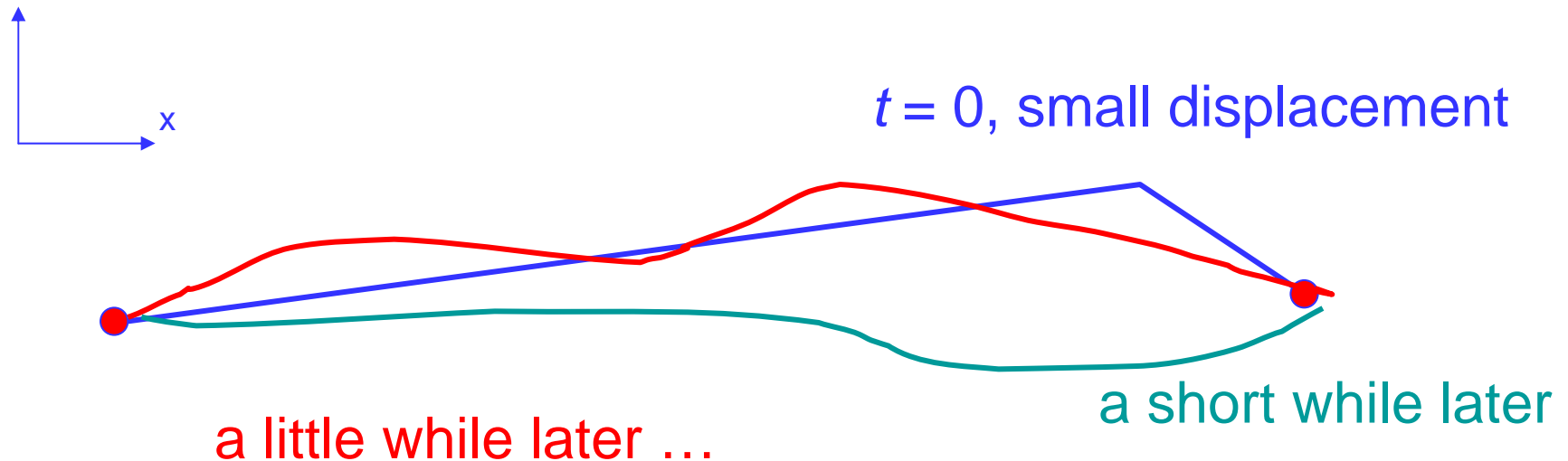


Wave equation in 1D (part 1)*

- Derivation of the 1D Wave equation
 - Vibrations of an elastic string
- Solution by separation of variables
 - Three steps to a solution
- Several worked examples
- Travelling waves
 - more on this in a later lecture
- d'Alembert's insightful solution to the 1D Wave Equation

A plucked string



These are snapshots, freezing time, and looking at the vertical displacement of the string:

$$u(x, t_i), i = 0, 1, \dots$$

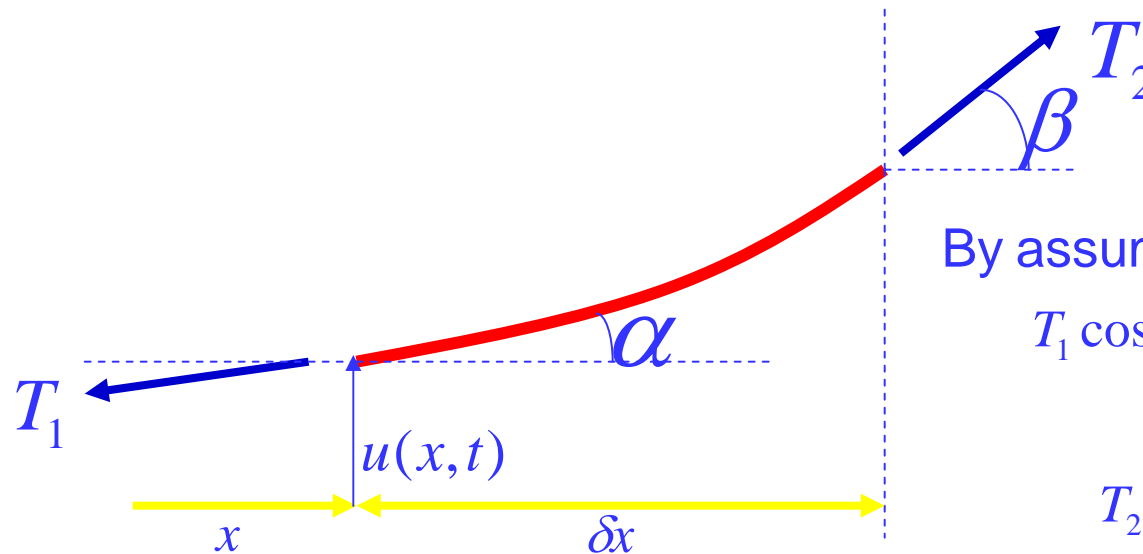
Equally, we could fix attention to one point on the string and watch it over time

Physical assumptions*

1. The mass of the string per unit length is constant (“*homogeneous string*”). The string is perfectly elastic and does not offer resistance to bending
2. The tension caused by stretching the string is so large that gravitational effects can be neglected.
3. The string performs small transverse motions – that is, every particle on the string moves strictly vertically and so that the deflection and the slope at every point of the string remain small.

Derivation of the 1D Wave Equation

Consider the transverse vibrations in a string held under tension



By assumption 3:

$$T_1 \cos \alpha = T_2 \cos \beta = T \text{ (a constant)}$$

$$T_2 \sin \beta - T_1 \sin \alpha = (\rho \Delta x) \frac{\partial^2 u}{\partial t^2}$$

Newton's Law of motion in the vertical direction

$$\tan \beta - \tan \alpha = \frac{(\rho \Delta x)}{T} \frac{\partial^2 u}{\partial t^2}$$

Since $\tan \alpha = \left(\frac{\partial u}{\partial x} \right) \Big|_x$ and $\tan \beta = \left(\frac{\partial u}{\partial x} \right) \Big|_{x+\Delta x}$

$$\frac{1}{\Delta x} \left(\left(\frac{\partial u}{\partial x} \right) \Big|_{x+\Delta x} - \left(\frac{\partial u}{\partial x} \right) \Big|_x \right) = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

Letting $\Delta x \rightarrow 0$, we obtain: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, where $c^2 = \frac{T}{\rho}$

Boundary conditions

To solve : $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Fixed ends : $u(0, t) = 0$ and $u(L, t) = 0$

Initial conditions :

$$u(x, 0) = f(x)$$

$$\left. \frac{\partial u}{\partial t} \right|_{t=0} = g(x)$$

Example :

$$g(x) = 0$$

$$f(x) = \begin{cases} \frac{2k}{L}x & \text{if } 0 < x \leq \frac{L}{2} \\ \frac{2k}{L}(L-x) & \text{if } \frac{L}{2} < x \leq L \end{cases}$$

Solution by *Separation of Variables*

1. Convert the PDE into two separate ODEs
2. Solve the two (well known) ODEs
3. Compose the solutions to the two ODEs into a solution of the original PDE
 - This uses Fourier series

Step 1: PDE \rightarrow 2 ODEs

The first step is to *assume* that the function of two variables has a very special form: the product of two separate functions, each of one variable, that is: Assume that : $u(x,t) = F(x)G(t)$

Differentiating, we find:

$$\frac{\partial^2 u}{\partial t^2} = F\ddot{G}$$

So that

$$F\ddot{G} = c^2 F''G$$

$$\frac{\partial^2 u}{\partial x^2} = F''G$$

equivalently

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k$$

where $\ddot{G} = \frac{d^2 G}{dt^2}$ and $F'' = \frac{d^2 F}{dx^2}$

The two (homogeneous) ODEs are:

$$F'' - kF = 0$$

$$\ddot{G} - kc^2 G = 0$$

Step 2a: boundary conditions

$$u(0,t) = F(0)G(t) = 0 \text{ and } u(L,t) = F(L)G(t) = 0$$

$G(t) \equiv 0$ is of no interest, and so :

$$F(0) = 0$$

$$F(L) = 0$$

Step 2b: solving for F

We have to solve: $F'' - kF = 0$

case $k = \mu^2 > 0$ the general solution is:

$$F(x) = Ae^{\mu x} + Be^{-\mu x}$$

Applying the boundary conditions

$$A + B = 0$$

$$Ae^{\mu L} + Be^{-\mu L} = 0$$

and so $A = B = 0$

case $k = -p^2 < 0$: the solution is

$$F(x) = A \cos px + B \sin px$$

Applying the boundary conditions, we find

$$F(0) = A = 0$$

$$F(L) = B \sin pL = 0, \text{ and so } \sin pL = 0$$

case $k = 0$:

$$F(x) = ax + b$$

Applying the boundary conditions: $a = b = 0$

$pL = n\pi$ that is:

$$p = n \frac{\pi}{L}$$

so that

$$F_n(x) = \sin n \frac{\pi}{L} x$$

Step 2c: solving for G

As a result of solving for F , we have restricted $k = -p^2 = -\left(n\frac{\pi}{L}\right)^2$

Denoting $\lambda_n = cn\frac{\pi}{L}$, the solutions for $G(t)$ are

$$G_n(t) = B_n \cos \lambda_n t + B_n^* \sin \lambda_n t$$

Solutions for the 1D Wave Equation are :

$$u_n(x,t) = F_n(x)G_n(t) = \left(B_n \cos \lambda_n t + B_n^* \sin \lambda_n t\right) \sin n\frac{\pi}{L}x$$

These functions are the ***eigenfunctions*** of the vibrating string, and the values $\lambda_n = cn\pi / L$ are called the ***eigenvalues***. The set of the eigenvalues $[\lambda_1, \dots, \lambda_n]$ is called the ***spectrum***.

Remembering that $c^2 = T / \rho$ there are several ways to change the spectrum of a string – change the material, the tension, and the length

Step 3: Initial conditions & Fourier

The solution we have found is $u(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin n \frac{\pi}{L} x$

We still have to apply the initial conditions :

$$u(x,0) = f(x)$$

$$\left. \frac{\partial u}{\partial t} \right|_{t=0} = g(x)$$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin n \frac{\pi}{L} x = f(x)$$

and so

$$B_n = \frac{2}{L} \int_0^L f(x) \sin n \frac{\pi}{L} x dx$$

$$\begin{aligned} \left. \frac{\partial u}{\partial t} \right|_{t=0} = g(x) &= \left[\sum_{n=1}^{\infty} (-\lambda_n B_n \sin \lambda_n t + \lambda_n B_n^* \cos \lambda_n t) \sin n \frac{\pi}{L} x \right]_{t=0} \\ &= \sum_{n=1}^{\infty} \lambda_n B_n^* \sin n \frac{\pi}{L} x \end{aligned}$$

and so

$$\lambda_n B_n^* = \frac{2}{L} \int_0^L g(x) \sin n \frac{\pi}{L} x dx$$

$$B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin n \frac{\pi}{L} x dx$$

Note that $B_n^* = 0$, if, as is often the case, $g(x) \equiv 0$

An alternative expression

Suppose that $g(x) \equiv 0$, so that $u(x,t) = \sum_{n=1}^{\infty} B_n \cos nc \frac{\pi}{L} t \sin n \frac{\pi}{L} x$

Remembering formulae for products of sines and cosines:

$$\cos nc \frac{\pi}{L} t \sin n \frac{\pi}{L} x = \frac{1}{2} \left[\sin n \frac{\pi}{L} (x - ct) + \sin n \frac{\pi}{L} (x + ct) \right]$$

Substituting this in the formula for $u(x,t)$ we see that:

$$u(x,t) = \frac{1}{2} (f_{\text{odd}}(x - ct) + f_{\text{odd}}(x + ct))$$

f_{odd} is the odd periodic extension of $f(x)$ with period $2L$

Later we will interpret these as travelling waves moving in opposite directions (the sign of $\pm c$)

Recall of the general solution to the 1D Wave Equation

To solve : $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Boundary conditions = fixed ends : $u(0, t) = 0$ and $u(L, t) = 0$

Initial conditions :

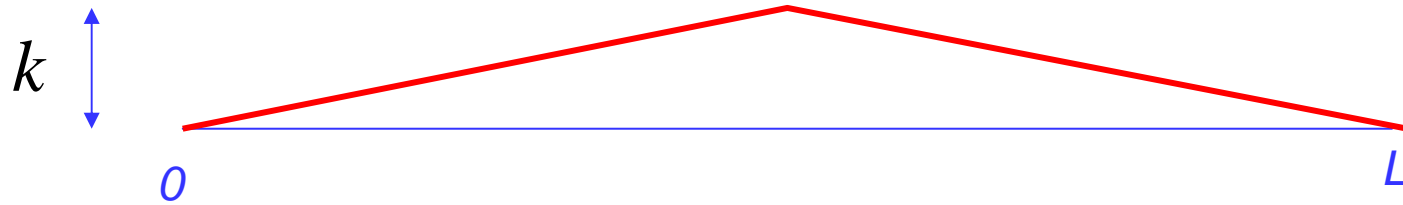
$$u(x, 0) = f(x)$$

$$\left. \frac{\partial u}{\partial t} \right|_{t=0} = g(x)$$

$$\text{Solution : } u(x, t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin n \frac{\pi}{L} x$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin n \frac{\pi}{L} x dx \quad B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \sin n \frac{\pi}{L} x dx$$

Example 1: plucked string*



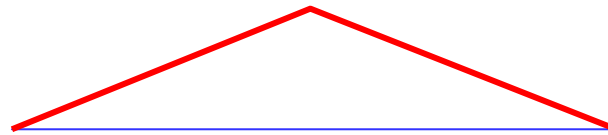
$$f(x) = \begin{cases} \frac{2k}{L}x & \text{if } 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x) & \text{if } \frac{L}{2} < x < L \end{cases}$$
$$g(x) \equiv 0$$

Since $g(x) \equiv 0$, $B_n^* = 0$, for all n

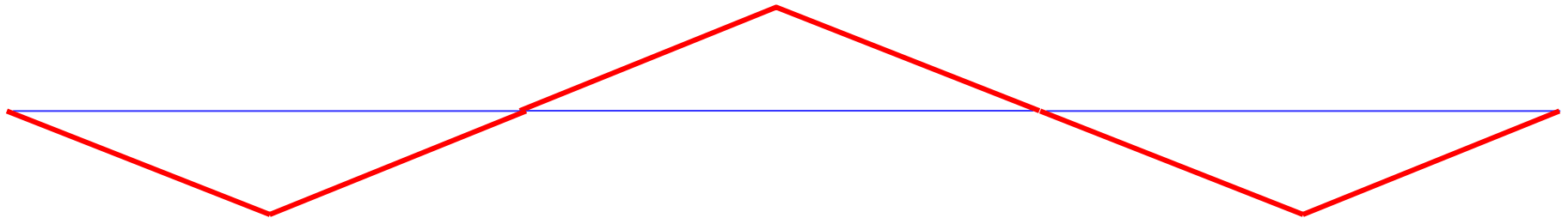
To solve for B_n either apply orthogonality relations or consult HLT

Fourier coefficients

We need to extend the string to a periodic function



We know that the solution is a sum of *sin* terms, so choose an odd function periodic extension:



Using HLT, orthogonality, or Kreysig, p545 - 6

$$B_n = (-1)^{2n-1} \frac{8k}{n^2 \pi^2}$$

$$u(x,t) = \frac{8k}{\pi^2} \left[\frac{1}{1^2} \sin \frac{\pi}{L} x \cos \frac{\pi c}{L} t - \frac{1}{3^2} \sin \frac{3\pi}{L} x \cos \frac{3\pi c}{L} t + \frac{1}{5^2} \sin \frac{5\pi}{L} x \cos \frac{5\pi c}{L} t - \dots \right]$$

d'Alembert's elegant solution to the 1D Wave Equation

To solve : $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ (Eqn 1)

Introduce two new variables :

$$v = x + ct$$

$$z = x - ct$$

Applying the Chain Rule :

$$u_x = u_v v_x + u_z z_x = u_v + u_z$$

$$u_{xx} = (u_v + u_z)_x = (u_v + u_z)_v v_x + (u_v + u_z)_z z_x$$

$$u_{xx} = u_{vv} + 2u_{vz} + u_{zz}$$

Similarly for t , we can derive

$$u_{tt} = c^2 (u_{vv} - 2u_{vz} + u_{zz})$$

Inserting these values in Eqn 1, we find :

$$u_{vz} = \frac{\partial^2 u}{\partial v \partial z} = 0$$

D'Alembert continued

Integrating $u_{vz} = 0$ with respect to z gives

$$u_v = h(v)$$

where $h(v)$ is some unknown function of v .

Integrating with respect to v gives :

$$u = \int h(v)dv + \psi(z)$$

for some unknown function $\psi(z)$.

Since the integral is a function $\phi(v)$ of v , we may conclude that :

$$u(x, t) = \phi(v) + \psi(z) = \phi(x + ct) + \psi(x - ct)$$

Jean Le Rond d'Alembert

Jean was the illegitimate son of a cavalry officer who was out of the country at his birth in 1717. His “socialite” mother left him on the steps of St Jean le Rond – hence his name. He invented the name d'Alembert for college entry.



He developed into one of the leading French mathematicians of the 18th Century, making particular contributions to mechanics, which is odd, since he believed mechanics is essentially metaphysics, and he rejected experimentation. His genes shaped his personality:

“D'Alembert was always surrounded by controversy. ... he was a lightning rod which drew sparks from all the foes of the philosophes. ... Unfortunately he carried this... pugnacity into his scientific research and once he had entered a controversy, he argued his cause with vigour and stubbornness. He closed his mind to the possibility that he might be wrong...”

Wave propagation

- Displacement of a string
- Compression waves
- Electrical waves
- Water waves
- Acoustic waves

The last couple of lectures will be dedicated to variants to the wave equation