# A categorical approach to the maximum theorem

Seerp Roald Koudenburg

Middle East Technical University - Northern Cyprus Campus

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### Maximum theorem

#### Let

- ▶  $J \subseteq A \times B$  be a relation between topological spaces
- ▶  $d: B \rightarrow [0, \infty]$  be a continuous map

Consider the function  $r \colon A \to [0, \infty]$  given by the suprema

$$rx = \sup d(Jx).$$

Berge's  $maximum\ theorem\ (1959)$  gives conditions on J ensuring the continuity of r.

#### Aims:

- give a categorical proof of the maximum theorem
- ▶ use this to obtain generalisations to e.g. closure spaces, approach spaces and metric closure spaces?

## Topological spaces as relational algebras

Let *U* be the "ultrafilter monad" on Set.

## Theorem (Barr, 1970)

Topologies  $\tau$  on A correspond precisely to relations  $\alpha\colon UA \to A$  satisfying certain associativity and unit axioms.

We can change the type of relations here...

## Theorem (Clementino-Hofmann, 2003)

Associative and unital  $[0,\infty]$ -relations  $\alpha\colon UA\times A\to [0,\infty]$  correspond to approach distances  $\delta\colon A\times PA\to [0,\infty]$  on A, making it into an approach space.

...or the monad.

#### **Theorem**

Associative and unital relations  $\alpha \colon PA \to A$  correspond to *closure* operations on A.



## $\mathcal{V}$ -relations

Consider relations  $J \colon A \times B \to \mathcal{V}$  with values in a *quantale*  $\mathcal{V} = (\mathcal{V}, \otimes, k)$ .

## Examples

- $\triangleright \ 2 = (\{\bot \leq \top\}, \land, \top)$
- ▶  $P_+ = (([0,\infty], \ge), +, 0)$

## Composition

$$(J\odot H)(x,z)=\sup_{y\in B}J(x,y)\otimes H(y,z)$$

#### Cells

$$A \xrightarrow{J} B$$

$$\exists ! \ f \downarrow \qquad \downarrow \qquad \downarrow g \qquad \Longleftrightarrow \qquad \bigvee_{\substack{x \in A \\ V \in B}} J(x, y) \leq K(fx, gy)$$

# $(T, \mathcal{V})$ -monoids

#### Definition

Let T be a Set-monad that "extends to V-relations".

- ▶ a (T, V)-monoid A is a set A equipped with an associative and unital V-relation  $\alpha \colon TA \to A$
- ▶ a morphism  $f: A \to C$  between (T, V)-monoids is a function  $f: A \to C$  such that the cell below exists

$$TA \xrightarrow{\alpha} A$$

$$Tf \downarrow \qquad \downarrow f$$

$$TC \xrightarrow{\gamma} C$$

If  $T=\operatorname{id}$  then we call  $(\operatorname{id},\mathcal{V})$ -monoids  $\mathcal{V}$ -monoids

- ▶ 2-monoids are preorders
- ► P<sub>+</sub>-monoids are generalised metric spaces (Lawvere)



# (T, V)-monoids from (T, V)-algebras

A  $(T, \mathcal{V})$ -algebra M is a triple  $M = (M, \overline{M}: M \to M, m: TM \to M)$  with  $(M, \overline{M})$  a  $\mathcal{V}$ -monoid and (M, m) a T-algebra, such that m is a map of  $\mathcal{V}$ -monoids.

#### Lemma

For any  $(T, \mathcal{V})$ -algebra  $M = (M, \overline{M}, m)$  the  $\mathcal{V}$ -relation  $\overline{M}(m, \mathrm{id}) \colon TM \to M$  is a  $(T, \mathcal{V})$ -monoid structure on M.

The case  $M = P_+$ 

- ▶ (Manes, 2002)  $P_+$  admits both  $U_-$  and  $P_-$ algebra structures, both of them a morphism of 2-monoids as well as  $P_+$ -monoids
- ▶ the topology on  $P_+$  is generated by the closed intervals [0,x]
- ▶ the approach distance on  $P_+$  is given by  $\delta(x, S) = x \ominus \sup S$
- ▶ for a topological/closure space A,  $f: A \rightarrow P_+$  is morphism of (T, V)-monoids precisely if it is *lower semicontinuous*:

$$\bigvee_{x \in P_{\perp}} f^{-1}([0,x]) \text{ is closed in } A$$

## Right Kan extensions into $\mathcal{V}$ -monoids

$$\begin{array}{ccc}
A & \xrightarrow{J} & B \\
r \downarrow & & \downarrow \varepsilon & \downarrow d \\
M & \xrightarrow{\bar{M}} & M
\end{array}$$

defines  $r: A \to M$  as the *right Kan extension* of d along J if every cell on the left below factors as shown.

# Back to the maximum theorem: hemicontinuity of relations

As at the start, let  $r: A \to P_+$  be the right Kan extension of a continuous map  $d: B \to P_+$  along a relation  $J: A \to B$ .

## "Lower maximum theorem" (Berge)

r is lower semicontinuous as soon as  $J: A \rightarrow B$  is lower hemicontinuous:  $J^{\circ}U$  is open in A for all  $U \subseteq B$  open.

#### Definition

A V-relation  $J: A \rightarrow B$  between (T, V)-monoids is called *lower hemicontinuous* if the cell on the left below exists.

$$TA \xrightarrow{\alpha} A \xrightarrow{J} B \qquad TA \xrightarrow{TJ} TB \xrightarrow{\beta} B$$

$$\parallel \qquad \qquad \parallel \qquad \qquad \parallel \qquad \qquad \parallel \qquad \parallel \qquad \parallel$$

$$TA \xrightarrow{TJ} TB \xrightarrow{\beta} B \qquad TA \xrightarrow{\alpha} A \xrightarrow{J} B$$

Dually J is upper hemicontinuous if the cell on the right exists.



## Examples

▶ for relations  $J: A \rightarrow B$  between topological spaces this recovers the notion of lower hemicontinuity just described:

 $J^{\circ}U$  is open in A for all  $U\subseteq B$  open

▶ a relation J: A → B between closure spaces is upper hemicontinuous if

JV is closed in B for all  $V \subseteq A$  closed

J: A → B between topological spaces is upper hemicontinuous if the condition above holds and

 $J^{\circ}y$  is compact in A for all  $y \in B$ 

▶ upper hemicontinuity of a  $P_+$ -relation  $J: A \rightarrow B$  between approach spaces is a "numerified generalisation" of topological upper hemicontinuity

### Generalised lower maximum theorem

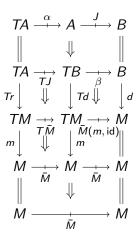
#### Let

- ▶  $M = (M, \overline{M}, m)$  be a (T, V)-algebra
- ▶  $d: B \to M$  be a morphism of (T, V)-monoids

The right Kan extension  $r: A \to M$  of d along J, if it exists, is a morphism of  $(T, \mathcal{V})$ -monoids

### Proof

Factor the following composite through the universal cell defining the right Kan extension r.



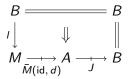
## Generalised upper maximum theorem

#### Let

- $M = (M, \overline{M}, m)$  be a (T, V)-algebra
- ▶  $d: A \rightarrow M$  be a morphism of (T, V)-monoids

The left Kan extension  $I: B \to M$  of d along  $J: A \to B$ , if it exists, is a morphism of  $(T, \mathcal{V})$ -monoids as soon as the following hold:

- ightharpoonup T preserves composites of  $\mathcal{V}$ -relations
- the cell below exists



#### Extreme value theorem

#### The second condition here means:

• for  $\mathcal{V}=2$ :

$$\forall_{y \in B} \exists_{x \in J^{\circ} y} ly \le dx$$

that is the supremum Iy of  $d(J^{\circ}y)$  is in fact a maximum

- ▶ if T = U then the above is implied by the theorem's assumptions on d and J, as long as the preorder M is total and  $J^{\circ}y \neq \emptyset$  for all  $y \in B$ : this is Weierstraß' extreme value theorem!
- ▶ if T = P then the above is implied by the assumption on d as long as M is total and  $J^{\circ}y$  is both non-empty and compact for all  $y \in B$

## Extreme value theorem for approach spaces

▶ for  $V = P_+$  the second condition means that, for all  $y \in B$ ,

$$\inf_{x\in A} \bar{M}(Iy, dx) + J(x, y) = 0$$

▶ if T = U and  $M = P_+$  then the above is implied by the theorem's assumptions on d and J, as long as J is discrete (i.e. im  $J \subseteq \{0, \infty\}$ ) with  $J_0^\circ y$  non-empty for all  $y \in B$