A categorical approach to the maximum theorem

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Maximum theorem

Let

- $J \subseteq A \times B$ be a relation between topological spaces
- $d \colon B \to [0, \infty]$ be a continuous map

Consider the function $r \colon A \to [0, \infty]$ given by the suprema

$$rx = \sup d(Jx).$$

Berge's *maximum theorem* (1959) gives conditions on J ensuring the continuity of r.

Aims:

- give a categorical proof of the maximum theorem
- use this to obtain generalisations to e.g. closure spaces, approach spaces and metric closure spaces?

Topological spaces as relational algebras

Let U be the "ultrafilter monad" on Set.

Theorem (Barr, 1970)

Topologies τ on A correspond precisely to relations $\alpha\colon UA \nrightarrow A$ satisfying certain associativity and unit axioms.

We can change the type of relations here...

Theorem (Clementino–Hofmann, 2003)

Associative and unital $[0,\infty]$-*relations* $\alpha\colon UA \times A \to [0,\infty]$ correspond to *approach distances* $\delta\colon A \times PA \to [0,\infty]$ on A, making it into an *approach space*.

...or the monad.

Theorem

Associative and unital relations $\alpha\colon PA \nrightarrow A$ correspond to *closure operations* on A.

$\mathcal{V}$-relations

Consider relations $J\colon A \times B \to \mathcal{V}$ with values in a *quantale* $\mathcal{V} = (\mathcal{V}, \otimes, k)$.

Examples

- $2 = (\{\bot \leq \top\}, \wedge, \top)$
- $P_+ = (([0,\infty], \geq), +, 0)$

Composition

$$(J \odot H)(x,z) = \sup_{y \in B} J(x,y) \otimes H(y,z)$$

Cells

$$\begin{tikzcd} A \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "\exists!\ f"'] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, "g"] \\ C \arrow[r, "K"', "\shortmid"{marking}] & D \end{tikzcd} \iff \forall_{\substack{x \in A \\ y \in B}} J(x,y) \leq K(fx, gy)$$

$(T, \mathcal{V})$-monoids

Definition

Let T be a Set-monad that "extends to $\mathcal{V}$-relations".

- a $(T, \mathcal{V})$-*monoid* A is a set A equipped with an associative and unital $\mathcal{V}$-relation $\alpha \colon TA \nrightarrow A$
- a *morphism* $f \colon A \to C$ between $(T, \mathcal{V})$-monoids is a function $f \colon A \to C$ such that the cell below exists

$$\begin{tikzcd} TA \arrow[r, "\alpha", "|" marking] \arrow[d, "Tf"'] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, "f"] \\ TC \arrow[r, "\gamma"', "|" marking] & C \end{tikzcd}$$

If $T = \mathrm{id}$ then we call $(\mathrm{id}, \mathcal{V})$-monoids $\mathcal{V}$*-monoids*

- 2-monoids are preorders
- P_+-monoids are generalised metric spaces (Lawvere)

$(T, \mathcal{V})$-monoids from $(T, \mathcal{V})$-algebras

A $(T, \mathcal{V})$-*algebra* M is a triple $M = (M, \bar{M} \colon M \nrightarrow M, m \colon TM \to M)$ with $(M, \bar{M})$ a $\mathcal{V}$-monoid and (M, m) a T-algebra, such that m is a map of $\mathcal{V}$-monoids.

Lemma

For any $(T, \mathcal{V})$-algebra $M = (M, \bar{M}, m)$ the $\mathcal{V}$-relation $\bar{M}(m, \mathrm{id}) \colon TM \nrightarrow M$ is a $(T, \mathcal{V})$-monoid structure on M.

The case $M = P_+$

- (Manes, 2002) P_+ admits both U- and P-algebra structures, both of them a morphism of 2-monoids as well as P_+-monoids
- the topology on P_+ is generated by the closed intervals $[0, x]$
- the approach distance on P_+ is given by $\delta(x, S) = x \ominus \sup S$
- for a topological/closure space A, $f \colon A \to P_+$ is morphism of $(T, \mathcal{V})$-monoids precisely if it is *lower semicontinuous*:

$$\bigforall_{x \in P_+} f^{-1}([0, x]) \text{ is closed in } A$$

Right Kan extensions into $\mathcal{V}$-monoids

Let $J\colon A \nrightarrow B$ be a $\mathcal{V}$-relation and $d\colon B \to M$ a function into a $\mathcal{V}$-monoid $M = (M, \bar{M}\colon M \nrightarrow M)$. A cell

$$\begin{tikzcd} A \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "r"'] \arrow[dr, phantom, "\Downarrow\varepsilon"] & B \arrow[d, "d"] \\ M \arrow[r, "\bar{M}"', "\shortmid"{marking}] & M \end{tikzcd}$$

defines $r\colon A \to M$ as the *right Kan extension* of d along J if every cell on the left below factors as shown.

$$\begin{tikzcd} C \arrow[r, "H", "\shortmid"{marking}] \arrow[d, "s"'] & A \arrow[r, "J", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow\phi"] & B \arrow[d, "d"] \\ M \arrow[rr, "\bar{M}"', "\shortmid"{marking}] & & M \end{tikzcd} = \begin{tikzcd} C \arrow[r, "H", "\shortmid"{marking}] \arrow[d, "s"'] \arrow[dr, phantom, "\Downarrow\phi'"] & A \arrow[r, "J", "\shortmid"{marking}] \arrow[d, "r"] \arrow[dr, phantom, "\Downarrow\varepsilon"] & B \arrow[d, "d"] \\ M \arrow[r, "\bar{M}"', "\shortmid"{marking}] \arrow[d, equal] & M \arrow[r, "\bar{M}"', "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow"] & M \arrow[d, equal] \\ M \arrow[rr, "\bar{M}"', "\shortmid"{marking}] & & M \end{tikzcd}$$

Back to the maximum theorem: hemicontinuity of relations

As at the start, let $r\colon A \to P_+$ be the right Kan extension of a continuous map $d\colon B \to P_+$ along a relation $J\colon A \nrightarrow B$.

"Lower maximum theorem" (Berge)

r is lower semicontinuous as soon as $J\colon A \nrightarrow B$ is *lower hemicontinuous*: $J^\circ U$ is open in A for all $U \subseteq B$ open.

Definition

A $\mathcal{V}$-relation $J\colon A \nrightarrow B$ between $(T, \mathcal{V})$-monoids is called *lower hemicontinuous* if the cell on the left below exists.

$$
\begin{tikzcd}
TA \arrow[r, "\alpha", "|" marking] \arrow[d, equal] & A \arrow[r, "J", "|" marking] \arrow[d, phantom, "\Downarrow"] & B \arrow[d, equal] \\
TA \arrow[r, "TJ"', "|" marking] & TB \arrow[r, "\beta"', "|" marking] & B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
TA \arrow[r, "TJ", "|" marking] \arrow[d, equal] & TB \arrow[r, "\beta", "|" marking] \arrow[d, phantom, "\Downarrow"] & B \arrow[d, equal] \\
TA \arrow[r, "\alpha"', "|" marking] & A \arrow[r, "J"', "|" marking] & B
\end{tikzcd}
$$

Dually J is *upper hemicontinuous* if the cell on the right exists.

Examples

- for relations $J\colon A \nrightarrow B$ between topological spaces this recovers the notion of lower hemicontinuity just described:

$$J^{\circ}U \text{ is open in } A \text{ for all } U \subseteq B \text{ open}$$

- a relation $J\colon A \nrightarrow B$ between closure spaces is upper hemicontinuous if

$$JV \text{ is closed in } B \text{ for all } V \subseteq A \text{ closed}$$

- $J\colon A \nrightarrow B$ between topological spaces is upper hemicontinuous if the condition above holds and

$$J^{\circ}y \text{ is compact in } A \text{ for all } y \in B$$

- upper hemicontinuity of a P_{+}-relation $J\colon A \nrightarrow B$ between approach spaces is a "numerified generalisation" of topological upper hemicontinuity

Generalised lower maximum theorem

Let

- $M = (M, \bar{M}, m)$ be a $(T, \mathcal{V})$-algebra
- $J \colon A \nrightarrow B$ be a lower hemicontinuous $\mathcal{V}$-relation between $(T, \mathcal{V})$-monoids
- $d \colon B \to M$ be a morphism of $(T, \mathcal{V})$-monoids

The right Kan extension $r \colon A \to M$ of d along J, if it exists, is a morphism of $(T, \mathcal{V})$-monoids

Proof

Factor the following composite through the universal cell defining the right Kan extension r.

$$
\begin{tikzcd}
TA \arrow[r, "\alpha", "|" marking] \arrow[d, equal] & A \arrow[r, "J", "|" marking] \arrow[d, phantom, "\Downarrow"] & B \arrow[d, equal] \\
TA \arrow[r, "TJ"', "|" marking] \arrow[d, "Tr"'] \arrow[dr, phantom, "\Downarrow"] & TB \arrow[r, "\beta"', "|" marking] \arrow[d, "Td"] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, "d"] \\
TM \arrow[r, "T\bar{M}"', "|" marking] \arrow[d, "m"'] \arrow[dr, phantom, "\Downarrow"] & TM \arrow[r, "\bar{M}(m, \mathrm{id})"', "|" marking] \arrow[d, "m"] & M \arrow[d, equal] \\
M \arrow[r, "\bar{M}"', "|" marking] \arrow[d, equal] & M \arrow[r, "\bar{M}"', "|" marking] \arrow[d, phantom, "\Downarrow"] & M \arrow[d, equal] \\
M \arrow[rr, "\bar{M}"', "|" marking] & & M
\end{tikzcd}
$$

Generalised upper maximum theorem

Let

- $M = (M, \bar{M}, m)$ be a $(T, \mathcal{V})$-algebra
- $J \colon A \nrightarrow B$ be a upper hemicontinuous $\mathcal{V}$-relation between $(T, \mathcal{V})$-monoids
- $d \colon A \to M$ be a morphism of $(T, \mathcal{V})$-monoids

The left Kan extension $l \colon B \to M$ of d along $J \colon A \nrightarrow B$, if it exists, is a morphism of $(T, \mathcal{V})$-monoids as soon as the following hold:

- T preserves composites of $\mathcal{V}$-relations
- the cell below exists

$$
\begin{tikzcd}
B \arrow[rr, equal] \arrow[d, "l"'] \arrow[drr, phantom, "\Downarrow"] & & B \arrow[d, equal] \\
M \arrow[r, "\bar{M}(\mathrm{id}, d)"', "\shortmid" marking] & A \arrow[r, "J"', "\shortmid" marking] & B
\end{tikzcd}
$$

Extreme value theorem

The second condition here means:

- for $\mathcal{V} = 2$:
  $$\forall_{y \in B} \exists_{x \in J^\circ y} ly \leq dx$$
  that is the supremum ly of $d(J^\circ y)$ is in fact a maximum
- if $T = U$ then the above is implied by the theorem's assumptions on d and J, as long as the preorder M is total and $J^\circ y \neq \emptyset$ for all $y \in B$: this is Weierstraß' extreme value theorem!
- if $T = P$ then the above is implied by the assumption on d as long as M is total and $J^\circ y$ is both non-empty and compact for all $y \in B$

Extreme value theorem for approach spaces

- for $\mathcal{V} = P_+$ the second condition means that, for all $y \in B$,

$$\inf_{x \in A} \bar{M}(Iy, dx) + J(x, y) = 0$$

- if $T = U$ and $M = P_+$ then the above is implied by the theorem's assumptions on d and J, as long as J is discrete (i.e. $\operatorname{im} J \subseteq \{0, \infty\}$) with $J_0^\circ y$ non-empty for all $y \in B$