CS 511, Fall 2020, Lecture Slides 23

Examples of Structural Induction

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All finite strings/words of decimal digits

▶ The set of all decimal digits: $A \stackrel{\text{def}}{=} \{\, 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9 \,\}$

▶ Definition of the set A^* of all finite *strings* (or *words*) over A by induction :

1. **Base step**: $\varepsilon \in A^*$ where ε denotes the empty string,
2. **Inductive step**: For all $s \in A^*$ and $x \in A$, the string $s \cdot x$ is a member of A^*.

Concatenation of two strings s and t is denoted $s \cdot t$.

For example, if $s \stackrel{\text{def}}{=} 4\,5\,6$ and $t \stackrel{\text{def}}{=} 2\,2$, then $s \cdot t = 4\,5\,6\,2\,2$.

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▶ Definition of the function <u>reverse</u> : $A^* \to A^*$ by structural induction :
   1. **Base step**: <u>reverse</u>$(\varepsilon) \stackrel{\text{def}}{=} \varepsilon$,
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▶ Definition of the function <u>foo</u> $: A^* \to \mathbb{R}$ by structural induction :
  1. **Base step**: <u>foo</u>$(\varepsilon) \stackrel{\text{def}}{=} 1.0$,
  2. **Inductive step**: <u>foo</u>$(s \cdot x) \stackrel{\text{def}}{=}$ <u>foo</u>$(s) \times 0.5$.

Properties of the function <u>reverse</u> : $A^* \to A^*$

▶ **Proposition**:

For all strings $s, t \in A^*$, it holds that $\underline{\text{reverse}}(s \cdot t) = \underline{\text{reverse}}(t) \cdot \underline{\text{reverse}}(s)$.

Proof: Use *structural induction* on $t \in A^*$ to prove the property $P(t)$ defined by:

$$P(t) \stackrel{\text{def}}{=} \text{"for all } s \in A^* \text{ it holds that } \underline{\text{reverse}}(s \cdot t) = \underline{\text{reverse}}(t) \cdot \underline{\text{reverse}}(s) \text{"}$$

This induction is on the structure of t as a string, there is no induction on natural numbers here.

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▶ **Proposition**:

For all strings $s \in A^*$, it holds that $\underline{\text{reverse}}(\underline{\text{reverse}}(s)) = s$.

Proof: By *structural induction*. (*Hint*: Use preceding proposition as a lemma.)

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▶ **Proposition**:

The function <u>reverse</u> : $A^* \to A^*$ is one-one and onto, *i.e.*, a $\boxed{\text{bijection}}$.

Proof: By *structural induction*.

<u>reverse</u> is *one-one*: for all $s_1, s_2 \in A^*$, if $\underline{reverse}(s_1) = \underline{reverse}(s_2)$ then $s_1 = s_2$.

<u>reverse</u> is *onto*: for every $t \in A^*$ there is $s \in A^*$ such that $\underline{reverse}(s) = t$.

Properties of the function <u>reverse</u> : $A^* \to A^*$

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Binary trees with labels in A

▶ Inductive definition of the set $\text{BN}(A)$ of binary trees with labels in A:

    1. **Base step**: if $x \in A$, then $x \in \text{BN}(A)$,

    2. **Inductive step**: if $x \in A$ and $t_1, t_2 \in \text{BN}(A)$, then $\langle x, t_1, t_2 \rangle \in \text{BN}(A)$.

▶ Different standard traversals of binary trees:
pre_order, post_order, in_order$^\text{L}$ (*from the left*), in_order$^\text{R}$ (*from the right*), etc.

▶ Definition of $\boxed{\text{in\_order}^\text{L} : \text{BN}(A) \to A^*}$ by structural induction:

    1. **Base step**: $\text{in\_order}^\text{L}(x) \overset{\text{def}}{=} x$,

    2. **Inductive step**: $\text{in\_order}^\text{L}(\langle x, t_1, t_2 \rangle) \overset{\text{def}}{=} \text{in\_order}^\text{L}(t_1) \cdot x \cdot \text{in\_order}^\text{L}(t_2)$.

▶ Definition of $\boxed{\text{in\_order}^\text{R} : \text{BN}(A) \to A^*}$ by structural induction:

    1. **Base step**: $\text{in\_order}^\text{R}(x) \overset{\text{def}}{=} x$,

    2. **Inductive step**: $\text{in\_order}^\text{R}(\langle x, t_1, t_2 \rangle) \overset{\text{def}}{=} \text{in\_order}^\text{R}(t_2) \cdot x \cdot \text{in\_order}^\text{R}(t_1)$.

in_order^L, in_order^R, and reverse

A nice property which is best proved by *structural induction* . . .

- ▶ **Proposition**:
  For all binary trees $t \in BN(A)$, it holds that $\underline{\text{reverse}}(\text{in\_order}^L(t)) = \text{in\_order}^R(t)$.

  Proof: We prove by *structural induction* on $t \in BN(A)$ the property $P(t)$ defined by:

  $$P(t) \stackrel{\text{def}}{=} \text{``for all } s \in BN(A) \text{ it holds that } \underline{\text{reverse}}(\text{in\_order}^L(t)) = \text{in\_order}^R(s) \text{''}$$

  This induction is on the structure of t as a binary tree, there is no induction on natural numbers here.

  1. **Base step**: Prove $P(t)$ when $t = x \in A$.
  2. **Inductive step**: Prove $P(t)$ when $t = \langle x, t_1, t_2 \rangle$, using the
     induction hypothesis (IH) which states that: $P(t_1)$ holds and $P(t_2)$ holds.

- ▶ **Exercise**: Fill in the missing details in the preceding proof.
  Hint: For the *inductive step*, you will need to use the definition of reverse on
  slide 2 and the first proposition on slide 3 .

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