CS 511, Fall 2020, Lecture Slides 15

Examples of First-Order Theories

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equivalence relations

1. $\forall x \quad (x \sim x)$ — reflexivity
2. $\forall x \, \forall y \quad (x \sim y \to y \sim x)$ — symmetry
3. $\forall x \, \forall y \, \forall z \quad (x \sim y \land y \sim z \to x \sim z)$ — transitivity

equality with uninterpreted functions (EUF)

1. $\forall x \quad (x \approx x)$ — reflexivity
2. $\forall x \, \forall y \quad (x \approx y \to y \approx x)$ — symmetry
3. $\forall x \, \forall y \, \forall z \quad (x \approx y \wedge y \approx z \to x \approx z)$ — transitivity

The three preceding axioms are identical to those in the theory of **equivalence relations** (preceding page).

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3. $\forall x\, \forall y\, \forall z \quad (x \approx y \wedge y \approx z \to x \approx z)$ transitivity

   The three preceding axioms are identical to those in the theory of **equivalence relations** (preceding page).

4. for every function symbol $f \in \mathcal{F}$ of arity $n \geqslant 1$:

$$\forall x_1 \cdots \forall x_n\ \forall y_1 \cdots \forall y_n$$
$$\Big(\bigwedge_{1 \leqslant i \leqslant n} x_i \approx y_i\Big) \to f(x_1, \ldots, x_n) \approx f(y_1, \ldots, y_n)$$

congruence

5. for every predicate symbol $P \in \mathcal{P}$ of arity $n \geqslant 1$:

$$\forall x_1 \cdots \forall x_n\ \forall y_1 \cdots \forall y_n$$
$$\Big(\bigwedge_{1 \leqslant i \leqslant n} x_i \approx y_i\Big) \to \big(P(x_1, \ldots, x_n) \leftrightarrow P(y_1, \ldots, y_n)\big)$$

congruence

orders

1. $\forall x\ \forall y\ \forall z \quad \big(x \leqslant y \wedge y \leqslant z \to x \leqslant z\big)$ transitive
2. $\forall x \quad \big(x \leqslant x\big)$ reflexive
3. $\forall x\ \forall y \quad \big(x \leqslant y \wedge y \leqslant x \to x \approx y\big)$ anti-symmetric

(1), (2) and (3) make "$\leqslant$" a **partial** order, which may not be **total**

(what is an example of a partial order which is not total?)

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4. $\forall x \, \forall y \quad \big(x \leqslant y \vee y \leqslant x\big)$ total (or linear) ordering
5. $\forall x \, \forall z \quad \Big(x \lessdot z \to \exists y \quad \big(x \lessdot y \wedge y \lessdot z\big)\Big)$ dense ordering

   (where "$x \lessdot y$" abbreviates "$(x \leqslant y) \wedge \neg(x \approx y)$")

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6. $\exists x\ \forall y\quad \big(x \leqslant y\big)$ smallest element
7. $\exists x\ \forall y\quad \big(y \leqslant x\big)$ largest element

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   can we express a **well-ordering** in first-order logic? i.e., "every non-empty subset has a smallest element"?

algebras with two binary operations

1. $\forall x \, \forall y \, \forall z \quad \Big(x \oplus (y \oplus z) \approx (x \oplus y) \oplus z \Big)$

   $\oplus$ is associative

2. $\forall x \, \forall y \quad \big(x \oplus y \approx y \oplus x \big)$

   $\oplus$ is commutative

3. $\forall x \, \forall y \, \forall z \quad \Big(x \otimes (y \oplus z) \approx (x \otimes y) \oplus (x \otimes z) \Big)$

   $\otimes$ distributes over $\oplus$

groups

1. $\forall x \quad \big(e \cdot x \approx x \ \wedge \ x \cdot e \approx x\big)$ — identity (or neutral element)
2. $\forall x \, \exists y \quad \big(x \cdot y \approx e \ \wedge \ y \cdot x \approx e\big)$ — inverse
3. $\forall x \, \forall y \, \forall z \quad \Big((x \cdot y) \cdot z \approx x \cdot (y \cdot z)\Big)$ — associative

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three preceding WFF's are true in every group,

does the following WFF φ follows from the preceding three:
$\varphi \stackrel{\text{def}}{=} \ \forall x \, \forall y \, \forall z \quad \big(x \cdot y \approx e \ \wedge \ x \cdot z \approx e \rightarrow y \approx z\big)$??

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some special cases of groups:

4. $\forall x\, \forall y \quad \big(x \cdot y \approx y \cdot x\big)$ — abelian group
5. $\forall x \quad \big(x \cdot x \approx e \rightarrow x \approx e\big)$, — torsion-free group
   $\forall x \quad \big(x \cdot x \cdot x \approx e \rightarrow x \approx e\big)$,
   $\forall x \quad \big(x \cdot x \cdot x \cdot x \approx e \rightarrow x \approx e\big), \ldots$

graphs

1. $\forall x\ \forall y\ \ \big(R(x,y) \to R(y,x)\big)$

   the graph is undirected

2. $\forall x\ \ \big(\neg R(x,x)\big)$

   there are no “loops” in the graph

graphs

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   the graph is undirected

2. $\forall x \quad \big(\neg R(x,x)\big)$

   there are no “loops” in the graph

assume there are two domains:
the domain V of vertices, the domain $\mathbb{R}$ of real numbers

assume there is a capacity function: $\quad c : V \times V \rightarrow \mathbb{R}$

a flow is a function $\quad f : V \times V \rightarrow \mathbb{R}$

3. $\forall f \; \forall x \; \forall y \quad \big(f(x,y) \leqslant c(x,y)\big)$

   is (3) a first-order WFF?

successor function over the natural numbers

1. $\forall x \quad \big(\neg(\mathsf{S}\, x \approx 0)\big)$
2. $\forall x\ \forall y \quad (\mathsf{S}\, x \approx \mathsf{S}\, y \to x \approx y)$
3. $\forall x \quad \big(\neg(x \approx 0)\ \to\ \exists y\ (\mathsf{S}\, y \approx x)\big)$

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3. $\forall x \quad \big(\neg(x \approx 0) \;\to\; \exists y\, (\mathsf{S}\,y \approx x)\big)$

4. for every WFF $\varphi(x)$ with a single free variable x, include the axiom $\varphi(0) \wedge \forall x\, \big(\varphi(x) \to \varphi(\mathsf{S}\,x)\big) \quad \to \quad \forall y\, \varphi(y)$

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   $\varphi(0) \wedge \forall x\,\big(\varphi(x) \to \varphi(\mathsf{S}\,x)\big) \quad \to \quad \forall y\,\varphi(y)$

   with **addition**:

5. $\forall x \quad (x \dot{+} 0 \approx x)$
6. $\forall x\, \forall y \quad \big(x \dot{+} \mathsf{S}\,y \approx \mathsf{S}\,(x \dot{+} y)\big)$

successor function over the natural numbers

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2. $\forall x\, \forall y \quad (\mathsf{S}\,x \approx \mathsf{S}\,y \rightarrow x \approx y)$
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   with **addition**:

5. $\forall x \quad (x \dot{+} 0 \approx x)$
6. $\forall x\, \forall y \quad \big(x \dot{+} \mathsf{S}\,y \approx \mathsf{S}\,(x \dot{+} y)\big)$

   with **addition** and **multiplication**:

7. $\forall x \quad (x \dot{\times} 0 \approx 0)$
8. $\forall x\, \forall y \quad \big(x \dot{\times} \mathsf{S}\,y \approx (x \dot{\times} y) \dot{+} x\big)$

linear integer arithmetic (LIA)

1. $\mathcal{P} = \{\dot{\leqslant}\}, \quad \mathcal{F} = \{\dot{+}, \dot{-}\}, \quad \mathcal{C} = \{0, 1\}$.
2. include all axioms for "$\dot{+}$" and "$\dot{-}$".
3. atomic WFF's are all of the form

$$a_1 x_1 \dot{+} a_2 x_2 \dot{+} \cdots \dot{+} a_n x_n \ \bowtie \ b$$

where $\bowtie \ \in \ \{\dot{\leqslant}, \dot{<}, \dot{\geqslant}, \dot{>}, \approx, \not\approx\}$ and $a_1, \ldots, a_n, b$ are integers.

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