### CS 511, Fall 2020, Lecture Slides 18

## First-Order Logic: Soundness and Completeness

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### consistency

#### $\Gamma$ is a set of WFF's.

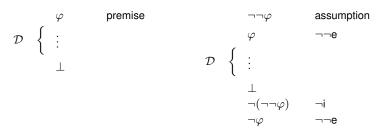
- ightharpoonup  $\Gamma$  is **consistent** iff  $\Gamma \not\vdash \bot$ .
  - ► **FACT.** The following three conditions are equivalent:
    - 1.  $\Gamma$  is consistent.
    - 2. For no WFF  $\varphi$  is it the case that both  $\Gamma \vdash \varphi$  and  $\Gamma \vdash \neg \varphi$ .
    - 3. There is at least one WFF  $\varphi$  such that  $\Gamma \not\vdash \varphi$ .
  - Contrapositive FACT. The following conditions are equivalent:
    - 4.  $\Gamma$  is inconsistent.
    - 5. There is a WFF  $\varphi$  such that both  $\Gamma \vdash \varphi$  and  $\Gamma \vdash \neg \varphi$ .
    - 6. For every WFF  $\varphi$ , it holds that  $\Gamma \vdash \varphi$ .

#### Proof.

- $(4)\Rightarrow (6)$ : Let  $\Gamma \vdash \bot$ . By the rule " $\bot$  elimination", we add one more step in the proof to obtain  $\Gamma \vdash \varphi$ , which holds for every  $\varphi$ .
- (6)  $\Rightarrow$  (5): Immediate.
- (5)  $\Rightarrow$  (4): By the rule " $\neg$  elimination", from the derivations  $\Gamma \vdash \varphi$  and  $\Gamma \vdash \neg \varphi$ , we get  $\Gamma \vdash \bot$ .

### consistency (continued)

- ▶ Theorem. Let  $\Gamma$  be a set of WFF's and  $\varphi$  a WFF. We then have the two following (equivalent) statements:
- 1.  $\Gamma \cup \{\varphi\}$  is inconsistent iff  $\Gamma \vdash \neg \varphi$ .
- 2.  $\Gamma \cup \{\varphi\}$  is consistent iff  $\Gamma \not\vdash \neg \varphi$ .
- ▶ **Proof.** It suffices to prove part 1 only. The simple right-to-left implication is left to you. For the left-to-right, suppose  $\Gamma \cup \{\varphi\}$  is inconsistent. Hence we are given a formal derivation of the form on the left, and we build a new one on the right. The new one starts by opening a box with assumption  $\neg\neg\varphi$ , then uses rule " $\neg\neg e$ " and copies the given derivation with no change, and closes the initial box with rule " $\neg i$ ":



The new formal derivation on the right shows that  $\Gamma \vdash \neg \varphi$  is a derivable sequent.

### soundness

▶ **Theorem.** Let  $\Gamma$  be a set of WFF's and  $\varphi$  a WFF.

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If \Gamma \vdash \varphi then \Gamma \models \varphi. (most common form for "soundness")
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Proof. Left to you. Consult also how each deduction rule is justified [LCS, Section 2.3].

### soundness

**Theorem.** Let  $\Gamma$  be a set of WFF's and  $\varphi$  a WFF.

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If \Gamma \vdash \varphi then \Gamma \models \varphi. (most common form for "soundness")
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▶ Proof. Left to you. Consult also how each deduction rule is justified [LCS, Section 2.3].

Another form for "soundness" is the following:

- **Corollary.** If  $\Gamma$  is satisfiable, then  $\Gamma$  is consistent.
- ▶ **Proof.** Suppose  $\Gamma$  is inconsistent. Then there is a WFF  $\varphi$  such that both  $\Gamma \vdash \varphi$  and  $\Gamma \vdash \neg \varphi$ , by part (5) on slide 3. By the previous theorem, both  $\Gamma \models \varphi$  and  $\Gamma \models \neg \varphi$ , which is a contradiction.

### completeness

One form of "completeness" is the following:

**Theorem.** Let  $\Gamma$  be a set of sentences (closed WFF's). If  $\Gamma$  is consistent, then  $\Gamma$  is satisfiable.

Proof. By the Model-Existence Lemma (not in the book [LCS], and not included in these notes, look up "model-existence" lemma or theorem on the Web).

### completeness

One form of "completeness" is the following:

If  $\Gamma$  is consistent, then  $\Gamma$  is satisfiable.

**Theorem.** Let  $\Gamma$  be a set of sentences (closed WFF's).

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Another form of "completeness", which is the most common:

look up "model-existence" lemma or theorem on the Web).

- **Corollary.** If  $\Gamma \models \varphi$  then  $\Gamma \vdash \varphi$ .
- ▶ **Proof.** Suppose  $\Gamma \not\vdash \varphi$ . Then  $\Gamma \not\vdash \neg \neg \varphi$ . So that  $\Gamma \cup \{\neg \varphi\}$  is consistent, by part 2 of theorem on slide 4. By the theorem on this slide, there is a model  $\mathcal{M}$  of  $\Gamma \cup \{\neg \varphi\}$ . Hence,  $\mathcal{M}$  is a model of  $\Gamma$  but not of  $\varphi$ . Hence,  $\Gamma \not\models \varphi$ .

# soundness and completeness – short form

For all WFF  $\varphi$ 



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