

CS 511, Fall 2020, Lecture Slides 14

Predicate Logic: Semantics

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first-order models (structures)

- ▶ given a **vocabulary**, aka **signature** or **similarity type**, $(\mathcal{F}, \mathcal{P})$:
 - ▶ a set \mathcal{F} of **function** symbols
(including **constant** symbols as zero-ary function symbols)
 - ▶ a set \mathcal{P} of **predicate** symbols
- ▶ a **model** \mathcal{M} for $(\mathcal{F}, \mathcal{P})$ consists of:
 - ▶ a non-empty set A , the **universe** or **domain** of concrete values
 - ▶ for every 0-ary $c \in \mathcal{F}$, a concrete element $c^{\mathcal{M}}$
 - ▶ for every n -ary $f \in \mathcal{F}$, with $n \geq 1$, a concrete function $f^{\mathcal{M}} : A^n \rightarrow A$
 - ▶ for every n -ary $P \in \mathcal{P}$, with $n \geq 1$, a concrete predicate $P^{\mathcal{M}} \subseteq A^n$

interpretation of open WFF's requires an environment

- ▶ We need an environment to interpret WFF's with **free variables**.
- ▶ an **environment** or **look-up table** for model $\mathcal{M} \triangleq (A, \mathcal{P}^{\mathcal{M}}, \mathcal{F}^{\mathcal{M}})$:

$$\ell : \{\text{all variables}\} \rightarrow A$$

- ▶ $\ell[x \mapsto a]$ denotes an adjustment of ℓ at variable x :

$$\ell[x \mapsto a](y) \triangleq \begin{cases} a & \text{if } x \text{ and } y \text{ are the same variable} \\ \ell(y) & \text{otherwise} \end{cases}$$

satisfaction of WFF's w.r.t. model \mathcal{M} and look-up table ℓ

► interpretation of terms:

$$t^{\mathcal{M}, \ell} \triangleq \begin{cases} \ell(x) & \text{if } t = x \\ c^{\mathcal{M}} & \text{if } t = c \text{ where } c \text{ is constant symbol} \\ f^{\mathcal{M}}(t_1^{\mathcal{M}, \ell}, \dots, t_n^{\mathcal{M}, \ell}) & \text{if } t = f(t_1, \dots, t_n) \text{ where } f \text{ is } n\text{-ary with } n \geq 1 \end{cases}$$

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- interpretation of WFF's:

- $\mathcal{M}, \ell \models (t_1 \approx t_2)$ iff $t_1^{\mathcal{M}, \ell} = t_2^{\mathcal{M}, \ell}$
- $\mathcal{M}, \ell \models P(t_1, \dots, t_n)$ iff $\langle t_1^{\mathcal{M}, \ell}, \dots, t_n^{\mathcal{M}, \ell} \rangle \in P^{\mathcal{M}}$
- $\mathcal{M}, \ell \models \varphi \vee \psi$ iff $\mathcal{M}, \ell \models \varphi$ **or** $\mathcal{M}, \ell \models \psi$
- $\mathcal{M}, \ell \models \varphi \wedge \psi$ iff $\mathcal{M}, \ell \models \varphi$ **and** $\mathcal{M}, \ell \models \psi$
- $\mathcal{M}, \ell \models \varphi \rightarrow \psi$ iff $\mathcal{M}, \ell \models \psi$ **whenever** $\mathcal{M}, \ell \models \varphi$
- $\mathcal{M}, \ell \models \neg \varphi$ iff it is **not** the case that $\mathcal{M}, \ell \models \varphi$
- $\mathcal{M}, \ell \models \forall x \varphi$ iff $\mathcal{M}, \ell[x \mapsto a] \models \varphi$ for **every** $a \in A$
- $\mathcal{M}, \ell \models \exists x \varphi$ iff $\mathcal{M}, \ell[x \mapsto a] \models \varphi$ for **some** $a \in A$

semantic entailment, semantic validity, satisfiability

- ▶ WFF φ is **satisfiable** iff
there is some \mathcal{M} and some ℓ such that $\mathcal{M}, \ell \models \varphi$
- ▶ WFF φ is **semantically valid** iff
for every \mathcal{M} and every ℓ it is the case that $\mathcal{M}, \ell \models \varphi$

let Γ be a set of WFF's:

- ▶ Γ is **satisfiable** iff
there is some \mathcal{M} and some ℓ such that $\mathcal{M}, \ell \models \Gamma$,
i.e., $\mathcal{M}, \ell \models \varphi$ for every $\varphi \in \Gamma$
- ▶ **semantic entailment**: $\Gamma \models \psi$ iff
for every \mathcal{M} and every ℓ , it holds that $\mathcal{M}, \ell \models \Gamma$ implies $\mathcal{M}, \ell \models \psi$

tautologies and (semantical) validities

- ▶ **validities** (or **semantical validities**) are WFF's that are satisfied by (or true in) every model \mathcal{M} and environment ℓ
- ▶ **tautologies** are a proper subset of the first-order **validities**
- ▶ in **propositional logic**, the two notions coincide
- ▶ in **first-order logic**, a **tautology** is a WFF that can be obtained by taking a tautology of propositional logic and uniformly replacing each propositional atom (or variable) by a first-order formula (one formula per propositional atom)

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- ▶ **example of a first-order validity which is not a tautology:**
$$(\forall x \varphi) \rightarrow (\neg \exists x \neg \varphi)$$

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