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Lecture 17: Introduction to Fourier Optics

Machine Learning and Imaging

BME 590L
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Let's take a step back: how does light propagate?

Maxwell's equations
without any charge

$$\nabla \times \vec{\mathcal{E}} = -\mu \frac{\partial \vec{\mathcal{H}}}{\partial t}$$

$$\nabla \times \vec{\mathcal{H}} = \epsilon \frac{\partial \vec{\mathcal{E}}}{\partial t}$$

$$\nabla \cdot \epsilon \vec{\mathcal{E}} = 0$$

$$\nabla \cdot \mu \vec{\mathcal{H}} = 0.$$

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$$\nabla \cdot \epsilon \vec{\mathcal{E}} = 0$$

$$\nabla \cdot \mu \vec{\mathcal{H}} = 0.$$

1. Take the curl of both sides of first equation
2. Substitute 2nd and 3rd equation
3. Arrive at the wave equation:

$$\nabla^2 \vec{\mathcal{E}} - \frac{n^2}{c^2} \frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2} = 0 \qquad n = \left(\frac{\epsilon}{\epsilon_0}\right)^{1/2} \qquad c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

Let's take a step back: how does light propagate?

Considering light that isn't pulsed over time, we can use the following solution:

$$u(P,t) = A(P)\cos[2\pi\nu t + \phi(P)]$$

$$u(P,t) = Re\{U(P)\exp(-j2\pi\nu t)\},$$

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With this particular solution, we get the following important time-independent equation:

Helmholtz Equation

$$\boxed{(\nabla^2 + k^2)U = 0.}$$

$$k = 2\pi n\frac{\nu}{c} = \frac{2\pi}{\lambda},$$

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Helmholtz Equation

$$\boxed{(\nabla^2 + k^2)U = 0.} \qquad k = 2\pi n\frac{\nu}{c} = \frac{2\pi}{\lambda},$$

This is an important equation in physics. We won't go into the details, but it leads to the Huygen-Fresnel principle:

$$U(P_2) = \frac{1}{j\lambda}\iint_{\Sigma} U(P_1)\frac{\exp(jkr_{21})}{r_{21}}\cos\theta\, ds$$

Plane-to-plane light propagation via the "paraxial approximation"

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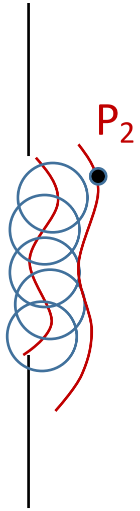
The Huygens-Fresnel Equation

$$U(P_2) = \frac{1}{j\lambda} \iint_{\Sigma} U(P_1) \frac{\exp(jkr_{21})}{r_{21}} \cos\theta \, ds$$



Aperture

Plane-to-plane light propagation via the ”paraxial approximation”

The Huygens-Fresnel Equation

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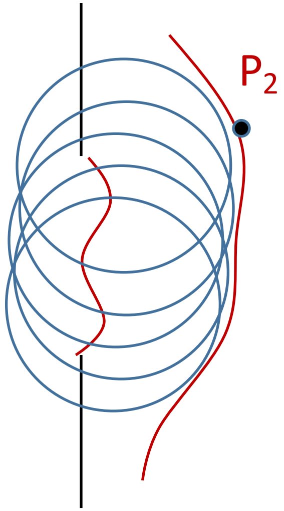


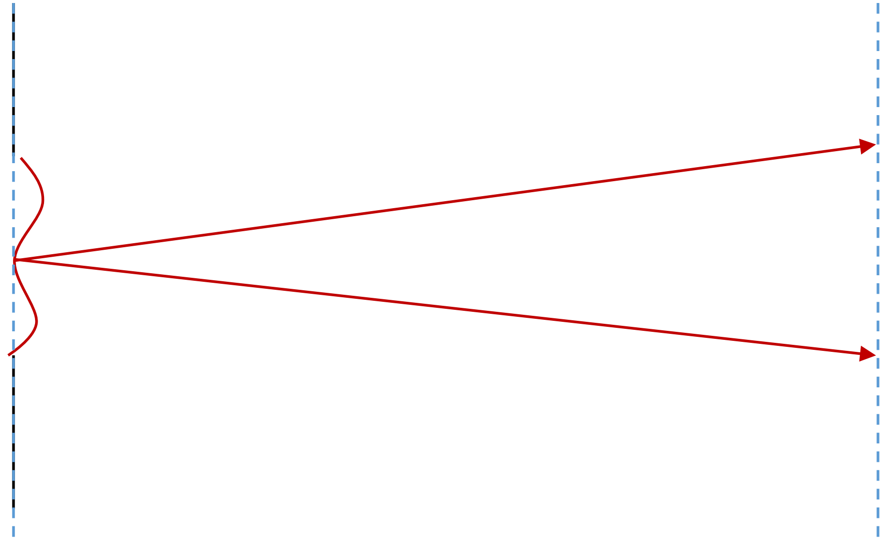
Generally connects two points in 3D:

$$U(P_1) = U(x_1, y_1, z_1)$$

$$U(P_2) = U(x_2, y_2, z_2)$$

Plane-to-plane light propagation via the "paraxial approximation"

We are usually concerned about propagation between two planes (almost always in an optical system):



$$U(P_1) = U(x_1, y_1, z_1 = z_{p1}) \qquad U(P_2) = U(x_2, y_2, z_2 = z_{p2})$$

$$U(P) = E(x, y, z)e^{ikz}$$

Plane-to-plane light propagation via the "paraxial approximation"

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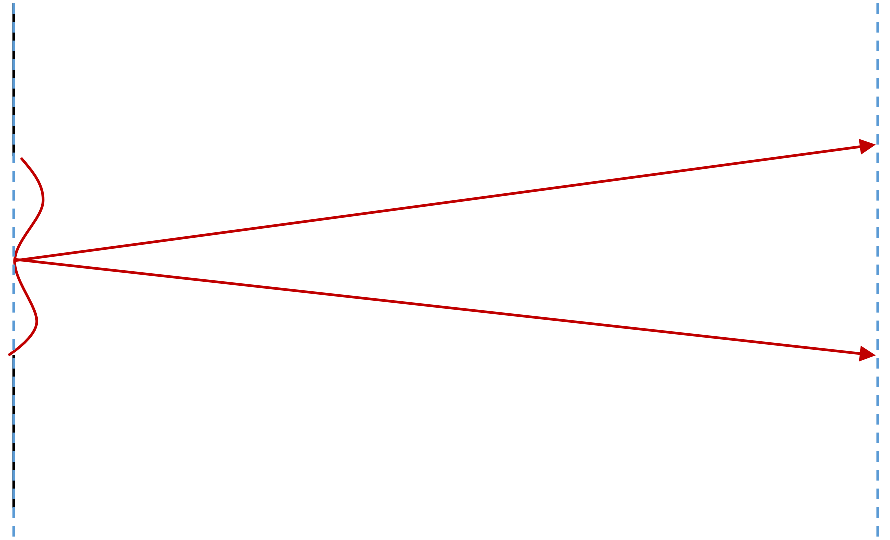

Paraxial approximation:

$$\boxed{\nabla_\perp^2 U + 2ik\frac{dU}{dz} = 0}$$

$$\nabla_\perp^2 \stackrel{\text{def}}{=} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$U(P_1) = U(x_1, y_1, z_1 = z_{p1}) \qquad U(P_2) = U(x_2, y_2, z_2 = z_{p2})$$

$$U(P) = E(x, y, z)e^{ikz}$$

Plane-to-plane light propagation via the ”paraxial approximation”

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Paraxial approximation:

$$\nabla_{\perp}^2 U + 2ik\frac{dU}{dz} = 0$$

Substitute in $U(P) = E(x,y,z)e^{ikz}$ and crank the wheel,

$$\nabla_{\perp}^2 E + 2ik\frac{dE}{dz} + 2k^2 E = 0$$

Paraxial Helmholtz Equation. This has an exact integral solution:

Plane-to-plane light propagation via the ”paraxial approximation”

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Paraxial Helmholtz Equation. This has an exact integral solution:

$$\boxed{E(x, y, z) = \frac{e^{ikz}}{i\lambda z}\iint_{-\infty}^{+\infty} E(x', y', 0)e^{\frac{ik}{2z}\left[(x-x')^2+(y-y')^2\right]}dx'dy'}$$

Fresnel diffraction integral

This is how light propagates from one plane to the next. It’s a convolution!

Fresnel light propagation as a convolution

$$E(x,y,z) = \frac{e^{ikz}}{i\lambda z} \iint_{-\infty}^{+\infty} E(x',y',0) e^{\frac{ik}{2z}\left[(x-x')^2+(y-y')^2\right]} dx' dy'$$

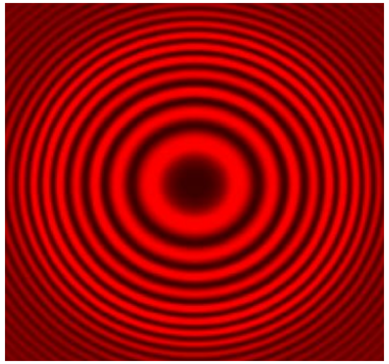
$$h(x,y,z) = \frac{e^{ikz}}{i\lambda z} e^{i\frac{k}{2z}(x^2+y^2)}$$

$$E(x,y,z) = E(x,y,0) * h(x,y,z)$$

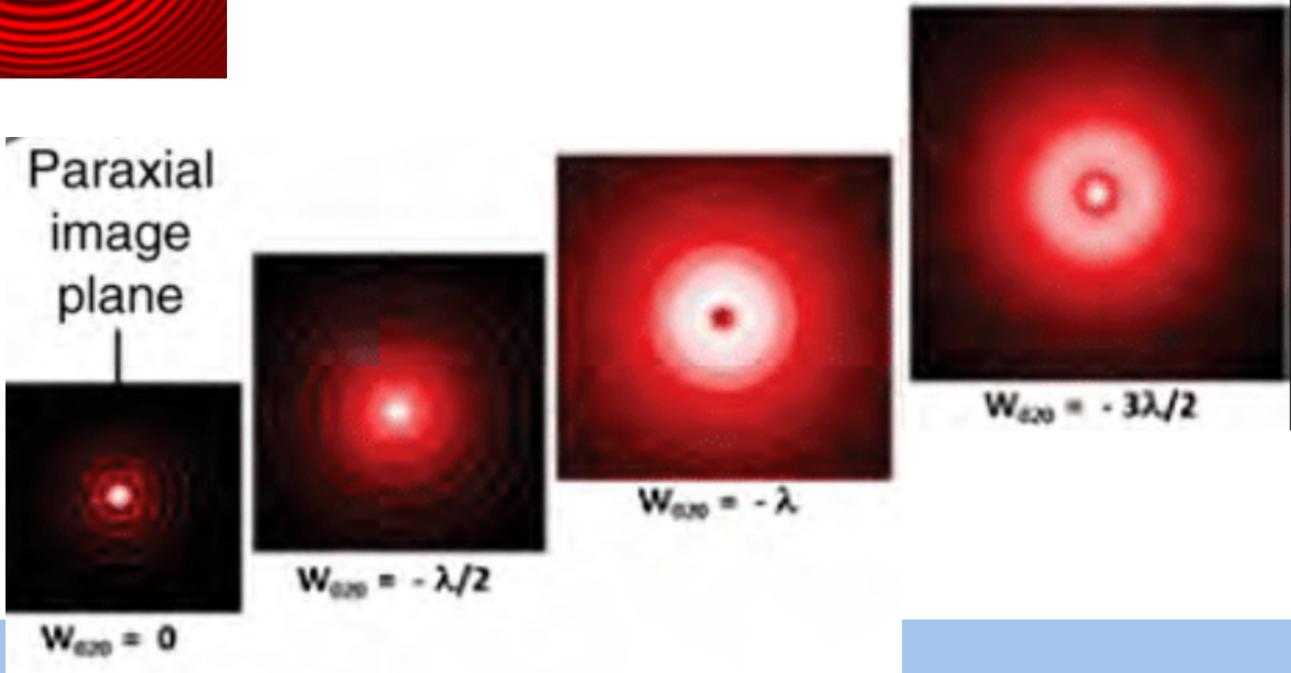
Fresnel light propagation as a convolution

$$E(x,y,z) = \frac{e^{ikz}}{i\lambda z}\iint_{-\infty}^{+\infty} E(x',y',0)\, e^{\frac{ik}{2z}\left[(x-x')^2+(y-y')^2\right]}\, dx'\, dy'$$

$$h(x,y,z) = \frac{e^{ikz}}{i\lambda z} e^{i\frac{k}{2z}(x^2+y^2)}$$



$$E(x,y,z) = E(x,y,0) * h(x,y,z)$$



From the Fresnel approximation to the Fraunhofer approximation

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Fresnel Approximation:

$$E(x,y,z) = \frac{e^{ikz}}{i\lambda z} \iint_{-\infty}^{+\infty} E(x',y',0) e^{\frac{ik}{2z}\left[(x-x')^2+(y-y')^2\right]} dx' dy'$$

Lets assume that the second plane is "pretty far away" from the first plane. Then,

$$\boxed{z > \frac{2D^2}{\lambda}}$$

From the Fresnel approximation to the Fraunhofer approximation

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Fresnel Approximation:

$$E(x,y,z) = \frac{e^{ikz}}{i\lambda z} \iint_{-\infty}^{+\infty} E(x',y',0) e^{\frac{ik}{2z}\left[(x-x')^2+(y-y')^2\right]} dx' dy'$$

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1. Expand the squaring

$$E(x,y,z) = \frac{e^{ikz}}{i\lambda z} \iint E(x',y',0) e^{\frac{ik}{2z}(x^2+y^2)} e^{\frac{ik}{2z}(x'^2+y'^2)} e^{\frac{ik}{2z}(xx'+yy')} dx' dy'$$

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From the Fresnel approximation to the Fraunhofer approximation

Fresnel Approximation:

$$E(x,y,z)=\frac{e^{ikz}}{i\lambda z}\iint_{-\infty}^{+\infty}E(x',y',0)e^{\frac{ik}{2z}\left[(x-x')^2+(y-y')^2\right]}dx'dy'$$

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$$E(x,y,z)=\frac{e^{ikz}}{i\lambda z}\iint E(x',y',0)e^{\frac{ik}{2z}(x^2+y^2)}e^{\frac{ik}{2z}(x'^2+y'^2)}e^{\frac{ik}{2z}(xx'+yy')}dx'dy'$$

2. Front term comes out, assume second term goes away, then,

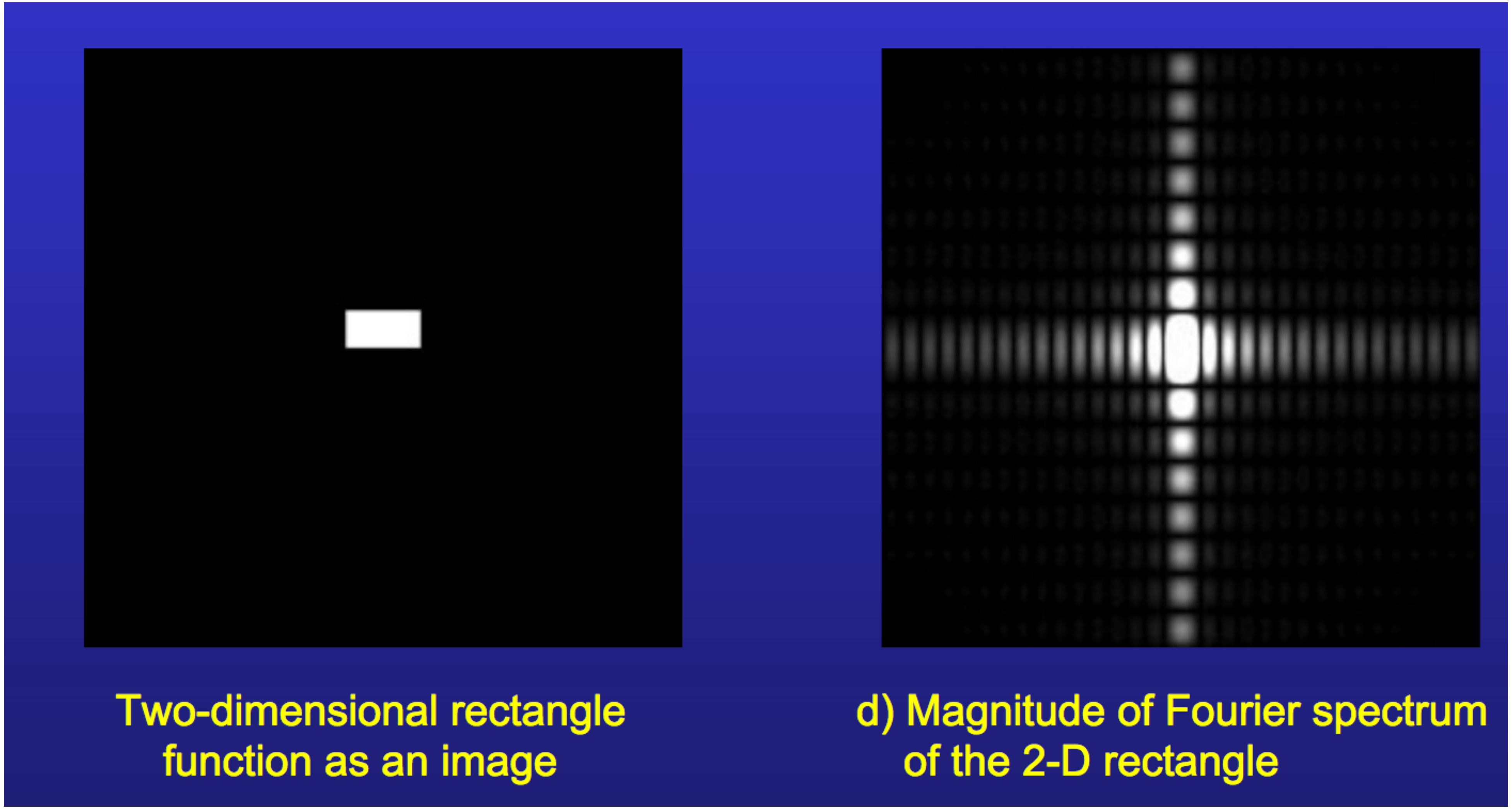
$$\boxed{E(x,y,z)=C\iint E(x',y',0)e^{\frac{ik}{2z}(xx'+yy')}dx'dy'}$$

$$C=\frac{e^{ikz}}{i\lambda z}e^{\frac{ik}{2z}(x^2+y^2)}$$

Fraunhofer diffraction is a Fourier transform!!!!!!!

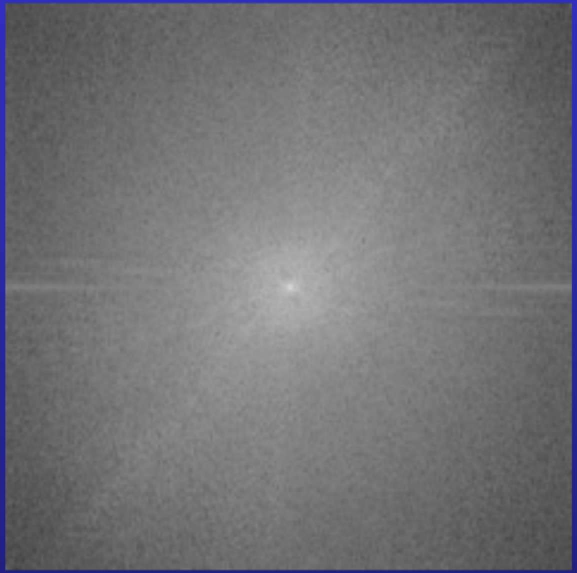
This is the aperture

This is what you see far away

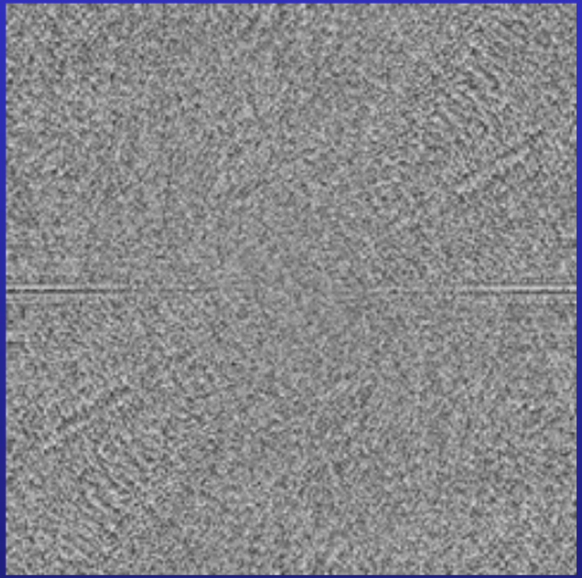


Cheetah

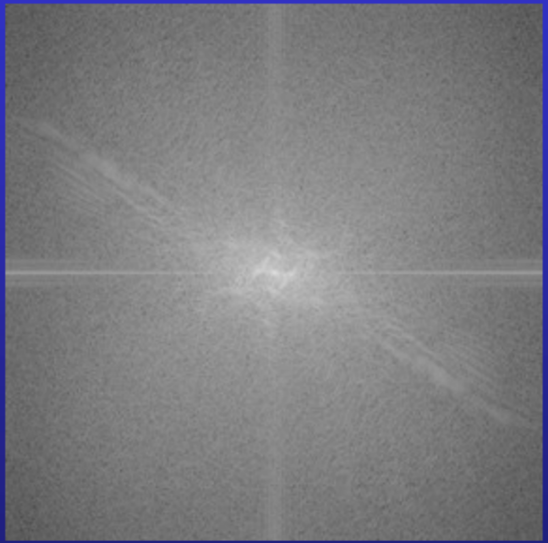
Zebra



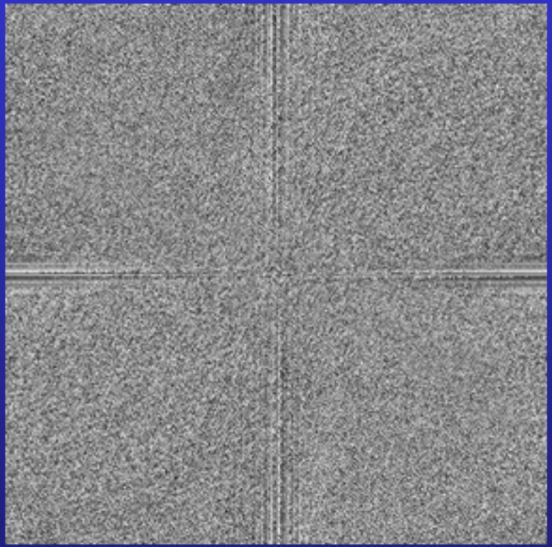
magnitude of cheetah



phase of cheetah

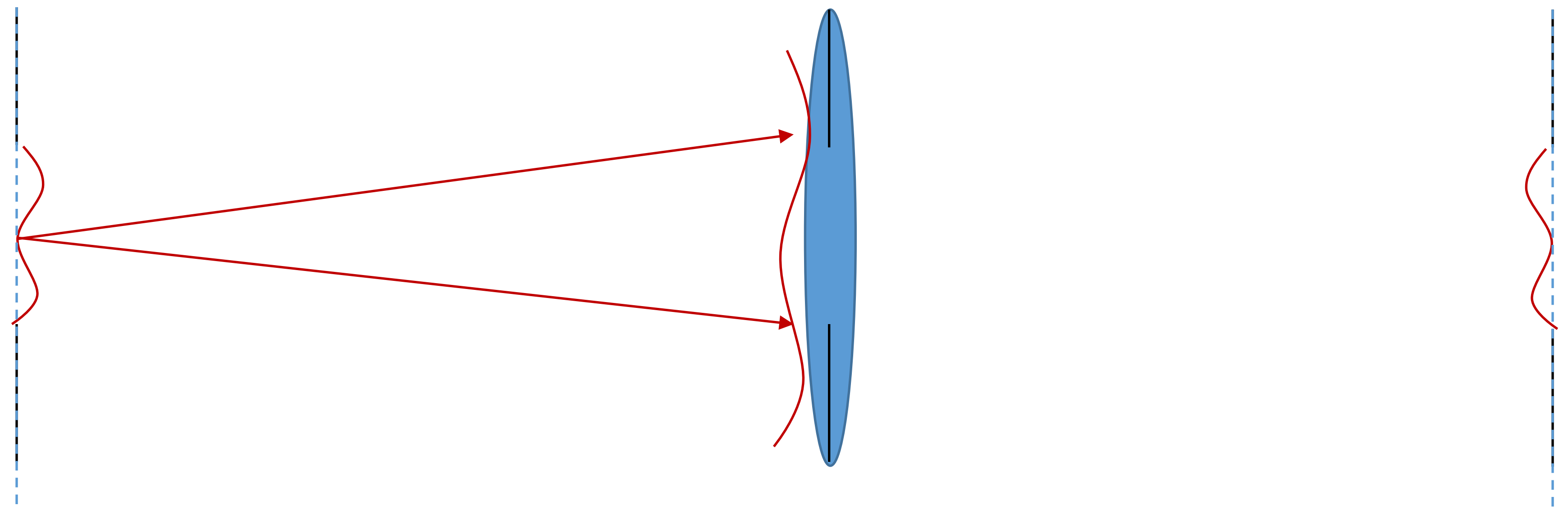


magnitude of zebra



phase of zebra

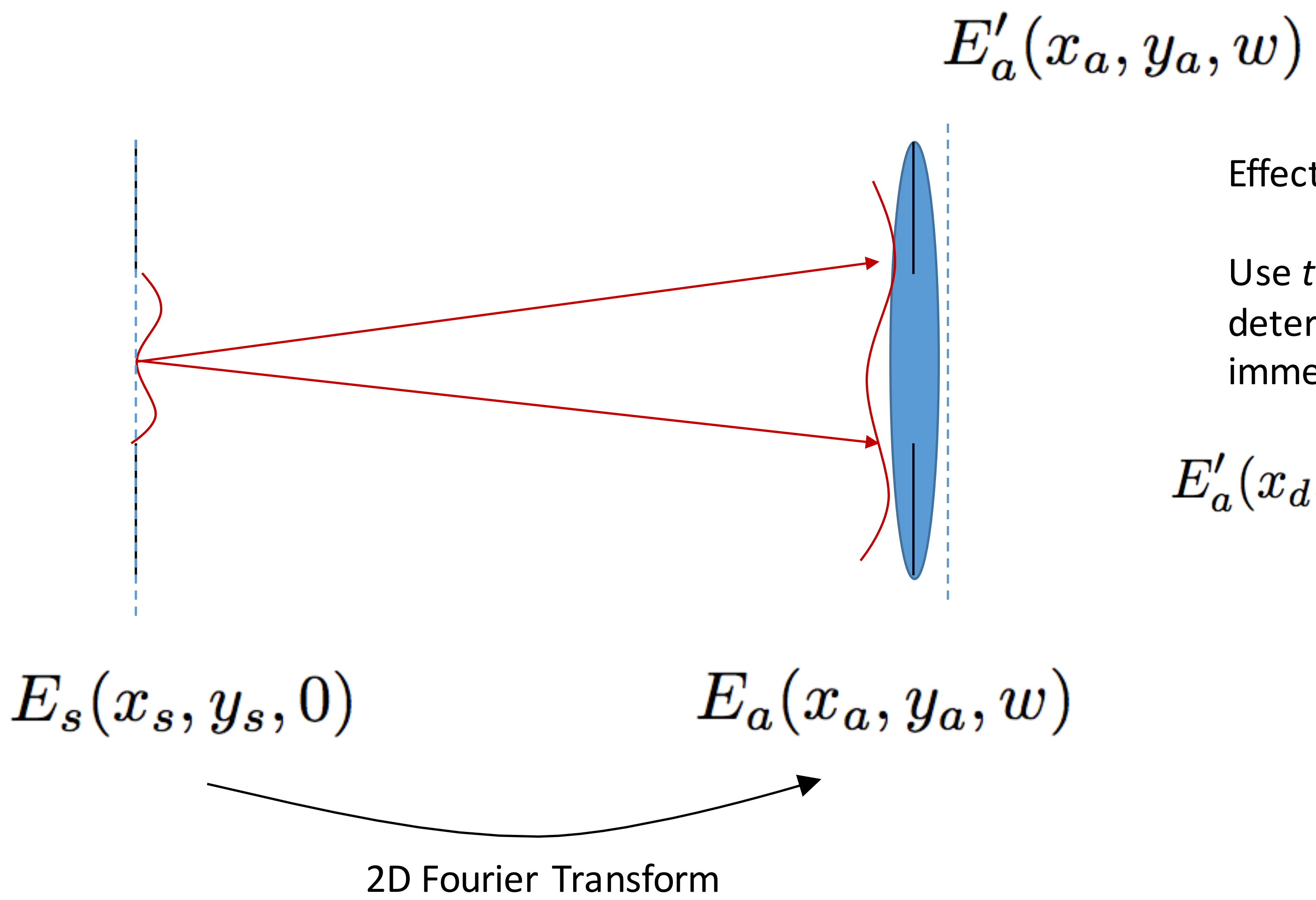
Model of a microscope (or camera) using Fourier transforms:



$$E_s(x_s, y_s, 0) \qquad\qquad E_a(x_a, y_a, w)$$

2D Fourier Transform

Model of a microscope (or camera) using Fourier transforms:



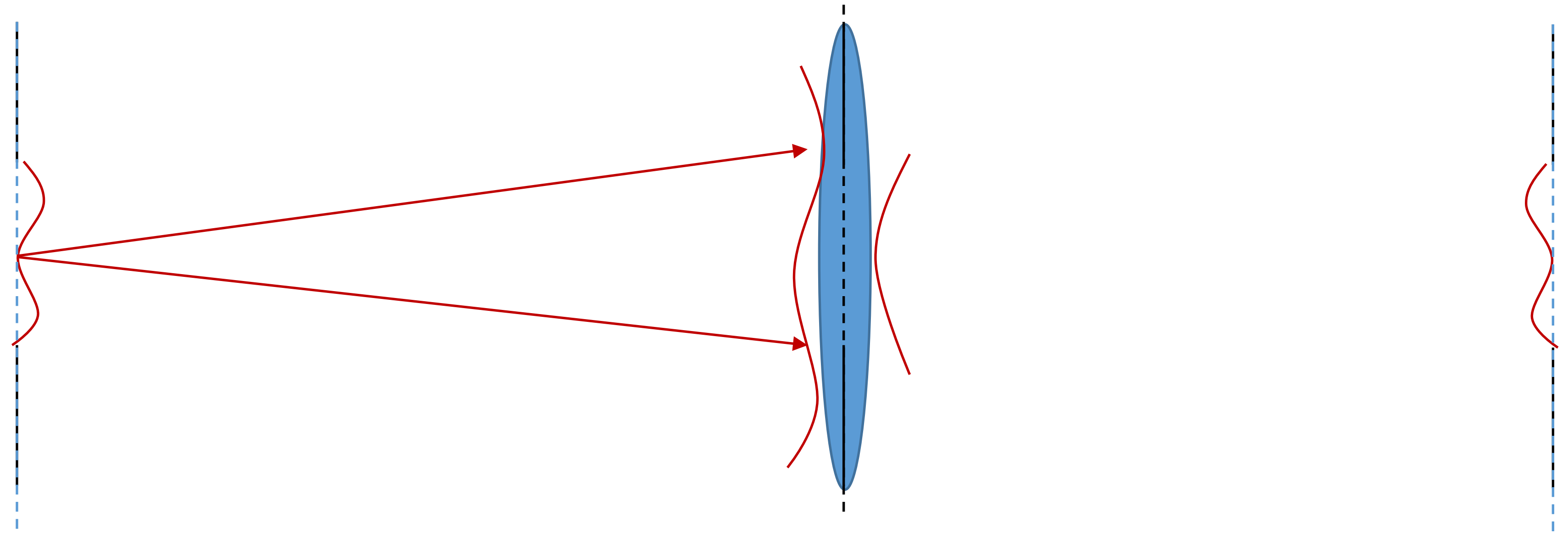
Effect of the lens is to block light.

Use *thin object approximation* to determine distribution of light on the immediate other side of the lens stop:

$$E'_a(x_d, y_d) = E_a(x_d, y_d)A(x_d, y_d)$$

Model of a microscope (or camera) using Fourier transforms:

Last piece of the puzzle: what happens from lens to sensor?



$$E_s(x_s, y_s, 0)$$

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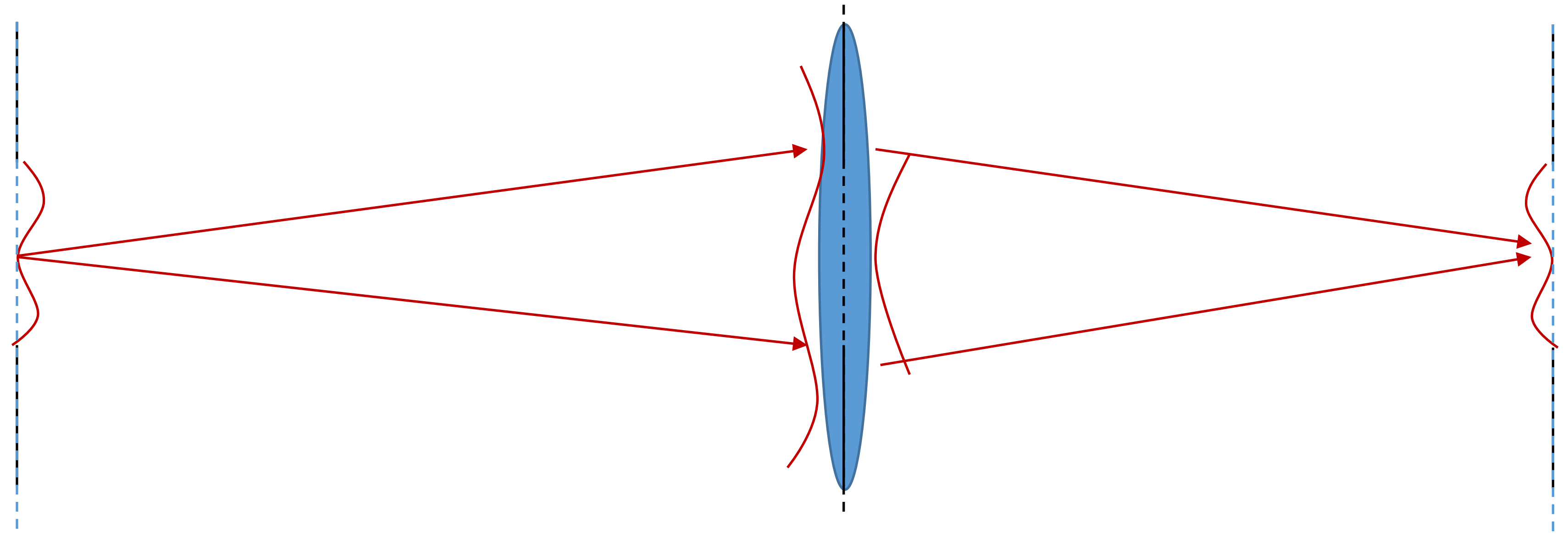
2D Fourier Transform

Model of a microscope (or camera) using Fourier transforms:

Last piece of the puzzle: what happens from lens to sensor?

inverse Fourier transform!



$$E_s(x_s, y_s, 0)$$

$$E'_a(x_d, y_d) = E_a(x_d, y_d)A(x_d, y_d)$$

2D Fourier Transform

2D inverse Fourier Transform

This process should sound familiar….

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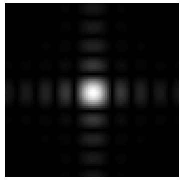
Input image

$U_1(x,y)$

Convolution filter h

$*$



$=$

Output image

$U_2(x,y)$

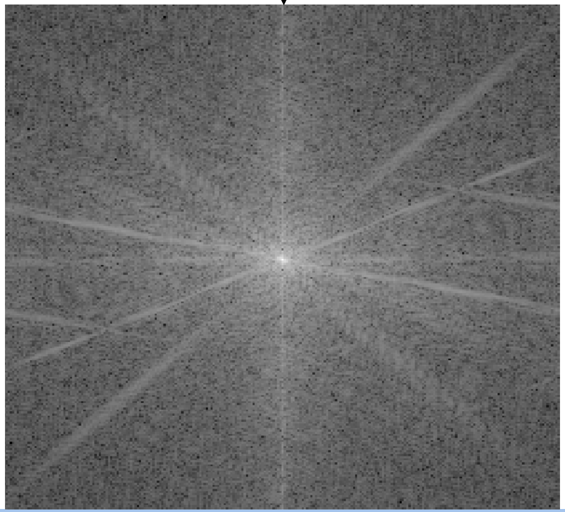
$F\,[U_1]$

$F\,[h]$

$F^{-1}[H\hat{U}_1]$

Input spectrum

$\hat{U}_1(f_x,f_y)$



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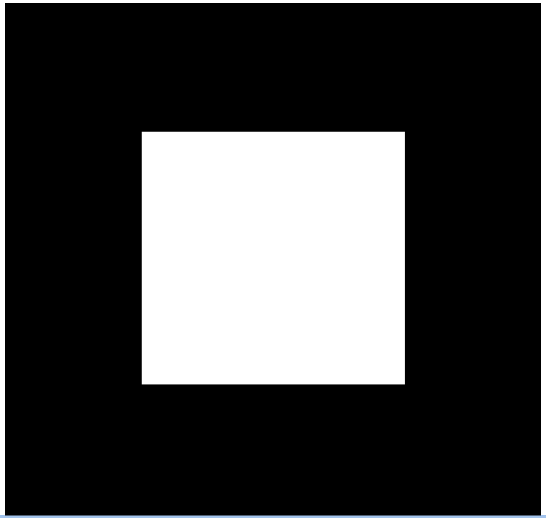

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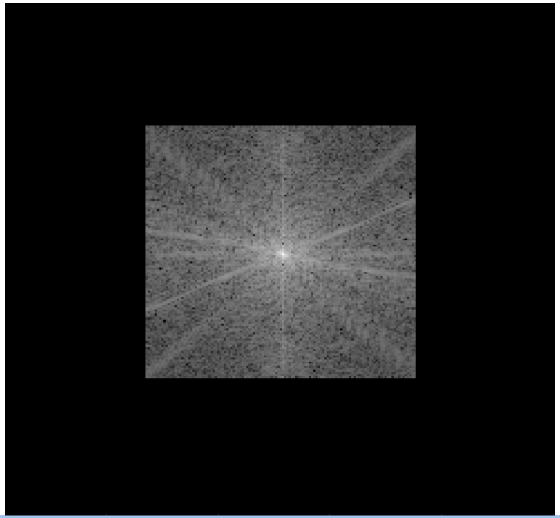
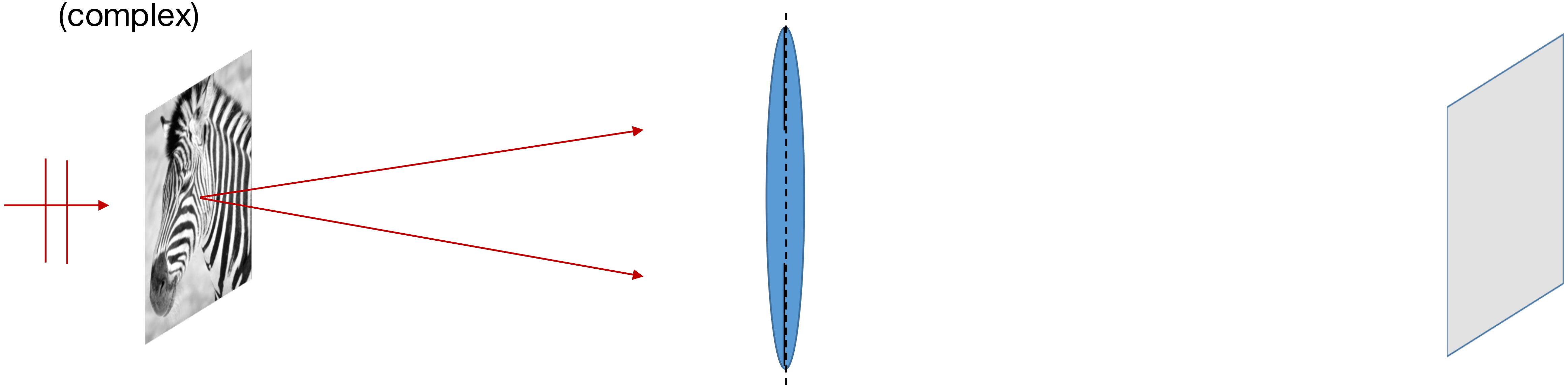


Model of image formation for wave optics (coherent light):

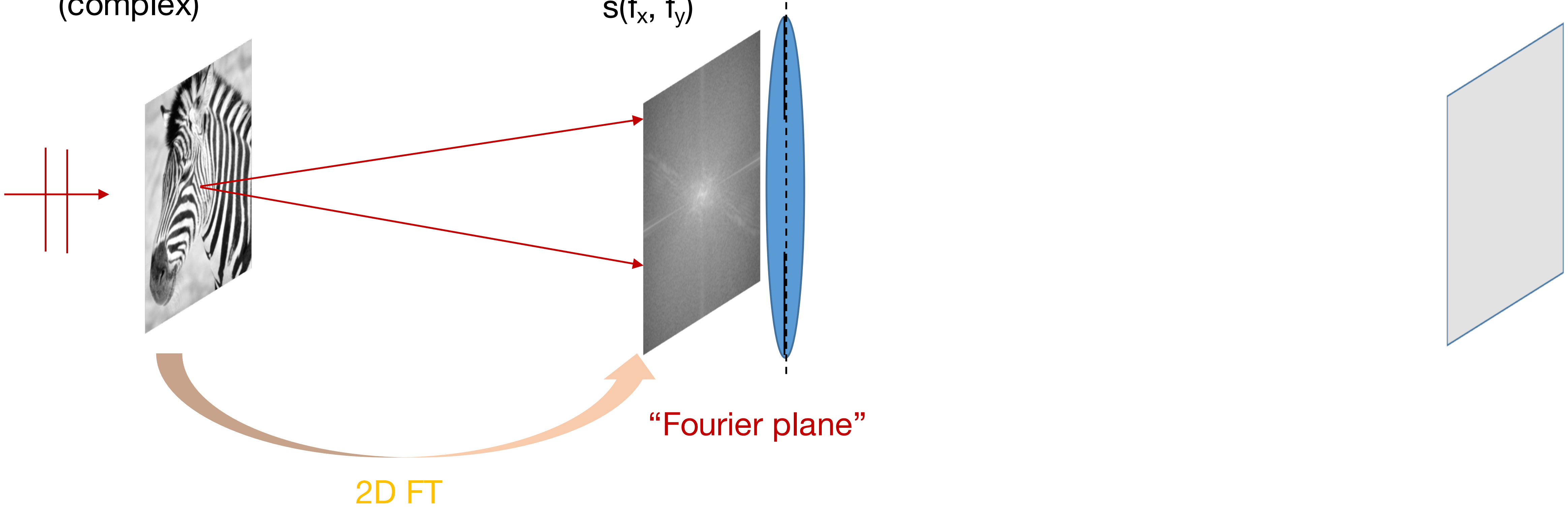
1. Discrete sample function s(x,y) (complex)



Model of image formation for wave optics (coherent light):

1. Discrete sample function s(x,y) (complex)

2. Compute its 2D Fourier transform
$\hat{s}(f_x, f_y)$

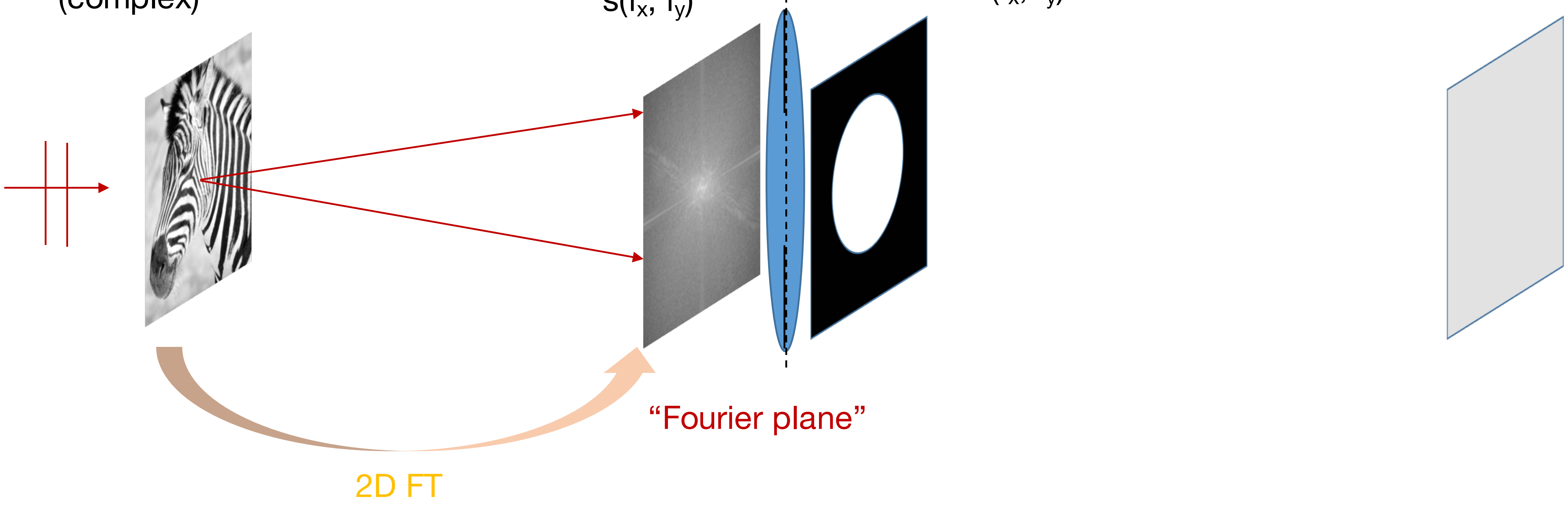


Model of image formation for wave optics (coherent light):

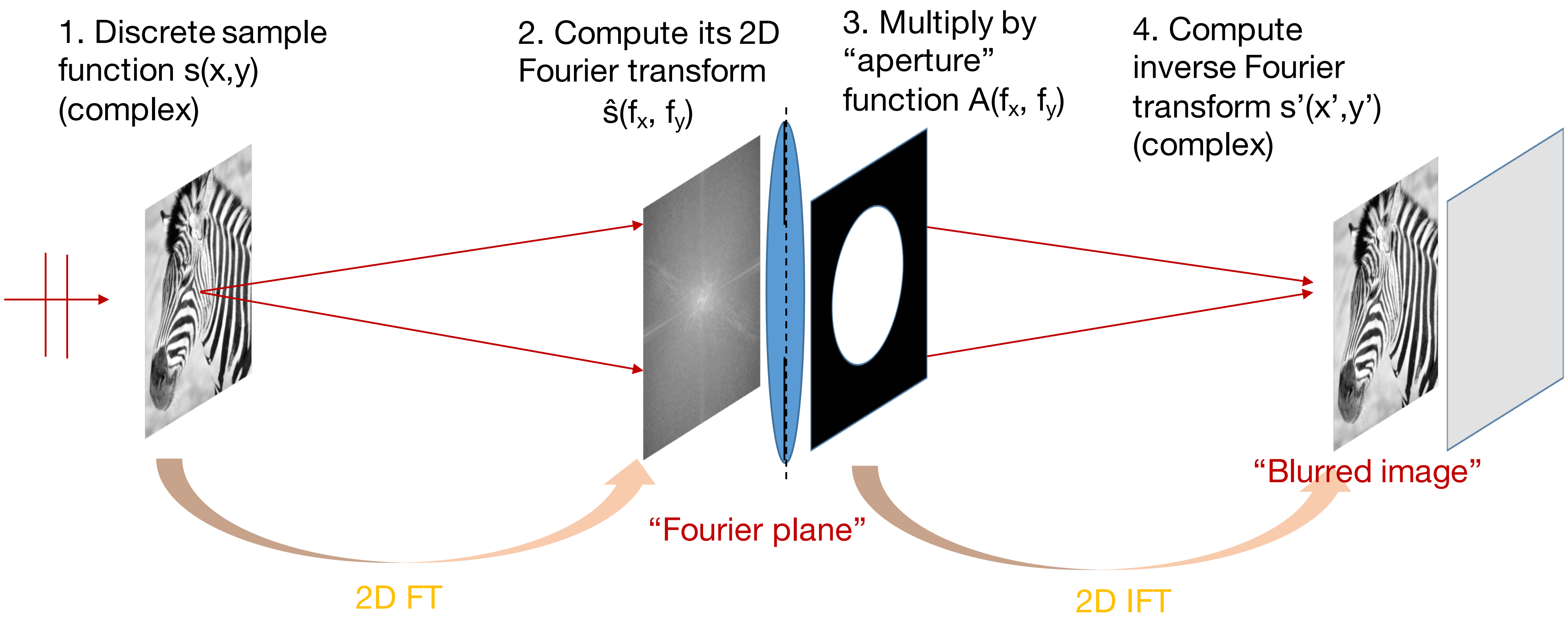
1. Discrete sample function s(x,y) (complex)

2. Compute its 2D Fourier transform $\hat{s}(f_x, f_y)$

3. Multiply by "aperture" function $A(f_x, f_y)$



Model of image formation for wave optics (coherent light):



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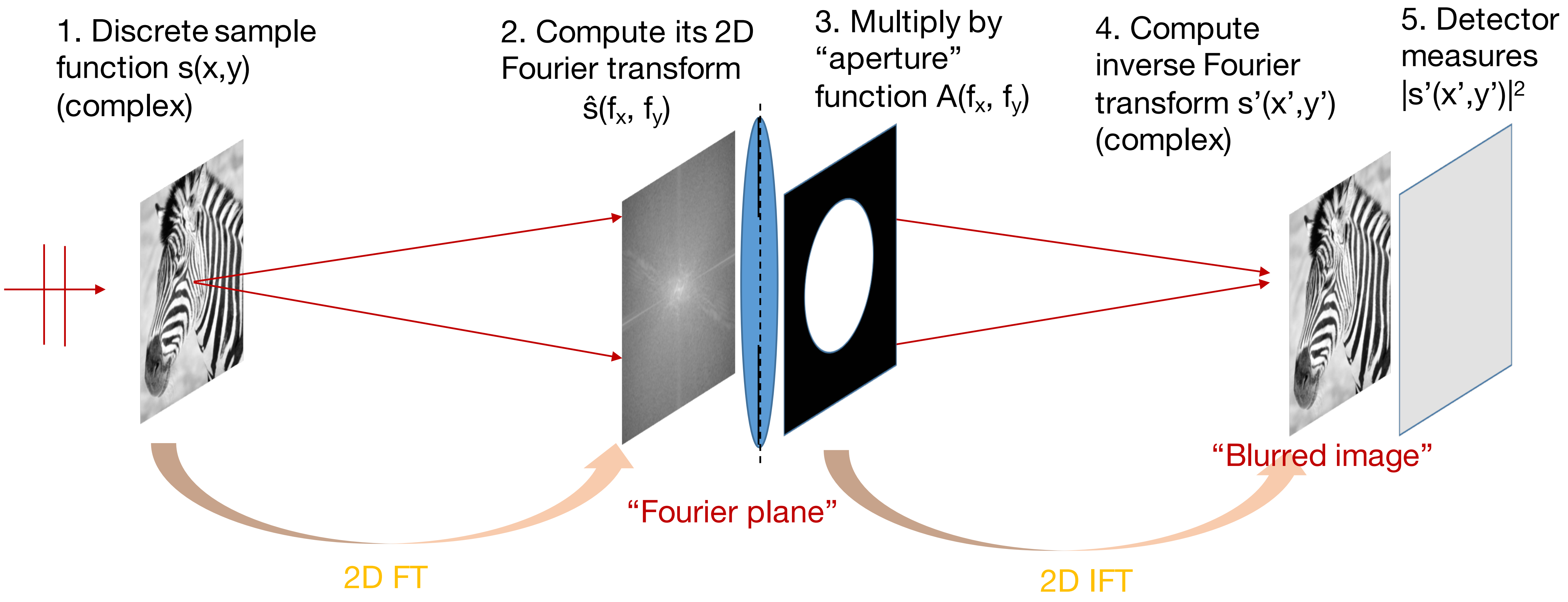
Model of image formation for wave optics (coherent light):



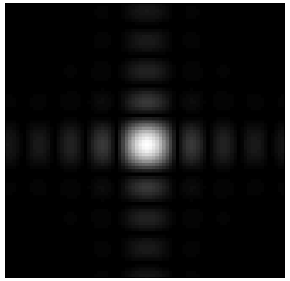
Can also model this using the Convolution Theorem

Aperture function (lens shape)

$A(f_x, f_y)$

Camera blur function (IFT of lens shape)

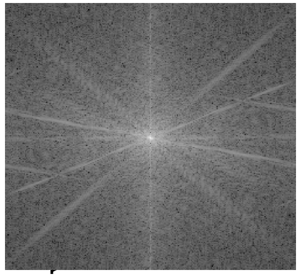
$h(x, y)$



2D IFT

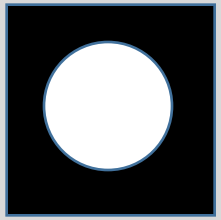
Two modeling choices for the camera:

Spatial Frequency Domain

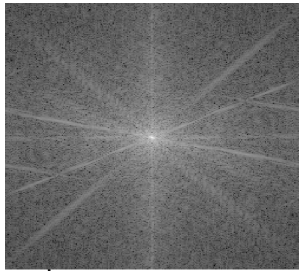


$\hat{s}_1(f_{xi},f_{yi})$

Multiply



$A\,(f_x,f_y)$

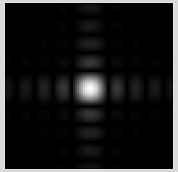


$\hat{s}_2(f_{xo},f_{yo})$

Spatial Domain

$s_1(x_i,y_i)$

Convolve = blur



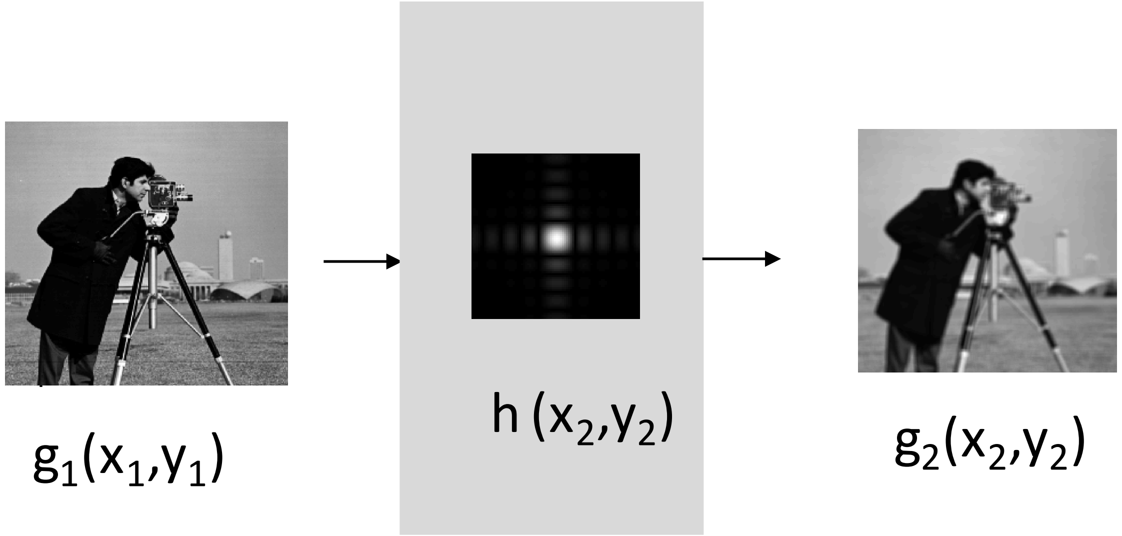
$h\,(x,y)$

$s_2(x_o,y_o)$

Linear systems and the black box

The optical black box system and the point-spread function:

Light $g_1(x_i,y_i)$ entering “black box” optical system modified by system point-spread function

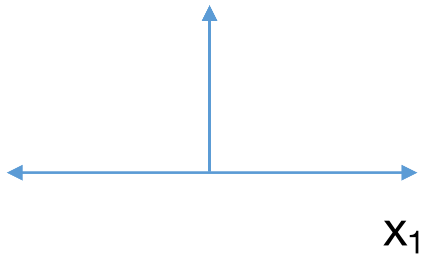


$$\boxed{g_2(x_2,y_2) = \iint_{-\infty}^{\infty} g_1(x_1,y_1)h(x_2-x_1,y_2-y_1)dx_1dy_1}$$

Assume shift invariance:
This is the system point-spread function

A little bit more detail about the convolution

Let's send a "spike" into our linear black box:

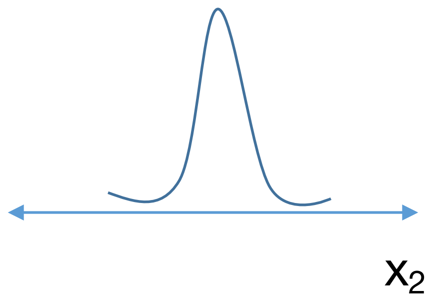


$$g_1(x_1) = \delta(x_1)$$

Linear system

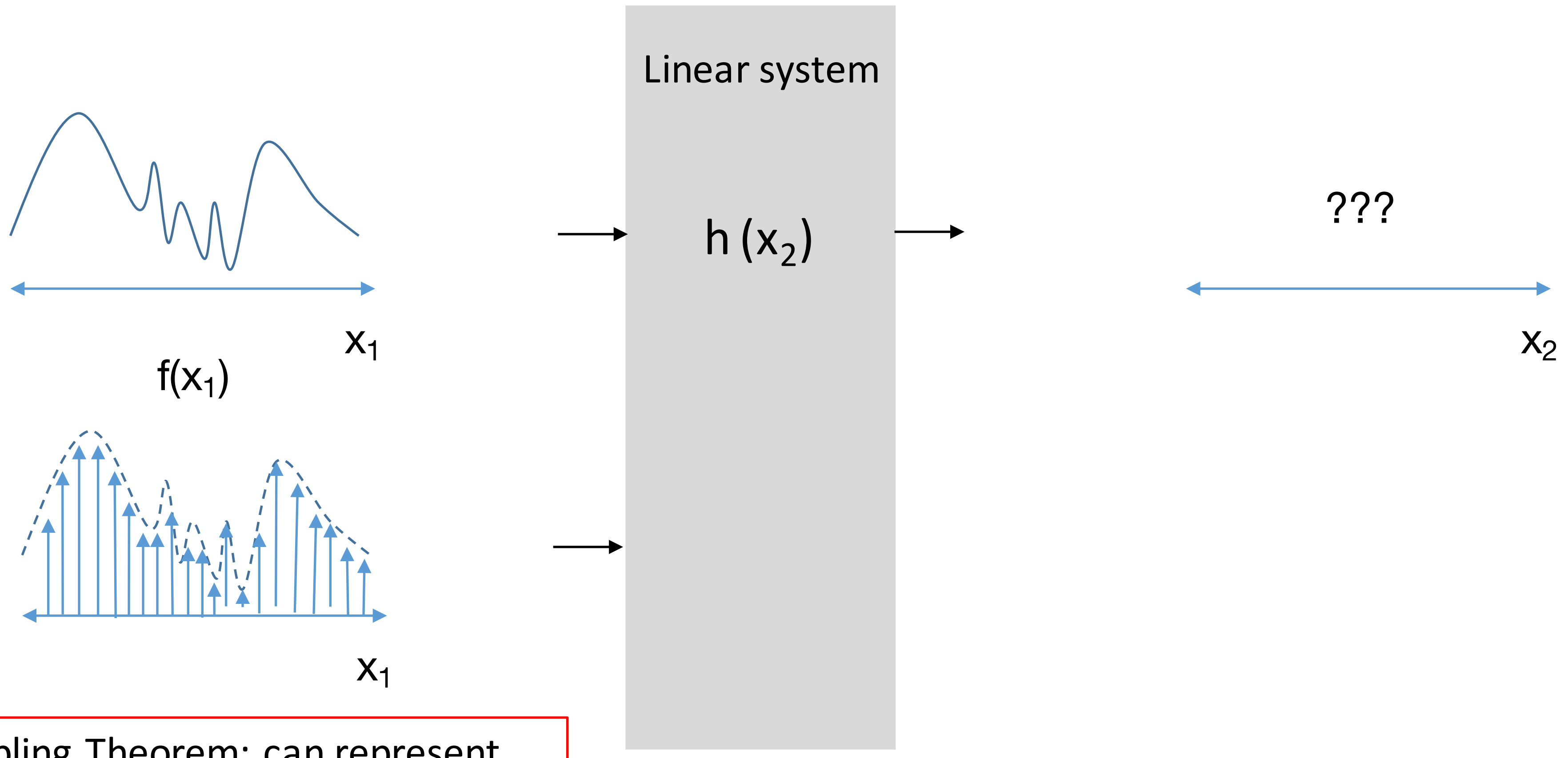
$h\,(x_2)$

For this black box, let's say it returns a Gaussian with a finite waist:



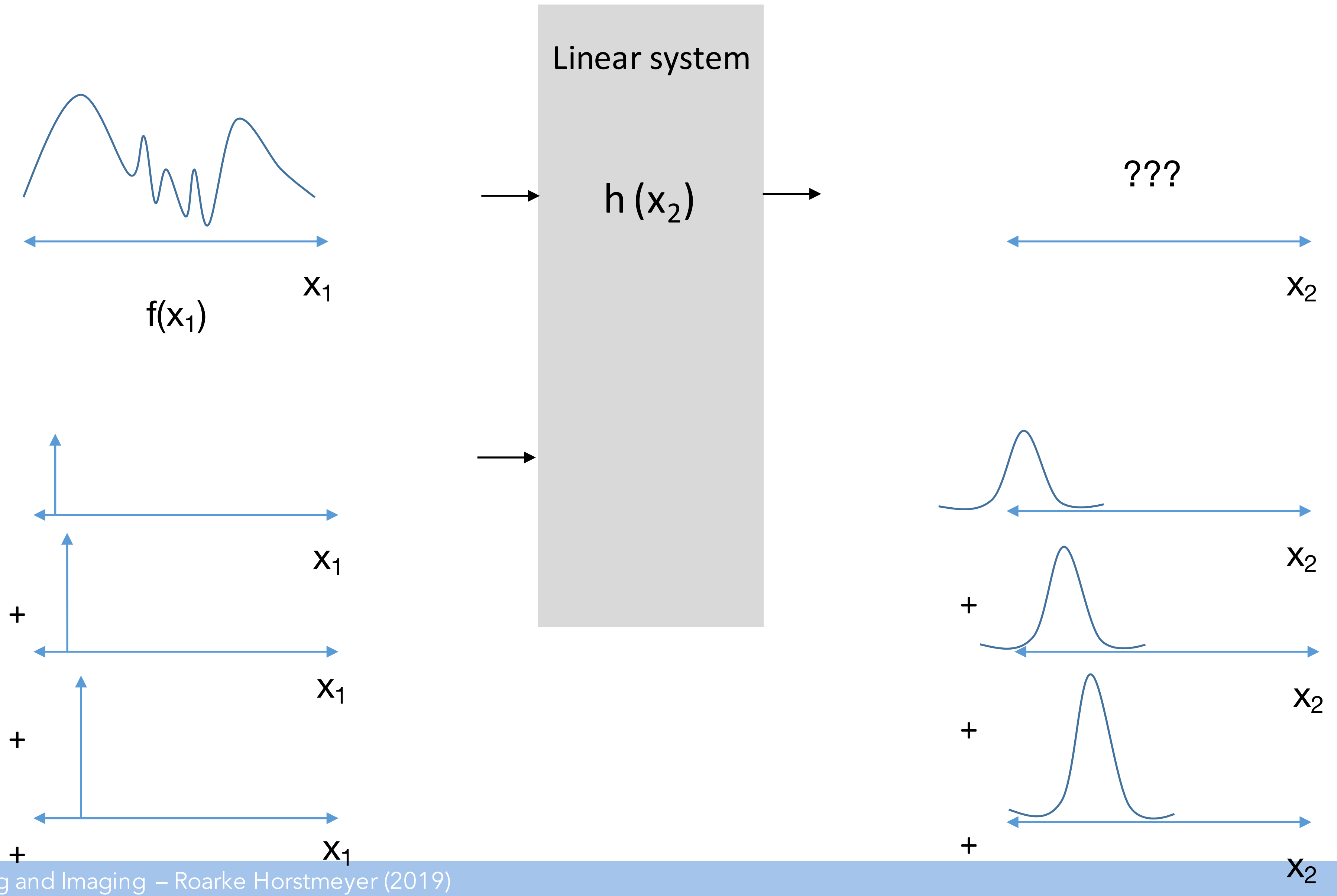
$$g_2(x_2) = e^{-x_2^2/\sigma^2}$$

Let's take these ideas and apply them to an arbitrary function



Sampling Theorem: can represent any signal as a discrete set of delta functions

Let's take these ideas and apply them to an arbitrary function



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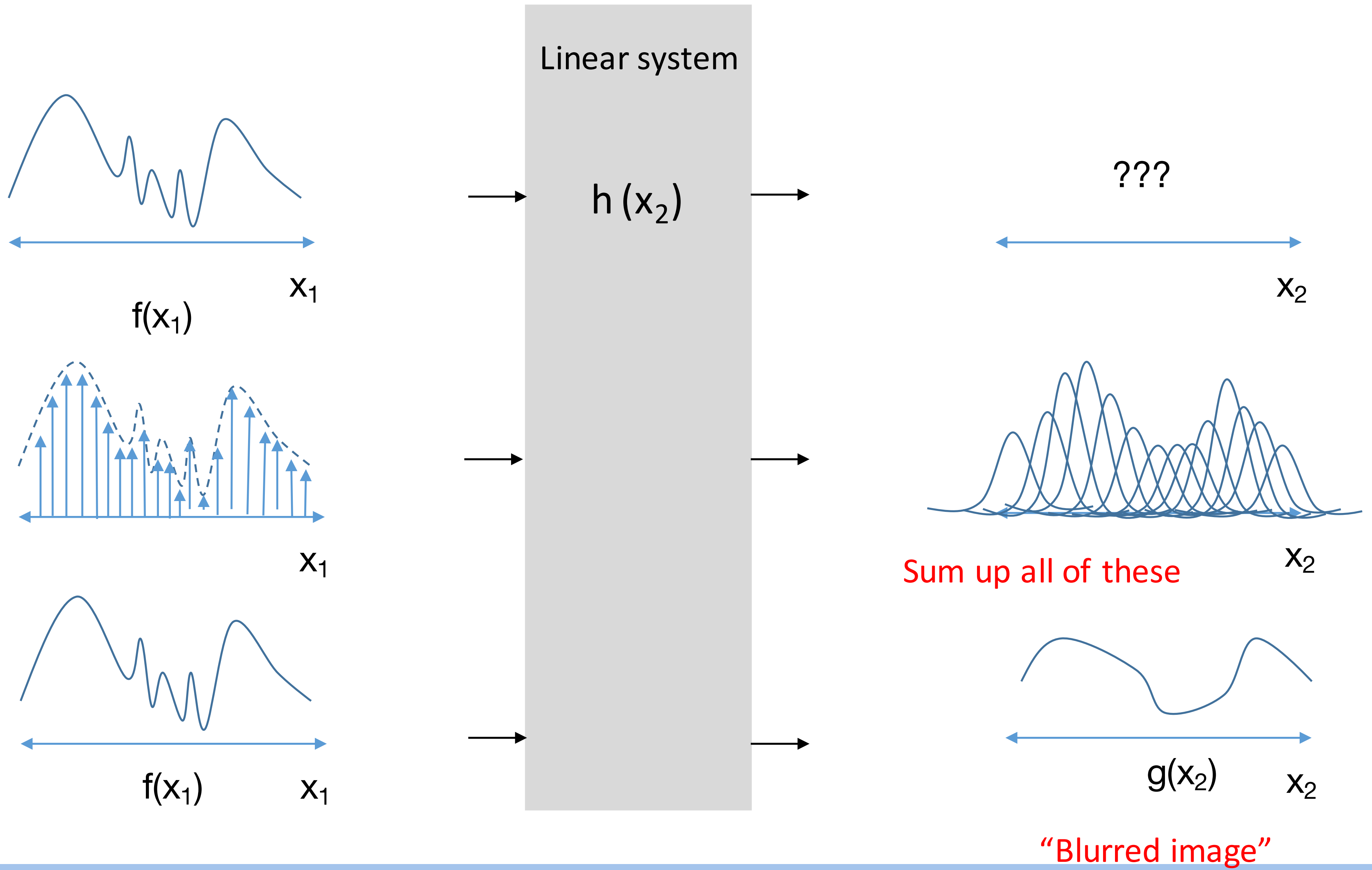

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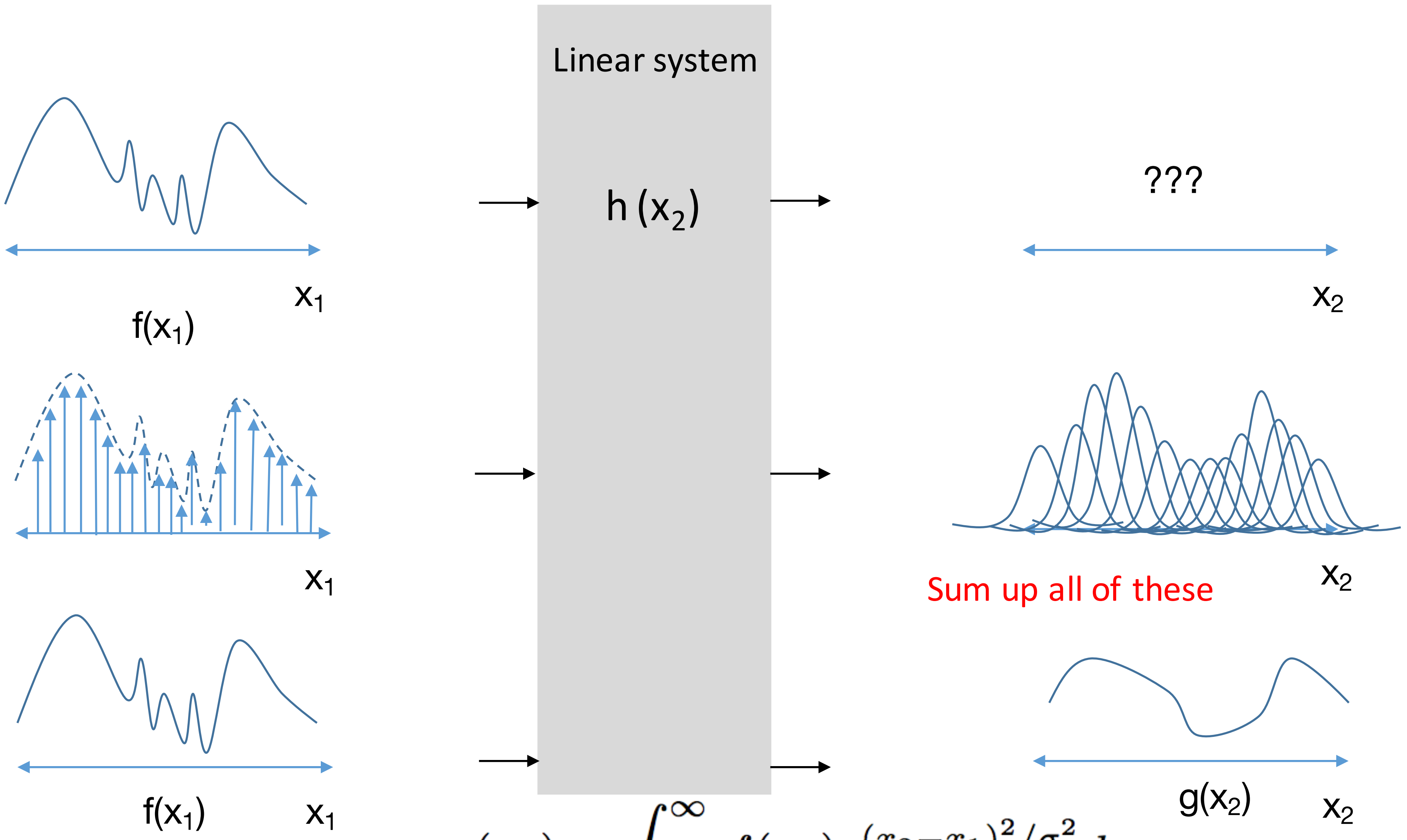

$$g_2(x_2) = \int_{-\infty}^{\infty} f(x_1) e^{(x_2 - x_1)^2/\sigma^2} dx_1$$

"Blurred image"

Model of image formation for wave optics (coherent light):

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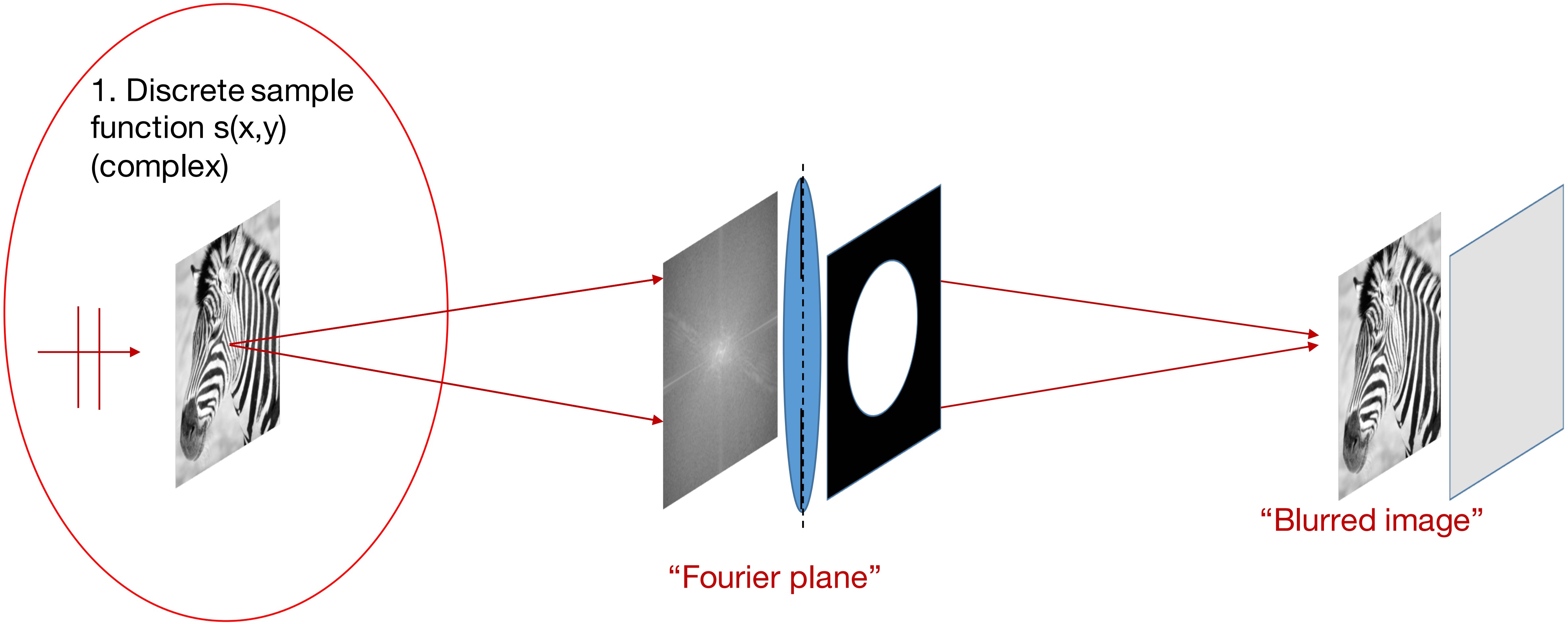
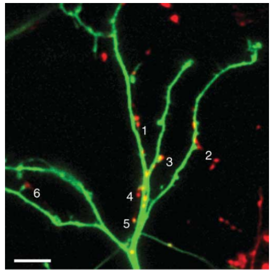
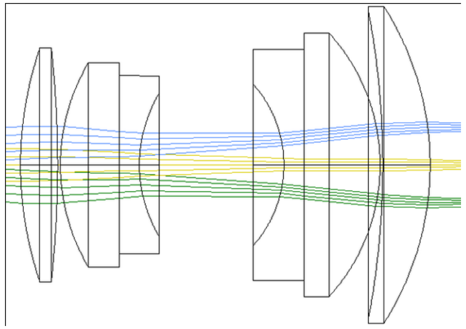


What's going on between the sample and the incident light?

Summary of two models for image formation

- **Interpretation #1: Radiation (*Incoherent*)**
- Model: Rays





- Real, non-negative
- Models absorption and brightness

$I_{tot} = I_1 + I_2$

Mathematical model of for incoherent image formation

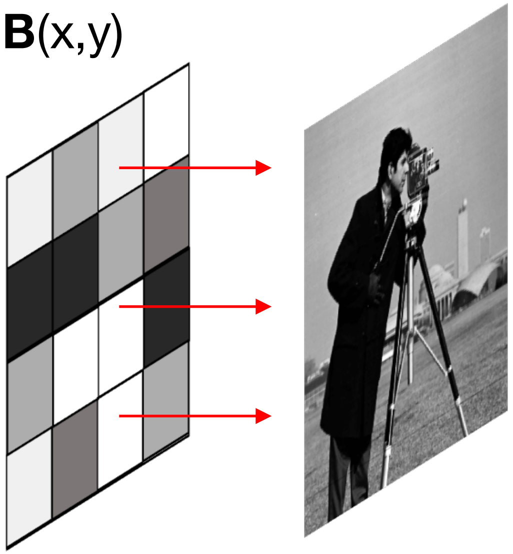
- All quantities are real, and non-negative

Object absorption:

$\mathbf{S_0}(x,y)$

Illumination brightness:

$\mathbf{B}(x,y)$



Mathematical model of for incoherent image formation

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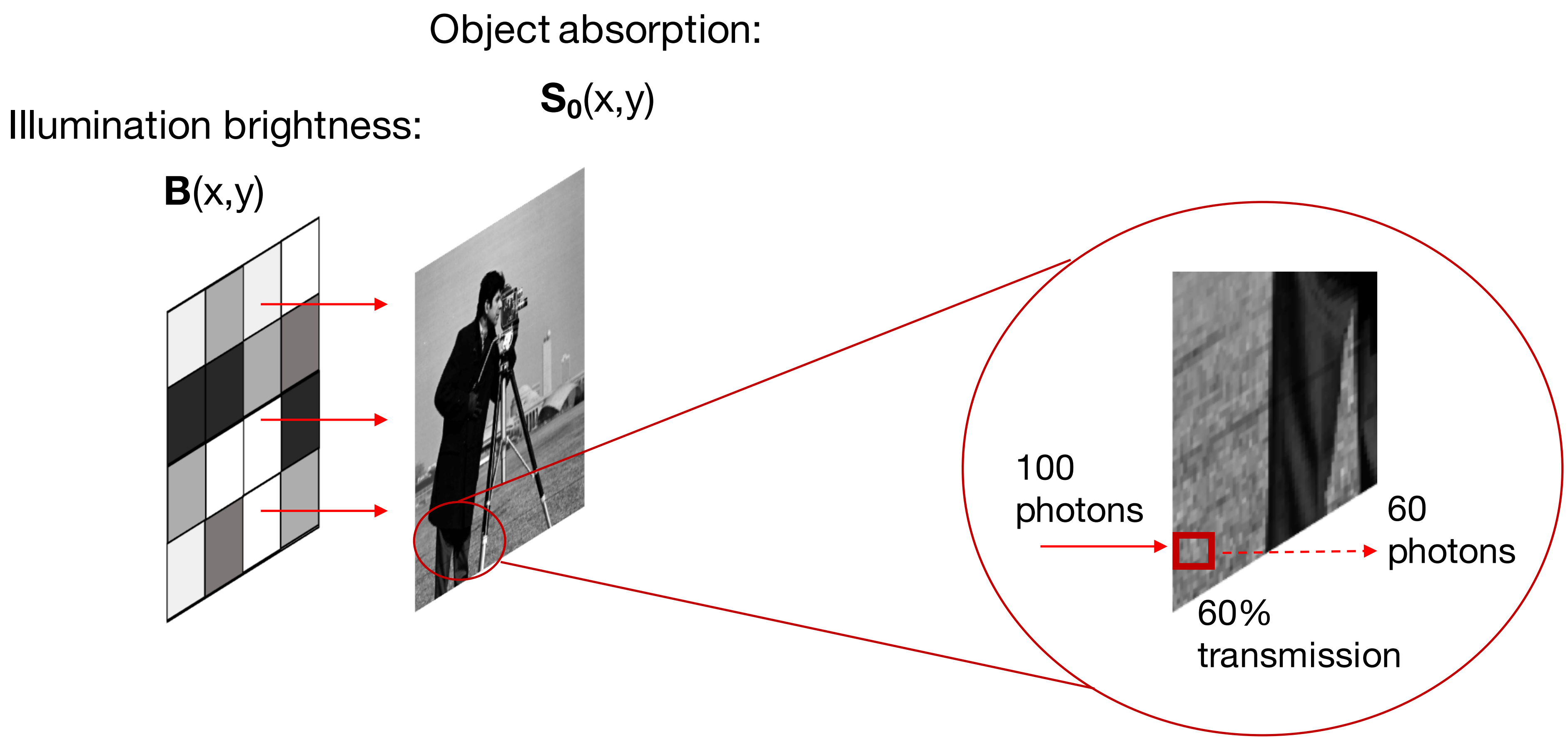

Mathematical model of for incoherent image formation

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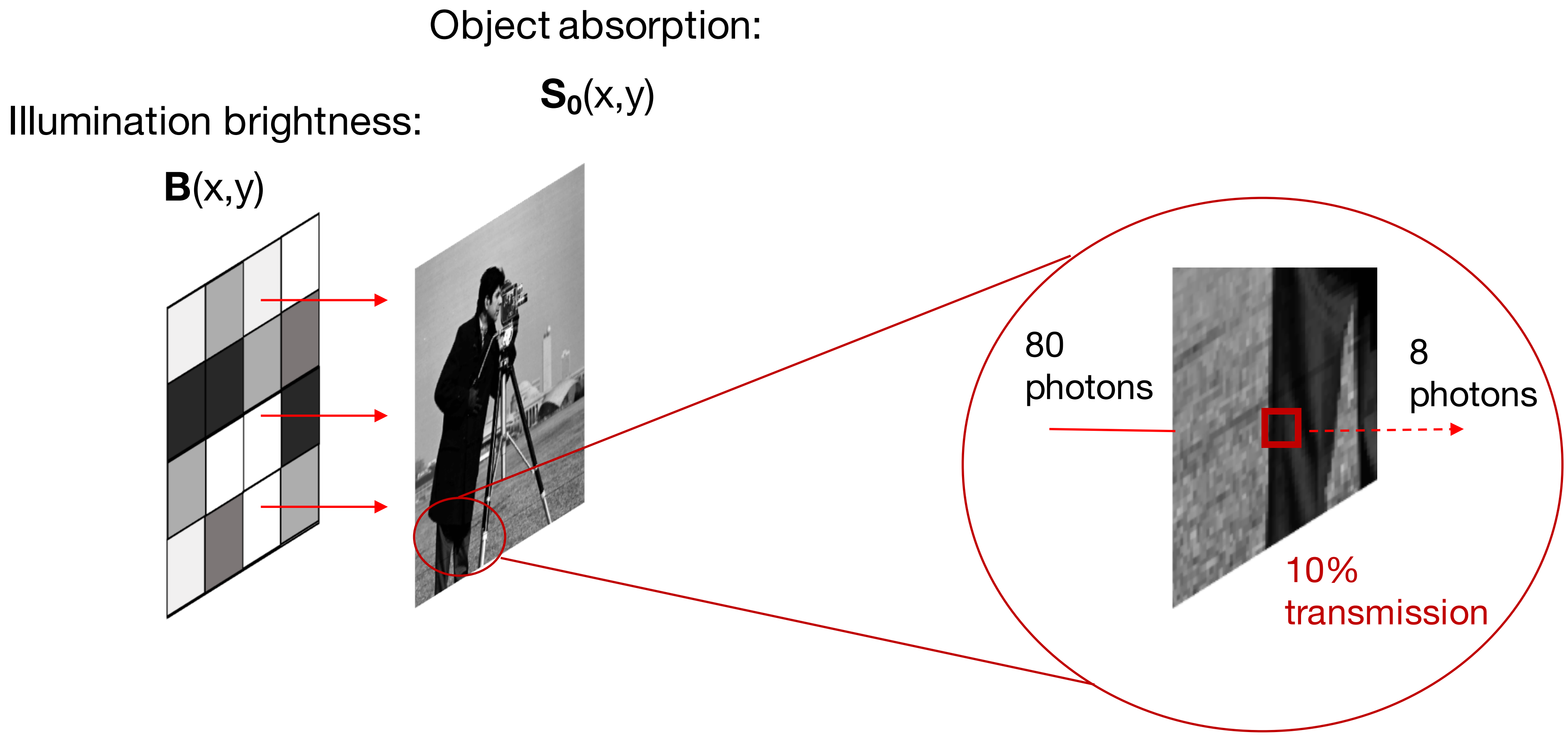

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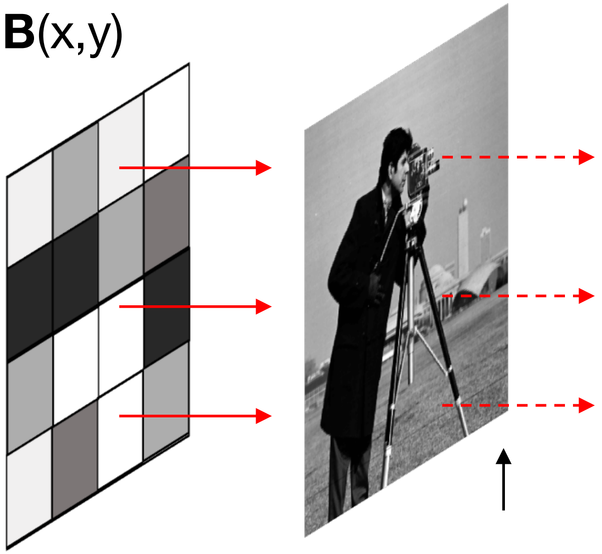
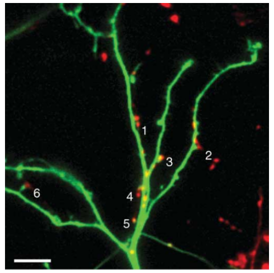


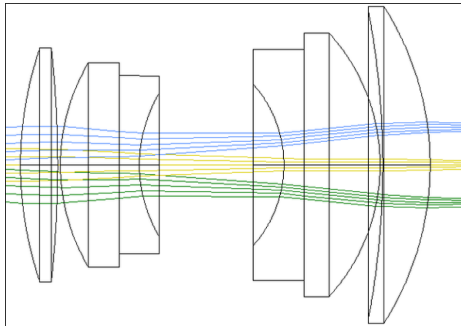
$\mathbf{B}\ \mathbf{S_0}$

multiplication

Summary of two models for image formation

- **Interpretation #1: Radiation (*Incoherent*)**
- Model: Rays





- Real, non-negative
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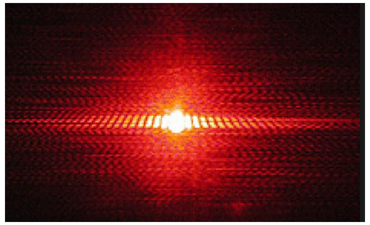
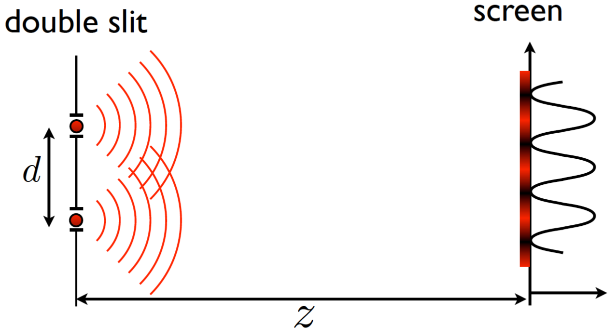
$$I_{tot} = I_1 + I_2$$

$$\mathbf{I_s = B\ S_0}$$

- **Interpretation #2: Electromagnetic wave (*Coherent*)**
- Model: Waves





- Complex field
- Models Interference

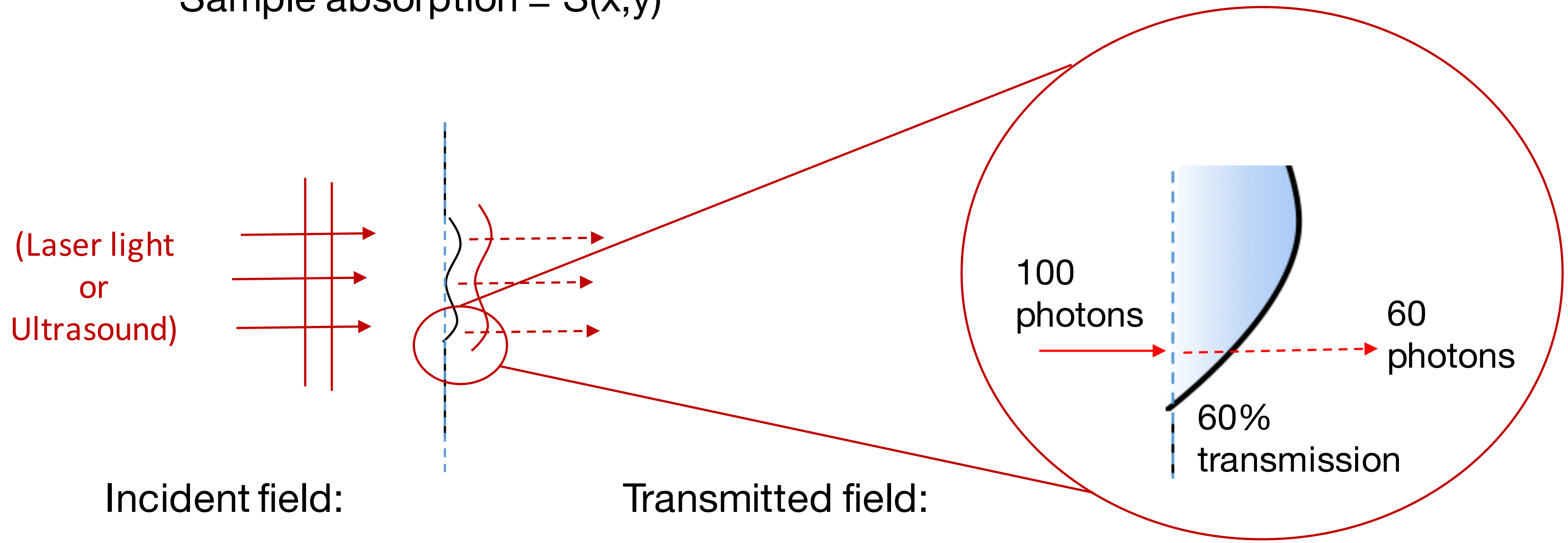
$$E_{tot} = E_1 + E_2$$

Mathematical model of for coherent image formation

- Pretty much the same thing, but now we have an amplitude and a complex phase

Sample absorption = S(x,y)

Point #1: Amplitudes behave just like before



$A_t(x,y) = A_i(x,y)\ S(x,y)$

Incident field:

$C(x,y) = A_i(x,y)$

Transmitted field:

$U(x,y) = A_t(x,y) = ?$

Mathematical model of for coherent image formation

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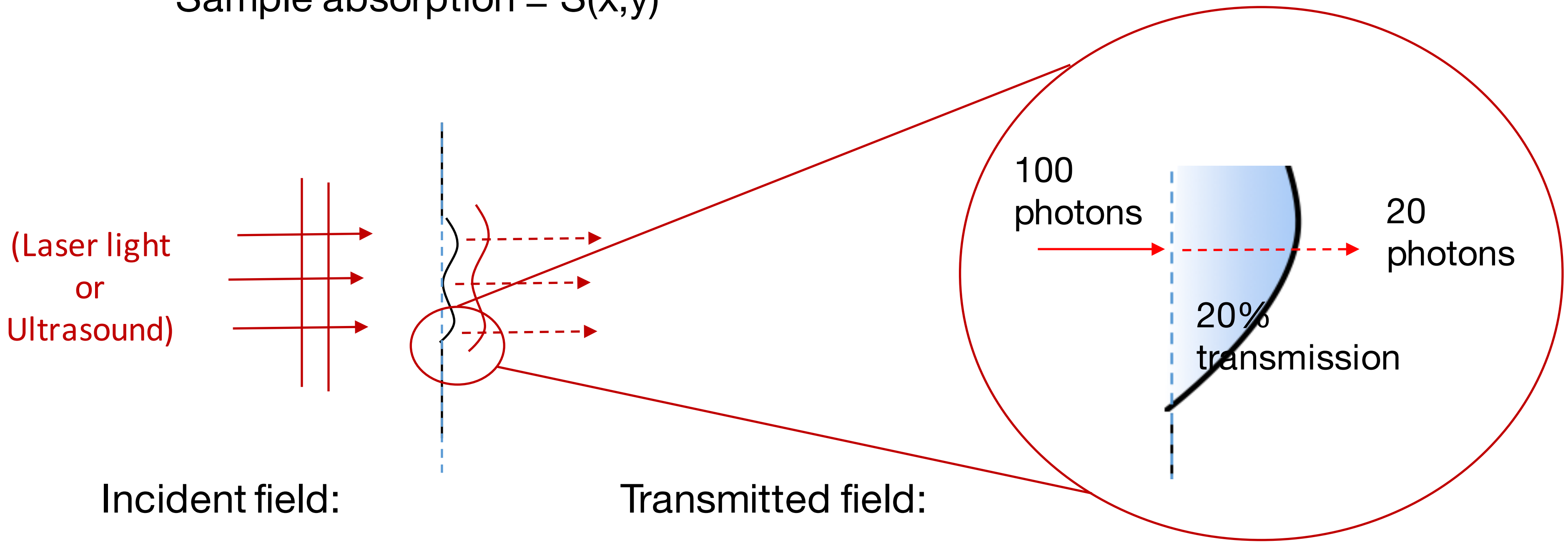

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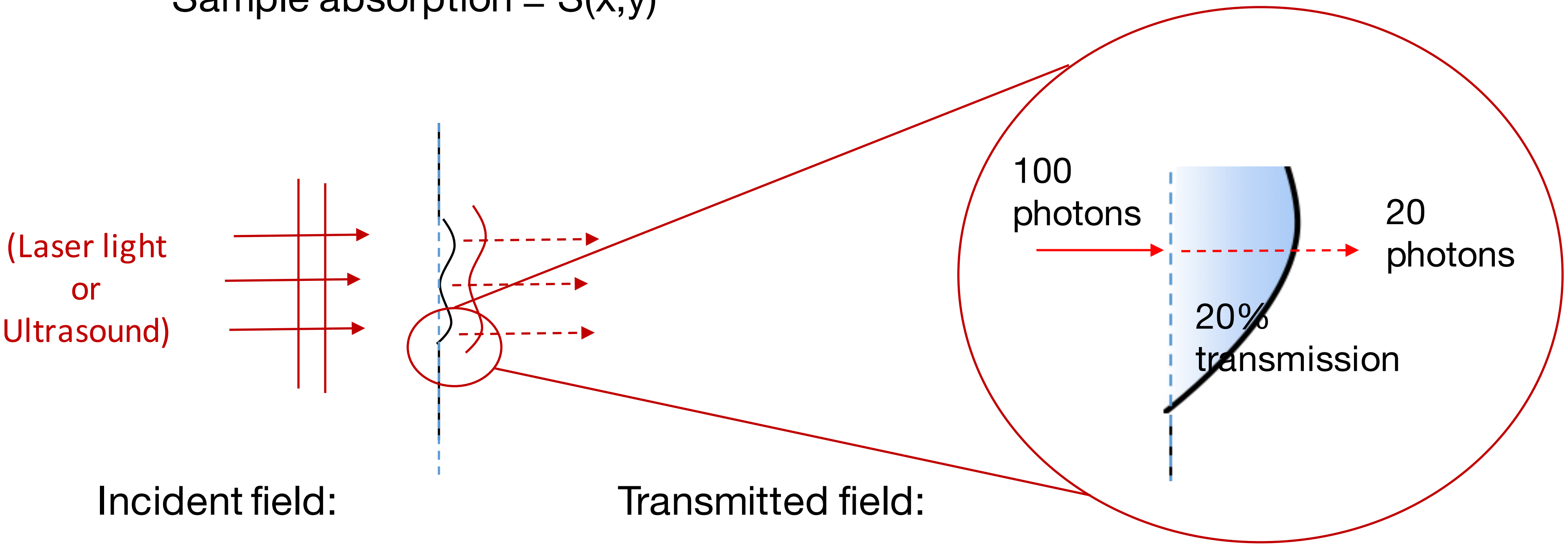

$\boxed{A_t(x,y) = A_i(x,y)\ S(x,y)}$

Incident field:

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Transmitted field:

$U(x,y) = A_t(x,y) = A_i(x,y)\ S(x,y)$

Mathematical model of for coherent image formation

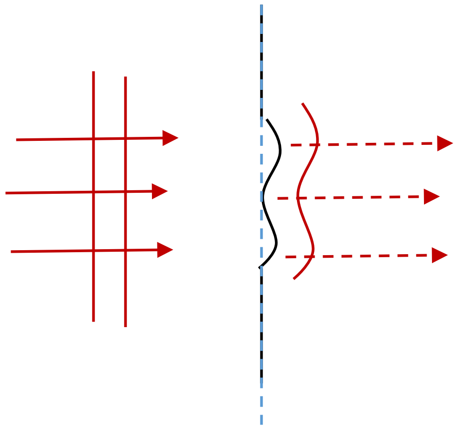
- Pretty much the same thing, but now we have an amplitude and a complex phase

Sample absorption = $S(x,y)$

Sample phase delay = $\exp[ik\varphi(x,y)]$

New: complex phase delay
- Needed to represent wave
- Represents wave delay across space

(Laser light or Ultrasound)



Incident field:

$C(x,y) = A_i(x,y) \exp[ik\varphi_i(x,y)]$

Transmitted field:

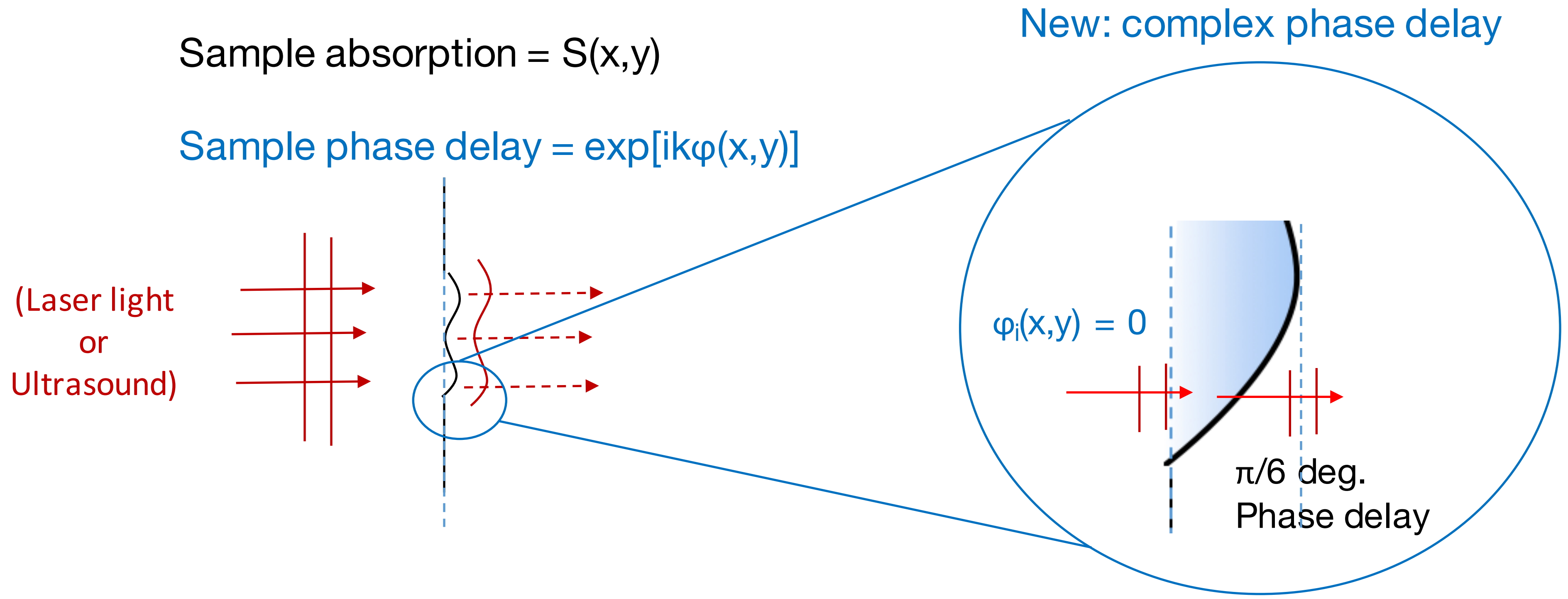
$U(x,y) = A_i(x,y)\, S(x,y) \exp[ik\varphi_t(x,y)]$

Mathematical model of for coherent image formation

- Pretty much the same thing, but now we have an amplitude and a complex phase

Sample absorption = S(x,y)

Sample phase delay = exp[ikφ(x,y)]



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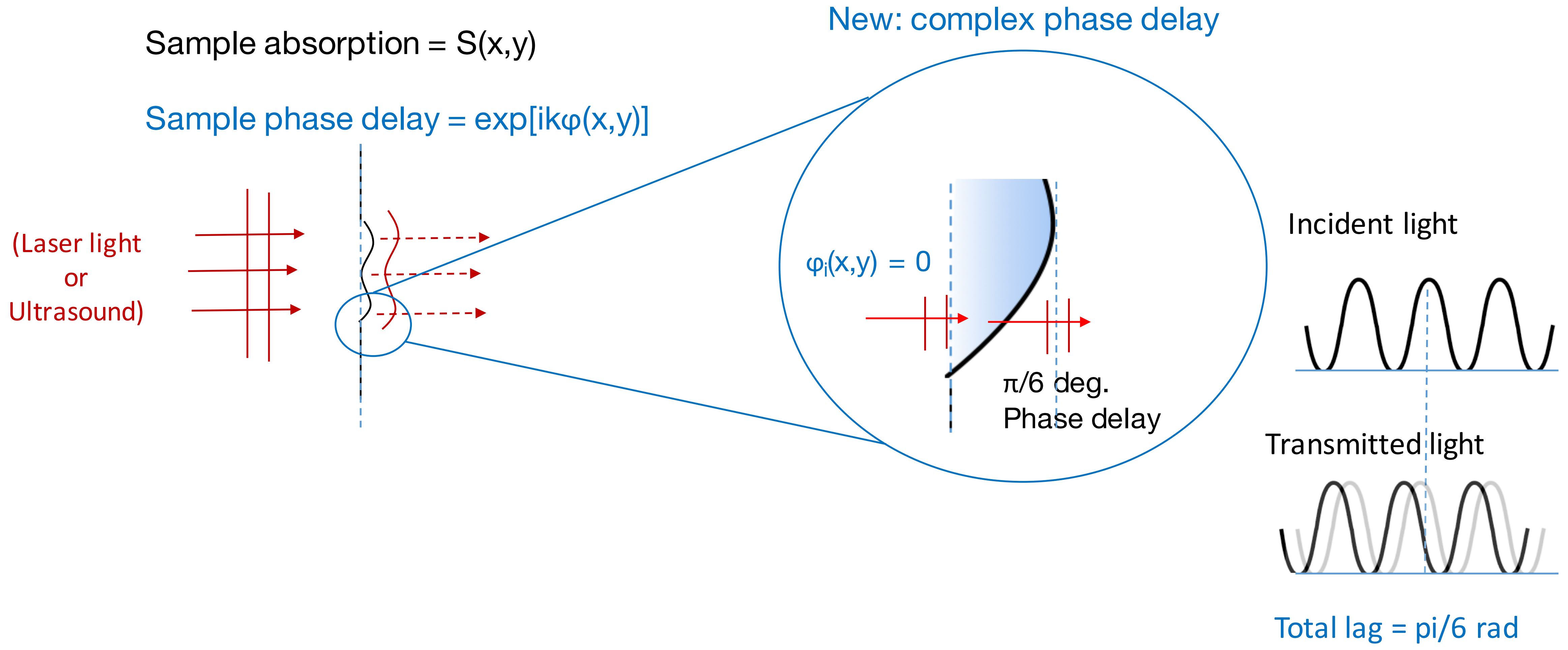

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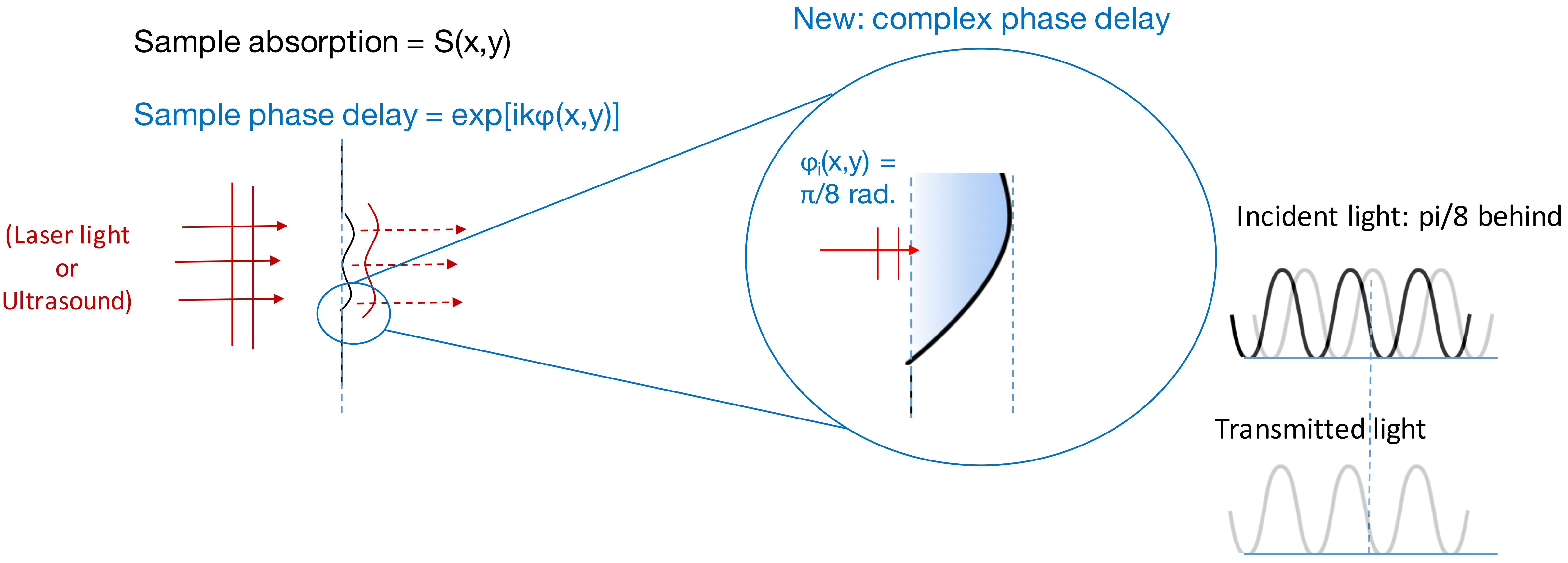

Mathematical model of for coherent image formation

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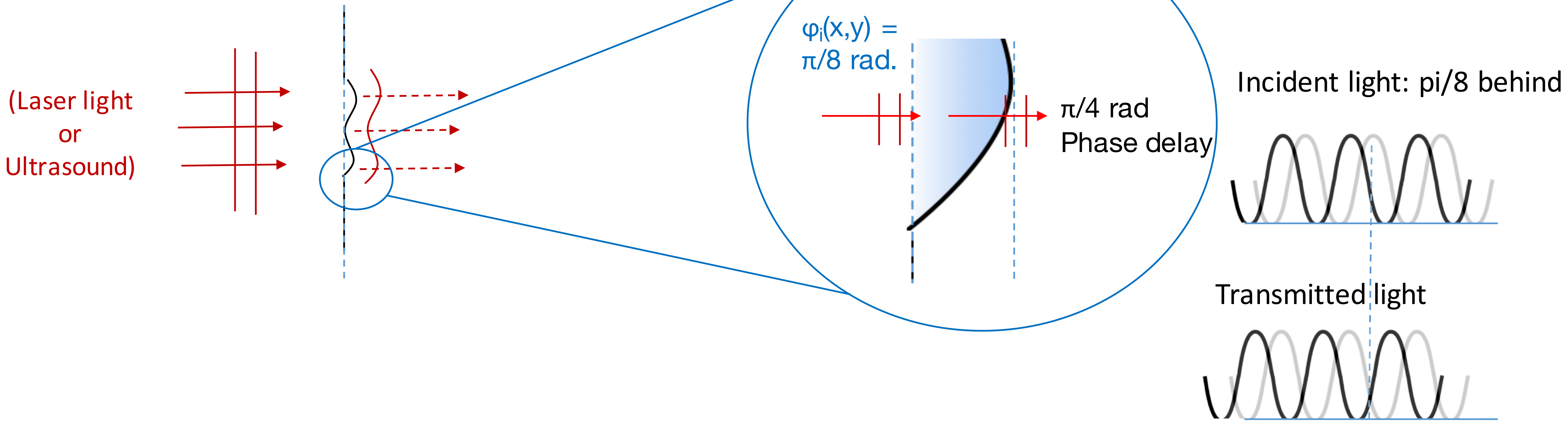
Sample absorption = S(x,y)

Sample phase delay = exp[ikφ(x,y)]

New: complex phase delay



Total lag = π/8+ π/4= 3π/8 rad

Mathematical model of for coherent image formation

- Pretty much the same thing, but now we have an amplitude and a complex phase

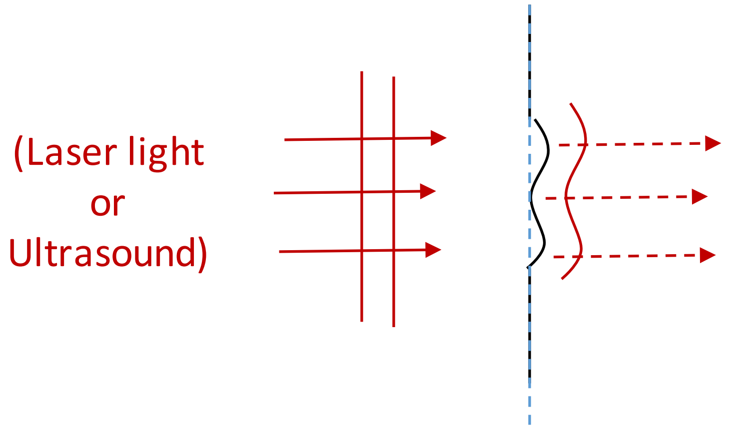
Sample absorption = S(x,y)

Sample phase delay = exp[ikφ(x,y)]

Output phase is sum of phase delays, product of phasors

$$\varphi_t(x,y) = \varphi(x,y) + \varphi_i(x,y)$$

$$\exp[ik\varphi_t(x,y)] = \exp[ik\varphi_i(x,y)]\ \exp[ik\varphi(x,y)]$$



Incident field:

$C(x,y) = A_i(x,y)\ \exp[ik\varphi_i(x,y)]$

Transmitted field:

$U(x,y) = A_i(x,y)\ S(x,y)\ \exp[ik\varphi_i(x,y)]\ \exp[ik\varphi(x,y)]$

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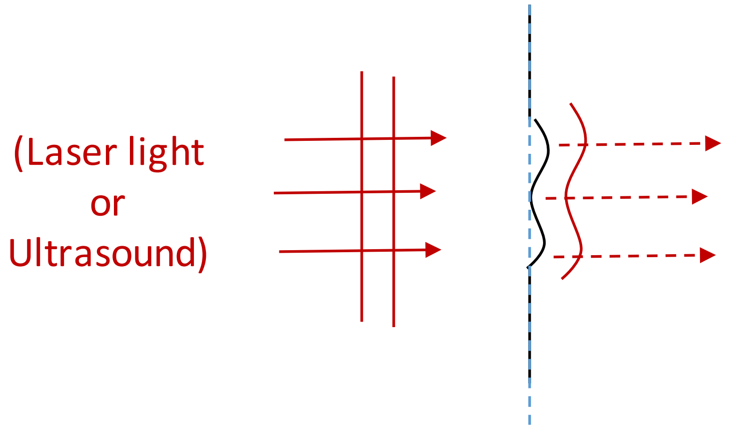

Conclusion:

Transmitted field = incident field x complex sample :

$$U(x,y) = C(x,y)\, S(x,y)\, \exp[ik\varphi(x,y)]$$

Incident field:

$$C(x,y) = A_i(x,y)\, \exp[ik\varphi_i(x,y)]$$

Transmitted field:

$$U(x,y) = A_i(x,y)\, S(x,y)\, \exp[ik\varphi_i(x,y)]\, \exp[ik\varphi(x,y)]$$

Summary of two models for image formation

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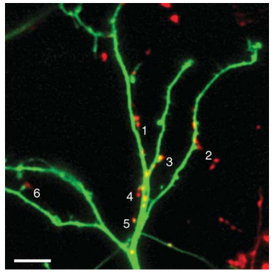

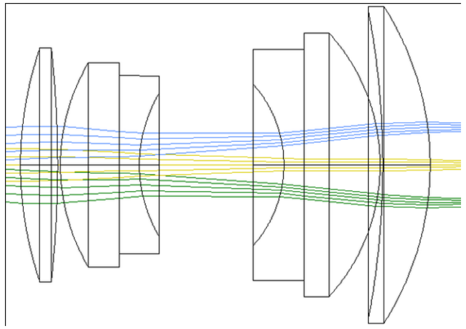

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$\mathbf{I_s = B\ S_0}$

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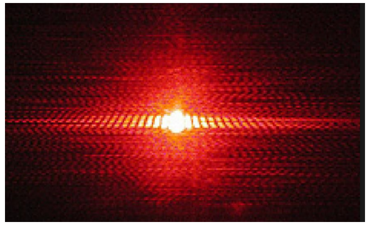
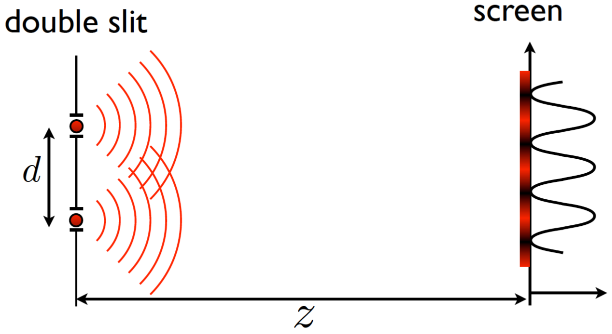




- Complex field
- Models Interference

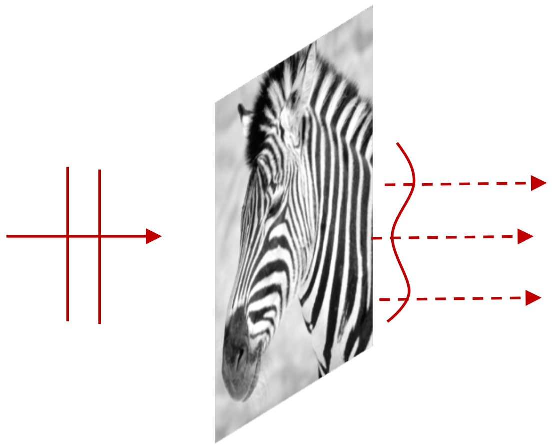
$E_{tot} = E_1 + E_2$

$\mathbf{U = C\ S_0}$

U, C and S are complex!

Model of image formation for wave optics (coherent light):

Discrete sample function s(x,y) (complex)

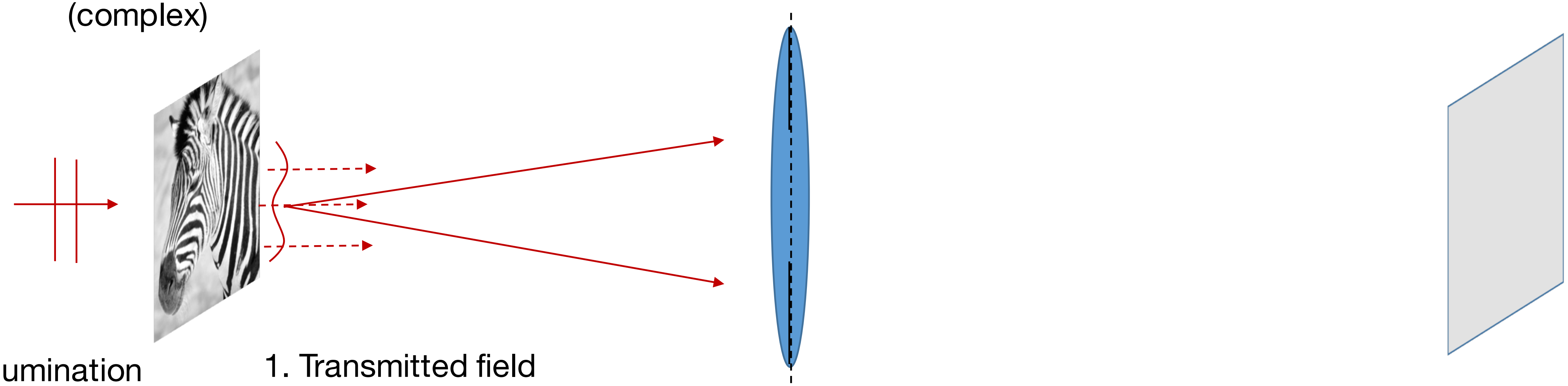


Illumination field C(x,y)

Transmitted field $s_c(x,y) = C(x,y)\, s(x,y)$

Model of image formation for wave optics (coherent light):

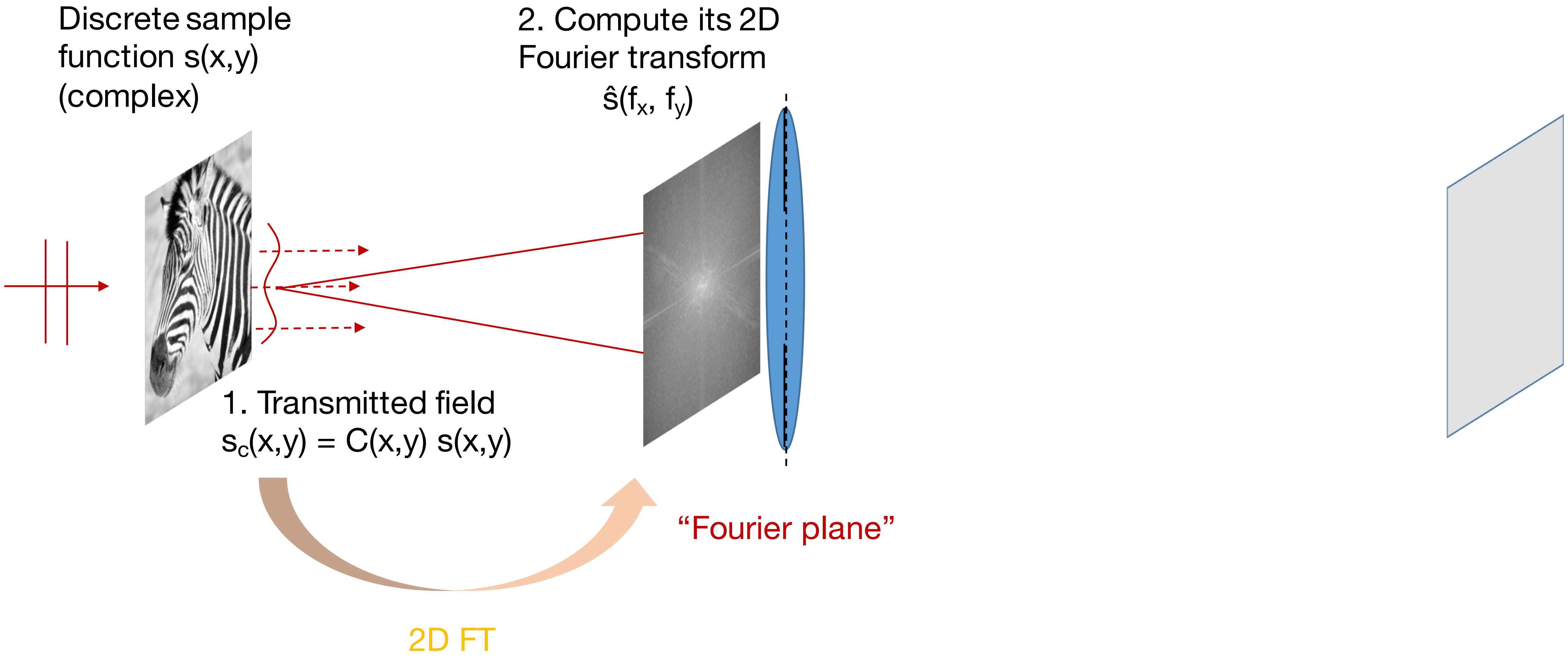
Discrete sample function s(x,y) (complex)



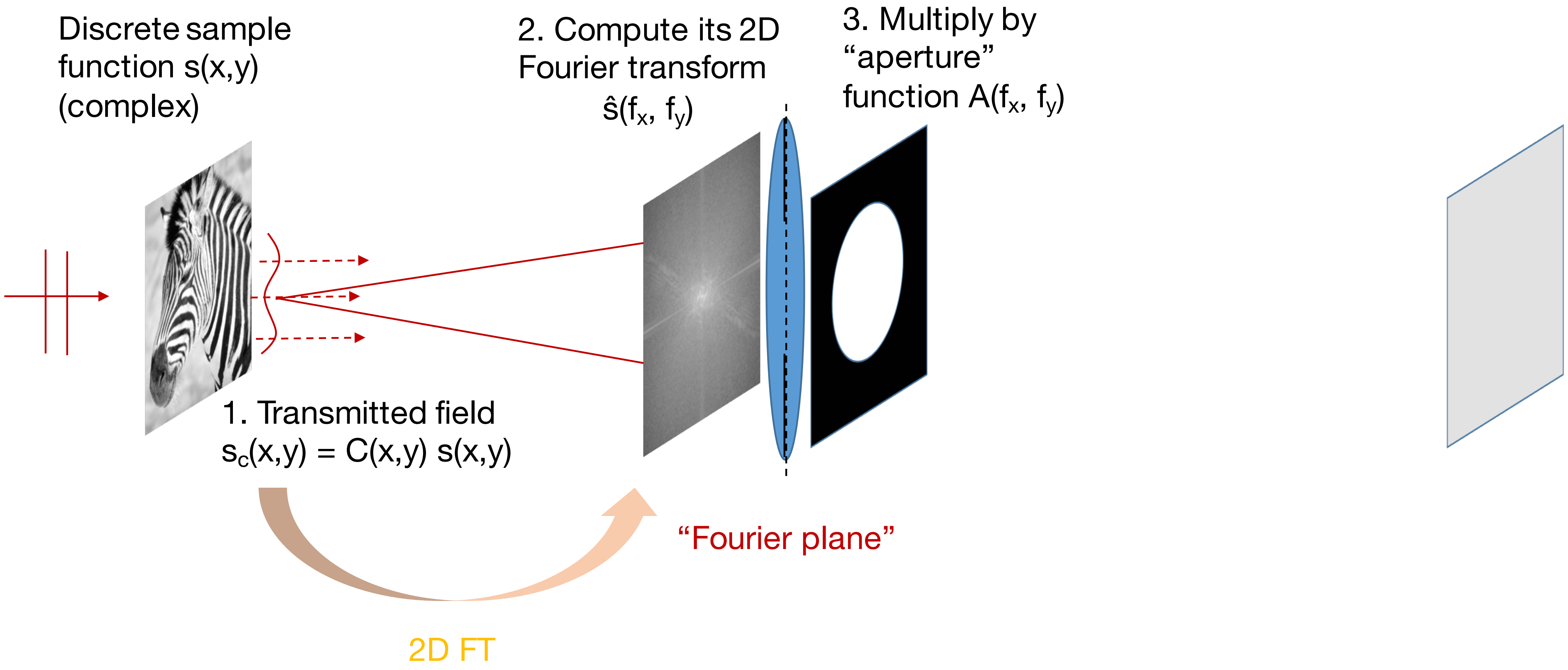
Illumination field C(x,y)

1. Transmitted field
$s_c(x,y) = C(x,y)\ s(x,y)$

Model of image formation for wave optics (coherent light):

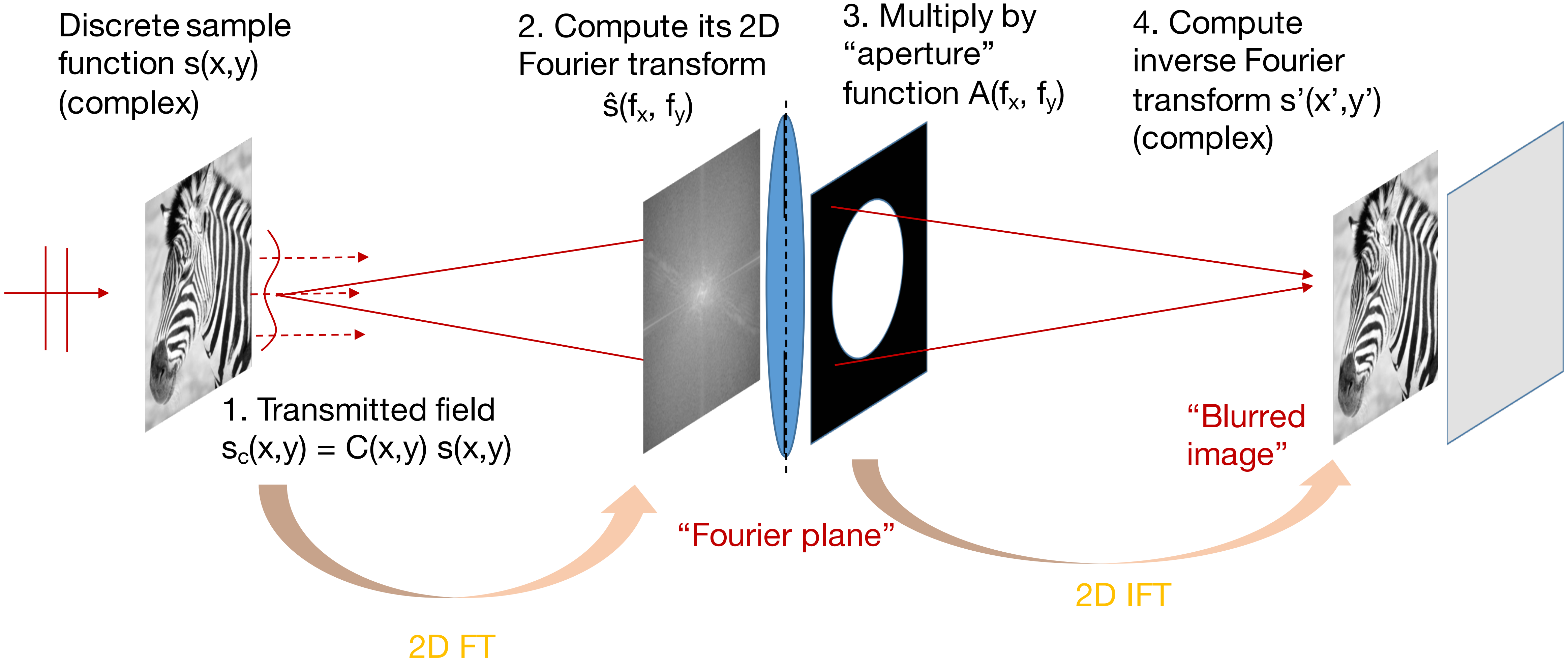


Model of image formation for wave optics (coherent light):



deep imaging

Model of image formation for wave optics (coherent light):



Model of image formation for wave optics (coherent light):