamma \models \varphi \;\Leftrightarrow\;$ "$\Gamma \cup \{\neg\varphi\}$ *is unsatisfiable" – or, equivalently,* $\Gamma \not\models \varphi \;\Leftrightarrow\;$ "$\Gamma \cup \{\neg\varphi\}$ *is satisfiable".*

Proof. We have the following sequence of equivalences:

$$\begin{aligned} \Gamma \models \varphi \quad &\Leftrightarrow \quad \mathsf{models}(\Gamma) \subseteq \mathsf{models}(\varphi) \\ &\Leftrightarrow \quad \mathsf{models}(\Gamma) \cap \mathsf{models}(\neg\varphi) = \varnothing \\ &\Leftrightarrow \quad \mathsf{models}(\Gamma \cup \{\neg\varphi\}) = \varnothing \\ &\Leftrightarrow \quad \Gamma \cup \{\neg\varphi\} \text{ is unsatisfiable,} \end{aligned}$$

which is the desired conclusion. □

Corollary 7 (Compactness for Propositional Logic, Version II)**.** *Let $\Gamma \subseteq \mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$ and $\varphi \in \mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$, both arbitrary. We then have: $\Gamma \models \varphi \Leftrightarrow$ there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models \varphi$.*

Proof. The implication "$\Leftarrow$" is immediate. For the implication "$\Rightarrow$", we prove the contrapositive. So, suppose $\Gamma_0 \not\models \varphi$ for every finite subset $\Gamma_0 \subseteq \Gamma$. We have the following equivalences:

$$\begin{aligned} \Gamma_0 \not\models \varphi \text{ for every finite } \Gamma_0 \subseteq \Gamma \quad &\Leftrightarrow \quad \Gamma_0 \cup \{\neg\varphi\} \text{ satisfiable for every finite } \Gamma_0 \subseteq \Gamma \text{ (by Lemma 6)} \\ &\Leftrightarrow \quad \Gamma \cup \{\neg\varphi\} \text{ finitely satisfiable (by definition)} \\ &\Leftrightarrow \quad \Gamma \cup \{\neg\varphi\} \text{ satisfiable (by Theorem 2)} \\ &\Leftrightarrow \quad \Gamma \not\models \varphi \text{ (by Lemma 6) ,} \end{aligned}$$

which is the desired conclusion. □

Exercise 8 shows one way in which Compactness breaks down. The exercise involves an extension of PL which is called the *infinitary propositional logic* (*Infinitary PL*).

Exercise 8. We can restrict the logical connectives to $\{\neg, \vee, \wedge\}$. The set of propositional variables is again $\mathcal{P} = \{p_0, p_1, p_2, \ldots\}$, which is countably infinite. Suppose we extend this syntax with two new connectives, denoted $\bigvee$ and $\bigwedge$, each taking as a single argument a countably infinite set of previously defined wff's. The resulting syntax is one version of *Infinitary PL*. If Γ is a countably infinite set of the form $\Gamma = \{\varphi_1, \varphi_2, \varphi_3, \ldots\}$, then:

$$\bigvee \Gamma \;=\; \varphi_1 \vee \varphi_2 \vee \varphi_3 \vee \cdots ,$$

and similarly for $\bigwedge \Gamma$. There are three parts in this exercise:

1. Define the syntax of *Infinitary PL*, preferably in an extended BNF (Backus-Naur Form). Try to be as precise as you can, paying special attention to the presence of ellipses "..." in the definition – or can you think of a mathematical formulation that avoids any mention of ellipses?
2. Define the semantics of *Infinitary PL*, by structural induction on the syntax in Part 1, starting from an assignment σ of truth values to every member of $\mathcal{P}$ (for the base case of the induction).
3. Show that Theorem 2 does not hold, and therefore nor does Corollary 7, for *Infinitary PL*.

   Hint: Define a countably infinite set Γ of wff's such that every finite $\Gamma_0 \subseteq \Gamma$ is satisfiable, but Γ is not. *Further Hint*: Include the wff $\varphi = \bigvee\{\neg p_0, \neg p_1, \neg p_2, \ldots\}$ in your proposed Γ. $\square$

Remark 9. The *infinitary propositional logic* (Infinitary PL) extends *finitary propositional logic*, which is just what is commonly called *propositional logic* (our PL here). The distinction between *infinitary* and *finitary* extends to other formal logics and their proof systems besides PL. You can take the phrase "finitary proof system" to qualify a system that generates new finite expressions (*e.g.*, the sequents of PL in natural-deduction style or the wff's of propositional logic in Hilbert style) from previously generated ones by means of finitely many rules that require each finitely many premises or antecedents – without using any notion of infinite sequence or any notion of infinite set.[5] $\square$

1.2 From Compactness in PL to Completeness in PL

We are now ready for the transition, from Compactness to Completeness. This is also an opportunity to present two fundamental concepts: the *Deduction Theorem* (here called a lemma) and *Consistency*.

[5] The words "finitary" and "infinitary" are also used elsewhere, in several areas of mathematics and theoretical computer science with different though related meanings, and sometimes a little too loosely. They are all intended to mean certain things are "finite" and other things "infinite", but more precisely in relation to particular aspects of mathematically defined notions in different contexts.

For example, closer to our concerns here, a *finitary operation* (resp. *relation*) is one which has *finite arity*, otherwise it is said to be an *infinitary operation* (resp. *infinitary relation*). So, if we say an algebra or a relational structure is *finite* (resp. *infinite*), we mean its domain or universe is finite (resp. infinite) – this convention is firmly established – but if some authors say an algebra or a relational structure is *finitary* (resp. *infinitary*), they mean its operations and relations have each finite arity (resp. at least one has infinite arity), regardless of the size of its universe.

In a different context, a *finitary* logic means one whose formulas and formal derivations all have finite length, otherwise the logic is said to be *infinitary*. Elsewhere, some authors have used the phrase *finitary mathematics* to mean mathematics that can be expressed without invoking infinite sets in any way. If you want to go deeper into the usages of "finite" versus "finitary", and "infinite" versus "infinitary", search the Web.

Lemma 10 (Deduction Theorem)**.** *Let* $\Gamma \subseteq \mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$ *and* $\varphi, \psi \in \mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$*, all arbitrary. It then holds that:* $\Gamma \vdash (\varphi \to \psi) \;\;\Leftrightarrow\;\; \Gamma \cup \{\varphi\} \vdash \psi$*.*

Exercise 11. Write the proof of Lemma 10.

Hint: Review the proof rules for PL in the Appendix C.1. This is a very easy exercise, especially when a formal proof is written as a natural deduction; all you need to consider are the two rules ($\to$I) and ($\to$E) and how they are used. □

A set of wff's $\Gamma \subseteq \mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$ is said to be *consistent* iff $\Gamma \nvdash \bot$. The set Γ is *inconsistent* iff it is not consistent. The next lemma is our first result connecting "$\vdash$" and "$\models$". Note how the earlier proof of Compactness (Theorem 2) dovetails with the next proof; it helps to understand the former before reading on.

Lemma 12. *Let* $\Gamma \subseteq \mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$ *be arbitrary. It then holds that if* $\Gamma \models \bot$*, then* $\Gamma \vdash \bot$*, i.e.,* Γ *is inconsistent.*

Proof. We prove the contrapositive: If $\Gamma \nvdash \bot$, then $\Gamma \not\models \bot$. In words, if Γ is consistent, then Γ is satisfiable.

First, observe that given an arbitrary wff φ, either $\Gamma \cup \{\varphi\}$ is consistent, or $\Gamma \cup \{\neg\varphi\}$ is consistent, or possibly both are consistent separately (but certainly not their union $\Gamma \cup \{\varphi, \neg\varphi\}$!). Indeed, if $\Gamma \cup \{\varphi\}$ is inconsistent, then $\Gamma \cup \{\varphi\} \vdash \bot$ so that, by Lemma 10, $\Gamma \vdash (\varphi \to \bot)$. From the latter sequent, we easily get the sequent $\Gamma \vdash \neg\varphi$ (review the proof rule ($\neg$I) in the Appendix C.1). Similarly, if $\Gamma \cup \{\neg\varphi\}$ is inconsistent, then $\Gamma \vdash \neg\neg\varphi$.

Hence, if both $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are inconsistent, then both $\Gamma \vdash \neg\varphi$ and $\Gamma \vdash \neg\neg\varphi$. The latter two sequents imply that Γ is inconsistent – contradicting our hypothesis that Γ is consistent. Hence, it cannot be that both $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{\neg\varphi\}$ are inconsistent, and at least one of the two is consistent.

As in the proof of Theorem 2, we consider a fixed, countably infinite, enumeration of all the formulas in $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$, say, $\varphi_1, \varphi_2, \varphi_3, \ldots$. We define a nested sequence of consistent sets:

$$\Gamma \overset{\text{def}}{=} \Delta_0 \subseteq \Delta_1 \subseteq \Delta_2 \subseteq \cdots, \quad \text{where}$$

$$\Delta_{i+1} \overset{\text{def}}{=} \begin{cases} \Delta_i \cup \{\varphi_{i+1}\} & \text{if } \Delta_i \cup \{\varphi_{i+1}\} \text{ is consistent,} \\ \Delta_i \cup \{\neg\varphi_{i+1}\} & \text{if } \Delta_i \cup \{\neg\varphi_{i+1}\} \text{ is inconsistent,} \end{cases}$$

for every $i \geqslant 0$. We now define:

$$\Delta = \bigcup\nolimits_i \Delta_i \qquad \text{(the limit of the } \Delta_i\text{'s)}$$

The set Δ is a *maximal consistent* set of wff's, in the sense that given an arbitrary wff φ, either $\varphi \in \Delta$ or $\neg\varphi \in \Delta$. In particular, for every propositional variable $p \in \mathcal{P}$, either $p \in \Delta$ or $\neg p \in \Delta$. We next define a truth assignment σ as follows:

$$\sigma(p_i) = \begin{cases} \textit{true} & \text{if } p_i \in \Delta, \\ \textit{false} & \text{if } \neg p_i \in \Delta. \end{cases}$$

As in the proof of Theorem 2, it is a straightforward exercise to show that σ satisfies a propositional wff φ iff $\varphi \in \Delta$. (See also Exercise 5.) Hence, $\sigma \models \Delta$ and, since $\Gamma \subseteq \Delta$, also $\sigma \models \Gamma$. Since Γ has a model σ and $\bot$ has no models, it follows that $\Gamma \not\models \bot$. □

Theorem 13 (Completeness for Propositional Logic)**.** *Let* Γ *be a set of propositional wff's (possibly infinite), and* ψ *a propositional wff. If* $\Gamma \models \psi$*, then* $\Gamma \vdash \psi$*.*

Proof. Suppose $\Gamma \models \psi$. From $\{\psi, \neg\psi\} \models \perp$, it follows that $\Gamma \cup \{\neg\psi\} \models \perp$. By Lemma 12, we thus have that $\Gamma \cup \{\neg\psi\} \vdash \perp$. It follows, by Lemma 10, that $\Gamma \vdash (\neg\psi \to \perp)$, which implies that $\Gamma \vdash \neg\neg\psi$ and again $\Gamma \vdash \psi$ (make sure you understand this last step, by reviewing the proof rules in the Appendix C.1). □

The crucial result in proving Completeness in the preceding theorem is Lemma 12, whose proof follows closely the steps of the proof of Compactness in Theorem 2. Note the interesting parallel: the proof of Theorem 2 builds a *maximal satisfiable* set, the proof of Lemma 12 builds a *maximal consistent* set.

There is another way of reaching Completeness from Compactness, through an intermediate result which we may call Weak Completeness and which does not depend on Compactness. The proof of Weak Completeness is far more involved technically than the proof of Lemma 12, and we only give an appropriate reference.[6]

Proposition 14 (Weak Completeness for Propositional Logic)**.** *Let $\varphi_1, \dots, \varphi_n, \psi$ be propositional wff's. If $\varphi_1, \dots, \varphi_n \models \psi$ then $\varphi_1, \dots, \varphi_n \vdash \psi$.*

Proof. This lemma is called the Completeness Theorem in the book by M. Huth and M. Ryan [5], which is strictly weaker than our (and commonly called) Completeness Theorem. The proof is in Section 1.4.4 of that book; specifically, this is the left-to-right implication in Corollary 1.39. □

Nevertheless, to reach Completeness via Weak Completeness, it is not possible to bypass Compactness, now in the form stated in Corollary 7. This is the next (alternative) proof of Completeness. Compactness remains the linchpin.

Alternative Proof of Theorem 13. If $\Gamma \models \psi$, there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models \psi$, by Corollary 7. Hence, by Proposition 14, $\Gamma_0 \vdash \psi$. Padding Γ_0 with the redundant premises in $(\Gamma - \Gamma_0)$, we conclude $\Gamma \vdash \psi$. □

1.3 Applications and Exercises

All the examples and exercises in this subsection are applications of Compactness. The final result in each of these applications can certainly be obtained by other means, but this is not immediately obvious and may require some tricky combinatorics. Compactness provides an elegant alternative. In each of the applications, most of the hard work is to formulate necessary and sufficient conditions for a solution in the form of an infinite set of propositional wff's; after which, satisfaction of those conditions is obtained by a relatively easy invocation of Compactness.

Example 15 (*Topological Sorting*)**.** A standard exercise in an undergraduate course on discrete algorithms is to show that every finite *directed acyclic graph* (dag) G can be *topologically sorted*, which means that the vertices of G can be linearly ordered on a horizontal line such that all the edges of G are drawn in the same direction, from left to right. We extend this result to infinite graphs: *Every infinite dag can be topologically sorted.*

Let $G \stackrel{\text{def}}{=} (V, E)$ be an infinite directed graph, where V is the set of vertices which we choose to name with the positive integers $\{1, 2, \dots\}$, and $E \subseteq V \times V$ is the set of edges. For convenience, we use two sets of doubly-indexed propositional variables, $\mathcal{Q}$ and $\mathcal{R}$, instead of $\mathcal{P}$:

$$\mathcal{Q} \stackrel{\text{def}}{=} \left\{ q_{i,j} \;\middle|\; i, j \in \{1, 2, \dots\} \right\} \quad \text{and} \quad \mathcal{R} \stackrel{\text{def}}{=} \left\{ r_{i,j} \;\middle|\; i, j \in \{1, 2, \dots\} \right\}.$$

[6] Many of the technical complications are the result of our choice of a proof system in natural-deduction style. It is possible to choose and adjust another proof system, notably in Hilbert-style rather than in natural-deduction style, which reduces the technical overhead in the proof of Weak Completeness.

The propositional wff's in this example are in $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{Q} \cup \mathcal{R})$. To facilitate our modeling of G's properties below, we purposely use names of vertices, such as i and j, as indices to identify variables $q_{i,j}$ and $r_{i,j}$. We consider initial finite fragments of the graph G, based on increasingly larger subsets of vertices:

$$V_1 \stackrel{\text{def}}{=} \{1\},\ V_2 \stackrel{\text{def}}{=} \{1,2\}, \ldots,\ V_n \stackrel{\text{def}}{=} \{1,2,\ldots,n\},\ \ldots \quad \text{so that also } V_1 \subseteq V_2 \subseteq\ \cdots\ \subseteq V_n \subseteq \cdots .$$

We write G_n for the finite subgraph of G induced by the vertices in V_n, *i.e.*, $G_n \stackrel{\text{def}}{=} (V_n, E_n)$ where $E_n = E \cap (V_n \times V_n)$. By this definition, G_n is a finite subgraph of $G_{n'}$, which is in turn a finite subgraph of the full graph G, for all $1 \leqslant n < n'$. Satisfaction of the following wff:

$$\pi_n \stackrel{\text{def}}{=} \bigwedge \Big\{ q_{i,j} \;\Big|\; (i,j) \in E_n \Big\} \;\wedge\; \bigwedge \Big\{ \neg q_{i,j} \;\Big|\; (i,j) \notin E_n \Big\}$$

determines subgraph G_n up to isomorphism, where "$\bigwedge$" stands for multiple conjunction, *i.e.*, $\bigwedge\{\varphi_1, \varphi_2, \ldots, \varphi_k\}$ means $(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_k)$. This is so because $q_{i,j}$ is satisfied (*i.e.*, assigned truth-value *true*) iff there is an edge from i to j, and $\neg q_{i,j}$ is satisfied (*i.e.*, assigned truth-value *true*) iff there is no edge from i to j.

Said differently, satisfaction of $q_{i,j}$ corresponds to the assertion "there is a path of length $= 1$ from vertex i to vertex j". We want to use satisfaction of variable $r_{i,j}$ to model the more general assertion "there is a finite path of length $\geqslant 1$ from vertex i to vertex j". We thus define another wff ρ_n as:

$$\rho_n \stackrel{\text{def}}{=} \bigwedge \Big\{ q_{i,j} \to r_{i,j} \;\Big|\; (i,j) \in E_n \Big\} \;\wedge\; \bigwedge \Big\{ q_{i,j} \wedge r_{j,k} \to r_{i,k} \;\Big|\; (i,j) \in E_n \text{ and } k \in \{1,\ldots,n\} \Big\}$$

Note carefully how the second part of ρ_n is defined. Informally in words, satisfaction of $r_{i,j}$ implies the existence of a path from i to j in G_n or, equivalently, the existence of an edge in the *transitive closure* of G_n.

We define one more wff θ_n, which models the assertion that "for one of the vertices i there is a path from i back to i in the transitive closure of G_n":

$$\theta_n \stackrel{\text{def}}{=} \bigvee \Big\{ r_{i,i} \;\Big|\; i \in \{1,\ldots,n\} \Big\}.$$

The wff θ_n is satisfied by G_n iff G_n contains a cycle. Hence, $\neg\theta_n$ is satisfied by G_n iff G_n is acyclic. Finally, consider the infinite set Γ of propositional wff's defined by:

$$\Gamma \stackrel{\text{def}}{=} \Big\{ \pi_n \;\Big|\; n \geqslant 1 \Big\} \cup \Big\{ (\pi_n \wedge \rho_n \to \neg\theta_n) \;\Big|\; n \geqslant 1 \Big\}.$$

If the full graph G is acyclic, then each of the finite subgraphs G_n is acyclic, which in turn implies that every finite subset of Γ is satisfiable. By Compactness, the full set of wff's Γ is satisfiable, which implies the full set of vertices $\{1, 2, \ldots, n, \ldots\}$ can be linearly ordered such that all the edges are drawn from left to right. $\square$

Exercise 16 (*Graph Coloring*)**.** An undirected graph G is said *k-colorable* if it is possible to assign only one of k colors to every vertex of G such that the two endpoints of every edge are assigned different colors. A famous (and very difficult to prove) result of graph theory is that *every finite planar graph is 4-colorable*. In this exercise you are asked to show that this result can be extended to *infinite* planar graphs using Compactness for PL.

We specify a graph $G \stackrel{\text{def}}{=} (V, E)$, finite or infinite, by its set of vertices V, which we take as an initial fragment of the positive integers $\{1, 2, \ldots\}$, and its set of edges $E \subseteq V \times V$. Graphs in this exercise are simple: undirected, with no self-loops and no multiple edges connecting the same two vertices.

Let $k \geqslant 1$ be fixed, the number of available colors. For convenience, use two separate sets of propositional variables, $\mathcal{Q}$ and $\mathcal{C}$, instead of $\mathcal{P}$:

$$\mathcal{Q} \stackrel{\text{def}}{=} \Big\{ q_{i,j} \;\Big|\; i,j \in \{1,2,\ldots\} \Big\} \quad \text{and} \quad \mathcal{C} \stackrel{\text{def}}{=} \Big\{ c_i^j \;\Big|\; i \in \{1,2,\ldots\} \text{ and } 1 \leqslant j \leqslant k \Big\}.$$

All wff's in this exercise should be in $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{Q} \cup \mathcal{C})$. Use variables in $\mathcal{Q}$ to model a given graph G: there is an edge connecting two distinct vertices i and j iff $q_{i,j}$ is set to truth value *true*. Use variables in $\mathcal{C}$ to model G's coloring: vertex i is assigned color $j \in \{1, \ldots, k\}$ iff c_i^j is set to truth value *true*. There are two parts in this exercise:

1. Let $G \stackrel{\text{def}}{=} (V, E)$ be finite, with $V = \{1, 2, \ldots, n\}$ for some $n \geqslant 1$. Write a wff φ_n which is satisfied iff G is k-colorable.

   Hint: Define φ_n in two parts, one part specifies the structure of G (including conditions that there are no self-loops and that G is undirected), and one part specifies that G is k-colorable.

2. Let $G \stackrel{\text{def}}{=} (V, E)$ be an infinite planar graph, with $V = \{1, 2, \ldots\}$, and let $k = 4$. Use Compactness for PL to write a rigorous argument showing that G is 4-colorable.

   Hint: If an infinite G is k-colorable, then so is every finite subgraph of G. Compactness should give you the converse. □

Exercise 17 (*Queens Problem*)**.** The n-*Queens Problem* is the problem of placing n queens on an $n \times n$ chessboard so that no two queens can attack each other. A solution of the problem when $n = 6$ is shown on the left of Figure 1 and three solutions are shown in Figure 2. In this exercise we specify the requirements of a solution for the n-Queens Problem as a propositional wff ψ_n, with one such wff for every $n \geqslant 4$. (There are no solutions for $n = 2$ and $n = 3$.) For convenience, we use a set $\mathcal{Q}$ of doubly-indexed propositional variables, instead of $\mathcal{P}$, where the indices range over the positive integers:

$$\mathcal{Q} \stackrel{\text{def}}{=} \left\{ q_{i,j} \;\middle|\; i, j \in \{1, 2, \ldots\} \right\}.$$

The desired wff ψ_n in this exercise is in $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{Q})$. We set the variable $q_{i,j}$ to truth value *true* (resp. *false*) if there is (resp. there is not) a queen placed in position (i, j) of the board, where we take the first index i (resp. the second index j) to range over the vertical axis downward (resp. the horizontal axis rightward); that is, i is a row number and j is a column number.[7] There are four parts in this exercise:

1. Write the wff ψ_n and justify how it accomplishes its task.

   Hint: Write ψ_n as a conjunction $\psi_n^{\text{row}} \wedge \psi_n^{\text{col}} \wedge \psi_n^{\text{diag1}} \wedge \psi_n^{\text{diag2}}$, where:

   (a) ψ_n^{row} is satisfied iff there is exactly one queen in each row,

   (b) ψ_n^{col} is satisfied iff there is exactly one queen in each column,

   (c) ψ_n^{diag1} is satisfied iff there is at most one queen in each diagonal,

   (d) ψ_n^{diag2} is satisfied iff there is at most one queen in each antidiagonal.

   Further Hint: Given any two distinct positions (i_1, j_1) and (i_2, j_2) along a diagonal, it is always the case that $i_1 - j_1 = i_2 - j_2$. And if the two positions are along an antidiagonal, then it is always the case that $i_1 + j_1 = i_2 + j_2$.

2. Imagine now an infinite chessboard, which occupies the entire south-east quadrant of the Cartesian plane. The coordinates along the vertical and horizontal axes are, respectively, i (increasing downward) and j (increasing rightward), both ranging over the positive integers $\{1, 2, \ldots\}$. In an attempt to repeat the argument in Example 15 and Exercise 16, someone once defined the set of wff's $\Gamma \stackrel{\text{def}}{=} \{ \psi_n \mid n \geqslant 4 \}$, and wrote the following (in outline here):

[7] This is the standard convention of identifying rows and columns in a two-dimensional matrix, which is not how we usually view the coordinates of the Cartesian plane, where the first coordinate is along the horizontal axis (going rightward) and the second coordinate is along the vertical axis (going upward). See Figure 1 for our convention. Also, following the conventions of two-dimensional matrices, a *diagonal* is a (-45^o)-*diagonal* directed downward starting from the west or north edge, and an *antidiagonal* is a $(+45^o)$-*diagonal* directed upward starting from the west or south edge.

The set Γ is finitely satisfiable and, therefore, satisfiable by Compactness. Hence, a solution of the *Infinite Queens Problem* exists, which satisfies conditions $\{$(a), (b), (c), (d)$\}$ for all $n \geqslant 4$.

What is wrong with the preceding argument? The answer is subtle and you need to be careful.

3. A solution for the *Infinite Queens Problem* can be inferred from a combinatorial game on the south-east quadrant of the Cartesian plane. Starting with 0, the positions are assigned natural numbers along successive upward antidiagonals, starting from the north-west corner, as shown in Figure 3. We call a traversal of the positions in increasing order of their assigned numbers a *good traversal*. In the game, a queen is initially placed anywhere on the board, and the players take turns moving it along a good traversal *in reverse*, *i.e.*, to a lower-numbered position, which is moreover a queen's move away. The first player unable to move loses. We now define a process of placing infinitely many queens, one at a time:

   Consider a good traversal of the south-east quadrant. When a position (i, j) is visited, if (i, j) is a queen's move away from all the previously placed queens, then place a queen in (i, j), else leave (i, j) empty and proceed to the next position.

   From the preceding game and the process just defined of placing infinitely many queens, it is possible to prove the following result:

   Every row and every column in the south-east quadrant is eventually occupied by exactly one queen. (By our definition of the process, every diagonal and every antidiagonal is necessarily occupied by at most one queen.)[8]

   Let the antidiagonals of the south-east quadrant be uniquely identified by the positive integers $1,\ 2,\ 3, \ldots$ starting from the top. (Antidiagonal 1 is empty, antidiagonal 2 is "*0*", antidiagonal 3 is "*1 2*", antidiagonal 4 is "*3 4 5*", etc., moving upwards along every antidiagonal.) Your task is to define, for every $k \geqslant 2$, a propositinal wff $\theta_k \in \mathsf{WFF}_{\mathsf{PL}}(\mathcal{Q})$ which is satisfied iff two conditions hold:

   (a) at most one queen occupies antidiagonal k, and

   (b) if a queen occupies antidiagonal k, then it cannot be attacked by any queen which is placed in a position to the north and/or west of antidiagonal k.

   We start at $k = 2$ because antidiagonal 1 is empty. Observe that antidiagonal k includes exactly $k - 1$ positions, namely $\{(k-1, 1),\ (k-2, 2),\ (k-3, 3),\ \ldots\ , (1, k-1)\}$, such that the two coordinates of each position add to k.

4. Let $\Theta \stackrel{\text{def}}{=} \{\, \theta_k \mid k \geqslant 2 \,\}$ be the set of wff's defined in the preceding part. Use Compactness for PL to give a rigorous argument that the *Infinite Queens Problem* has indeed a solution. $\square$

Exercise 18 (*Not-Three-In-Line Problem*)**.** This is an old problem of discrete geometry, not yet fully resolved as of this writing, which asks for the maximum number of pebbles that can be placed on an $n \times n$ chessboard so that no three pebbles are collinear, *i.e.*, not on the same row or column or diagonal. An upper bound on the number of pebbles is $2n$ because, by the Pigeonhole Principle, placing $2n + 1$ pebbles makes one row or one column necessarily contain three of them. But is $2n$ a reachable upper bound for all n? See Figure 6 for two solutions when $n = 10$, in which case the upper bound $2n = 20$ is reached.

We can take the *Not-Three-In-Line Problem* as a more complex variation on the *n-Queens Problem* in Exercise 17, with "pebbles" instead of "queens". Specifically, we use the same set $\mathcal{Q}$ of doubly-indexed propositional variables, with variable $q_{i,j}$ assigned truth-value *true* (resp. *false*) if there is (resp. there is not) a pebble placed in position (i, j) of the board, where the index i (resp. the index j) ranges over the vertical axis downward (resp. the horizontal axis rightward). There are two parts in this exercise:

[8] This follows from the analysis in Section 8 of the article "Queens in Exile: Non-attacking Queens on Infinite Chess Boards", by Dekking, Shallit, and Sloane [3].

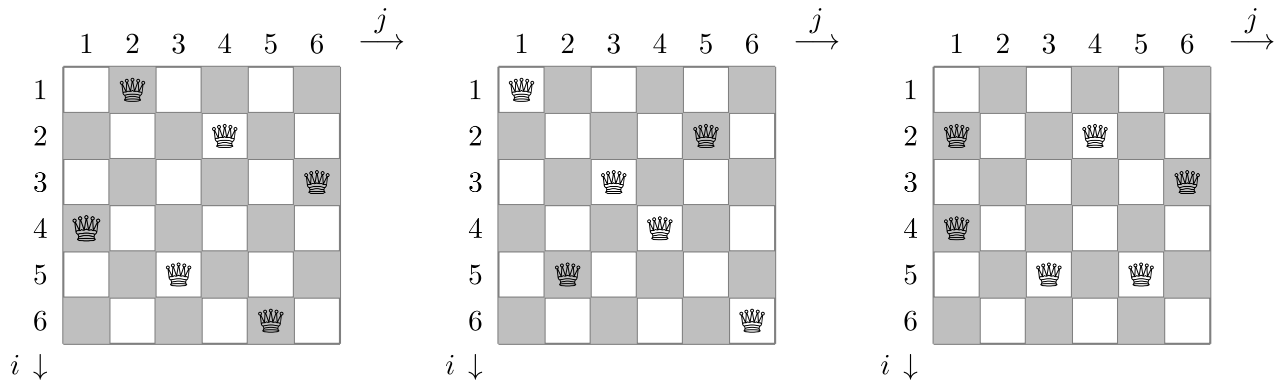

Figure 1: The 6-Queens Problem in Exercise 17: a solution *on the left*, a non-solution *in the middle* (satisfying conditions (a) and (b) only), a non-solution *on the right* (satisfying conditions (c) and (d) only).

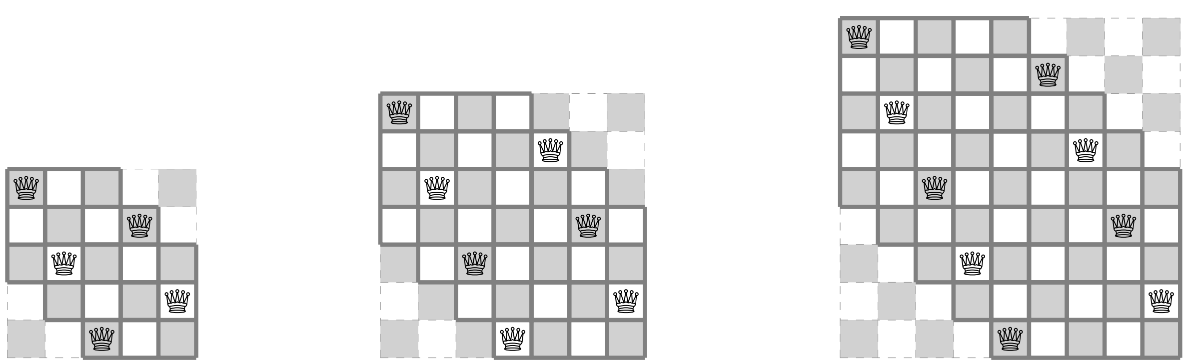

Figure 2: Solutions for the n-Queens Problem in Exercise 17 when $n \in \{5, 7, 9\}$. For every odd n, we use the same pattern to generate a solution, here called a *smooth board*, whereby the n queens are placed on n adjacent diagonals, and the $(n-1)$ empty diagonals are equally divided between the south-west and north-east corners.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	$\cdots$
1	$\boxed{0}$	*2*	*5*	*9*	*14*	*20*	*27*	*35*	*44*	*54*	$\cdots$	
2	*1*	*4*	*8*	$\boxed{13}$	*19*	*26*	*34*	*43*	*53*	$\cdots$		
3	*3*	$\boxed{7}$	*12*	*18*	*25*	*33*	*42*	*52*	$\cdots$			
4	*6*	*11*	*17*	*24*	$\boxed{32}$	*41*	*51*	$\cdots$				
5	*10*	*16*	$\boxed{23}$	*31*	*40*	*50*	$\cdots$					
6	*15*	*22*	*30*	*39*	*49*	$\cdots$						
7	*21*	*29*	*38*	*48*	$\cdots$							
8	*28*	*37*	*47*	$\cdots$								
9	*36*	*46*	$\cdots$									
10	*45*	$\cdots$										
11	$\cdots$											
$\vdots$												

Figure 3: Positions in the south-east quadrant of the Cartesian plane are identified by the natural numbers (in italics), starting with *0* at the north-west corner and continuing along successive upwards antidiagonals. The figure shows only antidiagonals 2, 3,. . ., 11 (antidiagonal 1 is empty). The boxed positions are the first 5 queens that are placed according to the traversal defined in part 3 of Exercise 17. In the shown fragment of the quadrant, antidiagonals 2, 5, 6, 8, and 9, are each occupied by a queen while antidiagonals 3, 4, 7, 10, and 11, are not.

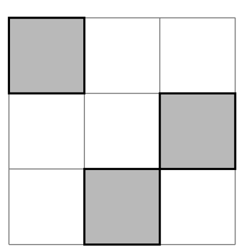

Figure 4: Shape of the *good board* B_k in part 3 of Exercise 17: Its size is $(5 \cdot 3^k) \times (5 \cdot 3^k)$, where $k \geqslant 1$. The non-empty sub-boards (shaded areas) are each a copy of B_{k-1} when $k \geqslant 2$; the *initial good board* B_1 is shown in Figure 5. For every $k \geqslant 1$, B_k is a solution of the $(5 \cdot 3^k)$-queens problem, with all queens placed on adjacent diagonals and with $\big((5 \cdot 3^k) - 1\big)$ empty diagonals equally divided between the south-west and north-east corners.

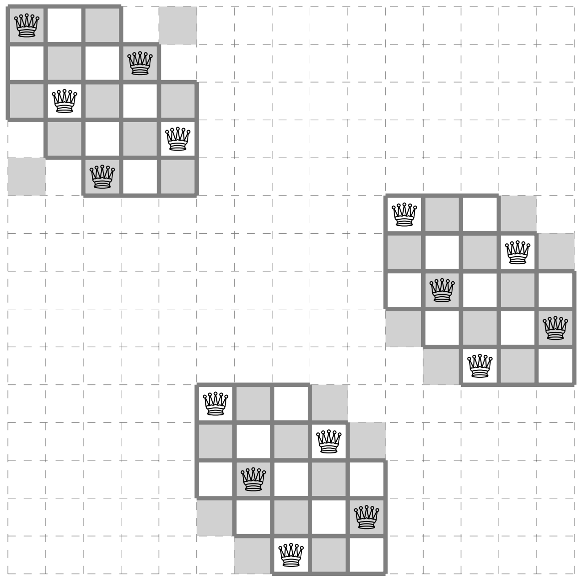

Figure 5: The *initial good board* B_1 of size 15×15 in Exercise 17, ~~which solves the 15-queens problem, with all queens on adjacent diagonals and all empty diagonals equally divided between the south-west and north-east corners~~.

1. Write a wff $\psi_n \stackrel{\text{def}}{=} \psi_n^{\text{row}} \wedge \psi_n^{\text{col}} \wedge \psi_n^{\text{diag1}} \wedge \psi_n^{\text{diag2}}$ in $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{Q})$ according to the following specification:
   (a) ψ_n^{row} is satisfied iff there are exactly two pebbles in each row,
   (b) ψ_n^{col} is satisfied iff there are exactly two pebbles in each column,
   (c) ψ_n^{diag1} is satisfied iff there are at most two pebbles in each diagonal,
   (d) ψ_n^{diag2} is satisfied iff there are at most two pebbles in each antidiagonal.

   Hint: Given two distinct positions (i_1, j_1) and (i_2, j_2) along a diagonal, it is always that $i_1 - j_1 = i_2 - j_2$. And if the two positions are along an antidiagonal, then it is always that $i_1 + j_1 = i_2 + j_2$.
2. We extend the *Not-Three-In-Line Problem* to an infinite chessboard, which occupies the entire south-east quadrant of the Cartesian plane. Give a precise argument, based on Compactness, showing that if the problem can be solved for each $n \geqslant 4$ (there is no solution for $n < 4$), then a solution exists for the *Infinite Not-Three-In-Line Problem*.[9] $\square$

[9]It is known that the *Not-Three-In-Line Problem* has a solution for every $n \leqslant 46$, but not for $n > 46$ as here formulated. It is conjectured that, for $n > 46$, no matter how you place $2n$ pebbles on the board, you are doomed to find three of them that are collinear. Put differently, for $n > 46$, it is conjectured that you are forced to place fewer than $2n$ pebbles to avoid three collinear pebbles. For more information on the *Not-Three-In-Line Problem*, search the Web.

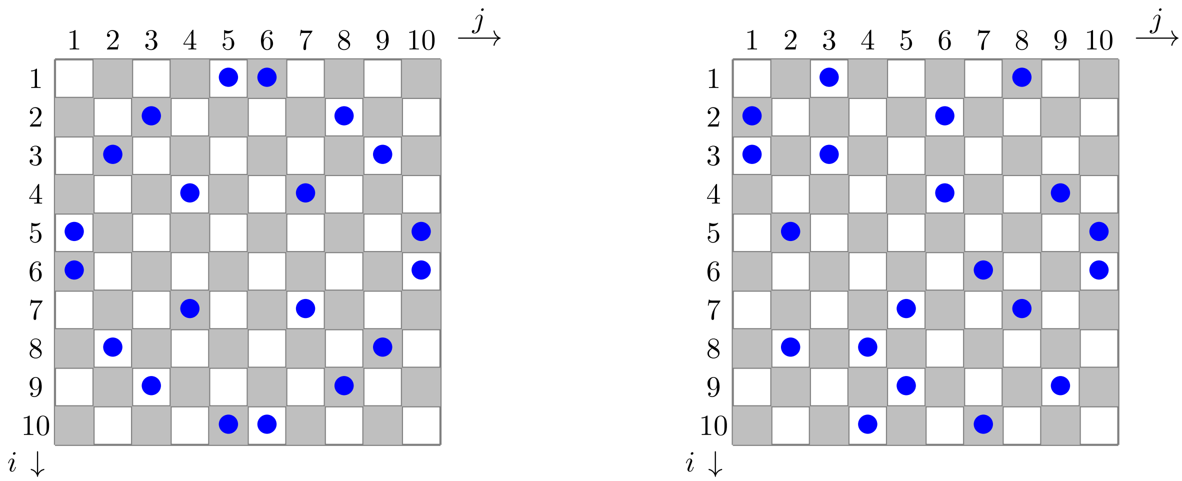

Figure 6: Two solutions of the Not-Three-In-Line Problem when $n = 10$ in Exercise 18, both satisfying conditions (a), (b), (c), and (d), in the exercise.

2 The Logic of QBF's (QBF)

We can prove Compactness for the logic of *quantified Boolean formulas* (QBF's) by reducing it to Compactness for *propositional logic*. We have already done much of the preliminary work in Section 1. We write $\mathsf{WFF}_{\mathsf{QBF}}(\mathcal{P})$ for the set of all QBF's over the set $\mathcal{P}$ of propositional variables.

Lemma 19. *Let Γ be a subset, finite or infinite, of $\mathsf{WFF}_{\mathsf{QBF}}(\mathcal{P})$. We can construct a set Γ' of quantifier-free formulas in $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$ such that:*

1. *Γ is finitely satisfiable iff Γ' is finitely satisfiable.*
2. *Γ is satisfiable iff Γ' is satisfiable.*

The construction in the proof below establishes a stronger result: Γ and Γ' are more than *finitely equisatisfiable* and *equisatisfiable*; they are in fact *equivalent* (informally, "they say the same thing"). Specifically, for every wff $\varphi \in \Gamma$ there is a wff $\varphi' \in \Gamma'$ such that φ and φ' are equivalent; and, similarly, for every propositional wff $\varphi' \in \Gamma'$ there is a wff $\varphi \in \Gamma$ such that φ and φ' are equivalent.[10]

Proof. If φ is a propositional wff, we write "$\varphi[p := \bot]$" and "$\varphi[p := \top]$" to denote the substitution of the symbols $\bot$ and $\top$, respectively, for every occurrence of variable p in φ.

[10] If so, why use QBF instead of PL? Applications and exercises in Subsection 2.3 illustrate some of the advantages – just a few out of many – of using QBF rather than PL. In particular, *quantified Boolean formulas* are central in the study of what is called the *polynomial-time hierarchy* in computational complexity, something outside the scope of these notes.

We define a translation from QBF to PL, named "$\boxed{\mathsf{QBF} \mapsto \mathsf{PL}}$" by structural induction:[11]

1. $\boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(p) \quad \stackrel{\text{def}}{=} \quad p \qquad$ (for every variable p)
2. $\boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(\neg\varphi) \quad \stackrel{\text{def}}{=} \quad \neg\ \boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(\varphi)$
3. $\boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(\varphi \wedge \psi) \quad \stackrel{\text{def}}{=} \quad \boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(\varphi) \wedge \boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(\psi)$
4. $\boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(\varphi \vee \psi) \quad \stackrel{\text{def}}{=} \quad \boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(\varphi) \vee \boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(\psi)$
5. $\boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(\varphi \to \psi) \quad \stackrel{\text{def}}{=} \quad \boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(\varphi) \to \boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(\psi)$
6. $\boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(\forall p\, \varphi) \quad \stackrel{\text{def}}{=} \quad \Big(\boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(\varphi)\Big)[p := \bot] \ \wedge\ \Big(\boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(\varphi)\Big)[p := \top]$
7. $\boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(\exists p\, \varphi) \quad \stackrel{\text{def}}{=} \quad \Big(\boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(\varphi)\Big)[p := \bot] \ \vee\ \Big(\boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(\varphi)\Big)[p := \top]$

Claim: For every QBF φ, the transformation $\boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(\varphi)$ satisfies the following properties:

(a) $\boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(\varphi)$ is a propositional wff,

(b) the set of free variables $\text{FV}(\varphi)$ in φ are exactly all the variables occurring in $\boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(\varphi)$, and

(c) if $X = \text{FV}(\varphi)$, then for every truth assignment σ to the members of X, it holds that σ satisfies φ iff σ satisfies $\boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(\varphi)$.

Part (c) in this claim shows that φ and $\boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(\varphi)$ are not only equisatisfiable, but also equivalent. We leave the proof of this claim as an exercise. Given an arbitrary subset $\Gamma \subseteq \mathsf{WFF}_{\mathsf{QBF}}(\mathcal{P})$, we now define Γ' by:

$$\Gamma' \stackrel{\text{def}}{=} \Big\{ \boxed{\mathsf{QBF} \mapsto \mathsf{PL}}(\varphi) \ \Big|\ \varphi \in \Gamma \Big\}$$

By the preceding claim, we conclude that for every truth assignment σ to $\mathcal{P}$:

- for every finite subset $\Delta \subseteq \Gamma$ there is a finite subset $\Delta' \subseteq \Gamma'$ s.t. σ satisfies Δ iff σ satisfies Δ',
- for every finite subset $\Delta' \subseteq \Gamma'$ there is a finite subset $\Delta \subseteq \Gamma$ s.t. σ satisfies Δ iff σ satisfies Δ',
- σ satisfies Γ iff σ satisfies Γ'.

We leave the missing details in the proof of the preceding three bullet points as an exercise. □

Exercise 20. Prove the **claim** in the proof of Lemma 19. *Hint*: Use structural induction on QBF's, following the seven steps in the definition of the transformation $\boxed{\mathsf{QBF} \mapsto \mathsf{PL}}$. □

Exercise 21. In the statement of Lemma 19 and its proof, the set Γ of QBF's and the set Γ' of propositional wff's are equivalent. Specify:

1. Conditions under which $\big|\Gamma\big| = \big|\Gamma'\big|$, and
2. Conditions under which $\big|\Gamma\big| > \big|\Gamma'\big|$,

where $\big|\Gamma\big|$ is the cardinality of the set Γ. *Hint*: Consider, for example, the case when all the QBF's in Γ are *closed*; what is Γ' in this case? □

Exercise 22. Supply the missing details in the proof of the three bullet points at the end of the proof of Lemma 19. *Hint*: This is subtler than at first blush; do Exercise 21 before you attempt this one. □

[11] I quickly run out of notation. To simplify my task, I denote translations of syntax in a particular way: Each is denoted by a framed box and what is inside the box, here "$\mathsf{QBF} \mapsto \mathsf{PL}$", suggests what the translation does. The box and its contents is a single name.

2.1 From Compactness in PL to Compactness in QBF

Theorem 23 (Compactness for the Logic of QBF's, Version I)**.** *Let $\Gamma \subseteq \mathsf{WFF}_{\mathsf{QBF}}(\mathcal{P})$. It then holds that Γ is satisfiable iff Γ is finitely satisfiable.*

Proof. The left-to-right implication is immediate. The non-trivial is the right-to-left implication, *i.e.*, we have to prove that if Γ is finitely satisfiable, then Γ is satisfiable. Let Γ' be the set of propositional wff's defined from Γ according to Lemma 19.

By Lemma 19, Γ is finitely satisfiable iff Γ' is finitely satisfiable. By Theorem 2, Γ' is finitely satisfiable iff Γ' is satisfiable. By Lemma 19 once more, Γ' is satisfiable iff Γ is satisfiable. Hence, if Γ is finitely satisfiable, then Γ is satisfiable, as desired. □

For the next lemma and its corollary, review the formal semantics of QBF's in Appendix B.

Lemma 24. *Let $\Gamma \subseteq \mathsf{WFF}_{\mathsf{QBF}}(\mathcal{P})$ and $\varphi \in \mathsf{WFF}_{\mathsf{QBF}}(\mathcal{P})$, both arbitrary. It then holds that $\Gamma \models \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable – or, equivalently, $\Gamma \not\models \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is satisfiable.*

Proof. Identical to the proof of Lemma 6, except that here Γ is a set of QBF's and φ is a QBF. □

Corollary 25 (Compactness for the Logic of QBF's, Version II)**.** *Let $\Gamma \subseteq \mathsf{WFF}_{\mathsf{QBF}}(\mathcal{P})$ and $\varphi \in \mathsf{WFF}_{\mathsf{QBF}}(\mathcal{P})$. It then holds that $\Gamma \models \varphi$ iff there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models \varphi$.*

Proof. Identical to the proof of Corollary 7, except that here Γ is a set of QBF's and φ is a QBF. Moreover, here we invoke Lemma 24 instead of Lemma 6, and Theorem 23 instead of Theorem 2. □

2.2 From Compactness in QBF to Completeness in QBF

The statements of Lemma 10 (the *Deduction Theorem*) and Lemma 12, as well as their respective proofs, hold verbatim for the logic of QBF's – except that "$\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$" has to be replaced by "$\mathsf{WFF}_{\mathsf{QBF}}(\mathcal{P})$" throughout.

Theorem 26 (Completeness for the Logic of QBF's)**.** *Let $\Gamma \subseteq \mathsf{WFF}_{\mathsf{QBF}}(\mathcal{P})$ and $\psi \in \mathsf{WFF}_{\mathsf{QBF}}(\mathcal{P})$, both arbitrary. If $\Gamma \models \psi$, then $\Gamma \vdash \psi$.*

Proof. This proof is identical to the proof of Theorem 13, except that all formulas are now QBF's, not just propositional wff's. □

2.3 Applications and Exercises

Example 27 (*Transition Systems*)**.** A *transition system* (sometime called a *state-transition system*) is specified as a structure $\mathcal{M} \stackrel{\text{def}}{=} (\mathsf{States}, R, \mathsf{Init}, \mathsf{End})$ where States is a finite or infinite set, $R \subseteq \mathsf{States} \times \mathsf{States}$ is a binary relation (*the transition relation*), and $\mathsf{Init} \subseteq \mathsf{States}$ and $\mathsf{End} \subseteq \mathsf{States}$ are the subsets of *initial states* and *end states*, respectively.

When States is a finite set, $\mathcal{M}$ is conveniently represented by a finite directed graph; an example is shown in Figure 7. Each state of the system is a *node* in the graph and each possible transition from a state to another is a directed *edge*.

We can uniquely identify each state by a bit vector, with $B = \{\bot, \top\}$ as the set of bits. For the system in Figure 7 with 4 states, 2-bit vectors suffice for this encoding; in this cae, we can model the *transition relation* by a propositional wff θ with propositional variables $\{p_1, p_2, p_3, p_4\}$ where we use the pairs (p_1, p_2) and (p_3, p_4) to encode the *from-state* and *to-state* of a transition, respectively. The setup in full generality is thus:

$$\begin{aligned}
\mathsf{encode} &: \mathsf{States} \to B^n && \text{(where } n = \lceil \log_2 \mathsf{size}(\mathsf{States}) \rceil\text{),}\\
\mathsf{init} &: B^n \to \{\mathit{false}, \mathit{true}\} && \text{(the set of initial states),}\\
\mathsf{end} &: B^n \to \{\mathit{false}, \mathit{true}\} && \text{(the set of end states),}\\
\theta &: B^n \times B^n \to \{\mathit{false}, \mathit{true}\} && \text{(the transition relation).}
\end{aligned}$$

Whichever is more convenient, we write $\{\mathsf{init}, \mathsf{end}, \theta\}$ as functions (as above) or sometimes as unary and binary relations; either way, they are translated into propositional wff's whose interpretations are values in $\{\mathit{false}, \mathit{true}\}$. Note that the symbol "$R$", "Init", and "End", are not part of the vocabulary of PL and QBF, which is why we need to write three wff's $\{\ \theta,\ \mathsf{init},\ \mathsf{end}\ \}$ in the syntax of PL and QBF to formally model these relations.

For the particular transition system in Figure 7 where $n = \log_2 4 = 2$, the setup is thus:

$$\begin{aligned}
\mathsf{encode}(\mathsf{States}) &\stackrel{\text{def}}{=} \Big\{(\bot,\bot),(\bot,\top),(\top,\bot),(\top,\top)\Big\},\\
\mathsf{init}(p_1,p_2) &\stackrel{\text{def}}{=} (p_1 \leftrightarrow \bot) \wedge (p_2 \leftrightarrow \bot) = (\neg p_1 \wedge \neg p_2) \quad \text{or also } \mathsf{init} \stackrel{\text{def}}{=} \{(\bot,\bot)\},\\
\mathsf{end}(p_1,p_2) &\stackrel{\text{def}}{=} (p_1 \leftrightarrow \top) \wedge (p_2 \leftrightarrow \top) = (p_1 \wedge p_2) \quad \text{or also } \mathsf{end} \stackrel{\text{def}}{=} \{(\top,\top)\},\\
\theta(p_1,p_2,p_3,p_4) &\stackrel{\text{def}}{=} \big((\neg p_1 \wedge \neg p_2) \ \to\ (\neg p_3 \wedge \neg p_4) \vee (\neg p_3 \wedge p_4)\big) && \text{(from } s_1\text{)}\\
&\wedge\ \big((\neg p_1 \wedge p_2) \ \to\ (\neg p_3 \wedge p_4) \vee (p_3 \wedge \neg p_4)\big) && \text{(from } s_2\text{)}\\
&\wedge\ \big((p_1 \wedge \neg p_2) \ \to\ (\neg p_3 \wedge p_4) \vee (\neg p_3 \wedge \neg p_4) \vee (p_3 \wedge p_4)\big) && \text{(from } s_3\text{)}\\
&\wedge\ \big((\neg p_3 \wedge \neg p_4) \ \to\ (\neg p_1 \wedge \neg p_2) \vee (p_1 \wedge \neg p_2)\big) && \text{(to } s_1\text{)}\\
&\wedge\ \big((\neg p_3 \wedge p_4) \ \to\ (\neg p_1 \wedge \neg p_2) \vee (\neg p_1 \wedge p_2) \vee (p_1 \wedge \neg p_2)\big) && \text{(to } s_2\text{)}\\
&\wedge\ \big((p_3 \wedge \neg p_4) \ \to\ (\neg p_1 \wedge p_2)\big) && \text{(to } s_3\text{)}\\
&\wedge\ \big((p_3 \wedge p_4) \ \to\ (p_1 \wedge \neg p_2)\big) && \text{(to } s_4\text{)}
\end{aligned}$$

Note how we write θ: it is a conjunction of seven implications, three implications for the *from-state* part of transitions (or the *tail* end of edges) and four implications for the *to-state* part of transitions (or the *head* end of edges). Convince yourself that θ faithfully models the behavior of the relation R (a little painstaking task!).

With the preceding we can now express problems of *reachability* in the transition system, namely, whether some states are reachable from other states. Consider the following wff's as an example, where we purposely use a different set of propositional variables $\{q_1, q_2, \ldots\}$:

$$\begin{aligned}
\varphi_1(q_1,\ldots,q_4) &\stackrel{\text{def}}{=} \mathsf{init}(q_1,q_2) \wedge \theta(q_1,q_2,q_3,q_4) \wedge \mathsf{end}(q_3,q_4),\\
\varphi_2(q_1,\ldots,q_6) &\stackrel{\text{def}}{=} \mathsf{init}(q_1,q_2) \wedge \theta(q_1,q_2,q_3,q_4) \wedge \theta(q_3,q_4,q_5,q_6) \wedge \mathsf{end}(q_5,q_6),\\
\varphi_3(q_1,\ldots,q_8) &\stackrel{\text{def}}{=} \mathsf{init}(q_1,q_2) \wedge \theta(q_1,q_2,q_3,q_4) \wedge \theta(q_3,q_4,q_5,q_6) \wedge \theta(q_5,q_6,q_7,q_8) \wedge \mathsf{end}(q_7,q_8).
\end{aligned}$$

The wff φ_1 (resp. φ_2, resp. φ_3) encodes the problem of whether it is possible to go from *initial state* s_1 to *end state* s_4 in one step (resp. two steps, resp. three steps). By inspection, the transition system $\mathcal{M}$ in Figure 7 does not satisfy φ_1 and φ_2, while $\mathcal{M}$ does satisfy φ_3. More succintly, using quantifiers allowed by the syntax of QBF, it holds that:[12]

$$\Gamma \not\models (\exists\, q_1 \cdots q_4.\, \varphi_1), \quad \Gamma \not\models (\exists\, q_1 \cdots q_6.\, \varphi_2), \quad \text{and } \Gamma \models (\exists\, q_1 \cdots q_8.\, \varphi_3),$$

$$\text{where } \Gamma \stackrel{\text{def}}{=} \Big\{ \forall\, p_1\, p_2.\, \mathsf{init}(p_1, p_2)\,, \forall\, p_1\, p_2.\, \mathsf{end}(p_1, p_2)\,, \forall\, p_1 \cdots p_4.\, \theta(p_1, p_2, p_3, p_4) \Big\}.$$

Things become more complicated when the transition relation expressed by θ has to account for many more states than only four in this example, or when the path from an *initial state* to an *end state* must satisfy some restriction. Exercises 28 and 29 pursue the analysis started in this example further. □

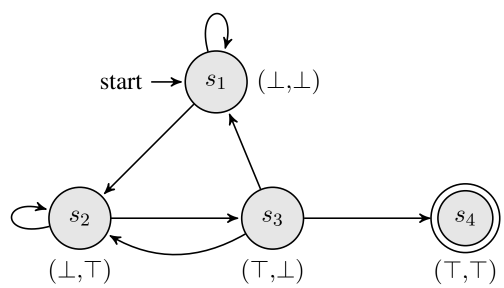

Figure 7: Graphical representation of a transition system with 4 states $\{s_1, s_2, s_3, s_4\}$ in Example 27, which can be encoded by 2-bit vectors $\{(\bot,\bot), (\bot,\top), (\top,\bot), (\top,\top)\}$, where s_1 is an *initial state* and s_4 is an *end state*.

Exercise 28 (*Reachability in Transition Systems*)**.** This is a continuation of the analysis in Example 27 in relation to the particular transition system in Figure 7. If we want to model reachability of s_4 from s_1 for some large number k of steps, the resulting wff will be unwieldy with k copies of θ as sub-wff's. A way out is to resort to a wff φ_k involving quantifiers and a single copy of θ, as follows:

$$\begin{aligned}\varphi_k \stackrel{\text{def}}{=} \exists\, q_1\, q_2\, \cdots\, q_{2k+1}\, q_{2k+2}\,.\, \mathsf{init}(q_1, q_2) \wedge \mathsf{end}(q_{2k+1}, q_{2k+2}) \wedge \\ \forall\, r_1\, r_2\, r_3\, r_4\,.\Big(\Big(\bigvee\nolimits_{0\leqslant i\leqslant k-1} r_1 = q_{2i+1} \wedge r_2 = q_{2i+2} \wedge r_3 = q_{2i+3} \wedge r_4 = q_{2i+4}\Big) \to \theta(r_1, r_2, r_3, r_4)\Big)\end{aligned}$$

where "$p = q$" is an abbreviation for "$p \leftrightarrow q$", and "$p \leftrightarrow q$" is an abbreviation for "$(p \to q) \wedge (q \to p)$". Give a precise argument showing that φ_k correctly models reachability of s_4 from s_1 in k steps. □

Exercise 29 (*The Unwind Property in Transition Systems*)**.** A finite transition system $\mathcal{M}$ is said to have the *unwind property* if there is a natural number n such that every execution path from an *initial state* to an *end state* halts within at most n steps (or n single-edge transitions in the graph representation of $\mathcal{M}$).

As it stands, the system in Figure 7 does not have the unwind property. From the *initial state* s_1 to the *end state* s_4, there are arbitrarily long executions paths, thus preventing any "unwinding" or "unrolling" of the system into an equivalent and finite loop-free transition system.

We now consider operating the system under two separate restrictions (assumed to be enforced by mechanisms not mentioned in our definitions here):

(a) An execution path from s_1 to s_4 is *valid* provided each of the states in $\{s_1, s_2, s_3\}$ is visited an equal number $n \geqslant 1$ of times.

[12]It is tempting to replace "$\Gamma \not\models \ldots$" and "$\Gamma \models \ldots$" by "$\mathcal{M} \not\models \ldots$" and "$\mathcal{M} \models \ldots$", respectively. Although the intent is clear, the latter notation is not permitted by the definition of "$\models$" in the semantics of QBF. $\mathcal{M}$ is not a model in quantified Boolean logic. The set of wff's Γ completely captures the behavior of the transition system.

(b) An execution path from s_1 to s_4 is *valid* provided s_1 is visited at most $n \leqslant 2$ times, and each of s_2 and s_3 is visited $2n$ times.

There are four parts in this exercise. For the last two parts, you may find it helpful to do Exercise 28 first, taking advantage of the succintness that quantifiers allow in writing wff's:

1. Write the propositional wff θ_a which models the transition relation when the system operates under restriction (a).

   Hint: θ_a is a restriction of θ defined in Example 27. The requirement that all the states in $\{s_1, s_2, s_3\}$ are visited an equal number of times precludes the use of the self-loops around s_1 and s_2, as well as the loop "$s_2 \to s_3 \to s_2$", but not the loop "$s_1 \to s_2 \to s_3 \to s_1$".

2. Write the propositional wff θ_b which models the transition relation when the system operates under restriction (b).

   Hint: The requirement that s_2 and s_3 are visited an equal number times precludes the use of the self-loop around s_2, but not the use of the other loops.

3. Give a formal logic-based argument (no "hand waving") showing the transition system under restriction (a) does not have the *unwind property*. Thus, under restriction (a), the existence of arbitrarily long (finite) valid executions implies the system does not always halt, *i.e.*, there are non-terminating valid executions.

   Hint: Exhibit an infinite set Δ of QBF wff's expressing the existence of an infinite path starting at s_1 with infinitely many valid finite prefixes. Show that Δ is finitely satisfiable and invoke Compactness.

4. Give a precise argument showing that the transition system under restriction (b) does have the *unwind property*. Thus, under restriction (b), the system always halts and all valid execution paths are finite.

   Hint: This will be (mostly) a counting argument based on what restriction (b) entails. □

3 Equality Logic (eL)

Equality logic (eL) is a restriction of *first-order logic* (FOL). We assume eL and FOL are over the same infinite set X of first-order variables. In eL, all atomic wff's are of the form $(x_i \approx x_j)$ for some $x_i, x_j \in X$. Though quite drastic, the restriction of FOL to eL is still capable of expressing non-trivial properties of first-order models.

The set of all eL wff's is denoted $\mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$. Our first task is to translate any set Γ of wff's in eL, *i.e.*, $\Gamma \subseteq \mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$, to a set $\Gamma' \subseteq \mathsf{WFF}_{\mathsf{PL}}(\mathcal{Q})$ where $\mathcal{Q}$ is a set of doubly-indexed propositional variables:

$$\mathcal{Q} \stackrel{\text{def}}{=} \{\, q_{i,j} \mid i, j \in \mathbb{N} \,\}.$$

We do this in Lemma 30. Our propositional wff's here are not over the set $\mathcal{P}$ of singly-indexed variables in Section 1. This is a convenience to make the translation in the lemma a little easier and more transparent.

The semantics of $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{Q})$ requires a truth assignment $\sigma : \mathcal{Q} \to \{\mathit{true}, \mathit{false}\}$, the semantics of $\mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$ requires a structure $\mathcal{A} \stackrel{\text{def}}{=} (A, =, \ldots)$ together with a valuation $\tau : X \to A$. The details are in Appendix B.3. We use propositional variable $q_{i,j}$ to represent the equality $(x_i \approx x_j)$ between first-order variables x_i and x_j. Thus, we want that $\sigma(q_{i,j}) = \mathit{true}$ iff $\tau(x_i) = \tau(x_j)$ in $\mathcal{A}$. To that end, we define a set of propositional wff's $\Delta(S)$ relative to a set $S \subseteq \mathbb{N}$ of indices as follows:

$$\begin{aligned}
\Delta(S) \stackrel{\text{def}}{=} & \{\, (\top \to q_{i,i}) \mid i \in S \,\} && \text{(“equality is reflexive”)} \\
& \cup \{\, (q_{i,j} \to q_{j,i}) \mid i, j \in S \,\} && \text{(“equality is symmetric”)} \\
& \cup \{\, (q_{i,j} \wedge q_{j,k} \to q_{i,k}) \mid i, j, k \in S \,\} && \text{(“equality is transitive”)}
\end{aligned}$$

Note how we use indices to model the properties of equality: reflexivity, symmetry, and transitivity.

Lemma 30. *Let Γ be a subset, finite or infinite, of* $\mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$. *We can construct a set Γ' of propositional formulas in* $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{Q})$ *such that:*

1. *Γ is finitely satisfiable iff Γ' is finitely satisfiable.*
2. *Γ is satisfiable iff Γ' is satisfiable.*

Proof. Let $S \stackrel{\text{def}}{=} \{\, i \in \mathbb{N} \mid x_i \in \mathrm{FV}(\Gamma) \,\}$. In words, S collects all the indices of first-order variables occurring in Γ. The set S may be finite or infinite.

The translation from Γ to Γ' is in two parts. We first transform each member of Γ using a function named $\boxed{\mathsf{eL} \mapsto \mathsf{PL}}$. The desired Γ' is $\Delta(S) \cup \boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\Gamma)$. The definition of $\boxed{\mathsf{eL} \mapsto \mathsf{PL}}$ is by structural induction, similar to that of $\boxed{\mathsf{QBF} \mapsto \mathsf{PL}}$ in the proof of Lemma 19:

1. $\boxed{\mathsf{eL} \mapsto \mathsf{PL}}(x_i \approx x_j) \stackrel{\text{def}}{=} q_{i,j}$ (for every $(x_i \approx x_j)$ in Γ)
2. $\boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\neg\varphi) \stackrel{\text{def}}{=} \neg\, \boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\varphi)$
3. $\boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\varphi \wedge \psi) \stackrel{\text{def}}{=} \boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\varphi) \wedge \boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\psi)$
4. $\boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\varphi \vee \psi) \stackrel{\text{def}}{=} \boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\varphi) \vee \boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\psi)$
5. $\boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\varphi \to \psi) \stackrel{\text{def}}{=} \boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\varphi) \to \boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\psi)$

In words, all that $\boxed{\mathsf{eL} \mapsto \mathsf{PL}}$ does is to replace every atomic wff of the form $(x_i \approx x_j)$ by the variable $q_{i,j}$. □

Exercise 31. Define a small subset $\Gamma \subseteq \mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$ – no more than two or three wff's – such that:

1. Γ is not satisfiable, *i.e.*, for every interpretation $(\mathcal{A}, \tau)$ we have $\mathcal{A}, \tau \not\models_{\mathsf{eL}} \Gamma$.
2. However, $\boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\Gamma)$ is satisfiable, *i.e.*, there is a truth assignment σ such that $\sigma \models_{\mathsf{PL}} \boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\Gamma)$.
3. But, as predicted by the preceding lemma, $\Delta(S) \cup \boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\Gamma)$ is not satisfiable, *i.e.*, for every truth assignment σ we have $\sigma \not\models_{\mathsf{PL}} \Delta(S) \cup \boxed{\mathsf{eL} \mapsto \mathsf{PL}}(\Gamma)$, where S is the set of variable indices occurring in Γ.

This shows that, in the proof of Lemma 30, we cannot omit $\Delta(S)$ in the definition of Γ'. $\square$

3.1 From Compactness in PL to Compactness in eL

We follow the same sequence as in Subsection 2.1.

Theorem 32 (Compactness for Equality Logic, Version I)**.** *Let* $\Gamma \subseteq \mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$. *It then holds that* Γ *is satisfiable iff* Γ *is finitely satisfiable.*

Proof. This proof is identical to the proof of Theorem 23, after replacing Lemma 19 by Lemma 30. $\square$

For the next lemma and its corollary, review the formal semantics of *equality logic* in Appendix B.

Lemma 33. *Let* $\Gamma \subseteq \mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$ *and* $\varphi \in \mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$*, both arbitrary. It then holds that* $\Gamma \models \varphi$ *iff* $\Gamma \cup \{\neg\varphi\}$ *is unsatisfiable – or, equivalently,* $\Gamma \not\models \varphi$ *iff* $\Gamma \cup \{\neg\varphi\}$ *is satisfiable.*

Proof. Identical to the proof of Lemma 6, except that here $\Gamma \cup \{\varphi\}$ is a set of wff's in *equality logic*. $\square$

Corollary 34 (Compactness for Equality Logic, Version II)**.** *Let* $\Gamma \cup \{\varphi\} \subseteq \mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$ *with* Γ *being possibly infinite. It then holds that* $\Gamma \models \varphi$ *iff there is a finite subset* $\Gamma_0 \subseteq \Gamma$ *such that* $\Gamma_0 \models \varphi$.

Proof. Identical to the proof of Corollary 7, except that here $\Gamma \cup \{\varphi\}$ is a set of wff's in *equality logic*. Moreover, here we invoke Lemma 33 instead of Lemma 6, and Theorem 32 instead of Theorem 2. $\square$

3.2 From Compactness in eL to Completeness in eL

The statements of Lemma 10 (the *Deduction Theorem*) and Lemma 12, as well as their respective proofs, hold verbatim for *equality logic* – except that "$\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$" has to be replaced by "$\mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$" throughout.

Theorem 35 (Completeness for Equality Logic)**.** *Let* $\Gamma \subseteq \mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$ *and* $\psi \in \mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$*, both arbitrary, with* Γ *being possibly infinite. If* $\Gamma \models \psi$*, then* $\Gamma \vdash \psi$.

Proof. Identical to the proof of Theorem 13, except that all the wff's are now wff's of *equality logic*. $\square$

3.3 Applications and Exercises

In many ways, eL is a very weak logic. Nonetheless, it can still express non-trivial properties of its models.

Example 36 (*Infiniteness is* eL*-Expressible*)**.** Can we write a set Γ of wff's in eL such that every interpretation $(\mathcal{A}, \sigma)$ satisfying Γ is infinite? The following Γ will do:

$$\Gamma \stackrel{\text{def}}{=} \{ \neg(x_i \approx x_j) \mid i, j \in \mathbb{N} \text{ and } i \neq j \}.$$

The justification is very simple. Take an arbitrary $(\mathcal{A}, \sigma)$ where the universe of $\mathcal{A}$ is a set A. If $\mathcal{A}, \sigma \models \Gamma$, then the valuation $\sigma : X \to A$ must assign a distinct element of A to every variable in X. Since X is infinite, we necessarily have that $\sigma(x_i) \neq \sigma(x_j)$ for all $i \neq j$, and the desired conclusion follows. $\square$

Remark 37. We have not invoked Compactness in Example 36, because it does not give us as much as we want. It is possible to invoke it by stating: *Every finite subset of* Γ *is satisfiable and therefore* Γ *is satisfiable*. But the conclusion that Γ is satisfiable only means that *there exists* an interpretation $(\mathcal{A}, \sigma)$ for Γ, not that *every* interpretation $(\mathcal{A}, \sigma)$ satisfies Γ. So, if we want to show that Γ is a formal specification of *all* infinite models, it does not help to invoke Compactness.

Can Γ in Example 36 distinguish between different infinite models? For example, can Γ distinguish between a model whose universe is $\mathbb{N}$ and another model whose universe is $\mathbb{R}$? No, it cannot. For our Γ here, all infinite cardinalities are the same. We consider this question again in later sections of these notes. $\square$

Exercise 38 (*Finiteness is* eL*-Ineffable*)**.** In contrast to Example 36, we have the following facts:

1. There does not exist a set Δ of wff's in eL such that, for every interpretation $(\mathcal{A}, \sigma)$, we have $\mathcal{A}, \sigma \models \Delta$ iff $\mathcal{A}$ is finite.

   Hint: Assume otherwise and invoke Compactness to get a contradiction. You may want to use Γ from Example 36.
2. Let $n \geqslant 1$, a fixed positive integer. Show there exists a set Δ_n of wff's in eL such that, for every interpretation $(\mathcal{A}, \sigma)$, we have $\mathcal{A}, \sigma \models \Delta_n$ iff the universe of $\mathcal{A}$ has n elements.

Give a precise argument for each of the two preceding facts. Thus, while finiteness in general is inexpressible in eL, finiteness of a fixed cardinality n is. $\square$

(MORE TO COME)

4 Zeroth-Order Logic (ZOL)

The *zeroth-order logic* (ZOL) is a fragment of *first-order logic* that mentions no quantifiers and no variables.[13] It is a restricted fragment, but which makes the transition to full first-order logic a little more gradual. The means to reduce Compactness for ZOL to Compactness for PL is what is known as *Herbrand theory*. In this section we need a limited version of *Herbrand theory*, the full version is used when we consider *first-order logic* with no restrictions.[14]

Given a first-order signature $\Sigma = \mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$, the set of wff's of ZOL over Σ is denoted $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$ when the symbol "$\approx$" for equality does not occur in wff's, and $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\}, \varnothing)$ when "$\approx$" is allowed to occur in wff's. Precise definitions of the syntax of ZOL is in Appendix A.4, the semantics of ZOL is in Appendix B.3, and a proof system for ZOL is in Appendix C.5.

Though far more limited than full *first-order logic*, the expressive power of *zeroth-order logic* is not trivial, as demonstrated by Examples 51 and 52 at the end of this section.[15]

4.1 Intermediate Herbrand Theory[16]

Let $\Sigma = \mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$ be a first-order signature, as specified in Appendix A.3. $\mathsf{Terms}(\Sigma, \varnothing)$ is the set of variable-free terms over Σ, also called *ground terms* over Σ. $\mathsf{Atoms}(\Sigma, \varnothing)$ is the set of variable-free atomic formulas over Σ, also called *ground atoms* over Σ, none mentioning the equality symbol "$\approx$".

In general, Σ is not empty, even though one or two of its three parts – $\mathcal{R}$, $\mathcal{F}$, and $\mathcal{C}$ – may be empty. If $\mathcal{C} = \varnothing$ we add a fresh constant symbol to it in order to be able to build a non-empty set of ground terms, *i.e.*, so that $\mathsf{Terms}(\Sigma, \varnothing) \neq \varnothing$. We denote a Σ-structure $\mathcal{A}$ by writing:

$$\mathcal{A} \stackrel{\text{def}}{=} \big(A,\, \mathcal{R}^{\mathcal{A}},\, \mathcal{F}^{\mathcal{A}},\, \mathcal{C}^{\mathcal{A}}\big)$$

where A is the universe of $\mathcal{A}$, always assumed not empty, and $\mathcal{R}^{\mathcal{A}}$, $\mathcal{F}^{\mathcal{A}}$, and $\mathcal{C}^{\mathcal{A}}$, are the interpretations of the symbols of Σ in $\mathcal{A}$. By definition, a set of wff's $\Gamma \subseteq \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$ is satisfiable iff it has a model; in symbols, iff there is a Σ-structure $\mathcal{A}$ such that $\mathcal{A} \models \Gamma$.

In general, an arbitrary Σ-structure $\mathcal{A}$ carries plenty of extra information unrelated to the satisfaction or non-satisfaction of Γ. A more economical notion is that of a *Herbrand Σ-structure* $\mathcal{H}$, which we specify as follows:

$$\mathcal{H} \stackrel{\text{def}}{=} \big(\mathsf{Terms}(\Sigma, \varnothing),\, \mathcal{R}^{\mathcal{H}},\, \mathcal{F},\, \mathcal{C}\big)$$

Note carefully how we have written the specification of $\mathcal{H}$:

- The universe of $\mathcal{H}$ is the set $\mathsf{Terms}(\Sigma, \varnothing)$ of ground terms.

[13] The phrase "zeroth-order logic" is not standard. If you search the Web for "zeroth-order logic", you will find that some authors have used the phrase to refer to *propositional logic*, but this is not our meaning here. I take *zeroth-order logic* in the sense defined by Terence Tao; see, for example, Tao's blog [9].

[14] Jacques Herbrand is a mathematician of the early twentieth century who laid out the foundation for this theory.

[15] And beyond the simple examples in these lecture notes, the powerful concept of the *diagram* of a Σ-structure $\mathcal{A} \stackrel{\text{def}}{=} (A, \ldots)$ in model theory consists of all variable-free atomic wff's and their negations satisfied by the expanded structure $\mathcal{A}' \stackrel{\text{def}}{=} (\mathcal{A}, a)_{a \in A}$ which adds new constant symbols to the signature Σ, one constant symbol for each element in the universe A. If Σ' is the signature of $\mathcal{A}'$, the diagram of $\mathcal{A}$ consists of all the wff's in $\mathsf{Atoms}(\Sigma' \cup \{\approx\}, \varnothing)$ and their negations, a subset of $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma' \cup \{\approx\}, \varnothing)$.

[16] I call it "intermediate" because it is not yet Herbrand theory in full generality. Understanding the "intermediate" case provides good intuition for how later steps are developed in Herbrand theory.

- The underlying functions and constants are the members of $\mathcal{F} \cup \mathcal{C}$, all left uninterpreted.[17]

The only part in $\mathcal{H}$ that needs to be specified further is $\mathcal{R}$. Thus, two Herbrand Σ-structures, $\mathcal{H}_1$ and $\mathcal{H}_2$, are distinguished only by the interpretation of the relations in $\mathcal{R}^{\mathcal{H}_1}$ and $\mathcal{R}^{\mathcal{H}_2}$.

Theorem 39 (Basic Herbrand Theorem)**.** *Let* $\varphi \in \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$ *and* $\Gamma \subseteq \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$*, both arbitrary. It then holds that:*

1. φ *is satisfiable* $\Leftrightarrow$ φ *has a Herbrand model.*
2. Γ *is satisfiable* $\Leftrightarrow$ Γ *has a Herbrand model.*

Proof. The implication "$\Leftarrow$", in both parts of the theorem, is immediate: If φ has a model, Herbrand or not, then φ is satisfiable, and likewise for Γ. The implication "$\Rightarrow$" is more delicate to prove.

Suppose φ is satisfiable, *i.e.*, there is a model $\mathcal{A} = \big(A,\, \mathcal{R}^{\mathcal{A}},\, \mathcal{F}^{\mathcal{A}},\, \mathcal{C}^{\mathcal{A}}\big)$ such that $\mathcal{A} \models \varphi$. Relative to this Σ-structure $\mathcal{A}$, we next define a Herbrand Σ-structure $\mathcal{H}$ such that $\mathcal{H} \models \varphi$. By definition, a Herbrand Σ-structure $\mathcal{H}$ is of the form:

$$\mathcal{H} \stackrel{\text{def}}{=} \big(\mathsf{Terms}(\Sigma, \varnothing),\, \mathcal{R}^{\mathcal{H}},\, \mathcal{F},\, \mathcal{C}\big)$$

where only the interpretation $\mathcal{R}^{\mathcal{H}}$ needs to be specified, which we do as follows. For every relation symbol $R \in \mathcal{R}$ of some arity $n \geqslant 0$ and all ground terms $t_1, \ldots, t_n \in \mathsf{Terms}(\Sigma, \varnothing)$, we set the truth value of the atom $R(t_1, \ldots, t_n) \in \mathsf{Atoms}(\Sigma, \varnothing)$ as follows:

$$R^{\mathcal{H}}(t_1, \ldots, t_n) \stackrel{\text{def}}{=} R^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}).$$

So far, we have not used any information about the given φ, except that it has a model $\mathcal{A}$. To reach the desired conclusion, we prove a stronger result, namely: *For all* $\psi \in \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$*, it holds that* $\mathcal{A} \models \psi \Leftrightarrow \mathcal{H} \models \psi$. Note that this assertion holds for all $\psi \in \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$, and not only for the given φ. We prove this assertion by induction on the number of connectives in $\{\neg, \wedge, \vee, \rightarrow\}$ occurring in ψ. Remaining details of Part 1, and all of Part 2, are left as an exercise. $\square$

Exercise 40. Complete the proof of Theorem 39: (1) Write the details of the induction to complete the proof of the implication "$\Rightarrow$" in Part 1 of the theorem. (2) Explain how the implication "$\Rightarrow$" in Part 1 implies the implication "$\Rightarrow$" in Part 2 of the theorem. $\square$

We extend our analysis to the case when the equality symbol "$\approx$" occurs in the syntax of wff's. By definition, the interpretation of a ground term $t \in \mathsf{Terms}(\Sigma, \varnothing)$ in a Herbrand Σ-structure is the term t itself. This poses a problem: Whereas two syntactically distinct ground terms t_1 and t_2 may be interpreted to the same element in the universe A of a Σ-structure $\mathcal{A}$, *i.e.*, so that $t_1^{\mathcal{A}} = t_2^{\mathcal{A}}$, they cannot be equated in the corresponding Herbrand Σ-structure.

So, how shall we construct a Herbrand Σ-structure $\mathcal{H}$ where satisfiability matches satisfiability in a given Σ-structure $\mathcal{A}$ with equality? The following suggests itself as a natural solution: In the $\mathcal{H}$ to be constructed, we add a new binary relation, denoted $\mathsf{eq}^{\mathcal{H}}$, which is a *congruence relation* on the universe $\mathsf{Terms}(\Sigma, \varnothing)$ whose congruence classes are each the set of all ground terms that are equated (*i.e.*, interpreted to the same element) in the given Σ-structure $\mathcal{A}$. To be precise, the augmented Herbrand structure is specified as:

$$\mathcal{H} \stackrel{\text{def}}{=} \big(\mathsf{Terms}(\Sigma, \varnothing), \mathsf{eq}^{\mathcal{H}},\, \mathcal{R}^{\mathcal{H}},\, \mathcal{F},\, \mathcal{C}\big)$$

and satisfies two conditions, the first of which is reproduced from the proof of Theorem 39:

[17] If you are familiar with notions of universal algebra, you will recognize that $\big(\mathsf{Terms}(\Sigma, \varnothing),\, =,\, \mathcal{F},\, \mathcal{C}\big)$ is what is called a *term algebra* or also an *absolutely free algebra*. In a term algebra, the signature is limited to $\mathcal{F} \cup \mathcal{C}$ where $\mathcal{C} \neq \varnothing$; there are no underlying relations other than equality "$=$"; and "$=$" always denotes syntactic equality between uninterpreted ground terms.

1. For all $R \in \mathcal{R}$ of arity $n \geqslant 0$ and $t_1, \ldots, t_n \in \mathsf{Terms}(\Sigma, \varnothing)$, $R^{\mathcal{H}}(t_1, \ldots, t_n) \stackrel{\text{def}}{=} R^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}})$;
2. For all $t_1, t_2 \in \mathsf{Terms}(\Sigma, \varnothing)$, $\quad \mathsf{eq}^{\mathcal{H}}(t_1, t_2) \Leftrightarrow t_1^{\mathcal{A}} = t_2^{\mathcal{A}}$.

Note that the signature of the constructed $\mathcal{H}$ is $\Sigma \cup \{\mathsf{eq}\}$, whereas the signature of the given $\mathcal{A}$ with equality is still Σ (or $\Sigma \cup \{\approx\}$, if we ignore our standing assumption that "$\approx$" is not part of a signature). Though our notation does not indicate it, purposely in order to avoid the clutter, the interpretation of eq and the interpretation of every $R \in \mathcal{R}$ in $\mathcal{H}$ depends on $\mathcal{A}$.[18]

To develop some intuition of how the congruence classes of $\mathsf{eq}^{\mathcal{H}}$ may behave, do Exercise 53 before proceeding further. Consider now a Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-structure, separate from any Σ-structure $\mathcal{A}$:

$$\mathcal{H} \stackrel{\text{def}}{=} \big(\mathsf{Terms}(\Sigma, \varnothing), \mathsf{eq}^{\mathcal{H}}, \mathcal{R}^{\mathcal{H}}, \mathcal{F}, \mathcal{C}\big)$$

We want the relation $\mathsf{eq}^{\mathcal{H}}$ to behave in a particular way, not as an arbitrary binary relation, but as a *congruence relation* on the universe $\mathsf{Terms}(\Sigma, \varnothing)$. Hence, for a Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-structure $\mathcal{H}$ to be *well-behaved*, we require that $\mathsf{eq}^{\mathcal{H}}$ satisfies the conditions of a congruence, namely:

1. *reflexivity*:
   $\mathsf{eq}^{\mathcal{H}}(t, t)$ for all $t \in \mathsf{Terms}(\Sigma, \varnothing)$,
2. *symmetry*:
   if $\mathsf{eq}^{\mathcal{H}}(t_1, t_2)$ then $\mathsf{eq}^{\mathcal{H}}(t_2, t_1)$ for all $t_1, t_2 \in \mathsf{Terms}(\Sigma, \varnothing)$,
3. *transitivity*:
   if $\mathsf{eq}^{\mathcal{H}}(t_1, t_2)$ and $\mathsf{eq}^{\mathcal{H}}(t_2, t_3)$ then $\mathsf{eq}^{\mathcal{H}}(t_1, t_3)$ for all $t_1, t_2, t_3 \in \mathsf{Terms}(\Sigma, \varnothing)$,
4. *compatible with* $\mathcal{F}$:
   if $\mathsf{eq}^{\mathcal{H}}(t_1, u_1), \ldots, \mathsf{eq}^{\mathcal{H}}(t_n, u_n)$ and $f \in \mathcal{F}$ has arity $n \geqslant 1$,
   then $\mathsf{eq}^{\mathcal{H}}\big(f(t_1, \ldots, t_n), f(u_1, \ldots, u_n)\big)$ for all $t_1, u_1, \ldots, t_n, u_n \in \mathsf{Terms}(\Sigma, \varnothing)$,
5. *compatible with* $\mathcal{R}$:
   if $\mathsf{eq}^{\mathcal{H}}(t_1, u_1), \ldots, \mathsf{eq}^{\mathcal{H}}(t_n, u_n)$ and $R \in \mathcal{R}$ has arity $n \geqslant 0$,
   then $R^{\mathcal{H}}(t_1, \ldots, t_n) \Leftrightarrow R^{\mathcal{H}}(u_1, \ldots, u_n)$ for all $t_1, u_1, \ldots, t_n, u_n \in \mathsf{Terms}(\Sigma, \varnothing)$.

Exercise 41. Let $\mathcal{A}$ be a Σ-structure and $\mathcal{H}$ a Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-structure. Show that if $\mathcal{H}$ is induced by $\mathcal{A}$, then $\mathcal{H}$ is well-behaved; *i.e.*, the relation $\mathsf{eq}^{\mathcal{H}}$ satisfies the preceding 5 conditions. $\square$

Definition 42 (*Enforcing* $\mathsf{eq}^{\mathcal{H}}$ *as a congruence relation*)**.** Instead of qualifying a Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-structure as being *well-behaved*, we can omit the qualifier and require instead that it satisfies the following set of ground atomic wff's $\Delta_{\mathsf{eq}} \subseteq \mathsf{Atoms}\big(\Sigma \cup \{\mathsf{eq}\}, \varnothing\big)$, which is the least set such that:

$$\begin{aligned}
\Delta_{\mathsf{eq}} \supseteq\ & \Big\{ \mathsf{eq}(t, t) \ \Big|\ t \in \mathsf{Terms}(\Sigma, \varnothing) \Big\} \cup \\
& \Big\{ \mathsf{eq}(t_1, t_2) \ \Big|\ \mathsf{eq}(t_2, t_1) \text{ and } t_i \in \mathsf{Terms}(\Sigma, \varnothing) \Big\} \cup \\
& \Big\{ \mathsf{eq}(t_1, t_3) \ \Big|\ \mathsf{eq}(t_1, t_2),\ \mathsf{eq}(t_2, t_3),\ \text{and } t_i \in \mathsf{Terms}(\Sigma, \varnothing) \Big\} \cup \\
& \Big\{ \mathsf{eq}\big(f(t_1, \ldots, t_n), f(u_1, \ldots, u_n)\big) \ \Big|\ \mathsf{eq}(t_1, u_1), \ldots, \mathsf{eq}(t_n, u_n),\ \text{and } t_i, u_i \in \mathsf{Terms}(\Sigma, \varnothing) \Big\} \cup \\
& \Big\{ R(t_1, \ldots, t_n) \ \Big|\ \mathsf{eq}(t_1, u_1), \ldots, \mathsf{eq}(t_n, u_n),\ R(u_1, \ldots, u_n),\ \text{and } t_i, u_i \in \mathsf{Terms}(\Sigma, \varnothing) \Big\}.
\end{aligned}$$

[18] A more precise notation would be therefore to write "$\mathcal{H}(\mathcal{A})$" instead of just "$\mathcal{H}$", making it clear that two distinct Σ-structures $\mathcal{A}_1$ and $\mathcal{A}_2$ induce two distinct Herbrand structures $\mathcal{H}(\mathcal{A}_1)$ and $\mathcal{H}(\mathcal{A}_2)$.

The five parts in the definition of Δ_{eq} correspond to the five conditions preceding Exercise 41. Because Δ_{eq} is the least such subset of ground atoms, if $\mathcal{H}$ satisfies no ground atoms other than Δ_{eq}, then $\mathsf{eq}^{\mathcal{H}}$ coincides with equality on the set of ground terms and each $\mathsf{eq}^{\mathcal{H}}$-congruence class will be a singleton set. In general, however, $\mathcal{H}$ satisfies additional ground atoms and some $\mathsf{eq}^{\mathcal{H}}$-congruence classes are not singleton sets. □

We define a translation $\boxed{\approx\,\mapsto \mathsf{eq}}$ from $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\}, \varnothing)$ to $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\mathsf{eq}\}, \varnothing)$ as follows:[19]

$$\boxed{\approx\,\mapsto \mathsf{eq}}(\varphi) \overset{\text{def}}{=} \begin{cases} \mathsf{eq}(t_1, t_2) & \text{if } \varphi = (t_1 \approx t_2), \\ \varphi & \text{if } \varphi \in \mathsf{Atoms}(\Sigma, \varnothing), \\ \neg\left(\boxed{\approx\,\mapsto \mathsf{eq}}(\psi)\right) & \text{if } \varphi = \neg\psi, \\ \left(\boxed{\approx\,\mapsto \mathsf{eq}}(\varphi_1)\right) \diamond \left(\boxed{\approx\,\mapsto \mathsf{eq}}(\varphi_2)\right) & \text{if } \varphi = \varphi_1 \diamond \varphi_2 \text{ and } \diamond \in \{\wedge, \vee, \to\}\ . \end{cases}$$

In words, all that $\boxed{\approx\,\mapsto \mathsf{eq}}$ does is to replace every atom $(t_1 \approx t_2)$ by $\mathsf{eq}(t_1, t_2)$.

We write $\boxed{\mathsf{eq} \mapsto\, \approx}$ for the inverse translation from $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\mathsf{eq}\}, \varnothing)$ to $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\}, \varnothing)$, which simply replaces every $\mathsf{eq}(t_1, t_2)$ by $(t_1 \approx t_2)$. We omit a formal definition of $\boxed{\mathsf{eq} \mapsto\, \approx}$. Both translations, $\boxed{\approx\,\mapsto \mathsf{eq}}$ and $\boxed{\mathsf{eq} \mapsto\, \approx}$ are needed to write the next theorem and its proof.

Theorem 43 (Intermediate Herbrand Theorem)**.** *Let φ be an arbitrary wff and Γ an arbitrary set of wff's in* $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\}, \varnothing)$. *Let* $\psi \overset{\text{def}}{=} \boxed{\approx\,\mapsto \mathsf{eq}}(\varphi)$ *and* $\Delta \overset{\text{def}}{=} \boxed{\approx\,\mapsto \mathsf{eq}}(\Gamma)$. *It then holds that:*

1. *φ is satisfiable $\Leftrightarrow$ ψ has a well-behaved Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-model.*
2. *Γ is satisfiable $\Leftrightarrow$ Δ has a well-behaved Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-model.*

Equivalently, using Δ_{eq} from Definition 42, it holds that:

1. *φ is satisfiable $\Leftrightarrow$ $\{\psi\} \cup \Delta_{\mathsf{eq}}$ has a Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-model.*
2. *Γ is satisfiable $\Leftrightarrow$ $\Delta \cup \Delta_{\mathsf{eq}}$ has a Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-model.*

Proof. For the "$\Rightarrow$" implication in both parts of the theorem, the steps here are very similar to those in the proof of Theorem 39, which we leave as an exercise.

In contrast to the proof of Theorem 39, the "$\Leftarrow$" implication here requires extra care. Consider the "$\Leftarrow$" implication in Part 1, the same issues applies to Part 2. Suppose that ψ has a well-behaved Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-model $\mathcal{H}$. To show φ is satisfiable, we want a model for it.

We recover φ from ψ by applying $\boxed{\mathsf{eq} \mapsto\, \approx}$ to ψ, *i.e.*, $\varphi = \boxed{\mathsf{eq} \mapsto\, \approx}(\psi)$, but for which $\mathcal{H}$ cannot be a model because the signature of φ is not $\mathcal{H}$'s signature: φ uses "$\approx$", which is not in $\mathcal{H}$'s signature. One case is simple, however, which occurs when every congruence class defined by the congruence $\mathsf{eq}^{\mathcal{H}}$ consists of a single element, in which case it suffices to replace $\mathsf{eq}^{\mathcal{H}}$ by equality "$=$" to make $\mathcal{H}$ a model of φ.

The general case is when some of the congruence classes of $\mathsf{eq}^{\mathcal{H}}$ are not singleton sets. In this case, we first define a homomorphic image (or quotient) of $\mathcal{H}$ modulo $\mathsf{eq}^{\mathcal{H}}$, which produces a $(\Sigma \cup \{\mathsf{eq}\})$-structure $\mathcal{H}'$ where every congruence class is a singleton set. It is then easy to argue that $\mathcal{H}'$ is a model of ψ. Finally, by replacing $\mathsf{eq}^{\mathcal{H}'}$ by equality "$=$" in $\mathcal{H}'$, we obtain a model $\mathcal{H}''$ of φ. □

[19] See footnote 11 for our convention of naming transformations of syntax.

Exercise 44. As much as you can, write the details of the proof of Theorem 43. For the the implication "$\Rightarrow$", use the proof of Theorem 39 and Exercise 40 as a guide for what you need to do. For the converse implication "$\Leftarrow$", you will need to brush up your knowledge of what a *homomorphic image*, or *quotient structure*, modulo a congruence relation is. $\square$

4.2 From Compactness in PL to Compactness in ZOL

A close examination of the Herbrand theory so far shows that the ground atoms in $\mathsf{Atoms}(\Sigma \cup \{\mathsf{eq}\}, \varnothing)$ are all that matters in the specification of a Herbrand model $\mathcal{H}$, more precisely, in the interpretation of the relations in $\{\mathsf{eq}\} \cup \mathcal{R}$. The interpretations of the symbols in $\mathcal{F} \cup \mathcal{C}$ are themselves and the same in all Herbrand models, and thus play no role in differentiating Herbrand models.

And since there are only propositional connectives, but no variables and no quantifiers in ground atoms, each ground atom plays the role of a propositional variable. This suggests the introduction of fresh variables, one for every ground atom; more precisely, we introduce a fresh set $\mathcal{Y}$ of propositional variables by:

$$\mathcal{Y} \;=\; \Big\{ Y_\alpha \;\Big|\; \alpha \in \mathsf{Atoms}\big(\Sigma \cup \{\mathsf{eq}\}, \varnothing\big) \Big\}.$$

Each member of $\mathcal{Y}$ is named by the upper-case letter "Y" subscripted with a ground atom α. (We use upper-case "Y" to keep these new variables separate from other variables in these notes.) For the rest of this section we consider only $\mathsf{Atoms}(\Sigma \cup \{\mathsf{eq}\}, \varnothing)$, which includes $\mathsf{Atoms}(\Sigma, \varnothing)$ as a proper subset.

We define a translation from ZOL to PL, named "$\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}$" suggestively:

$$\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}} \;:\; \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\mathsf{eq}\}, \varnothing) \;\to\; \mathsf{WFF}_{\mathsf{PL}}(\mathcal{Y})$$

such that for all $\varphi \in \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\mathsf{eq}\}, \varnothing)$:

$$\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\varphi) \;\stackrel{\text{def}}{=}\; \begin{cases} Y_\alpha & \text{if } \varphi = \alpha \in \mathsf{Atoms}(\Sigma \cup \{\mathsf{eq}\}, \varnothing), \\ \neg\,\Big(\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\psi)\Big) & \text{if } \varphi = \neg\psi, \\ \Big(\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\varphi_1)\Big) \;\diamond\; \Big(\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\varphi_2)\Big) & \text{if } \varphi = \varphi_1 \diamond \varphi_2 \text{ and } \diamond \in \{\wedge, \vee, \to\}\;. \end{cases}$$

Informally in words, all that $\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}$ does is to replace every atomic wff $\alpha \in \mathsf{Atoms}(\Sigma \cup \{\mathsf{eq}\}, \varnothing)$ by the propositional variable Y_α.

We write $\boxed{\mathsf{PL} \mapsto \mathsf{ZOL}}$ for the inverse transformation from $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{Y})$ to $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\mathsf{eq}\}, \varnothing)$ which simply replaces every propositional variable Y_α by the corresponding ground atom $\alpha \in \mathsf{Atoms}(\Sigma \cup \{\mathsf{eq}\}, \varnothing)$. We omit a formal definition of $\boxed{\mathsf{PL} \mapsto \mathsf{ZOL}}$.

As usual, we extend the definitions of $\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}$ and $\boxed{\mathsf{PL} \mapsto \mathsf{ZOL}}$ to sets of wff's, in the obvious way:

$$\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\Gamma) \;\stackrel{\text{def}}{=}\; \Big\{ \boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\varphi) \;\Big|\; \varphi \in \Gamma \Big\},$$

$$\boxed{\mathsf{PL} \mapsto \mathsf{ZOL}}(\Delta) \;\stackrel{\text{def}}{=}\; \Big\{ \boxed{\mathsf{PL} \mapsto \mathsf{ZOL}}(\psi) \;\Big|\; \psi \in \Delta \Big\}.$$

We are now ready to state a *transfer principle* from ZOL to PL.

Lemma 45 (Transfer Principle)**.** *Let Γ be an arbitrary set, possibly finite, of wff's in $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\}, \varnothing)$, and let $\Delta \stackrel{\text{def}}{=} \boxed{\approx \mapsto \mathsf{eq}}(\Gamma)$. It then holds that:*

1. Γ *is satisfiable (in the sense of* zeroth-order logic*)* $\Leftrightarrow$
   $\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\Delta \cup \Delta_{\mathsf{eq}})$ *is satisfiable (in the sense of* propositional logic*).*
2. Γ *is finitely satisfiable (in the sense of* zeroth-order logic*)* $\Leftrightarrow$
   $\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\Delta \cup \Delta_{\mathsf{eq}})$ *is finitely satisfiable (in the sense of* propositional logic*).*

Δ_{eq} *is the set of ground atoms from Definition 42 which enforce that a Herbrand* $(\Sigma \cup \{\mathsf{eq}\})$*-structure is well-behaved.*

Proof. It suffices to prove Part 1 since Γ is possibly a finite set. For the proof of Part 1, it suffices to show, by Theorem 43:

$\Delta \cup \Delta_{\mathsf{eq}}$ has a Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-model $\Leftrightarrow$
$\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\Delta \cup \Delta_{\mathsf{eq}})$ is satisfiable (in the sense of *propositional logic*).

Let $\mathcal{Y}'$, a subset of $\mathcal{Y}$, be the set of propositional variables occurring in $\boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\Delta \cup \Delta_{\mathsf{eq}})$. For the "$\Rightarrow$" implication, we assume there is a Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-model for $\Delta \cup \Delta_{\mathsf{eq}}$, and then derive from it a truth assignment $\sigma : \mathcal{Y} \to \{\mathit{false}, \mathit{true}\}$ such that $\sigma \models \boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\Delta \cup \Delta_{\mathsf{eq}})$. We omit the easy details.

For the "$\Leftarrow$" implication, from a truth assignment $\sigma : \mathcal{Y} \to \{\mathit{false}, \mathit{true}\}$ such that $\sigma \models \boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\Delta \cup \Delta_{\mathsf{eq}})$, we define a Herbrand $(\Sigma \cup \{\mathsf{eq}\})$-model $\mathcal{H}$ for $\Delta \cup \Delta_{\mathsf{eq}}$. All we need to specify is the interpretation of every symbol in $\mathcal{R} \cup \{\mathsf{eq}\}$ in $\mathcal{H}$. We specify the interpretation of $R \in \mathcal{R}$ by assigning a truth value to every member of the set:

$$\{\, R^{\mathcal{H}}(t_1, \ldots, t_n) \mid R \text{ has arity } n \text{ and } t_1, \ldots, t_n \in \mathsf{Terms}(\Sigma, \varnothing) \,\}.$$

An expression of the form $R(t_1, \ldots, t_n)$ is an atom, call it α, in the set $\mathsf{Atoms}(\Sigma \cup \{\mathsf{eq}\}, \varnothing)$, to which corresponds a variable $Y_\alpha \in \mathcal{Y}$. We now define:

$$R^{\mathcal{H}}(t_1, \ldots, t_n) \stackrel{\text{def}}{=} \begin{cases} \mathit{false} & \text{if } \sigma(Y_\alpha) = \mathit{false}, \\ \mathit{true} & \text{if } \sigma(Y_\alpha) = \mathit{true}. \end{cases}$$

Because $\sigma \models \boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\Delta)$, we next use structural induction (easy details omitted) on an arbitrary $\varphi \in \Delta$, to conclude that $\mathcal{H} \models \Delta$.

So far, we have defined a Herbrand Σ-model $\mathcal{H}$ for Δ. We need to expand $\mathcal{H}$ to $(\Sigma \cup \{\mathsf{eq}\})$-model for Δ_{eq}. We proceed in the same way as for $R \in \mathcal{R}$, by assigning a truth value to every member of the set:

$$\{\, \mathsf{eq}^{\mathcal{H}}(t_1, t_2) \mid t_1, t_2 \in \mathsf{Terms}(\Sigma, \varnothing) \,\},$$

guaranteeing the $\mathsf{eq}^{\mathcal{H}}$ is a congruence relation. We use the fact that $\sigma \models \boxed{\mathsf{ZOL} \mapsto \mathsf{PL}}(\Delta_{\mathsf{eq}})$, together with structural induction on wff's in Δ_{eq}, to conclude that $\mathcal{H} \models \Delta_{\mathsf{eq}}$. All remaining details omitted. □

Theorem 46 (Compactness for Zeroth-Order Logic, Version I)**.** *Let* $\Gamma \subseteq \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\}, \varnothing)$*, an arbitrary set of wff's. We then have:* Γ *is satisfiable* $\Leftrightarrow \Gamma$ *is finitely satisfiable.*

Proof. This follows from Theorem 2 and Lemma 45. □

Corollary 47 (Compactness for Zeroth-Order Logic, Version II)**.** *Let* $\Gamma \subseteq \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\}, \varnothing)$*, an arbitrary set of wff's, and* $\varphi \in \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\}, \varnothing)$*, an arbitrary wff. We then have:* $\Gamma \models \varphi \Leftrightarrow$ *there is a finite subset* $\Gamma_0 \subseteq \Gamma$ *such that* $\Gamma_0 \models \varphi$*.*

Proof. This follows from Corollary 7, Lemma 45, and Theorem 46. Some of the details reproduce details in the proof of Corollary 7, all left as an exercise. □

Exercise 48. As much as you can, supply the details in the proof of Corollary 47. □

4.3 From Compactness in ZOL to Completeness in ZOL

Lemma 49 is a weak form of the Completeness Theorem; it does not need Compactness for its proof.

Lemma 49. *Let* $\varphi_1, \ldots, \varphi_n, \psi \in \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\}, \varnothing)$*. If* $\varphi_1, \ldots, \varphi_n \models \psi$ *then* $\varphi_1, \ldots, \varphi_n \vdash \psi$*.*

Proof. **(MORE TO COME)** □

Theorem 50 (Completeness for Zeroth-Order Logic)**.** *Let* $\Gamma \subseteq \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\}, \varnothing)$*, an arbitrary set of wff's, and* $\psi \in \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\}, \varnothing)$*, an arbitrary wff. It holds that if* $\Gamma \models \psi$*, then* $\Gamma \vdash \psi$*.*

Proof. A straightforward consequence of Corollary 47 and Lemma 49. The details here are almost identical to the details in the proof of Theorem 13, which is Completeness for PL. □

4.4 Applications and Exercises

Example 51 (*Arithmetic*)**.** A zeroth-order language for arithmetic over the set $\mathbb{N}$ of natural numbers may include the equality relation $=$, the order relation $<$, the binary operations $+$ and $\times$, the successor operation S, and the constants 0 and 1 (or all of them $0, 1, 2, 3, \ldots$ with 2 being a shorthand for $S(1)$, 3 a shorthand for $S(S(1))$, $\ldots$). The structure under consideration is therefore $\mathcal{N} \stackrel{\text{def}}{=} (\mathbb{N}, =, <, +, \times, S, 0, 1)$. An example of zeroth-order wff's are the wff's expressing the Pythagorean property $i \times i + j \times j = k \times k$, which are satisfied in $\mathcal{N}$ for some triples but not all triples (i, j, k) of natural numbers. We ignore here the difference between the equation and the uninterpreted wff corresponding to it.

If we want to express other facts of airthmetic, we may expand the signature with other relations and operations. We may for example add a unary relation $\mathit{prime}(\,)$ which is *true* or *false* depending on whether its argument is a prime or not. We can now formally express that (i, j) is a *Twin-Prime Pair* by writing $\mathit{prime}(i) \wedge \mathit{prime}(j) \wedge (i + 2 = j)$ which says "there are two prime numbers whose difference is 2", or that (i, j) is a *Bertrand-Chebyshev Pair* by writing $\mathit{prime}(i) \wedge \mathit{prime}(j) \wedge (i + i > j)$ which says "there are two prime numbers whose difference is less than the smaller of the two".[20] □

Example 52 (*Groups*)**.** In algebra, a group $\mathcal{G}$ is said to be a 2-generated group if its universe can be generated by at most two of its elements, using the binary group operation. Such a group can be specified as:

$$\mathcal{G} \stackrel{\text{def}}{=} \big(G, \approx^{\mathcal{G}}, f^{\mathcal{G}}, i^{\mathcal{G}}, \mathbf{e}^{\mathcal{G}}, a^{\mathcal{G}}, b^{\mathcal{G}}\big),$$

where $f^{\mathcal{G}}$ is the binary group operation, $i^{\mathcal{G}}$ is the inverse operation, $\mathbf{e}^{\mathcal{G}}$ is the group identity, and $\{a^{\mathcal{G}}, b^{\mathcal{G}}\}$ are the two generators. The signature of $\mathcal{G}$ is therefore $\{f, i, \mathbf{e}, a, b\}$ where f is a binary function symbol, i is a unary function symbol, and $\{\mathbf{e}, a, b\}$ are three constant symbols.

[20] There are infinitely many *Pythagorean Triples*, a relatively easy fact which can be proved by hand. The related problem of the *Boolean Pythagorean Triples*, which asks whether $\mathbb{N}$ can be divided into two parts such that neither part contains a triple, is notoriously difficult and has – so far – required the use of automated theorem provers (to prove that it is impossible to so divide $\mathbb{N}$). There are infinitely many *Bertrand-Chebyshev Pairs*, a fact for which a proof has been automated using different interactive proof assistants, notably in Coq and Isabelle. As for *Twin-Prime Pairs*, it is still not known whether there are infinitely many of them!

$\otimes$	**e**	1	2	3
e	**e**	1	2	3
1	1	**e**	3	2
2	2	3	**e**	1
3	3	2	1	**e**

$(\,)^{-1}$	
e	**e**
1	1
2	2
3	3

Figure 8: Group $\mathcal{K}$ in Example 52. I omit the superscript "$\mathcal{K}$" on **e** for clarity, $\mathcal{K}$'s group identity.

A particular 2-generated group is the so-called Klein group $\mathcal{K}$ whose universe K has 4 elements, $K = \{\mathbf{e}^{\mathcal{K}}, 1, 2, 3\}$. $\mathcal{K}$ can be specified as $\big(K, =, \otimes, (\,)^{-1}, \mathbf{e}^{\mathcal{K}}, 1, 2\big)$. Because K is finite, its two operations $\{\otimes, (\,)^{-1}\}$ can be conveniently specified by the tables in Figure 8. Examples of *zeroth-order* wff's that are satisfied by the Klein group are the following three atomic wff's:

$$f(a,a) \approx \mathbf{e}, \quad f(b,b) \approx \mathbf{e}, \quad f\big(f(a,b), f(a,b)\big) \approx \mathbf{e},$$

or as interpreted equations and writing $\otimes = f^{\mathcal{K}}$ in infix position:

$$1 \otimes 1 \;=\; \mathbf{e}^{\mathcal{K}}, \qquad 2 \otimes 2 \;=\; \mathbf{e}^{\mathcal{K}}, \qquad (1 \otimes 2) \otimes (1 \otimes 2) \;=\; \mathbf{e}^{\mathcal{K}},$$

which turn out to fully specify the Klein group; more precisely, those three atomic wff's turn out to imply every other *zeroth-order* wff satisfied by $\mathcal{K}$ (not shown here).

Another particular and more common 2-generated group is the set $\mathbb{Z}$ of all integers under addition, which can be specified as $\mathcal{Z} \stackrel{\text{def}}{=} (\mathbb{Z}, =, +, -, 0, 1, -1)$, where binary addition $+$, unary negation $-$, and constants 0, 1, and -1, are the respective interpretations of the symbols f, i, **e**, a, and b. □

Exercise 53 (*Groups*)**.** There are two parts in this exercise.

1. Consider the Klein group $\mathcal{K}$ in Example 52. Its universe K has 4 elements $\{\mathbf{e}^{\mathcal{K}}, 1, 2, 3\}$, whereas the universe $\mathsf{Terms}(\Sigma, \varnothing)$ of ground terms is infinite. The latter is generated from the constants in $\{\mathbf{e}, a, b\}$ by applying functions f and i repeatedly. The induced congruence $\mathsf{eq}^{\mathcal{H}}$ on $\mathsf{Terms}(\Sigma, \varnothing)$ has therefore 4 congruence classes, one for each element in $\{\mathbf{e}^{\mathcal{K}}, 1, 2, 3\}$. Note there is no constant symbol in the signature which corresponds to the element "3"$\in K$. Compute a few ground terms (at least three, say) in each of the congruence classes.
2. Consider the additive group $\mathcal{Z}$ of all integers in Example 52. Its universe $\mathbb{Z}$ is infinite, and so is the universe of uninterpreted ground terms $\mathsf{Terms}(\Sigma, \varnothing)$ generated from constant symbols $\{\mathbf{e}, a, b\}$ by applying function symbols f and i. Relative to $\mathcal{Z}$, the induced congruence $\mathsf{eq}^{\mathcal{H}}$ on $\mathsf{Terms}(\Sigma, \varnothing)$ has infinitely many congruence classes, one class for each integer. Select a few integers (say, $\{0, 1, 2, 3\}$) and compute a few ground terms (at least three) in each of the congruence classes corresponding to the selected integers. □

5 Equational Logic and Quasi-Equational Logic (EL and QEL)

Section 4 on *zeroth-order logic* covers enough background material for a relatively smooth transition to Section 6 on *first-order logic*, which can be read without referring back to the present section. However, besides making the transition a little more gradual, a good reason for including this section is that computer scientists working on *automated theorem proving* and *interactive proof assistants* have been great contributors to *equational logic* and *quasi-equational logic*.

(MORE TO COME)

6 First-Order Logic (FOL)

For our plan to reduce Compactness for FOL to Compactness for PL, we develop *Herbrand theory* further – beyond our presentation in Section 4.1, which you now need to review before you go on to the next section. Throughout, we assume the signature $\Sigma = \mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$ is arbitrary, with the proviso that at least one of its three parts is not empty; if the set of constant symbols $\mathcal{C}$ is empty, we add a fresh constant symbol to it in order to be able to build a non-empty set of ground terms, *i.e.*, so that $\mathsf{Terms}(\Sigma, \varnothing) \neq \varnothing$.

6.1 Herbrand Theory

We take advantage of the transformations $\boxed{\mathsf{prenex}}$ and $\boxed{\mathsf{skolem}}$, both defined in Appendix E. Given an arbitrary first-order wff φ, we denote the application of those two transformations to φ in sequence by defining:

$$\boxed{\mathsf{sko,pre}}(\varphi) \overset{\text{def}}{=} \boxed{\mathsf{skolem}}\big(\boxed{\mathsf{prenex}}(\varphi)\big).$$

We present Herbrand's theorem (Theorem 59) gradually. We start with a lemma which proves the theorem for a single first-order sentence φ with the restriction that it does not contain "$\approx$".

Lemma 54. *Let φ be a first-order sentence which does not contain any subformula of the form $(t_1 \approx t_2)$. Then φ is satisfiable iff $\boxed{\mathsf{sko,pre}}(\varphi)$ has a Herbrand model.*

Proof. Let $\psi \overset{\text{def}}{=} \boxed{\mathsf{sko,pre}}(\varphi)$. If ψ has a model, Herbrand or not, then ψ is satisfiable. By Lemma 94, if ψ is satisfiable, then φ is satisfiable. The converse is more delicate to prove.

Suppose φ is satisfiable. By Lemma 94 again, ψ is satisfiable. Let $\Sigma = \mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$ be the signature of ψ, which in general expands the signature of φ. Hence, there is a structure $\mathcal{M}$ with signature Σ satisfying ψ, *i.e.*, $\mathcal{M} \models \psi$. We need to show there is a Herbrand structure satisfying ψ, *i.e.*, ψ has a Herbrand model. We proceed by first specifying a Herbrand structure $\mathcal{H}$ with signature Σ, and then by showing that $\mathcal{H}$ satisfies ψ. By definition, the signature Σ of $\mathcal{H}$ is is also the signature of $\mathcal{M}$. By definition again, the universe of $\mathcal{H}$ and the interpretations of every $f \in \mathcal{F}$ and every $c \in \mathcal{C}$ are already fixed, namely:

- the universe of $\mathcal{H}$ is $\mathsf{Terms}(\Sigma, \varnothing)$, which is the set of ground terms over Σ,
- $f^{\mathcal{H}}(t_1, \ldots, t_n) \overset{\text{def}}{=} f(t_1, \ldots, t_n)$ for every n-ary $f \in \mathcal{F}$ and $t_1, \ldots, t_n \in \mathsf{Terms}(\Sigma, \varnothing)$,
- $c^{\mathcal{H}} \overset{\text{def}}{=} c$ for every $c \in \mathcal{C}$.

Only the interpretation of the relation symbols in $\mathcal{R}$ need to be specified, which we set as follows:

- $(t_1, \ldots, t_n) \in R^{\mathcal{H}}$ iff $(t_1^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}}) \in R^{\mathcal{M}}$
  for every n-ary $R \in \mathcal{R}$ and $t_1, \ldots, t_n \in \mathsf{Terms}(\Sigma, \varnothing)$.

To conclude the proof, we prove a stronger assertion, namely: *For every sentence α in Skolem form over the signature Σ which does not mention the symbol "$\approx$", it holds that if $\mathcal{M} \models \alpha$ then $\mathcal{H} \models \alpha$*, which we prove by induction on the number $k \geqslant 0$ of universal quantifiers in α:

1. *Basis step*: $k = 0$, in which case α has no quantifiers, *i.e.*, α is a propositional combination of elements in $\mathsf{Atoms}(\Sigma, \varnothing)$, which is the set of ground atoms. For this basis step, we proceed by induction on the number of propositional connectives in $\{\neg, \wedge, \vee, \rightarrow\}$ occurring in α. Remaining details of this induction are straightforward and left to you.

2. *Induction hypothesis*: The assertion holds for every sentence α in Skolem form with k universal quantifiers, for some $k \geqslant 0$.
3. *Induction step*: Let $\beta \stackrel{\text{def}}{=} \forall x\, \alpha(x)$ be an arbitrary Skolem form where $\alpha(x)$ has one free variable x and $k \geqslant 0$ universal quantifiers, and β has $k+1$ universal quantifiers.

We prove the *induction step* by a sequence of implications. Let U be the universe of $\mathcal{M}$. We write "$[x \mapsto u]$" to denote the part of a valuation that maps the free variable x to the element $u \in U$:

$$\begin{array}{ll} & \mathcal{M} \models \forall x\, \alpha(x) \\ \Rightarrow & \text{for all } u \in U \text{, it holds that } \mathcal{M}, [x \mapsto u] \models \alpha \\ \Rightarrow & \text{for all } u \in U \text{ of the form } u = t^{\mathcal{M}} \text{ where } t \in \mathsf{Terms}(\Sigma, \varnothing) \text{, it holds that } \mathcal{M}, [x \mapsto u] \models \alpha \\ \Rightarrow & \text{for all } t \in \mathsf{Terms}(\Sigma, \varnothing) \text{, it holds that } \mathcal{M}, [x \mapsto t^{\mathcal{M}}] \models \alpha \\ \Rightarrow & \text{for all } t \in \mathsf{Terms}(\Sigma, \varnothing) \text{, it holds that } \mathcal{M} \models \alpha[x := t] \quad (\alpha[x := t] \text{ is a sentence}) \\ \Rightarrow & \text{for all } t \in \mathsf{Terms}(\Sigma, \varnothing) \text{, it holds that } \mathcal{H} \models \alpha[x := t] \quad (\text{by the \textit{induction hypothesis}}) \\ \Rightarrow & \text{for all } t \in \mathsf{Terms}(\Sigma, \varnothing) \text{, it holds that } \mathcal{H}, [x \mapsto t^{\mathcal{H}}] \models \alpha \\ \Rightarrow & \text{for all } t \in \mathsf{Terms}(\Sigma, \varnothing) \text{, it holds that } \mathcal{H}, [x \mapsto t] \models \alpha \quad (\mathcal{H} \text{ is a Herbrand structure}) \\ \Rightarrow & \mathcal{H} \models \forall x\, \alpha \end{array}$$

This completes the induction and the proof of the lemma. □

Exercise 55. Two parts, for a better understanding of the preceding proof:

1. What goes wrong in the proof of Lemma 54 if φ (and therefore ψ too) contains free variables?
2. And what goes wrong if ψ is not in Skolem form?

Hint for 1: As a warm-up, try Exercise 95 first, which may be a little easier.

Hint for 2: Consider the sentence $\varphi \stackrel{\text{def}}{=} R(a) \wedge \exists x \neg R(x)$ which is not in Skolem form, where R is a unary relation symbol and a is a constant symbol. Show there is a structure $\mathcal{M}$ satisfying φ but that $\mathcal{M}$ cannot be a Herbrand structure. □

The following theorem is more general than Lemma 54 but still restricted to first-order sentences without "$\approx$".

Theorem 56 (Herbrand)**.** *Let Γ be a set of first-order sentences, none containing a subformula of the form $(t_1 \approx t_2)$, and $\Gamma' \stackrel{\text{def}}{=} \left\{ \boxed{\mathsf{sko,pre}}(\varphi) \;\middle|\; \varphi \in \Gamma \right\}$. Then Γ is satisfiable iff Γ' has a Herbrand model.*

Proof Sketch. This is a simple variation on the proof of Lemma 54. We should be careful in making the Skolem functions distinct for each sentence $\varphi \in \Gamma$: Specifically, every time we introduce a Skolem function symbol for φ, we have to make it distinct from all Skolem function symbols generated before, whether for φ or for all other sentences in Γ. Hence, if Γ is infinite, so is the set of generated Skolem functions infinite. We omit all the straightforward details. □

Theorem 59 below is a stronger version of Theorem 56: In Theorem 59, first-order sentences are unrestricted and may contain subformulas of the form $(t_1 \approx t_2)$. Before we do this, we need to take a closer look at the presence of the equality relation in Herbrand structures.

Definition 57 (*Enforcing* $\mathsf{eq}^{\mathcal{H}}$ *as a congruence relation*)**.** We adjust the set Δ_{eq} in Definition 42, which enforces that a Herbrand structure $\mathcal{H}$ be *well-behaved*, *i.e.*, that $\mathsf{eq}^{\mathcal{H}}$ is a congruence relation. We now write $\Delta_{\mathsf{eq}}(\Sigma)$ to make explicit the signature over which it is written. $\Delta_{\mathsf{eq}}(\Sigma)$ is the following set of axioms (closed universal wff's) over the signature $\Sigma \cup \{\mathsf{eq}\}$:

1. $\forall x.\, \mathsf{eq}(x, x)$ (*reflexivity*)
2. $\forall x\, \forall y.\, \mathsf{eq}(x, y) \to \mathsf{eq}(y, x)$ (*symmetry*)
3. $\forall x\, \forall y\, \forall z.\, \mathsf{eq}(x, y) \wedge \mathsf{eq}(y, z) \to \mathsf{eq}(x, z)$ (*transitivity*)
4. $\forall x_1 \cdots x_n\, \forall y_1 \cdots y_n.\, \mathsf{eq}(x_1, y_1) \wedge \cdots \wedge \mathsf{eq}(x_n, y_n) \to \mathsf{eq}(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n))$
   (*congruence*, one such axiom for every function symbol $f \in \mathcal{F}$ of arity $n \geqslant 1$)
5. $\forall x_1 \cdots x_n\, \forall y_1 \cdots y_n.\, \mathsf{eq}(x_1, y_1) \wedge \cdots \wedge \mathsf{eq}(x_n, y_n) \to (R(x_1, \ldots, x_n) \to R(y_1, \ldots, y_n))$
   (*congruence*, one such axiom for every relation symbol $R \in \mathcal{R}$ of arity $n \geqslant 1$)

Note the difference with the earlier Definition 42, where quantifiers are not available and cannot be used. The five parts here corresponds to the five parts in the earlier definition. The first three axioms make eq an *equivalence relation*, and the last two turn this equivalence into a *congruence relation*. □

All the axioms in $\Delta_{\mathsf{eq}}(\Sigma)$ are already universal first-order sentences in prenex form and, therefore, do not need to be Skolemized. Hence, $\Delta_{\mathsf{eq}}(\Sigma) = \boxed{\text{sko,pre}}(\Delta_{\mathsf{eq}}(\Sigma))$.

Let $\mathcal{M}$ be a structure for the signature Σ whose universe is M. If we take the interpretation of eq in $\mathcal{M}$ to be the equality relation on the universe M, *i.e.* if we interpret $\mathsf{eq}^{\mathcal{M}}$ as just "=", then it is easy to check that $\mathcal{M} \models \Delta_{\mathsf{eq}}(\Sigma)$. (We write eq in prefix position, whereas $=$ is used in infix position; this is a minor adjustment in the syntax which causes no problem.) However, it is important to note there are other models $\mathcal{M}'$ of $\Delta_{\mathsf{eq}}(\Sigma)$, with signature $\Sigma \cup \{\mathsf{eq}\}$ such that $\mathsf{eq}^{\mathcal{M}'}$ is not as restrictive as the equality relation $=$.

Exercise 58. Let Σ be a signature and let $\mathcal{M} \stackrel{\text{def}}{=} (M, \ldots)$ be a Σ-structure with universe M. There are two parts in this exercise:

1. Starting from $\mathcal{M}$, construct a new structure $\mathcal{M}'$ for the expanded signature $\Sigma \cup \{\mathsf{eq}\}$ such that $\mathcal{M}' \models \Delta_{\mathsf{eq}}(\Sigma)$ and the interpretation of eq in $\mathcal{M}'$ does not coincide with the equality relation $=$.
2. Characterize the equality relation $=$ in contrast to the relation $\mathsf{eq}^{\mathcal{M}'}$ in any structure $\mathcal{M}'$ for the signature $\Sigma \cup \{\mathsf{eq}\}$. Is one "smaller" than the other?

Hint for 1: Pick an arbitrary element $m \in M$. Define the structure $\mathcal{M}' \stackrel{\text{def}}{=} (M \cup \{m'\}, \ldots)$ for the signature $\Sigma \cup \{\mathsf{eq}\}$ where m' is a fresh element such that $\mathcal{M}'$ acts on m' exactly like $\mathcal{M}$ on m. The elements m and m' cannot be distinguished by the relation $\mathsf{eq}^{\mathcal{M}'}$, whereas $m \neq m'$ and thus the two elements are distinguishable by the equality relation $=$.

Hint for 2: For all closed terms t_1 and t_2 over the signature Σ, if $\mathcal{M}' \models (t_1 \approx t_2)$ then $\mathcal{M}' \models \mathsf{eq}(t_1, t_2)$, while the converse may or may not hold. □

We need the transformation $\boxed{\approx \mapsto \mathsf{eq}}$, defined in Subsection 4.1, which replaces every atom $(t_1 \approx t_2)$ for some terms t_1 and t_2 by the atom $\mathsf{eq}(t_1, t_2)$. Note that t_1 and t_2 may now contain variables.

Theorem 59 (Herbrand Theorem)**.** *Let φ be a wff, and Γ a set of wff's, in $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma \cup \{\approx\}, X)$. Define:*

$$\psi \stackrel{\text{def}}{=} \boxed{\approx \mapsto \mathsf{eq}}(\boxed{\text{sko,pre}}(\varphi)),$$
$$\Delta \stackrel{\text{def}}{=} \boxed{\approx \mapsto \mathsf{eq}}(\boxed{\text{sko,pre}}(\Gamma)).$$

Let $\Sigma' \supseteq \Sigma$ be the signature of $\boxed{\text{sko,pre}}(\varphi)$ and $\boxed{\text{sko,pre}}(\Gamma)$, where $\Sigma' - \Sigma$ is the set of Skolem functions introduced in the Skolemization of $\boxed{\text{prenex}}(\varphi)$ and $\boxed{\text{prenex}}(\Gamma)$. The signature of ψ and Δ is therefore $\Sigma' \cup \{\mathsf{eq}\}$. It then holds that:

1. *φ is satisfiable $\Leftrightarrow$ $\psi \cup \Delta_{\mathsf{eq}}(\Sigma')$ has a Herbrand $(\Sigma' \cup \{\mathsf{eq}\})$-model.*
2. *Γ is satisfiable $\Leftrightarrow$ $\Delta \cup \Delta_{\mathsf{eq}}(\Sigma')$ has a Herbrand $(\Sigma' \cup \{\mathsf{eq}\})$-model.*

Proof. By Proposition 94, we can assume that φ is in Skolem form and Γ is a set of wff's all in Skolem form, in which case $\varphi = \boxed{\mathsf{sko,pre}}(\varphi)$ and $\Gamma = \boxed{\mathsf{sko,pre}}(\Gamma)$. With this assumption, we have $\Sigma' = \Sigma$. All the wff's under consideration are *universal* wff's. The wff's in $\Delta_{\mathsf{eq}}(\Sigma) = \Delta_{\mathsf{eq}}(\Sigma')$ are already in Skolem form.

The rest of the proof is a straightforward and simple adjustment to the proofs of Lemma 54 and Theorem 56. We first consider the case of a single wff φ as in Lemma 54, then generalize to an arbitrary set of wff's Γ as in Theorem 56.

The adaptation of Lemma 54 for the present proof is the only part that needs some non-trivial attention. Here, $\psi = \boxed{\approx \mapsto \mathsf{eq}}(\varphi)$ The non-trivial part is about constructing a Herbrand model $\mathcal{H}$ for $\psi \cup \Delta_{\mathsf{eq}}(\Sigma)$ from a model $\mathcal{M}$ for φ. The universe of $\mathcal{H}$ and the interpretation of the function symbols in $\mathcal{F}$ and constant symbols in $\mathcal{C}$ is the same as in the proof of Lemma 54. For the interpretation of the relation symbols in $\mathcal{R}$ which now includes eq, we define:

- $(t_1, \ldots, t_n) \in R^{\mathcal{H}}$ iff $(t_1^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}}) \in R^{\mathcal{M}}$
  for every n-ary $R \in \mathcal{R}$ and $t_1, \ldots, t_n \in \mathsf{Terms}(\Sigma, \varnothing)$,
- $(t_1, t_2) \in \mathsf{eq}^{\mathcal{H}}$ iff $t_1^{\mathcal{M}} = t_2^{\mathcal{M}}$
  for every $t_1, t_2 \in \mathsf{Terms}(\Sigma, \varnothing)$.

The first of these two bullet points is identical to the corresponding bullet point in the proof of Lemma 54; the second bullet point is new. The rest of the proof proceeds as the proof of Lemma 54, and for the case of a set of wff's Γ, as the proof of Theorem 56. This establishes the "$\Rightarrow$" implications in Part 1 and Part 2 in the theorem statement. We leave the "$\Leftarrow$" implications as a straightforward exercise. $\square$

Exercise 60. Consider φ, ψ, Γ, and Δ as defined in Theorem 59, all over a signature Σ. Prove:

1. $\psi \cup \Delta_{\mathsf{eq}}(\Sigma')$ has a Herbrand $(\Sigma' \cup \{\mathsf{eq}\})$-model $\Rightarrow$ φ is satisfiable.
2. $\Delta \cup \Delta_{\mathsf{eq}}(\Sigma')$ has a Herbrand $(\Sigma' \cup \{\mathsf{eq}\})$-model $\Rightarrow$ Γ is satisfiable.

As in the proof of Theorem 59, you can assume that φ is in Skolem form and Γ is a set of wff's all in Skolem form, so that also $\Sigma' = \Sigma$.

Hint: The only question here is how to recover a model for φ, which is a Σ-structure with "$=$" among its underlying relations, from a Herbrand model for $\psi \cup \Delta_{\mathsf{eq}}(\Sigma)$, which is a $(\Sigma \cup \{\mathsf{eq}\})$-structure without "$=$" among its underlying relations. See the proof of Theorem 43 for a similar situation. $\square$

6.2 From Compactness in PL to Compactness in FOL

Let φ be a first-order sentence in Skolem form, $\varphi \stackrel{\text{def}}{=} \forall x_1 \cdots \forall x_n.\varphi_0$ over signature Σ, where φ_0 is the quantifier-free matrix and $\mathsf{FV}(\varphi_0) \subseteq \{x_1, \ldots, x_n\}$. The *Herbrand expansion*, also called *ground expansion*, of φ is:

$$\mathsf{H\_Expansion}(\varphi) \stackrel{\text{def}}{=} \Big\{ \varphi_0[x_1 := t_1] \cdots [x_n := t_n] \;\Big|\; t_1, \ldots, t_n \in \mathsf{Terms}(\Sigma, \varnothing) \Big\}.$$

In words, the set $\mathsf{H\_Expansion}(\varphi)$ is obtained by deleting all universal quantifiers and replacing all variables by atomic terms in all possible ways. While φ is one sentence, $\mathsf{H\_Expansion}(\varphi)$ is a set of (quantifier-free) sentences, which is infinite if $\mathsf{Terms}(\Sigma, \varnothing)$ is infinite. If Γ is a set of first-order sentences in Skolem form, then:

$$\mathsf{H\_Expansion}(\Gamma) \stackrel{\text{def}}{=} \bigcup \left\{ \mathsf{H\_Expansion}(\varphi) \;\middle|\; \varphi \in \Gamma \right\}.$$

The next lemma pursues the analysis of Theorem 59.

Lemma 61. *Let φ be a sentence (closed wff) in $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma \cup \{\approx\}, X)$ and let:*

$$\psi \stackrel{\text{def}}{=} \boxed{\approx \mapsto \mathsf{eq}}(\boxed{\mathsf{sko,pre}}(\varphi)).$$

The signature of $\boxed{\mathsf{sko,pre}}(\varphi)$ is some $\Sigma' \supseteq \Sigma$, where $\Sigma' - \Sigma$ is the set of Skolem functions introduced in the Skolemization of $\boxed{\mathsf{prenex}}(\varphi)$, and the signature of ψ is $\Sigma' \cup \{\mathsf{eq}\}$. Then φ is satisfiable iff the Herbrand expansion $\mathsf{H\_Expansion}\big(\{\psi\} \cup \Delta_{\mathsf{eq}}(\Sigma')\big)$ is satisfiable.

Proof. Straightforward consequence of Theorem 59, according to which: φ is satisfiable iff $\{\psi\} \cup \Delta_{\mathsf{eq}}(\Sigma')$ has a Herbrand model. The universe of the Herbrand structure $\mathcal{H}$ is $\mathsf{Terms}(\Sigma', \varnothing)$. Deletion of the universal quantifiers corresponds to replacing the variables in $\{\psi\} \cup \Delta_{\mathsf{eq}}(\Sigma')$ by elements of the universe $\mathsf{Terms}(\Sigma', \varnothing)$ in all possible ways. All details omitted. □

Exercise 62. Supply the missing details in the proof of Lemma 61. □

Let $\Delta \stackrel{\text{def}}{=} \mathsf{H\_Expansion}\big(\{\psi\} \cup \Delta_{\mathsf{eq}}(\Sigma')\big)$, the set of quantifier-free sentences over the signature $\Sigma' \cup \{\mathsf{eq}\}$ in the conclusion of Lemma 61. Every wff in Δ is a propositional combination of wff's in $\mathsf{Atoms}\big(\Sigma' \cup \{\mathsf{eq}\}, \varnothing\big)$. Proceeding as in Subsection 4.2, we introduce a set $\mathcal{Y}$ of propositional variables by:

$$\mathcal{Y} = \left\{ Y_\alpha \;\middle|\; \alpha \in \mathsf{Atoms}\big(\Sigma' \cup \{\mathsf{eq}\}, \varnothing\big) \right\}.$$

Each member of $\mathcal{Y}$ is named by the upper-case letter "Y" subscripted with a ground atom α. We can now translate the set Δ of first-order wff's into a set of propositional wff's according to the transformation:

$$\boxed{\mathsf{FOL} \mapsto \mathsf{PL}} \;:\; \mathsf{WFF}_{\mathsf{FOL}}(\Sigma' \cup \{\mathsf{eq}\}, \varnothing) \;\to\; \mathsf{WFF}_{\mathsf{PL}}(\mathcal{Y})$$

such that for every $\varphi \in \mathsf{WFF}_{\mathsf{FOL}}(\Sigma' \cup \{\mathsf{eq}\}, \varnothing)$:

$$\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\varphi) \stackrel{\text{def}}{=} \begin{cases} Y_\alpha & \text{if } \varphi = \alpha \in \mathsf{Atoms}(\Sigma' \cup \{\mathsf{eq}\}, \varnothing), \\ \neg\left(\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\psi)\right) & \text{if } \varphi = \neg\psi, \\ \left(\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\varphi_1)\right) \diamond \left(\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\varphi_2)\right) & \text{if } \varphi = \varphi_1 \diamond \varphi_2 \text{ and } \diamond \in \{\wedge, \vee, \to\}\ . \end{cases}$$

The next lemma is a continuation of Lemma 61.

Lemma 63. *Let φ be a sentence (closed wff) in $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma \cup \{\approx\}, X)$ and let:*

$$\Delta \stackrel{\text{def}}{=} \mathsf{H\_Expansion}\left(\left\{\boxed{\approx \mapsto \mathsf{eq}}(\boxed{\mathsf{sko,pre}}(\varphi))\right\} \cup \Delta_{\mathsf{eq}}(\Sigma')\right)$$

The signature of $\boxed{\mathsf{sko,pre}}(\varphi)$ is some $\Sigma' \supseteq \Sigma$, with $\Sigma' - \Sigma$ being the set of Skolem functions introduced in the Skolemization of $\boxed{\mathsf{prenex}}(\varphi)$, and the signature of Δ is $\Sigma' \cup \{\mathsf{eq}\}$. Then φ is satisfiable (in the sense of FOL) iff $\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\Delta)$ is satisfiable (in the sense of PL).

Proof. By Lemma 61, φ is satisfiable iff Δ' is satisfiable. It suffices therefore to show that: Δ' is satisfiable (in the sense of first-order logic) iff $\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\Delta')$ is satisfiable (in the sense of propositional logic). Keep in mind that Δ' is a set of quantifier-free sentences.

Let $\{\alpha_1, \alpha_2, \ldots\} = \mathsf{Atoms}(\Sigma' \cup \{\mathsf{eq}\}, \varnothing)$ the countable set, finite or infinite, of ground atoms occuring in Δ', and $\{Y_{\alpha_1}, Y_{\alpha_2}, \ldots\}$ the corresponding set of propositional variables occurring in $\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\Delta')$. For the left-to-right implication, assume there is a first-order structure $\mathcal{M}$ such that $\mathcal{M} \models \Delta'$, and derive a truth-value assignment σ from $\mathcal{M}$ such that $\sigma \models \boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\Delta')$. For the right-to-left implication, assume there is a truth-value assignment σ such that $\sigma \models \boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\Delta')$, and derive a first-order structure $\mathcal{M}$ from σ such that $\mathcal{M} \models \Delta'$. All straightforward details omitted. $\square$

Exercise 64. Supply the missing details in the proof of Lemma 63. $\square$

We are now ready to state our *transfer principle* from FOL to PL.

Lemma 65 (Transfer Principle)**.** *Let Γ be a set, possibly infinite, of sentences in $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma \cup \{\approx\}, X)$, and let:*

$$\Delta \stackrel{\text{def}}{=} \mathsf{H\_Expansion}\left(\left\{\boxed{\approx \mapsto \mathsf{eq}}(\boxed{\mathsf{sko,pre}}(\Gamma))\right\} \cup \Delta_{\mathsf{eq}}(\Sigma')\right)$$

where $\Sigma' \supseteq \Sigma$ is the signature of $\boxed{\mathsf{sko,pre}}(\Gamma)$, with $\Sigma' - \Sigma$ being the set of Skolem functions introduced in the Skolemization of $\boxed{\mathsf{prenex}}(\Gamma)$ to obtain $\boxed{\mathsf{sko,pre}}(\Gamma)$. It then holds that:

1. *Γ is satisfiable (in the sense of FOL) $\Leftrightarrow$*
   $\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\Delta)$ is satisfiable (in the sense of PL).
2. *Γ is finitely satisfiable (in the sense of FOL) $\Leftrightarrow$*
   $\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\Delta)$ is finitely satisfiable (in the sense of PL).

Proof. Part 1 is already established, when Γ is a singleton set, in Lemma 63. For the case when Γ is not a singleton set, we need to repeat and generalize the proof of Lemma 63, as well as the proofs preceding it on which it depends. This is the same generalization that we use in going from the proof of Lemma 54 to the proof of Theorem 56.

Part 2 follows from Part 1, which covers the case when Γ is a finite set. $\square$

We first prove Compactness for first-order logic by invoking results of Herbrand theory. Then, in steps almost identical to the steps in Section 2, we prove Completeness as a consequence of Compactness.

Theorem 66 (Compactness for First-Order Logic, Version I)**.** *Let Γ be a set of first-order sentences. Then Γ is satisfiable iff Γ is finitely satisfiable.*

Proof. The left-to-right implication is immediate. For the converse, let Γ be finitely satisfiable. We use the *transfer principle* expressed by Lemma 65 and its notation.

If Γ is finitely satisfiable, then $\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\Delta)$ is finitely satisfiable (in PL), by Part 2 in the *transfer principle*. If $\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\Delta)$ is finitely satisfiable (in PL), then $\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\Delta)$ is satisfiable (in PL) by Theorem 2, which is Compactness for PL. If $\boxed{\mathsf{FOL} \mapsto \mathsf{PL}}(\Delta)$ is satisfiable (in PL), then Γ is satisfiable (in FOL), by Part 1 in the *transfer principle*. $\square$

Corollary 67 (Compactness for First-Order Logic, Version II)**.** *Let Γ be a set of first-order sentences and φ an arbitrary first-order sentence. Then $\Gamma \models \varphi$ iff there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models \varphi$.*

Proof. This follows from Corollary 7, Lemma 65, and Theorem 66. Some of the details reproduce details in the proof of Corollary 7 as a consequence of Lemma 6, all left as an exercise. $\square$

Exercise 68. Supply the details in the proof of Corollary 67. $\square$

6.3 From Compactness in FOL to Completeness in FOL

The next lemma is a weaker form of the Completeness Theorem for first-order logic. The Completeness Theorem for first-order logic in full generality is Theorem 70.

Lemma 69. *Let $\varphi_1, \ldots, \varphi_n, \psi$ be first-order sentences. If $\varphi_1, \ldots, \varphi_n \models \psi$ then $\varphi_1, \ldots, \varphi_n \vdash \psi$.*

Proof. The book [LCS] omits this lemma and its proof, though it mentions in passing that the natural-deduction proof system is "sound and complete" with respect to the formal semantics it discusses in Section 2.4 in details.[21] The proof can be carried out along the lines of the proof of Lemma 14, although the semantics of first-order logic are more involved than the semantics of propositional logic. $\square$

Theorem 70 (Completeness for First-Order Logic)**.** *Let Γ be a set (possibly infinite) of first-order sentences, and ψ a first-order sentence. If $\Gamma \models \psi$, then $\Gamma \vdash \psi$.*

Proof. Straightforward consequence of Corollary 67 and Lemma 69. The details here are almost identical to the details in the proof of Theorem 13, except that all formulas are now first-order sentences. $\square$

6.4 Applications and Exercises

(MORE TO COME)

[21] See page 96 in Michael Huth and Mark Ryan, *Logic in Computer Science*, Second Edition, Cambridge University Press, 2004.

7 Concluding Remarks

What have we missed? Plenty.

The material we have covered in these lecture notes is but a tiny fraction of a much larger body of knowledge, which has developed into a sophisticated and very robust area of mathematics over more than a century. It can be presented from many equally valid angles. It is then unavoidable that many elegant arguments that you will find elsewhere are ignored from the perspective of these notes, which gives precedence to semantic notions over proof-theoretic notions.

Chief among these is perhaps Henkin's proof of Completeness, which bypasses Compactness to reach its end and, of course, makes the latter a corollary (provided Soundness is also available). Henkin's proof works like magic, from the syntactic raw material, it finds a way to build a model.[22]

(MORE TO COME)

[22] And it is a little surprising the first time you see it, as it gives you the feeling of a bootstrapping that may not work – or at least that was my reaction when I first encountered it as a student.

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A Syntax of Well-Formed Formulas

This appendix is a compendium of syntactic conventions we use in the main body of these lecture notes. It is intended as a handy reference, which can be quickly consulted whenever you need clarification on notations in the main body.

We first cover the syntax of *well-formed formulas* (wff's) of the following: *propositional logic*, the *logic of quantified boolean formulas*, and *first-order logic*. We thus define the sets $\mathsf{WFF}_{\mathsf{PL}}$, $\mathsf{WFF}_{\mathsf{QBF}}$ and $\mathsf{WFF}_{\mathsf{FOL}}$ first. The syntax of $\mathsf{WFF}_{\mathsf{FOL}}$ includes that of *zeroth-order logic*, *equality logic*, *equational logic*, and *quasi-equational logic*, as four special cases which are therefore left to the end of this appendix. The resulting sets of wff's are denoted $\mathsf{WFF}_{\mathsf{ZOL}}$, $\mathsf{WFF}_{\mathsf{eL}}$, $\mathsf{WFF}_{\mathsf{EL}}$, and $\mathsf{WFF}_{\mathsf{QEL}}$.

Throughout, we use lower-case Greek letters from the end of the alphabet (mostly φ and ψ) and occasionally from the beginning of the alphabet (α, β, and γ) as metavariables denoting well-formed formulas (wff's). We use upper-case Greek letters $\Gamma, \Delta, \ldots$ as metavariables denoting sets of wff's.

A.1 Well-Formed Formulas of PL

The syntax of *propositional logic* (PL) is built up from a set $\mathcal{P}$ of variables and a few logical connectives:

- $\mathcal{P} = \{p_0, p_1, \ldots\}$ is a countably infinite set of *propositional variables* (also called *propositional* or *Boolean atoms*). We use p and lower-case Roman letters nearby $\{q, r, s, \ldots\}$, possibly decorated, as metavariables ranging over $\mathcal{P}$.
- The set of *logical connectives* is $\{\neg, \wedge, \vee, \rightarrow\}$. We use the symbol "$\diamond$" as a metavariable ranging over the binary connectives $\wedge$, $\vee$, and $\rightarrow$.

The set $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$ of well-formed propositional formulas over $\mathcal{P}$ is the least set such that:

$$\begin{aligned}\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P}) \;\supseteq\; & \mathcal{P} \,\cup\, \{\bot, \top\} \,\cup\, \left\{ (\neg\varphi) \,\middle|\, \varphi \in \mathsf{WFF}_{\mathsf{PL}}(\mathcal{P}) \right\} \\ & \cup \left\{ (\varphi \diamond \psi) \,\middle|\, \varphi, \psi \in \mathsf{WFF}_{\mathsf{PL}}(\mathcal{P}) \text{ and } \diamond \in \{\wedge, \vee, \rightarrow\} \right\}.\end{aligned}$$

It is customary to omit parentheses whenever possible, using the following precedences:

- $\{\neg\}$ binds more tightly than binary connectives $\{\wedge, \vee, \rightarrow\}$, *e.g.*, $\neg\varphi_1 \wedge \varphi_2$ means $((\neg\varphi_1) \wedge \varphi_2)$.
- binary connectives $\{\wedge, \vee\}$ associate to the left, *e.g.*, $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ means $((\varphi_1 \wedge \varphi_2) \wedge \varphi_3)$.
- the binary connective $\{\rightarrow\}$ associates to the right, *e.g.*, $\varphi_1 \rightarrow \varphi_2 \rightarrow \varphi_3$ means $(\varphi_1 \rightarrow (\varphi_2 \rightarrow \varphi_3))$.
- $\{\wedge, \vee\}$ have higher precedence than $\{\rightarrow\}$, *e.g.*, $\varphi_1 \rightarrow \varphi_2 \wedge \varphi_3$ means $(\varphi_1 \rightarrow (\varphi_2 \wedge \varphi_3))$.

Whenever in doubt about the conventions, insert matching parentheses to disambiguate wff's. Also, to break precedences of logical connectives, insert parentheses; for example, if the intended wff is $((\varphi_1 \rightarrow \varphi_2) \rightarrow \varphi_3)$, we can omit the outer matching parentheses as in $(\varphi_1 \rightarrow \varphi_2) \rightarrow \varphi_3$, but not the inner matching parentheses as in $\varphi_1 \rightarrow \varphi_2 \rightarrow \varphi_3$, otherwise the wff is understood to mean $(\varphi_1 \rightarrow (\varphi_2 \rightarrow \varphi_3))$.

Exercise 71. There is an implicit induction in our definitions of $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$ above. Make this induction explicit in an alternative definition of $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$, using BNF or extended BNF notation. $\square$

A.2 Well-Formed Formulas of QBF

The *logic of quantified boolean formulas* (QBF) extends PL by introducing the quantifiers $\{\forall, \exists\}$. The set $\mathsf{WFF}_{\mathsf{QBF}}(\mathcal{P})$ of well-formed formulas of QBF over $\mathcal{P}$ is the least set such that:

$$\begin{aligned}\mathsf{WFF}_{\mathsf{QBF}}(\mathcal{P}) \;\supseteq\; \mathcal{P} \;\cup\; \{\bot, \top\} \;\cup\; &\Big\{ (\neg\varphi) \;\Big|\; \varphi \in \mathsf{WFF}_{\mathsf{QBF}}(\mathcal{P}) \Big\}\\ \cup\; &\Big\{ (\varphi \diamond \psi) \;\Big|\; \varphi, \psi \in \mathsf{WFF}_{\mathsf{QBF}}(\mathcal{P}) \text{ and } \diamond \in \{\wedge, \vee, \to\} \Big\}\\ \cup\; &\Big\{ (\forall p\, \varphi) \;\Big|\; \varphi \in \mathsf{WFF}_{\mathsf{QBF}}(\mathcal{P}) \text{ and } p \in \mathcal{P} \Big\} \cup \Big\{ (\exists p\, \varphi) \;\Big|\; \varphi \in \mathsf{WFF}_{\mathsf{QBF}}(\mathcal{P}) \text{ and } p \in \mathcal{P} \Big\}.\end{aligned}$$

To omit parentheses for better readability, we use the same precedences as in PL, in addition to the following conventions for quantifiers:

- $\forall p.\, \varphi$ means $(\forall p\, \varphi)$ and $\exists p.\, \varphi$ means $(\exists p\, \varphi)$.
- $\forall p\, q.\, \varphi$ means $(\forall p\, (\forall q\, \varphi))$ and $\exists p\, q.\, \varphi$ means $(\exists p\, (\exists q\, \varphi))$.

A.3 Well-Formed Formulas of FOL

Let $\Sigma = \mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$ be a first-order signature, where:

- $\mathcal{R} = \{R_1, R_2, \ldots\}$ is a countable set, possibly infinite, of relation symbols, each with an arity $\geqslant 1$. We use R and upper-case Roman letters nearby $\{P, Q, S, \ldots\}$, possibly decorated, as metavariables ranging over $\mathcal{R}$.[23]
- $\mathcal{F} = \{f_1, f_2, \ldots\}$ is a countable set, possibly infinite, of function symbols, each with an arity $\geqslant 1$. We use f and lower-case Roman letters nearby $\{f, g, h, \ldots\}$, possibly decorated, as metavariables ranging over $\mathcal{F}$.
- $\mathcal{C} = \{c_1, c_2, \ldots\}$ is a countable set, possible infinite, of constant symbols. We use c and lower-case Roman letters nearby $\{d, e, \ldots\}$, possibly decorated, as metavariables ranging over $\mathcal{C}$.

Following most presentations of *first-order logic* (FOL), wff's may include a symbol for the equality relation, which is here denoted "$\approx$" and used in infix position, as in "$t_1 \approx t_2$". We consider the symbol "$\approx$" to be outside the signature Σ.

Besides symbols from the signature, wff's of FOL may contain variables:

- $X = \{x_0, x_1, x_2, \ldots\}$ is a countably infinite set of variables. We use letters from the end of the Roman alphabet $\{x, y, z, \ldots\}$, possibly decorated, as metavariables ranging over X.

We build up wff's gradually, starting with the set of terms $\mathsf{Terms}(\Sigma, X)$, followed by the set of atomic formulas $\mathsf{Atoms}(\Sigma, X)$, followed by the full set $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$ of wff's. These are the three stages:

1. $\mathsf{Terms}(\Sigma, X)$ is the least set satisfying the condition:

$$\mathsf{Terms}(\Sigma, X) \;\supseteq\; \mathcal{C} \;\cup\; X \;\cup \Big\{ f(t_1, \ldots, t_n) \;\Big|\; f \in \mathcal{F} \text{ has arity } n \geqslant 1,\; t_1, \ldots, t_n \in \mathsf{Terms}(\Sigma, X) \Big\}.$$

   Since there are no relation symbols in terms, we may write $\mathsf{Terms}(\mathcal{F} \cup \mathcal{C}, X)$ instead of $\mathsf{Terms}(\Sigma, X)$.

[23]Some authors prefer the words "predicate" and "predicate symbol" to what we call "relation" and "relation symbol".

2. $\mathsf{Atoms}(\Sigma, X)$ is the set defined by:

$$\mathsf{Atoms}(\Sigma, X) \stackrel{\text{def}}{=} \{\,\bot, \top\,\} \cup \Big\{\, R(t_1, \ldots, t_n) \;\Big|\; R \in \mathcal{R} \text{ has arity } n \geqslant 0,\ t_1, \ldots, t_n \in \mathsf{Terms}(\Sigma, X) \,\Big\}.$$

3. $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$ is the least set satisfying the condition:

$$\begin{aligned}
\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X) \;\supseteq\; & \mathsf{Atoms}(\Sigma, X) \;\cup\; \Big\{\, (\neg\varphi) \;\Big|\; \varphi \in \mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X) \,\Big\} \\
& \cup\; \Big\{\, (\varphi \diamond \psi) \;\Big|\; \varphi, \psi \in \mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X) \text{ and } \diamond \in \{\wedge, \vee, \to\} \,\Big\} \\
& \cup\; \Big\{\, (\forall x\, \varphi) \;\Big|\; \varphi \in \mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X) \text{ and } x \in X \,\Big\} \\
& \cup\; \Big\{\, (\exists x\, \varphi) \;\Big|\; \varphi \in \mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X) \text{ and } x \in X \,\Big\}.
\end{aligned}$$

We follow standard practice of omitting parentheses whenever possible, using the following conventions, which extend those already mentioned for PL and QBF:

- $\{\neg\}$ binds more tightly than binary connectives $\{\wedge, \vee, \to\}$, *e.g.*, $\neg\varphi_1 \wedge \varphi_2$ means $((\neg\varphi_1) \wedge \varphi_2)$.
- binary connectives $\{\wedge, \vee\}$ associate to the left, *e.g.*, $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ means $((\varphi_1 \wedge \varphi_2) \wedge \varphi_3)$.
- binary connective $\{\to\}$ associates to the right, *e.g.*, $\varphi_1 \to \varphi_2 \to \varphi_3$ means $(\varphi_1 \to (\varphi_2 \to \varphi_3))$.
- $\{\wedge, \vee\}$ have higher precedence than $\{\to\}$, *e.g.*, $\varphi_1 \to \varphi_2 \wedge \varphi_3$ means $(\varphi_1 \to (\varphi_2) \wedge \varphi_3))$.
- $\forall x.\, \varphi$ means $(\forall x\, \varphi)$ and $\exists x.\, \varphi$ means $(\exists x\, \varphi)$.
- $\forall x\, y.\, \varphi$ means $(\forall x\, (\forall y\, \varphi))$ and $\exists x\, y.\, \varphi$ means $(\exists x\, (\exists y\, \varphi))$.

Whenever in doubt about the conventions, insert matching parentheses to disambiguate the formula.

Exercise 72. There is an implicit induction in our definitions of $\mathsf{Terms}(\Sigma, X)$ and $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$ above. Make this induction explicit in alternative definitions of $\mathsf{Terms}(\Sigma, X)$ and $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$, using BNF or extended BNF notation. $\square$

If we expand the signature Σ with fresh function symbols or relation symbols, then the sets $\mathsf{Terms}(\Sigma, X)$, $\mathsf{Atoms}(\Sigma, X)$, and $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$ are extended in the obvious way. For example, If we introduce a new relation symbol $R \notin \mathcal{R}$ of some arity $n \geqslant 0$, the set $\mathsf{Atoms}(\Sigma, X)$ is extended as follows:

$$\mathsf{Atoms}\big(\Sigma \cup \{R\}, X\big) \stackrel{\text{def}}{=} \mathsf{Atoms}(\Sigma, X) \;\cup\; \{\, R(t_1, \ldots, t_n) \mid t_1, \ldots, t_n \in \mathsf{Terms}(\Sigma, X) \,\}.$$

and the set $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$ is extended to $\mathsf{WFF}_{\mathsf{FOL}}\big(\Sigma \cup \{R\}, X\big)$, defined by substituting $\mathsf{Atoms}\big(\Sigma \cup \{R\}, X\big)$ for $\mathsf{Atoms}(\Sigma, X)$ in the definition of $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$.

An important instance of the preceding is when we allow the equality symbol "$\approx$" to occur in wff's. In this case the set of atomic wff's is now denoted $\mathsf{Atoms}\big(\Sigma \cup \{\approx\}, X\big)$ and the corresponding $\mathsf{WFF}_{\mathsf{FOL}}\big(\Sigma \cup \{\approx\}, X\big)$ is obtained by substituting $\mathsf{Atoms}\big(\Sigma \cup \{\approx\}, X\big)$ for $\mathsf{Atoms}(\Sigma, X)$ in the definition of $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$.

A.4 Well-Formed Formulas of ZOL

We write $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$ for the set of wff's of *zeroth-order logic* (ZOL), a proper subset of $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$ that mention no variables in X and no quantifiers in $\{\forall, \exists\}$. The definition is in three stages:

1. $\mathsf{Terms}(\Sigma, \varnothing)$ is the same as the set of variable-free terms of FOL:

$$\mathsf{Terms}(\Sigma, \varnothing) \stackrel{\text{def}}{=} \Big\{ t \;\Big|\; t \in \mathsf{Terms}(\Sigma, X) \text{ and } \mathrm{FV}(t) = \varnothing \Big\}.$$

   Since there are no relation symbols in terms, we may write $\mathsf{Terms}(\mathcal{F} \cup \mathcal{C}, \varnothing)$ instead of $\mathsf{Terms}(\Sigma, \varnothing)$.

2. $\mathsf{Atoms}(\Sigma, \varnothing)$ is the same as the set of variable-free atomic formulas of FOL:

$$\mathsf{Atoms}(\Sigma, \varnothing) \stackrel{\text{def}}{=} \{\bot, \top\} \cup \Big\{ \varphi \;\Big|\; \varphi \in \mathsf{Atoms}(\Sigma, X) \text{ and } \mathrm{FV}(\varphi) = \varnothing \Big\}.$$

3. $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$ is the least set satisfying the condition:

$$\begin{aligned} \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing) \supseteq \mathsf{Atoms}(\Sigma, \varnothing) &\cup \Big\{ (\neg\varphi) \;\Big|\; \varphi \in \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing) \Big\} \\ &\cup \Big\{ (\varphi \diamond \psi) \;\Big|\; \varphi, \psi \in \mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing) \text{ and } \diamond \in \{\wedge, \vee, \to\} \Big\}. \end{aligned}$$

In words, $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$ is the set of all variable-free and quantifier-free formulas of *first-order logic* over signature Σ.

It is possible to define a *zeroth-order logic* which allows variables but disallows quantifiers. The set of wff's of such a logic is $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, X)$, indicated by the second argument $X \neq \varnothing$. $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, X)$ is intermediate between $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$ and $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$, more expressive than $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$ but less expressive than $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$. For all of its interesting properties, we do not examine $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, X)$ in these lecture notes.

Of particular interest for our presentation is the extension $\mathsf{WFF}_{\mathsf{ZOL}}\big(\Sigma \cup \{\approx\}, \varnothing\big)$ which allows the equality symbol "$\approx$" to occur in wff's, but still precludes variables and quantifiers. See our examination in Section 4.

A.5 Well-Formed Formulas of eL, EL, and QEL

We write:

- $\mathsf{WFF}_{\mathsf{eL}}\big(\{\approx\}, X\big)$ for the set of wff's of *equality logic* (eL),
- $\mathsf{WFF}_{\mathsf{EL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big)$ for the set of wff's of *equational logic* (EL),
- $\mathsf{WFF}_{\mathsf{QEL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big)$ for the set of wff's of *quasi-equational logic* (QEL),

which are all proper subsets of $\mathsf{WFF}_{\mathsf{FOL}}\big(\Sigma \cup \{\approx\}, X\big)$, the set of first-order wff's where the equality symbol $\approx$ may occur. In these three logics, relation symbols are precluded from wff's.[24]

Starting with $\mathsf{WFF}_{\mathsf{eL}}\big(\{\approx\}, X\big)$, the definition is once more in three stages:

1. $\mathsf{Terms}\big(\varnothing, X\big)$ is simply the set of all variables, as no symbols from the signature are allowed:

$$\mathsf{Terms}\big(\varnothing, X\big) \stackrel{\text{def}}{=} X.$$

2. $\mathsf{Atoms}\big(\{\approx\}, X\big)$ is restricted to the equality symbol "$\approx$" and its members are called *equalities* (between first-order variables):

$$\mathsf{Atoms}\big(\{\approx\}, X\big) \stackrel{\text{def}}{=} \{\bot, \top\} \cup \Big\{ (x \approx y) \;\Big|\; x, y \in X \Big\}.$$

[24] The distinction between "*equality*" and "*equation*" is a little confusing and does not conform to how we use the same words in other contexts. We use two different words here so that we can name differently two distinct logics, eL and EL. To confuse the matter a little more, what we here call *equations* and *quasi-equations* are called elsewhere *identities* and *quasi-identities*.

3. $\mathsf{WFF}_{\mathsf{eL}}\big(\{\approx\}, X\big)$ is the least set satisfying the condition:

$$\mathsf{WFF}_{\mathsf{eL}}\big(\{\approx\}, X\big) \;\supseteq\; \mathsf{Atoms}(\{\approx\}, X) \;\cup\; \Big\{\, (\neg\varphi) \;\Big|\; \varphi \in \mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)\, \Big\}$$
$$\cup\; \Big\{\, (\varphi \diamond \psi) \;\Big|\; \varphi, \psi \in \mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X) \text{ and } \diamond \in \{\wedge, \vee, \to\}\, \Big\}.$$

In words, $\mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$ is the set of all (quantifier-free) propositional combinations of equalities between variables.

We define $\mathsf{WFF}_{\mathsf{EL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big)$ and $\mathsf{WFF}_{\mathsf{QEL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big)$ simultaneously in three stages:

1. $\mathsf{Terms}\big(\mathcal{F} \cup \mathcal{C}, X\big)$ is the same as the set of terms of FOL, the least satisfying the condition:

$$\mathsf{Terms}(\mathcal{F} \cup \mathcal{C}, X) \;\supseteq\; \mathcal{C} \cup X \cup \Big\{\, f(t_1, \ldots, t_n) \;\Big|\; f \in \mathcal{F} \text{ has arity } n \geqslant 1,\ t_1, \ldots, t_n \in \mathsf{Terms}(\Sigma, X)\, \Big\}.$$

2. $\mathsf{Atoms}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big)$ is restricted to the equality symbol "$\approx$" and its members are called *equations* (between terms):

$$\mathsf{Atoms}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big) \;\stackrel{\text{def}}{=}\; \{\bot, \top\} \;\cup\; \Big\{\, (t_1 \approx t_2) \;\Big|\; t_1, t_2 \in \mathsf{Terms}\big(\mathcal{F} \cup \mathcal{C}, X\big)\, \Big\}.$$

3. $\mathsf{WFF}_{\mathsf{EL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big)$ and $\mathsf{WFF}_{\mathsf{QEL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big)$ are defined by:

$$\mathsf{WFF}_{\mathsf{EL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big) \;\stackrel{\text{def}}{=}\; \mathsf{Atoms}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big)$$

$$\mathsf{WFF}_{\mathsf{QEL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big) \;\stackrel{\text{def}}{=}\; \Big\{\, (\varphi_1 \wedge \cdots \wedge \varphi_k \to \psi) \;\Big|$$
$$\varphi_1, \ldots, \varphi_k, \psi \in \mathsf{Atoms}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big) \text{ and } k \geqslant 0\, \Big\}.$$

We call *equations* and *quasi-equations* the wff's in $\mathsf{WFF}_{\mathsf{EL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big)$ and $\mathsf{WFF}_{\mathsf{QEL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big)$, respectively. Every equation is a quasi-equation, but not conversely, so that $\mathsf{WFF}_{\mathsf{EL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big)$ is a proper subset of $\mathsf{WFF}_{\mathsf{QEL}}\big(\mathcal{F} \cup \mathcal{C} \cup \{\approx\}, X\big)$.

In these lecture notes, every *equality* is an *equation*, but not vice-versa. Moreover, although every equality is an equation, it is not the case that every wff of *equality logic* (eL) is a wff of *equational logic* (EL); wff's of eL can include all the logical connectives in $\{\neg, \wedge, \vee, \to\}$, wff's of EL do not mention any of these logical connectives.

B Semantics of Well-Formed Formulas

We try as much as possible to give a uniform presentation of the semantics of all the formal logics considered in these lecture notes. We start with the semantics of *propositional logic*, then follow with the semantics of *quantified Boolean formulas*, and then with the semantics of *first-order logic*. The semantics of the other logics are special cases of the semantics of *first-order logic* and do not need a separate treatment.

B.1 Semantics of $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$

We interpret the wff's of *propositional logic* in a 2-element Boolean algebra $\mathcal{B}$, which we can take in the form:

$$\mathcal{B} \stackrel{\text{def}}{=} (B, \mathsf{Not}, \mathsf{And}, \mathsf{Or}, \mathsf{Implies}, \mathit{false}, \mathit{true}) \qquad \text{with } B \stackrel{\text{def}}{=} \{\mathit{false}, \mathit{true}\},$$

where Not is a unary operation, and each of the operations in $\{\mathsf{And}, \mathsf{Or}, \mathsf{Implies}\}$ is binary. Since the domain B is finite, these operations can be conveniently defined in tabular forms as follows:[25]

	Not
false	*true*
true	*false*

And	*false*	*true*
false	*false*	*false*
true	*false*	*true*

Or	*false*	*true*
false	*false*	*true*
true	*true*	*true*

Implies	*false*	*true*
false	*true*	*true*
true	*false*	*true*

A *truth assignment* for the set $\mathcal{P}$ of propositional variables is any map $\sigma : \mathcal{P} \to \{\mathit{false}, \mathit{true}\}$. Having fixed the interpretations of the symbols $\{\neg, \wedge, \vee, \to\}$ as the operations $\{\mathsf{Not}, \mathsf{And}, \mathsf{Or}, \mathsf{Implies}\}$ of the Boolean algebra $\mathcal{B}$, the satisfaction of a propositional wff φ, *i.e.*, the truth value of φ, depends only on the assignment σ.

We next lift the truth assignment σ to all propositional wff's. We use a notation favored by computer scientists: the meaning of a syntactic object φ is denoted by inserting it between double brackets, as in "$[\![\varphi]\!]$" or, more precisely here, "$[\![\varphi]\!]_\sigma$" since it depends on σ. The definition of $[\![\varphi]\!]_\sigma$ is by *structural induction*, *i.e.*, on the "shape" of φ:[26]

$$\begin{aligned}
[\![p]\!]_\sigma &\stackrel{\text{def}}{=} \sigma(p) \\
[\![\bot]\!]_\sigma &\stackrel{\text{def}}{=} \mathit{false} \\
[\![\top]\!]_\sigma &\stackrel{\text{def}}{=} \mathit{true} \\
[\![\neg\varphi]\!]_\sigma &\stackrel{\text{def}}{=} \mathsf{Not}\big([\![\varphi]\!]_\sigma\big) \\
[\![\varphi \wedge \psi]\!]_\sigma &\stackrel{\text{def}}{=} \mathsf{And}\big([\![\varphi]\!]_\sigma, [\![\psi]\!]_\sigma\big) \\
[\![\varphi \vee \psi]\!]_\sigma &\stackrel{\text{def}}{=} \mathsf{Or}\big([\![\varphi]\!]_\sigma, [\![\psi]\!]_\sigma\big) \\
[\![\varphi \to \psi]\!]_\sigma &\stackrel{\text{def}}{=} \mathsf{Implies}\big([\![\varphi]\!]_\sigma, [\![\psi]\!]_\sigma\big)
\end{aligned}$$

[25] These are not what are usually called the *truth-tables* of the Boolean operations, which are typically written as:

p	$\neg p$
false	*true*
true	*false*

p	q	$p \wedge q$
false	*false*	*false*
false	*true*	*false*
true	*false*	*false*
true	*true*	*true*

p	q	$p \vee q$
false	*false*	*false*
false	*true*	*true*
true	*false*	*true*
true	*true*	*true*

p	q	$p \to q$
false	*false*	*true*
false	*true*	*true*
true	*false*	*false*
true	*true*	*true*

Our tabular forms for the Boolean operations here is the same tabular forms we use elsewhere in these notes whenever we deal with unary and binary operations over finite domains. In particular for binary operations, they are more compact than truth-tables, but their generalization to higher-arity functions lose their graphical appeal and are practically useless.

[26] Or, as computer scientists often like to say, the definition is *syntax-directed*.

Following convention:

- we write $\sigma \models \varphi$ and say σ *satisfies* φ iff $[\![\varphi]\!]_\sigma = \mathit{true}$,
- we write $\sigma \not\models \varphi$ and say σ *does not satisfy* φ iff $[\![\varphi]\!]_\sigma = \mathit{false}$,
- if for every assignment σ we have $\sigma \models \varphi$, we may write $\models \varphi$ and say φ is *valid* or is a *tautology*,
- if for some assignment σ we have $\sigma \not\models \varphi$, we may write $\not\models \varphi$ and say φ is *falsifiable*,
- if for every assignment σ we have $\sigma \not\models \varphi$, we may say φ is *unsatisfiable* or is a *contradiction*.

It is worth noting that the double-bracket notation is a convenient visual aid to separate syntax from semantics: everything inside the pair "$[\![$" and "$]\!]$" is a piece of syntax, and everything outside is about its semantics. Similarly, "$\models$" conveniently separates syntax from semantics: what is to the right of "$\models$" is a piece of syntax and what is to the left of "$\models$" is something that determines the semantics of the former. These are by now firmly established notational conventions.[27]

B.2 Semantics of $\mathsf{WFF}_{\mathsf{QBF}}(\mathcal{P})$

Consider the definition of $[\![\varphi]\!]_\sigma$ by *structural induction* when $\varphi \in \mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$. We want to extend it to wff's in $\mathsf{WFF}_{\mathsf{QBF}}(\mathcal{P})$ which mention the quantifiers $\forall$ and $\exists$. There are two steps that are missing for this extension and, before supplying them, we agree on how to write a *(one-point) adjustment* of a truth assignment σ. If $p \in \mathcal{P}$, an adjustment of σ at p is denoted "$\sigma[p \mapsto \mathit{false}]$" or "$\sigma[p \mapsto \mathit{true}]$". The precise definition is, for all $q \in \mathcal{P}$:

$$\sigma[p \mapsto \mathit{false}]\ (q) \ \stackrel{\text{def}}{=} \begin{cases} \mathit{false} & \text{if } p = q, \\ \sigma(q) & \text{if } p \neq q, \end{cases}$$

$$\sigma[p \mapsto \mathit{true}]\ (q) \ \stackrel{\text{def}}{=} \begin{cases} \mathit{true} & \text{if } p = q, \\ \sigma(q) & \text{if } p \neq q. \end{cases}$$

Now for the two missing steps in the structural induction:

$$[\![\forall p\, \varphi]\!]_\sigma \ \stackrel{\text{def}}{=}\ \mathsf{And}\left([\![\varphi]\!]_{\sigma[p\mapsto \mathit{false}]}\,,[\![\varphi]\!]_{\sigma[p\mapsto \mathit{true}]}\right)$$

$$[\![\exists p\, \varphi]\!]_\sigma \ \stackrel{\text{def}}{=}\ \mathsf{Or}\left([\![\varphi]\!]_{\sigma[p\mapsto \mathit{false}]}\,,[\![\varphi]\!]_{\sigma[p\mapsto \mathit{true}]}\right)$$

Exercise 73. Based on the preceding definition, show that the following are equivalent assertions:

- $\sigma \models \forall p\, \varphi$,
- $\sigma \models \varphi[p := \bot] \ \wedge\ \varphi[p := \top]$.

And similarly, show that the following are equivalent assertions:

- $\sigma \models \exists p\, \varphi$,
- $\sigma \models \varphi[p := \bot] \ \vee\ \varphi[p := \top]$.

We write $\varphi[p := \bot]$ and $\varphi[p := \top]$ to denote the substitution of $\bot$ and $\top$ for every free occurrence of p in φ. □

[27] The double-bracket notation "$[\![\ldots]\!]$" was probably first used by computer scientists working on the denotational semantics of programming languages in the early 1970's. The double-turnstile notation "$\models$" was first introduced by mathematical logicians at least a decade earlier. The symbol "$\models$" appears throughout the classic book *Model Theory* by C.C. Chang and H. Jerome Keisler [2]; the authors point out, in the Preface of the first edition, that their book grew out of lecture notes in circulation since the early 1960's.

Exercise 74. A wff $\varphi \in \mathsf{WFF}_{\mathsf{QBF}}(\mathcal{P})$ is *closed* if $\mathrm{FV}(\varphi) = \varnothing$, *i.e.*, every occurrence of a variable p in φ falls in the scope of some "$\forall p$" or "$\exists p$". Use structural induction to show that, if φ is closed, then for every assignment σ it holds that either $\sigma \models \varphi$ or $\sigma \not\models \varphi$.

In words, if φ is closed, then φ is either a tautology or a contradiction.

Hint: This is subtle. In the structural induction, keep track of variables that occur free in a wff, there are finitely many of them in any wff. Formalize the idea that only a finite part of an assignment σ is relevant for the truth-value returned by $[\![\varphi]\!]_\sigma$, namely, the part that assigns a truth value to a variable occurring free in φ. $\square$

B.3 Semantics of the Other Logics

All the other logics considered in these lecture notes are: *equality logic*, *zeroth-ary logic*, *equational logic*, *quasi-equational logic*, and *first-order logic*. The first four in this list are fragments of the last one, *first-order logic*. So, we restrict attention to the semantics of $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$ and $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma \cup \{\approx\}, X)$.

Given signature $\Sigma = \mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$, a Σ-*structure* has the form:

$$\mathcal{A} \stackrel{\text{def}}{=} (A, \mathcal{R}^{\mathcal{A}}, \mathcal{F}^{\mathcal{A}}, \mathcal{C}^{\mathcal{A}}) \quad \text{where}$$

A is a non-empty set, finite or infinite, called the *universe* or *domain* of $\mathcal{A}$,

$$\mathcal{R}^{\mathcal{A}} \stackrel{\text{def}}{=} \{\, R^{\mathcal{A}} \subseteq \underbrace{A \times \cdots \times A}_{n} \mid R \in \mathcal{R} \text{ has arity } n \geqslant 1 \,\},$$

$$\mathcal{F}^{\mathcal{A}} \stackrel{\text{def}}{=} \{\, f^{\mathcal{A}} : \underbrace{A \times \cdots \times A}_{n} \to A \mid f \in \mathcal{F} \text{ has arity } n \geqslant 1 \,\},$$

$$\mathcal{C}^{\mathcal{A}} \stackrel{\text{def}}{=} \{\, c^{\mathcal{A}} \in A \mid c \in \mathcal{C} \,\}.$$

In words, a Σ-*structure* $\mathcal{A}$ assigns an interpretation to every symbol in Σ over some set of elements A. If the equality symbol $\approx$ occurs in wff's, we need to expand $\mathcal{A}$ to include an interpretation for it and write:

$$\mathcal{A} \stackrel{\text{def}}{=} (A, \approx^{\mathcal{A}}, \mathcal{R}^{\mathcal{A}}, \mathcal{F}^{\mathcal{A}}, \mathcal{C}^{\mathcal{A}}) \text{ or also } (A, =, \mathcal{R}^{\mathcal{A}}, \mathcal{F}^{\mathcal{A}}, \mathcal{C}^{\mathcal{A}})$$

since $\approx^{\mathcal{A}}$ is always interpreted as the equality "$=$" on the universe A.

A *valuation* for the set X of variables in the Σ-*structure* $\mathcal{A}$ is a map $\sigma : X \to A$. Note that σ maps every member of X, which is an infinite set, to an element of A. In case A is finite, σ necessarily maps many members of X to the same element of A.[28] A *(one-point) adjustment* of σ at variable x is a new valuation denoted $\sigma[x \mapsto a]$ where $a \in A$:

$$\sigma[x \mapsto a]\ (y) \stackrel{\text{def}}{=} \begin{cases} a & \text{if } x = y, \\ \sigma(y) & \text{if } x \neq y. \end{cases}$$

We use a Σ-structure $\mathcal{A}$ together with a valuation σ to give a meaning to every $\varphi \in \mathsf{WFF}_{\mathsf{FOL}}(\Sigma \cup \{\approx\}, X)$. Below is the *structural induction* we use to interpret every such wff, it includes some of the steps already used for wff's in $\mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$ and $\mathsf{WFF}_{\mathsf{QBF}}(\mathcal{P})$. Following the three stages in the definition of $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma \cup \{\approx\}, X)$, we start with the interpretation of *terms*, continue with the interpretation of *atomic wff's*, and conclude with the interpretation of first-order wff's in general:

[28] Some authors call a valuation an *assignment* or also an *environment*. We reserve the word "assignment" to appear in the expessions "truth assignment" and "assignment of truth values", a way of distinguishing it from what the word "valuation" is used for.

1. Interpretation of terms:

$$\llbracket x \rrbracket_{\mathcal{A},\sigma} \stackrel{\text{def}}{=} \sigma(x)$$

$$\llbracket c \rrbracket_{\mathcal{A},\sigma} \stackrel{\text{def}}{=} c^{\mathcal{A}} \qquad \text{for every } c \in \mathcal{C}$$

$$\llbracket f(t_1, \dots, t_n) \rrbracket_{\mathcal{A},\sigma} \stackrel{\text{def}}{=} f^{\mathcal{A}}\big(\llbracket t_1 \rrbracket_{\mathcal{A},\sigma}, \dots, \llbracket t_n \rrbracket_{\mathcal{A},\sigma}\big) \qquad \text{for every } f \in \mathcal{F} \text{ of arity } n \geqslant 1$$

2. Interpretation of atomic wff's:

$$\llbracket \bot \rrbracket_{\mathcal{A},\sigma} \stackrel{\text{def}}{=} \textit{false}$$

$$\llbracket \top \rrbracket_{\mathcal{A},\sigma} \stackrel{\text{def}}{=} \textit{true}$$

$$\llbracket R(t_1, \dots, t_n) \rrbracket_{\mathcal{A},\sigma} \stackrel{\text{def}}{=} R^{\mathcal{A}}\big(\llbracket t_1 \rrbracket_{\mathcal{A},\sigma}, \dots, \llbracket t_n \rrbracket_{\mathcal{A},\sigma}\big) \qquad \text{for every } R \in \mathcal{R} \text{ of arity } n \geqslant 1$$

$$\llbracket t_1 \approx t_n \rrbracket_{\mathcal{A},\sigma} \stackrel{\text{def}}{=} \begin{cases} \textit{false} & \text{if } \llbracket t_1 \rrbracket_{\mathcal{A},\sigma} \neq \llbracket t_2 \rrbracket_{\mathcal{A},\sigma} \\ \textit{true} & \text{if } \llbracket t_1 \rrbracket_{\mathcal{A},\sigma} = \llbracket t_2 \rrbracket_{\mathcal{A},\sigma} \end{cases}$$

3. Interpretation of first-order wff's in general:

$$\llbracket \neg\varphi \rrbracket_{\mathcal{A},\sigma} \stackrel{\text{def}}{=} \mathsf{Not}\big(\llbracket \varphi \rrbracket_{\mathcal{A},\sigma}\big)$$

$$\llbracket \varphi \wedge \psi \rrbracket_{\mathcal{A},\sigma} \stackrel{\text{def}}{=} \mathsf{And}\big(\llbracket \varphi \rrbracket_{\mathcal{A},\sigma}, \llbracket \psi \rrbracket_{\mathcal{A},\sigma}\big)$$

$$\llbracket \varphi \vee \psi \rrbracket_{\mathcal{A},\sigma} \stackrel{\text{def}}{=} \mathsf{Or}\big(\llbracket \varphi \rrbracket_{\mathcal{A},\sigma}, \llbracket \psi \rrbracket_{\mathcal{A},\sigma}\big)$$

$$\llbracket \varphi \to \psi \rrbracket_{\mathcal{A},\sigma} \stackrel{\text{def}}{=} \mathsf{Implies}\big(\llbracket \varphi \rrbracket_{\mathcal{A},\sigma}, \llbracket \psi \rrbracket_{\mathcal{A},\sigma}\big)$$

$$\llbracket \forall x\, \varphi \rrbracket_{\mathcal{A},\sigma} \stackrel{\text{def}}{=} \begin{cases} \textit{false} & \text{if } \llbracket \varphi \rrbracket_{\mathcal{A},\sigma[x \mapsto a]} = \textit{false} \quad \text{for some } a \in A \\ \textit{true} & \text{if } \llbracket \varphi \rrbracket_{\mathcal{A},\sigma[x \mapsto a]} = \textit{true} \quad \text{for every } a \in A \end{cases}$$

$$\llbracket \exists x\, \varphi \rrbracket_{\mathcal{A},\sigma} \stackrel{\text{def}}{=} \begin{cases} \textit{false} & \text{if } \llbracket \varphi \rrbracket_{\mathcal{A},\sigma[x \mapsto a]} = \textit{false} \quad \text{for every } a \in A \\ \textit{true} & \text{if } \llbracket \varphi \rrbracket_{\mathcal{A},\sigma[x \mapsto a]} = \textit{true} \quad \text{for some } a \in A \end{cases}$$

We can view $\llbracket \ldots \rrbracket_{\mathcal{A},\sigma}$ as a two-sorted function, from a two-sorted domain $\{\text{terms}\} \cup \{\text{wff's}\}$ to a two-sorted co-domain $A \cup \{\textit{false}, \textit{true}\}$: It maps every term t to an element in the universe A, and every wff φ to a truth value.

Following convention:

- we write $\mathcal{A}, \sigma \models \varphi$ and say $(\mathcal{A}, \sigma)$ *satisfies* φ iff $\llbracket \varphi \rrbracket_{\mathcal{A},\sigma} = \textit{true}$,
- we write $\mathcal{A}, \sigma \not\models \varphi$ and say $(\mathcal{A}, \sigma)$ *does not satisfy* φ iff $\llbracket \varphi \rrbracket_{\mathcal{A},\sigma} = \textit{false}$,
- if for every valuation σ we have $\mathcal{A}, \sigma \models \varphi$, we write $\mathcal{A} \models \varphi$ and say $\mathcal{A}$ *satisfies* φ or φ is *true in* $\mathcal{A}$,
- if for every Σ-structure $\mathcal{A}$ and valuation σ we have $\mathcal{A}, \sigma \models \varphi$, we write $\models \varphi$ and say φ is *valid*.

Exercise 75. This continues Exercise 74. Let $\mathcal{A}$ be a fixed Σ-structure. Use structural induction to show that, if $\varphi \in \mathsf{WFF}_{\mathsf{FOL}}(\Sigma, X)$ is closed, then for every valuation σ it holds that either $\mathcal{A}, \sigma \models \varphi$ or $\mathcal{A}, \sigma \not\models \varphi$.

Hence, when φ is closed, we can ignore σ and write $\mathcal{A} \models \varphi$ ("φ is true in $\mathcal{A}$") or $\mathcal{A} \not\models \varphi$ ("φ is false in $\mathcal{A}$"). $\square$

Exercise 76. Let $\varphi \in \mathsf{WFF}_{\mathsf{FOL}}(\Sigma \cup \{\approx\}, X)$ and $\mathrm{FV}(\varphi) = \{x_1, \ldots, x_n\}$. The *existential closure* of φ is the closed wff $\exists x_1 \cdots \exists x_n.\, \varphi$ and the *universal closure* of φ is the closed wff $\forall x_1 \cdots \forall x_n.\, \varphi$.

1. Show that φ is satisfiable iff the existential closure of φ is satisfiable.
2. Show that φ is valid iff the universal closure of φ is valid. $\square$

Let Γ and Δ be sets, possibly infinite, of wff's in $\mathsf{WFF}_{\mathsf{FOL}}(\Sigma \cup \{\approx\}, X)$.

- We say a Σ-structure $\mathcal{A}$ is a *model* of Γ iff for every $\varphi \in \Gamma$ it holds that $\mathcal{A} \models \varphi$, in which case we may also write $\mathcal{A} \models \Gamma$.
- We say Γ *semantically entails* or *implies* (others say *logically entails* or *implies*) Δ iff every model of Γ is a model of Δ (but not necessarily the converse), and we may write $\mathsf{models}(\Gamma) \subseteq \mathsf{models}(\Delta)$.

Sometimes we may want to make explicit the signature Σ of a model $\mathcal{A}$ of Γ, in which case we may say that $\mathcal{A}$ is a Σ-model. If the equality symbol $\approx$ occurs in Γ, we may say $\mathcal{A}$ is a $(\Sigma \cup \{\approx\})$-model.

Let $\varphi \in \mathsf{WFF}_{\mathsf{FOL}}(\Sigma \cup \{\approx\}, X)$ and let $\mathrm{FV}(\varphi) = \{x_1, \ldots, x_n\}$. By a slight abuse of notation, we may write $\varphi(x_1, \ldots, x_n)$ to indicate which variables have free occurrences in φ. Let $(\mathcal{A}, \sigma)$ be an interpretation for φ, consisting of a Σ-structure $\mathcal{A}$ and a valuation $\sigma : X \to A$, and let:

$$\sigma(x_1) = a_1, \ldots, \sigma(x_n) = a_n.$$

Then, instead of writing $\mathcal{A}, \sigma \models \varphi$, we may write:

$$\mathcal{A}, a_1, \ldots, a_n \models \varphi \quad \text{or also} \quad \mathcal{A} \models \varphi[a_1, \ldots, a_n]$$

with the understanding that the elements $a_1, \ldots, a_n \in A$ are substituted for the variables $x_1, \ldots, x_n$ (in the same order) in φ. Note the additional abuse of notation when we write "$\varphi[a_1, \ldots, a_n]$" (what is it?).

C Systems of Formal Proofs

The syntax and semantics of all the formal logics in these lecture notes are basically the same that you will find elsewhere in the published literature. The differences are unessential, mostly in the notation, in the presentation, and sometimes in the terminology.

This is no longer the case when we consider their proof systems. For each of our formal logics, there are many proof systems, and each system has its own advantages and disadvantages – though at the end, they all fulfill the requirement of the same Completeness Theorem, which asserts, in a nutshell: "whatever is true according to the semantics is also formally provable." But that is not the only requirement by which we choose a proof system, and there are indeed other requirements fulfilled by some but not all proof systems.

The profusion of proof systems for the same formal logic is always a little bewildering for newcomers to this material. It takes time and effort to understand and appreciate the reasons for their differences, all related to different aspects of formal proofs (*e.g.*, the efficient implementation of procedures for deciding validity, or an examination of what is called *cut-elimination* and its implications regarding consistency, or the existence of what are called *interpolation theorems*, and other proof-theoretic matters). These are all important topics beyond the scope of these lecture notes.

But we still have to select at least one proof system to round off our presentation. We choose here a particular way of setting it up, so-called *natural deduction*, and a particular way of defining its formal rules and organizing its formal derivations. There is no overarching reason to choose *natural deduction* over the many other alternatives, except that it is a little easier to present and seems to be favored by computer scientists, particularly by researchers in areas related to automated theorem provers and interactive proof assistants.

In each of our logics, if we can formally derive a wff ψ (the *conclusion*, also called *consequent*) from a finite set of wff's $\{\varphi_1, \ldots, \varphi_n\}$ (the *premises*, also called *antecedents* or *hypotheses*) according to the rules of natural deduction, we can assert this fact by writing:

$$\varphi_1, \ldots, \varphi_n \;\vdash\; \psi.$$

If we want to make explicit the logic we use, we may add a subscript "$\mathcal{L}$":

$$\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{L}} \psi,$$

where $\mathcal{L} \in \{\mathsf{PL}, \mathsf{QBF}, \mathsf{eL}, \mathsf{ZOL}, \mathsf{EL}, \mathsf{QEL}, \mathsf{FOL}\}$. Such an expression is called a *sequent* in these lecture notes, even though the word is not used by all authors and with the same intention. In our setup, the symbol "$\vdash$" is outside the formalism of natural deduction, in contrast to other proof systems that are called *sequent calculi*. It is only after a natural-deduction proof is completed that we use "$\vdash$", to separate wff's that are assumed to hold with no justification necessary (these are the premises) from the wff appearing on the last line (the conclusion).[29] Several examples for how to use the proof rules are in Appendix D and Appendix E.

C.1 Rules for PL

Following tradition, rules are given suggestive names. For the logical connectives, here limited to $\{\wedge, \vee, \to, \neg\}$, rules come in pairs. Each pair has one *introduction* rule and one *elimination* rule, indicated by the letters "I"

[29] The symbol "$\vdash$" is often called *turnstile* because of its resemblance to a typical turnstile if viewed from above. You may read it as "formally yields", or "formally proves", or "formally derives". According to Wikipedia, "$\vdash$" was first introduced towards the end of the 19th Century, by the mathemaical logician Gottlob Frege in 1879. It thus preceded its companion "$\models$" by many decades, which also reflects how concerns of mathematical logicians evolved over time, initially focusing on proof-theoretic issues and subsequently adding model-theoretic issues. See footnote 27 for comments on "$\models$".

and "E", respectively. Sometimes the *introduction* rule has two parts, *e.g.*, rule $(\vee\mathrm{I})$ has two parts: $(\vee\mathrm{I}_1)$ and $(\vee\mathrm{I}_2)$; and sometimes the *elimination* rule has two parts, *e.g.*, rule $(\wedge\mathrm{E})$ has two parts: $(\wedge\mathrm{E}_1)$ and $(\wedge\mathrm{E}_2)$.

C.1.1 *Introduction* and *elimination* rules for each of $\{\wedge, \vee, \to, \neg\}$:

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi}\ (\wedge\mathrm{I}) \qquad\qquad \frac{\varphi \wedge \psi}{\varphi}\ (\wedge\mathrm{E}_1) \qquad\qquad \frac{\varphi \wedge \psi}{\psi}\ (\wedge\mathrm{E}_2)$$

$$\frac{\varphi}{\varphi \vee \psi}\ (\vee\mathrm{I}_1) \qquad\qquad \frac{\psi}{\varphi \vee \psi}\ (\vee\mathrm{I}_2) \qquad\qquad \frac{\varphi \vee \psi \quad \boxed{\begin{array}{c}\varphi\\ \vdots\\ \theta\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \theta\end{array}}}{\theta}\ (\vee\mathrm{E})$$

$$\frac{\boxed{\begin{array}{c}\varphi\\ \vdots\\ \psi\end{array}}}{\varphi \to \psi}\ (\to\mathrm{I}) \qquad\qquad \frac{\varphi \quad \varphi \to \psi}{\psi}\ (\to\mathrm{E})$$

$$\frac{\boxed{\begin{array}{c}\varphi\\ \vdots\\ \bot\end{array}}}{\neg\varphi}\ (\neg\mathrm{I}) \qquad\qquad \frac{\varphi \qquad \neg\varphi}{\bot}\ (\neg\mathrm{E})$$

Remark: In the rules with boxes $\{\,(\to\mathrm{I}),\ (\vee\mathrm{E}),\ (\neg\mathrm{I})\,\}$, the ellipsis points '...' may be empty. Thus, in $(\to\mathrm{I})$, it is possible that $\varphi = \psi$, and similarly in the rule $(\vee\mathrm{E})$, it is possible that $\varphi = \theta$ or $\psi = \theta$ (or both). In the rule $(\neg\mathrm{I})$, if the ellipsis points are empty, then $\varphi = \bot$, in which case the conclusion of the rule is $\neg\varphi = \neg\bot$.

C.1.2 An *elimination* rule for $\bot$, an *introduction* rule for $\top$:

$$\frac{\bot}{\varphi}\ (\bot\mathrm{E}) \qquad \text{(also called } \textit{ex falso quodlibet} \text{ or just } \textit{ex falso}\text{)}$$

$$\frac{}{\top}\ (\top\mathrm{I})$$

There is no introduction rule for $\bot$ and no elimination rule fo $\top$. So far, there are 10 rules, not all of equal importance: You will be right in guessing that you get more traction from $(\to\mathrm{I})$ and $(\to\mathrm{E})$ in our proof system than from the other rules, while $(\top\mathrm{I})$ is useless (why?). But this question ("given a subset of the rules, what can be said about the wff's that are in its deductive closure?") is for another study outside these lecture notes.

C.1.3 Rules (LEM), (PBC), ($\neg\neg$E), and (Peirce's) :

These four rules have a special status. Without any of these four, the preceding rules define a proof system for what is called *intuitionistic propositional logic*; such a proof system is complete for a semantics of *propositional logic* based on what are called *Heyting algebras*, which include as a special case the familiar *Boolean algebras*. This is another matter outside the scope of these notes.

Adding anyone of the four rules in $\{(\text{LEM}), (\text{PBC}), (\neg\neg\text{E}), (\text{Peirce's})\}$ augments the deductive power of the proof system and makes it complete for the semantics we use for *propositional logic* in these notes, one based on *Boolean algebras*. The system so augmented is sometimes called *classical propositional logic*, the qualifier "classical" being use to make explicit the distinction with "intuitionistic".

LEM is a shorthand for *Law of Excluded Middle*. **PBC** is a shorthand for *Proof by Contradiction*. As its name indicates, **$\neg\neg$E** means *elimination of double negation*. **Peirce's** stands for *Peirce's Law* and is named for the 19th Century mathematical logician Charles Sanders Peirce. Here are the precise formulations of the four rules:

$$\frac{}{\varphi \vee \neg\varphi}\ (\text{LEM}) \qquad\qquad \frac{\boxed{\begin{array}{c}\neg\varphi\\ \vdots\\ \bot\end{array}}}{\varphi}\ (\text{PBC})$$

$$\frac{\neg\neg\varphi}{\varphi}\ (\neg\neg\text{E}) \qquad\qquad \frac{}{((\varphi \to \psi) \to \varphi) \to \varphi}\ (\text{Peirce's})$$

Exercise 77. Show that any two of the four rules $\{(\text{LEM}), (\text{PBC}), (\neg\neg\text{E}), (\text{Peirce's})\}$ are inter-derivable. In fact, they are inter-derivable using only two rules, ($\to$I) and ($\to$E).

Hint: One way is to consider $\binom{4}{2} = 6$ cases, one for each pair from the set of four rules, with each pair involving two derivations, for a total of 12 derivations. A much simpler approach requires only 4 derivations:

(a) (Peirce's) is derivable from (PBC),

(b) (LEM) is derivable from (Peirce's),

(c) ($\neg\neg$E) is derivable from (LEM),

(d) (PBC) is derivable from ($\neg\neg$E).

Schematically, you have to show that (PBC) $\Rightarrow$ (Peirce's) $\Rightarrow$ (LEM) $\Rightarrow$ ($\neg\neg$E) $\Rightarrow$ (PBC). $\square$

Exercise 78. In some accounts of natural deduction, the rule for *disjunction elimination* is given as ($\vee$E*):

$$\frac{\boxed{\begin{array}{c}\varphi\\ \vdots\\ \theta\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \theta\end{array}}}{(\varphi \vee \psi) \to \theta}\ (\vee\text{E}^*)$$

which is often more convenient to use than the standard ($\vee$E). You have to show that ($\vee$E) and ($\vee$E*) are inter-derivable. Specifically, there are two parts:

1. Show ($\vee$E) is derivable from ($\vee$E*) and ($\to$E).
2. Show ($\vee$E*) is derivable from ($\vee$E) and ($\to$I). $\square$

C.2 Rules for QBF

All the rules in Subsection C.1 for *propositional logic* can be used, in addition to *introduction* and *elimination* rules for the quantifiers, namely, two rules $\{(\forall\text{I}), (\forall\text{E})\}$ for "$\forall$" and two rules $\{(\exists\text{I}), (\exists\text{E})\}$ for "$\exists$":

$$\frac{\boxed{\begin{array}{ll} q & \text{fresh var} \\ \vdots & \\ \varphi[p := q] & \end{array}}}{\forall p\, \varphi}\ (\forall\text{I}) \qquad\qquad \frac{\forall p\, \varphi}{\varphi[p := t]}\ (\forall\text{E})$$

$$\frac{\varphi[p := t]}{\exists p\, \varphi}\ (\exists\text{I}) \qquad\qquad \frac{\exists p\, \varphi \qquad \boxed{\begin{array}{ll} q & \text{fresh var} \\ \varphi[p := q] & \text{assumption} \\ \vdots & \\ \psi & \end{array}}}{\psi}\ (\exists\text{E})$$

The rules for quantifiers must be used with extra care, in a way to respect the following side conditions:

- "$\varphi[p := q]$" (resp. "$\varphi[p := t]$") means that q (resp. t) is substituted for every *free occurrence* of the propositional variable p in φ.
- In the rules $(\forall\text{E})$ and $(\exists\text{I})$, we use t as a metavariable ranging over $\{\bot, \top\} \cup \mathcal{P}$. Moreover, if t is the propositional variable $q \in \mathcal{P}$, then q must be *substitutable for* p in φ, *i.e.*, every occurrence of the substituted q must be outside the scope of a pre-existing binding quantifier, "$\forall q$" or "$\exists q$", in φ.

It takes some practice to use the quantifier rules without tripping on many pattern-matching complications. It is helpful to keep in mind informal justifications for the rules, at least the two rules with boxes:

- Informal justification for $(\forall\text{I})$: *If we can derive $\varphi[p := q]$ with a fresh propositional variable q substituted for p in φ, then we can derive $(\forall p\, \varphi)$. The crucial qualification is that q is* fresh*, i.e., it does not occur anywhere outside the box. Thus, since we assume nothing about this q, the derivation works for any propositional variable substituted for q.*
- Informal justification for $(\exists\text{E})$: *If we can derive $(\exists p\, \varphi)$, then φ must hold for at least one value. We then proceed by case analysis over possible values, writing q for a generic value representing them all. If we can derive ψ, which does not mention q, from the assumption $\varphi[p := q]$, then ψ must hold regardless of the value q stands for.*

C.3 Rules for FOL

All the rules in Subsection C.1 for *propositional logic* can be used, in addition to the four rules for quantifiers below. In fact, these four have the same exact form as the four quantifier rules in Subsection C.2, except that now the quantification is over first-order variables rather than propositional variables. The context making clear which are intended, I choose to identify them with the same four names $\{(\forall\text{I}), (\forall\text{E}), (\exists\text{I}), (\exists\text{E})\}$:

$$
\frac{\boxed{\begin{array}{ll} y & \text{fresh var} \\ \vdots & \\ \varphi[x := y] & \end{array}}}{\forall x\ \varphi}\ (\forall\mathrm{I}) \qquad\qquad \frac{\forall x\ \varphi}{\varphi[x := t]}\ (\forall\mathrm{E})
$$

$$
\frac{\varphi[x := t]}{\exists x\ \varphi}\ (\exists\mathrm{I}) \qquad\qquad \frac{\exists x\ \varphi \qquad \boxed{\begin{array}{ll} y & \text{fresh var} \\ \varphi[x := y] & \text{assumption} \\ \vdots & \\ \psi & \end{array}}}{\psi}\ (\exists\mathrm{E})
$$

The side conditions for the rules of first-order quantifiers here are nearly the same as those in Subsection C.2:

- "$\varphi[x := y]$" (resp. "$\varphi[x := t]$") means that first-order variable y (resp. term t) is substituted for every *free occurrence* of the first-order variable x in φ.
- In the rules ($\forall$E) and ($\exists$I), the term t must be *substitutable for* x in φ, *i.e.*, every variable $y \in \mathrm{FV}(t)$ must be outside the scope of a pre-existing binding quantifier, "$\forall y$" or "$\exists y$", in φ.

If we allow the equality symbol "$\approx$" in the syntax of first-order logic, then we need two additional rules: rule ($\approx$I) which introduces one occurrence of $\approx$, and rule ($\approx$E) which eliminates one occurrence of $\approx$. They read as follows:

$$
\frac{}{t \approx t}\ (\approx\mathrm{I}) \qquad\qquad \frac{t_1 \approx t_2 \qquad \varphi[x := t_1]}{\varphi[x := t_2]}\ (\approx\mathrm{E})
$$

subject to the following side conditions:

- t, t_1, t_2 range over the set of first-order terms.
- In the rule ($\approx$E), terms t_1 and t_2 must be *substitutable for* x, *i.e.*, every variable $y \in \mathrm{FV}(t_1) \cup \mathrm{FV}(t_2)$ must be outside the scope of a pre-existing binding quantifier, "$\forall y$" or "$\exists y$", in φ.

The rule ($\approx$I) guarantees that $\approx$ is reflexive. The other usual properties of equality, *symmetry* and *transitivity*, follow from ($\approx$I) and ($\approx$E), as shown in the next exercise.

Exercise 79. Alternative suggestive names for ($\approx$I) and ($\approx$E) are ($\approx$*reflexive*) and ($\approx$*congruent*), respectively. Show that both of the following rules:

$$
\frac{t_1 \approx t_2}{t_2 \approx t_1}\ (\approx\textit{symmetric})
$$

$$
\frac{t_1 \approx t_2 \qquad t_2 \approx t_3}{t_1 \approx t_3}\ (\approx\textit{transitive})
$$

are derivable from ($\approx$I) and ($\approx$E). □

Exercise 80. Show that the following rule is derivable from ($\approx$E):

$$
\frac{t_1 \approx u_1 \ \cdots \ t_n \approx u_n \qquad \varphi[x_1 := t_1, \ldots, x_n := t_n]}{\varphi[x_1 := u_1, \ldots, x_n := u_n]}
$$

Particular cases of ($\approx$E*) is when φ is $R(x_1, \ldots, x_n)$ where R is a n-ary relation symbol or when φ is $f(x_1, \ldots, x_n) \approx y$ where f is a n-ary function symbol. □

C.4 Rules for eL

The set of wff's of *equality logic* is $\mathsf{WFF}_{\mathsf{eL}}(\{\approx\}, X)$. These wff's do not include quantifiers, which implies that the rules in $\{(\forall\mathrm{I}), (\forall\mathrm{E}), (\exists\mathrm{I}), (\exists\mathrm{E})\}$ do not apply to them.

The rules of natural deduction for *equality logic* are therefore all the rules in Subsection C.1 for *propositional logic* in addition to the rules ($\approx$I) and ($\approx$E) in Subsection C.3 and the rules derived from the preceding, namely: ($\approx$ *symmetric*), ($\approx$ *transitive*), and ($\approx$E*). See Exercises 79 and 80. Keep in mind that t, t_i, and u_i, in these rules are limited to range over variables in X when used for *equality logic*.

C.5 Rules for ZOL

The set of wff's of *zeroth-order logic* is $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma, \varnothing)$ or, when $\approx$ is allowed, $\mathsf{WFF}_{\mathsf{ZOL}}(\Sigma \cup \{\approx\}, \varnothing)$. These wff's do not include quantifiers, which implies that the rules in $\{(\forall\mathrm{I}), (\forall\mathrm{E}), (\exists\mathrm{I}), (\exists\mathrm{E})\}$ do not apply to them.

The rules of natural deduction for *zeroth-order logic* are therefore all the rules in Subsection C.1 for *propositional logic* in addition to the rules ($\approx$I) and ($\approx$E) in Subsection C.3 and the rules derived from the preceding, namely: ($\approx$*symmetric*), ($\approx$*transitive*), and ($\approx$E*). Keep in mind that t, t_i, and u_i, in these rules are limited to range over variable-free first-order terms, *i.e.*, over the set $\mathsf{Atoms}(\Sigma, \varnothing)$, when used for *zeroth-order logic*.

C.6 Rules for EL and QEL (not yet completed)

C.7 Soundness

Soundness is a minimal requirement for any system of formal proofs: it means that formal *deducibility* (others say *derivability*) implies *semantic validity* (others say *logical validity* or also *truth*). We want a proof system to be as strong as possible, *i.e.*, to formally deduce as many semantically valid wff's as possible, without deriving a contradiction.

Theorem 81 (Soundness)**.** *For any of the logics $\mathcal{L} \in \{\mathsf{PL}, \mathsf{QBF}, \mathsf{eL}, \mathsf{ZOL}, \mathsf{EL}, \mathsf{QEL}, \mathsf{FOL}\}$ defined in these lecture notes, and for any set $\Gamma \cup \{\varphi\}$ of wff's in $\mathcal{L}$, it holds that:*

$$\Gamma \vdash_{\mathcal{L}} \varphi \quad \textit{implies} \quad \Gamma \models_{\mathcal{L}} \varphi.$$

In words, if φ is formally deducible from Γ, then φ is semantically entailed/implied by Γ. The proof system for $\mathcal{L}$ is thus not too strong.

Proof. For every logic $\mathcal{L}$ in these notes, the proof of soundness follows the same pattern: a straightforward (though somewhat laborious) induction on the length $\geqslant 1$ of the natural deduction for the sequent $\Gamma \vdash_{\mathcal{L}} \varphi$. We take the length of a natural deduction to be the number of lines containing a wff, with one wff per line. We do *not* include in the count a line which introduces a fresh variable, as "fresh var" in the rules ($\forall$I) and ($\forall$E); so, whenever we say "line" in this proof, we mean "wff" and we view a natural deduction as written top-down as a sequence of wff's.

We restrict attention to the case when $\mathcal{L}$ is FOL, which is the most involved logic in these notes. We thus omit the subscript "$\mathcal{L}$" on "$\vdash$" and "$\models$" in the rest of the proof and omit many of the obvious details. Given a natural deduction $\mathcal{D}$ for the sequent $\Gamma \vdash \varphi$, we say that $\mathcal{D}$ *satisfies the conclusion of the theorem* iff $\Gamma \models \varphi$.

The basis of the induction is when the natural deduction $\mathcal{D}$ for $\Gamma \vdash \varphi$ consists of a single line, *i.e.*, the single wff φ. In this case, $\varphi \in \Gamma$, which also implies that $\Gamma \models \varphi$, *i.e.*, $\mathcal{D}$ satisfies the conclusion of the theorem.

The induction proceeds by considering a natural deduction $\mathcal{D}$ with $k+1$ lines, with the induction hypothesis being that every natural deduction $\mathcal{D}'$ with at most k lines satisfies the conclusion of the theorem, where $k \geqslant 1$. If the natural deduction $\mathcal{D}$ has $k+1$ lines, we consider the proof rule according to which the last line in $\mathcal{D}$ is obtained – for each of the possible proof rules. There are the proof rules of PL, which can be used again in FOL, and there are the proof rules specifically belonging to FOL, namely, $\{(\forall\mathrm{I}), (\forall\mathrm{E}), (\exists\mathrm{I}), (\exists\mathrm{E}), (\approx\mathrm{I}), (\approx\mathrm{E})\}$.

Consider the rules inherited from PL first. So, suppose the last line in the natural deduction $\mathcal{D}$ with $k+1$ lines is obtained by applying the rule $(\wedge\mathrm{I})$. Thus, $\mathcal{D}$ is a natural deduction for a sequent of the form $\Gamma \vdash (\varphi_1 \wedge \varphi_2)$. This implies there are two natural deductions $\mathcal{D}_1$ and $\mathcal{D}_2$ whose respective last lines are φ_1 and φ_2. Let the respective sets of premises in $\mathcal{D}_1$ and $\mathcal{D}_2$ be Γ_1 and Γ_2, so that $\mathcal{D}_1$ and $\mathcal{D}_2$ are natural deductions for the sequents $\Gamma_1 \vdash \varphi_1$ and $\Gamma_2 \vdash \varphi_2$. It also follows that $\Gamma \supseteq \Gamma_1 \cup \Gamma_2$.

By the induction hypothesis, we have $\Gamma_1 \models \varphi_1$ and $\Gamma_2 \models \varphi_2$. Let $(\mathcal{A}, \sigma)$ be an interpretation such that $\mathcal{A}, \sigma \models \Gamma$, which implies that both $\mathcal{A}, \sigma \models \Gamma_1$ and $\mathcal{A}, \sigma \models \Gamma_2$, because Γ_1 and Γ_2 are subsets of Γ. Hence, both $\mathcal{A}, \sigma \models \varphi_1$ and $\mathcal{A}, \sigma \models \varphi_2$, because $\Gamma_1 \models \varphi_1$ and $\Gamma_2 \models \varphi_2$. Hence, $\mathcal{A}, \sigma \models (\varphi_1 \wedge \varphi_2)$. Hence, $\Gamma \models (\varphi_1 \wedge \varphi_2)$, as desired.

We omit the cases when the last line in the natural deduction $\mathcal{D}$ with $k+1$ lines is a wff of the form $(\varphi_1 \vee \varphi_2)$ or $(\varphi_1 \to \varphi_2)$ or $(\neg\varphi)$, which are totally similar to the case when the last line is $(\varphi_1 \wedge \varphi_2)$.

For one more rule inherited from PL, consider the case when the last line in $\mathcal{D}$, which has $k+1$ lines, is obtained by applying the rule (PBC) and showing that a sequent $\Gamma \vdash \varphi$ holds. Thus, just before the last line in $\mathcal{D}$, there is a closed box, call it B, whose first line is $\neg\varphi$ (it is an "assumption" or "local hypothesis") and whose last line is $\perp$. The rule (PCB) is invoked to close B and to write the last line of $\mathcal{D}$ which is φ. If we add $\neg\varphi$ as a premise to the entire deduction, we can open the box B (*i.e.*, remove the frame of B but not its contents!) and obtain a natural deduction with k lines for the sequent $\Gamma \cup \{\neg\varphi\} \vdash \perp$. By the induction hypothesis, $\Gamma \cup \{\neg\varphi\} \models \perp$. Since for all interpretations $(\mathcal{A}, \sigma)$ we have that $\mathcal{A}, \sigma \not\models \perp$, it follows that for all interpretations $(\mathcal{A}, \sigma)$ we also have $\mathcal{A}, \sigma \not\models \Gamma \cup \{\neg\varphi\}$, *i.e.*, $\Gamma \cup \{\neg\varphi\}$ is unsatisfiable. Hence, $\Gamma \models \varphi$, as desired.[30]

We next consider proof rules that are specific to FOL, $\{(\forall\mathrm{I}), (\forall\mathrm{E}), (\exists\mathrm{I}), (\exists\mathrm{E}), (\approx\mathrm{I}), (\approx\mathrm{E})\}$, and examine a natural deduction $\mathcal{D}$ with $k+1$ lines whose last line is obtained by applying one of those rules. We limit our examination to the case of rules $(\forall\mathrm{I})$ and $(\forall\mathrm{E})$, the case of the other rules being totally similar.

So, suppose the last line in $\mathcal{D}$ is obtained by applying rule $(\forall\mathrm{E})$ to show $\Gamma \vdash \varphi[x := t]$. Thus, the last line is the wff $\varphi[x := t]$, and the last but one line is $(\forall x\ \varphi)$. Let $\mathcal{D}'$ be the natural deduction consisting of the first k lines in $\mathcal{D}$, which establishes the sequent $\Gamma \vdash (\forall x\ \varphi)$. By the induction hypothesis, it holds that $\Gamma \models (\forall x\ \varphi)$. This means that for all interpretations $(\mathcal{A}, \sigma)$, if $\mathcal{A}, \sigma \models \Gamma$ then $\mathcal{A}, \sigma \models (\forall x\ \varphi)$, which in turn implies that $\mathcal{A}, \big(\sigma[x \mapsto a]\big) \models \varphi$ for all $a \in A$ where A is the universe of $\mathcal{A}$. In the notation of Appendix B.3, this is the same as $[\![\varphi]\!]_{\mathcal{A},\sigma[x\mapsto a]} = \textit{true}$ for all $a \in A$. Now observe that:

$$\Big\{\, [\![t]\!]_{\mathcal{A},\sigma[x\mapsto a]} \;\Big|\; a \in A \Big\} \;\subseteq\; \Big\{\, [\![x]\!]_{\mathcal{A},\sigma[x\mapsto a]} \;\Big|\; a \in A \Big\} \;=\; A.$$

Hence, $[\![\varphi[x := t]]\!]_{\mathcal{A},\sigma[x\mapsto a]} = \textit{true}$ for all $a \in A$. Hence, $\mathcal{A}, \big(\sigma[x \mapsto a]\big) \models \varphi[x := t]$, for every interpretation $(\mathcal{A}, \sigma)$ and every $a \in A$. This in turn implies $\mathcal{A}, \sigma \models \varphi[x := t]$ for every interpretation $(\mathcal{A}, \sigma)$, as desired.

Finally, consider the case when the last line in $\mathcal{D}$ is obtained by applying rule $(\forall\mathrm{I})$ to show $\Gamma \vdash (\forall x\ \varphi)$. Thus, the last line is the wff $(\forall x\ \varphi)$. We invoke $(\forall\mathrm{I})$ to close a box, call it B. B starts with a fresh variable y (which

[30] For additional details for this last step, see Lemma 6 and its proof.

does not count as a separate line in $\mathcal{D}$) and ends with the wff $\varphi[x := y]$ where y occurs free. We therefore have $\Gamma \vdash \varphi[x := y]$. By the induction hypothesis, $\Gamma \models \varphi[x := y]$, *i.e.*, for every interpretation $(\mathcal{A}, \sigma)$ it holds that if $\mathcal{A}, \sigma \models \Gamma$ then $\mathcal{A}, \sigma \models \varphi[x := y]$. Equivalently, for every $a \in A$, if $\mathcal{A}, \big(\sigma[x \mapsto a]\big) \models \Gamma$ then $\mathcal{A}, \big(\sigma[x \mapsto a]\big) \models \varphi[x := y]$. Since y does not occur in Γ, we also have for every $(\mathcal{A}, \sigma)$, if $\mathcal{A}, \sigma \models \Gamma$ then for every $a \in A$, it holds that $\mathcal{A}, \big(\sigma[x \mapsto a]\big) \models \varphi[x := t]$. Hence, $\Gamma \models (\forall x\ \varphi)$, as desired. $\square$

D De Morgan's Laws: Semantically and Proof-Theoretically

De Morgan's Laws can be asserted as four semantically valid wff's:

1. $\models \neg(p \vee q) \quad \rightarrow (\neg p \wedge \neg q)$
2. $\models (\neg p \wedge \neg q) \rightarrow \neg(p \vee q)$
3. $\models (\neg p \vee \neg q) \rightarrow \neg(p \wedge q)$
4. $\models \neg(p \wedge q) \quad \rightarrow (\neg p \vee \neg q)$

Their semantic validity can be established using *truth tables*. For example, for the first and fourth laws we can write the following tables:

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$(\neg p \wedge \neg q)$	$\neg(p \vee q) \rightarrow (\neg p \wedge \neg q)$
false	*false*	*false*	*true*	*true*	*true*	*true*	*true*
false	*true*	*true*	*false*	*true*	*false*	*false*	*true*
true	*false*	*true*	*false*	*false*	*true*	*false*	*true*
true	*true*	*true*	*false*	*false*	*false*	*false*	*true*

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$(\neg p \vee \neg q)$	$\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$
false	*false*	*false*	*true*	*true*	*true*	*true*	*true*
false	*true*	*false*	*true*	*true*	*false*	*true*	*true*
true	*false*	*false*	*true*	*false*	*true*	*true*	*true*
true	*true*	*true*	*false*	*false*	*false*	*false*	*true*

The two leftmost columns in the two tables list all possible truth assignments to the pair (p, q). The rightmost column in the first table assigns a truth-value to the wff $\neg(p \vee q) \rightarrow (\neg p \wedge \neg q)$ for each of the assignments of (p, q), and since every entry in the rightmost column is *true*, the wff is semantically valid. And similarly for the rightmost column in the second table, which establishes the semantic validity of the wff $\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$.

Exercise 82. Write the truth tables for the remaining de Morgan's Laws: 2 and 3, to show that they are all semantically valid. □

De Morgan's Laws can also be asserted in the form of four formally deducible sequents according to the proof rules in Section C.1:

1. $\vdash \neg(p \vee q) \quad \rightarrow (\neg p \wedge \neg q)$
2. $\vdash (\neg p \wedge \neg q) \rightarrow \neg(p \vee q)$
3. $\vdash (\neg p \vee \neg q) \rightarrow \neg(p \wedge q)$
4. $\vdash \neg(p \wedge q) \quad \rightarrow (\neg p \vee \neg q)$

Below are natural-deduction formal proofs for the first and fourth de Morgan's Laws:

1	$\neg(p \vee q)$	assume
2	p	assume
3	$p \vee q$	$\vee$I 2
4	$\bot$	$\neg$E $1, 3$
5	$\neg p$	$\neg$I 2-4
6	q	assume
7	$p \vee q$	$\vee$I 6
8	$\bot$	$\neg$E $1, 7$
9	$\neg q$	$\neg$I 6-8
10	$\neg p \wedge \neg q$	$\wedge$I $5, 9$
11	$\neg(p \vee q) \rightarrow (\neg p \wedge \neg q)$	$\rightarrow$I 1-10

1	$\neg(p \wedge q)$	assume
2	$\neg(\neg p \vee \neg q)$	assume
3	$\neg p$	assume
4	$(\neg p \vee \neg q)$	$\vee$I 3
5	$\bot$	$\neg$E $2, 4$
6	$\neg\neg p$	$\neg$I 3-5
7	$\neg q$	assume
8	$\neg p \vee \neg q$	$\vee$I 7
9	$\bot$	$\neg$E $2, 8$
10	$\neg\neg q$	$\neg$I 7-9
11	p	$\neg\neg$E 6
12	q	$\neg\neg$E 10
13	$p \wedge q$	$\wedge$I $11, 12$
14	$\bot$	$\neg$E $1, 13$
15	$\neg\neg(\neg p \vee \neg q)$	$\neg$I 2-14
16	$(\neg p \vee \neg q)$	$\neg\neg$E 15
17	$\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$	$\rightarrow$I 1-16

Remark: Our formal proof of de Morgan's Law 1 does not use any rule in $\{$(LEM), (PBC), $(\neg\neg$E$)$, (Peirce's)$\}$, in contrast to our formal proof of de Morgan's Law 4, which uses $(\neg\neg$E$)$ twice. It turns out that any formal proof for de Morgan's Law 4 must use one of the rules in $\{$(LEM), (PBC), $(\neg\neg$E$)$, (Peirce's)$\}$; this is not a trivial result and requires a deeper examination of formal-proof systems (not in these lecture notes).

See Section C.1.3 and Exercise 77 for an explanation of how the rules in $\{$(LEM), (PBC), $(\neg\neg$E$)$, (Peirce's)$\}$ are related.

Exercise 83. Write natural-deduction proofs for the remaining de Morgan's Laws: 2 and 3, to show that they are all formally derivable. For full credit, avoid using any of the four rules in $\{$(LEM), (PBC), $(\neg\neg$E$)$, (Peirce's)$\}$. *Hint*: For de Morgan's Laws 2 and 3 this is possible, though it may require a little more care. $\square$

From the preceding exercise and earlier remark, de Morgan's Law 4 has a special status: the first three de Morgan's Laws are valid intuitionistically, while de Morgan's Law 4 is not.

It should be clear by now that writing formal proofs is a tedious task, generally requiring many failed attempts to add a line in the deduction and causing as many backtrackings. The search for a legal deduction, *i.e.*, a deduction which is produced according to the proof rules, is therefore a process of repeated backtrackings in general. If this process does not terminate, it may be because our starting premises do not in fact imply our conjectured conclusion – not because we have not tried hard enough. More examples and exercises of natural-deduction proofs are in Appendix E.

E Prenex Form and Skolemization

The process of transforming a wff with quantifiers into its *prenex form*, and the additional process of *skolemizing* it, applies equally well to quantified Boolean wff's and first-order wff's. The transformation of the two kinds of wff's being entirely similar, we restrict our presentation to first-order wff's.

E.1 Prenex Form

A first-order wff $\varphi \in \mathsf{WFF}_{\mathsf{FOL}}(\Sigma \cup \{\approx\}, X)$ is in *prenex form* (or *prenex normal form*) iff φ consists of a (possibly empty) string of quantifiers followed by a quantifier-free wff. The string of quantifiers in φ is its *prefix* and the quantifier-free subformual of φ is its *matrix*.

We call our transformation "$\boxed{\mathsf{prenex}}$", and the result of applying it to an arbitrary wff φ is denoted "$\boxed{\mathsf{prenex}}(\varphi)$". In what follows, we use Q, possibly subscripted, to range over $\{\forall, \exists\}$. Moreover, if Q is $\forall$ (resp. $\exists$), then $\overline{Q}$ is $\exists$ (resp. $\forall$), *i.e.*, $\overline{\forall}$ denotes $\exists$ and $\overline{\exists}$ denotes $\forall$. For an arbitrary first-order wff φ, the wff $\boxed{\mathsf{prenex}}(\varphi)$ is therefore of the form $(Q_1 x_1 \cdots Q_n x_n .\, \psi)$ where $Q_1, \ldots, Q_n \in \{\forall, \exists\}$ and ψ is quantifier-free.

The definition of $\boxed{\mathsf{prenex}}(\varphi)$ is by structural induction on φ. This can be done in one of two ways: *top-bottom* (as in the definition of $\boxed{\mathsf{QBF} \mapsto \mathsf{PL}}$ in the proof of Lemma 19), or *bottom-up*. In *top-bottom*, we start with φ fully given and we think of $\boxed{\mathsf{prenex}}$ as being pushed down recursively through the sub-wff's of φ. In *bottom-up*, we define $\boxed{\mathsf{prenex}}(\varphi)$ simultaneously with φ, as the latter is being built up inductively. Our induction here is *bottom-up*, which is a bit simpler:

1. If φ is quantifier-free, then $\boxed{\mathsf{prenex}}(\varphi) \stackrel{\text{def}}{=} \varphi$.

2. If $\varphi \stackrel{\text{def}}{=} (\neg\psi)$ and $\boxed{\mathsf{prenex}}(\psi) = (Q_1 x_1 \cdots Q_n x_n .\, \theta)$ where θ is quantifier-free, then:

$$\boxed{\mathsf{prenex}}(\varphi) \stackrel{\text{def}}{=} \big(\overline{Q}_1 x_1 \cdots \overline{Q}_n x_n .\, \neg\theta\big).$$

   In the next two cases, let:

$$\boxed{\mathsf{prenex}}(\varphi_1) = (Q_1 y_1 \cdots Q_n y_n .\, \theta_1),$$

$$\boxed{\mathsf{prenex}}(\varphi_2) = (Q_1 z_1 \cdots Q_p z_p .\, \theta_2),$$

   where θ_1 and θ_2 are quantifier-free. By renaming bound variables in $\boxed{\mathsf{prenex}}(\varphi_1)$ and $\boxed{\mathsf{prenex}}(\varphi_2)$, we can assume that the variables in $\{y_1, \ldots, y_n, z_1, \ldots, z_p\}$ are all distinct and that:

$$\{y_1, \ldots, y_n\} \cap \mathrm{FV}\big(\boxed{\mathsf{prenex}}(\varphi_2)\big) = \varnothing \quad \text{and} \quad \{z_1, \ldots, z_p\} \cap \mathrm{FV}\big(\boxed{\mathsf{prenex}}(\varphi_1)\big) = \varnothing.$$

3. If $\varphi \stackrel{\text{def}}{=} (\varphi_1 \diamond \varphi_2)$ where $\diamond \in \{\wedge, \vee\}$, then:

$$\boxed{\mathsf{prenex}}(\varphi) \stackrel{\text{def}}{=} \big(Q_1 y_1 \cdots Q_n y_n\; Q_1 z_1 \cdots Q_p z_p .\, (\theta_1 \diamond \theta_2)\big).$$

4. If $\varphi \stackrel{\text{def}}{=} (\varphi_1 \to \varphi_2)$, then:

$$\boxed{\mathsf{prenex}}(\varphi) \;\stackrel{\text{def}}{=}\; \big(\; \overline{Q}_1 y_1 \cdots \overline{Q}_n y_n\; Q_1 z_1 \cdots Q_p z_p .\, (\theta_1 \to \theta_2)\; \big)$$

5. If $\varphi \stackrel{\text{def}}{=} (Qx.\,\psi)$ where $Q \in \{\forall, \exists\}$, then:

$$\boxed{\mathsf{prenex}}(\varphi) \stackrel{\text{def}}{=} \big(Qx.\,\boxed{\mathsf{prenex}}(\psi)\big).$$

We can show, for an arbitrary first-order wff φ, that φ and $\boxed{\mathsf{prenex}}(\varphi)$ are equivalent in one of two ways:

- semantically, *i.e.*, $\models\ \big(\varphi \to \boxed{\mathsf{prenex}}(\varphi)\big)\ \wedge\ \big(\boxed{\mathsf{prenex}}(\varphi) \to \varphi\big)$, or
- proof-theoretically, *i.e.*, $\vdash\ \big(\varphi \to \boxed{\mathsf{prenex}}(\varphi)\big)\ \wedge\ \big(\boxed{\mathsf{prenex}}(\varphi) \to \varphi\big)$.

Either way, we can follow the bottom-up induction which we used to define φ and $\boxed{\mathsf{prenex}}(\varphi)$ simultaneously. All we need for this are Lemma 84 and Lemma 91.

Lemma 84. *Let φ be an arbitrary first-order wff. Then:*

1. $\neg(\exists x.\,\varphi)$ *and* $\big(\forall x.\,\neg\varphi\big)$ *are equivalent wff's.*
2. $\neg(\forall x.\,\varphi)$ *and* $\big(\exists x.\,\neg\varphi\big)$ *are equivalent wff's.*

Proof. We give formal natural-deduction proofs, and we ask you to give (much easier) semantic proofs in Exercise 85. For part 1, it suffices to show (why?):

$\neg\exists x.\ \varphi(x)\ \vdash\ \forall x.\ \neg\varphi(x)$ instead of $\vdash\ \big(\neg\exists x.\ \varphi(x)\big) \to \big(\forall x.\ \neg\varphi(x)\big)$, and
$\forall x.\ \neg\varphi(x)\ \vdash\ \neg\exists x.\ \varphi(x)$ instead of $\vdash \forall x.\ \neg\varphi(x)\ \to\ \neg\exists x.\ \varphi(x)$.

1.	$\neg\exists x\ \varphi(x)$	premise
2.	y	fresh variable
3.	$\varphi(y)$	assumption
4.	$\exists x\ \varphi(x)$	∃I 3
5.	$\bot$	¬E 1, 4
6.	$\neg\varphi(y)$	¬I 3–5
7.	$\forall x\ \neg\varphi(x)$	∀I 2–6

1.	$\forall a\ \neg\varphi(a)$	premise
2.	$\exists a\ \varphi(a)$	assumption
3.	a_0	fresh variable
4.	$\varphi(a_0)$	assumption
5.	$\neg\varphi(a_0)$	∀E 1
6.	$\bot$	¬E 4,5
7.	$\bot$	∃E 2, 3–6
8.	$\neg\exists a\ \varphi(a)$	PBC 2–7

The natural deduction on the left says "$\neg\exists x.\ \varphi(x)\ \vdash\ \forall x.\ \neg\varphi(x)$", and the natural deduction on the right says "$\forall x.\ \neg\varphi(x)\ \vdash\ \neg\exists x.\ \varphi(x)$". For part 2 of the lemma, we can write the following:

1.	$\exists x\ \neg\varphi(x)$	premise
2.	y	fresh variable
3.	$\neg\varphi(y)$	assumption
4.	$\forall x\ \varphi(x)$	assumption
5.	$\varphi(y)$	∀E 4
6.	$\bot$	¬E 3, 5
7.	$\neg\forall x\ \varphi(x)$	¬I 4–6
8.	$\neg\forall x\ \varphi(x)$	∃E 1, 2–7

1.	$\neg\forall x\ \varphi(x)$	premise
2.	$\neg\exists x\ \neg\varphi(x)$	assumption
3.	y	fresh variable
4.	$\neg\varphi(y)$	assumption
5.	$\exists x\ \neg\varphi(x)$	∃I 4
6.	$\bot$	¬E 2, 5
7.	$\varphi(y)$	PBC 4–6
8.	$\forall x\ \varphi(x)$	∀I 3–7
9.	$\bot$	¬E 1, 8
10.	$\exists x\ \neg\varphi(x)$	PBC 2–9

From the left, we conclude $\big(\exists x.\, \neg\varphi(x)\big) \vdash \neg\big(\forall x.\, \varphi(x)\big)$, and from the right, $\neg\big(\forall x.\, \varphi(x)\big) \vdash \big(\exists x.\, \neg\varphi(x)\big)$. $\square$

Exercise 85. Write semantic proofs for the two equivalences in Lemma 84, noting that each equivalence consists of two implications. For example, for the first equivalence, you have to show both of the following:

- $\mathcal{A} \models \neg(\exists x.\, \varphi) \to \big(\forall x.\, \neg\varphi\big)$,
- $\mathcal{A} \models \big(\forall x.\, \neg\varphi\big) \to \neg(\exists x.\, \varphi)$,

where $\mathcal{A}$ is an arbitrary Σ-structure and, for simplicity, you can assume $\mathrm{FV}(\varphi) = \{x\}$. Do the same for the second equivalence in Lemma 84. $\square$

As a warm-up for the proof of Lemma 91 and the exercises following it, you may try the following examples. They all involve natural deductions showing that a sequent of the form $\varphi \vdash \psi$ holds; we omit the (typically easier) proof of the corresponding semantic validity $\varphi \models \psi$.

Example 86. We write two natural deductions showing that: $\neg\varphi \vee \psi \vdash \varphi \to \psi$ and $\varphi \to \psi \vdash \neg\varphi \vee \psi$.

1.	$\neg\varphi \vee \psi$	premise
2.	$\neg\varphi$	assumption
3.	φ	assumption
4.	$\bot$	$\neg$E 2, 3
5.	ψ	$\bot$E 4
6.	$\varphi \to \psi$	$\to$I 3–5
7.	ψ	assumption
8.	φ	assumption
9.	ψ	repeat 7
10.	$\varphi \to \psi$	$\to$I 8–9
11.	$\varphi \to \psi$	$\vee$E 1, 2–6, 7–10

1.	$\varphi \to \psi$	premise
2.	$\varphi \vee \neg\varphi$	LEM
3.	$\neg\varphi$	assumption
4.	$\neg\varphi \vee \psi$	$\vee$I$_1$ 3
5.	φ	assumption
6.	ψ	$\to$E 1, 5
7.	$\neg\varphi \vee \psi$	$\vee$I$_2$ 6
8.	$\neg\varphi \vee \psi$	$\vee$E 2, 3–4, 5–7

The two preceding natural deductions show that $(\varphi \to \psi)$ and $(\neg\varphi \vee \psi)$ are equivalent wff's. Note that the deduction on the left does not use any of the four rules in $\{$(LEM), (PBC), ($\neg\neg$E), (Peirce's)$\}$ which is therefore legal intuitionistically, whereas the deduction on the right uses (LEM). It can be shown (not in these notes) that it is not possible to write a deduction for $(\varphi \to \psi) \to (\neg\varphi \vee \psi)$ without invoking one of those four rules. $\square$

Example 87. Permuting two adjacent universal quantifiers does not change the meaning of a wff, *i.e.*, the following sequent holds: $\forall x\, \forall y\, \varphi(x, y) \;\vdash\; \forall y\, \forall x\, \varphi(x, y)$, as confirmed by the following natural deduction.

1.	$\forall x\, \forall y\, \varphi(x, y)$	premise
2.	y_0	fresh y_0
3.	x_0	fresh x_0
4.	$\forall y\, \varphi(x_0, y)$	$\forall$E 1
5.	$\varphi(x_0, y_0)$	$\forall$E 4
6.	$\forall x\, \varphi(x, y_0)$	$\forall$I 3–5
7.	$\forall y\, \forall x\, \varphi(x, y)$	$\forall$I 2–6

where only the rules ($\forall$E) and ($\forall$I) are used. $\square$

Example 88. An existential quantifier can be distributed over a logical or "$\vee$", *i.e.*, the following sequent holds: $\exists x\ (\varphi(x) \vee \psi(x)) \ \vdash\ \exists x\ \varphi(x) \ \vee\ \exists x\ \psi(x)$, as confirmed by the following natural deduction:

1.	$\exists x\ (\varphi(x) \vee \psi(x))$	premise
2.	x_0	fresh variable
3.	$\varphi(x_0) \vee \psi(x_0)$	assumption
4.	$\varphi(x_0)$	assumption
5.	$\exists x\ \varphi(x)$	$\exists$I 4
6.	$\exists x\ \varphi(x) \vee \exists x\ \psi(x)$	$\vee$I 5
7.	$\psi(x_0)$	assumption
8.	$\exists x\ \psi(x)$	$\exists$I 7
9.	$\exists x\ \varphi(x) \vee \exists x\ \psi(x)$	$\vee$I 8
10.	$\exists x\ \varphi(x) \vee \exists x\ \psi(x)$	$\vee$E 3, 4–6, 7–9
11.	$\exists x\ \varphi(x) \vee \exists x\ \psi(x)$	$\exists$E 1, 2–10

where only rules for '$\vee$' and '$\exists$' are used, both for introduction and elimination. $\square$

Exercise 89. Write a natural deduction to establish the converse of the sequent in Example 88, to formally prove that the following sequent holds: $\exists x\ \varphi(x) \ \vee\ \exists x\ \psi(x) \ \vdash\ \exists x\ (\varphi(x) \vee \psi(x))$. This shows that we can "push" to the "left" existential quantifiers out of the scope of a "$\vee$" immediately preceding them. $\square$

Exercise 90. Read Example 88 and do Exercise 89 before attempting this exercise. Write natural deductions to establish the two following sequents:

- $\forall x\ \varphi(x) \ \wedge\ \forall x\ \psi(x) \ \vdash\ \forall x\ (\varphi(x) \wedge \psi(x))$.
- $\forall x\ (\varphi(x) \wedge \psi(x)) \ \vdash\ \forall x\ \varphi(x) \ \wedge\ \forall x\ \psi(x)$.

We can "push" to the "left" universal quantifiers out of the scope of a "$\wedge$" immediately preceding them. $\square$

Lemma 91. *Let φ and ψ be arbitrary first-order wff's, such that $x \notin \mathrm{FV}(\psi)$. Then:*

1. $\big((\forall x\, \varphi) \wedge \psi\big)$ *and* $\big(\forall x\, (\varphi \wedge \psi)\big)$ *are equivalent wff's.*
2. $\big((\exists x\, \varphi) \wedge \psi\big)$ *and* $\big(\exists x\, (\varphi \wedge \psi)\big)$ *are equivalent wff's.*
3. $\big((\forall x\, \varphi) \vee \psi\big)$ *and* $\big(\forall x\, (\varphi \vee \psi)\big)$ *are equivalent wff's.*
4. $\big((\exists x\, \varphi) \vee \psi\big)$ *and* $\big(\exists x\, (\varphi \vee \psi)\big)$ *are equivalent wff's.*
5. $\big((\forall x\, \varphi) \to \psi\big)$ *and* $\big(\exists x\, (\varphi \to \psi)\big)$ *are equivalent wff's.*
6. $\big((\exists x\, \varphi) \to \psi\big)$ *and* $\big(\forall x\, (\varphi \to \psi)\big)$ *are equivalent wff's.*
7. $\big(\psi \to (\forall x\, \varphi)\big)$ *and* $\big(\forall x\, (\psi \to \varphi)\big)$ *are equivalent wff's.*
8. $\big(\psi \to (\exists x\, \varphi)\big)$ *and* $\big(\exists x\, (\psi \to \varphi)\big)$ *are equivalent wff's.*

In parts $\{1, 2, 3, 4\}$, we omit the cases when the two components of the logical connectives are permuted, as in $\big(\psi \wedge (\exists x\, \varphi)\big)$ instead of $\big((\exists x\, \varphi) \wedge \psi\big)$, because "$\wedge$" and "$\vee$" are commutative binary connectives.

Proof. Left as an exercise, which should be straightforward after studying Examples 86, 87, and 88, and doing Exercises 89 and 90. $\square$

Exercise 92. Prove each of the four odd-numbered (or the four even-numbered) equivalences in Lemma 91 twice: once by writing natural deductions (more tedious), and once by providing rigorous semantic arguments (simpler and easier). $\square$

Proposition 93. *Let φ be an arbitrary first-order wff and $\psi \stackrel{\text{def}}{=} \boxed{\text{prenex}}(\varphi)$. Then φ and ψ are equivalent wff's.*

Proof. We repeat the *bottom-up* induction that defines φ and $\boxed{\text{prenex}}(\varphi)$ simultaneously. But now, at every step of the induction, we also show that φ and $\boxed{\text{prenex}}(\varphi)$ are equivalent wff's. At step 2 of the induction, you need to us Lemma 84 repeatedly to "move" quantifiers to the left past the logical negation "$\neg$". At steps 3 and 4, you need to use Lemma 91 to "move" quantifiers outside the logical connectives "$\wedge$", "$\vee$", and "$\to$". All obvious details omitted. $\square$

E.2 Skolem Form

Let ψ be a first-order wff in prenex form. Again here, as in Section E.1, ψ may be a quantified Boolean wff or a first-order wff. We limit our examination to the first-order case, the case of quantified Boolean wff's being totally similar.

The *Skolemization* of ψ produces another first-order wff, call it θ, in prenex form where the prefix of quantifiers mentions only the universal "$\forall$". Our name for the transformation from ψ to θ is "$\boxed{\text{skolem}}$". The wff θ is obtained by initially setting θ to ψ and then repeatedly applying the following three-step sequence to it:[31]

1. Find the leftmost $\exists$ in the quantifier prefix of ψ, which binds a variable x and appears as "$\exists\, x$",
2. Introduce a fresh function symbol f_x of arity equal to the number of $\forall$'s to the left of "$\exists\, x$",
3. If the $\forall$'s to the left of "$\exists\, x$" are "$\forall y_1 \cdots \forall y_n$", then cross out "$\exists\, x$" from the quantifier prefix and replace all occurrences of x in the matrix of ψ by the term $f_x(y_1, \ldots, y_n)$.

This process is bound to terminate because the initial prefix of quantifiers in ψ has finite length. We denote the resulting θ by writing "$\boxed{\text{skolem}}(\psi)$", and refer to it as the *Skolem form* of ψ.

Note that there are as many new fresh function symbols "f_x" in θ as there are existential quantifiers "$\exists\, x$" in the prefix of the initial wff ψ in prenex form. These fresh function symbols are called *Skolem functions*. Note also that if the leftmost "$\exists\, x$" in the initial ψ is not preceded by any $\forall$, the associated Skolem function f_x has arity $= 0$, *i.e.*, f_x is a constant symbol.

If φ is an arbitrary first-order wff, not necessarily in prenex form, then we write $\boxed{\text{sko,pre}}(\varphi)$ to denote the two-stage transformation of φ – first, into prenex form and, second, into Skolemized form – and we also call $\boxed{\text{sko,pre}}(\varphi)$ the Skolemization of φ.

While φ and $\boxed{\text{prenex}}(\varphi)$ are logically equivalent ("they say the same thing"), it does not make sense to talk about the equivalence (or non-equivalence) of φ and $\boxed{\text{sko,pre}}(\varphi)$ because the signature of the latter is different from the signature of φ. Nevertheless, we have the following result. Recall that a *sentence* φ is a closed formula, *i.e.*, $\text{FV}(\varphi) = \varnothing$.

[31] The words *Skolemize* and *Skolemization* are derived from the name of the mathematical logician Thoralf Skolem. If you want to find out more about the many uses of *Skolemization*, click here.

Proposition 94. *Let φ and Γ be an arbitrary first-order sentence and set of first-order sentences. We then have:*

1. *φ is satisfiable iff* $\boxed{\textsf{sko,pre}}(\varphi)$*,*
2. *Γ is satisfiable iff* $\boxed{\textsf{sko,pre}}(\Gamma)$*.*

In Part 2, we have to be careful that, when we Skolemize distinct wff's φ_1 and φ_2 of Γ, we introduce distinct Skolem functions for each wff, i.e., the Skolem functions of φ_1 do not interfere with the Skolem functions of φ_2.

Proof. We leave the proof of Part 2 as an easy exercise implied by Part 1. For Part 1, we can assume that φ is already in prenex form, by Proposition 93. It suffices to show how the elimination of the leftmost existential quantifier from the prefix of φ produces another prenex form, say θ, which is equisatisfiable with φ, and then the same process can be repeated for the elimination of all the other existential quantifers in the prefix of φ. Let then φ be of the form:

$$\varphi \stackrel{\text{def}}{=} \forall x_1 \cdots \forall x_n \exists y\, \varphi_0$$

where $n \geqslant 0$ and φ_0 is a a prenex form such that $\text{FV}(\varphi_0) \subseteq \text{FV}(\varphi) \cup \{x_1, \ldots, x_n, y\}$ and, because φ is closed, in fact $\text{FV}(\varphi_0) = \{x_1, \ldots, x_n, y\}$. According to the Skolemization process, θ is of the form:

$$\theta \stackrel{\text{def}}{=} \forall x_1 \cdots \forall x_n \,\big(\varphi_0[y := f_y(x_1, \ldots, x_n)]\big)$$

where f_y is a fresh n-ary function symbol. Σ and $\Sigma' \stackrel{\text{def}}{=} \Sigma \cup \{f_y\}$ are the signatures of φ and θ, respectively.

Let $\mathcal{A}$ be a Σ-structure. The expansion $\mathcal{A}' \stackrel{\text{def}}{=} (\mathcal{A}, f_y^{\mathcal{A}'})$ of $\mathcal{A}$ is a Σ'-structure. Let A be the universe of $\mathcal{A}$, which is also the universe of $\mathcal{A}'$. If $\mathcal{A}' \models \theta$, it is easy to check that $\mathcal{A} \models \varphi$. Hence, if θ is satisfiable, so is φ.

Conversely, let $\mathcal{A} \models \varphi$ and let $\sigma : X \to A$ be an arbitrary valuation where A is the universe of $\mathcal{A}$. We construct a Σ'-structure $\mathcal{A}'$ by expanding $\mathcal{A}$ so that for every $a_1, \ldots, a_n \in A$, the function $f_y^{\mathcal{A}'}$ maps $(a_1, \ldots, a_n)$ to b where:[32]

$$\mathcal{A},\ \big(\sigma[x_1 \mapsto a_1, \ldots, x_n \mapsto a_n, y \mapsto b]\big)\ \models\ \ \varphi_0.$$

We choose the interpretation $f_y^{\mathcal{A}'}$ of the Skolem function f_y precisely so that the preceding satisfaction holds. It is now easy to check that $\mathcal{A}' \models \theta$. Hence, if φ is satisfiable, then so is θ. □

Exercise 95. What goes wrong in the proof of Proposition 94 if φ is an open wff?

Hint: Try the open wff $\varphi(y) \stackrel{\text{def}}{=} \exists! v \forall w \big(R(a, w) \wedge R(v, w)\big) \to \exists x \big(R(a, y) \wedge R(x, y)\big)$, where "$\exists!$" means "there exists exactly one", R is a binary relation symbol and a is a constant symbol. Show that $\models \varphi(y)$, but the construction in the proof of Proposition 94 produces an open wff $\theta(y)$ not satisfied by any structure $\mathcal{A}$, unless we introduce additional constraints at the meta-level on $\mathcal{A}$. □

Exercise 96. Let R be a binary relation symbol and f a unary function symbol.

1. Show that the sentence $\varphi \stackrel{\text{def}}{=} \forall x\, R\big(x, f(x)\big) \to \forall x \exists y\, R(x, y)$ is valid. Do it in two different ways:
   (a) proof-theoretically, $\vdash \varphi$, using natural deduction, and
   (b) semantically, $\models \varphi$.

[32] Review the definition of $\sigma[x \mapsto a]$ in Section B.3.

2. Show that the sentence $\psi \stackrel{\text{def}}{=} \forall x \exists y\, R(x,y) \rightarrow \forall x\, R\big(x, f(x)\big)$ is not valid. Note that ψ is just the converse implication of φ.

   Hint: Try a semantic approach, *i.e.*, show $\not\models \psi$. You need to define a structure $\mathcal{A}$ so that the left-hand side of "$\rightarrow$" in ψ is true in $\mathcal{A}$ but the right-hand side of "$\rightarrow$" is false in $\mathcal{A}$.

3. Conclude that $\forall x \exists y\, R(x,y)$ and $\forall x\, R\big(x, f(x)\big)$ are not equivalent first-order wff's.

Remark: Despite the conclusion in part 3, Proposition 94 asserts that $\forall x \exists y\, R(x,y)$ and $\forall x\, R\big(x, f(x)\big)$ are equisatisfiable, *i.e.*, if there is a model for one, then there is a model for the other, and vice-versa. $\square$

F Alternative Proofs of Compactness

We present two alternative proofs of Compactness for *propositional logic*. At bottom, these are not "new" proofs, but different presentations of the same fundamental idea (or topological core, if you will) underlying the proof of Theorem 2 in Section 1. This fundamental idea is what *König's Lemma* asserts. The difference here is that they make the connection with topology a little more explicit by naming and presenting the same key concepts differently. One can read the first alternative proof as an elaboration of the proof in Section 1, and the second alternative proof as an elaboration of the first.[33]

Lemma 97 (König's Lemma)**.** *Every infinite, finitely branching, tree $\mathcal{T}$ has an infinite path.*

Proof. Using induction, we define an infinite sequence of nodes $\alpha_0, \alpha_1, \ldots$, forming an infinite path in $\mathcal{T}$. At stage 0 of the induction, let α_0 be the root node of $\mathcal{T}$, which has infinitely many successors by the hypothesis that $\mathcal{T}$ is infinite. At every stage $n \geqslant 1$, assume we have already selected nodes $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ so far, forming a path of length $(n-1)$, such that α_{n-1} has infinitely many successors. By hypothesis, $\mathcal{T}$ is finitely branching, which implies α_{n-1} has only finitely many immediate successors. Hence, one of the immediate successors of α_{n-1}, say β, must have infinitely many successors. Define α_n to be β, which has infinitely many successors in $\mathcal{T}$, and proceed to stage $n+1$ of the induction. $\square$

The preceding proof is not constructive: We do not have an algorithm to select the next node β at stage n after having already selected nodes $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$. We only know that one of the immediate successors of α_{n-1} is the root node of an infinite subtree; we know that it exists, but we do not know which one it is. So, at every stage we invoke what is called the Axiom of Choice to select the next node β.

For the next alternative proof of Compactness for PL, we specialize *König's Lemma* (KL) to the case of binary trees, where every node has exactly two successors. The *full binary tree* is a tree without leaf nodes, which therefore has $2^{\aleph_0}$ distinct infinite paths. Every binary tree can be viewed as an initial fragment of the full binary tree, *i.e.*, by inserting a copy of the full binary tree at every leaf node of the former.

Another way of stating KL relative to binary trees is to say: *If a binary tree has arbitrarily long full finite paths, then it has an infinite path*, which is the form we use in the next proof. By a "full finite path" we mean a path that starts at the root node and ends at a leaf node. This form of KL specialized to binary trees is sometimes called *Weak König's Lemma* (WKL).

The next exercise is a little application of WKL, which has a distinctly topological flavor.

Exercise 98 (*Sequential Compactness*)**.** We write $A = [0, 1]$ for the closed interval of all real numbers between 0 and 1. Let $\boldsymbol{a} \stackrel{\text{def}}{=} (a_n \mid n \in \mathbb{N})$ be an infinite sequence of elements in A. Show there is an infinite subsequence $\boldsymbol{a}'$ of $\boldsymbol{a}$, say $\boldsymbol{a}' \stackrel{\text{def}}{=} (a_k \mid k \in K)$ where $K \subseteq \mathbb{N}$, such that $\boldsymbol{a}'$ converges to an element $b \in A$. (Take the elements of the sequence $\boldsymbol{a}$ and subsequence $\boldsymbol{a}'$ to be listed in order of increasing indices.)

Alternative Proof I of Theorem 2 (Compactness for Propositional Logic). As in the earlier proof of Theorem 2 in Section 1, we only need to consider the non-trivial implication "$\Leftarrow$": If Γ is finitely satisfiable, then Γ is satisfiable.

[33] And there are still other proofs with a decidedly algebraic or topological content. A particular construction nicely complementing the material in this appendix is Łoś's Theorem which proves Compactness using what are called *ultrafilters* and *ultraproducts*. Search the Web for "propositional compactness via ultraproducts" and "first-order compactness via ultraproducts".

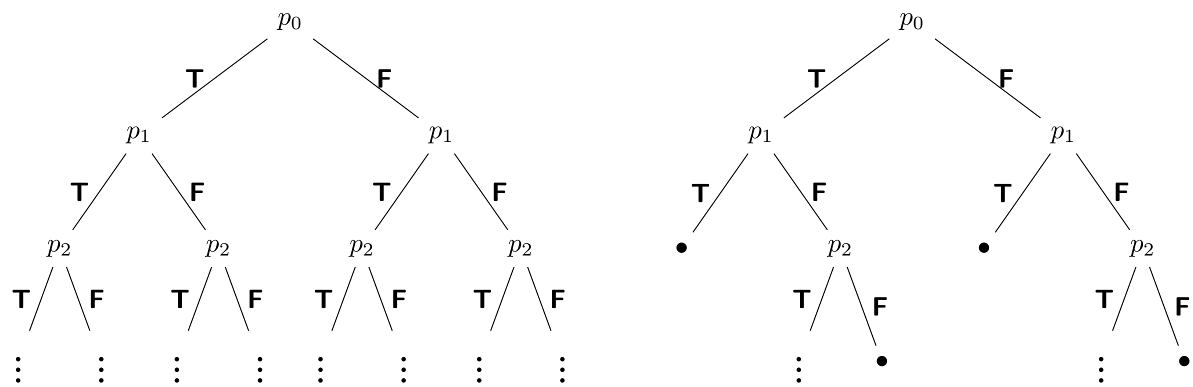

Figure 9: *On the left*: The top three levels of the full binary tree $\mathcal{T}_{\text{full}}$, with each left/right edge labelled T/F. *On the right*: An example of how $\mathcal{T}_{\text{full}}$ is pruned if p_0 occurs nowhere in Γ (and therefore has no effect on the satisfiability of Γ) and Γ contains the wff $(\neg p_1 \wedge p_2)$. Starting from the root, if a path reaches a leaf node "•", the corresponding truth assignment is doomed to falsify Γ.

The set $\mathcal{P}$ of propositional variables is countably infinite: $\{p_0, p_1, p_2, \ldots\}$. Assume a fixed ordering of $\mathcal{P}$, in the order of their indices $0, 1, 2, \ldots$. We view a truth assignment $\sigma : \mathcal{P} \to \{\mathsf{T}, \mathsf{F}\}$ as defining an infinite path in the full binary tree, call it $\mathcal{T}_{\text{full}}$, by using T as the label of every left edge and F as the label of every right edge. See left of Figure 9 for a partial graphic representation of $\mathcal{T}_{\text{full}}$.

From $\mathcal{T}_{\text{full}}$ we define another binary tree $\mathcal{T}(\Gamma)$ by pruning some of the infinite paths as follows: Given infinite path $\sigma \overset{\text{def}}{=} t_0 t_1 t_2 \cdots t_n \cdots$ in $\mathcal{T}_{\text{full}}$ where $t_n \in \{\mathsf{T}, \mathsf{F}\}$ for every $n \geqslant 0$, let k be the smallest integer (if any) such the truth assignment corresponding to σ falsifies some wff in Γ; if such a k exists, delete from $\mathcal{T}_{\text{full}}$ all paths extending the finite path $t_0 t_1 \cdots t_k$. At the node where $\mathcal{T}_{\text{full}}$ is pruned, we replace p_{k+1} by a leaf node denoted "•". See right of Figure 9 for an example of how $\mathcal{T}_{\text{full}}$ is pruned when p_0 occurs nowhere in Γ and Γ includes the wff $(\neg p_1 \wedge p_2)$. By this definition of $\mathcal{T}(\Gamma)$, note that $\mathcal{T}_{\text{full}}$ is none other than $\mathcal{T}(\varnothing)$, which is $\mathcal{T}_{\text{full}}$ without any pruning.

The resulting $\mathcal{T}(\Gamma)$ contains some full finite paths (possibly none) and some infinite paths (possibly none). Γ is satisfiable iff $\mathcal{T}(\Gamma)$ contains an infinite path, so that also Γ is not satisfiable iff $\mathcal{T}(\Gamma)$ does not contain an infinite path. By WKL, if $\mathcal{T}(\Gamma)$ does not contain an infinite path, then $\mathcal{T}(\Gamma)$ does not contain arbitrarily long full finite paths, *i.e.*, there is a finite bound $k \geqslant 1$ such that all full finite paths have length $\leqslant k$. But this implies there is a finite subset of Γ which is not satisfiable. □

Exercise 99. This exercise is couched in a language a little more familiar to computer scientists. Given a set $\Gamma \subseteq \mathsf{WFF}_{\mathsf{PL}}(\mathcal{P})$, the binary tree $\mathcal{T}(\Gamma)$ induced by Γ is defined in the proof above. An arbitrary binary tree $\mathcal{U}$ can be represented by a subset of $\{\mathsf{T}, \mathsf{F}\}^*$, which denotes the set of all finite strings over the alphabet $\{\mathsf{T}, \mathsf{F}\}$, satisfying two conditions:

- $\mathcal{U}$ is prefix-closed, *i.e.*, for all $\pi_1, \pi_2 \in \{\mathsf{T}, \mathsf{F}\}^*$, if $\pi_1 \in \mathcal{U}$ and π_2 is a prefix of π_1, then $\pi_2 \in \mathcal{U}$.
- For all $\pi \in \{\mathsf{T}, \mathsf{F}\}^*$, it holds that $\pi\mathsf{T} \in \mathcal{U}$ iff $\pi\mathsf{F} \in \mathcal{U}$, *i.e.*, every non-leaf node has two successors.

Let Γ_0, Γ_1, and Γ_2, be defined as follows:

$$\Gamma_0 \overset{\text{def}}{=} \{\, \neg p_1 \wedge p_2 \},$$

$$\Gamma_1 \overset{\text{def}}{=} \Big\{\, p_1 \to \neg(p_2 \wedge \cdots \wedge \neg p_k) \;\Big|\; k \geqslant 2 \,\Big\},$$

$$\Gamma_2 \overset{\text{def}}{=} \Big\{\, \neg p_1 \to \neg(p_2 \wedge \cdots \wedge \neg p_k) \;\Big|\; k \geqslant 2 \,\Big\}.$$

There are two parts in this exercise:

1. Define the binary trees $\mathcal{T}(\Gamma_0)$, $\mathcal{T}(\Gamma_1 \cup \Gamma_2)$, and $\mathcal{T}(\Gamma_0 \cup \Gamma_1 \cup \Gamma_2)$, as subsets of $\{\mathsf{T}, \mathsf{F}\}^*$.
2. For each of the three binary trees defined in part 1, explain how the tree indicates whether the corresponding set of wff's is satisfiable or not.

You will find it useful to consult Figure 9. $\square$

The next proof makes explicit reference to notions in topology. As indicated in the preceding alternative proof, a truth assignment σ can be denoted by a path in the full binary tree $\mathcal{T}_{\text{full}}$, now viewed as an ω-sequence in the product space $\{\mathsf{T}, \mathsf{F}\}^\omega$. (We write ω for the first infinite ordinal, which is the set of natural numbers listed in their standard order.)

We view $\{\mathsf{T}, \mathsf{F}\}^\omega$ as the underlying space of a product topology $\big(\{\mathsf{T}, \mathsf{F}\}^\omega, \mathcal{O}\big)$, thus making every truth assignment a "point" in that topology. $\mathcal{O}$ is a family of *open sets* that define the topology, which are in this case subsets of points in $\{\mathsf{T}, \mathsf{F}\}^\omega$ and satisfy the usual requirements of a topology:

- The empty set $\varnothing$ and the full space $\{\mathsf{T}, \mathsf{F}\}^\omega$ are in $\mathcal{O}$,
- $\mathcal{O}$ is closed under *arbitrary* unions,
- $\mathcal{O}$ is closed under *finite* intersections.

We can define a subset $U \subseteq \{\mathsf{T}, \mathsf{F}\}^\omega$ to be *open* iff there is a finite set of indices $I \subseteq \omega$ such that:

$$U \;=\; \prod \Big\{\, A_i \;\Big|\; i \in \omega \text{ and } A_i \subseteq \{\mathsf{T}, \mathsf{F}\} \,\Big\} \quad \text{where } A_i = \{\mathsf{T}, \mathsf{F}\} \text{ for every } i \in \omega - I.$$

In words, in the infinite product $U = A_0 \times A_1 \times \cdots \times A_i \times \cdots$, for all but finitely many indices i it is the case that $A_i = \{\mathsf{T}, \mathsf{F}\}$. A set $U \subseteq \{\mathsf{T}, \mathsf{F}\}^\omega$ is *closed* iff it is the complement of an open set. In the case of the product topology, every open subset $U \subseteq \{\mathsf{T}, \mathsf{F}\}^\omega$ is also closed, and thus called *clopen*.

Let A be a subset of $\{\mathsf{T}, \mathsf{F}\}^\omega$. An *open covering of* A is a family of open sets $\{\, U_i \,|\, i \in I \,\} \subseteq \mathcal{O}$ such that $A \subseteq \bigcup\{\, U_i \,|\, i \in I \,\}$. And A is said *compact* if every open covering of A has a finite subcovering; *i.e.*, there exists a finite subfamily $U_{i_1}, U_{i_2}, \ldots, U_{i_n}$ of $\{\, U_i \,|\, i \in I\}$ such that $A \subseteq (U_{i_1} \cup U_{i_2} \cup \ldots \cup U_{i_n})$. By Tychonoff's Theorem in topology, the product topology $\big(\{\mathsf{T}, \mathsf{F}\}^\omega, \mathcal{O}\big)$ is *compact*, which means that every open covering of a subset of points $A \subseteq \{\mathsf{T}, \mathsf{F}\}^\omega$ has a finite subcovering.

Alternative Proof II of Theorem 2 (Compactness for Propositional Logic). Again here, we only need to consider the non-trivial implication "$\Leftarrow$": If Γ is finitely satisfiable, then Γ is satisfiable.

For every propositional wff φ, let $A_\varphi \subseteq \{\mathsf{T}, \mathsf{F}\}^\omega$ be the collection of all points/truth assignments that satisfy φ. The set A_φ is a closed (and open) subset of $\{\mathsf{T}, \mathsf{F}\}^\omega$ in the topology $\big(\{\mathsf{T}, \mathsf{F}\}^\omega, \mathcal{O}\big)$, which follows from the fact that φ only mentions finitely many propositional variables. It is easy to check that, for every finite subset Δ of Γ, if Δ is satisfiable, then $\bigcap\{\, A_\varphi \,|\, \varphi \in \Delta \,\}$ is not empty. Hence, the family of closed sets $\{\, A_\varphi \,|\, \varphi \in \Gamma \,\}$ satisfies the *finite intersection property*. Moreover, the product topology $\big(\{\mathsf{T}, \mathsf{F}\}^\omega, \mathcal{O}\big)$ is compact, as noted above. Hence, $\bigcap\{\, A_\varphi \,|\, \varphi \in \Gamma \,\}$ is not empty, which is the desired conclusion. $\square$