Likelihoods

Most information in this exercise comes from Yudi Pawitan's book 'In all likelihood: Statistical modelling and inference using likelihood' and Alex Etz's blog posts on likelihoods.

Likelihood approaches to statistical inferences can be seen as a third way to draw inferences from data, separate from Frequentist and Bayesian statistics. At the same time, likelihood functions are an important part of Bayesian statistics, so a better understanding of likelihoods will also make it easier to understand Bayesian statistics later. Where Frequentist and Bayesian statistics only allow probability-based inferences, the likelihood approach suggests that inference is possible directly from the likelihood function.

We can use likelihood functions to say something about unknown quantities. Let's imagine you flip a coin 10 times, and it turns up heads 8 times. What is the true probability (which we will indicate by the Greek letter theta, θ) of this coin landing on heads?

The **binomial probability** of observing x successes in n studies is:

$$P(\theta) = \frac{n!}{x!\,(n-x)!} * \theta^{x} * (1-\theta)^{n-x}$$

where θ is the probability of a success. The first term indicates the number of possible combinations of results (e.g., you could start out with eight successes, end with eight successes, or any of the other possible combinations), which is multiplied by the probability of observing one success in each of the trials, which is then multiplied by the probability of observing no success in the remaining trials.

Q1: Let's assume you expect this is a fair coin. What is the binomial probability of observing 8 heads out of 10 coin flips, when θ = 0.5?

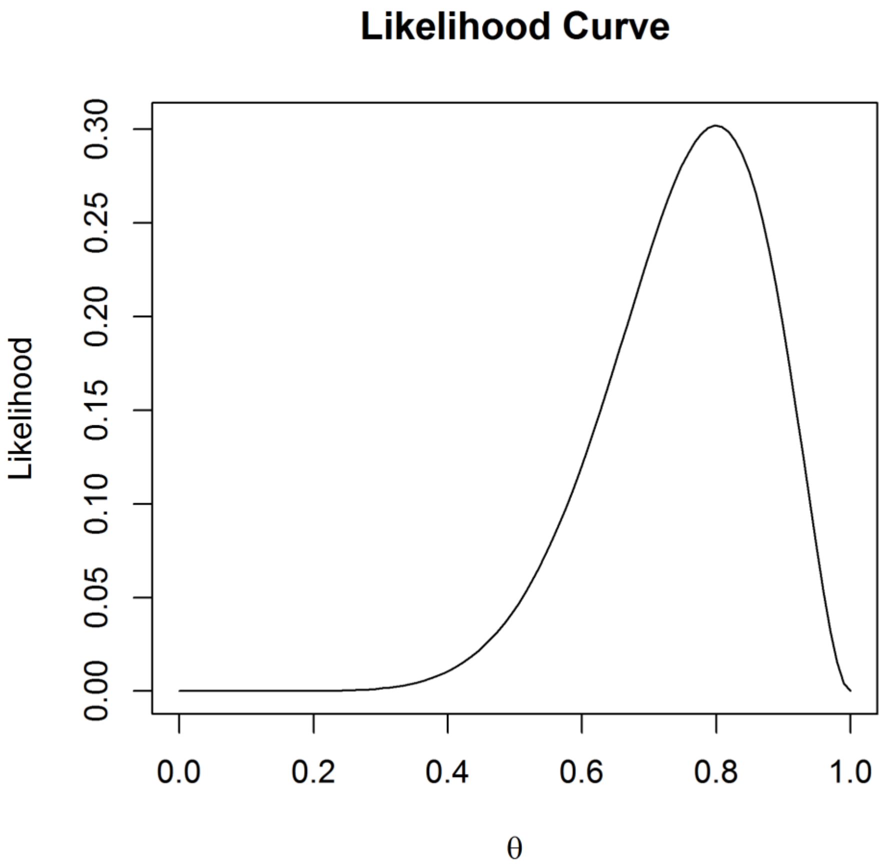
A) 0.044

B) 0.05

C) 0.5

D) 0.8

Let's assume we don't have any other information about this coin. (You might believe most coins are fair; such priors will be discussed when we talk about Bayesian statistics). Based on the data we have observed, which value for θ has the **maximum likelihood**? Fisher calls this maximum likelihood estimation, and published it when he was 22 as a third year undergraduate (in addition to contributing to a huge number of areas in statistics, he is also one of the greatest biologists since Darwin). Since θ can be any value between 0 and 1, it is common to plot all values in what is known as the *likelihood curve*.



All possible values for θ from 0 to 1 are on the x-axis, and the likelihood is on the y-axis. It should not be surprising that the best guess we have, or the most likely value for θ, is that the true parameter is 8 out of 10, or θ = 0.8, with a likelihood of 0.30 (the highest point on the y-axis). It is important to know that the value of the likelihood itself has no meaning in isolation. In this sense, it differs from a probability. The likelihood of 0.30 does not mean much in isolation, but we can compare likelihoods of the same curve, and compare different values of θ. You can read off any other value for any other θ, and see that very low values (e.g., 0.2) are not very likely.

Probabilities and likelihoods are related, but different. In probability, you start with a given parameter (e.g., the probability a coin is fair, or the probability of heads being θ = 0.5) and you estimate the probability of a specific sample (e.g., getting 5 heads out of 10 coin tosses. Likelihoods start with a specific sample (e.g., observing 5 heads out of 10 coin tosses), and ask the likelihood of different parameters (e.g., the likelihood of θ = 0.5). Likelihoods are a statistical inference: We have observed some data, and we use this data to draw an inference about the likelihood of different parameters. More formally, the likelihood function is the (joint) density function evaluated at the observed data. Likelihood functions can be calculated for many different models (binomial distributions, normal distributions, etc., see Millar, 2011).

Q2: The likelihood curve rises up and falls down, except at the extremes, when 0 heads or only heads are observed. Open the PlotLikelihood.R script, and plot the likelihood curves for 0 heads by changing the number of successes in line 3 to 0, and running the script. What does the likelihood curve look like?

A) The likelihood curve is a horizontal line.

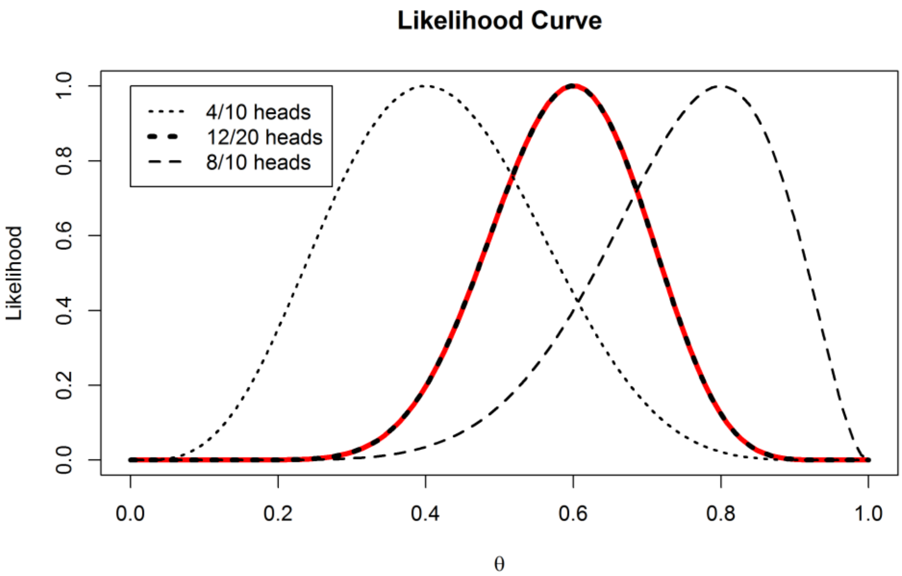
B) The script returns and error message: it is not possible to plot the likelihood curve for 0 heads.

C) The curve starts at its highest point at θ = 0, and then the likelihood decreases as θ increases.

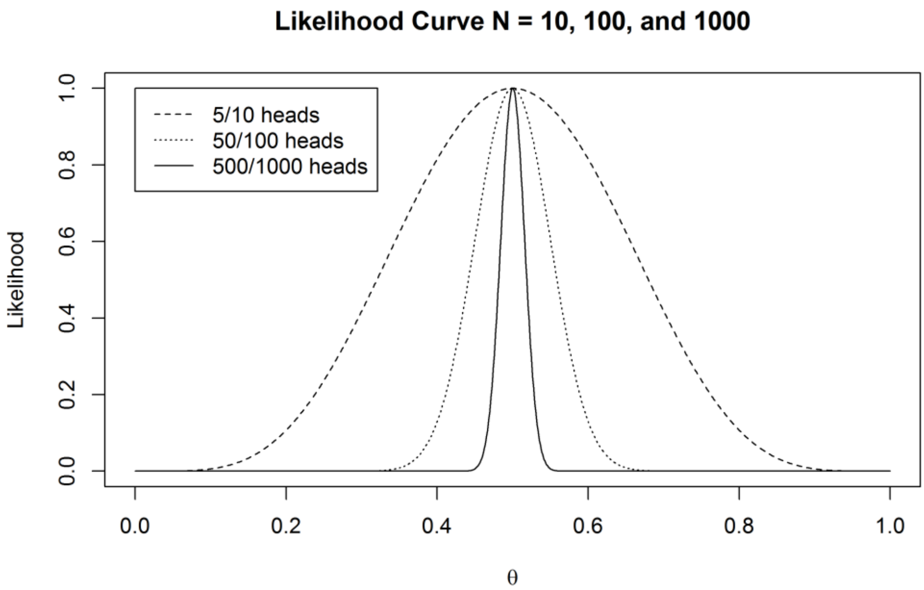
D) The curve starts at its lowest point at θ = 0, and then the likelihood increases as θ increases.

Likelihoods can easily be combined. Imagine we have two people flipping the same coin independently. One person observes eight heads out of 10 flips, and the other observes 4 heads out of 10 flips. You might believe that this should give the same likelihood curve as one person flipping a coin 20 times, and observing 12 heads, and indeed, it does. In the plot below, all likelihood curves are standardized by dividing the curve by the maximum of each likelihood curve. This is why all curves now have a maximum of 1, and we can more easily compare different likelihood curves.

The curve on left is for 4 out of 10 heads, the one on the right is for 8 out of 10 heads. The black dotted curve in the middle is for 12 out of 20 heads. The red curve, exactly underneath the 12 out of 20 heads curve, is calculated by multiplying the likelihood curves: $L(\theta_{combined}) = L(\theta = 0.8) * L(\theta = 0.4)$. In the plot below, you can see that multiplying the likelihood curves for 4/10 heads and 8/10 heads (the red line) gives the same likelihood curve as that of 12/20 heads (black dotted line).



In the plot below, 10, 100, and 1000 coin flips are plotted, which yield 5, 50, and 500 heads, respectively. The likelihood curves are again standardized to make them more easily comparable. As the sample size increases, the curves become more narrow (the dashed line is for n = 10, the dotted line is for n = 100, and the solid line is for n = 1000). This means that as the sample size increases, all other values than 0.5 become increasingly less likely. Or, in other words, we have collected increasingly strong evidence for θ = 0.5, compared to most other possible values.

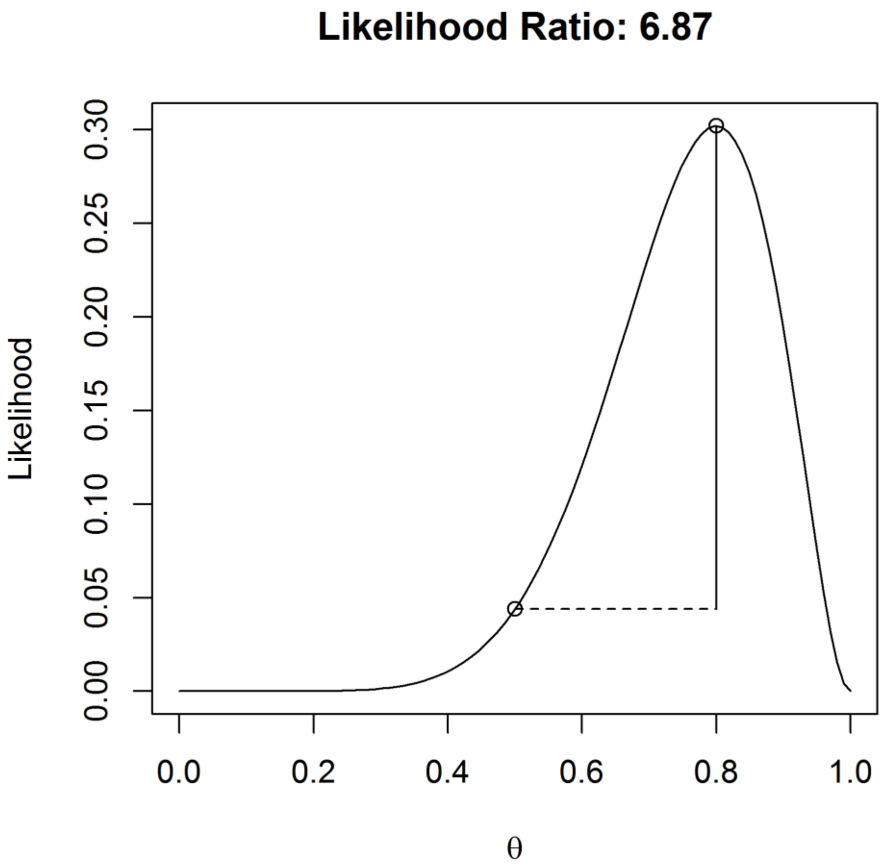


We can use the likelihood to compare possible values of θ. For example, we might believe the coin we flipped was fair, even though we flipped eight out of ten heads. A fair coin will have θ = 0.5, while we observed θ = 0.8. The likelihood tells us the relative preference we might have for different possible parameters. Given the result that we have observed, how much more likely is that this is an unfair coin that will on average give heads 80% of the time, compared to the alternative theory that this is a fair coin which should give heads 50% of the time?

We can calculate the likelihood ratio:

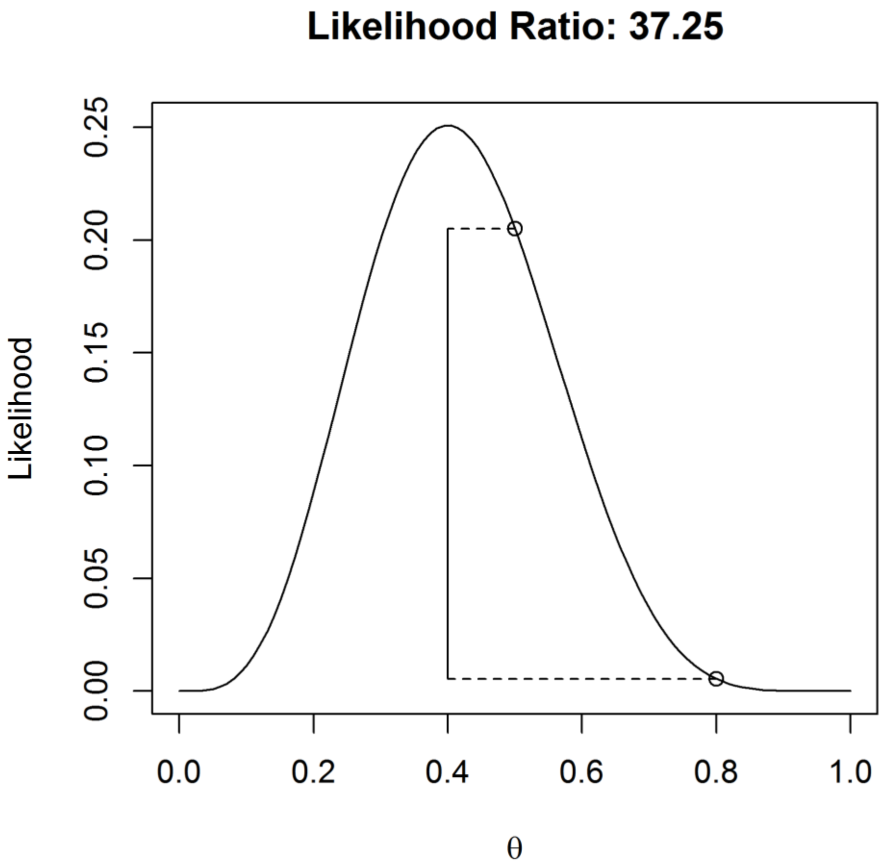
$$\frac{L(\theta = 0.8)}{L(\theta = 0.5)}$$

Which is 0.302/0.044=6.87. In the plot below, both circles show the points on the likelihood curve for L(θ = 0.5) and L(θ = 0.8).



We can subjectively interpret this likelihood ratio, which tells us an unfair coin that will give 80% heads is 6.87 times more likely than a fair coin. How convincing is this? Let's round the likelihood ratio to 7, and imagine two bags of marbles. One bag contains 7 blue marbles. The second contains 7 marbles, each one a different color of the rainbow, so violet, indigo, blue, green, yellow, orange, and red. Someone randomly picks one of the two bags, draws a marble, and shows it to you. The marble is blue: How certain are you this is the bag with all blue marbles, compared to the bag with rainbow coloured marbles? This is how strong the likelihood ratio tells us to believe an unfair coin with 80% heads is over a fair coin with 50% heads, given that we have flipped 8 heads in 10 tosses.

Note that likelihood ratios give us the relative evidence for one specified hypothesis, over another specified hypothesis. The likelihood ratio can be calculated for any two hypothesized values. For example, in the graph below, the likelihood ratio is calculated that compares the hypothesis for a fair coin (θ = 0.5) with the alternative hypothesis that the coin produces 80% heads (θ = 0.8), when we have observed 4 heads out of 10 coin flips. We see the hypothesis that this is a fair coin is 0.2050/0.0055=37.25 times (ignoring rounding differences – and try to calculate these numbers by hand using the formula on page 1) more likely than a coin that gives 80% heads, based on the observed data.



A likelihood ratio of 1 means both hypotheses are equally likely. Values further away from 1 indicate one hypothesis is more likely than the other. The ratio can be expressed in favor of one hypothesis over the other (for example L(θ = 0.5)/L(θ = 0.8) or vice versa (L(θ

= 0.8)/[L(θ = 0.5). This means the likelihood ratio of 37.25 is equivalent to a likelihood ratio of 1/37.25 = 0.02685. Likelihood ratios range from 0 to infinity, and the closer to zero or infinity, the stronger the relative evidence for one over the other.

Q3: Get a coin out of your wallet. Flip it 13 times, and count the number of heads. Open the R file CalculateLikelihoodRatio.R to calculate the likelihood that your coin is fair, compared to the likelihood that the coin is not fair, and will give the % of heads you observed. In line 3, set the number of successes to the number of heads you observed. In line 5, change the 0 in 0/13 to the number of heads you have observed (or leave it to 0 if you didn't observe any heads at all!). Run the script to calculate the likelihood ratio. What is the likelihood ratio of a fair compared to a non-fair coin (or H0/H1) that flips heads as often as you have observed, based on the observed data?

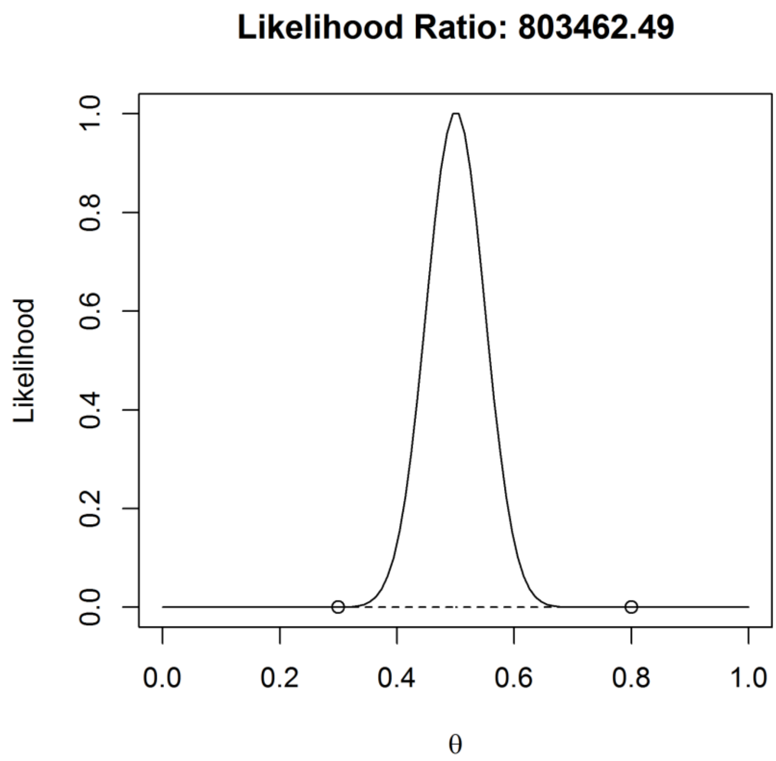
Earlier we mentioned that with increasing sample sizes, we had collected stronger relative evidence. Let's say we would want to compare L(θ = 0.4) with L(θ = 0.5).

Q4: What is the likelihood ratio for 5 out of 10 heads?

Q5: What is the likelihood ratio for 50 out of 100 heads?

Q6: What is the likelihood ratio for 500 out of 1000 heads?

Likelihoods are relative evidence. Just because one possible value of θ is more likely than another value, doesn't mean that it is likely. Other values might be even more likely. For example, consider the situation where we flip a coin 100 times, and observe 50 heads. We compare θ = 0.3 versus θ = 0.8, and find that the likelihood ratio is 803462, implying that θ = 0.3 is 803461 times more likely than θ = 0.8. That might sound pretty conclusive evidence for θ = 0.3. But it is only relative evidence for θ = 0.3 compared to θ = 0.8. If we look at the likelihood function, we clearly see that, non-surprisingly, θ = 0.5 is much more likely than both other values considered (you can recreate this plot in R if you want to).



Q7: When comparing two hypotheses (θ = X vs θ = Y), a likelihood ratio of:

A) 0.02 means that neither of the two hypotheses is very likely.

B) 5493 means that hypothesis θ = X is very likely to be true.

C) 5493 means that hypothesis θ = X is much more likely than θ = Y.

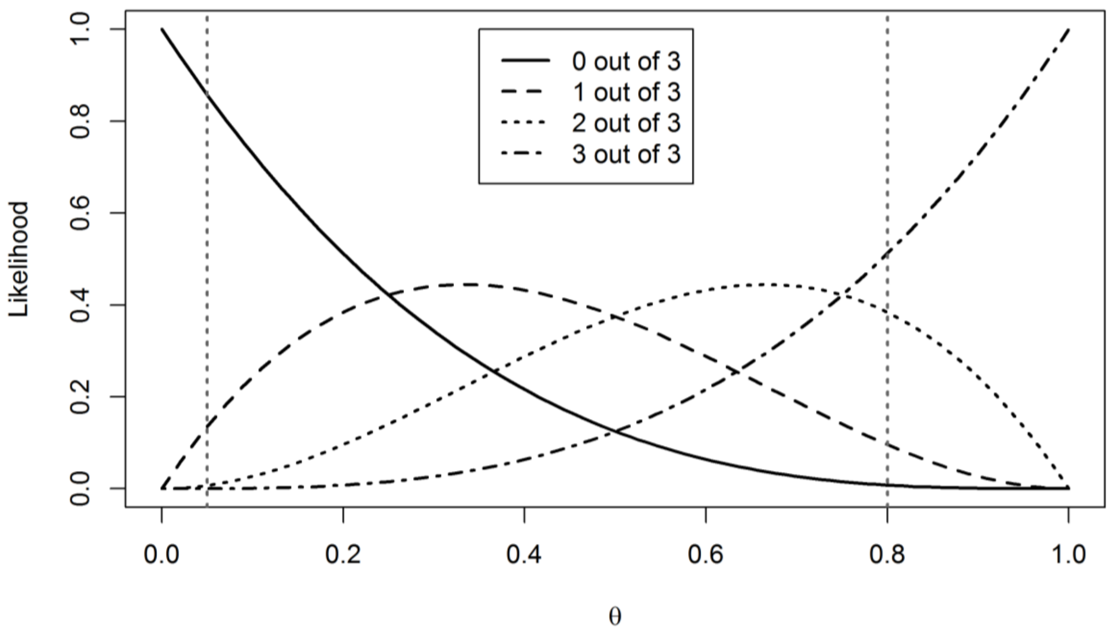
D) 0.02 means that the hypothesis that θ = X is 2% more likely to be true than θ = Y.

Likelihoods of sets of studies

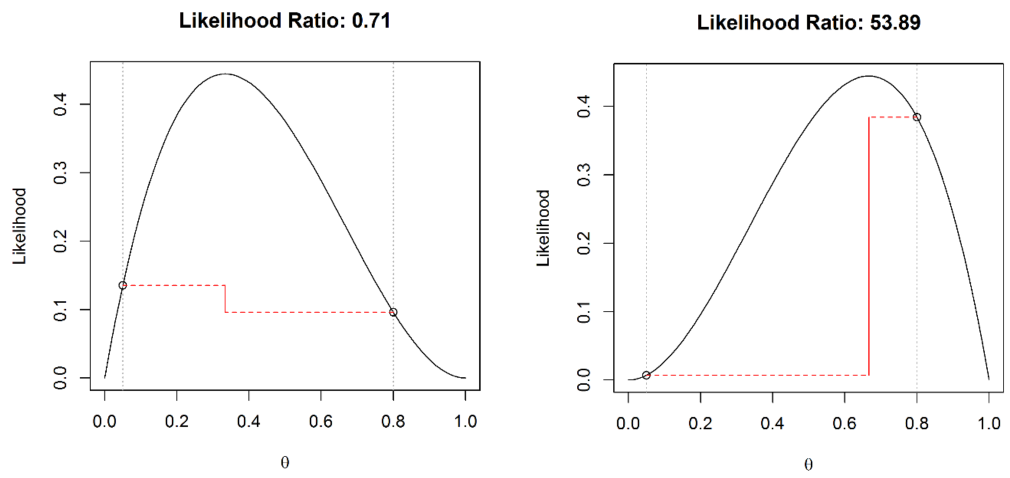
Let's imagine we have two bags. There are twenty marbles in each bag. In one bag, we know there are 19 blue marbles, and one red marble. The red marble represents a Type 1 error, the blue marbles represent true negatives, and this bag represents a situation where we perform a statistical test where the null-hypothesis is true. In the other bag, there is a number of blue and red marbles. The red marbles represent true positives, blue marbles represent false negatives, and this bag represents a situation where the alternative hypothesis is true. We don't know the true power, or the percentage of red marbles, but we can make a guess. For example, we might believe 16 out of 20 (or 80%) of the marbles are red.

We perform 3 studies, by drawing a ball three times (with replacement) from one of the two bags. We don't know which of the bags we are drawing from. We could be drawing from the bag where the null hypothesis is true or the bag where the alternative hypothesis is true. There are 4 possible outcomes. Either 0 out of 3, 1 out of 3, 2 out of 3, or 3 out of 3 red balls are drawn. We can plot these four likelihood curves (see the figure below – you can plot these curves one at a time using the R script).

Each curve has the maximum likelihood estimate at the outcome: θ = 0 for 0 out of 3 red balls, θ = 0.33 for 1 out of 3, θ = 0.66 for 2 out of 3, and θ = 1 from 3 out of 3. We know that for the bag where the null hypothesis is true, θ = 0.05, or the Type 1 error rate. If we believe our studies would have 80% power when the alternative hypothesis is true, we can assume θ = 0.8. In the plot below, we see the four likelihood curves, and two vertical lines at θ = 0.05 and θ = 0.8.



We can calculate likelihood ratios for θ = 0.05 vs. θ = 0.80 for the different outcomes. Below, the likelihood ratios are visualized for 1 out of 3 and 2 out of 3 red balls, or significant results. We see that only 1 out of 3 findings (below, left) is slightly more likely when the null hypothesis is true, but the likelihood ratio of 0.71 is not very far from 1. However, with 2 out of 3 significant results (below, right), it is clear that this result is much more likely when studies are performed with 80% power than if we assume we observed two out of three Type 1 errors. Indeed, when two out of three studies are significant, the likelihood ratio provides pretty strong relative evidence for a situation where the alternative hypothesis is true, even when the assumed power is much lower.



Although we can't formally evaluate the probability that the alternative hypothesis is true based on these likelihood ratios (we instead need Bayesian statistics), we can see that when multiple studies are performed, but not all studies are statistically significant, it might very well be more likely that a true effect is examined, than that all significant studies are Type 1 errors.

The graphs above with the four likelihood curves also shows when a researcher is more likely to observe mixed results than consistent results. This occurs for all values of θ where the likelihood curve for mixed results is higher than the likelihood curves for 0 out of *n*, or *n* out of *n* significant results. More formally, mixed results are more likely than consistent results when power drops below *n*/(*n*+1) percent, or increases above 1-(*n*/(*n*+1)). For example, when performing three studies, it is more likely to observe mixed results than only significant or only non-significant effects when power is lower than 3/(3+1)=0.75, and higher than 1-(3/(3+1))=0.25.

We have seen how likelihood functions allow us to evaluate the relative likelihood of different possible true values, given the data we have collected. In this assignment, we have focused on binomial likelihoods, but likelihood functions exist for many different distributions (e.g., Poisson, normal, etc.). We have applied this basic understanding of binomial likelihoods to the likelihood that a set of studies with mixed results (e.g., 2 out of 3 significant results) occurs when we are making Type 1 errors compared to when we have a specific level of power. We will continue using binomial likelihoods when we learn about Bayesian statistics.