CS 511, Fall 2020, Lecture Slides 06

Propositional Logic:

Soundness, Completeness, Compactness

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Soundness

- Let Γ a (possibly infinite) set of propositional wff's.

  If, for every model (or assignment of truth values) it holds that:

  - whenever all the wff's in Γ evaluate to **T**,
  - it is also the case that ψ evaluates to **T**,

  then we write:

  $\Gamma \models \psi$ in words, "Γ semantically entails ψ"

- **Theorem (Soundness):** If $\Gamma \vdash \psi$ then $\Gamma \models \psi$.

  (Slightly stronger than the statement of Soundness in [LCS, Theorem 1.35, p 46]:
  If $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ then $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$.)

Soundness once more

- **Theorem (Soundness)** – as in [LCS, Theorem 1.35, p 46]:
  If $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ then $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$.
- **Proof: Course-of-values induction**
  (sometimes called **strong induction**) on $k \geqslant 1$,
  where k is number of lines in a formal proof.
- Base step: Consider $k = 1$ (quite trivial!).
  In this case $n = k = 1$.

  From a given sequent $\varphi_1 \vdash \psi$, we want to show $\varphi_1 \models \psi$.

  Such a sequent implies $\varphi_1 = \psi$, i.e., $\psi \vdash \psi$.

  Hence, $\psi \models \psi$,

  which is the same as $\varphi_1 \models \psi$. (QED for base case)
- Inductive step: Consider arbitrary $k \geqslant 2$.
  (Actually for $1 \leqslant k \leqslant n$, it is trivial again. Interesting case: $k > n$.)

  Induction hypothesis (IH): Soundness holds for every $k' < k$.

Structure of a formal proof with n premises:

1	φ_1	**premise**	
2	φ_2	**premise**	
$\vdots$	$\vdots$		
n	φ_n	**premise**	
$\vdots$	$\vdots$		
k	ψ	"justification"	(a proof rule, or **assumption**, or **repeat**)

- Last line in the proof, line k, is the result of 1 or 2 or ... preceding it.
- Consider each possible "justification" separately: finitely many.
- Suppose "justification" is "$\wedge$i".
  This means line k uses lines k_1 and k_2, with $k_1, k_2 < k$.
- Use IH on $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi_{k_1}$ and $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi_{k_2}$.

Completeness

- **Theorem (Completeness):** If $\Gamma \models \psi$ then $\Gamma \vdash \psi$.

  (Stronger than the statement of Completeness in [LCS, Corollary 1.39, p 53]: If $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$ then $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$.)

- **Proof idea in [LCS]** (which works if Γ is a finite set $\{\varphi_1, \ldots, \varphi_n\}$):

  Establish 3 preliminary results. From $\varphi_1, \ldots, \varphi_n \models \psi$, show that:
  1. $\models \varphi_1 \to (\varphi_2 \to (\varphi_3 \to (\cdots (\varphi_n \to \psi) \cdots)))$ holds.
  2. $\vdash \varphi_1 \to (\varphi_2 \to (\varphi_3 \to (\cdots (\varphi_n \to \psi) \cdots)))$ is a valid sequent.
  3. $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ is a valid sequent.

- If Γ is infinite, we need another preliminary result: Compactness.

Compactness

- Γ is said to be **satisfiable** if there is a model which satisfies/makes true every φ in Γ.

- **Theorem (Compactness)** (not in [LCS]):
  Γ is satisfiable iff every finite subset of Γ is satisfiable.

- **Corollary** (not in [LCS]):
  If $\Gamma \models \psi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models \psi$.

- For proof of Compactness above and its corollary, is in my lecture notes [fCtC], posted on Piazza.

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