CS 511, Fall 2020, Lecture Slides 25

Analytic Tableaux for Classical First-Order Logic (Part 1)

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October 15, 2020

REVIEW and PRELIMINARIES

- This handout continues Lecture Slides 08, which introduced tableaux for propositional logic.
- All the comments in Lecture Slides 08, including comments about extensions to non-classical logics, apply again to tableaux for first-order logic.
- We only need to introduce expansion rules for quantifiers in this handout.
- **GOAL**: We use tableaux to check the validity of first-order sentences, *i.e.*, closed first-order WFF's.

  This is not a limitation because, if there are free variables in the WFF's to be checked, these free variables are implicitly universally quantified. We can therefore add universal quantifiers over the free variables in the WFF's to turn them into first-order sentences.

REVIEW and PRELIMINARIES

- There are two approaches for presenting first-order tableaux:
  1. using ground instantiation (henceforth called ***ground tableaux***),
  2. using free variables (henceforth called ***free-variable tableaux***).
- The first approach is simpler to explain than the second approach, as it involves some of the material (***Herbrand theory***) already used to present Gilmore's algorithm in Lecture Slides 24.
- The second approach requires a prior study of ***unification theory***.
- In both approaches, we assume we start from a finite set Γ of first-order sentences which are all in ***prenex normal form***.
- The set Γ is ***unsatisfiable*** iff there is a tableau that starts from Γ at its root and all its paths are closed.

first TABLEAU method: INSTANTIATION WITH GROUND TERMS

- We use the expansion rules for the simple version of tableaux for classical PL (in Lecture Slides 08), and add two new expansion rules:
  - rule $(\forall)$ for sentences that start with a universal quantifier, and
  - rule $(\exists)$ for sentences that start with an existential quantifier.

  $$(\forall) \quad \frac{\forall x\, \varphi(x)}{\varphi[x := t]} \qquad\qquad (\exists) \quad \frac{\exists x\, \varphi(x)}{\varphi[x := c]}$$

  where t is an arbitrary ground term and c is a fresh constant symbol.

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- Rule $(\forall)$ is non-deterministic : In general, there are many ground terms to chose from, and rule $(\forall)$ can be used in as many different ways.

- Rule $(\exists)$ is deterministic : It is used the same way every time it is applied.

  Caution: Every time it is applied, even to the same WFF, a new fresh constant is introduced.

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- Repeated applications of $(\forall)$ to the same sentence along the same path involve instantiations with *different* gound terms.

  Caution: The set of ground terms (available for substitution) grows every time $(\exists)$ is applied.

first TABLEAU method: example

We apply the method to a finite set of first-order sentences

$$\Gamma \stackrel{\text{def}}{=} \Big\{\forall x\,(\neg P(x) \vee P(f\,x)),\ P(b),\ \neg P(f\,f\,b)\Big\}$$

over the signature $\Sigma = \{P, f, b\}$. We write "$f\,x$" instead of "$f(x)$".

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$$\forall x\,(\neg P(x) \vee P(f\,x))$$

$$\neg P(b) \vee P(f\,b)$$

$$\begin{array}{ccc}
\neg P(b) & & P(f\,b) \\
| & & | \\
P(b) & & \neg P(f\,b) \vee P(f\,f\,b) \\
| & & \begin{array}{cc} \neg P(f\,b) & P(f\,f\,b) \\ | & | \\ \text{X} & \neg P(f\,f\,b) \\ & | \\ & \text{X} \end{array} \\
\text{X} & &
\end{array}$$

EXPANSION RULES

$$(\wedge)\quad \frac{\varphi \wedge \psi}{\begin{array}{c}\varphi \\ \psi\end{array}}$$

$$(\neg\wedge)\quad \frac{\neg(\varphi \wedge \psi)}{\neg\varphi \mid \neg\psi}$$

$$\vdots$$

$$(\neg\neg)\quad \frac{\neg\neg\varphi}{\varphi}$$

$$(\forall)\quad \frac{\forall x\,\varphi(x)}{\varphi[x := t]}$$

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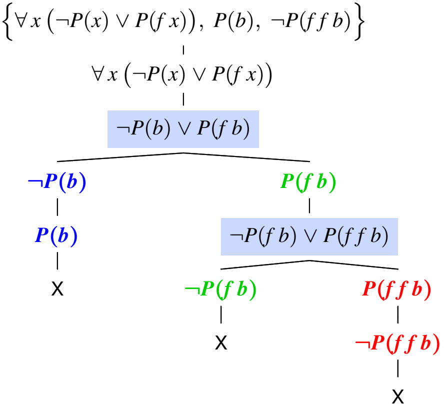

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The tableau is **closed** because every path includes a ground atom and its negation (*e.g.*, $\neg P(b)$ and $P(b)$). Γ is **unsatisfiable**.

first TABLEAU method: INSTANTIATION WITH GROUND TERMS

- ***Problem***. For the same initial finite set Γ of first-order sentences, a tableau is ***not uniquely defined***, because:
  - order in which the expansion rules are applied is not fixed (already a problem with tableaux for propositional logic),
  - instantiation by ground terms in rule $(\forall)$ is not fixed (in general, infinitely many ground terms are available),
  - we cannot enforce strictness of the tableaux, *i.e.*, rule $(\forall)$ has to be applied to the same WFF more than once along the same path.

(In the example, the two shaded WFF's result from applying $(\forall)$ to the same WFF.)

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- ***Soundness*** of rules $(\forall)$ and $(\exists)$ (together with the rules for propositional tableaux) is immediate: *If we can generate a closed tableau from an initial set Γ of sentences (in prenex normal form), then Γ is unsatisfiable.*

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- ***Completeness*** of rules $(\forall)$ and $(\exists)$ (together with the rules for propositional tableaux) can also be proved (using Herbrand theory): *If a set Γ of sentences (in prenex normal form) is unsatisfiable, then there exists a closed tableau generated from Γ by these rules.*

first TABLEAU method: example

We apply the method to the set Γ over the signature $\Sigma = \{P, f, b\}$

$$\Gamma \stackrel{\text{def}}{=} \Big\{\forall x\, P(x),\ \neg P(f\, b)\Big\}\ .$$

$$\Big\{\forall x\, P(x),\ \neg P(f\, b)\Big\}$$

$$|$$

$$\forall x\, P(x)$$

$$|$$

$$P(b)$$

$$|$$

$$P(f\, b)$$

$$|$$

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$$|$$

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Although Γ is unsatisfiable, we applied rule $(\forall)$ in the tableau carelessly and never closed it. Caution needs to be used to avoid such situations.

first TABLEAU method: exercises

1. **Exercise**. Use first-order ground tableaux to show that:

   $$\Gamma \models \varphi \qquad \text{where}$$

   $$\Gamma \overset{\text{def}}{=} \Big\{\forall x \forall y \forall z\, \big(P(x,y) \wedge P(y,z) \to P(x,z)\big),\ \forall x \forall y\, \big(P(x,y) \to P(y,x)\big)\Big\}$$

   $$\varphi \overset{\text{def}}{=} \forall x \forall y \forall z\, \big(P(x,y) \wedge P(z,y) \to P(x,z)\big)$$

   where P is a binary predicate symbol. Note there is no contant symbol and no function symbol in $\Gamma \cup \{\varphi\}$.

2. **Exercise**. Use first-order ground tableaux to show that:

   $$\Gamma \models \varphi \qquad \text{where}$$

   $$\begin{aligned}\Gamma \overset{\text{def}}{=} \Big\{&\forall x\, Q(a,x,x),\\ &\forall x \forall y \forall z\, \big(Q(x,y,z) \to Q(x,s(y),s(z))\big),\\ &\forall x \forall y \forall z\, \big(Q(x,y,z) \to Q(y,x,z)\big)\Big\}\end{aligned}$$

   $$\varphi \overset{\text{def}}{=} \exists x\, Q\big(s^{(2)}(a), s^{(3)}(a), x\big)$$

   where Q is a ternary predicate symbol, s is a unary function symbol, and a is a constant symbol. We write $s^{(2)}(a)$ and $s^{(3)}(a)$ for $s(s(a))$ and $s(s(s(a)))$.

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