CS 511, Fall 2020, Handout 02

Natural Deduction and Examples of Natural Deduction in Propositional Logic

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from **informal/common** reasoning to **formal** reasoning:

- **IF** the train arrives late **AND** there are **NO** taxis
  THEN John is late for the meeting
- John is **NOT** late for the meeting
- the train did arrive late
- **THEREFORE** there were taxis

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again symbolically:

- $(\ P \ \wedge \ \neg Q\ ) \rightarrow \ R$
- $\neg R$
- P
- $\vdash \quad Q$

more succinctly:

$P \wedge \neg Q \rightarrow R, \ \ \neg R, \ \ P \ \ \vdash \ \ Q$

Formal Proof of the Sequent $* * *$

- a **sequent** (also called a **judgment**) is an expression of the form:

$$\varphi_1, \ldots, \varphi_n \ \vdash \ \psi$$

where:

1. $\varphi_1, \ldots, \varphi_n, \psi$ are **well-formed formulas** (also called **wff's**)

2. the symbol "$\vdash$" is pronounced **turnstile**

3. the wff's $\varphi_1, \ldots, \varphi_n$ to the left of "$\vdash$" are called the **premises** (also called **antecedents** or **hypotheses**)

4. the wff ψ to the right of "$\vdash$" is called the **conclusion** (also called **succedent**)

- a sequent is said to be **valid** (also **deducible** or **derivable**) if there is a **formal proof** for it

- a **formal proof** (also called **deduction** or **derivation**) is a sequence of wff's which starts with the **premises** of the sequent and finishes with the **conclusion** of the sequent:

$$\begin{array}{ll} \varphi_1 & \text{premise} \\ \varphi_2 & \text{premise} \\ \vdots & \\ \varphi_n & \text{premise} \\ \vdots & \\ \psi & \text{conclusion} \end{array}$$

where every wff in the deduction is obtained from the wff's preceding it using a **proof rule**

Examples of Proof Rules

- $$\frac{\varphi \qquad \psi}{\varphi \wedge \psi} \quad \wedge\mathsf{I}$$
- $$\frac{\varphi \wedge \psi}{\varphi} \quad \wedge\mathsf{E}_1$$
- $$\frac{\varphi \wedge \psi}{\psi} \quad \wedge\mathsf{E}_2$$
- $$\frac{\varphi}{\neg\neg\varphi} \quad \neg\neg\mathsf{I}$$
- $$\frac{\neg\neg\varphi}{\varphi} \quad \neg\neg\mathsf{E}$$ (cannot be used in **intuitionistic logic**)

Examples of Proof Rules

- $$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$ $\to$E (or **MP** for **Modus Ponens**)
- $$\frac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi}$$ **MT** (for **Modus Tollens**)

Examples of Proof Rules

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- $$\frac{\varphi \rightarrow \psi \qquad \neg\psi}{\neg\varphi}$$ **MT** (for Modus Tollens)
- $$\frac{\boxed{\begin{array}{c}\varphi \\ \vdots \\ \psi\end{array}}}{\varphi \rightarrow \psi}$$ $\rightarrow$I

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open a box when you *introduce* an **assumption** (wff φ in rule $\to$I)

close the box when you *discharge* the **assumption**

you must close every **box** and discharge every **assumption**
in order to complete a formal proof

Proof Rules Associated with Only One "$\neg$" and with "$\bot$"

So far, we have an **elimination** rule and an **introduction** rule for double negation "$\neg\neg$", namely $\neg\neg$E and $\neg\neg$I, but not for single negation "$\neg$". We now compose for this lack:

- $$\frac{\varphi \qquad \neg\varphi}{\bot}\ \neg\text{E}$$ (or **LNC** for **Law of Non-Contradiction**)

  where "$\bot$" (a single symbol) stands for "**contradiction**"

Proof Rules Associated with Only One "$\neg$" and with "$\perp$"

So far, we have an **elimination** rule and an **introduction** rule for double negation "$\neg\neg$", namely $\neg\neg$E and $\neg\neg$I, but not for single negation "$\neg$". We now compensate for this lack:

- $$\frac{\varphi \qquad \neg\varphi}{\perp}\ \neg\text{E}$$ (or **LNC** for **Law of Non-Contradiction**)

  where "$\perp$" (a single symbol) stands for "**contradiction**"

- $$\frac{\boxed{\begin{array}{c}\varphi\\ \vdots\\ \perp\end{array}}}{\neg\varphi}\ \neg\text{I}$$

- $$\frac{\perp}{\varphi}\ \perp\text{E}$$ ("if you can prove $\perp$, you can prove every wff")

Two Derived Proof Rules

The two following rules are derived rules –
the first from rules $\rightarrow$I, $\neg$I, $\rightarrow$E, and $\neg\neg$E (see [LCS, pp 24-25]);
the second from rules $\vee$I, $\neg$I, $\neg$E, and $\neg\neg$E (see [LCS, pp 25-26]):

- $$\frac{\boxed{\begin{array}{l}\neg\varphi \\ \vdots \\ \bot\end{array}}}{\varphi}$$ **PBC** (for **Proof by Contradiction**)

- $$\frac{}{\varphi \vee \neg\varphi}$$ **LEM** (for **Law of Excluded Middle**)

Because $\neg\neg$E is rejected in **intuitionistic logic**, so are **PBC** and **LEM**

(a summary of all **proof rules** and some **derived rules** in [LCS, p. 27])

Examples of Natural Deductions

formal proof of the sequent $P \vdash Q \to (P \wedge Q)$

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formal proof of the sequent $P \vdash Q \to (P \wedge Q)$

1	P	
2	Q	
3	$P \wedge Q$	$\wedge$I 1, 2
4	$Q \to (P \wedge Q)$	$\to$I

Examples of Natural Deductions

formal proof of the sequent $P \to (Q \to R) \vdash P \land Q \to R$

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formal proof of the sequent $P \to (Q \to R) \vdash P \land Q \to R$

1	$P \to (Q \to R)$	
2	$P \land Q$	
3	P	$\land\mathsf{E}_1$ 2
4	$Q \to R$	$\to\mathsf{E}$ 1, 3
5	Q	$\land\mathsf{E}_2$ 2
6	R	$\to\mathsf{E}$ 4, 5
7	$P \land Q \to R$	$\to\mathsf{I}$

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formal proof of the sequent $P \wedge Q \to R \vdash P \to (Q \to R)$

1	$P \wedge Q \to R$	
2	P	
3	Q	
4	$P \wedge Q$	$\wedge$I $2, 3$
5	R	$\to$E $1, 4$
6	$Q \to R$	$\to$I
7	$P \to (Q \to R)$	$\to$I

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formal proof of the sequent $P \to (Q \to R) \vdash (P \to Q) \to (P \to R)$

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1	$P \to (Q \to R)$	
2	$P \to Q$	
3	P	
4	Q	$\to$E $2, 3$
5	$Q \to R$	$\to$E $1, 3$
6	R	$\to$E $5, 4$
7	$P \to R$	$\to$I
8	$(P \to Q) \to (P \to R)$	$\to$I

Formal Proof of the Initial Sequent:

Initial Sequent

1	$P \wedge \neg Q \to R$	premise
2	$\neg R$	premise
3	P	premise
4	$\neg Q$	assume
5	$P \wedge \neg Q$	$\wedge$I $3, 4$
6	R	$\to$E $1, 5$
7	$\perp$	$\neg$E $6, 2$
8	$\neg\neg Q$	$\neg$I
9	Q	$\neg\neg$E 8

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