1 Di-lepton production in e^+e^- in the SM

Using the Feynman rules for scalar QED we get an amplitude

$$i\mathcal{M} = \bar{v}^{s'}(p')(-ie\gamma^\mu)u^s(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}^r(k)(-ie\gamma^\nu)v^{r'}(k') \tag{1}$$

Where p and p' are the momentum for the incoming e^+e^-,and k and k' are the momentums of the outgoing $\tau^+\tau^-$. s and r are the spin indicies, q is the momentum of the force mediator and e is the elementary charge. On a more compact form leaving the spin superscripts implicit

$$i\mathcal{M} = \frac{ie^2}{q^2}\left(\bar{v}(p')\gamma^\mu u(p)\right)\left(\bar{u}(k)\gamma_\mu v(k')\right) \tag{2}$$

Then using $(\bar{v}\gamma^\mu u)^* = \bar{u}\gamma^\mu v$ to get the $|\mathcal{M}|^2$ leads to

$$|\mathcal{M}|^2 = \frac{e^4}{q^4}\Big(\bar{v}(p')\gamma^\mu u(p)\bar{u}(p)\gamma^\nu v(p')\Big)\Big(\bar{u}(k)\gamma_\mu v(k')\bar{v}(k')\gamma_\nu u(k)\Big) \tag{3}$$

Now taking the spins into account we have an expression on the form

$$\frac{1}{4}\sum_s\sum_{s'}\sum_r\sum_{r'}|\mathcal{M}(s,s'\to r,r')|^2. \tag{4}$$

With the 2 completeness relations

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p}+m\,; \qquad \sum_s v^s(p)\bar{v}^s(p) = \not{p}-m \tag{5}$$

used in the the first parenthesis in equation (3) written out in spinor indicies, making it possible to move v and $\bar{v}$ next to each other we get

$$\begin{aligned}\sum_{s,s'}\bar{v}_a^{s'}(p')\gamma^\mu_{ab}u^s_b(p)\bar{u}^s_c(p)\gamma^\nu_{cd}v^{s'}_d(p') &= (\not{p}'-m)_{da}\gamma^\mu_{ab}(\not{p}+m)_{bc}\gamma^\nu_{cd}\\ &= \mathrm{tr}[(\not{p}'-m)\gamma^\mu(\not{p}+m)\gamma^\nu]\end{aligned}$$

thus our squared matrix element becomes the product of 2 traces

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{4q^4}\mathrm{tr}\left[(\not{p}'-m_e)\gamma^\mu(\not{p}+m_e)\gamma^\nu\right]\mathrm{tr}\left[(\not{k}+m_\tau)\gamma_\mu(\not{k}'-m_\tau)\gamma_\nu\right] \tag{6}$$

Insert trace technology here Using the trace relations we get for the e part

$$\mathrm{tr}\left[(\not{p}'-m_e)\gamma^\mu(\not{p}+m_e)\gamma^\nu\right] = 4\left[p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p\cdot p' + m_e^2)\right] \tag{7}$$

and for the τ

$$\mathrm{tr}\left[(\not{k}+m_\tau)\gamma_\mu(\not{k}'+m_\tau)\gamma_\nu\right] = 4\left[k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}(k\cdot k' + m_\tau^2)\right] \tag{8}$$

There is a large difference in $m_e << m_\tau$ which makes it reasonable to set $m_e = 0$, multiplying the traces gives us a square matrix element of

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^4}{q^4}\left[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_\tau^2(p\cdot p')\right] \tag{9}$$

Furterhmore we choose the center of mass reference frame and translates our momentums into kinematic variables instead, energies and angles.

$$\begin{array}{ccc} q^2 = (p+p')^2 = 4E^2 & ; & p\cdot p' = 2E^2 \\ p\cdot k = E^2 - E|\mathbf{k}|\cos\theta & ; & p\cdot k' = E^2 + E|\mathbf{k}|\cos\theta \end{array} \tag{10}$$

Rewriting eq. (9) in terms of E and θ we get

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^4}{16E^4}\Big[E^2(E-|\mathbf{k}|\cos\theta)^2 + E^2(E+|\mathbf{k}|\cos\theta)^2 + 2m_\tau^2E^2\Big] \tag{11}$$

$$= e^4\left[\left(1+\frac{m_\tau^2}{E^2}\right) + \left(1-\frac{m_\tau^2}{E^2}\right)\cos^2\theta\right] \tag{12}$$

Now that we have $|\mathcal{M}|^2$ we can put it into a formula for $d\sigma/d\cos\theta$ derived in Peskin[1]

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{2E_A 2E_B|v_p - v_{p'}|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{CM}}|\mathcal{M}|^2 \tag{13}$$

In the center of mass frame the relative speed $|v_p - v_{p'}|$ becomes 2, and $E_p = E_{p'} = E_{CM}/2 = \sqrt{s}/2$. With a symmetry about the longitudinal direction we can make the differential cross section

$$\frac{d\sigma}{d\cos\theta} = \frac{|\mathbf{k}|}{32\pi^2 s^{3/2}}|\mathcal{M}|^2 = \frac{1}{32\pi^2 s}\sqrt{1-4\frac{m_\tau^2}{s}}|\mathcal{M}|^2 \tag{14}$$

Integrating with respect to $\cos\theta$ we aquire the total cross section. Keeping the prefactors out of the calculation since they are not dependent on $cos\theta$ and integrate our expression for the squared matrix element.

$$\begin{aligned} \int_{-1}^{1} d\cos\theta e^4|\mathcal{M}|^2 &= e^4\left[\left(1+4\frac{m_\tau^2}{s}\right)\cos\theta + \frac{1}{3}\left(1-4\frac{m_\tau^2}{s}\right)\cos^3\theta\right]_{-1}^{1} \\ &= \frac{8}{3}\left(1+2\frac{m_\tau^2}{s}\right) \end{aligned}$$

Combining combining this with our differential cross section we get an expression for the total cross section

$$\begin{aligned} \sigma &= \frac{4}{3}\frac{e^4}{32\pi^2 s}\sqrt{1-4\frac{m_\tau^2}{s}}\left(1+2\frac{m_\tau^2}{s}\right) \\ &= \frac{2}{3}\frac{\alpha^2}{s}\sqrt{1-4\frac{m_\tau^2}{s}}\left(1+2\frac{m_\tau^2}{s}\right) \end{aligned}$$

with $\alpha = e^2/(4\pi)$.

[1] Page 107, Equation 4.84