CS 511, Fall 2020: Take-Home Mid-Term Examination

Out: Thursday, 22 October 2020, by 12:30 pm
Due: Friday, 23 October 2020, by 12:29 pm

Submission Guidelines:

- There are 6 problems in this exam. Each problem is worth 8 points, for a total of 48 points for the whole exam.
- You should submit a single "`.pdf`" file to Gradescope.
- **Be careful with your submission**: You will not be able to submit you exam past the deadline on Friday, October 23, past 12:29 pm.

Honor Code:

- You are expected to do the exam entirely on your own, without consultation with anyone else, as you must be the sole responsible for your answers.
- If you use external material that you found on the Web, or in a book, or in an article, or in some old solution set, **you are required to acknowledge that external material and what you took from it**.

Problem 1. ***Formal Proofs in Propositional Logic.*** A *Hilbert-style proof system* is an alternative to a *natural-deduction proof system*. A Hilbert-style proof system for the propositional logic can be formulated by specifying only three *axiom schemes*:

A1: $(\varphi \to (\psi \to \varphi))$

A2: $((\varphi \to (\psi \to \theta)) \to ((\varphi \to \psi) \to (\varphi \to \theta)))$

A3: $((\neg\varphi \to \neg\psi) \to (\psi \to \varphi))$

together with a single *rule of inference*, called **modus ponens** (MP), specified as follows:

$$\frac{\varphi \qquad (\varphi \to \psi)}{\psi}$$

The rule MP corresponds to the *arrow-elimination* rule ($\to$E) of natural deduction. Your task in this problem is to show that any wff of propositional logic which is an instance of A1, or A2, or A3, is deducible in natural deduction, *i.e.*, for arbitrary wff's φ, ψ, and θ, it is the case that the following sequents hold in natural deduction:

1. $\vdash\ (\varphi \to (\psi \to \varphi))$
2. $\vdash\ ((\varphi \to (\psi \to \theta)) \to ((\varphi \to \psi) \to (\varphi \to \theta)))$
3. $\vdash\ ((\neg\varphi \to \neg\psi) \to (\psi \to \varphi))$

Problem 2. ***Modeling with Propositional Logic.*** Consider three persons A, B, and C who need to be seated in a row, next to each other, in three separate chairs. However:

1. A does not want to sit next to C.
2. A does not want to sit in the left chair.

(a) Write a propositional WFF that is satisfiable iff there is a seat assignment for the three persons that satisfies constraints 1 and 2. Is the WFF satisfiable? Justify your answer in a line or two. In particular, if it is, give an assignment of truth values that satisfies it.

Hint 1: The seat assignment "$A\ C\ B$" violates both constraints. The seat assignment "$B\ C\ A$" violates constraint 1, but not constraint 2. The seat assignment "$C\ B\ A$" satisfies both constraints.

Hint 2: Introduce six propositional atoms, each of the form P_{xy}, where $x, y \in \{A, B, C\}$ and $x \neq y$. Let P_{xy} be *true* iff "x sits to the left of y" (or, equivalently, "y sits to the right of x").

Introduce a third constraint:

3. B wants to sit to the right of C.

(b) Write a propositional WFF that is satisfiable iff there is a seat assignment for the three persons that satisfies constraints 1, 2, and 3. Is the WFF satisfiable? Justify your answer in a line or two. In particular, if it is, give an assignment of truth values that satisfies it.

For some reason, B changed his mind, introducing a new constraint $3'$ instead of 3:

$3'$. B does not want to sit to the right of C.

(c) Write a propositional WFF that is satisfiable iff there is a seat assignment for the three persons that satisfies constraints 1, 2, and $3'$. Is the WFF satisfiable? Justify your answer in a line or two. In particular, if it is, give an assignment of truth values that satisfies it.

Problem 3. ***Compactness in Propositional Logic.*** Download the set of lecture notes **2020-10-18.fCfC.pdf (partial notes)**, posted on October 18, 2020, on the Piazza website under *Resources*. Go to page 74, do Exercise 99.

Problem 4. ***Structural Induction and Binary Decision Diagrams.*** There are two parts in this problem. Both parts depend on the ROBDD's for the 2-bit comparator which are analyzed on slides 21-24 in Lecture Slides 12. Your task is to extend this analysis to arbitrary n-bit comparators, for arbitrary $n \geqslant 2$, when the ordering on the variables interleaves the x_i's and the y_i's, as in $x_1 < y_1 < x_2 < y_2 < \cdots < x_n < y_n$. In fact, the analysis is a little simpler if you adopt two conventions, with no loss of generality:

- use a reverse ordering on the variables $x_n < y_n < x_{n-1} < y_{n-1} < \cdots < x_2 < y_2 < x_1 < y_1$,
- let 'ROBDD$_n$' be the name of the ROBDD relative to this reverse ordering for the n-bit comparator.

You should try to solve part 1 before you solve part 2:

1. Define ROBDD$_n$ by structural induction for arbitrary $n \geqslant 2$. The base case ROBDD$_2$ is already available to you, it is the ROBDD on slide 22 of Lecture Slides 12. Your task now is to show how to define ROBDD$_n$ from ROBDD$_{n-1}$ when $n \geqslant 3$.

   Hint 1: Every ROBDD D has exactly one root node, call it $\mathsf{root\_node}(D)$, and exactly to leaf nodes, call them $\mathsf{zero\_leaf}(D)$ and $\mathsf{one\_leaf}(D)$ which are respectively labelled with 0 and 1.

   Hint 2: If N is a non-leaf node in ROBDD D, then N has exactly two children, which you can call $\mathsf{zero\_child}(N)$ and $\mathsf{one\_child}(N)$. Following conventions, the edge from N to $\mathsf{zero\_child}(N)$ is a dashed edge, and the edge from N to $\mathsf{one\_child}(N)$ is a plain edge.

2. Based on your answer for part 1, show that the number of nodes in ROBDD$_n$ is $3n + 2$, where $n \geqslant 2$.

Problem 5. ***Gilmore's Algorithm.*** Download Lecture Slides 24, entitled "Gilmore's Algorithm" in the file named **HD24.gilmores_algorithm.pdf** and posted on the Piazza website for this course, under *Resources*. Do the exercise at the top of page 12.

Problem 6. ***Modeling with First-Order Logic.*** You will show in this problem that every infinite planar graph is four-colorable. The theory of simple undirected graphs can be taken as a set Γ of two axioms over signature $\Sigma \triangleq \{R, =\}$ consisting of one binary predicate symbol and the equality symbol:

$$\Gamma \;\triangleq\; \Big\{ \;\forall x.\forall y.\; R(x,y) \to R(y,x), \quad \forall x.\; \neg R(x,x)\; \Big\}$$

Expand the signature Σ to $\Sigma' = \Sigma \cup \{B, G, P, Y\}$ where B, G, P, and Y are unary predicate symbols (for 'blue', 'green', 'purple', and 'yellow').

(a) Write a first-order sentence φ_1 which, in any Σ'-structure $\mathcal{M}$ satisfying Γ (*i.e.*, $\mathcal{M}$ is a simple undirected graph), asserts "*every vertex has at least one of the colors: blue, green, purple, yellow*":

(b) Write a first-order sentence φ_2 which, in any Σ'-structure $\mathcal{M}$ satisfying Γ, asserts "*every vertex has at most one color*":

(c) Write a first-order sentence φ_3 which, in any Σ'-structure $\mathcal{M}$ satisfying Γ, asserts "*no two adjacent vertices have the same color*":

(d) Show that if $\mathcal{M}$ is an infinite planar graph, *i.e.*,

- $\mathcal{M}$ is a Σ-structure satisfying Γ,
- the domain of $\mathcal{M}$ is infinite, and
- $\mathcal{M}$ is planar as a graph,

then there is a Σ-structure $\mathcal{M}'$, which expands $\mathcal{M}$ with four unary relations $B^{\mathcal{M}'}$, $G^{\mathcal{M}'}$, $P^{\mathcal{M}'}$, and $Y^{\mathcal{M}'}$, and which satisfies $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$, *i.e.*, $\mathcal{M}'$ is four-colorable and, thus, $\mathcal{M}$ is also four-colorable.

Hint 1: Find a way to make use of the following fact: Every *finite* planar graph is four-colorable. (Do not try to prove this fact, which is difficult, but you are allowed to invoke it.)

Hint 2: If $\mathcal{M}$ is a planar graph, then every subgraph of $\mathcal{M}$ is also planar. A *subgraph* of $\mathcal{M}$ is a graph whose vertices are a subset of the vertices of $\mathcal{M}$ and whose adjacency relation is a subset of the adjacency relation of $\mathcal{M}$ restricted to this subset.