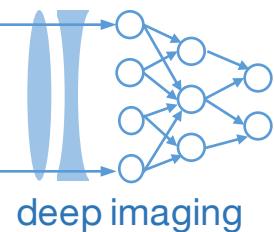


Lecture 2: Mathematical preliminaries for continuous functions

Machine Learning and Imaging

BME 590L
Roarke Horstmeyer

- Light as a continuous wave
- Light as a complex field
- Light transformations as a black box
- Linear black-box systems
- Convolutions in 1D and 2D

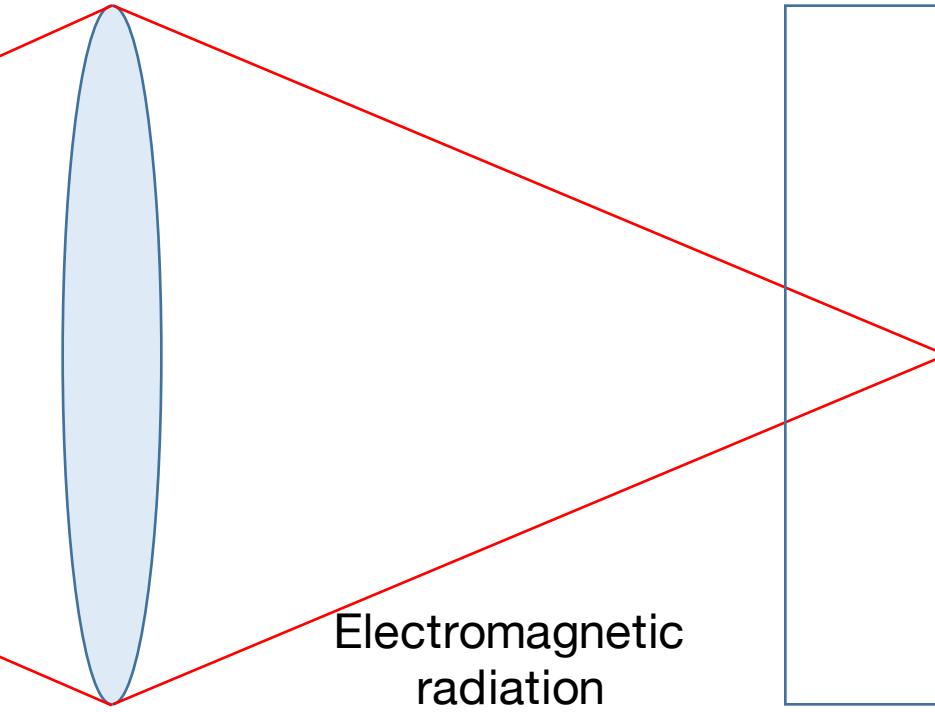


Last lecture: what is an image?

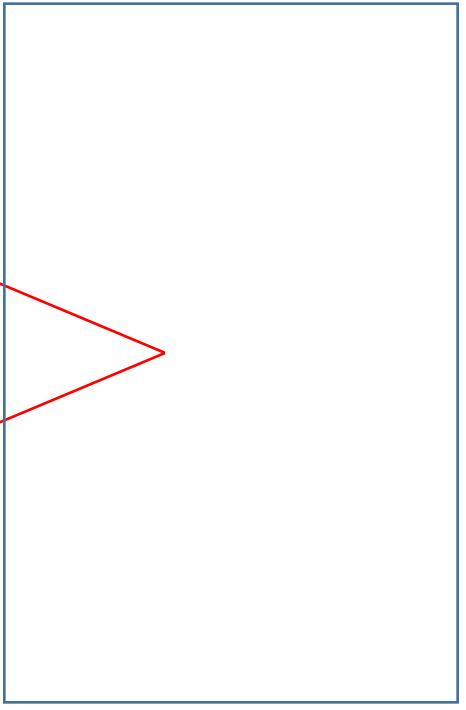
2. “Physical” Interpretation



Image plane

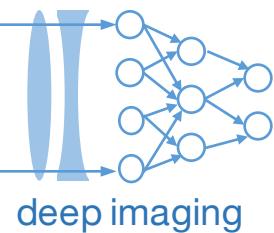


“Collection”
Element



Physical world
(Object plane)

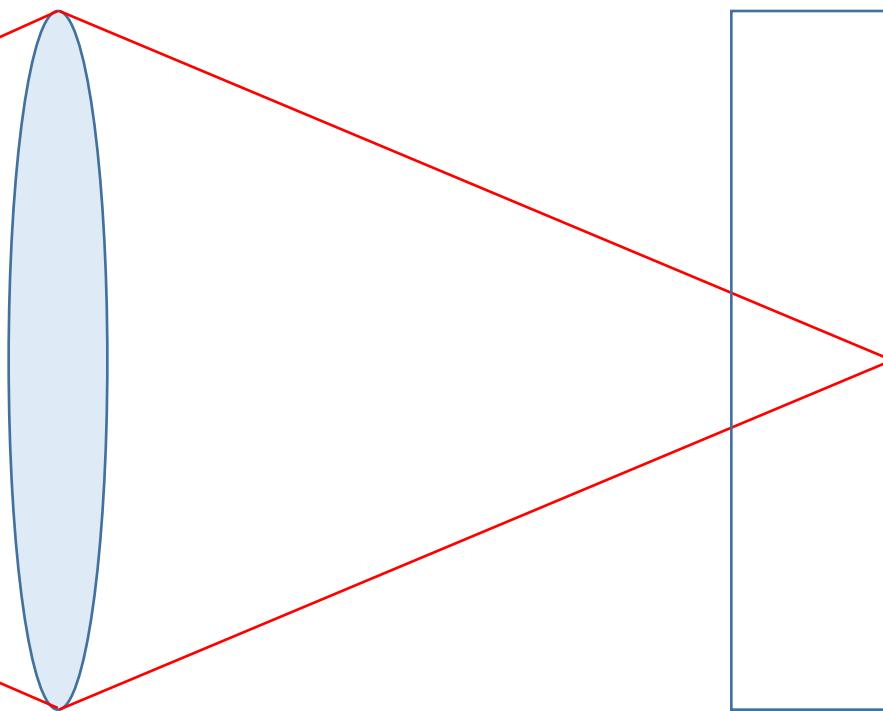
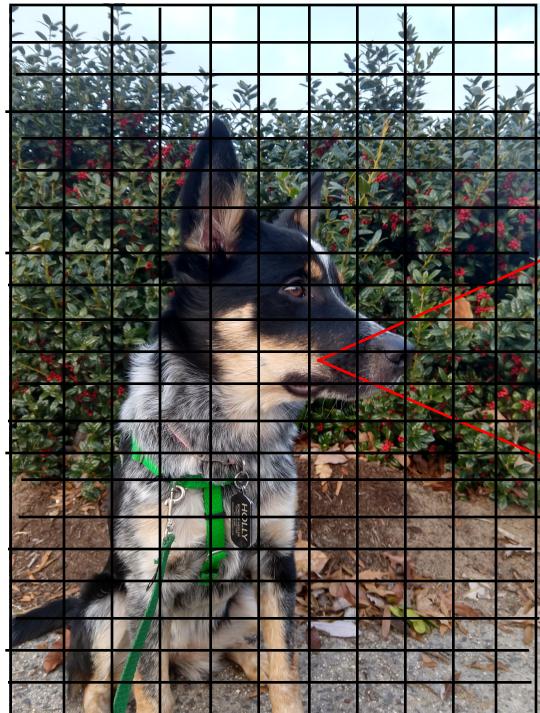
Continuous signal: $I(x, y), (x, y) \in (-\infty, \infty)$



Last lecture: what is an image?

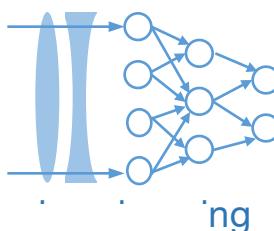
$n \times m$ array

3. “Digital” Interpretation

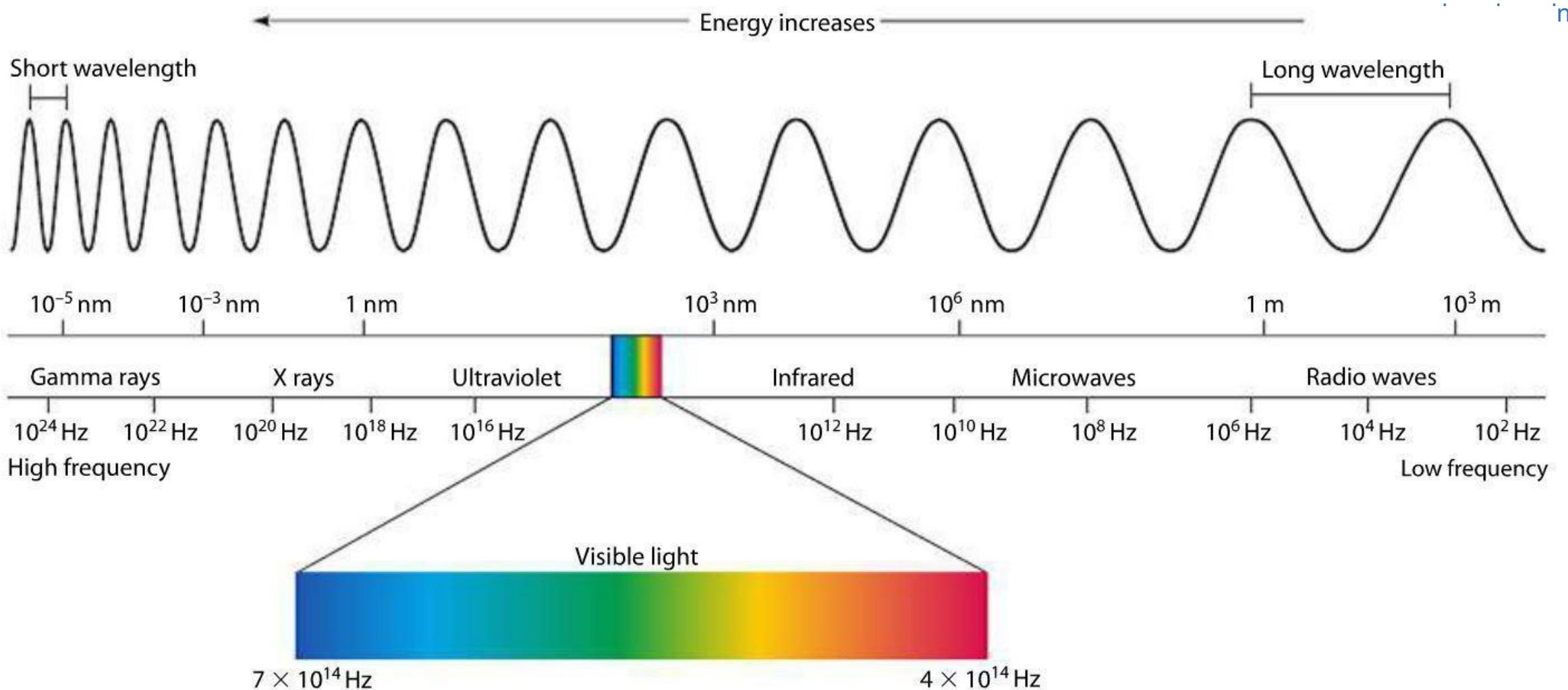


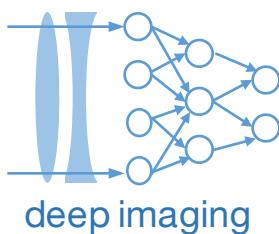
Photons to electrons → Digitazation → *Discrete signal*

$$I_s(x, y), (x, y) \in Z^{n \times m}$$

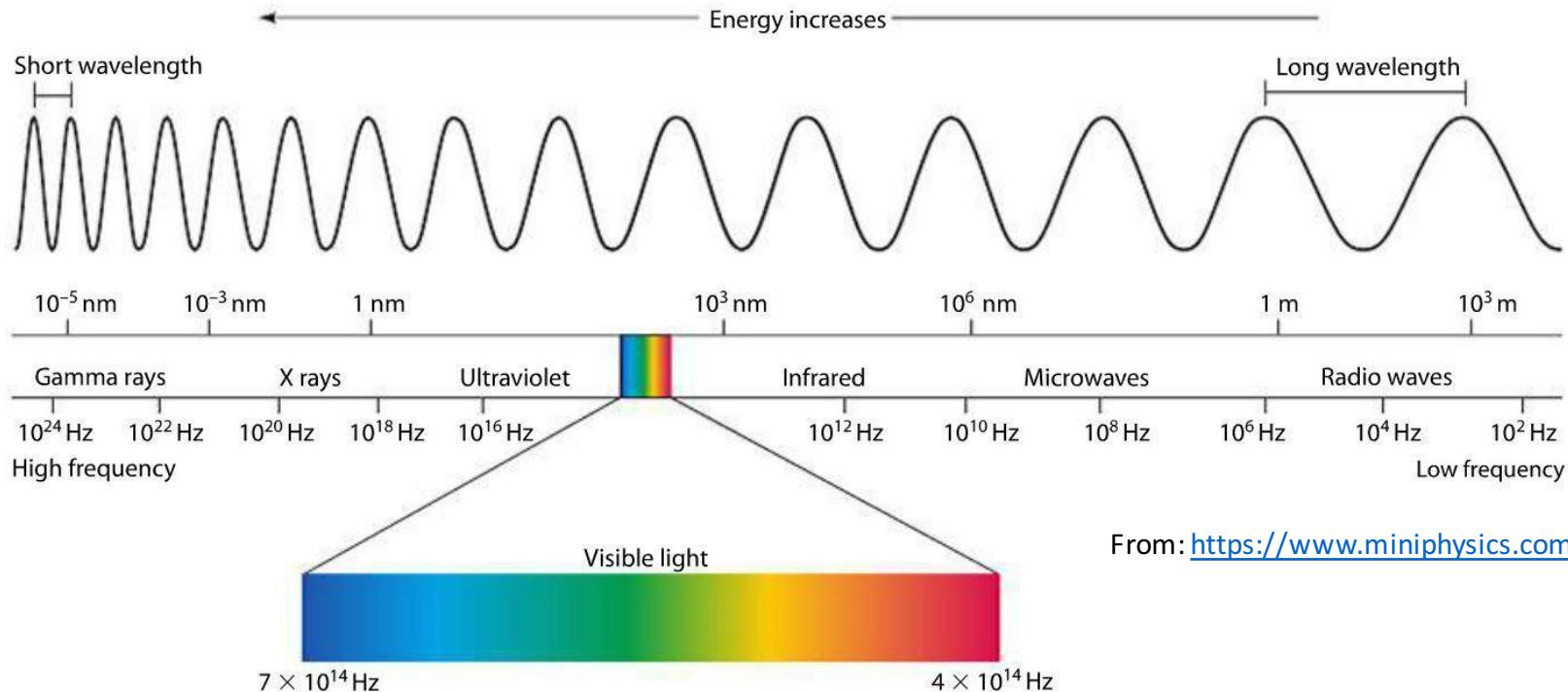


Start at the beginning: Electromagnetic waves





Start at the beginning: Electromagnetic waves



From: https://www.miniphysics.com/electromagnetic-spectrum_25.html

Maxwell's equations

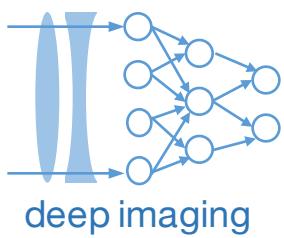
$$\begin{aligned}\nabla \times \vec{\mathcal{E}} &= -\mu \frac{\partial \vec{\mathcal{H}}}{\partial t} \\ \nabla \times \vec{\mathcal{H}} &= \epsilon \frac{\partial \vec{\mathcal{E}}}{\partial t} \\ \nabla \cdot \epsilon \vec{\mathcal{E}} &= 0 \\ \nabla \cdot \mu \vec{\mathcal{H}} &= 0.\end{aligned}$$

Free-space propagation

Scalar solution, 1 freq.

$$\nabla^2 \vec{\mathcal{E}} - \frac{n^2}{c^2} \frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2} = 0$$

$$A(\mathbf{r}_1) \cos(\mathbf{k}\mathbf{r}_1 - \omega t)$$

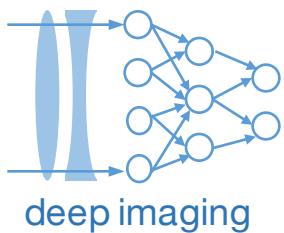


Start at the beginning: EM fields and the black box

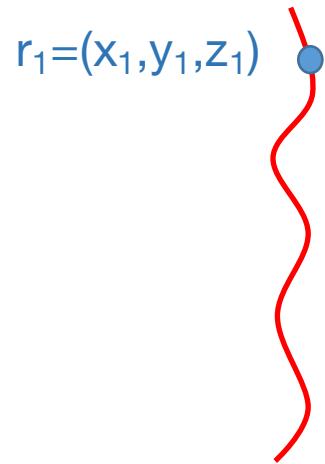


The general idea:

1. We will treat light as a wave (an “optical field”)



Start at the beginning: EM fields and the black box

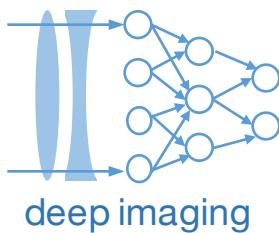


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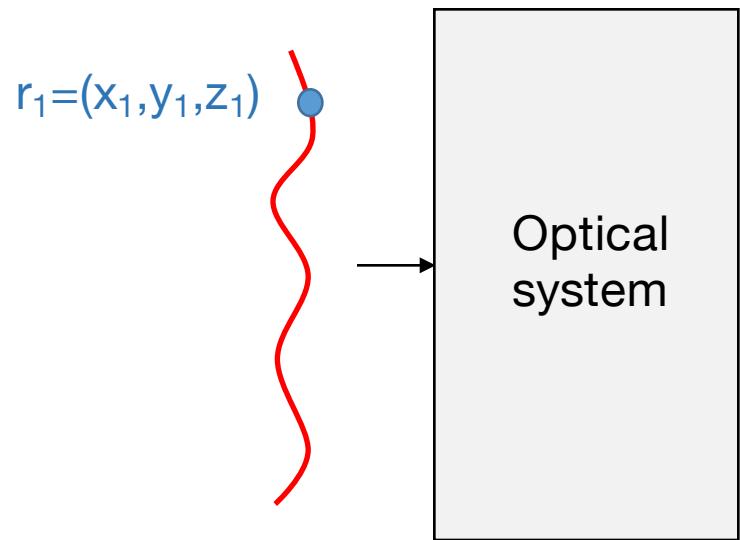
1. We will treat light as a wave (an “optical field”)

$$U(r_1) = A(r_1) \cos(kr_1 - \omega t)$$

(We will get into the details of optical fields in a few weeks)



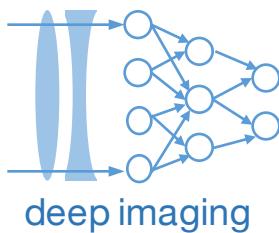
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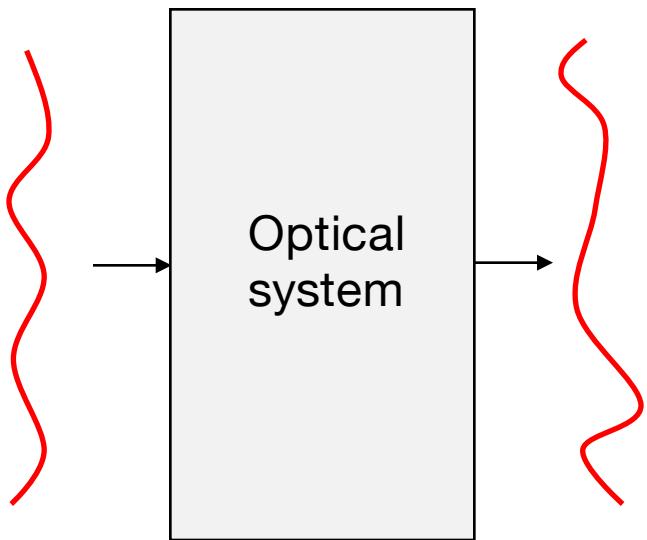
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The general idea:

1. We will treat light as a wave (an “optical field”)
2. It enters an optical system, which we treat as a black box
3. This black box has a number of useful properties



Start at the beginning: EM fields and the black box

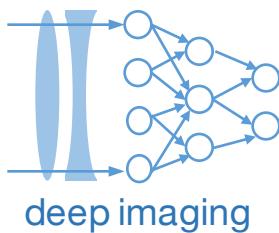


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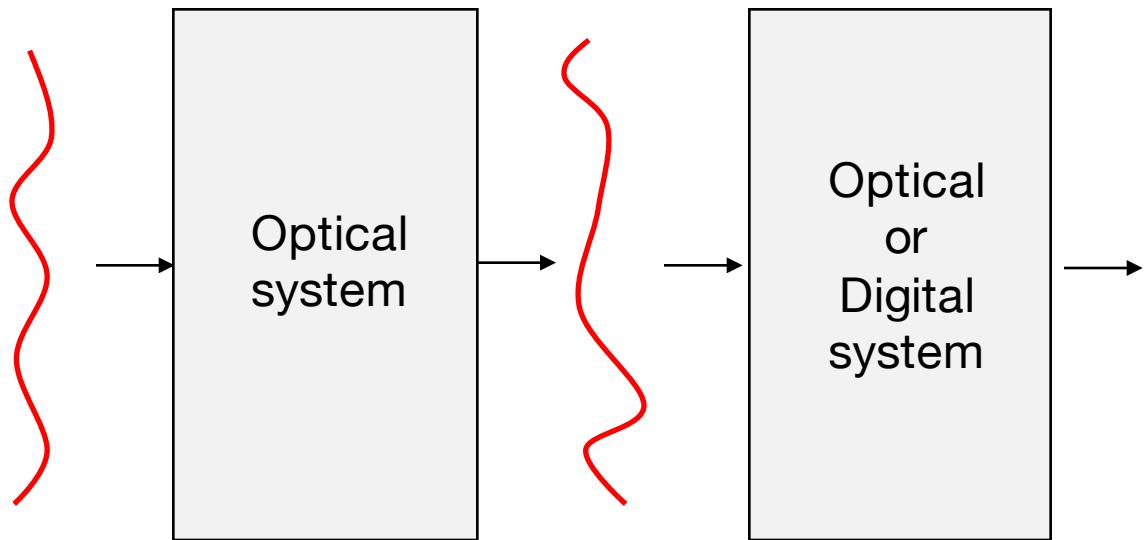
1. We will treat light as a wave (an “optical field”)
2. It enters an optical system, which we treat as a black box
3. This black box has a number of useful properties
4. The black box outputs an optical field

$$A(\mathbf{r}_1) \cos(\mathbf{k}\mathbf{r}_1 - \omega t)$$

$$A(\mathbf{r}_2) \cos(\mathbf{k}\mathbf{r}_2 - \omega t)$$



Start at the beginning: EM fields and the black box

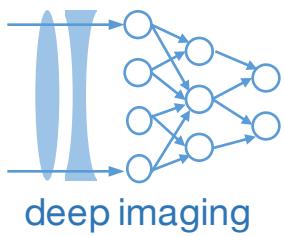


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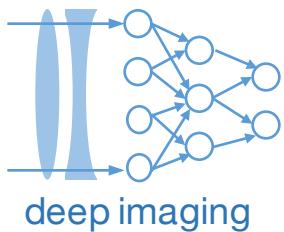
1. We will treat light as a wave (an “optical field”)
2. It enters an optical system, which we treat as a black box
3. This black box has a number of useful properties
4. The black box outputs an optical field, which then enters another optical system or a digital system
5. We can cascade these boxes...



Linear systems and the black box

Simplification #1: Let's forget about light changing as a function of time. It does so way too fast, and way too slow:

$$A(\mathbf{r}) \cos(k\mathbf{r} - \omega t) \rightarrow A(\mathbf{r}) \cos(k\mathbf{r})$$



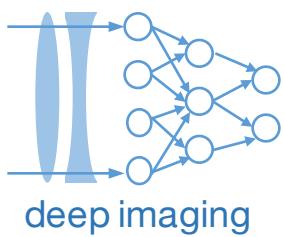
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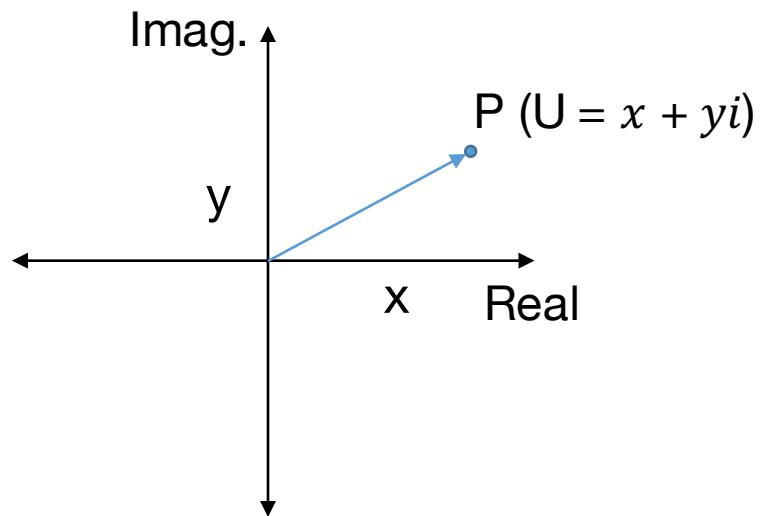
Simplification #2: We'll use complex numbers when required, it'll make our lives easier. This leads to the *complex field*, $U(\mathbf{r})$:

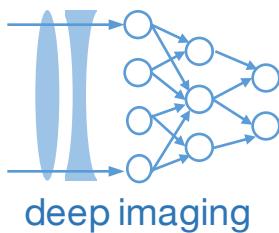
$$A(\mathbf{r}) \cos(\mathbf{k}\mathbf{r}) \leftrightarrow A(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} = U(\mathbf{r})$$



Some things you need to recall about complex numbers

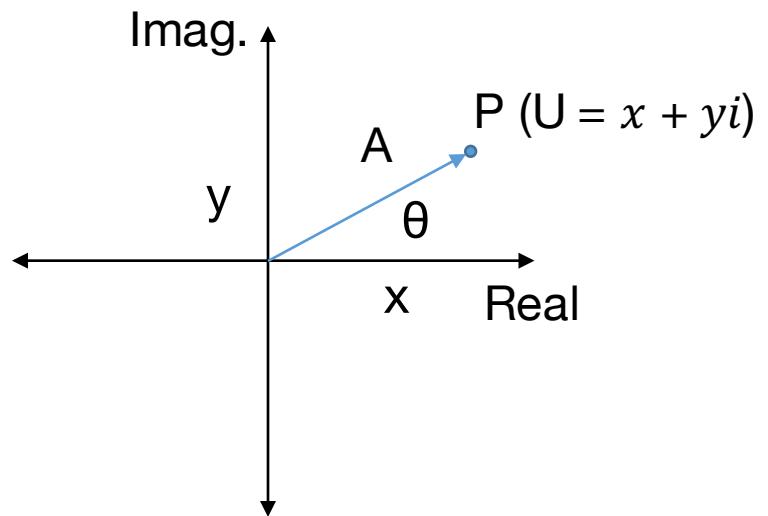
$$U = x + iy, i = \sqrt{-1}$$





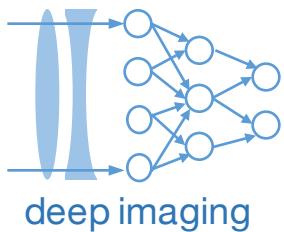
Some things you need to recall about complex numbers

$$U = x + iy, i = \sqrt{-1}$$



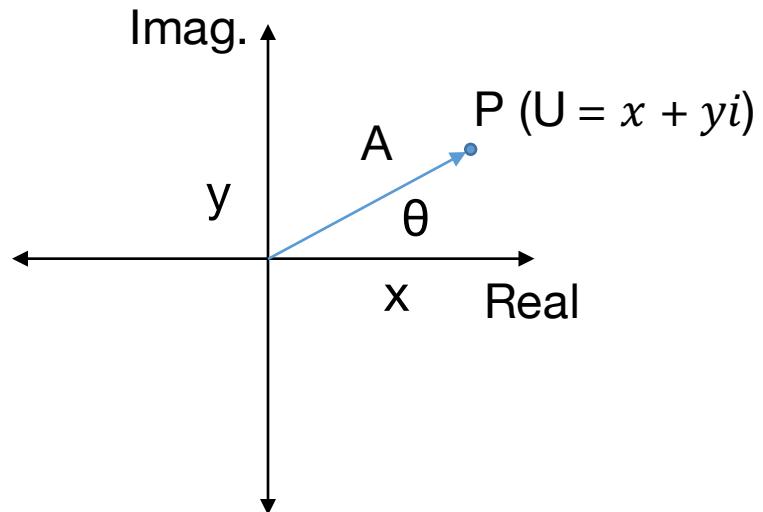
$$A = \sqrt{x^2 + y^2}$$

$$\theta = \text{atan}(y/x)$$



Some things you need to recall about complex numbers

$$U = x + iy, i = \sqrt{-1}$$



$$A = \sqrt{x^2 + y^2}$$

$$\theta = \text{atan}(y/x)$$

More useful representation:

$$x = A \cos\theta$$

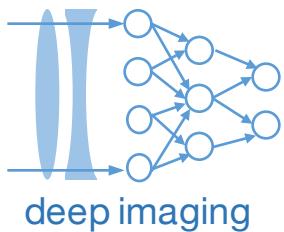
$$y = A \sin\theta$$

$$U = A (\cos\theta + i \sin\theta)$$

$$U = A e^{i\theta}$$

A = Amplitude of field

θ = Phase of field



Linear systems and the black box

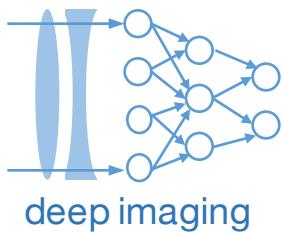
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We'll work with complex signals of this form



Linear systems and the black box

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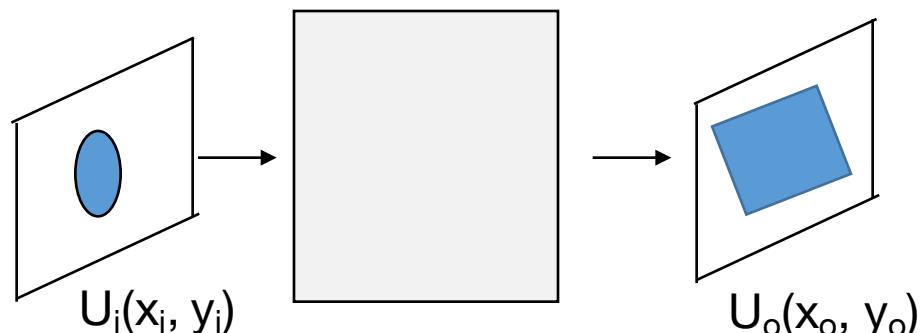
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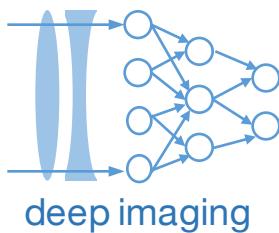
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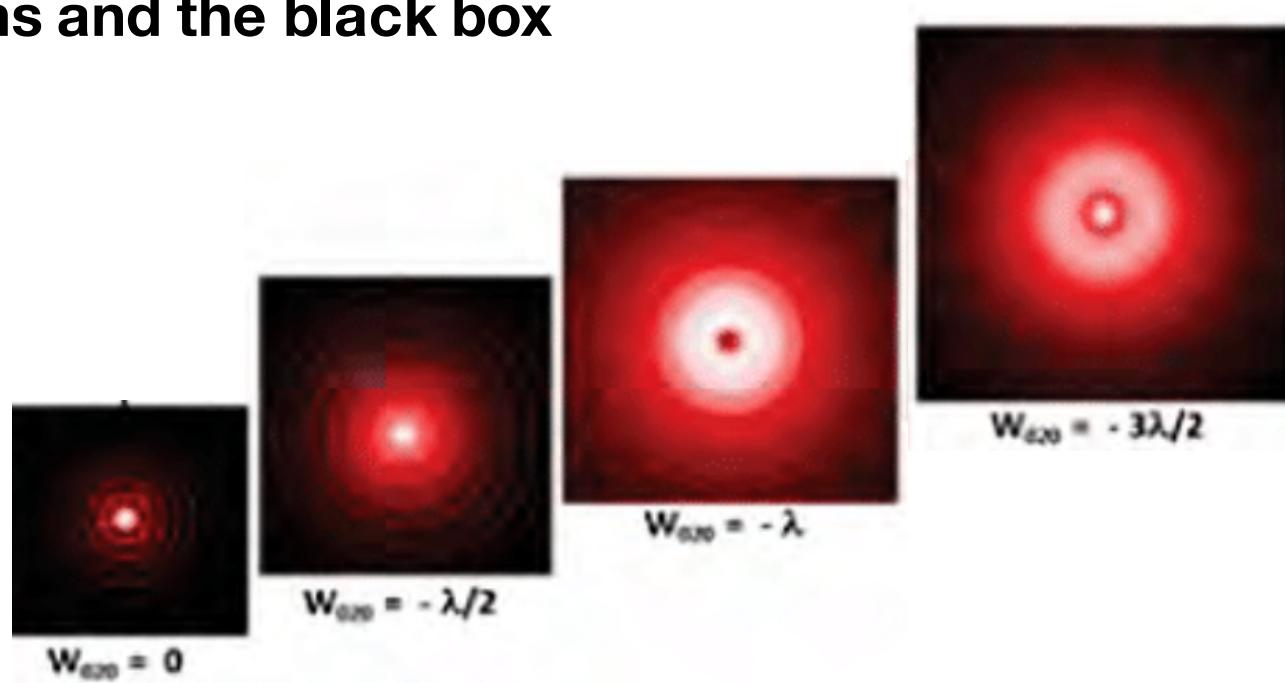
Simplification #3: Just consider mappings between planes across space. This is a critically important way of thinking for optics. Think “index card 1 to index card 2”.

$$U(\mathbf{r}) \rightarrow U(x, y)$$





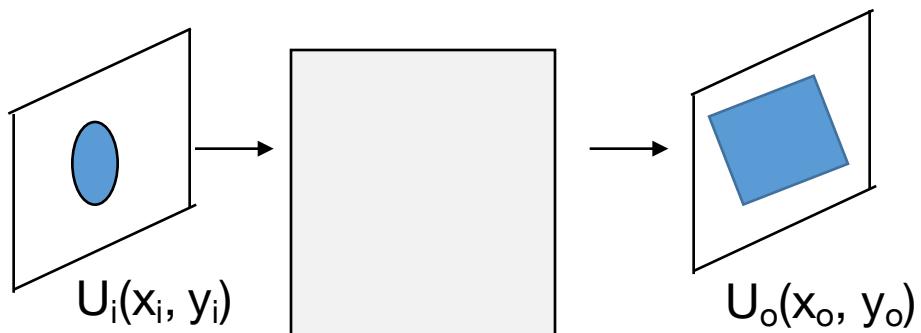
Linear systems and the black box

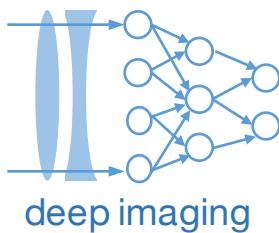


Propagation of monochromatic light

Simplification #3: Just consider mappings between planes across space. This is a critically important way of thinking for optics. Think “index card 1 to index card 2”.

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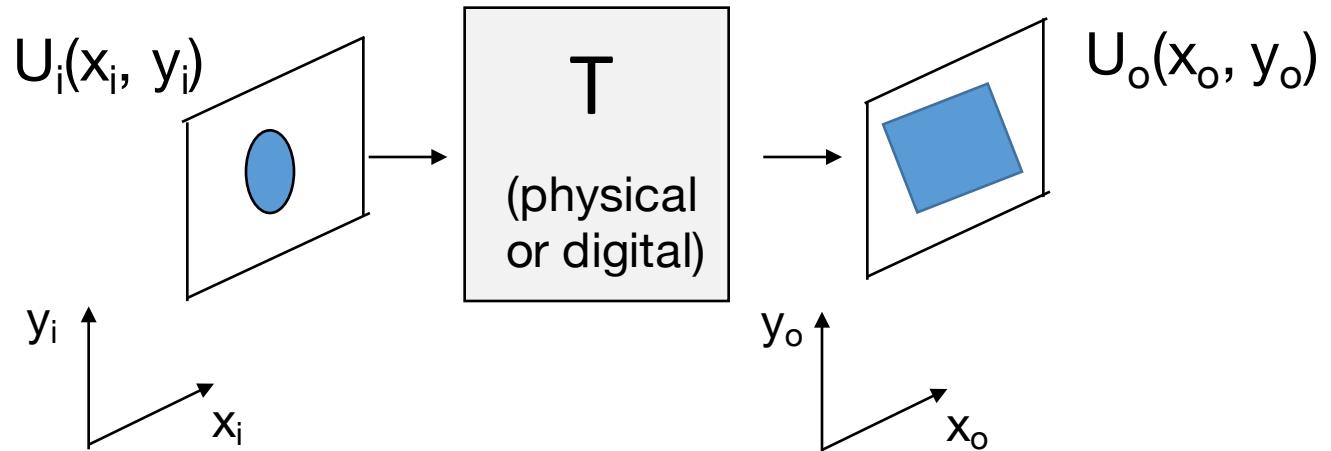
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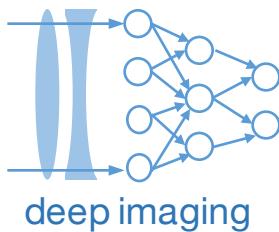
The “optical” black box system:

An optical black box system maps an input function $U_i(x_i, y_i)$ to an output function $U_o(x_o, y_o)$ via a transform T :

$$U_o(x_o, y_o) = T [U_i(x_i, y_i)]$$

Where $T[]$ denotes the optical black box transformation





Linear systems and the black box

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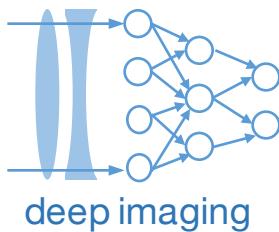
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Important properties of linear systems:

1. Homogeneity and additivity (superposition):

$$T [aU_1(x, y) + bU_2(x, y)] = aT [U_1(x, y)] + bT [U_2(x, y)]$$



Linear systems and the black box

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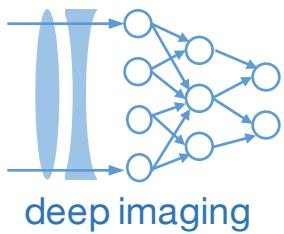
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1. Homogeneity and additivity (superposition):

$$T [aU_1(x, y) + bU_2(x, y)] = aT [U_1(x, y)] + bT [U_2(x, y)]$$

2. Shift invariance: for shift distances d_x and d_y , we assume that,

$$U_o(x_o - d_x, y_o - d_y) = T [U_i(x_i - d_x, y_i - d_y)]$$

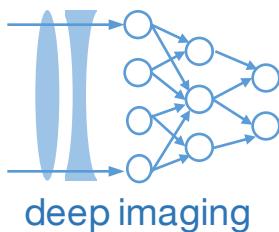


Black box transforms as a convolution

Assuming 1) linearity and 2) shift-invariance, we can model any black box with 1 piece of information:

Input Dirac delta function into the black box:

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

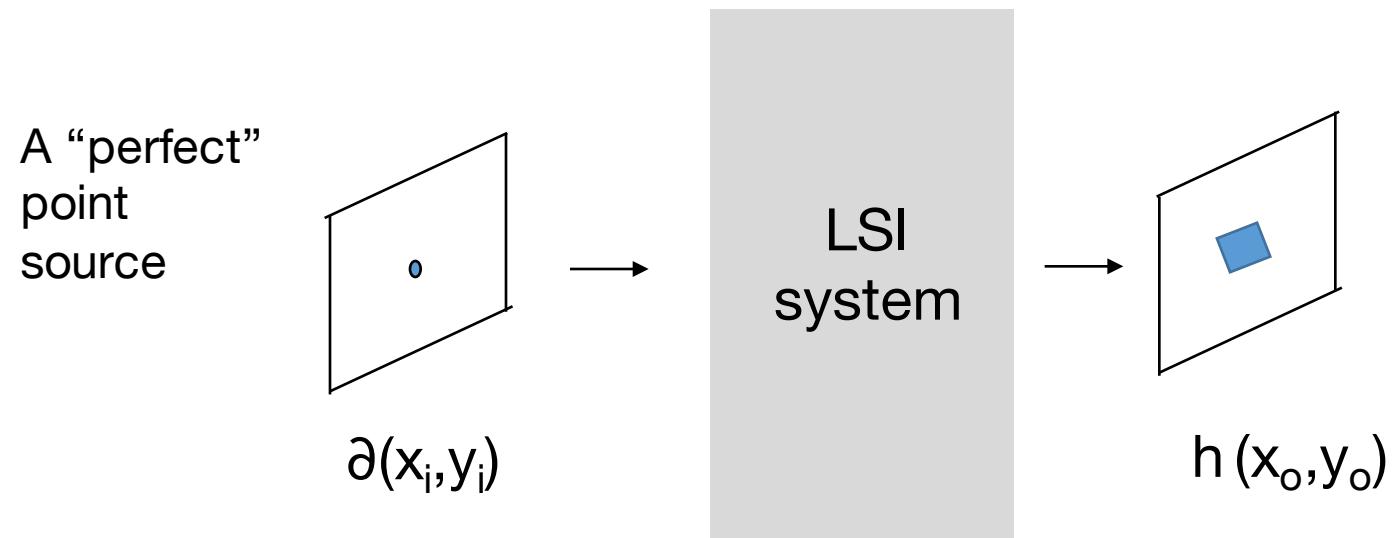


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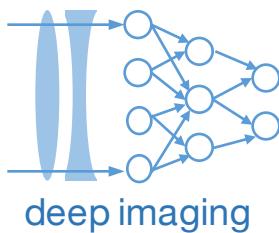
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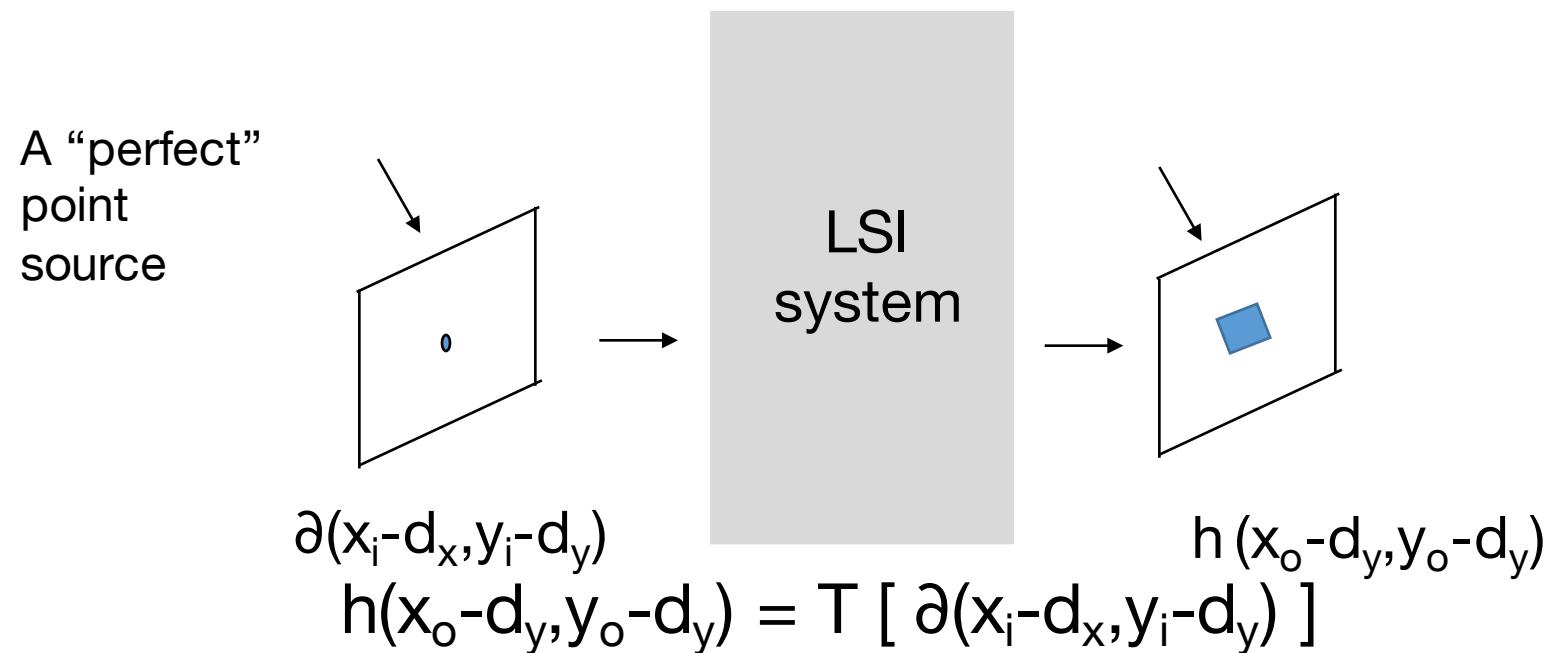
$$h(x_o, y_o) = T [\delta(x_i, y_i)]$$

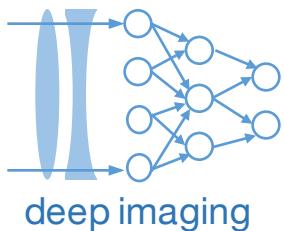


Black box transforms as a convolution

Assuming 1) linearity and 2) shift-invariance, we can model any black box with 1 piece of information:

We know the system is shift invariant:





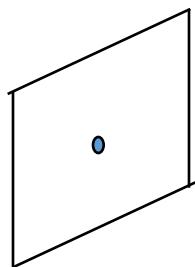
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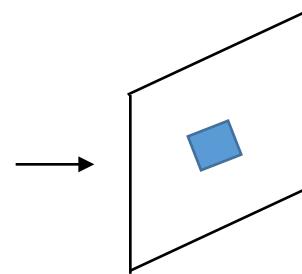
$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

A “perfect”
point
source



$$\delta(x_i, y_i)$$

LSI
system

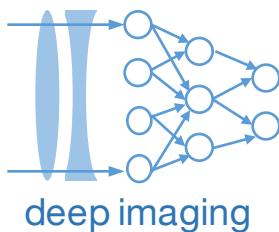


$$h(x_o, y_o)$$

$h(x_o, y_o)$ is the
system’s point-
spread function

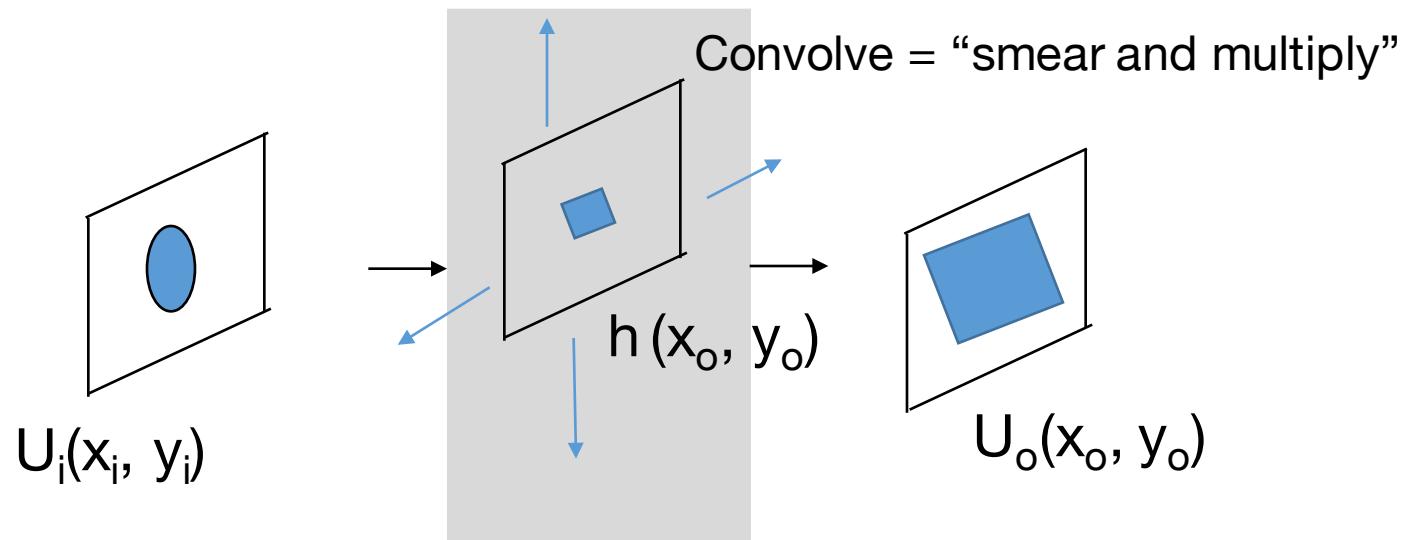
Point-spread function

$$h(x_o, y_o) = T [\delta(x_i, y_i)]$$



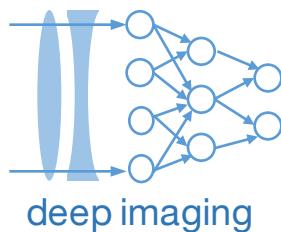
Black box transforms as a convolution

Knowing the point-spread function, it is direct to model any output of the black box, given an input:

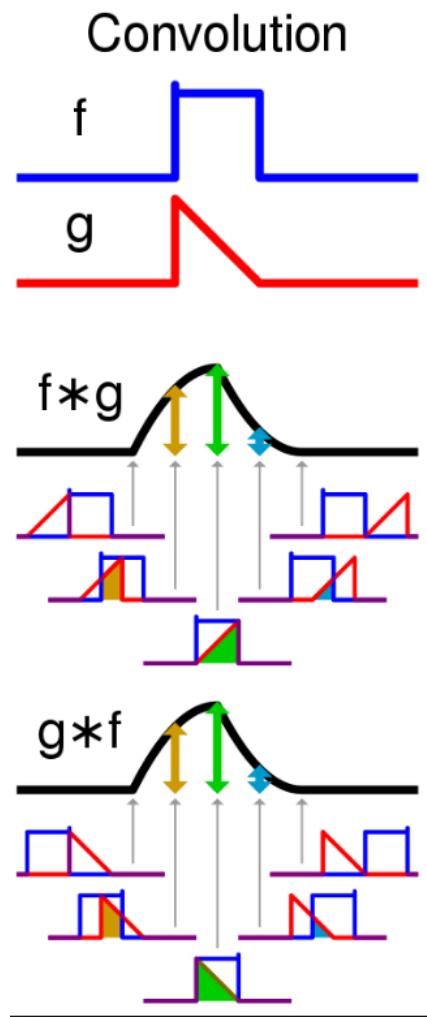


$$U_o(x_o, y_o) = \iint_{-\infty}^{\infty} U_i(x_i, y_i) h(x_o - x_i, y_o - y_i) dx_i dy_i$$

Output of linear system is a convolution of the input with its point-spread function

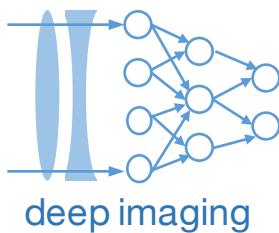


1D convolution example



Steps to perform a convolution:

1. Flip one signal (the second one = the PSF)
2. Position PSF right before overlap
- With incremental steps:
3. Step PSF over to position x_o
4. Compute *area* of overlap of two functions
5. Convolution value at x_o = area of overlap
6. Repeat 3-5 until signals do not overlap



2D convolution example

- Direct extension of 1D concept to 2D functions
- Note – it is effectively the same with discrete functions = matrices

$$U_1(x,y)$$



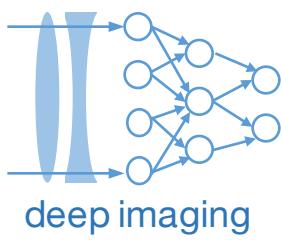
$$U_0(x,y)$$



$$U_1(x,y) * h(x,y) = U_0(x,y)$$

x

x



Next Lecture: Analyzing light and image formation via Fourier transforms!