deep imaging

Lecture 3: From continuous to discrete functions

Machine Learning and Imaging

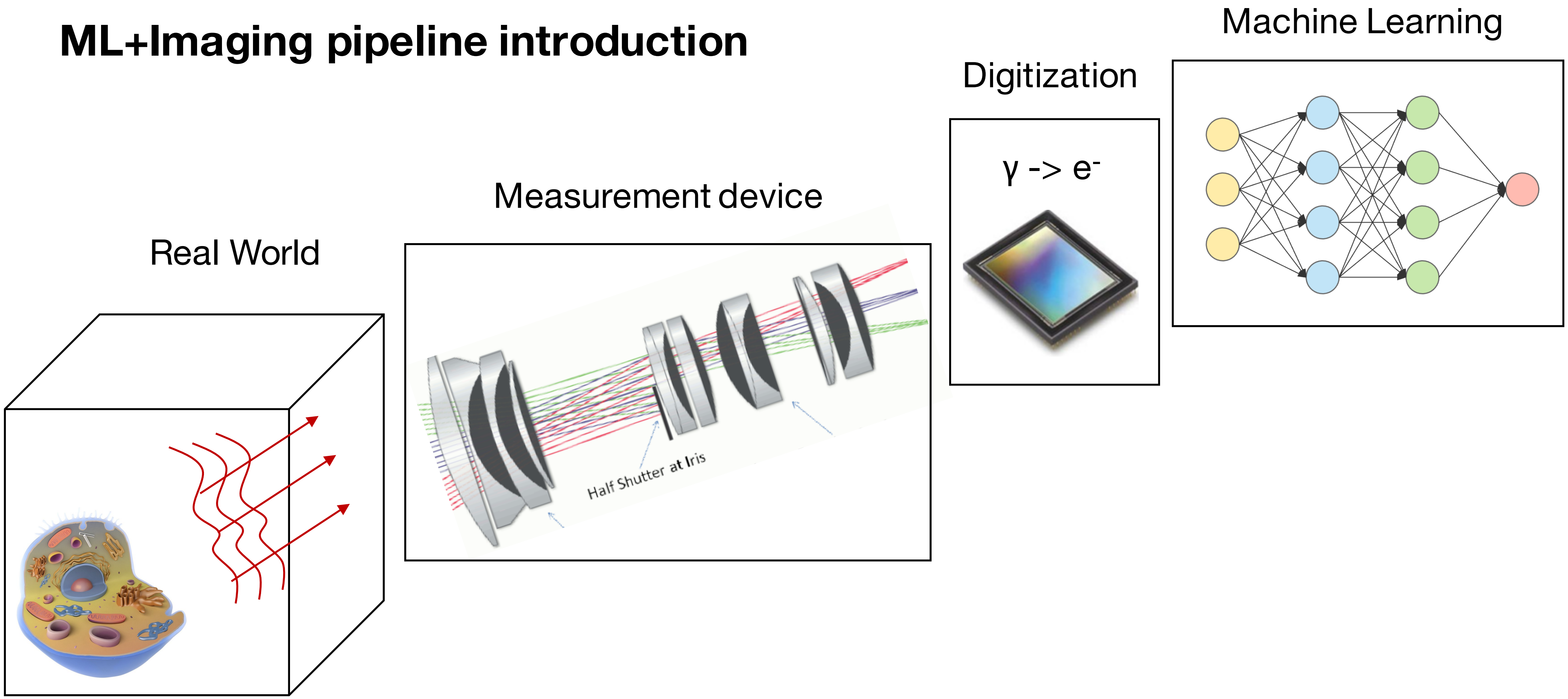
BME 590L

Roarke Horstmeyer

- Linear black-box systems
- Convolutions in 1D and 2D
- Fourier transforms
- Convolution theorem
- Sampling theorem

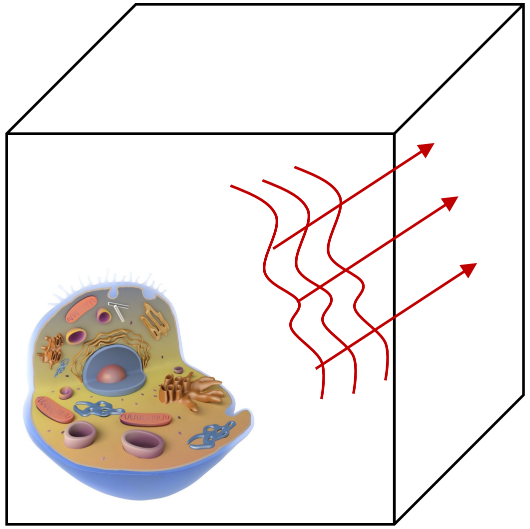
ML+Imaging pipeline introduction

deep imaging



ML+Imaging pipeline introduction

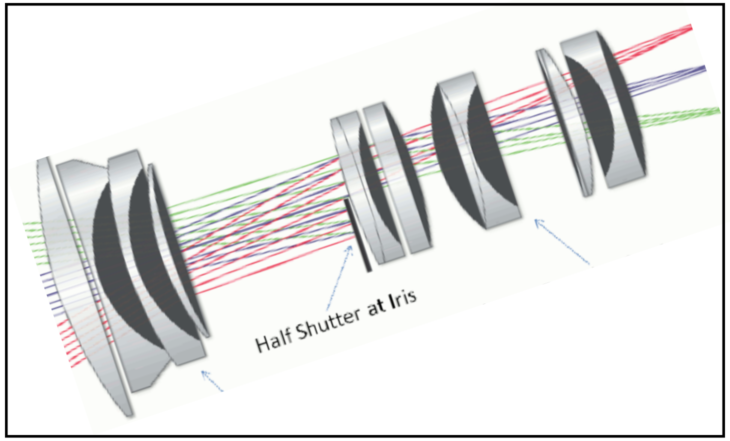
Real World



Continuous
complex fields

(last class)

Measurement device

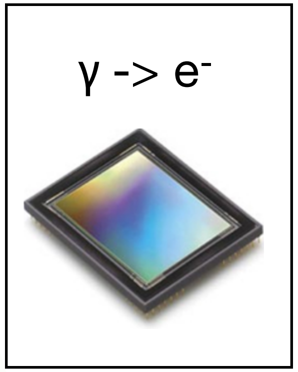


Black box transformations

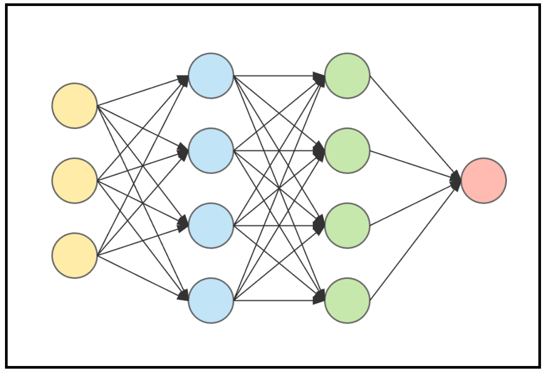
- Convolution
- Fourier Transform

(last class, this class)

Digitization



Machine Learning



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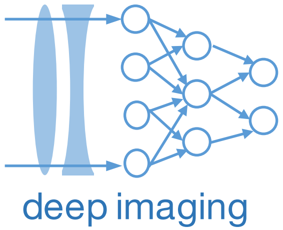
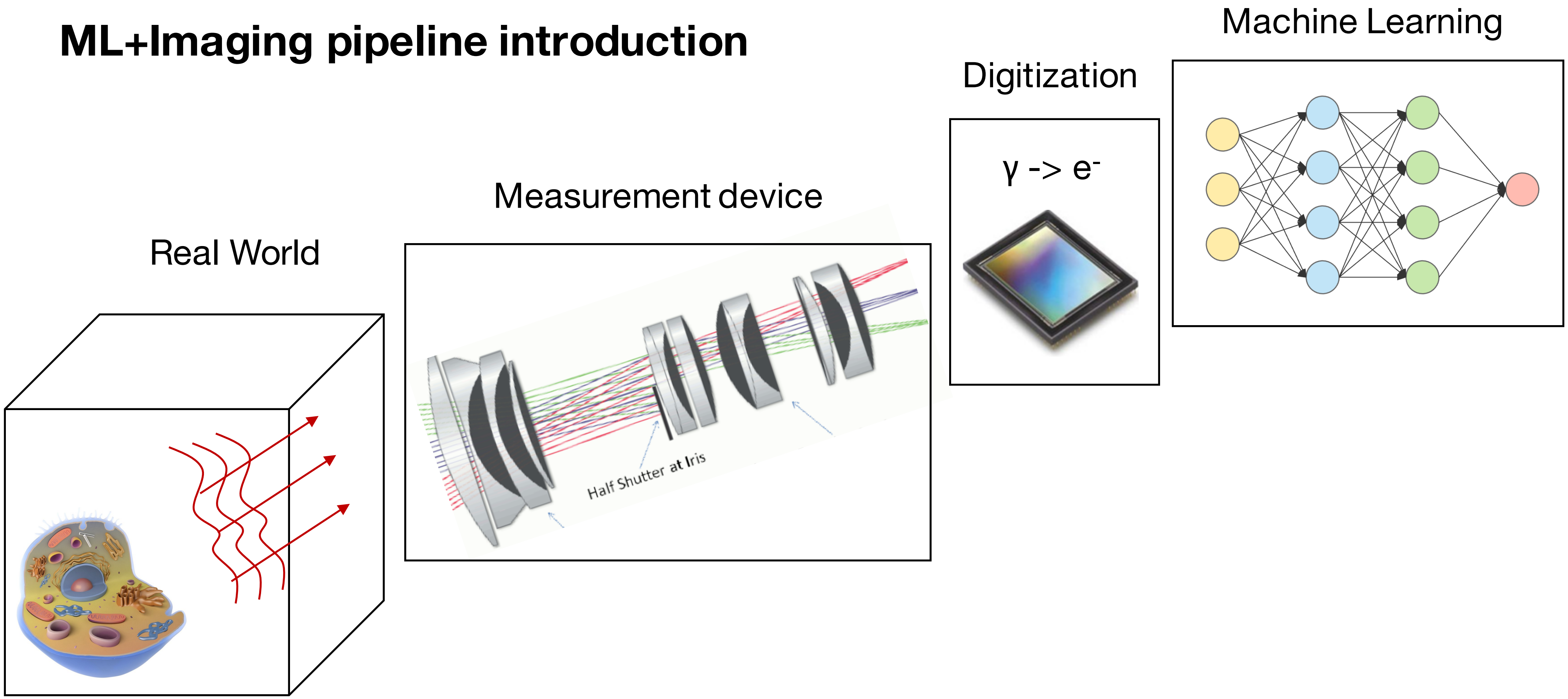




(last class) → (last class, this class) → (this class, next class)

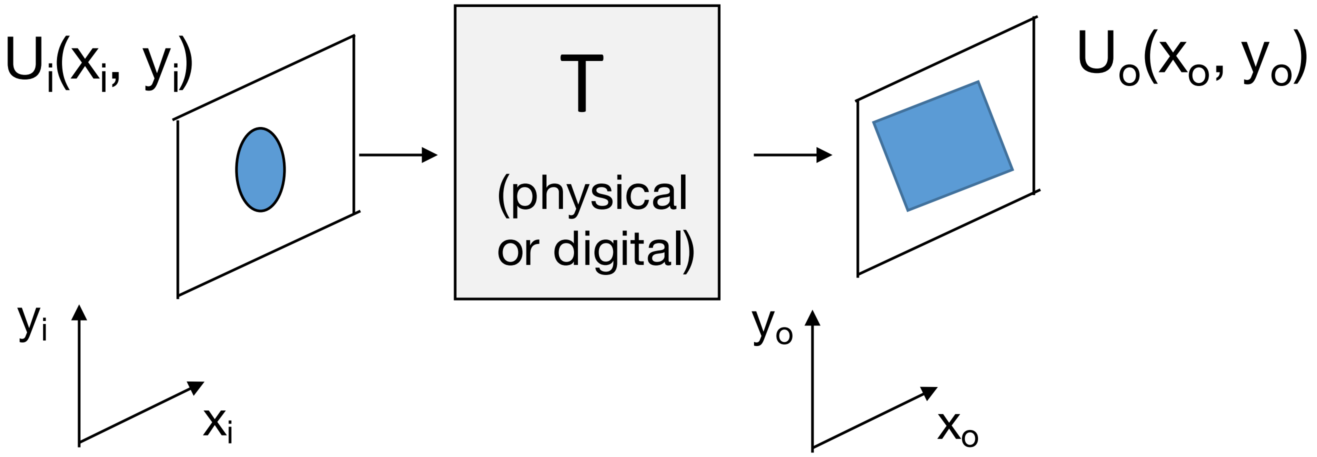
Linear systems and the black box

The "optical" black box system:

An optical black box system maps an input function $U_i(x_i,y_i)$ to an output function $U_o(x_o,y_o)$ via a transform T:

$$U_o(x_o,y_o) = T\,[\,U_i(x_i,y_i)\,]$$

Where T[ ] denotes the optical black box transformation



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Important properties of linear systems:

1. Homogeneity and additivity (superposition):

$$T\,[aU_1(x,y) + bU_2(x,y)] = aT\,[U_1(x,y)] + bT\,[U_2(x,y)]$$

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2. Shift invariance: for shift distances d_x and d_y, we assume that,

$$U_o(x_o - d_x, y_o - d_y) = T\,[U_i(x_i - d_x, y_i - d_y)]$$

Black box transforms as a convolution

Assuming 1) linearity and 2) shift-invariance, we can model any black box with 1 piece of information:

Input Dirac delta function into the black box:

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

Black box transforms as a convolution

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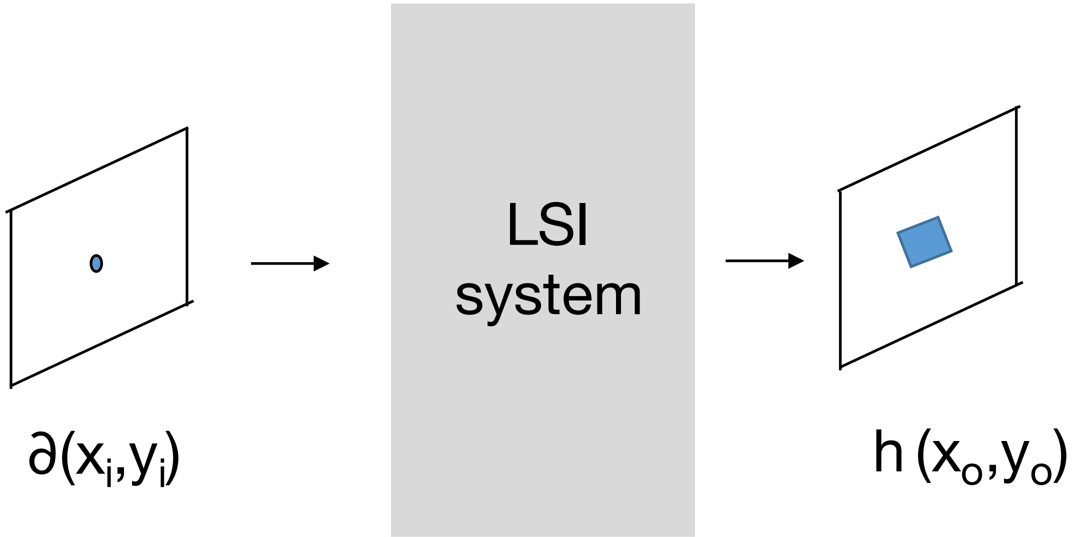
A "perfect" point source



$$h(x_o,y_o) = T\,[\,\partial(x_i,y_i)\,]$$

Black box transforms as a convolution

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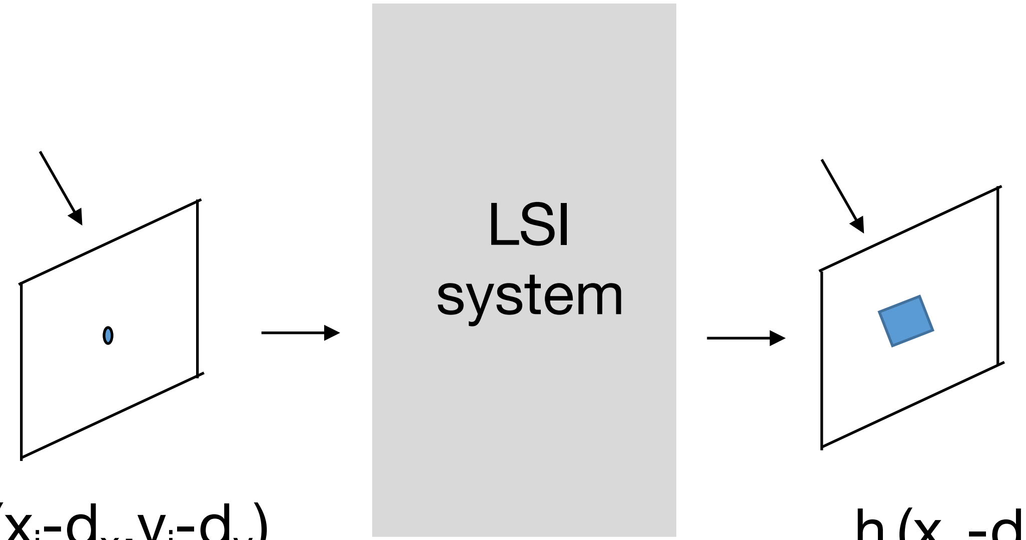
Assuming 1) linearity and 2) shift-invariance, we can model any black box with 1 piece of information:

We know the system is shift invariant:

A “perfect” point source



$\partial(x_i-d_x,y_i-d_y)$ $h\,(x_o-d_y,y_o-d_y)$

$$h(x_o-d_y,y_o-d_y) = T\,[\,\partial(x_i-d_x,y_i-d_y)\,]$$

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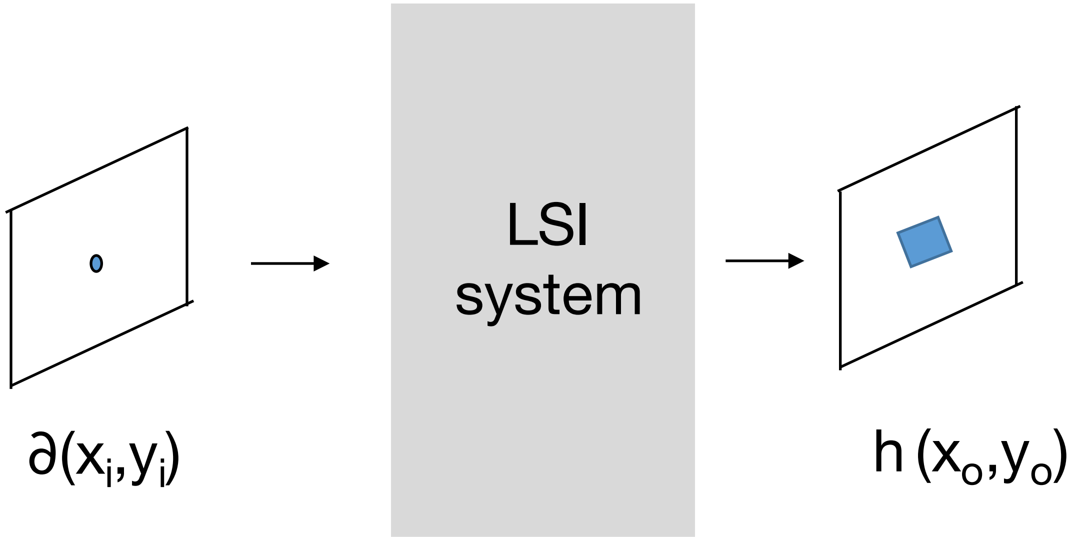
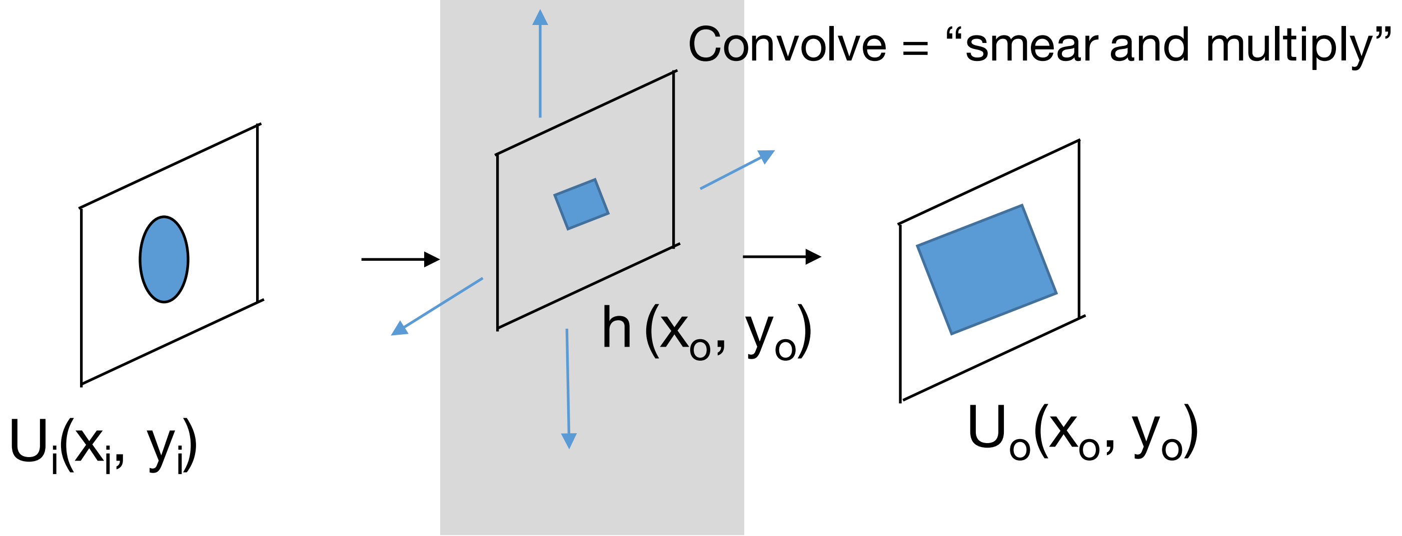


$h(x_o,y_o)$ is the system's point-spread function

Point-spread function $h(x_o,y_o) = T\,[\,\partial(x_i,y_i)\,]$

Black box transforms as a convolution

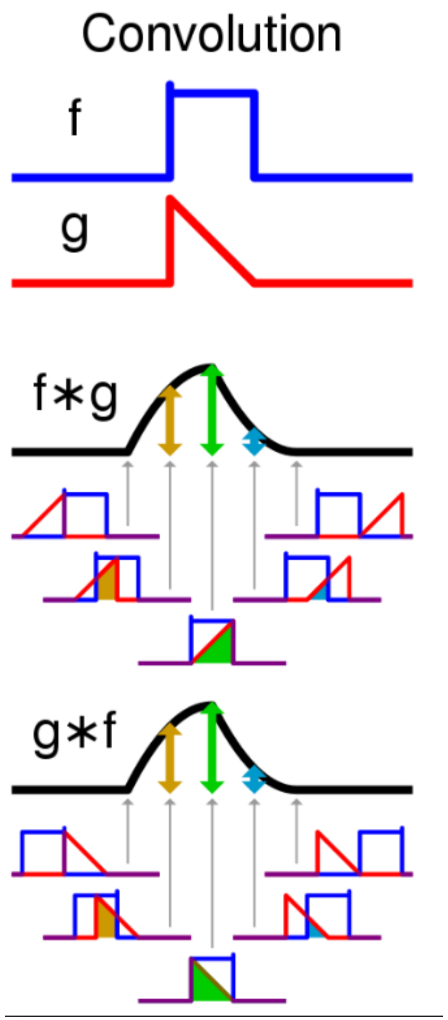
Knowing the point-spread function, it is direct to model any output of the black box, given an input:



$$U_o(x_o, y_o) = \iint_{-\infty}^{\infty} U_i(x_i, y_i) h(x_o - x_i, y_o - y_i) dx_i dy_i$$

Output of linear system is a convolution of the input with its point-spread function

1D convolution example



Steps to perform a convolution:

1. Flip one signal (the second one = the PSF)

2. Position PSF right before overlap

With incremental steps:

3. Step PSF over to position x_o

4. Compute *area* of overlap of two functions

5. Convolution value at x_o= area of overlap

6. Repeat 3-5 until signals do not overlap

https://en.wikipedia.org/wiki/Convolution

2D convolution example

- Direct extension of 1D concept to 2D functions
- Note – it is effectively the same with discrete functions = matrices

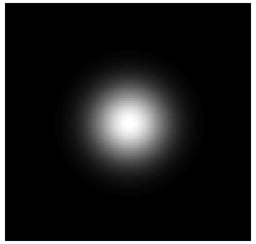
$U_1(x,y)$

y

x

$*$

y2



x2

=

$U_0(x,y)$

y

x

2D convolution example

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High-res. real-world object

Blur caused by camera lens

Image at camera sensor plane

$U_1(x,y)$

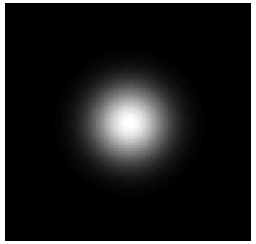
$U_0(x,y)$

y

x

$*$

y2



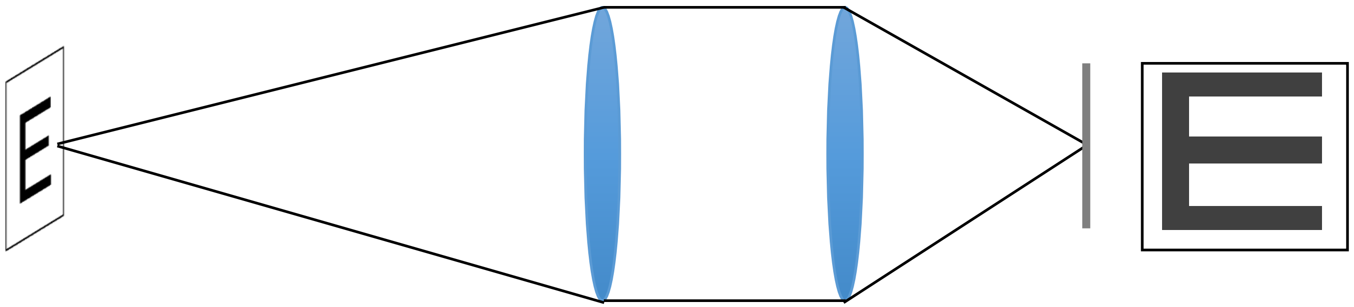
x2

=

y

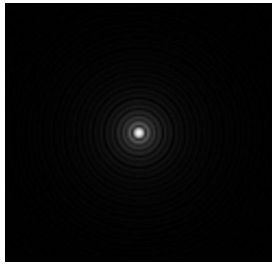
x

Optical modification Ex. #1: The cubic phase mask



in focus

Standard camera
Point-spread function



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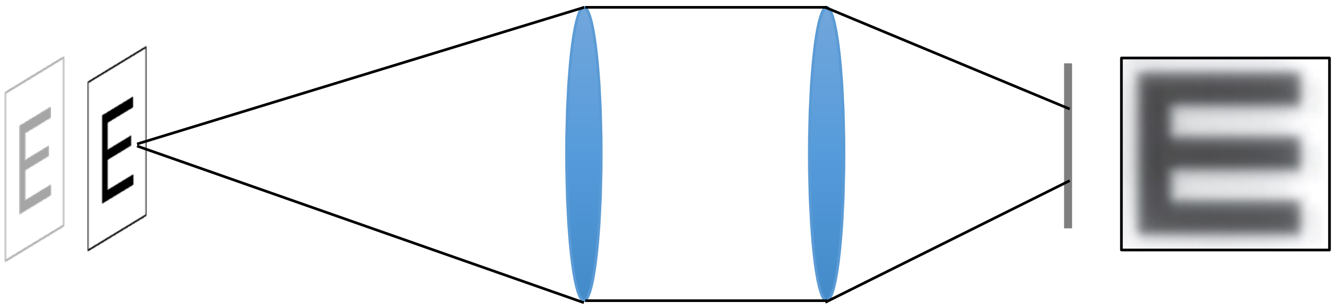
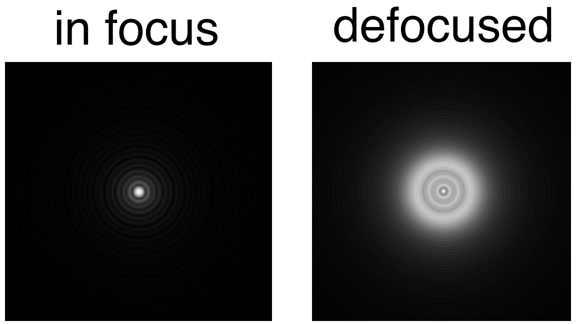
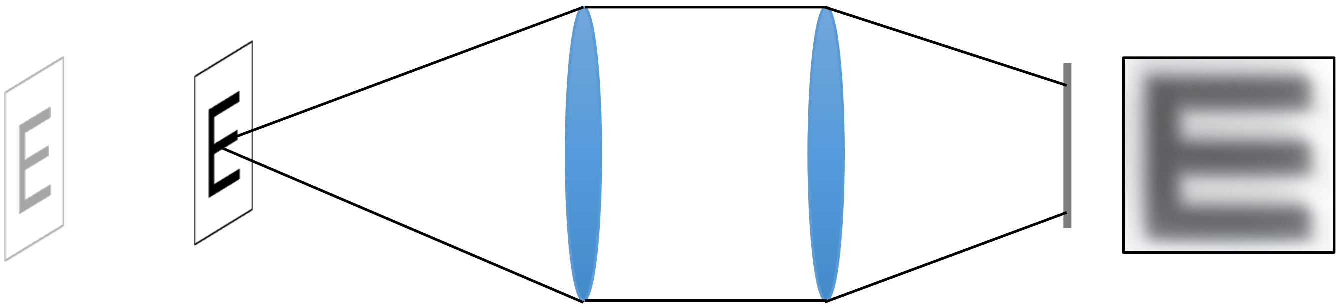
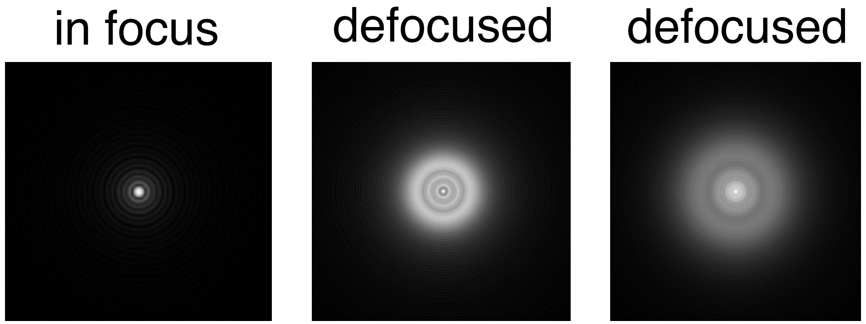




Standard camera:
Limited depth-of-field

Optical modification Ex. #1: The cubic phase mask





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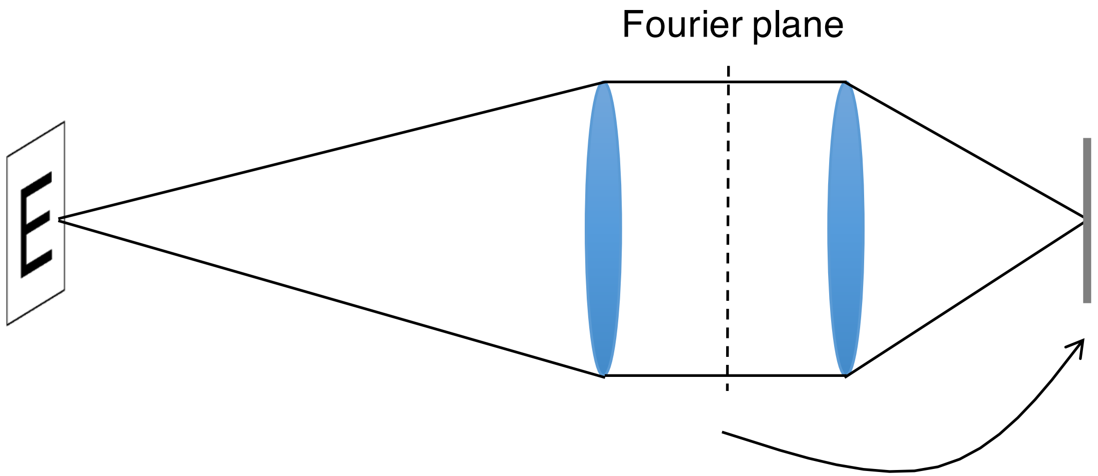
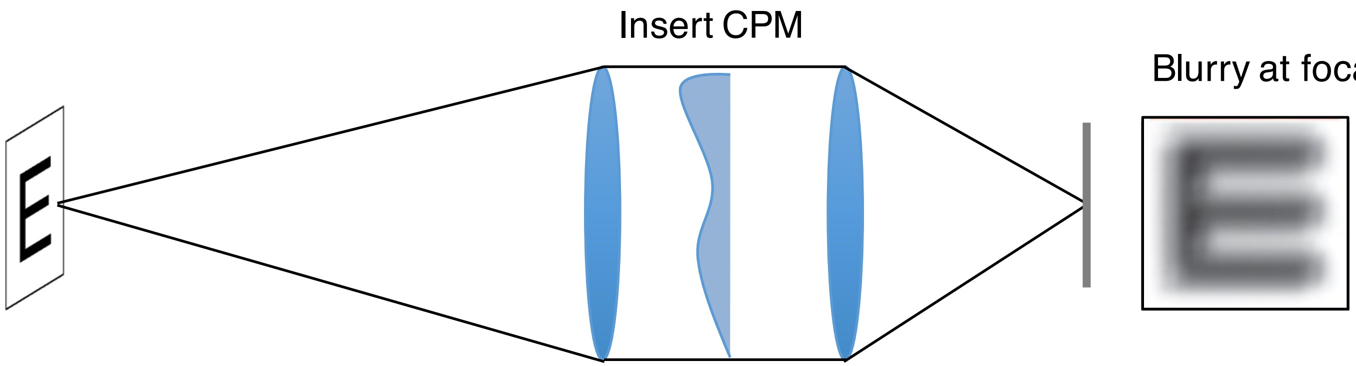
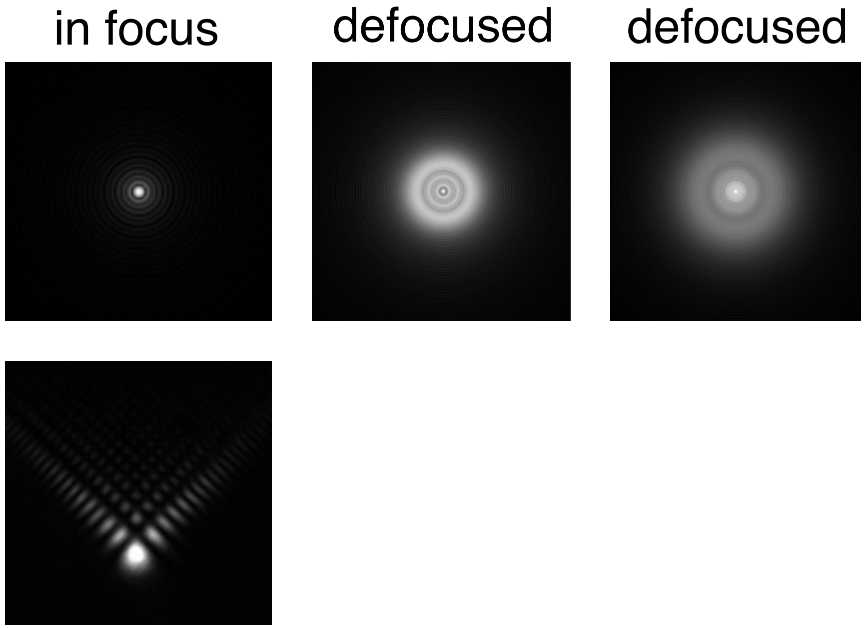


Blur proportional to Fourier transform of shape of aperture

Optical modification Ex. #1: The cubic phase mask





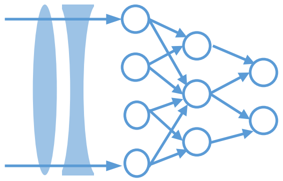
Standard camera:
Limited depth-of-field

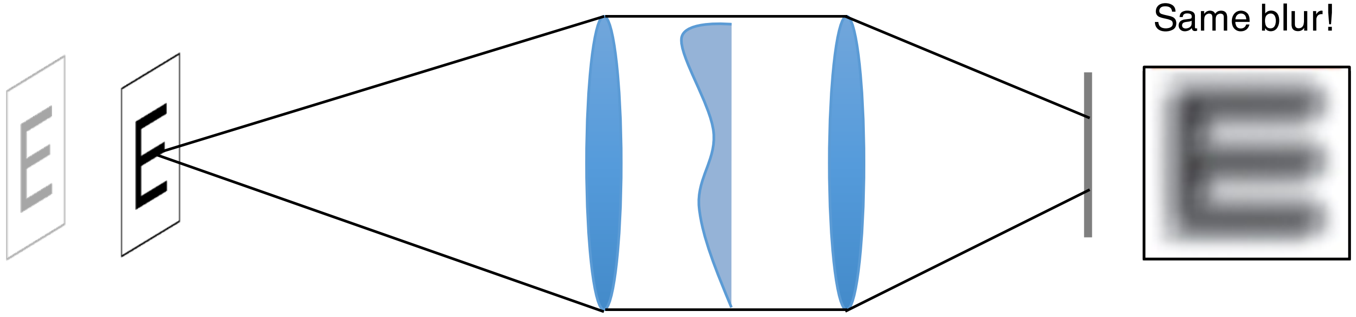
Cubic phase mask:
extended depth-of-field

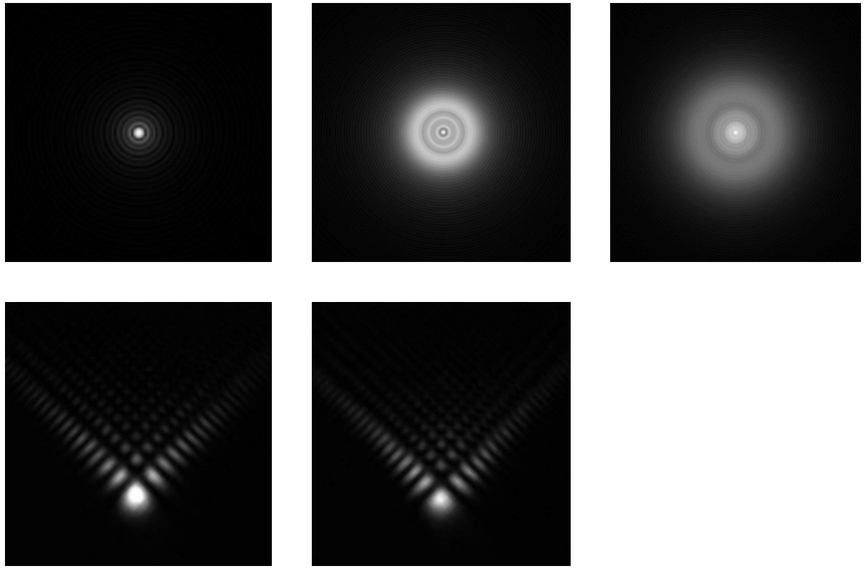
CPM Phase profile

E. Dowski and W. T Cathey, "Extended depth of field through wave-front coding," Appl Opt. 1994

Optical modification Ex. #1: The cubic phase mask







Standard camera:
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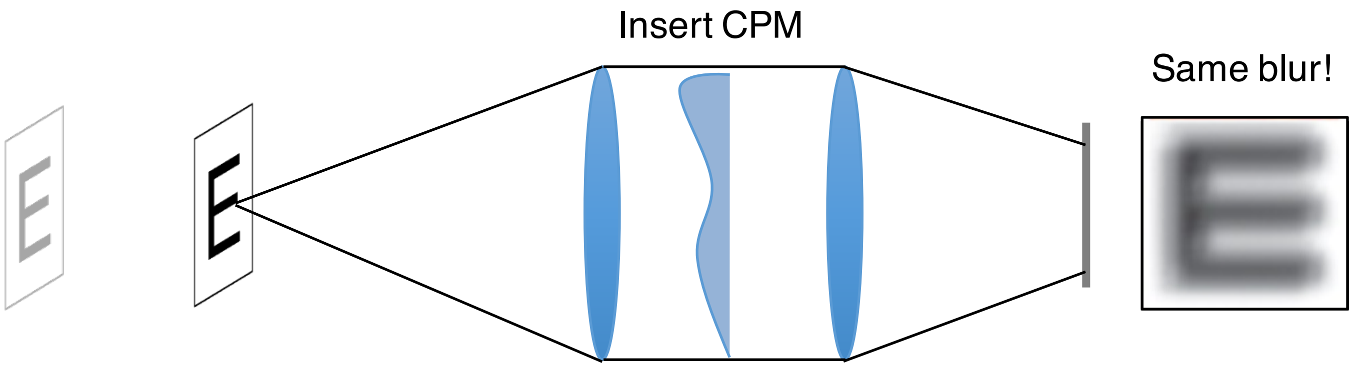

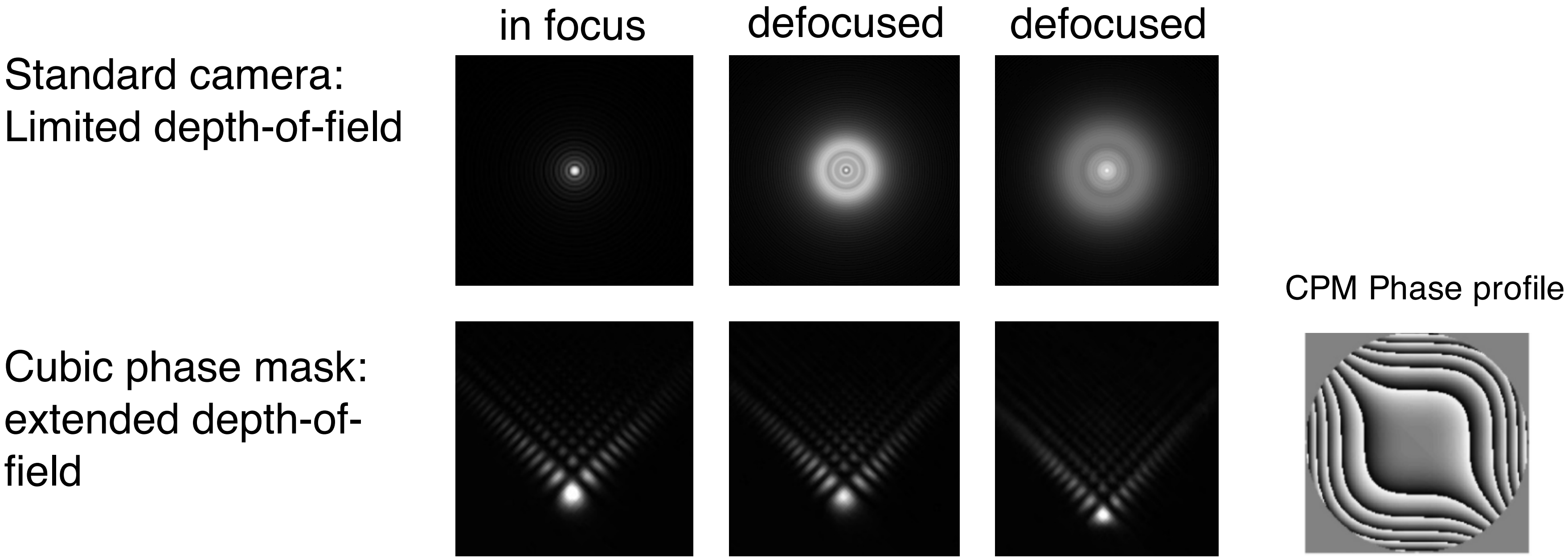

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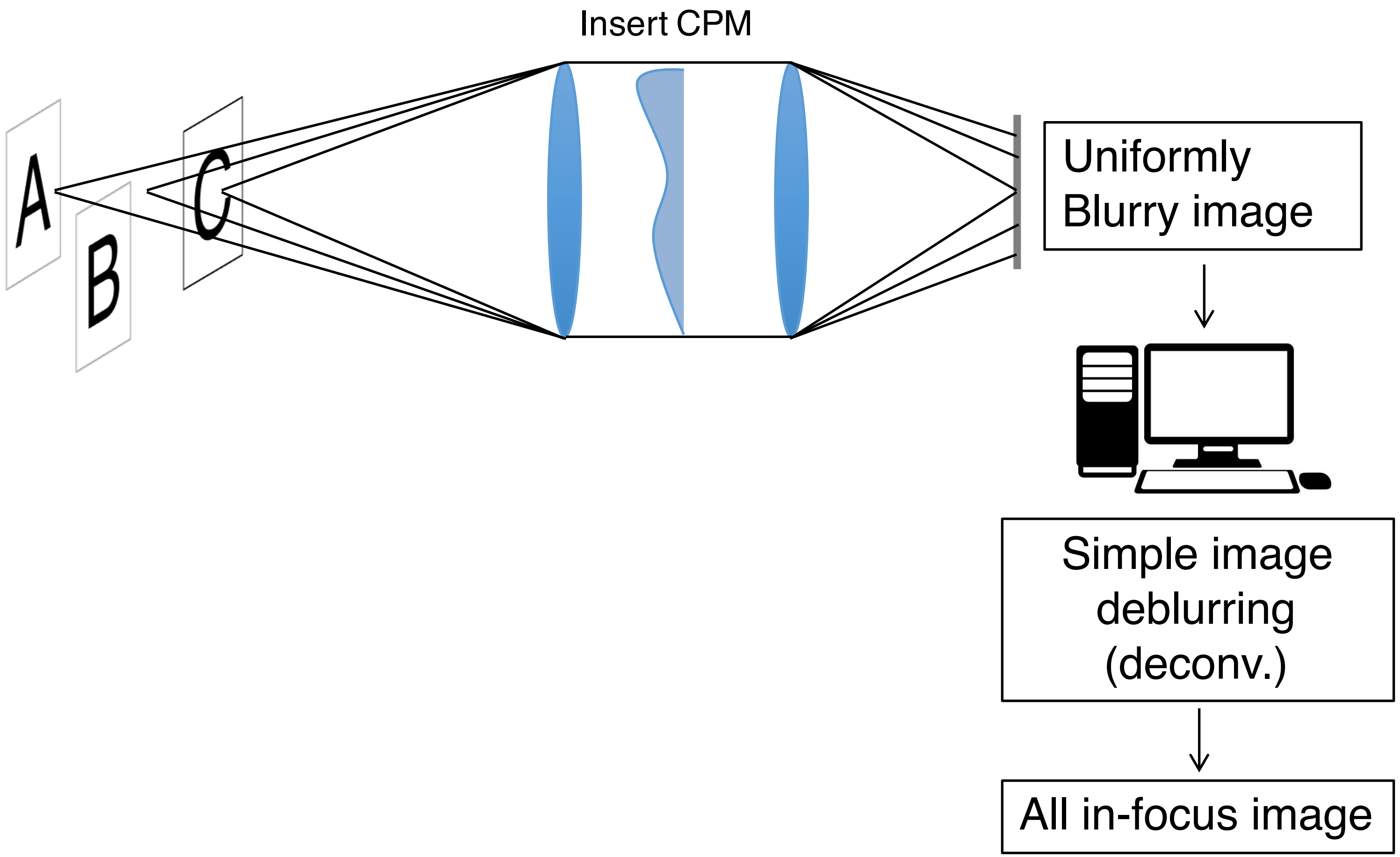

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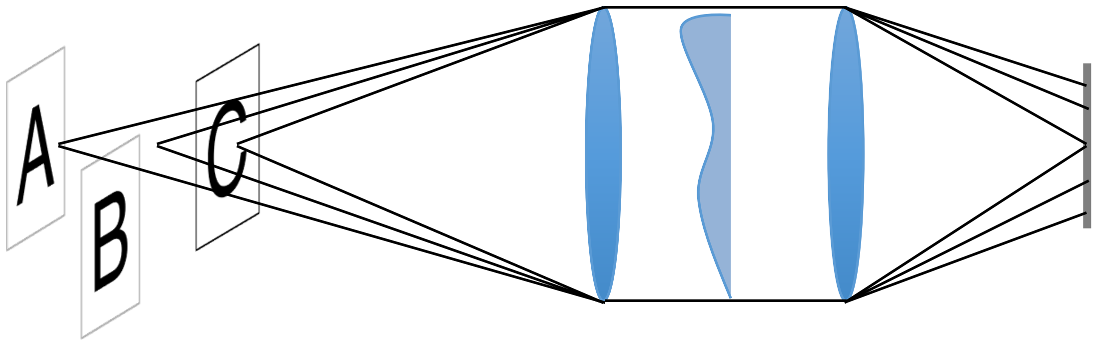
Optical modification Ex. #1: The cubic phase mask

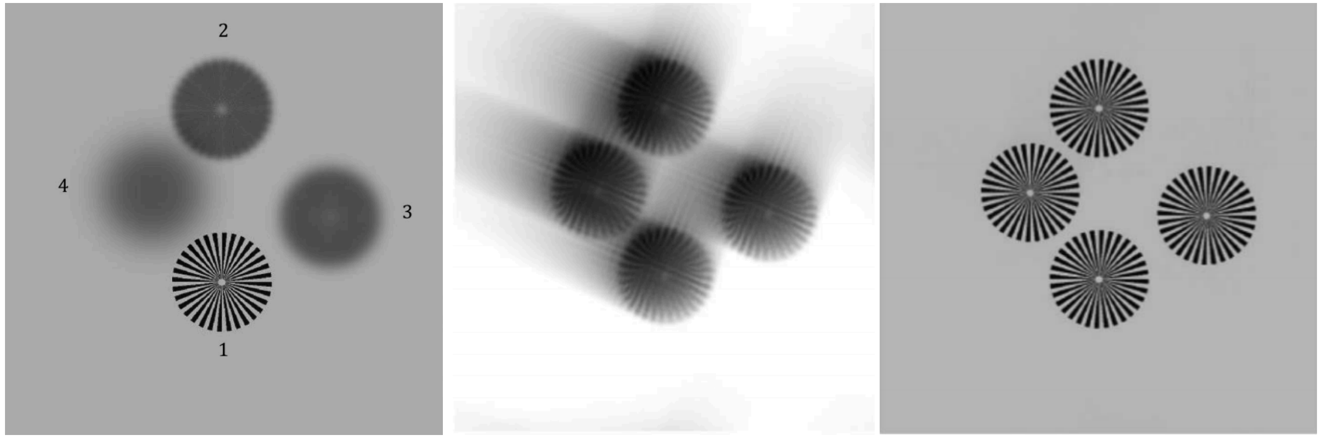
Insert CPM



Uniformly Blurry image

Simple image deblurring (deconv.)

All in-focus image

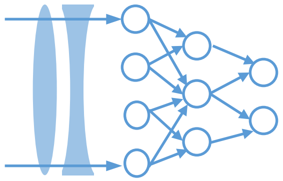


Standard camera image

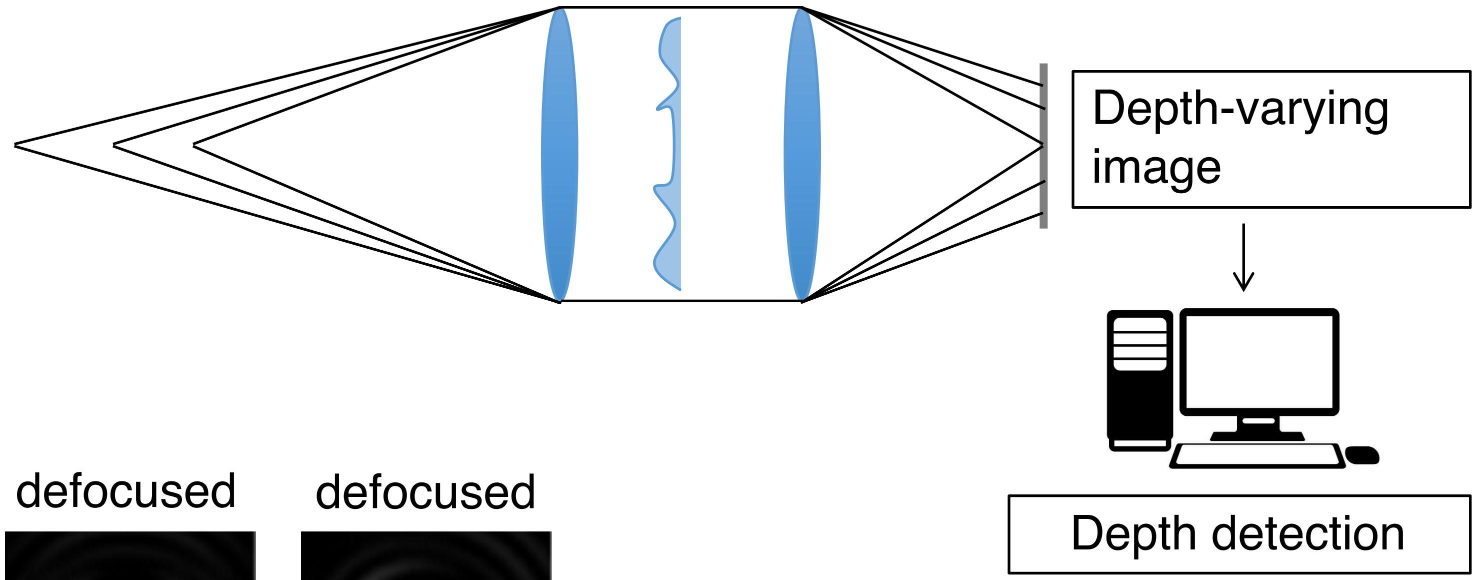
CPM image, raw

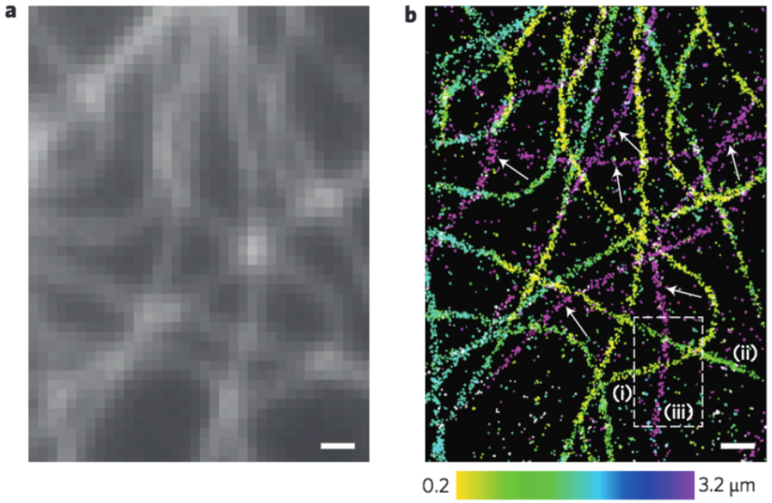
CPM image, deconvolved

E. Dowski and W. T Cathey, "Extended depth of field through wave-front coding," Appl Opt. 1994



Optical modification Ex. #1b: Double helix mask





Moerner Lab
Nobel Prize in Chemistry, 2014

Jia et al., Nature Photonics 2014

Useful properties of the convolution

1. <u>Commutativity</u> $$U(x) * h(x) = h(x) * U(x)$$

$\Rightarrow$ You can choose which signal to "flip"

Useful properties of the convolution

deep imaging

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2. Associativity $$U(x) * [V(x) * W(x)] = [U(x) * V(x)] * W(x)$$

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3. Distributivity $$U(x) * [h_1(x) * h_2(x)] = U(x) * h_1(x) + U(x) * h_2(x)$$

Signals in space and spatial frequency

- What we have so far:
    - Continuous & (possibly) complex function for images across space
    - Black-box linear transformation from one domain to the next via convolution

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  - Audio signal passing through a filter

Complex function of time -> frequency

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Fourier Transforms

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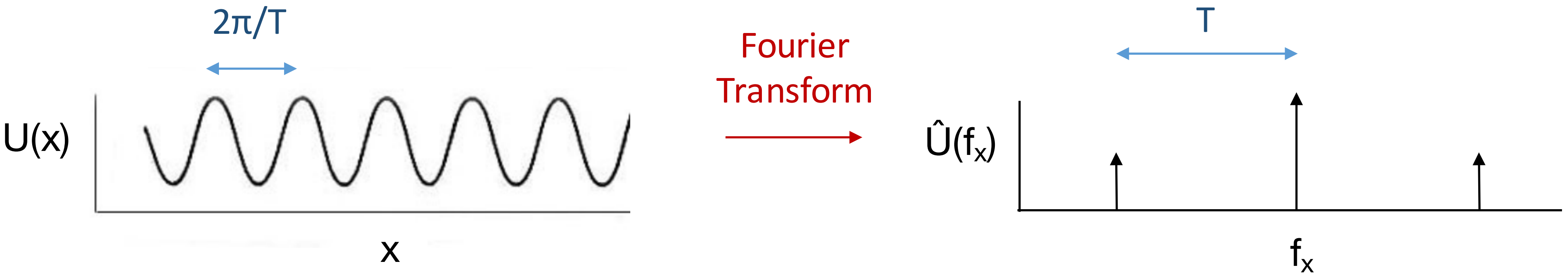
Complex function of time -> frequency

- Here, we have 2D (complex) function across space (x,y) -> *spatial* frequency (f_x, f_y)



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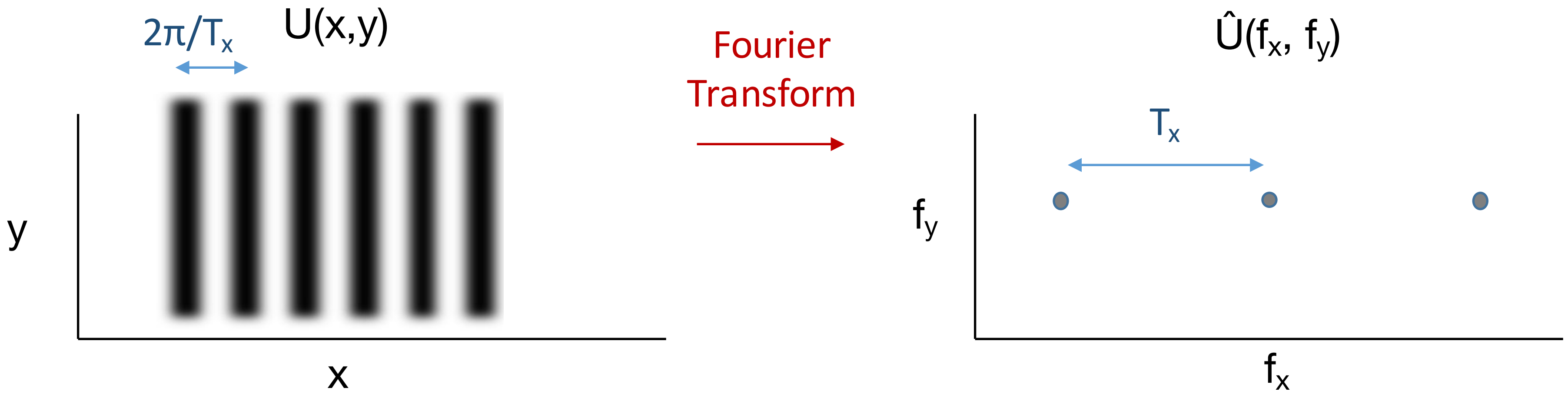

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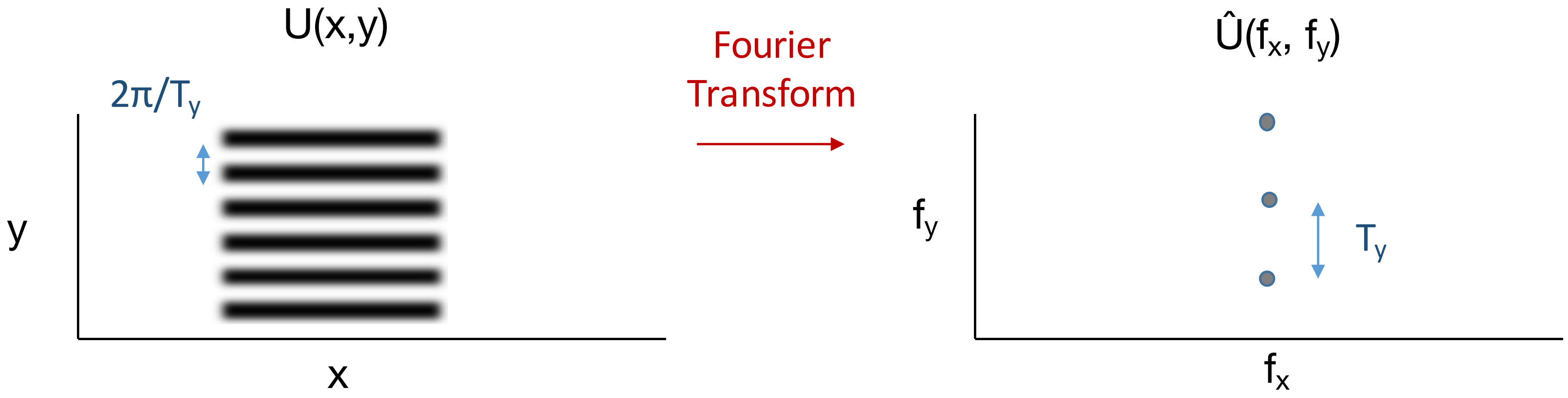

Continuous Fourier transforms – for 2D images

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Decomposition of a signal into elementary functions of form, $\exp\left(-2\pi i(f_x x + f_y y)\right)$:

$$\mathcal{F}\{U(x,y)\} = \hat{U}(f_x, f_y) = \iint_{-\infty}^{\infty} U(x,y) \exp\left(-2\pi i(f_x x + f_y y)\right) dx\, dy$$

Continuous Fourier transforms – for 2D images

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U is absolutely integrable & no infinite discontinuities. The inverse Fourier transform is,

$$\mathcal{F}^{-1}\{\hat{U}(f_x, f_y)\} = U(x,y) = \iint_{-\infty}^{\infty} \hat{U}(f_x, f_y)\exp\left(2\pi i(f_x x + f_y y)\right) df_x\, df_y$$

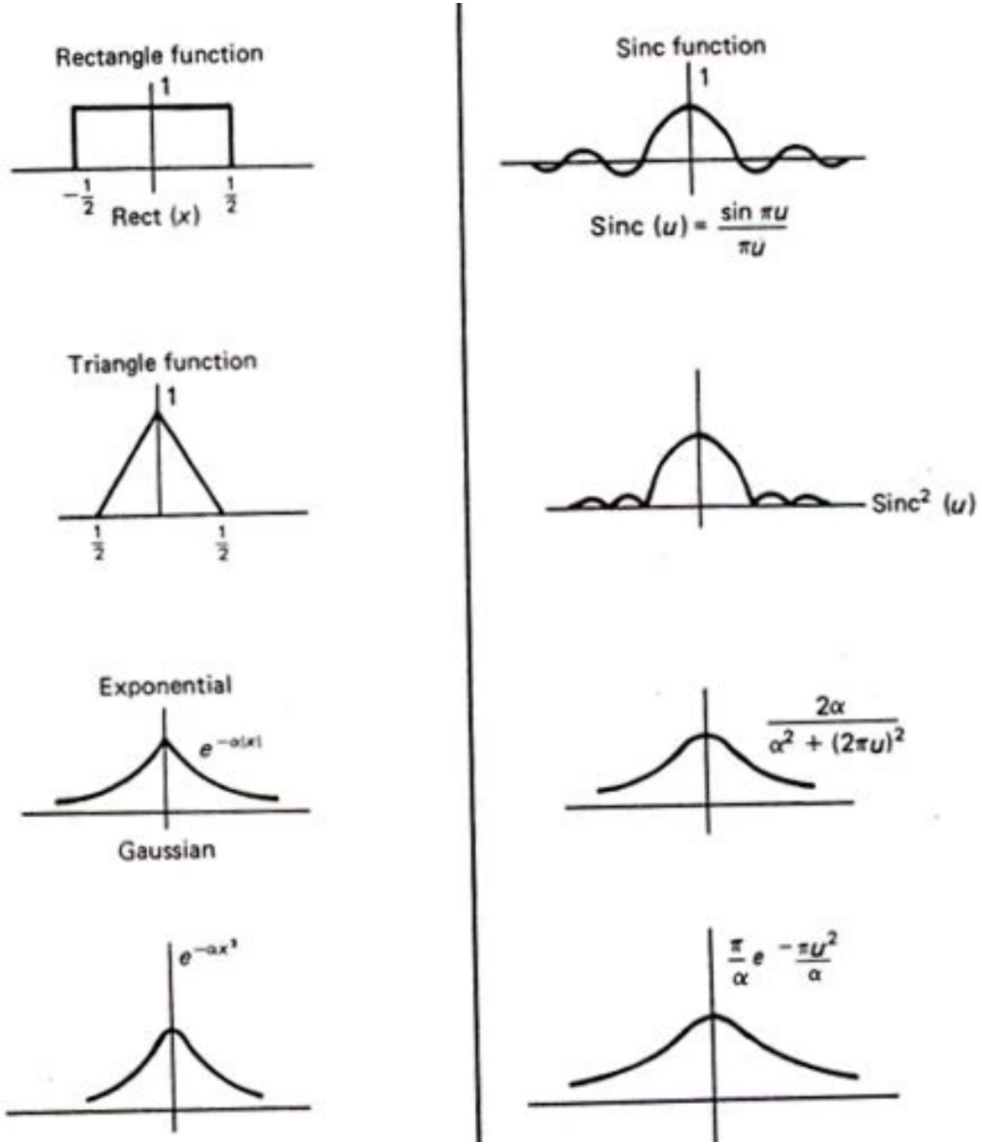
Additional Details:

- Goodman Chapter 2.1
- Mathworld/Wikipedia, Fourier Transform

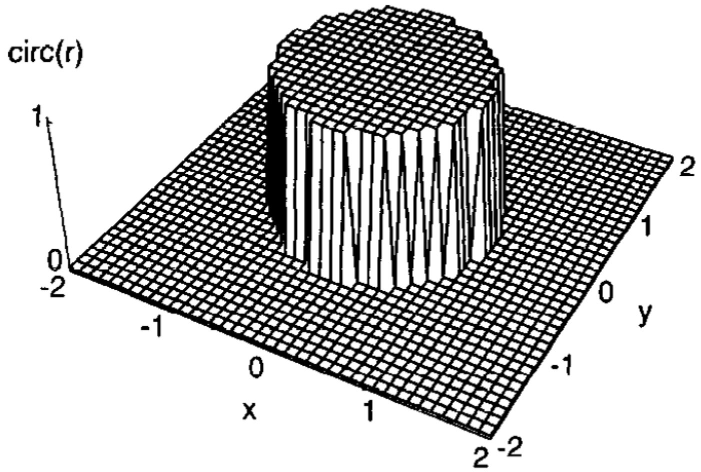
A few examples of Fourier transform pairs, 1D

$U(x)$

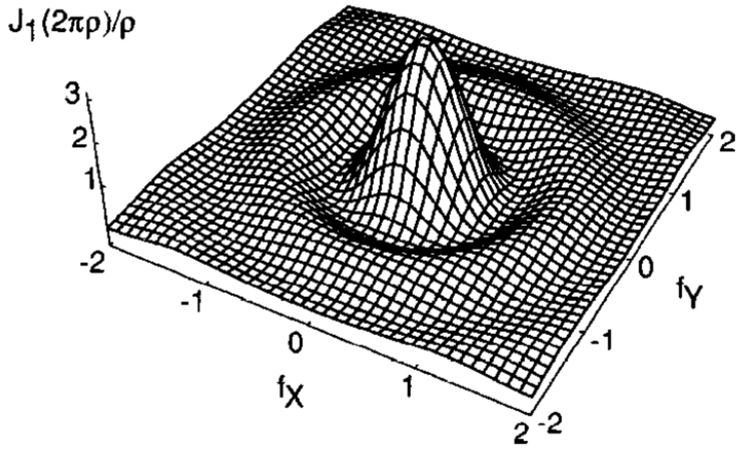
$\hat{U}(f_x)$



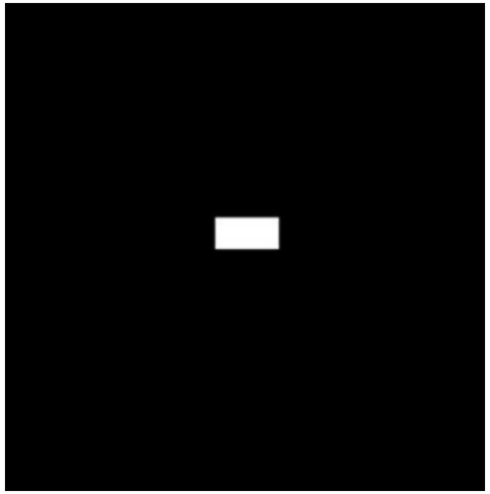
Examples of Fourier transform pairs, 2D

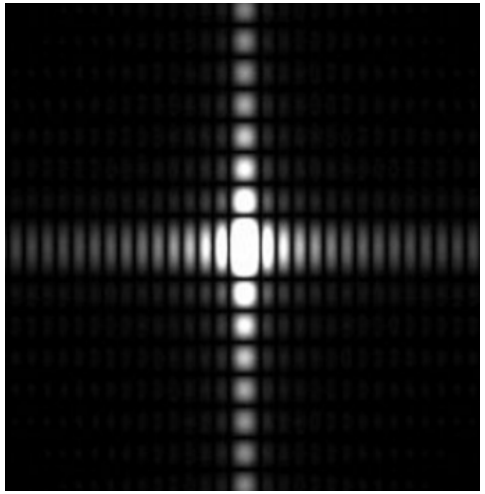


(a)



(b)

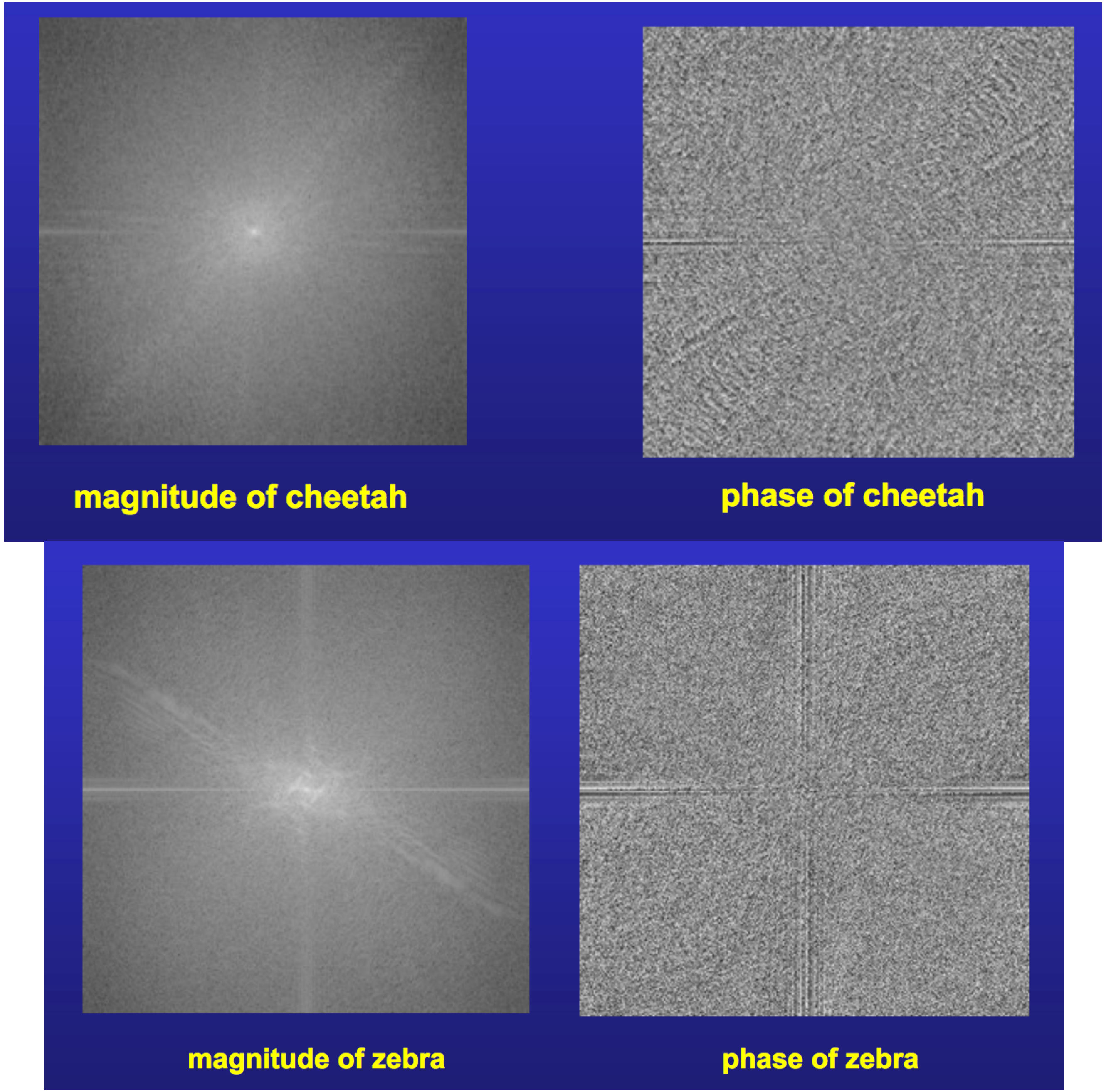




$U_1(x,y)$ $U_2(x,y)$

$\hat{U}_1(f_x, f_y)$

$\hat{U}_2(f_x, f_y)$



Important properties of the Fourier transform

- Linearity
- Scaling
- Shift
- Parseval's Theorem (energy conservation)
- Fourier integral theorem

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- Mathworld/Wikipedia, Fourier Transform

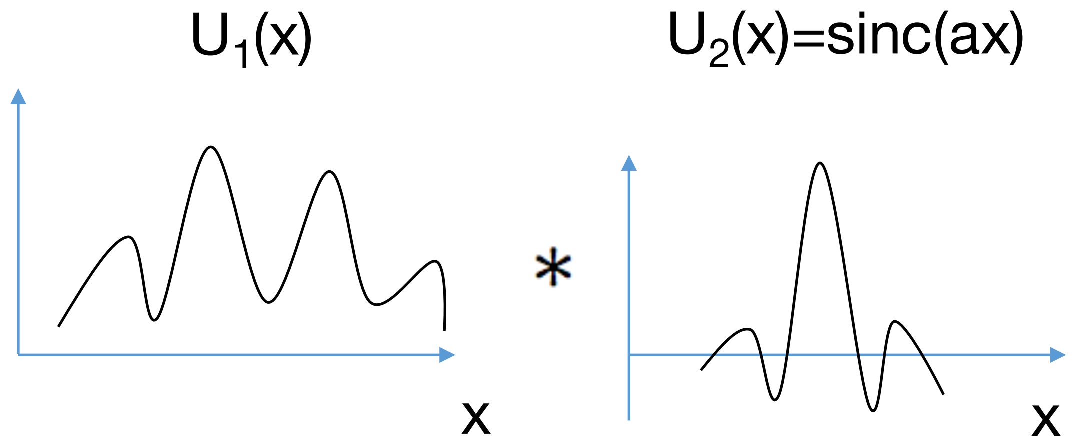
Convolution - Fourier Transform relationship: Convolution Theorem

Convolution theorem. If $\mathcal{F}\{g(x, y)) = G(f_X, f_Y)$ and $\mathcal{F}\{h(x, y)\} = H(f_X, f_Y)$, then

$$\mathcal{F}\left\{\iint_{-\infty}^{\infty} g(\xi, \eta)\, h(x-\xi, y-\eta)\, d\xi\, d\eta\right\} = G(f_X, f_Y)\, H(f_X, f_Y).$$

"The convolution of two functions in space can be performed by a multiplication in the Fourier domain (spatial frequency domain)"

Example of convolution theorem, 1D



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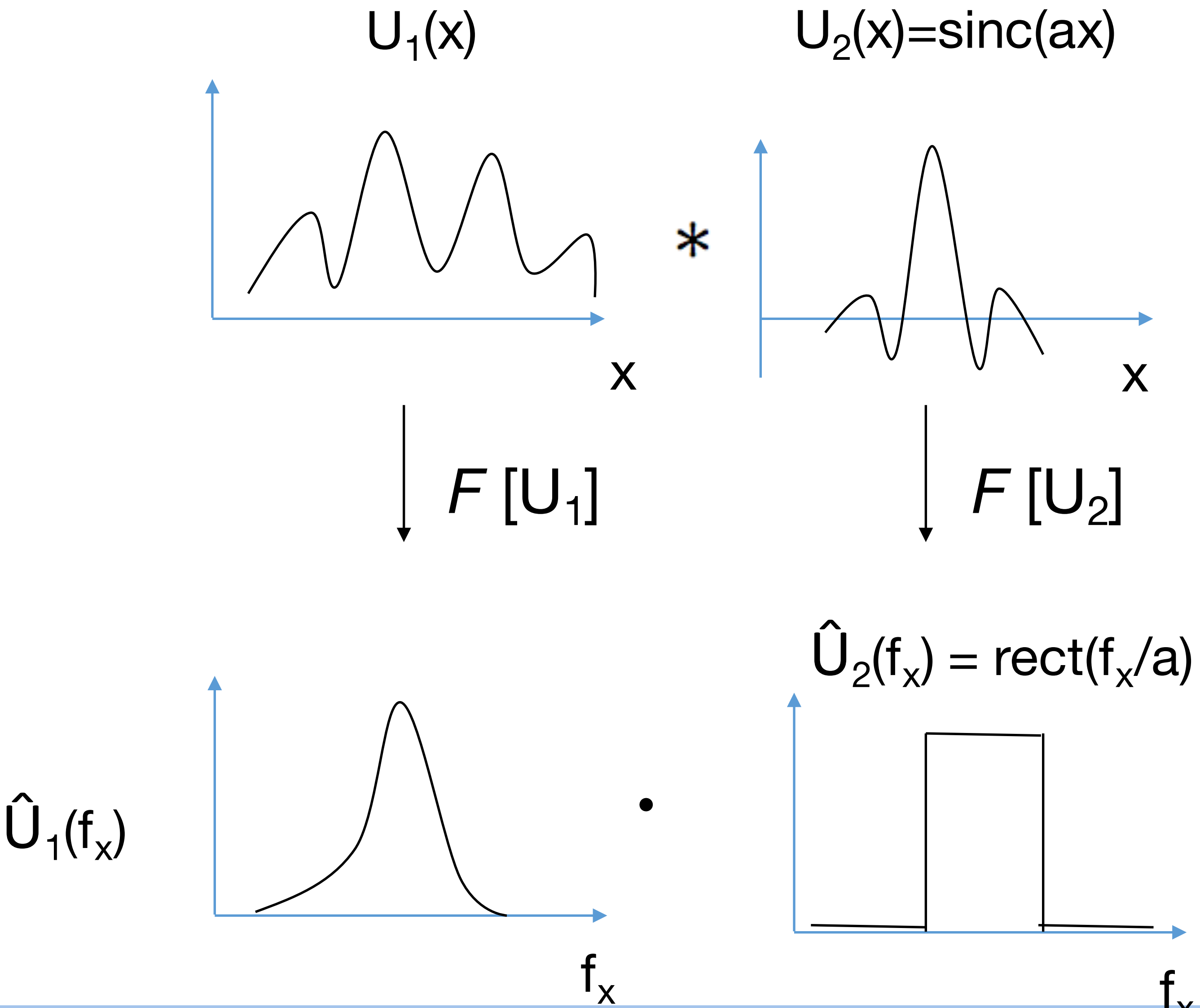
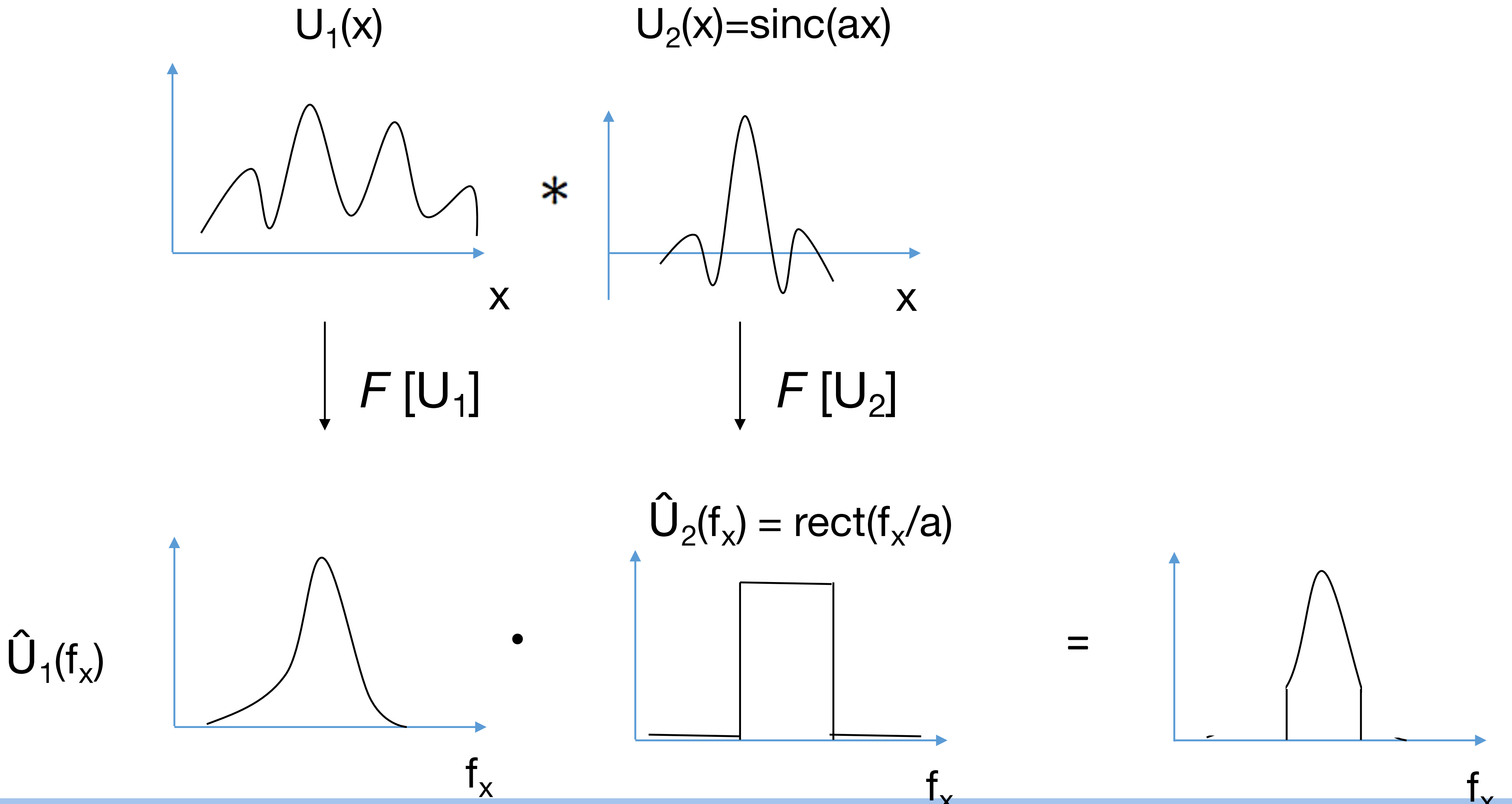
Example of convolution theorem, 1D



Example of convolution theorem, 1D



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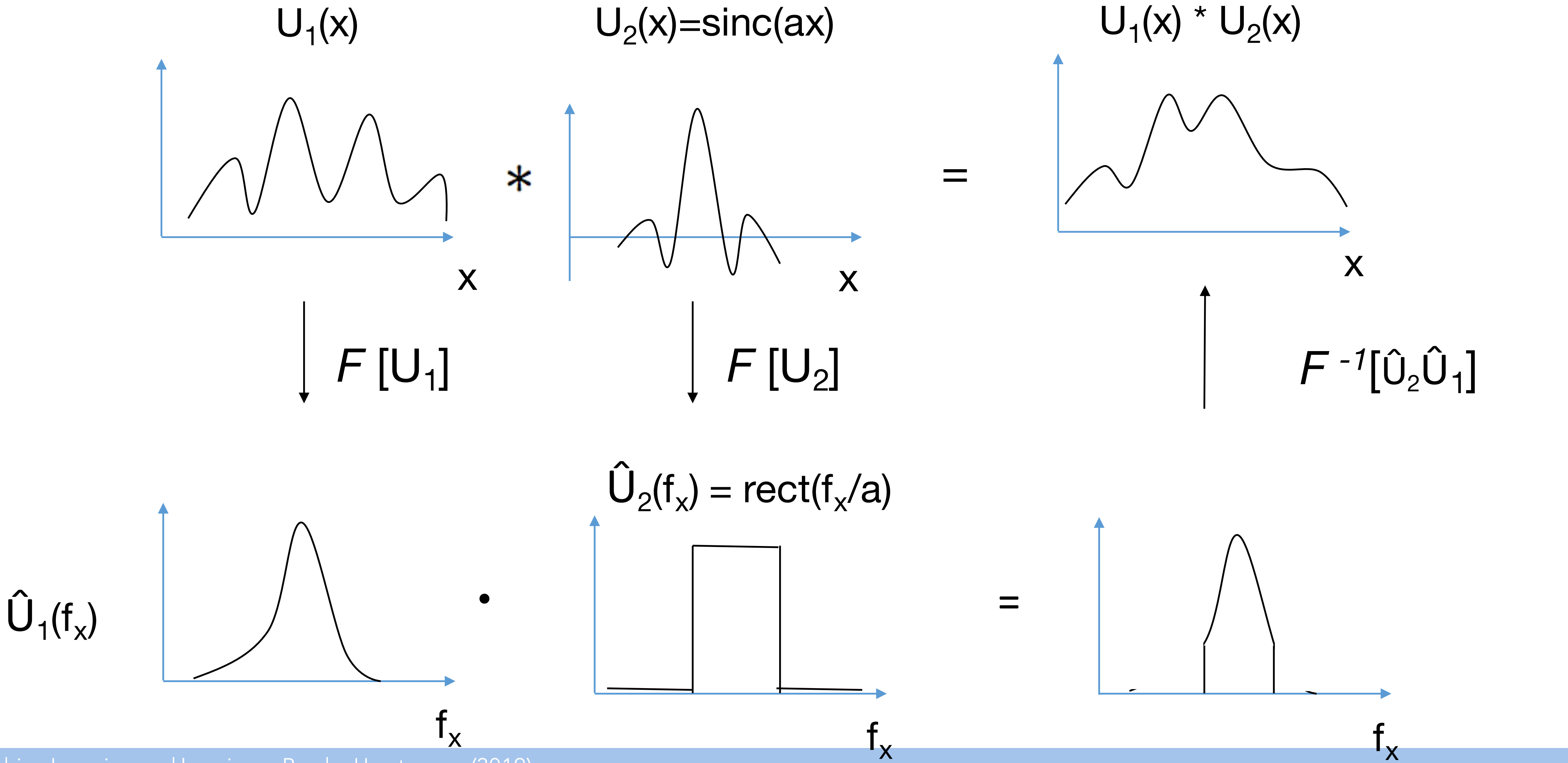
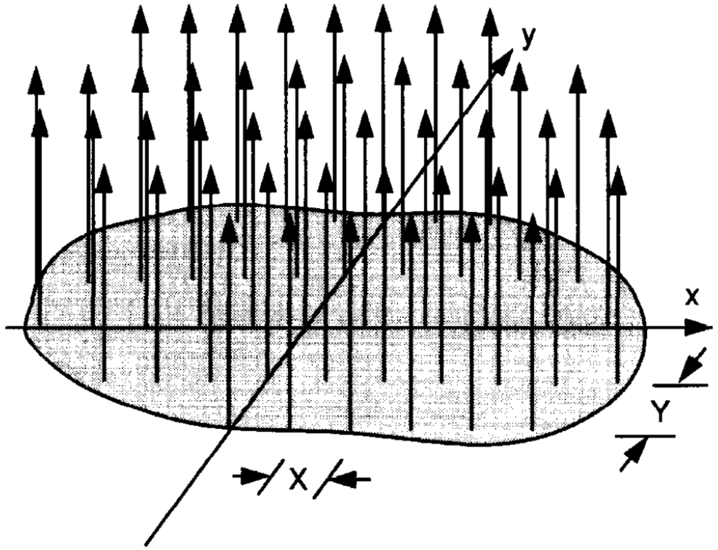
Example of convolution theorem, 1D



The Sampling Theorem – from Goodman Section 2.4.1

$$U_s(x, y) = \mathrm{comb}(x/X)\mathrm{comb}(y/Y)U(x, y)$$



Signal sampling occurs with:

- CMOS (pixel) sensors, PMTs, SPADs
- A-to-D after antennas
- A-to-D after acoustic transducers

Sampling interval width X and Y

The Sampling Theorem – from Goodman Section 2.4.1

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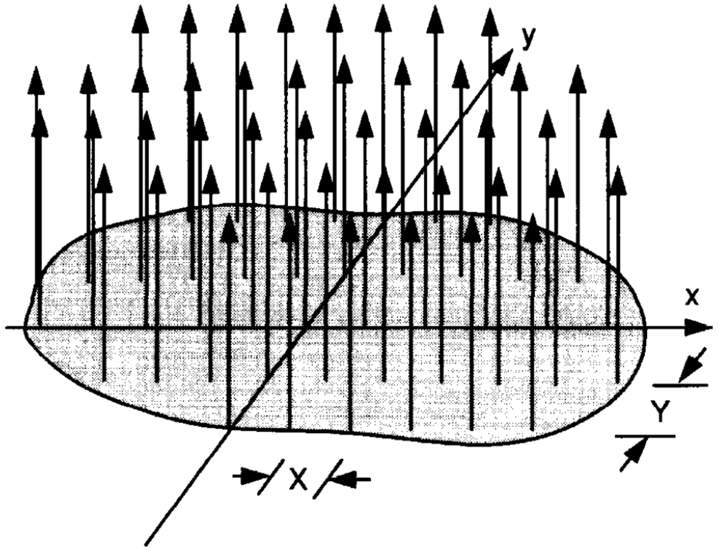

Sampling interval width X and Y

$$\hat{U}_s(f_x, f_y) = \mathcal{F}\left[\text{comb}(x/X)\text{comb}(y/Y)\right] * \hat{U}(f_x, f_y)$$

The Sampling Theorem – from Goodman Section 2.4.1

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$$\mathcal{F}\left[\mathrm{comb}(x/X)\mathrm{comb}(y/Y)\right] = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta\left(f_x - \frac{n}{X}, f_y - \frac{m}{Y}\right)$$

The Sampling Theorem – from Goodman Section 2.4.1

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The Sampling Theorem – from Goodman Section 2.4.1

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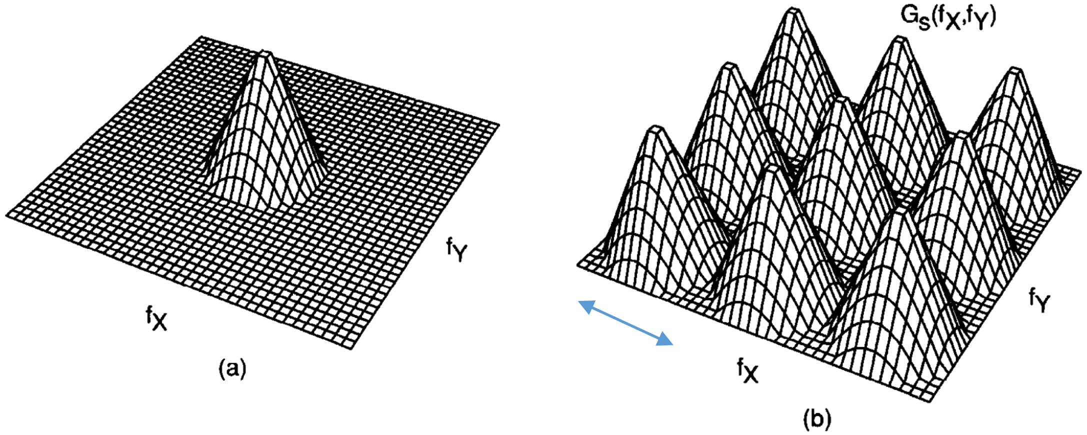

$$\text{rect}\left(\frac{f_x}{2B_x}\right)\text{rect}\left(\frac{f_y}{2B_y}\right)\hat{U}_s(f_x, f_y) = \hat{U}(f_x, f_y)$$

Signal extends from ($-B_x$, $-B_y$) to (B_x, B_y) in Fourier domain

Bandwidth (B_x, B_y) of signal

$$\mathrm{rect}\left(\frac{f_x}{2B_x}\right)\mathrm{rect}\left(\frac{f_y}{2B_y}\right)\hat{U}_s(f_x, f_y) = \hat{U}(f_x, f_y)$$

$F\,[\bullet]$

$$h(x, y) = 4B_xB_y\mathrm{sinc}(2B_x x)\mathrm{sinc}(2B_y y)$$

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$$\mathrm{rect}\left(\frac{f_x}{2B_x}\right)\mathrm{rect}\left(\frac{f_y}{2B_y}\right)\hat{U}_s(f_x, f_y) = \hat{U}(f_x, f_y)$$

$F\,[\bullet]$

$$h(x, y) = 4B_xB_y\mathrm{sinc}(2B_xx)\mathrm{sinc}(2B_yy)$$

$$h(x, y) * (U(x, y)\mathrm{comb}(x/X)\mathrm{comb}(y/Y)) = U(x, y)$$

$$\mathrm{rect}\left(\frac{f_x}{2B_x}\right)\mathrm{rect}\left(\frac{f_y}{2B_y}\right)\hat{U}_s(f_x, f_y) = \hat{U}(f_x, f_y)$$

$$\xrightarrow{F[\bullet]} \quad h(x, y) = 4B_xB_y\mathrm{sinc}(2B_xx)\mathrm{sinc}(2B_yy)$$

$$h(x, y) * (U(x, y)\mathrm{comb}(x/X)\mathrm{comb}(y/Y)) = U(x, y)$$

$$U(x, y)\mathrm{comb}(x/X)\mathrm{comb}(y/Y) = XY\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}U(nX, mY)\delta(x - nX, y - mY)$$

$$\mathrm{rect}\left(\frac{f_x}{2B_x}\right)\mathrm{rect}\left(\frac{f_y}{2B_y}\right)\hat{U}_s(f_x, f_y) = \hat{U}(f_x, f_y)$$

$F\,[\bullet]$

$$h(x,y) = 4B_xB_y\mathrm{sinc}(2B_xx)\mathrm{sinc}(2B_yy)$$

$$h(x,y) * (U(x,y)\mathrm{comb}(x/X)\mathrm{comb}(y/Y)) = U(x,y)$$

$$U(x,y)\mathrm{comb}(x/X)\mathrm{comb}(y/Y) = XY\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}U(nX,mY)\delta(x-nX,y-mY)$$

$$U(x,y) = 4B_xB_yXY\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}U(nX,mY)\mathrm{sinc}\left[2B_x(x-nX)\right]\mathrm{sinc}\left[2B_y(y-mY)\right]$$

The Sampling Theorem

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When sampled appropriately, a discrete signal can *exactly* reproduce a continuous signal:

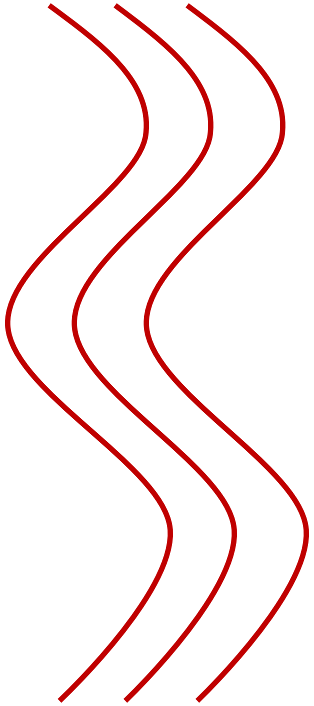
$$\underline{U(x,y)} = 4B_x B_y XY \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \underline{U(nX, mY)}\text{sinc}\left[2B_x(x-nX)\right] \text{sinc}\left[2B_y(y-mY)\right]$$

Continuous signal:
- EM field
- Sound wave
- MR signal

Discretized signal:
- Detected EM field
- Sampled sound wave
- Sampled MR signal

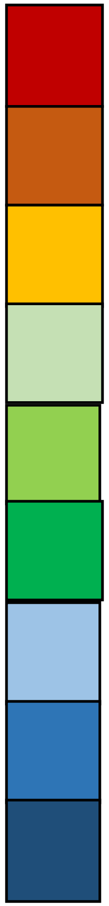
What does the Sampling Theorem mean for us?

Continuous fields



*conditions

Discretize vectors
(and matrices)



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