CS 511, Fall 2020, Addendum 06 (B)

Modeling the Queens Problem in Propositional Logic

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the finite case

For convenience, we use the following set of propositional variables:

$$\mathcal{Q} \stackrel{\text{def}}{=} \left\{ q_{i,j} \;\middle|\; i,j \in \{1,2,\dots\} \right\}.$$

- For every $n \geqslant 4$, propositional wff ψ_n models all the solutions of the n-Queens Problem, where $\psi_n \stackrel{\text{def}}{=} \psi_n^{\text{row}} \wedge \psi_n^{\text{col}} \wedge \psi_n^{\text{diag1}} \wedge \psi_n^{\text{diag2}}$ and:

$$\psi_n^{\text{row}} \stackrel{\text{def}}{=} \bigwedge_{1\leqslant i\leqslant n} \Big(\bigvee_{1\leqslant j\leqslant n} \big(q_{i,j} \wedge \bigwedge_{1\leqslant k\leqslant n, k\neq j} \neg q_{i,k}\big)\Big)$$

$$\psi_n^{\text{col}} \stackrel{\text{def}}{=} \bigwedge_{1\leqslant j\leqslant n} \Big(\bigvee_{1\leqslant i\leqslant n} \big(q_{i,j} \wedge \bigwedge_{1\leqslant k\leqslant n, k\neq i} \neg q_{k,j}\big)\Big)$$

$$\psi_n^{\text{diag1}} \stackrel{\text{def}}{=} \bigwedge_{1\leqslant i,j\leqslant n} \Big\{ q_{i,j} \to \bigwedge \{ \neg q_{i',j'} \mid i+j = i'+j' \text{ and } (i,j) \neq (i',j') \} \Big\}$$

$$\psi_n^{\text{diag2}} \stackrel{\text{def}}{=} \bigwedge_{1\leqslant i,j\leqslant n} \Big\{ q_{i,j} \to \bigwedge \{ \neg q_{i',j'} \mid i-j = i'-j' \text{ and } (i,j) \neq (i',j') \} \Big\}$$

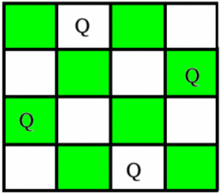
- For every truth assignment $\sigma : \mathcal{Q} \to \{\mathbf{F},\mathbf{T}\}$ such that $\sigma \models \psi_n$, the following set:

$$\left\{ (i,j) \;\middle|\; i,j \in \{1,\dots,n\} \text{ and } \sigma(q_{i,j}) = \mathbf{T} \right\}$$

specifies the positions of n queens on the $n \times n$ chessboard.

two solutions of the 4-queens problem (copied from the Web)

Solution A

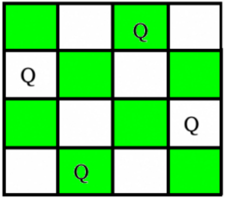


Every $\sigma : \mathcal{Q} \to \{\mathbf{F}, \mathbf{T}\}$ such that

$\sigma(q_{1,2}) = \sigma(q_{2,4}) = \sigma(q_{3,1}) = \sigma(q_{4,3}) = \mathbf{T}$,
and $\sigma(q_{i,j}) = \mathbf{F}$ for every (i, j) without "Q",

corresponds to Solution A and satisfies ψ_4
(there are infinitely many such σ)

Solution B



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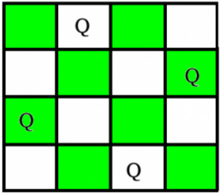

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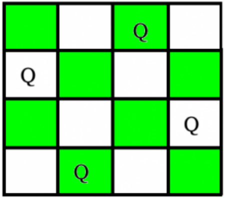
Solution B



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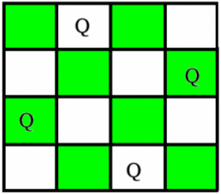
and $\sigma(q_{i,j}) = \mathbf{F}$ for every (i, j) without “Q”,

corresponds to Solution B and satisfies ψ_4
(there are infinitely many such σ)

Question: Which of the σ’s above satisfy all the wff’s in $\{\psi_5, \psi_6, \psi_7, \ldots\}$?

two solutions of the 4-queens problem (copied from the Web)

Solution A



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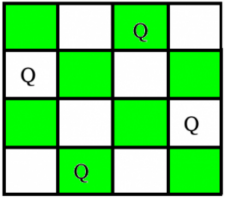
Solution B



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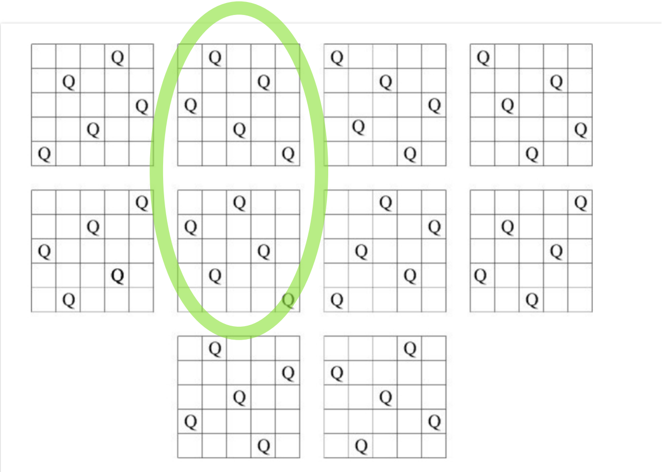
and $\sigma(q_{i,j}) = \mathbf{F}$ for every (i, j) without "Q",

corresponds to Solution B and satisfies ψ_4
(there are infinitely many such σ)

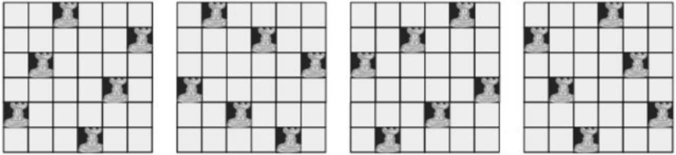
Question: Which of the σ's above satisfy all the wff's in $\{\psi_5, \psi_6, \psi_7, \ldots\}$?
Answer: **None!**

ten solutions of the 5-queens problem (copied from the Web)



Two solutions only of the 5-Queens Problem extend Solution A and Solution B.

four solutions of the 6-queens problem (copied from the Web) – out of several dozens



No solutions of the 6-Queens Problem extend Solution A and Solution B !

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