9.4 3-23 odd

3) a) $\sum_{k=1}^{\infty} \frac{1}{k^3}$ $p=3$ $\therefore$ $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is $\boxed{\text{convergent}}$ (p-series)

b) $\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$ $p=1/2$ $\therefore$ $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ is $\boxed{\text{divergent}}$ (p-series)

c) $\sum_{k=1}^{\infty} k^{-1}$ $p=1$ $\therefore$ $\sum_{k=1}^{\infty} k^{-1}$ is $\boxed{\text{divergent}}$ (p-series)

d) $\sum_{k=1}^{\infty} k^{-2/3} = \sum_{k=1}^{\infty} \frac{1}{k^{2/3}}$ $p=2/3$ $\therefore$ $\sum_{k=1}^{\infty} k^{-2/3}$ is $\boxed{\text{divergent}}$ (p-series)

5) a) $\sum_{k=1}^{\infty} \frac{k^2+k+3}{2k^2+1}$

$$\lim_{k\to\infty} \frac{k^2+k+3}{2k^2+1} \cdot \frac{1/k^2}{1/k^2} = \lim_{k\to\infty} \frac{1+\cancel{\frac{1}{k}}^{0}+\cancel{\frac{3}{k^2}}^{0}}{2+\cancel{\frac{1}{k^2}}^{0}} = \boxed{1/2}$$

$\therefore \sum_{k=1}^{\infty} \frac{k^2+k+3}{2k^2+1}$ diverges by divergence test

b) $\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^k$

$$\lim_{k\to\infty} \left(1+\cancel{\frac{1}{k}}^{0}\right)^k = \lim_{k\to\infty} 1^k = 1$$

$\therefore \sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^k$ diverges by divergence test

c) $\sum_{k=1}^{\infty} \cos k\pi$

$\lim_{k\to\infty} \cos k\pi = \cos(\infty \cdot \pi)$ $\therefore \sum_{k=1}^{\infty} \cos k\pi$ diverges via divergence test

$\boxed{\text{Oscillating, therefore DNE}}$

d) $\sum_{k=1}^{\infty} \frac{1}{k!}$

$\lim_{k\to\infty} \frac{1}{\cancel{k!}}^{0} = 0$, $\therefore$ Divergence test is inconclusive

7.

a) $\sum_{k=1}^{\infty} \frac{1}{5k+2}$

let $f(x) = \frac{1}{5x+2}$

1) $f(x)$ is positive for all $x \geq 1$
2) $f(x)$ is decreasing for all $x \geq 1$
3) $f(x)$ is continuous for all $x \geq 1$

$\lim_{t\to\infty} \frac{1}{5}\int_1^t \frac{5}{5x+2}\,dx$ let $u = 5x+2$
$du = 5dx$

$= \lim_{t\to\infty} \frac{1}{5}\int_7^{5t+2} \frac{1}{u}\,du = \lim_{t\to\infty} \frac{1}{5}\Big[\ln(u)\Big]_7^{5t+2}$

$= \lim_{t\to\infty} \frac{1}{5}\Big[\ln(5t+2) - \ln(7)\Big]$

$= +\infty$ $\therefore \sum_{k=1}^{\infty} \frac{1}{5k+1}$ diverges via integral test

b) $\sum_{k=1}^{\infty} \frac{1}{1+9k^2}$ let $f(x) = \frac{1}{1+9k^2}$

1) $f(x)$ is positive $x \geq 1$
2) $f(x)$ is continuous $x \geq 1$
3) $f(x)$ is decreasing $x \geq 1$

$\sec^2 x - \tan^2 x = 1$

$$\lim_{t\to\infty} \int_1^t \frac{1}{1+9x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{1+(3x)^2}\,dx$$

let $u = 3x$
$du = 3dx$

$$= \lim_{t\to\infty} \frac{1}{3} \int_3^{3t} \frac{1}{1+u^2}\,du = \lim_{t\to\infty} \frac{1}{3} \left[\tan^{-1}(u)\right]_3^{3t}$$

$$= \lim_{t\to\infty} \frac{1}{3}\left[\tan^{-1}(3t) - \tan^{-1}(3)\right]$$

$$= \frac{1}{3}\left[\frac{\pi}{2} - \tan^{-1}(3)\right]$$

$$= \frac{\pi}{6} - \frac{\tan^{-1}(3)}{3}$$ $\Rightarrow$ convergent by integral test

9) $\sum_{k=1}^{\infty} \frac{1}{k+6}$

let $f(x) = \frac{1}{x+6}$

Divergence test

$$\lim_{x\to\infty} \frac{1}{x+6} = 0 \quad \therefore \text{Inconclusive}$$

Integral test 1 ✓ 2 ✓ 3 ✓

$$\lim_{t\to\infty} \int_1^t \frac{1}{x+6}\,dx$$ let $u = x+6$, $du = dx$

$$\lim_{t\to\infty} \int_7^{t+6} \frac{1}{u}\,du = \lim_{t\to\infty} \Big[\ln|u|\Big]_7^{t+6}$$

$$= \lim_{t\to\infty} \left(\ln(t+6) - \ln(7)\right)$$ DNE $\therefore$ diverges by integral test

11) $$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+5}}$$ let $f(x) = \frac{1}{\sqrt{x+5}}$

$$\lim_{x\to\infty} \frac{1}{\sqrt{x+5}} = 0$$ $\therefore$ Inconclusive by divergence test

Integral test

$$\lim_{t\to\infty} \int_1^t \frac{1}{(x+5)^{1/2}}\,dx$$ let $u = x+5$, $du = dx$

$$\lim_{t\to\infty} \int_6^{t+5} \frac{1}{(u)^{1/2}}\,du = \lim_{t\to\infty} \left[2u^{1/2}\right]_6^{t+5}$$

$$= \lim_{t\to\infty} \left[2(t+5)^{1/2} - 2(6)^{1/2}\right]$$

DNE $\therefore \sum_{k=1}^{\infty} \frac{1}{\sqrt{k+5}}$ is divergent by integral test

13) $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{2k-1}}$ let $f(x) = \frac{1}{\sqrt[3]{2x-1}}$

$$\lim_{x\to\infty} \frac{1}{(2x-1)^{3/2}} = 0 \quad \therefore \text{Inconclusive}$$

$$\lim_{t\to\infty} \int_1^{t} \frac{1}{(2x-1)^{3/2}}\,dx$$

let $u = 2x-1$

$du = 2dx$

$$\lim_{t\to\infty} \frac{1}{2} \int_1^{2t-1} \frac{1}{u^{3/2}}\,du = \lim_{t\to\infty} \frac{1}{2} \int_1^{2t-1} u^{-\frac{3}{2}} = \lim_{t\to\infty} \frac{1}{2} \left[2u^{-\frac{1}{2}}\right]_1^{2t-1}$$

$$= \lim_{t\to\infty} \frac{1}{2} \left[2(2t-1)^{-1/2} - 2\right]$$ $+\infty$, DNE

$\therefore \sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{2k-1}}$ diverges by integral test

15) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{\ln(k+1)}$

Divergergence Test

let $f(x) = \dfrac{x}{\ln(x+1)}$

$$\lim_{x\to\infty} \frac{x}{\ln(x+1)} \overset{H}{=} \frac{1}{\frac{1}{x+1}} = \overset{\infty}{\cancel{x}}+1,\ +\infty,\ \text{DNE}$$

$$\boxed{\therefore \sum_{k=1}^{\infty} \frac{k}{\ln(k+1)} \text{ is divergent by divergence test}}$$

17) $\displaystyle\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^{-k}$

Divergence Test

let $f(x) = \left(1+\dfrac{1}{x}\right)^{-x}$

$$\lim_{x\to\infty} \frac{1}{\left(1+\overset{0}{\cancel{\frac{1}{x}}}\right)^{x}} = 1 \qquad \therefore \sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^{-k} \text{ is divergent by divergence test}$$

19) $\sum_{k=1}^{\infty} \frac{\tan^{-1}(k)}{1+k^2}$

Divergence test

let $f(x) = \frac{\tan^{-1}(x)}{1+x^2}$

$\lim_{x\to\infty} \frac{\tan^{-1}(x)}{1+x^2} \quad \frac{\frac{\pi}{2}}{1+\cancel{x^2}\,\infty} = 0$, Inconclusive

Integral Test

$\lim_{t\to\infty} \int_1^t \frac{\tan^{-1}(x)}{1+x^2}dx$ let $u = \tan^{-1}(x)$

$du = \frac{1}{1+x^2}dx$

$= \lim_{t\to\infty} \int_{\tan^{-1}(1)}^{\tan^{-1}(t)} u\,du = \lim_{t\to\infty} \left[\frac{u^2}{2}\right]_{\tan^{-1}(1)}^{\tan^{-1}(t)} = \lim_{t\to\infty} \left(\frac{(\tan^{-1}(t))^2}{2} - \frac{(\tan^{-1}(1))^2}{2}\right)$

$= \frac{(\frac{\pi}{2})^2}{2} - \frac{(\frac{\pi}{4})^2}{2} = \frac{\frac{\pi^2}{4}}{2} - \frac{\frac{\pi^2}{16}}{2} = \frac{\pi^2}{2} - \frac{\pi^2}{8} = \frac{4\pi^2}{8} - \frac{\pi^2}{8} = \boxed{\frac{3\pi^2}{8}}$

21) $\sum_{k=1}^{\infty} k^2\sin^2(\frac{1}{k})$

$\lim_{k\to\infty} k^2\sin^2(\frac{1}{k})$

let $u = 1/k \Rightarrow k = \frac{1}{u}$

$du =$

as $k\to\infty$, $u\to 0^+$

$$= \lim_{u \to 0^+} \frac{1}{2} \sin^2(u)$$

$$= \lim_{u \to 0^+} \frac{\sin^2(u)}{u^2} = \lim_{u \to 0^+} \left(\frac{\sin u}{u}\right)^2 = 1^2 = 1$$

23) $\sum_{k=1}^{\infty} 7k^{-1.01} = \sum_{k=1}^{\infty} \frac{1}{7k^{1.01}}$

$$\boxed{\therefore \sum_{k=1}^{\infty} 7k^{-1.01} \text{ is a convergent p-series, } p = 1.01}$$