1 Di-lepton production in e^+e^- in the SM

1.1 QED

Using the Feynman rules for scalar QED we get an amplitude

$$i\mathcal{M} = \bar{v}^{s'}(p')(-ie\gamma^\mu)u^s(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}^r(k)(-ie\gamma^\nu)v^{r'}(k') \tag{1}$$

Where p and p' are the momentum for the incoming e^+e^-,and k and k' are the momentums of the outgoing $\tau^+\tau^-$. s and r are the spin indicies, q is the momentum of the force mediator and e is the elementary charge. On a more compact form leaving the spin superscripts implicit

$$i\mathcal{M} = \frac{ie^2}{q^2}\left(\bar{v}(p')\gamma^\mu u(p)\right)\left(\bar{u}(k)\gamma_\mu v(k')\right) \tag{2}$$

Then using $(\bar{v}\gamma^\mu u)^* = \bar{u}\gamma^\mu v$ to get the $|\mathcal{M}|^2$ leads to

$$|\mathcal{M}|^2 = \frac{e^4}{q^4}\Big(\bar{v}(p')\gamma^\mu u(p)\bar{u}(p)\gamma^\nu v(p')\Big)\Big(\bar{u}(k)\gamma_\mu v(k')\bar{v}(k')\gamma_\nu u(k)\Big) \tag{3}$$

Now taking the spins into account we have an expression on the form

$$\frac{1}{4}\sum_s\sum_{s'}\sum_r\sum_{r'}|\mathcal{M}(s,s'\to r,r')|^2. \tag{4}$$

With the 2 completeness relations

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m\,; \qquad \sum_s v^s(p)\bar{v}^s(p) = \not{p} - m \tag{5}$$

used in the the first parenthesis in equation (3) written out in spinor indicies, making it possible to move v and $\bar{v}$ next to each other we get

$$\begin{aligned}\sum_{s,s'}\bar{v}_a^{s'}(p')\gamma^\mu_{ab}u_b^s(p)\bar{u}_c^s(p)\gamma^\nu_{cd}v_d^{s'}(p') &= (\not{p}' - m)_{da}\gamma^\mu_{ab}(\not{p} + m)_{bc}\gamma^\nu_{cd} \\ &= \mathrm{tr}[(\not{p}' - m)\gamma^\mu(\not{p} + m)\gamma^\nu]\end{aligned}$$

thus our squared matrix element becomes the product of 2 traces

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{4q^4}\mathrm{tr}\left[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\right]\mathrm{tr}\left[(\not{k} + m_\tau)\gamma_\mu(\not{k}' - m_\tau)\gamma_\nu\right] \tag{6}$$

Insert trace technology here Using the trace relations we get for the e part

$$\mathrm{tr}\left[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\right] = 4\left[p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p\cdot p' + m_e^2)\right] \tag{7}$$

and for the τ

$$\mathrm{tr}\left[(\not{k}+m_\tau)\gamma_\mu(\not{k}'+m_\tau)\gamma_\nu\right] = 4\left[k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}(k\cdot k' + m_\tau^2)\right] \tag{8}$$

There is a large difference in $m_e << m_\tau$ which makes it reasonable to set $m_e = 0$, multiplying the traces gives us a square matrix element of

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^4}{q^4}\left[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_\tau^2(p\cdot p')\right] \tag{9}$$

Furterhmore we choose the center of mass reference frame and translates our momentums into kinematic variables instead, energies and angles.

$$\begin{array}{ccc} q^2 = (p+p')^2 = 4E^2 & ; & p\cdot p' = 2E^2 \\ p\cdot k = E^2 - E|\mathbf{k}|\cos\theta & ; & p\cdot k' = E^2 + E|\mathbf{k}|\cos\theta \end{array} \tag{10}$$

Rewriting eq. (9) in terms of E and θ we get

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^4}{16E^4}\left[E^2(E-|\mathbf{k}|\cos\theta)^2 + E^2(E+|\mathbf{k}|\cos\theta)^2 + 2m_\tau^2E^2\right] \tag{11}$$

$$= e^4\left[\left(1+\frac{m_\tau^2}{E^2}\right) + \left(1-\frac{m_\tau^2}{E^2}\right)\cos^2\theta\right] \tag{12}$$

Now that we have $|\mathcal{M}|^2$ we can put it into a formula for $d\sigma/d\cos\theta$ derived in Peskin[1]

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{2E_A 2E_B|v_p - v_{p'}|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{CM}}|\mathcal{M}|^2 \tag{13}$$

In the center of mass frame the relative speed $|v_p - v_{p'}|$ becomes 2, and $E_p = E_{p'} = E_{CM}/2 = \sqrt{s}/2$. With a symmetry about the longitudinal direction we can make the differential cross section

$$\frac{d\sigma}{d\cos\theta} = \frac{|\mathbf{k}|}{32\pi^2 s^{3/2}}|\mathcal{M}|^2 = \frac{1}{32\pi^2 s}\sqrt{1-4\frac{m_\tau^2}{s}}|\mathcal{M}|^2 \tag{14}$$

Integrating with respect to $\cos\theta$ we aquire the total cross section. Keeping the prefactors out of the calculation since they are not dependent on $cos\theta$ and integrate our expression for the squared matrix element.

$$\int_{-1}^{1} d\cos\theta e^4|\mathcal{M}|^2 = e^4\left[\left(1+4\frac{m_\tau^2}{s}\right)\cos\theta + \frac{1}{3}\left(1-4\frac{m_\tau^2}{s}\right)\cos^3\theta\right]_{-1}^{1}$$

$$= \frac{8}{3}\left(1+2\frac{m_\tau^2}{s}\right)$$

[1] Page 107, Equation 4.84

Combining combining this with our differential cross section we get an expression for the total cross section

$$\sigma = \frac{4}{3}\frac{e^4}{32\pi^2 s}\sqrt{1-4\frac{m_\tau^2}{s}}\left(1+2\frac{m_\tau^2}{s}\right)$$
$$= \frac{2}{3}\frac{\alpha^2}{s}\sqrt{1-4\frac{m_\tau^2}{s}}\left(1+2\frac{m_\tau^2}{s}\right)$$

with $\alpha = e^2/(4\pi)$.

1.2 Electroweak

This unifcation requires both the $\mathcal{M}_{QED}$ and $\mathcal{M}_{WI}$ matrix elements. Using the feynman-rules from the weak model our $\mathcal{M}_{WI}$ will take the form

$$\mathcal{M}_{WI} = -\frac{g_Z^2}{s-m_Z^2+im_z\Gamma_Z}g_{\mu\nu}\left[\bar{v}(p')\gamma^\mu\frac{1}{2}(c_V^e - c_A^e\gamma^5)u(p)\right]\left[\bar{u}(k)\gamma^\mu\frac{1}{2}(c_V^\tau - c_A^\tau\gamma^5)v(k')\right] \tag{15}$$

where $1/(s-m_Z^2+im_Z\Gamma_Z) = P_Z(s)$ is the Z propogator, $c_{V/A}^{e/\tau}$ are the vector and axial-vector couplings of the Z to our leptons. It will be handy to rewrite those couplings to left- and right-handed chiral states as $c_V = c_L + c_R$ and $c_A = c_L - c_R$

$$-P_Z g_Z^2 g_{\mu\nu}\left[c_L^e\bar{v}(p')\gamma^\mu P_L u(p) + c_R^e\bar{v}(p')\gamma^\mu P_R u(p)\right]\left[c_L^\tau\bar{u}(k)\gamma^\nu P_L v(k') + c_R^\tau\bar{u}(k)\gamma^\nu P_R v(k')\right] \tag{16}$$

where P_L and P_R are the chiral projection operators $\frac{1}{2}(1\mp\gamma^5)$. Using these chiral projection operators on a particle state give the result

$$P_L u = u_\downarrow,\ P_R u = u_\uparrow,\ P_L v = v_\uparrow,\ P_R v = v_\downarrow$$

And with helicity combinations like $\bar{u}_\uparrow\gamma^\mu v_\uparrow$ giving zero matrix elements we are left with four helicity combinations

$$\mathcal{M}_{RR} = -P_Z g_Z^2 c_R^e c_R^\tau g^{\mu\nu}\left[\bar{v}_\downarrow(p')\gamma^\mu u_\uparrow(p)\right]\left[\bar{u}_\uparrow(k)\gamma^\nu v_\downarrow(k')\right] \tag{17}$$
$$\mathcal{M}_{RL} = -P_Z g_Z^2 c_R^e c_L^\tau g^{\mu\nu}\left[\bar{v}_\downarrow(p')\gamma^\mu u_\uparrow(p)\right]\left[\bar{u}_\downarrow(k)\gamma^\nu v_\uparrow(k')\right] \tag{18}$$
$$\mathcal{M}_{LR} = -P_Z g_Z^2 c_L^e c_R^\tau g^{\mu\nu}\left[\bar{v}_\uparrow(p')\gamma^\mu u_\downarrow(p)\right]\left[\bar{u}_\uparrow(k)\gamma^\nu v_\downarrow(k')\right] \tag{19}$$
$$\mathcal{M}_{LL} = -P_Z g_Z^2 c_L^e c_L^\tau g^{\mu\nu}\left[\bar{v}_\uparrow(p')\gamma^\mu u_\downarrow(p)\right]\left[\bar{u}_\uparrow(k)\gamma^\nu v_\uparrow(k')\right] \tag{20}$$

The combinations of these four-vector currents can be shown to come to a simpler form as with this example

$$g_{\mu\nu}[\bar{v}_\downarrow(p')\gamma^\mu u_\uparrow(p)][\bar{u}_\uparrow(k)\gamma^\nu v_\downarrow(k')] = s(1+\cos\theta)$$

using this on each helicity state we get

$$|\mathcal{M}_{RR}|^2 = |P_Z|^2 g_Z^4 (c_R^e)^2 (c_R^\tau)^2 s^2 (1+\cos\theta)^2 \tag{21}$$
$$|\mathcal{M}_{RL}|^2 = |P_Z|^2 g_Z^4 (c_R^e)^2 (c_L^\tau)^2 s^2 (1-\cos\theta)^2 \tag{22}$$
$$|\mathcal{M}_{LR}|^2 = |P_Z|^2 g_Z^4 (c_L^e)^2 (c_R^\tau)^2 s^2 (1-\cos\theta)^2 \tag{23}$$
$$|\mathcal{M}_{LL}|^2 = |P_Z|^2 g_Z^4 (c_L^e)^2 (c_L^\tau)^2 s^2 (1+\cos\theta)^2 \tag{24}$$

The spin average of the full matrix elemnt $|\mathcal{M}|^2$ is then the sum of each helicity state squared

$$< |\mathcal{M}|^2 >= \frac{1}{4}\left(|\mathcal{M}_{RR}|^2 + |\mathcal{M}_{LL}|^2 + |\mathcal{M}_{RL}|^2 + |\mathcal{M}_{LR}|^2\right)$$

writing the terms out gives us

$$< |\mathcal{M}|^2 >= \frac{1}{4}|P_Z|^2 g_Z^4 s^2 \Big(\left[(c_R^e)^2(c_R^\tau)^2 + (c_L^e)^2(c_L^\tau)^2\right](1+\cos\theta)^2 \\ + \left[(c_R^e)^2(c_L^\tau)^2 + (c_L^e)^2(c_R^\tau)^2\right](1-\cos\theta)^2\Big)$$