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Machine Learning and Imaging

BME 590L

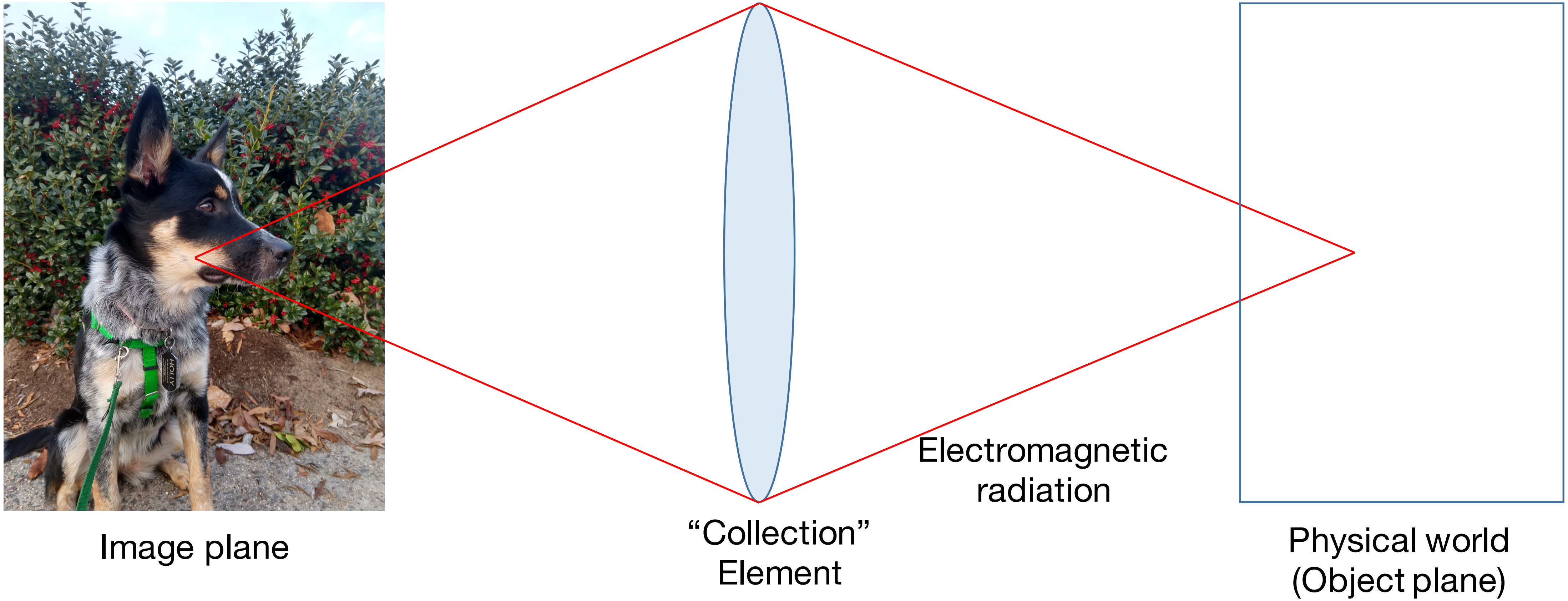
Roarke Horstmeyer

Lecture 2: Mathematical preliminaries for continuous functions

- Light as a continuous wave
- Light as a complex field
- Light transformations as a black box
- Linear black-box systems
- Convolutions in 1D and 2D

Last lecture: what is an image?

2. “Physical” Interpretation



Continuous signal: $\boxed{I(x,y), (x,y) \in (-\infty, \infty)}$

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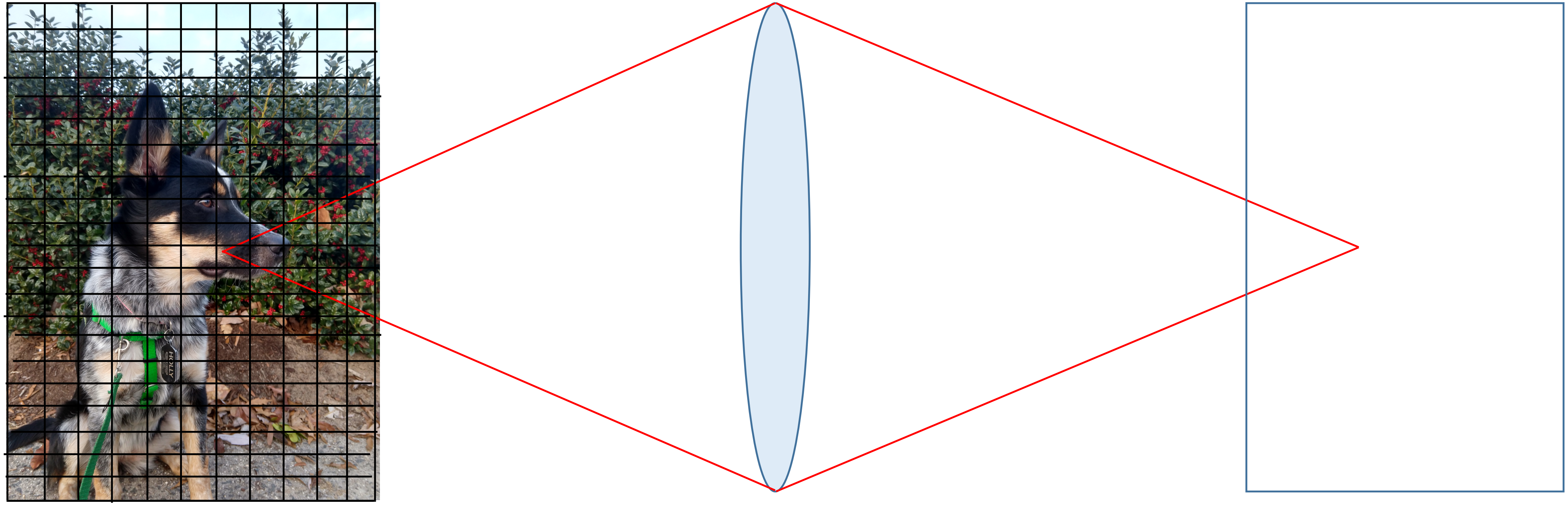
Last lecture: what is an image?

3. “Digital” Interpretation

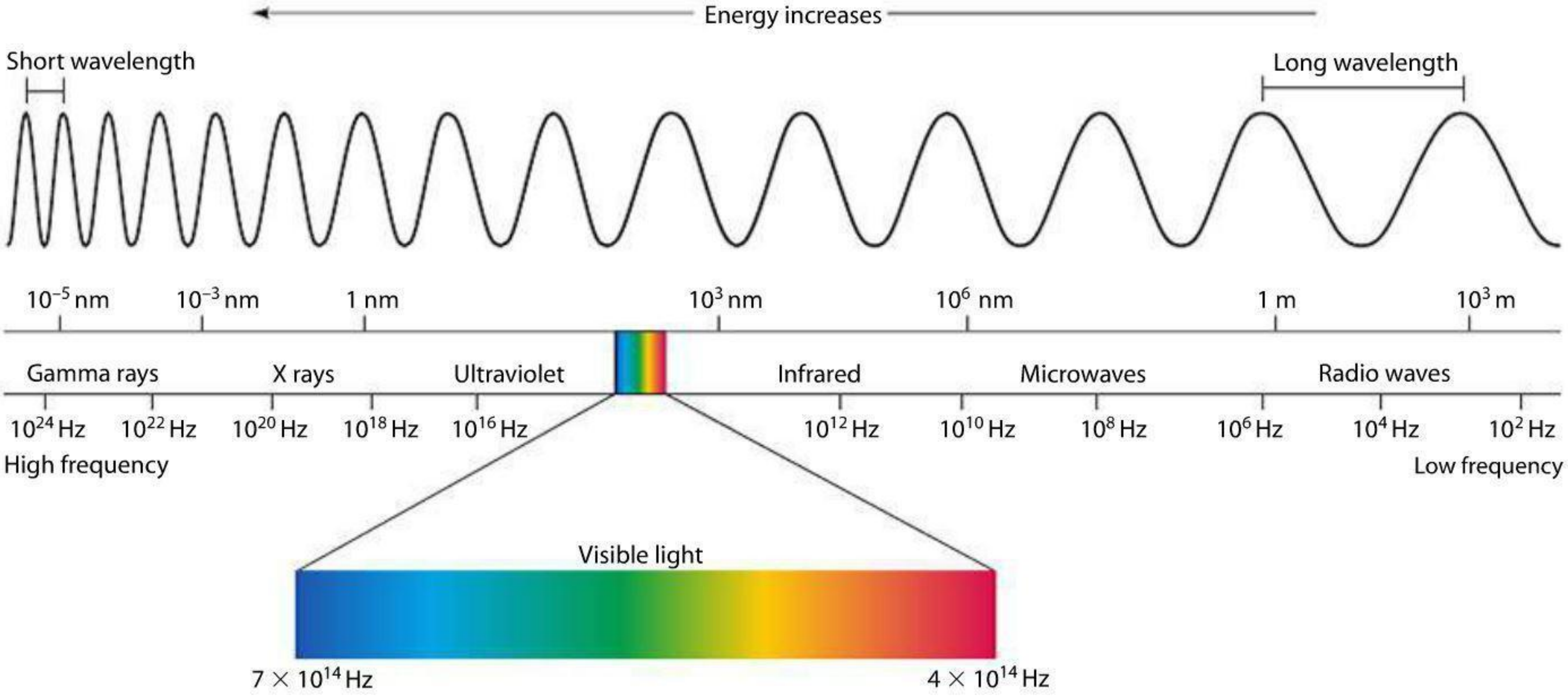
n x *m* array



Photons to electrons → Digitazation → *Discrete signal*

$$\boxed{I_s(x,y), (x,y) \in Z^{n\times m}}$$

Start at the beginning: Electromagnetic waves



From: https://www.miniphysics.com/electromagnetic-spectrum_25.html

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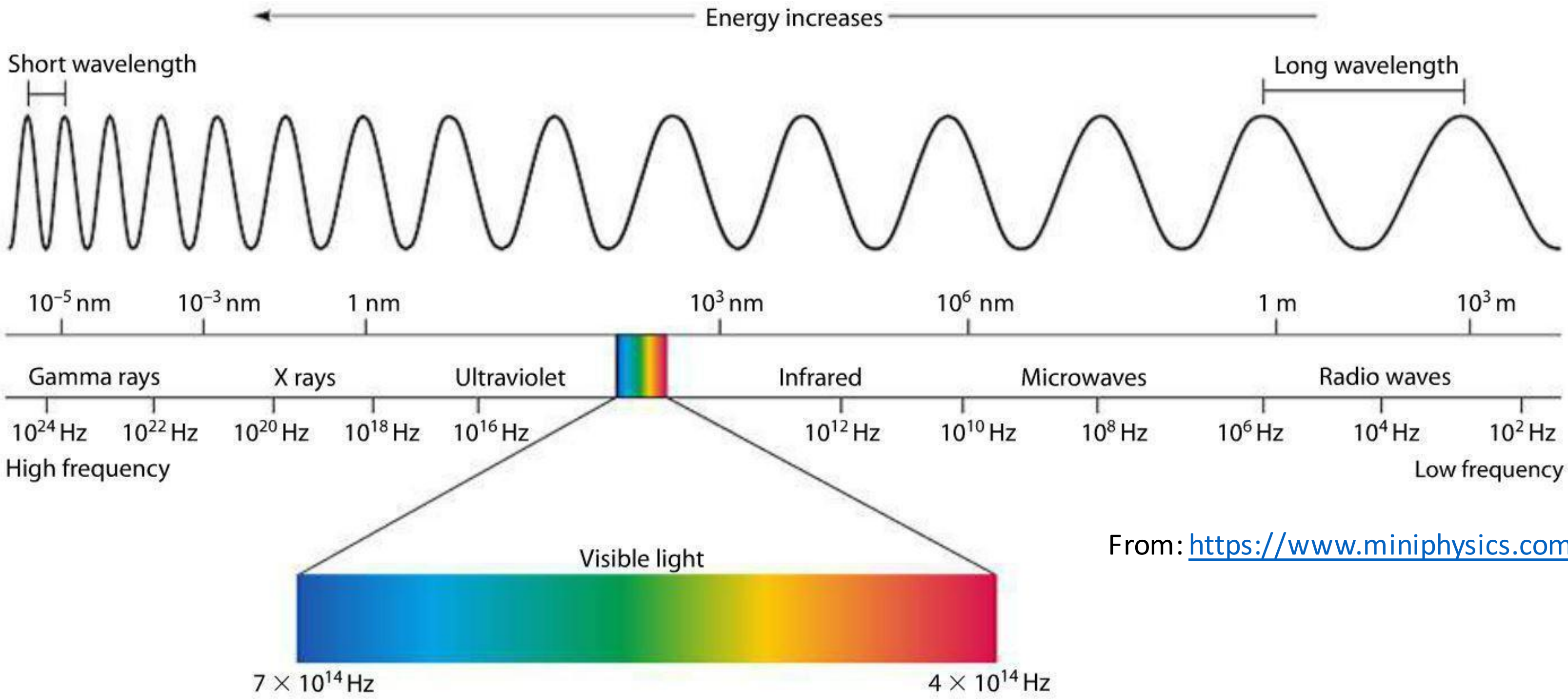

From: https://www.miniphysics.com/electromagnetic-spectrum_25.html

Maxwell's equations

$$
\begin{aligned}
\nabla \times \vec{\mathcal{E}} &= -\mu \frac{\partial \vec{\mathcal{H}}}{\partial t} \\
\nabla \times \vec{\mathcal{H}} &= \epsilon \frac{\partial \vec{\mathcal{E}}}{\partial t} \\
\nabla \cdot \epsilon \vec{\mathcal{E}} &= 0 \\
\nabla \cdot \mu \vec{\mathcal{H}} &= 0.
\end{aligned}
$$

$\rightarrow$

Free-space propagation

$$\nabla^2 \vec{\mathcal{E}} - \frac{n^2}{c^2} \frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2} = 0$$

$\rightarrow$

Scalar solution, 1 freq.

$A(\mathbf{r_1}) \cos(\mathbf{kr_1} - \omega t)$

Start at the beginning: EM fields and the black box

The general idea:

1. We will treat light as a wave (an "optical field")

Start at the beginning: EM fields and the black box

$$U(\mathbf{r_1}) = A(\mathbf{r_1}) \cos(\mathbf{kr_1} - \omega t)$$

(We will get into the details of optical fields in a few weeks)

The general idea:

1. We will treat light as a wave (an "optical field")

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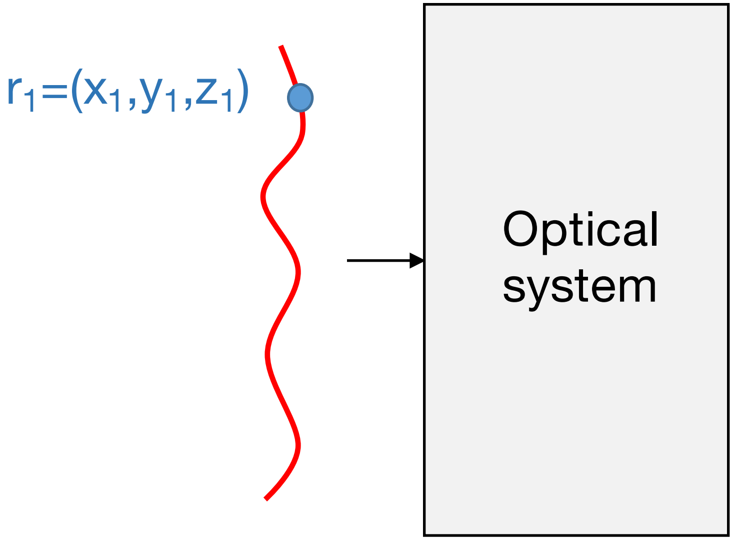
Start at the beginning: EM fields and the black box



$$U(\mathbf{r}_1) = A(\mathbf{r}_1)\cos(\mathbf{k}\mathbf{r}_1 - \omega t)$$

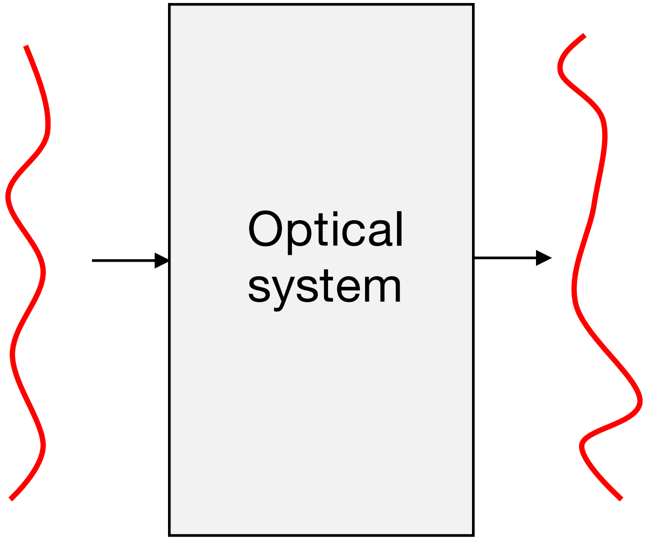
The general idea:

1. We will treat light as a wave (an "optical field")

2. It enters an optical system, which we treat as a black box

3. This black box has a number of useful properties

Start at the beginning: EM fields and the black box



$A(\mathbf{r}_1)\cos(\mathbf{kr}_1 - \omega t)$ $\qquad$ $A(\mathbf{r}_2)\cos(\mathbf{kr}_2 - \omega t)$

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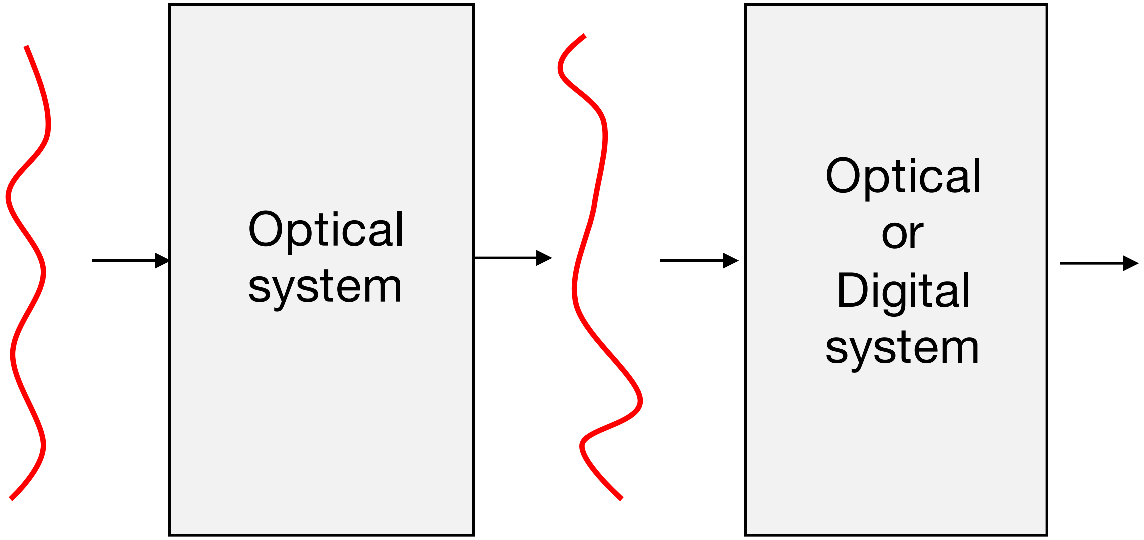
4. The black box outputs an optical field

Start at the beginning: EM fields and the black box



$A(\mathbf{r}_1)\cos(\mathbf{k}\mathbf{r}_1 - \omega t)$ $\quad$ $A(\mathbf{r}_2)\cos(\mathbf{k}\mathbf{r}_2 - \omega t)$

The general idea:

1. We will treat light as a wave (an "optical field")
2. It enters an optical system, which we treat as a black box
3. This black box has a number of useful properties
4. The black box outputs an optical field, which then enters another optical system or a digital system
5. We can cascade these boxes…

Linear systems and the black box

Simplification #1: Let's forget about light changing as a function of time. It does so way too fast, and way too slow:

$$A(\mathbf{r})\cos(\mathbf{kr} - \omega t) \rightarrow A(\mathbf{r})\cos(\mathbf{kr})$$

Linear systems and the black box

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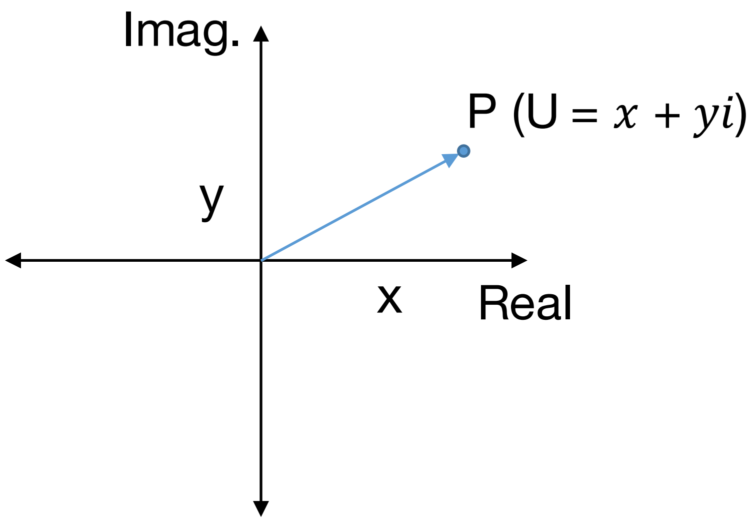
$$A(\mathbf{r}) \cos(\mathbf{kr} - \omega t) -> A(\mathbf{r}) \cos(\mathbf{kr})$$

Simplification #2: We'll use complex numbers when required, it'll make our lives easier. This leads to the *complex* field, U(**r**):

$$A(\mathbf{r}) \cos(\mathbf{kr}) <-> A(\mathbf{r})\, e^{i\mathbf{k}\cdot\mathbf{r}} = U(\mathbf{r})$$

Some things you need to recall about complex numbers

$$U = x + iy, \; i = \sqrt{-1}$$



Some things you need to recall about complex numbers

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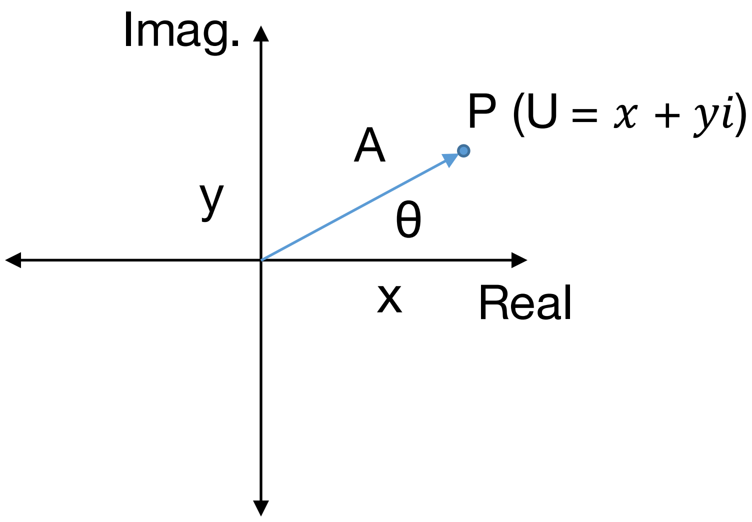

$A = \sqrt{(x^2 + y^2)}$

$\theta = \mathrm{atan}(y/x)$

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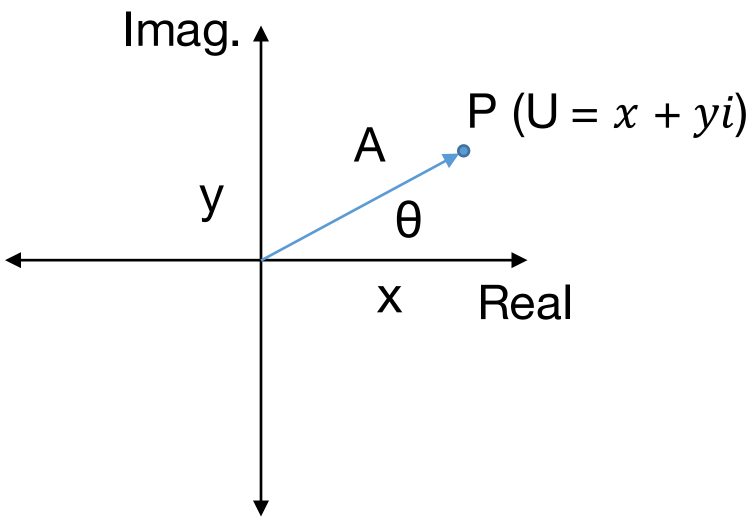

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More useful representation:

$x = A \cos\theta$
$y = A \sin\theta$
$U = A (\cos\theta + i \sin\theta)$

$\boxed{U = A e^{i\theta}}$

A = Amplitude of field

θ = Phase of field

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We'll work with complex signals of this form

Linear systems and the black box

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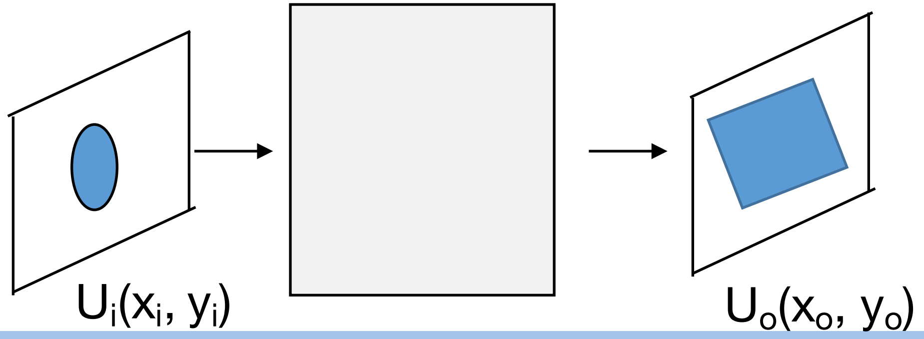
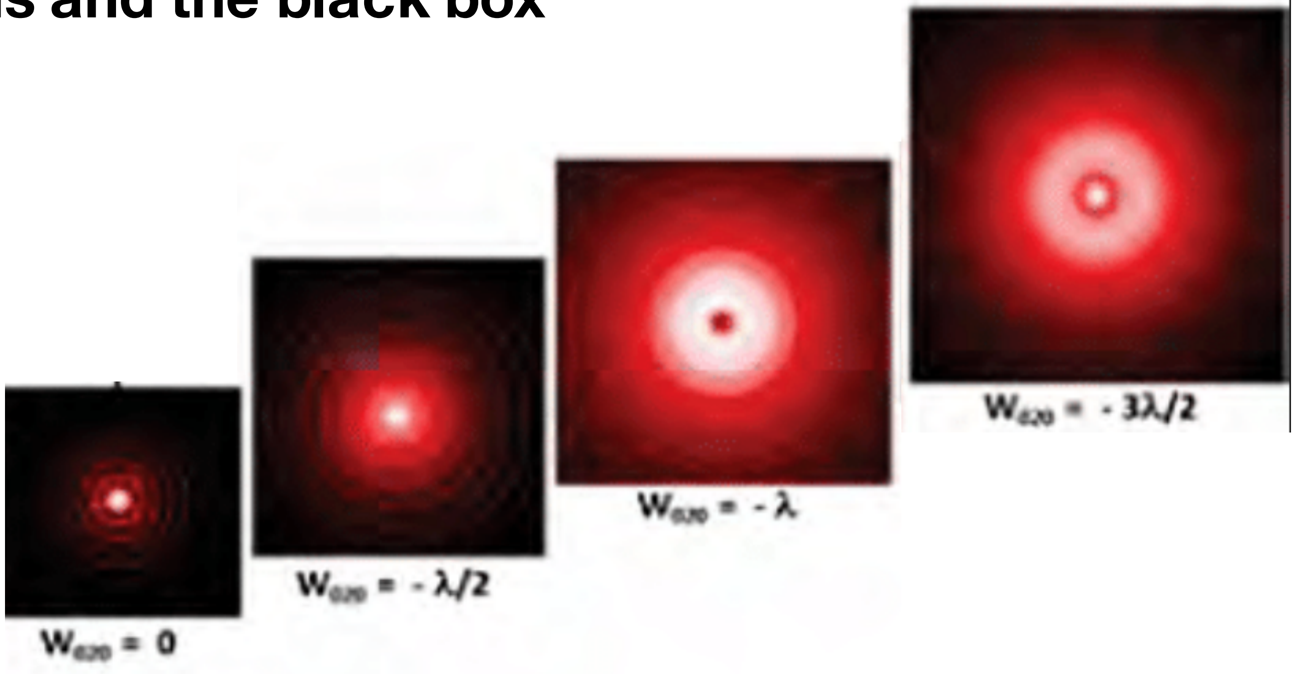
$$U(\mathbf{r}) \text{ -> } U(x,y)$$



Linear systems and the black box



Propagation of monochromatic light

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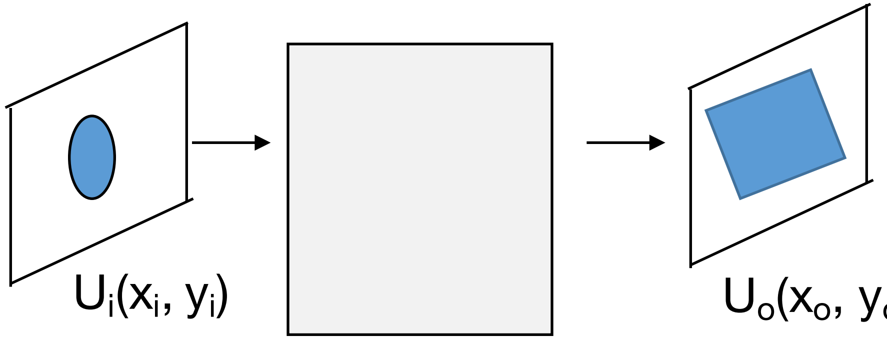


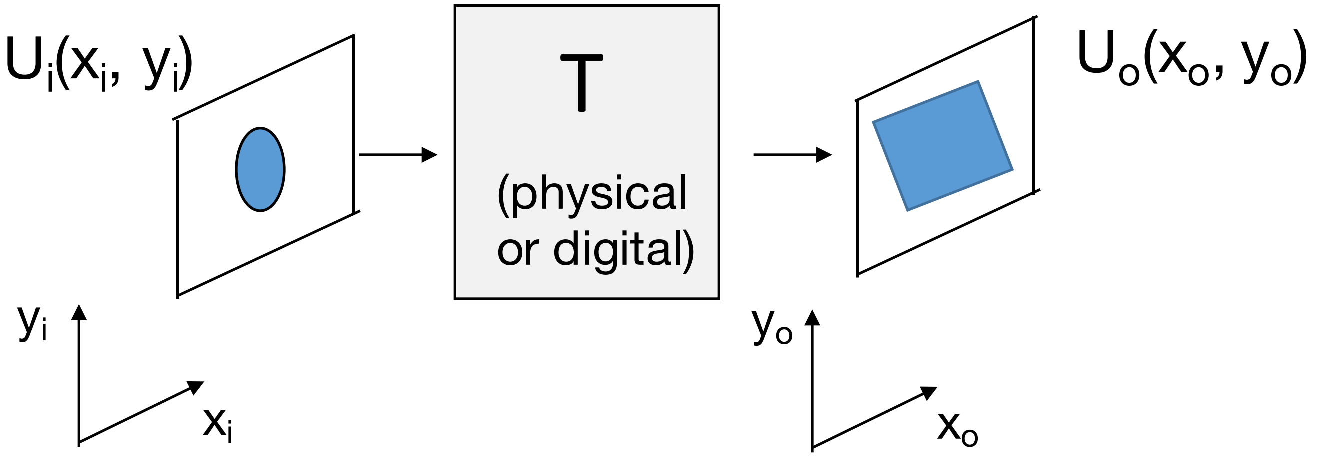
Linear systems and the black box

The “optical” black box system:

An optical black box system maps an input function $U_i(x_i,y_i)$ to an output function $U_o(x_o,y_o)$ via a transform T:

$$U_o(x_o,y_o) = T\,[\,U_i(x_i,y_i)\,]$$

Where T[ ] denotes the optical black box transformation



Linear systems and the black box

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Important properties of linear systems:

1. Homogeneity and additivity (superposition):

$$T\,[aU_1(x,y) + bU_2(x,y)] = aT\,[U_1(x,y)] + bT\,[U_2(x,y)]$$

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Linear systems and the black box

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Important properties of linear systems:

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$$T\,[aU_1(x,y) + bU_2(x,y)] = aT\,[U_1(x,y)] + bT\,[U_2(x,y)]$$

2. Shift invariance: for shift distances d_x and d_y, we assume that,

$$U_o(x_o - d_x, y_o - d_y) = T\,[U_i(x_i-d_x,\, y_i-d_y)]$$

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Black box transforms as a convolution

Assuming 1) linearity and 2) shift-invariance, we can model any black box with 1 piece of information:

Input Dirac delta function into the black box:

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

Black box transforms as a convolution

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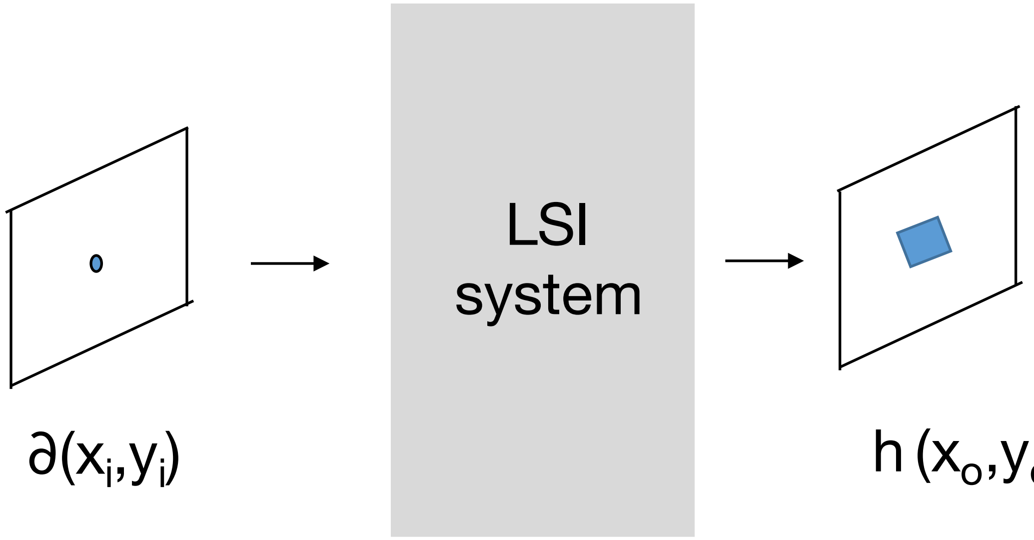
A "perfect" point source



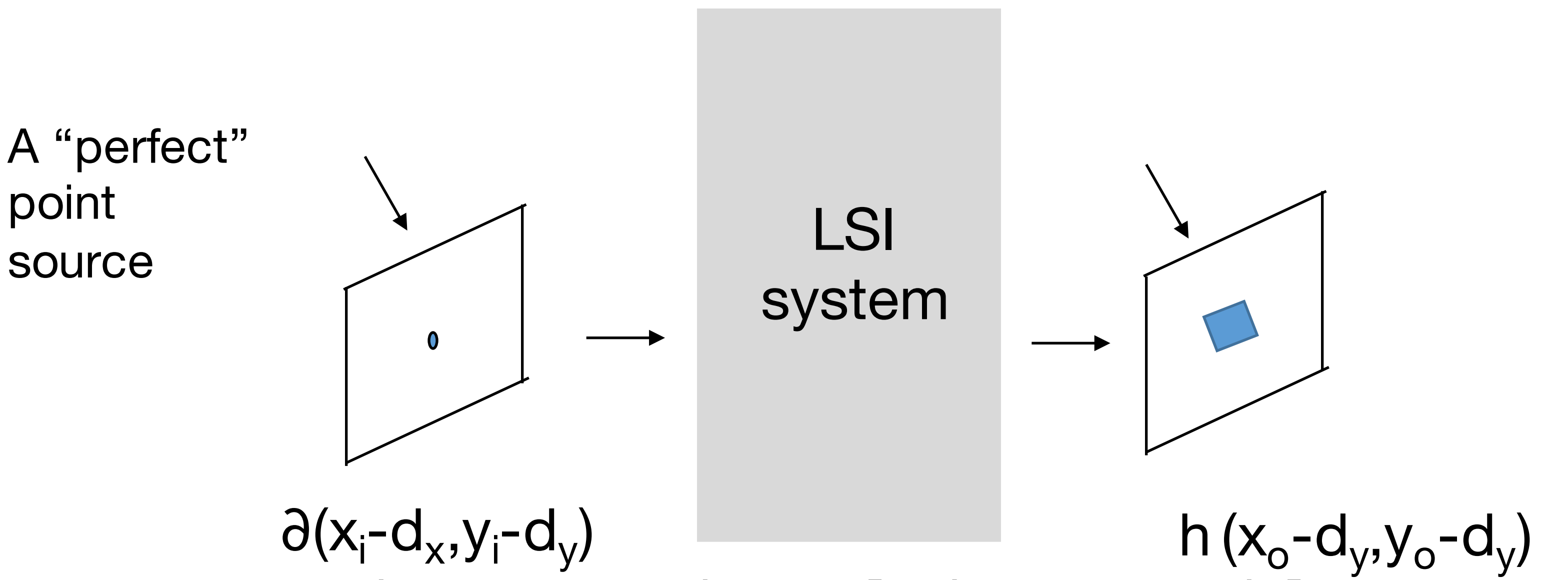
$\partial(x_i,y_i)$ $h\,(x_o,y_o)$

$$h(x_o,y_o) = T\,[\,\partial(x_i,y_i)\,]$$

Black box transforms as a convolution

Assuming 1) linearity and 2) shift-invariance, we can model any black box with 1 piece of information:

We know the system is shift invariant:



$$h(x_o-d_y,y_o-d_y) = T\,[\,\partial(x_i-d_x,y_i-d_y)\,]$$

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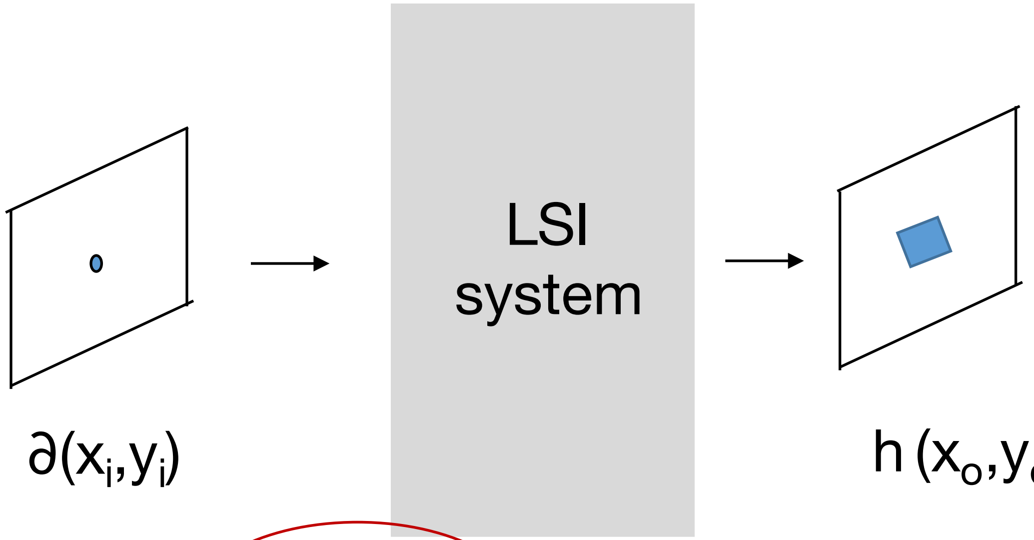
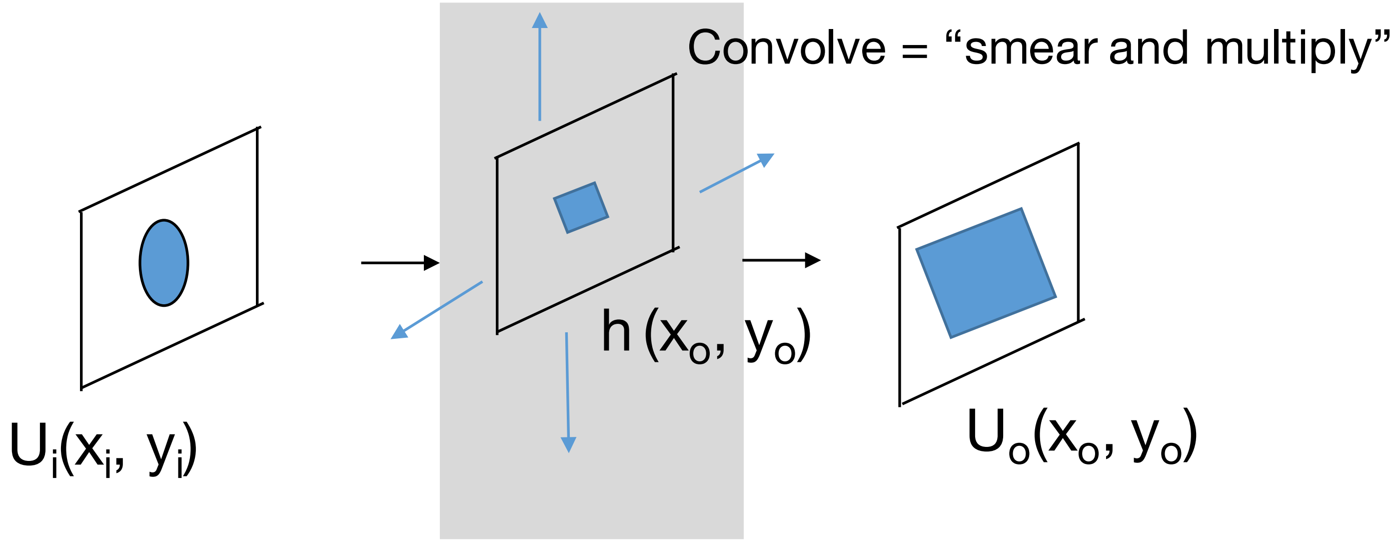


$h(x_o,y_o)$ is the system's point-spread function

Point-spread function $h(x_o,y_o) = T\,[\,\partial(x_i,y_i)\,]$

Black box transforms as a convolution

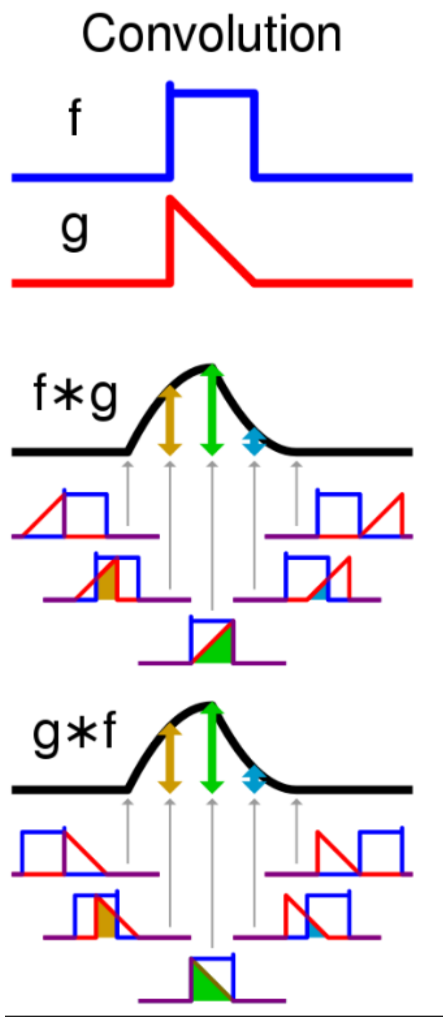
Knowing the point-spread function, it is direct to model any output of the black box, given an input:



$$U_o(x_o, y_o) = \iint_{-\infty}^{\infty} U_i(x_i, y_i) h(x_o - x_i, y_o - y_i) dx_i dy_i$$

Output of linear system is a convolution of the input with its point-spread function

1D convolution example



https://en.wikipedia.org/wiki/Convolution

Steps to perform a convolution:

1. Flip one signal (the second one = the PSF)
2. Position PSF right before overlap

With incremental steps:

3. Step PSF over to position x_o
4. Compute *area* of overlap of two functions
5. Convolution value at x_o= area of overlap
6. Repeat 3-5 until signals do not overlap

2D convolution example

- Direct extension of 1D concept to 2D functions

- Note – it is effectively the same with discrete functions = matrices

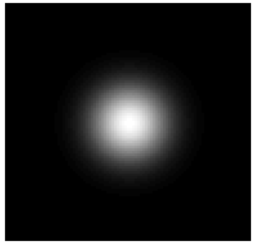
$U_1(x,y)$

y

x

$*$

y2



x2

$=$

$U_0(x,y)$

y

x

Next Lecture: Analyzing light and image formation via Fourier transforms!