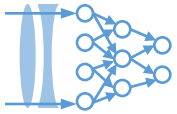


Machine Learning in Imaging

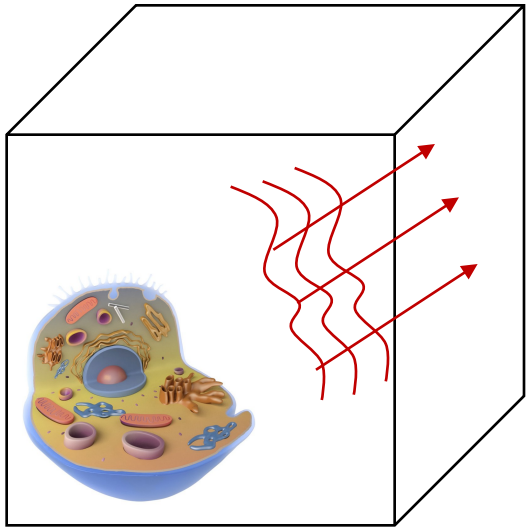
BME 590L
Roarke Horstmeyer

Lecture 4: Mathematical preliminaries for discrete functions

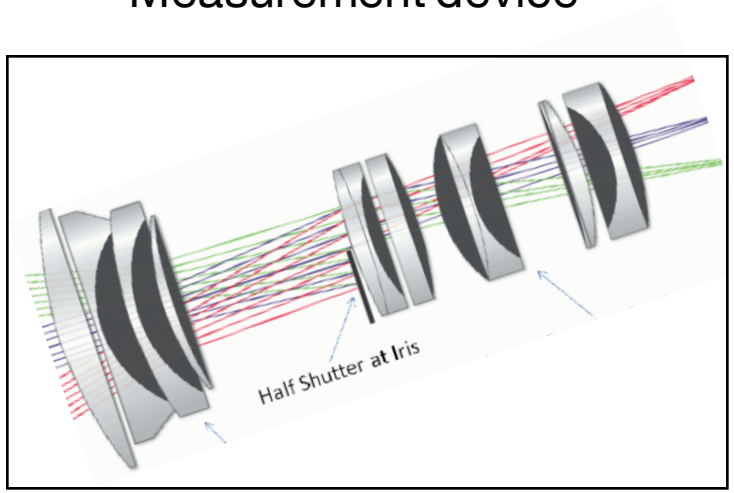
ML+Imaging pipeline introduction

deep imaging

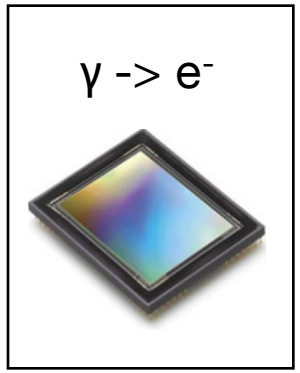
Real World



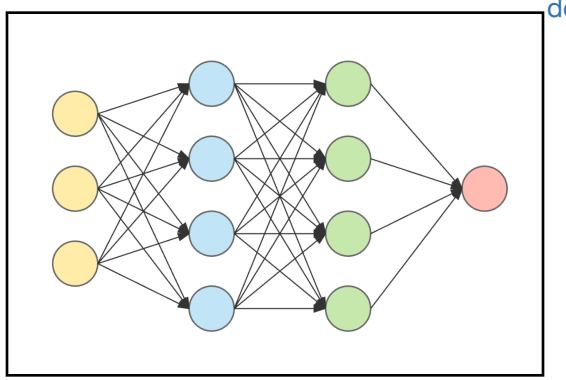
Measurement device



Digitization

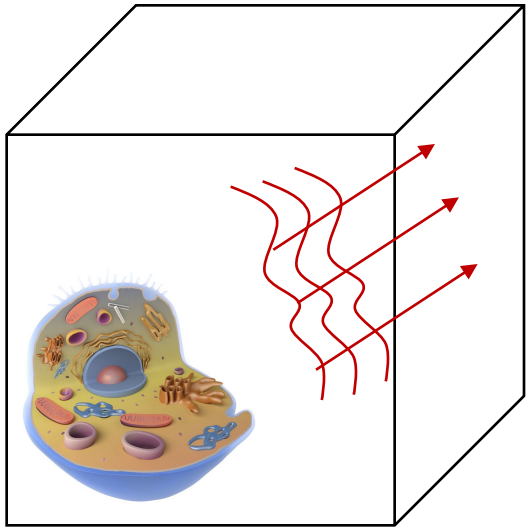


Machine Learning



ML+Imaging pipeline introduction

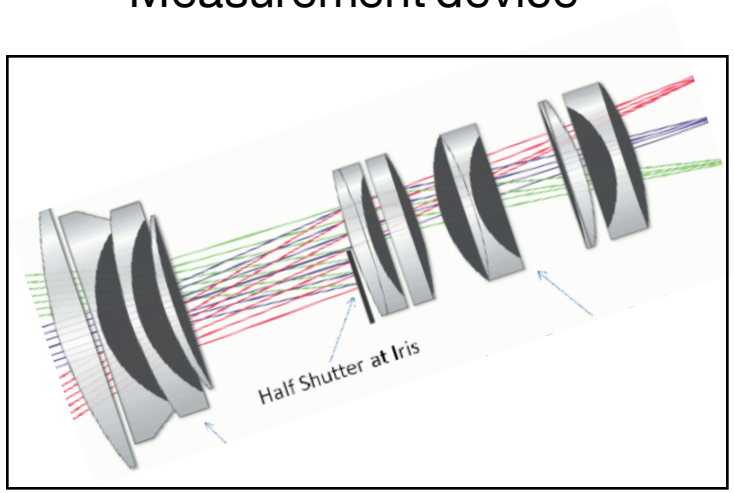
Real World



Continuous complex fields

(last class)

Measurement device

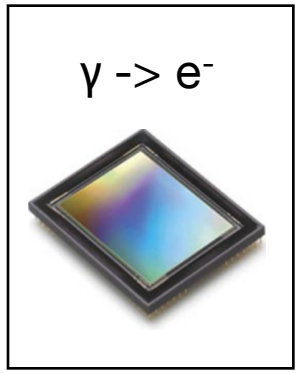


Black box transformations

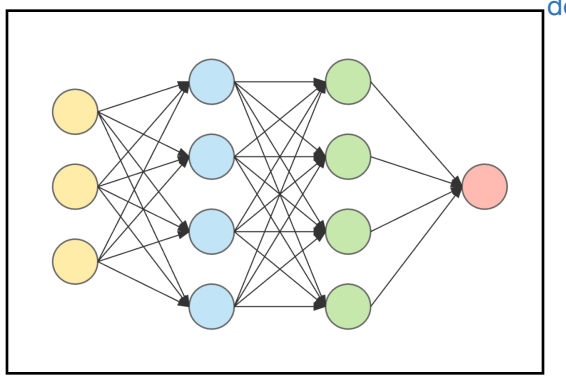
- Convolution
- Fourier Transform

(last class, this class)

Digitization



Machine Learning

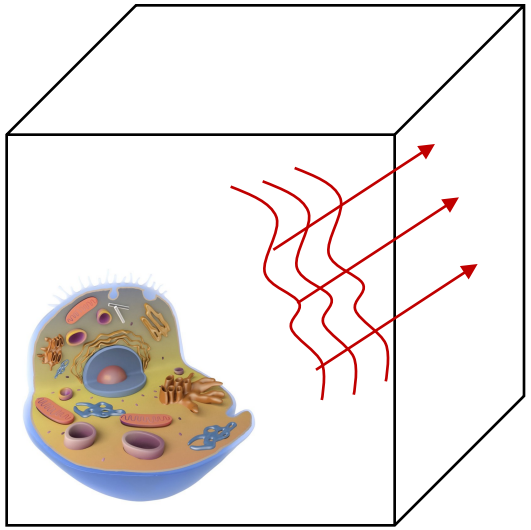


deep imaging

ML+Imaging pipeline introduction

deep imaging

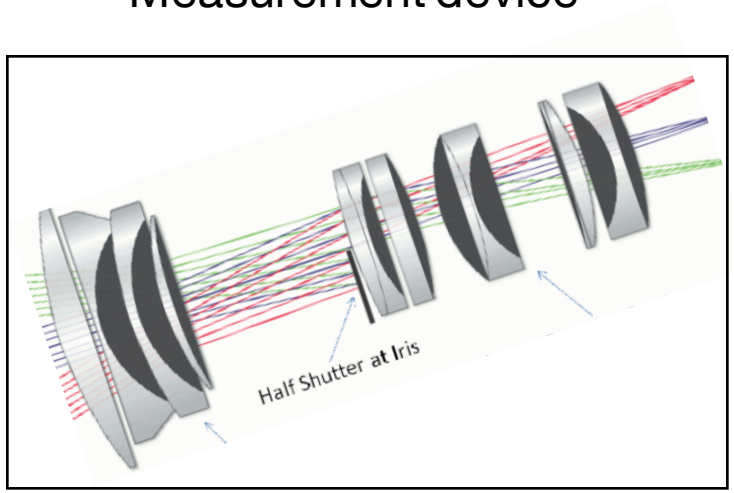
Real World



Continuous complex fields

(last class)

Measurement device

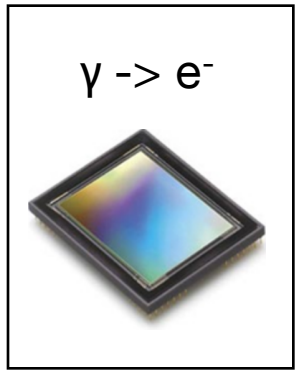


Black box transformations

- Convolution
- Fourier Transform

(last class, this class)

Digitization

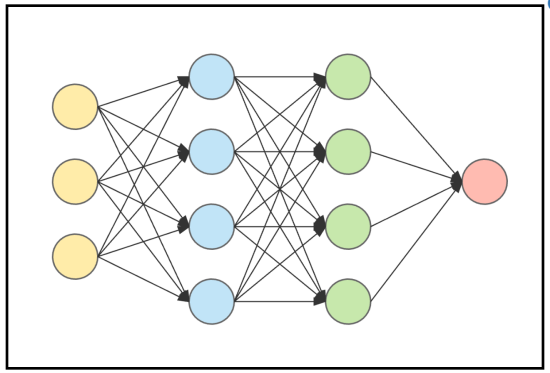


Sampling Theorem

Discrete math &
Linear algebra

(this class)

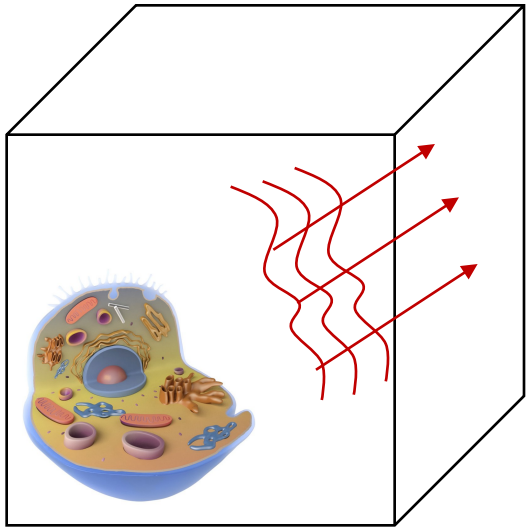
Machine Learning



ML+Imaging pipeline introduction

deep imaging

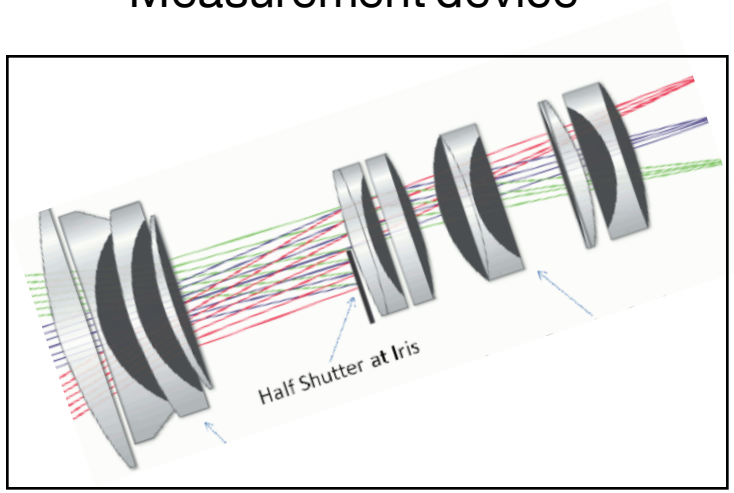
Real World



Continuous complex fields

(last class)

Measurement device

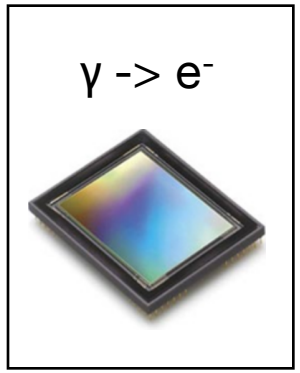


Black box transformations

- Convolution
- Fourier Transform

(last class, this class)

Digitization

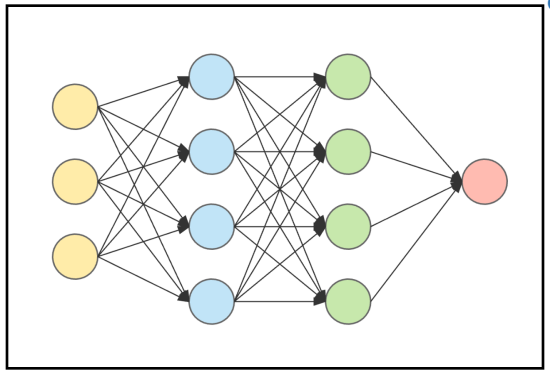


Sampling Theorem

Discrete math & Linear algebra

(this class)

Machine Learning



Optimization

Linear classification

Logistic classifier

Neural networks

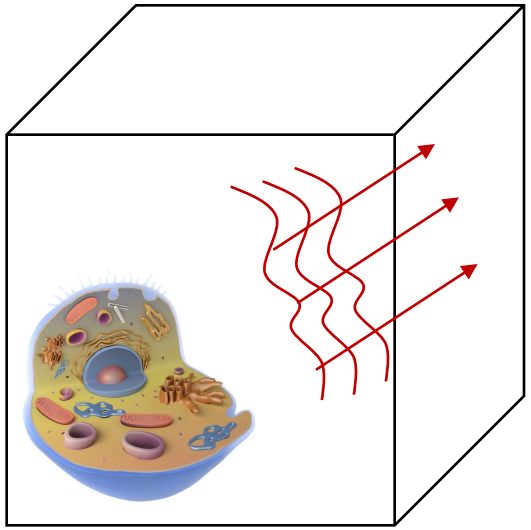
Convolutional NN's

(next few weeks)

ML+Imaging pipeline introduction

deep imaging

Real World



Continuous complex fields

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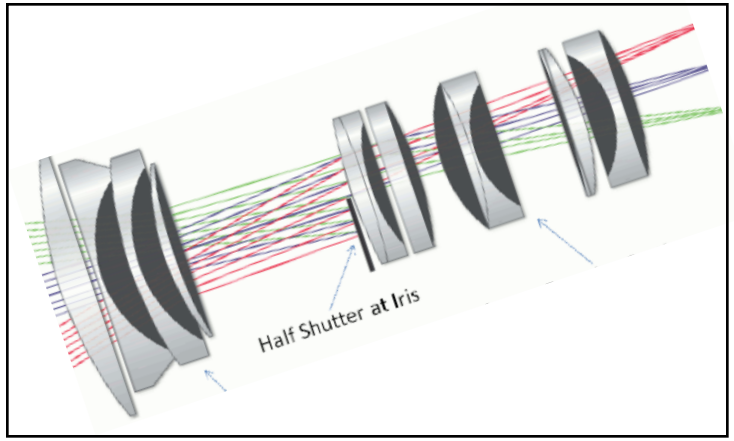
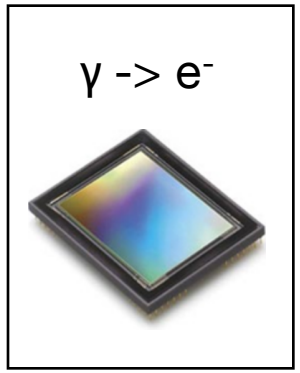


Black box transformations

- Convolution
- Fourier Transform

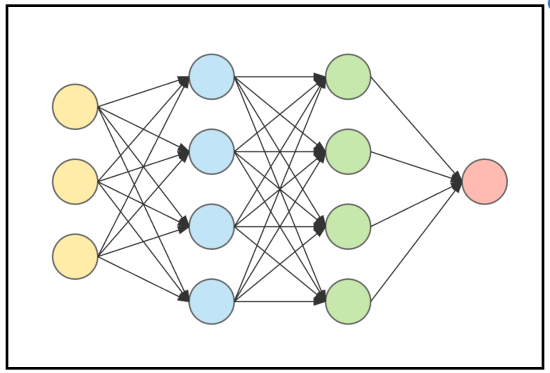
Digitization



Sampling Theorem

Discrete math & Linear algebra

Machine Learning



Optimization

Linear classification

Logistic classifier

Neural networks

Convolutional NN's

(last class) → (last class, this class) → (this class) → (next few weeks)

October → November

Review: signals as complex fields

Simplification #1: Let's forget about light changing as a function of time. It does so way too fast, and way too slow:

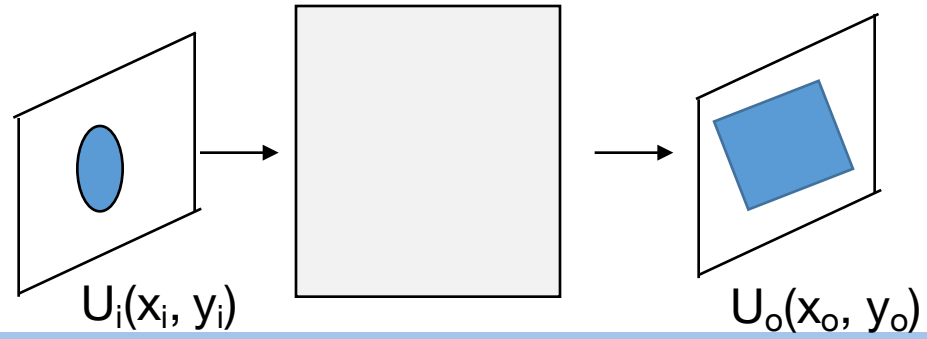
$$A(\mathbf{r})\cos(\mathbf{kr} - \omega t) \rightarrow A(\mathbf{r})\cos(\mathbf{kr})$$

Simplification #2: We'll use complex numbers when required, it'll make our lives easier. This leads to the *complex* field, U(**r**):

$$A(\mathbf{r})\cos(\mathbf{kr}) \leftrightarrow A(\mathbf{r})\, e^{i\mathbf{k}\cdot\mathbf{r}} = U(\mathbf{r})$$

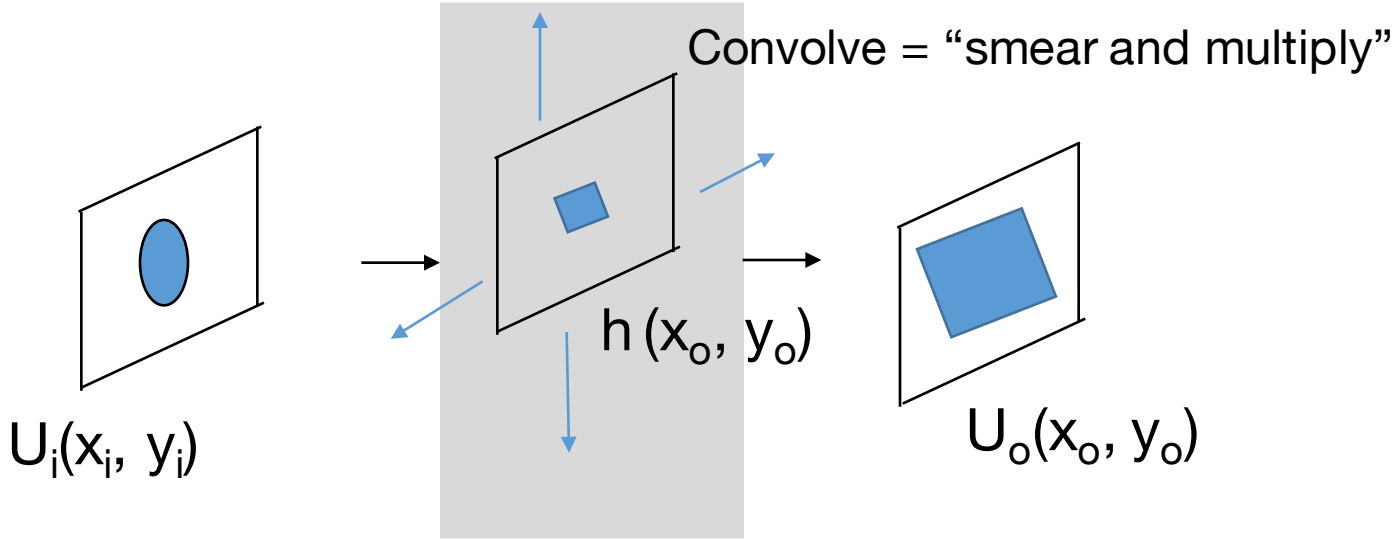
Simplification #3: Just consider mappings between planes across space. This is a critically important way of thinking for optics. Think "index card 1 to index card 2".

$$U(\mathbf{r}) \rightarrow U(x,y)$$



Review: black box transforms as a convolution

Knowing the point-spread function, it is direct to model any output of the black box, given an input:



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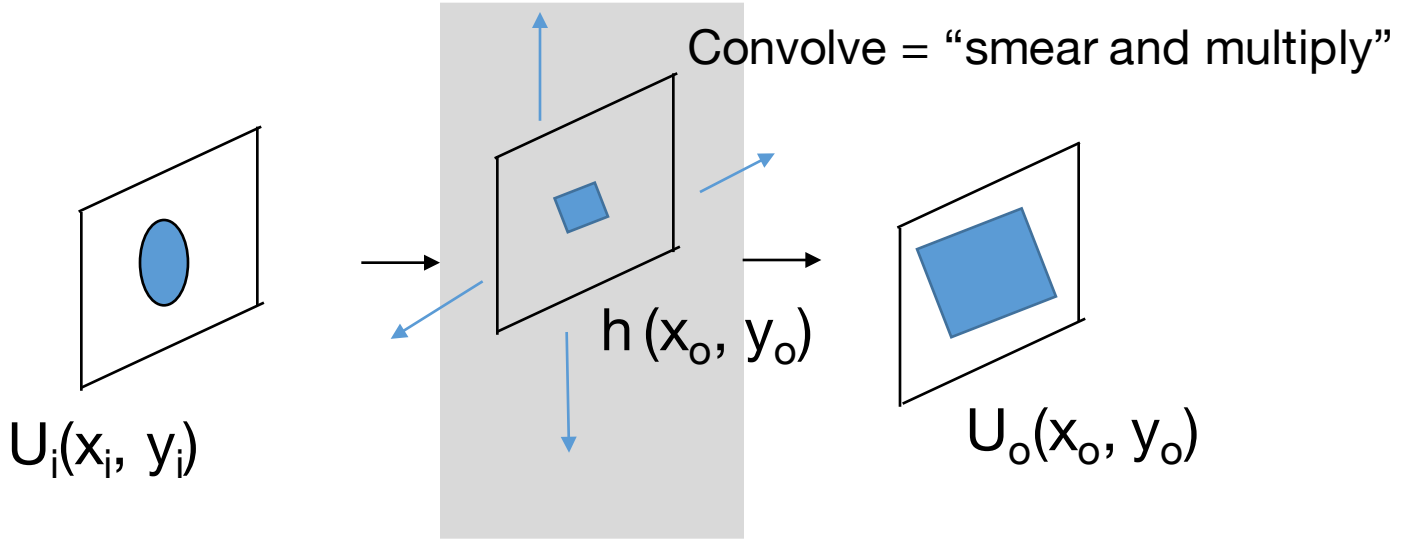
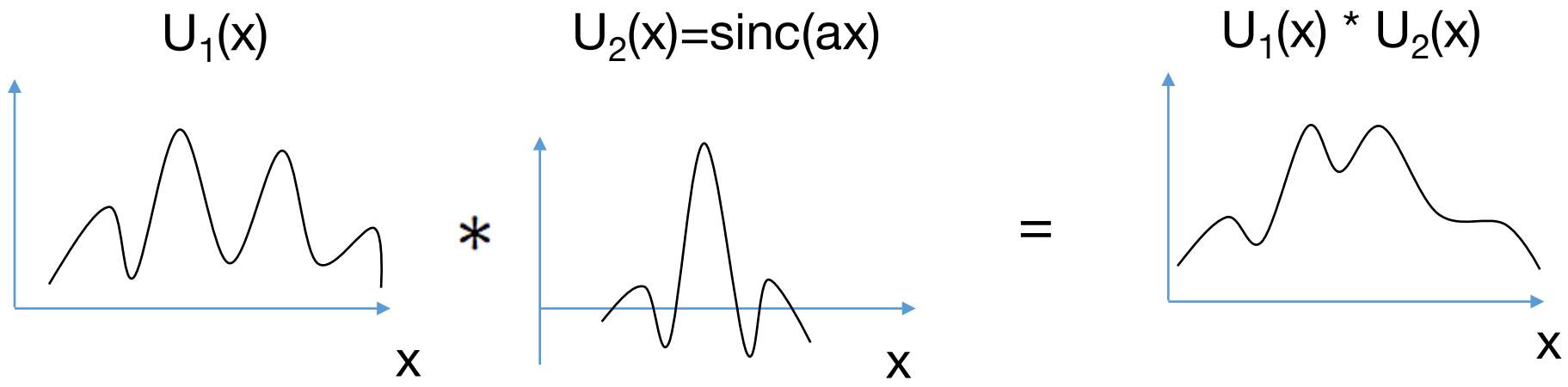


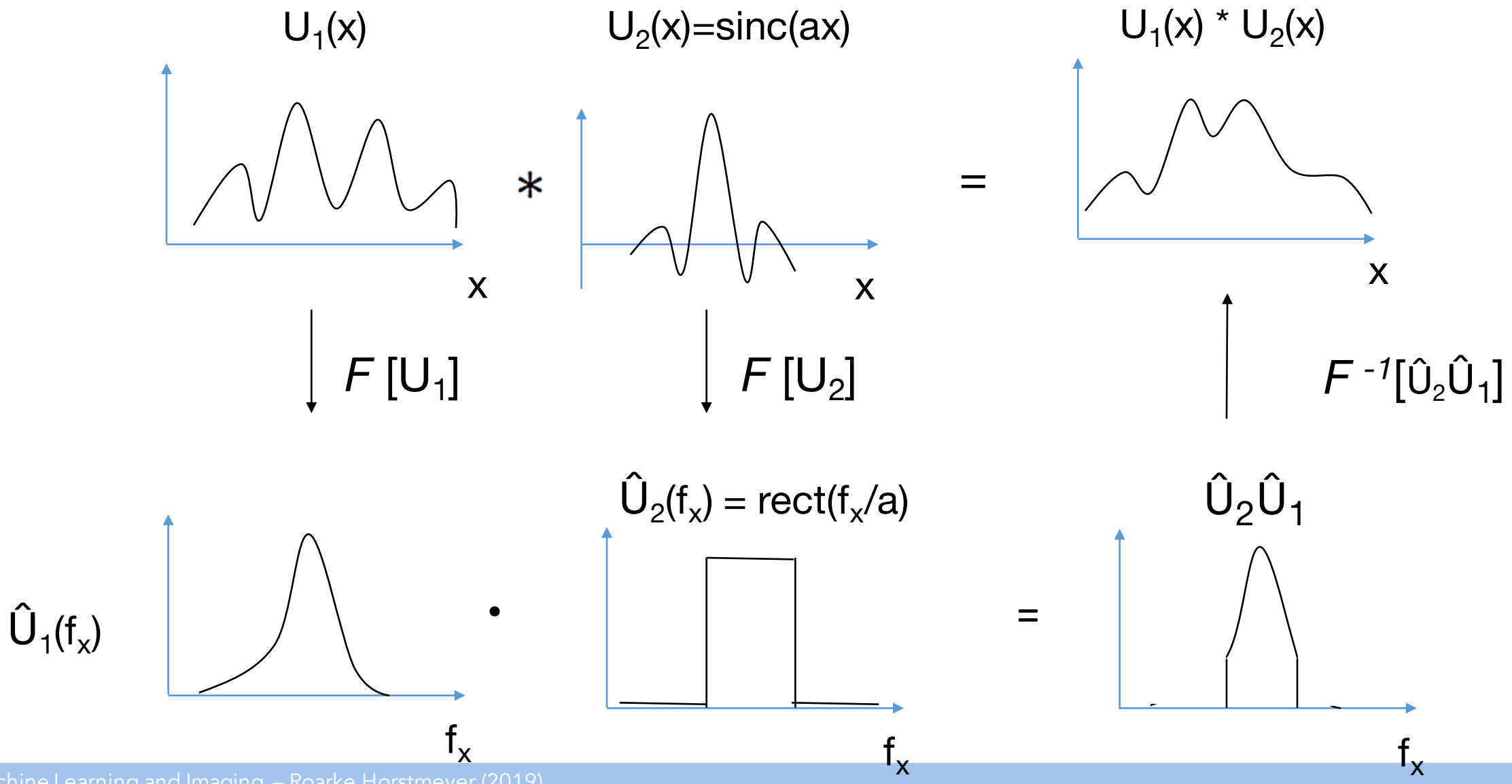
$$U_o(x_o, y_o) = \iint_{-\infty}^{\infty} U_i(x_i, y_i) h(x_o - x_i, y_o - y_i) dx_i dy_i$$

Output of linear system is a convolution of the input with its point-spread function

Review: Convolution theorem



Review: Convolution theorem



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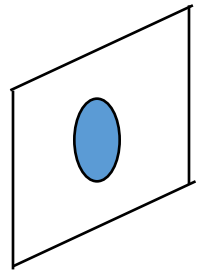
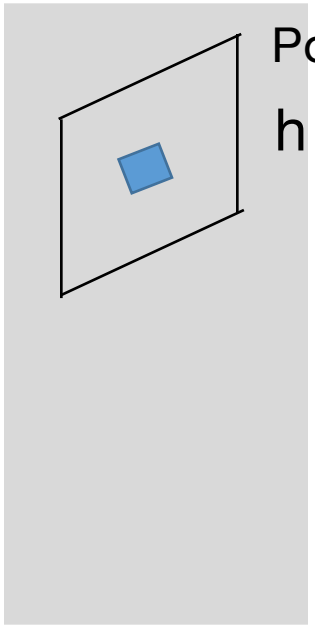


$U_i(x_i, y_i)$



Review: black box transforms as a convolution

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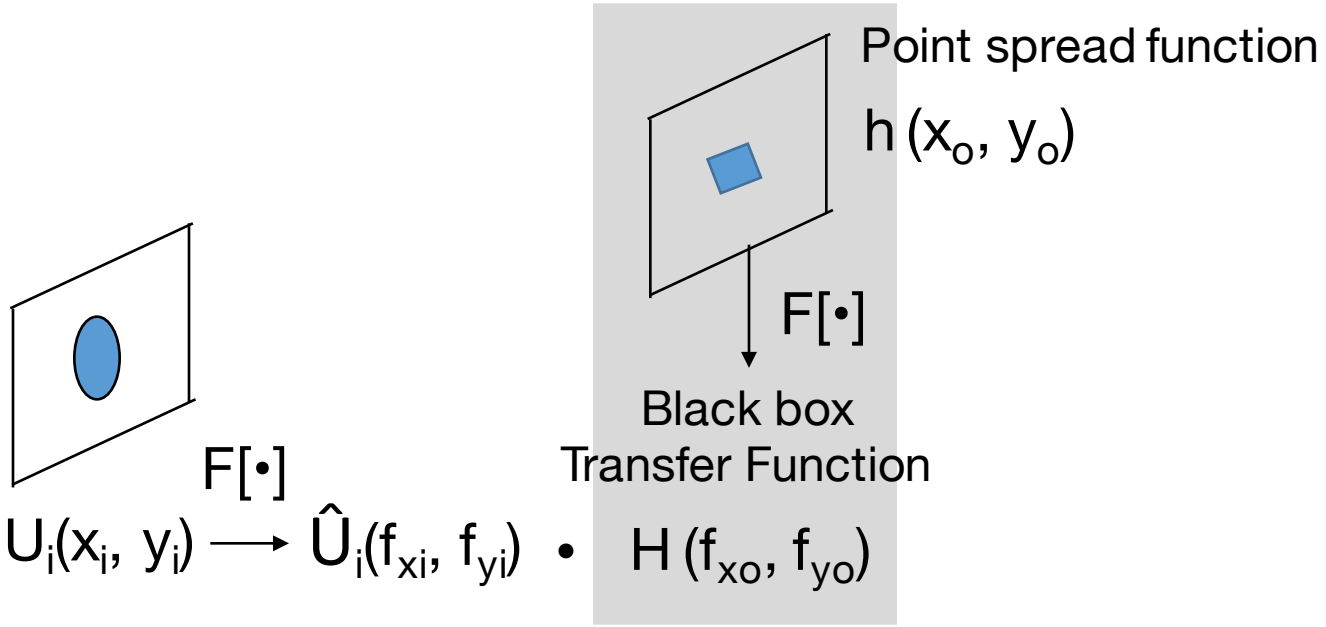

Review: black box transforms as a convolution

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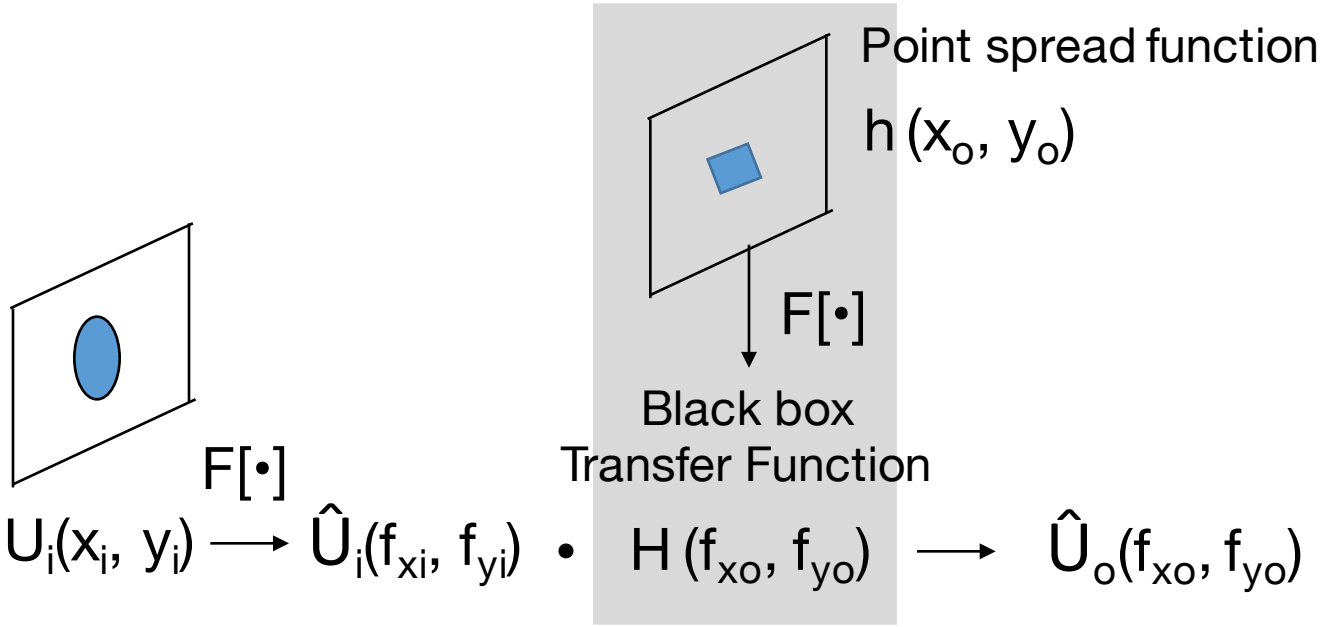

$$\hat{U}_o(f_x, f_y) = \hat{U}_i(f_x, f_y) H(f_x, f_y)$$

Can also multiply Fourier transform of input with transfer function H to obtain Fourier transform of output

Review: black box transforms as a convolution

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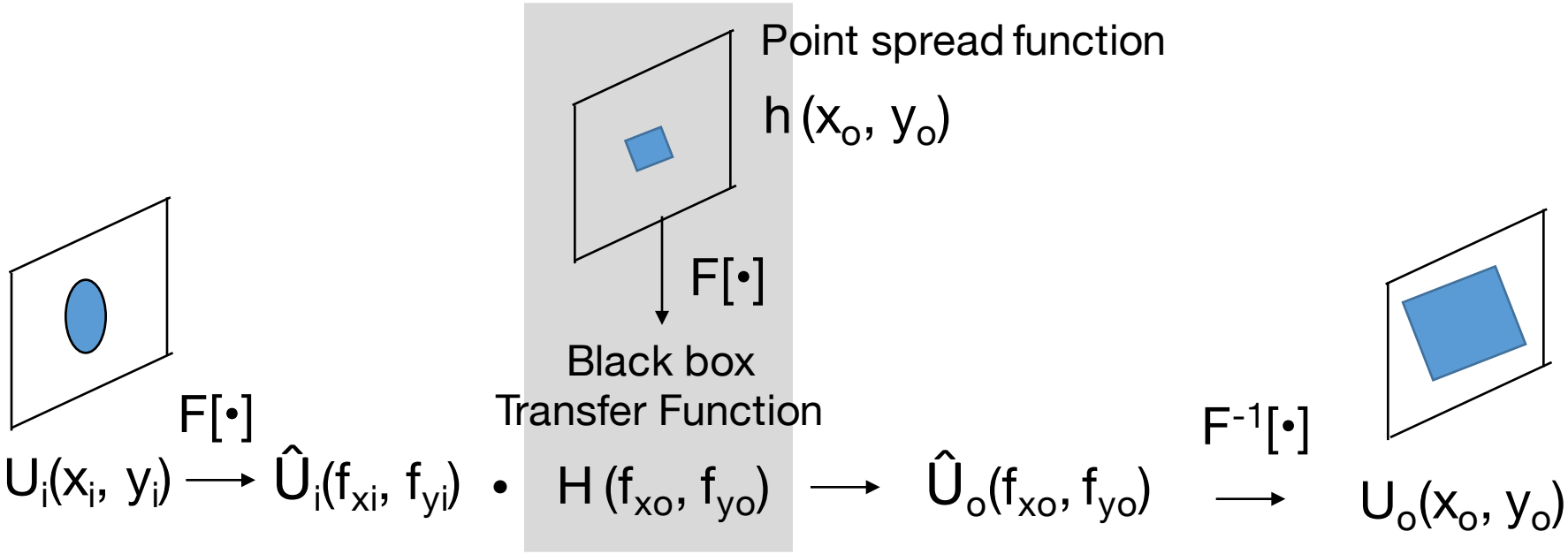

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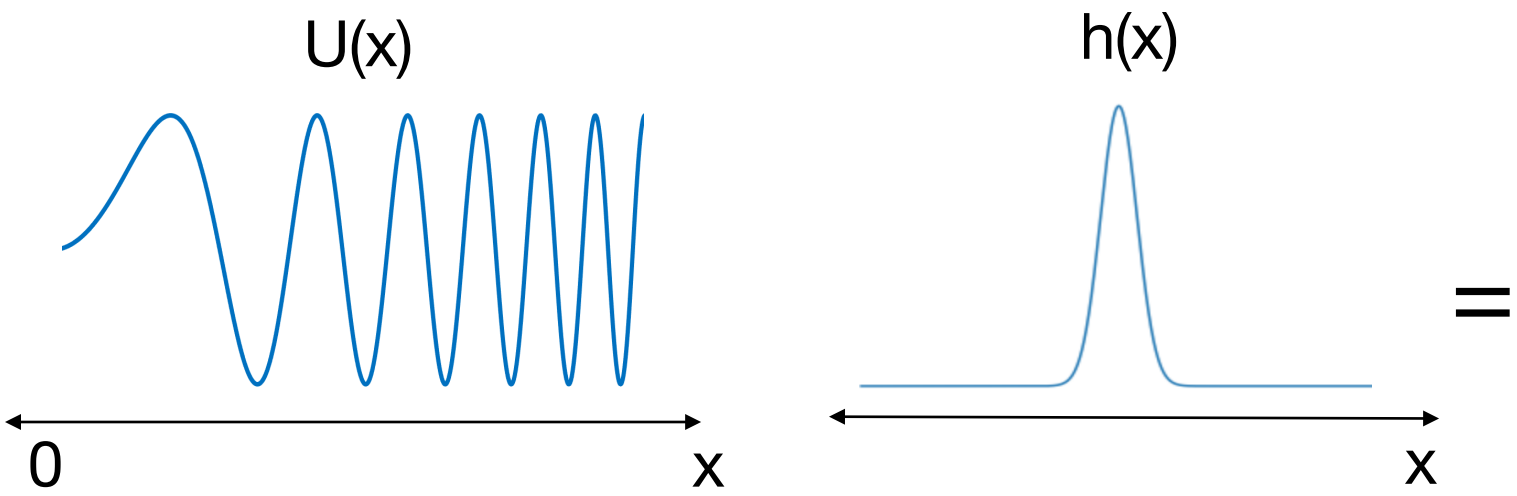
Can also multiply Fourier transform of input with transfer function H to obtain Fourier transform of output

Conceptual questions for the class:

1. Draw what you think the convolution of these two functions looks like:



deep imaging

Conceptual questions for the class:

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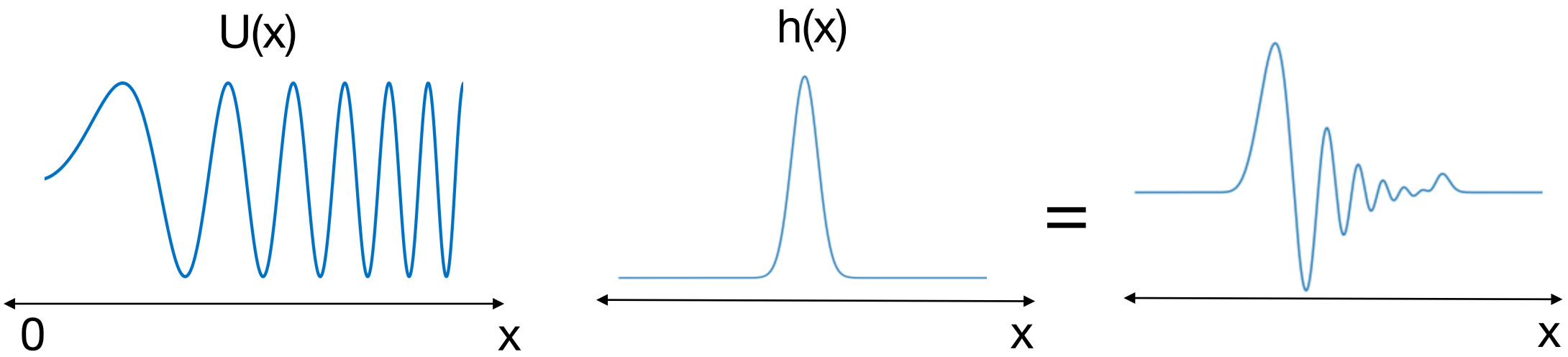
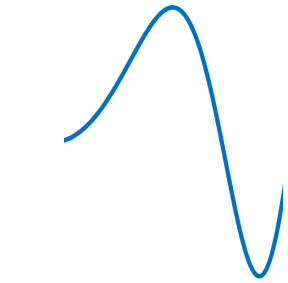
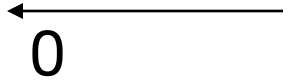


Conceptual questions for the class:

deep imaging

1. Dra





In [1]:
```
import numpy as np
```

In [50]:
```
x = np.linspace(0,1,1000)
```

In [72]:
```
U=np.sin(40*np.square(x))
V=np.exp(-300*np.square(x-0.5))
```

In [73]:
```
W=np.convolve(U,V)
```

In [74]:
```
import matplotlib.pylab as plt
```

In [75]:
```
plt.plot(np.linspace(0,1,1999),W)
```

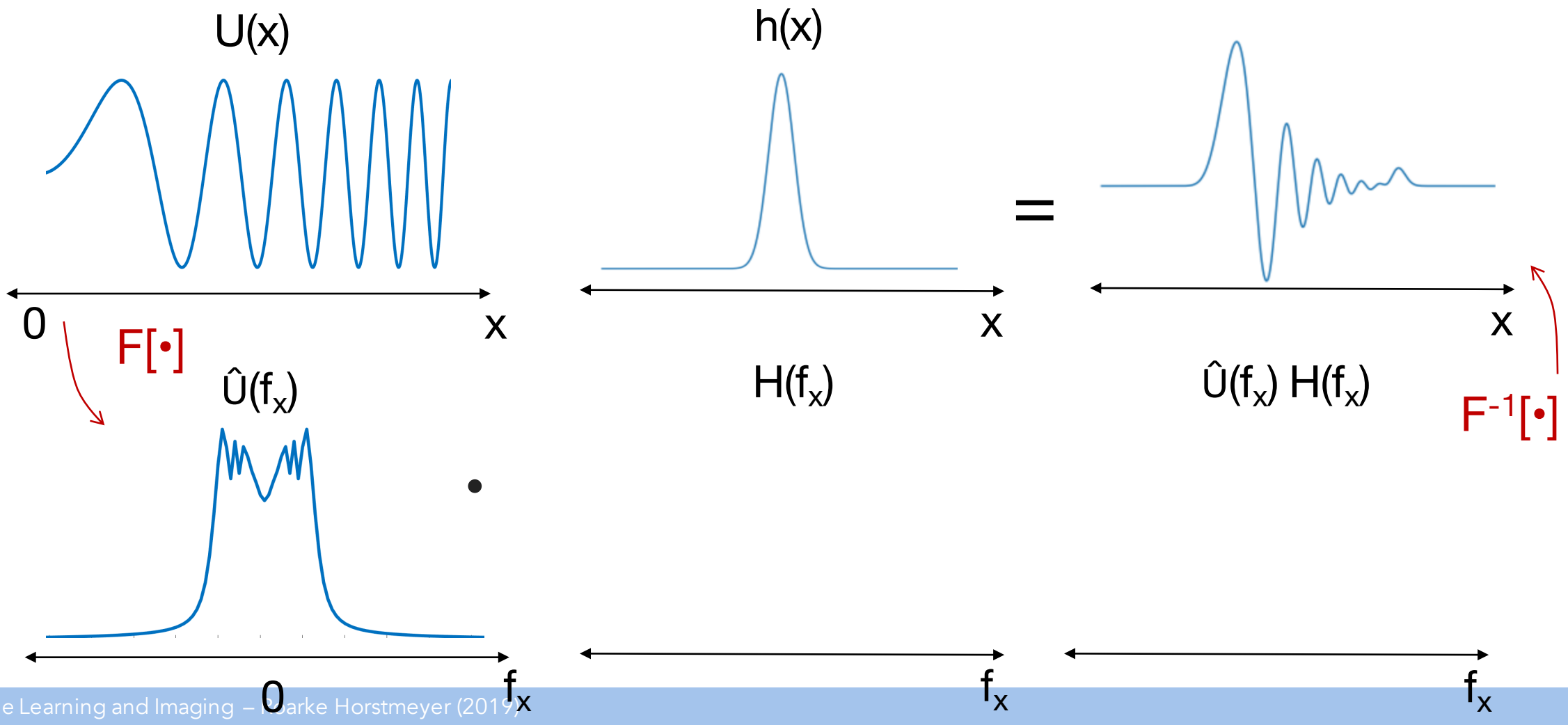
Out[75]:
```
[<matplotlib.lines.Line2D at 0x108f42828>]
```

In [76]:
```
plt.show()
```

Conceptual questions for the class:

deep imaging

1. Draw what you think the convolution of these two functions looks like:



Conceptual questions for the class:

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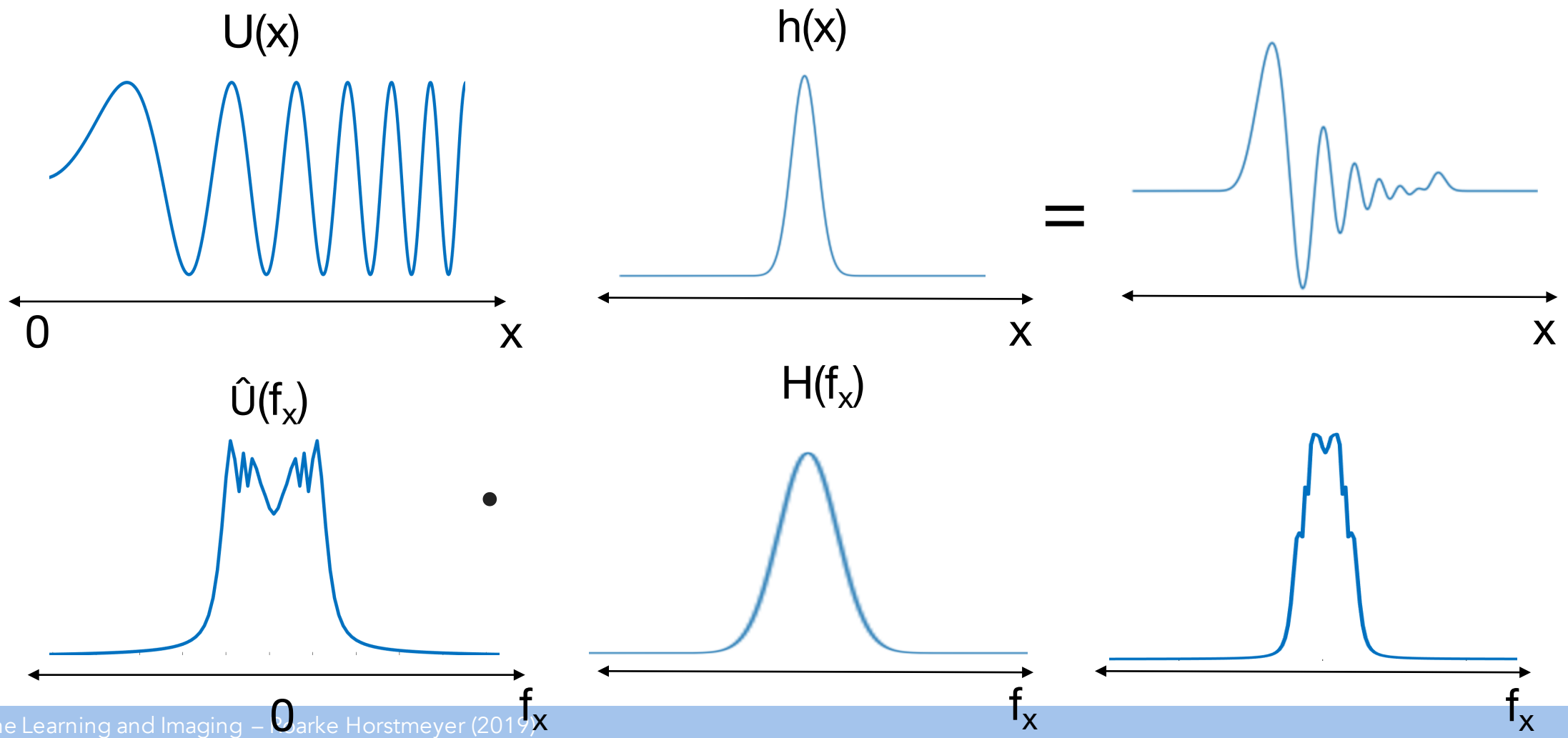
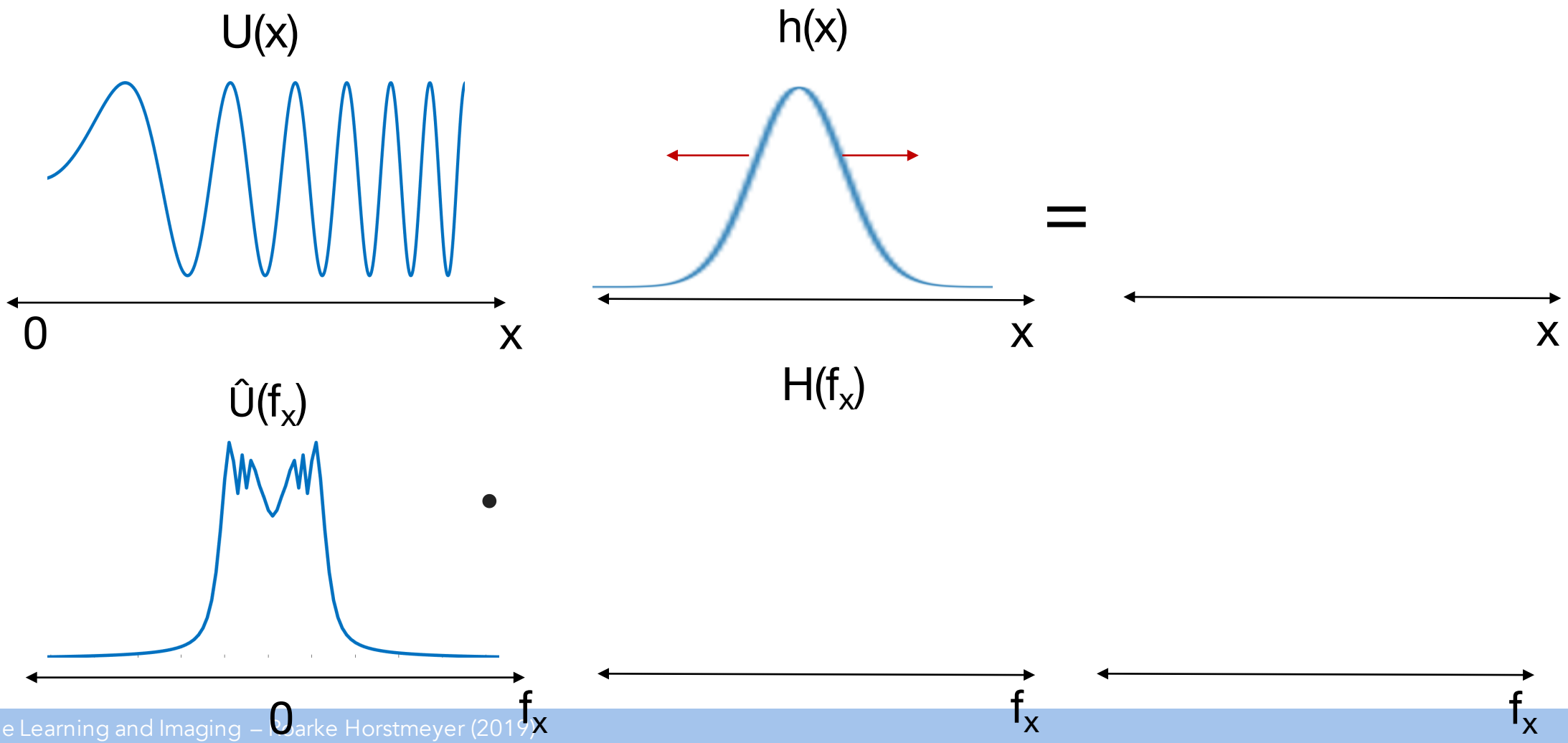


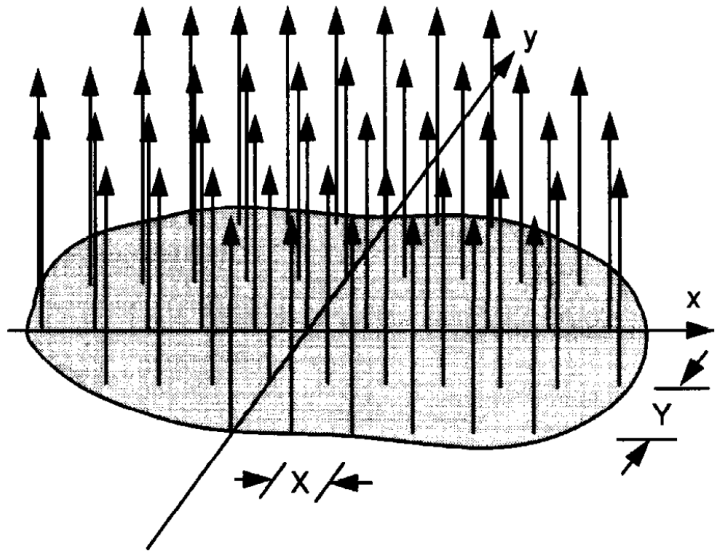
Conceptual questions for the class:

2. Repeat with wider convolution filter:



The Sampling Theorem – from Goodman Section 2.4.1

$$U_s(x,y) = \mathrm{comb}(x/X)\mathrm{comb}(y/Y)U(x,y)$$



Signal sampling occurs with:

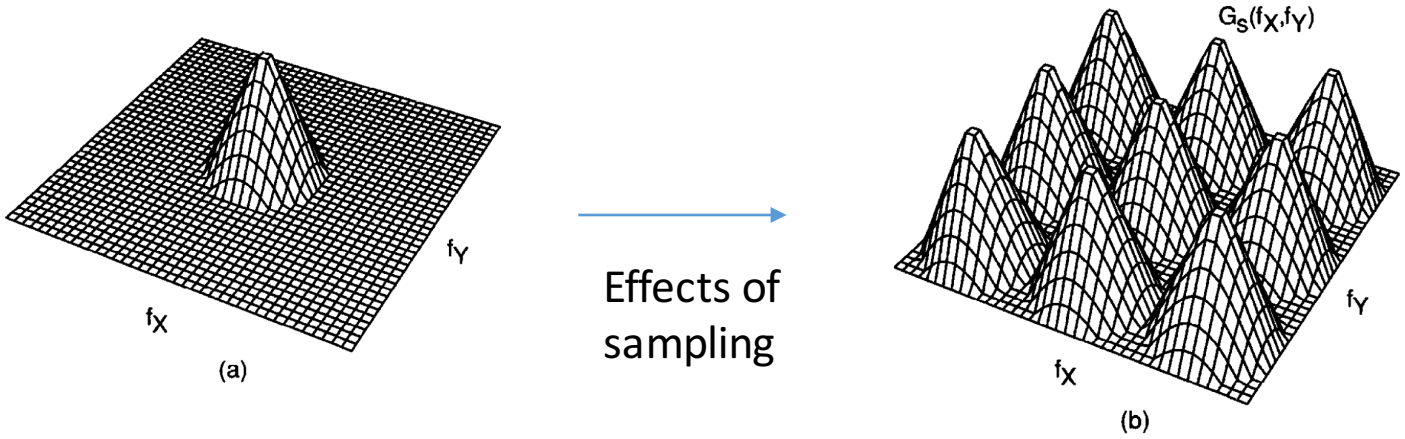
- CMOS (pixel) sensors, PMTs, SPADs
- A-to-D after antennas
- A-to-D after acoustic transducers

Sampling interval width X and Y

The Sampling Theorem – from Goodman Section 2.4.1

$$\mathcal{F}\left[\mathrm{comb}(x/X)\mathrm{comb}(y/Y)\right] = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\delta\left(f_x - \frac{n}{X}, f_y - \frac{m}{Y}\right)$$

$$\hat{U}_s(f_x, f_y) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\hat{U}\left(f_x - \frac{n}{X}, f_y - \frac{m}{Y}\right)$$



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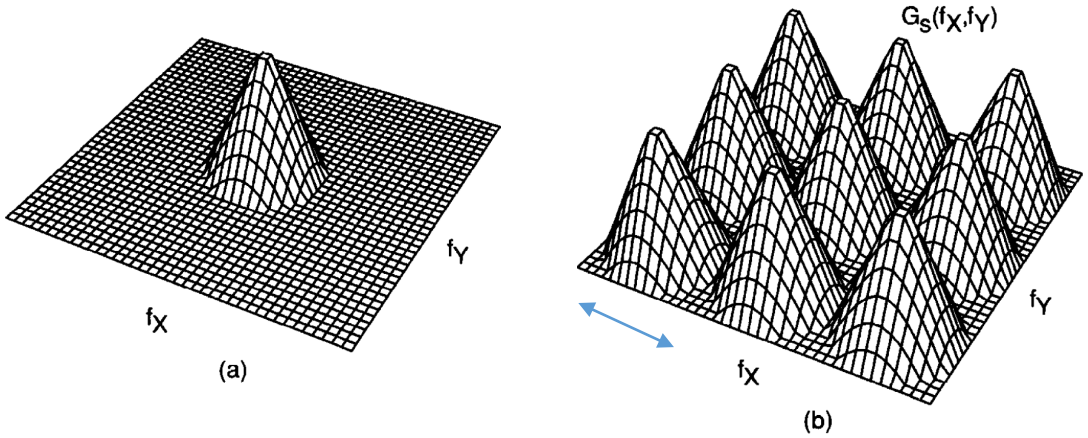


Signal extends from (-B_x, -B_y) to (B_x, B_y) in Fourier domain

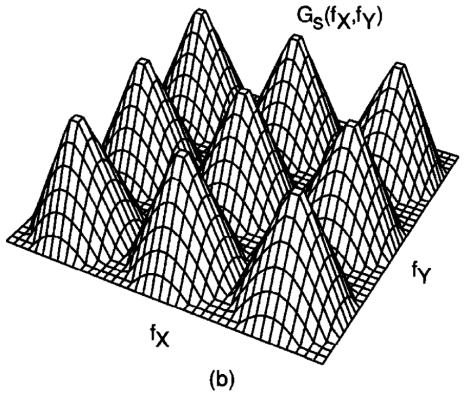
Filter out extra copies:

$$\mathrm{rect}\left(\frac{f_x}{2B_x}\right)\mathrm{rect}\left(\frac{f_y}{2B_y}\right)\hat{U}_s(f_x, f_y) = \hat{U}(f_x, f_y)$$

Bandwidth (B_x, B_y) of signal

The Sampling Theorem – from Goodman Section 2.4.1

$$\text{rect}\left(\frac{f_x}{2B_x}\right)\text{rect}\left(\frac{f_y}{2B_y}\right)\hat{U}_s(f_x,f_y)=\hat{U}(f_x,f_y)$$



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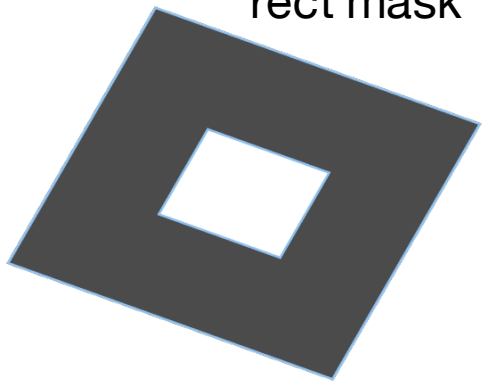


rect mask

$F[\bullet]$

$$h(x,y)=4B_xB_y\text{sinc}(2B_xx)\text{sinc}(2B_yy)$$

Can manipulate the above masking to show,

$$U(x,y)=4B_xB_yXY\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}U(nX,mY)\text{sinc}\left[2B_x(x-nX)\right]\text{sinc}\left[2B_y(y-mY)\right]$$

The Sampling Theorem

When sampled appropriately, a discrete signal can *exactly* reproduce a continuous signal:

$$U(x,y) = 4B_xB_yXY \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U(nX,mY)\,\mathrm{sinc}\left[2B_x(x-nX)\right]\mathrm{sinc}\left[2B_y(y-mY)\right]$$

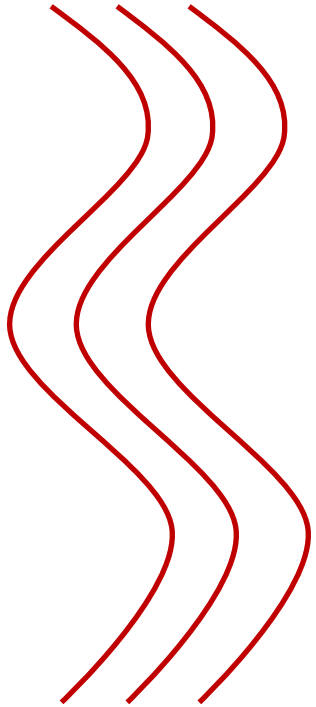
Continuous signal:
- EM field
- Sound wave
- MR signal

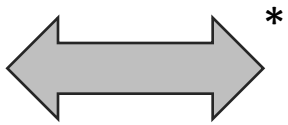
Discretized signal:
- Detected EM field
- Sampled sound wave
- Sampled MR signal

What does the Sampling Theorem mean for us?

deep imaging

Continuous fields

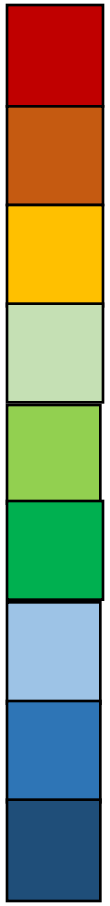


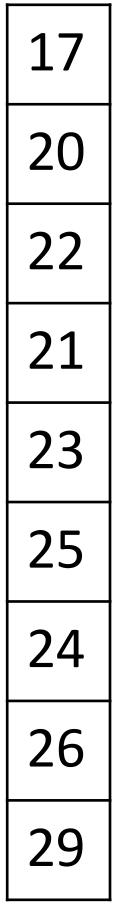


*

(*) Under certain conditions

Discretize vectors (and matrices)





Conditions to safely apply the sampling theorem

$$U(x,y) = 4B_x B_y XY \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U(nX, mY) \operatorname{sinc}\left[2B_x(x - nX)\right] \operatorname{sinc}\left[2B_y(y - mY)\right]$$

- Sampling must be proportional to bandwidth ($2B_x$ and $2B_y$)
  - "Nyquist" sampling: $X = 1/2B_x$, $Y = 1/2B_y$

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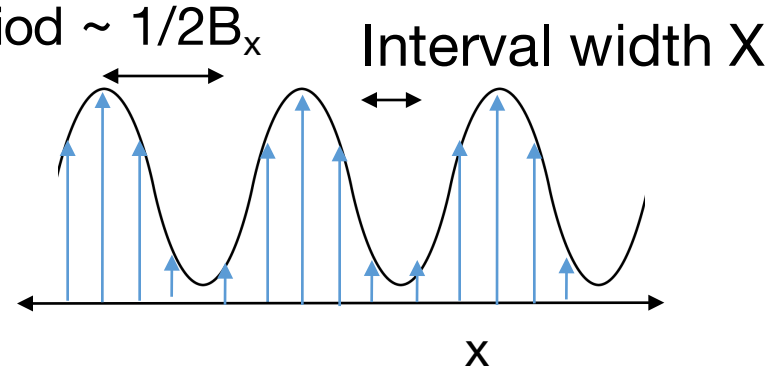
(Assume largest spatial freq. in signal here)



U(x)comb(x/X)

$X < 1/2B_x$ ✔

Conditions to safely apply the sampling theorem

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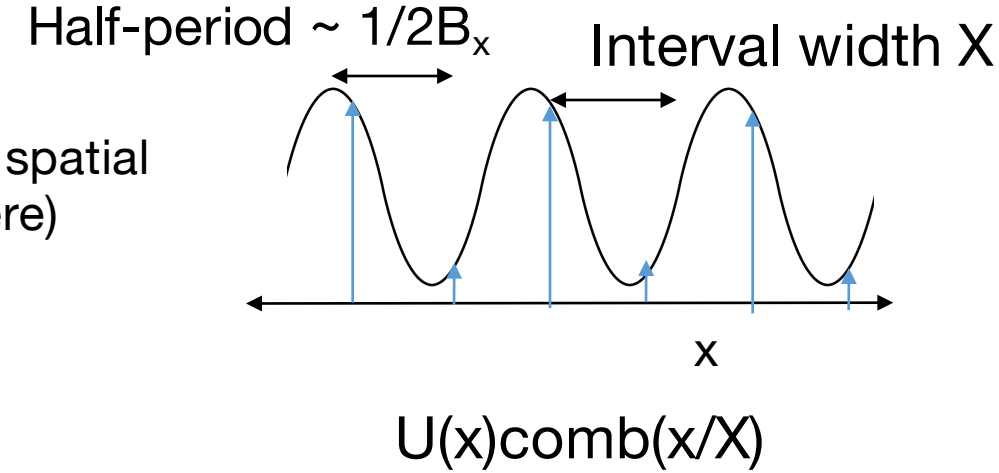

$X = 1/2B_x$ ✔

Nyquist sampling – still sampling peak and trough

Conditions to safely apply the sampling theorem

$$U(x,y) = 4B_xB_yXY \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U(nX,mY)\mathrm{sinc}\left[2B_x(x-nX)\right]\mathrm{sinc}\left[2B_y(y-mY)\right]$$

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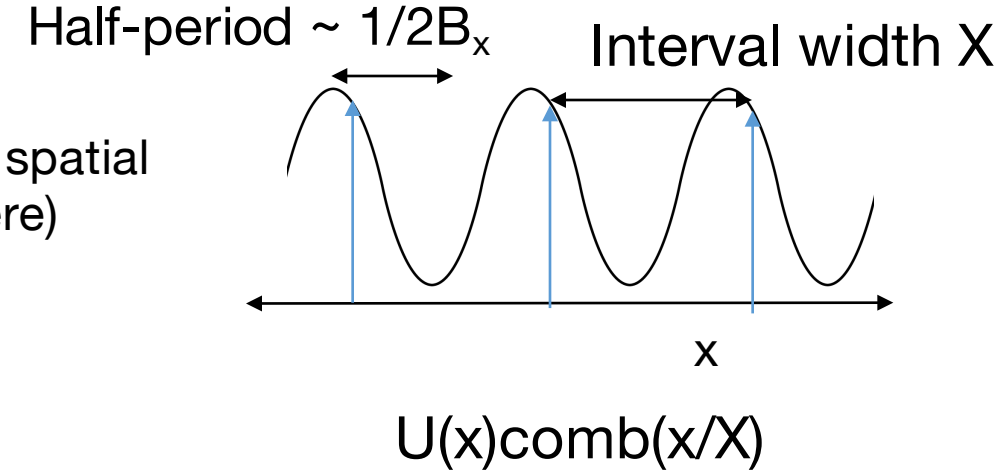


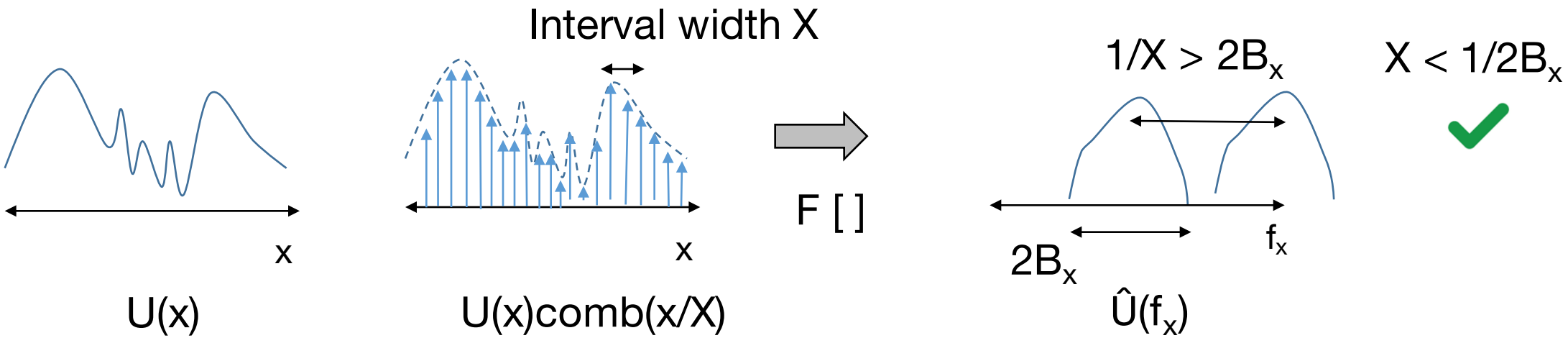
$X > 1/2B_x$ 

Can't detect the frequency anymore!

Conditions to safely apply the sampling theorem

$$U(x,y) = 4B_xB_yXY \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U(nX,mY)\text{sinc}\left[2B_x(x-nX)\right]\text{sinc}\left[2B_y(y-mY)\right]$$

- Sampling must be proportional to bandwidth ($2B_x$ and $2B_y$)
  - "Nyquist" sampling: $X = 1/2B_x$, $Y = 1/2B_y$
  - Needed to avoid aliasing



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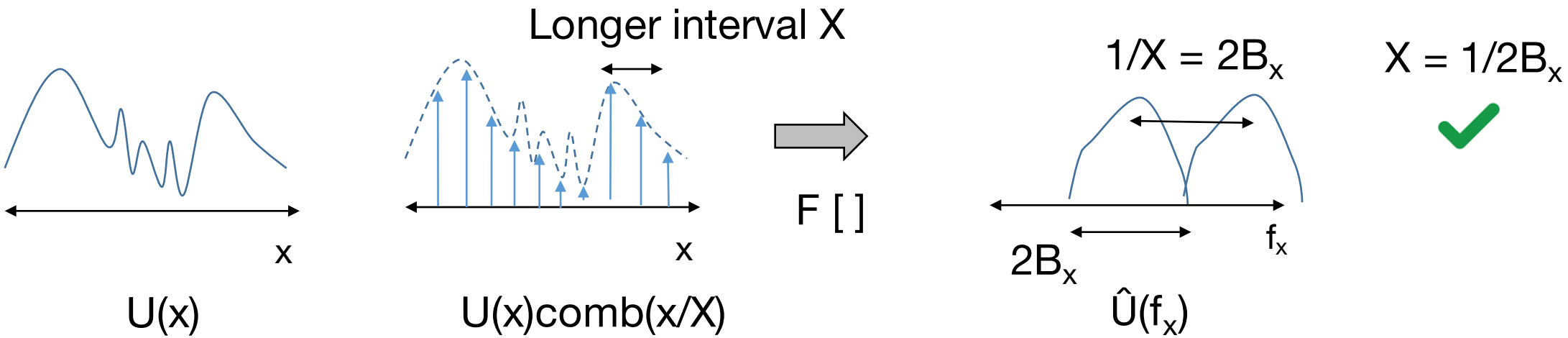

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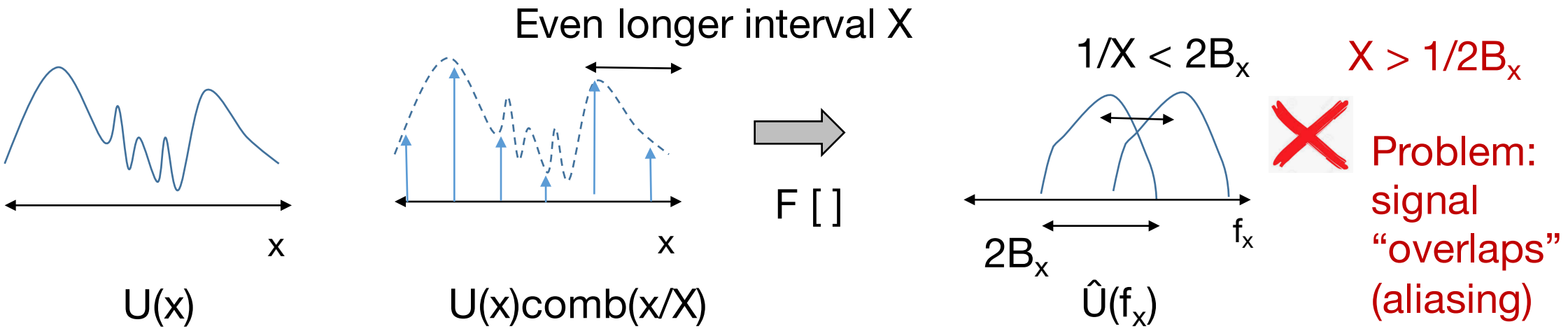


Linear Algebra – notation and basics

- We'll (try to) write *column* vectors as lower case variables
- Row vectors will be denoted as the transpose
- We'll try to write matrices as upper case variables
- We'll try to denote if a matrix/vector is real, complex etc. and its size with a certain notation

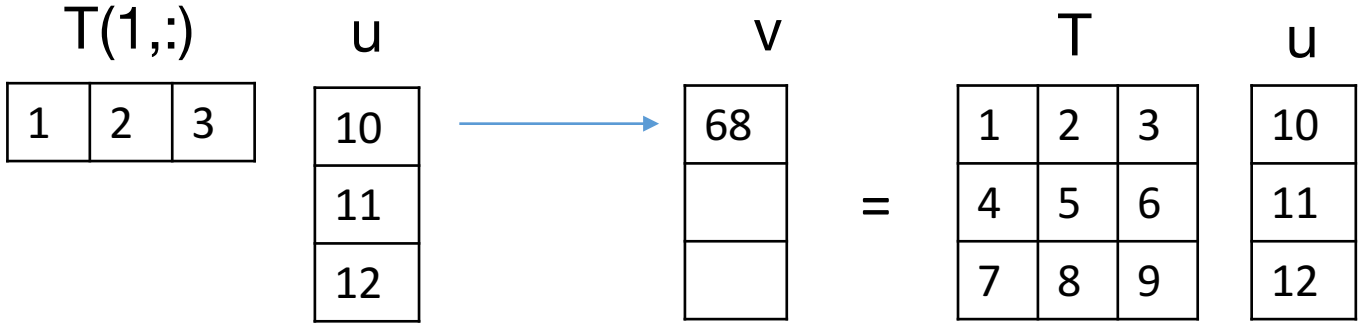
Linear Algebra – notation and basics

Some basic vector operations you should know:

- Conjugate, transpose, conjugate transpose
- Inner product
- Hadamard (element-wise, dot-times) product
- outer product
- Vector (matrix) addition
- matrix-vector product
- convolution

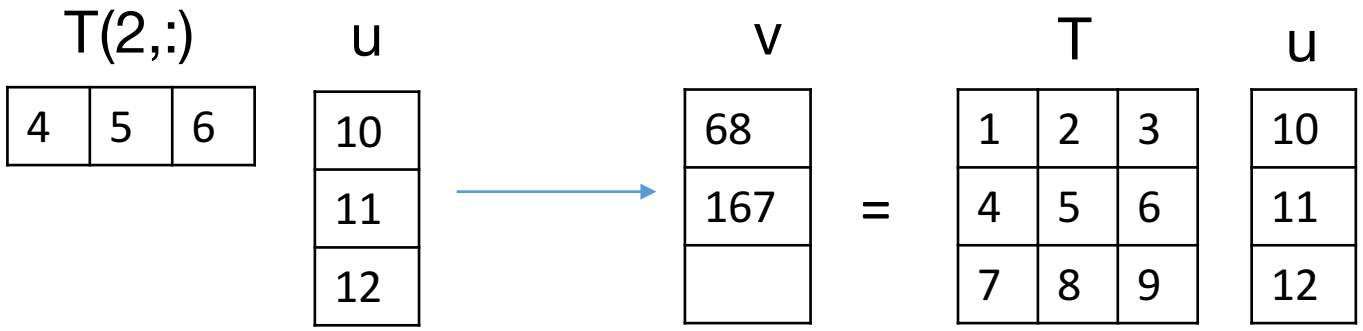
Matrix-vector products – two useful interpretations

1. Inner products per entry:



Matrix-vector products – two useful interpretations

1. Inner products per entry:



Matrix-vector products – two useful interpretations

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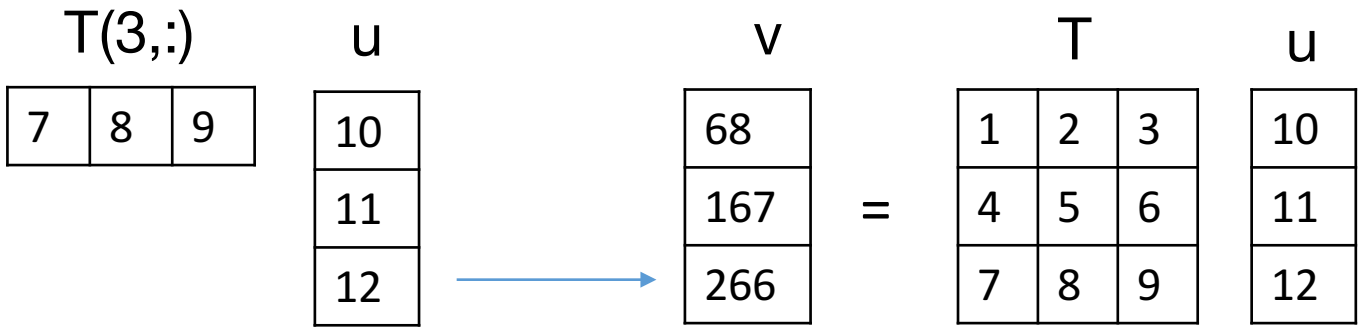
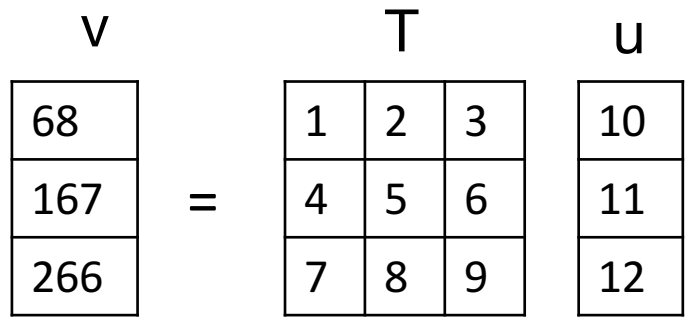
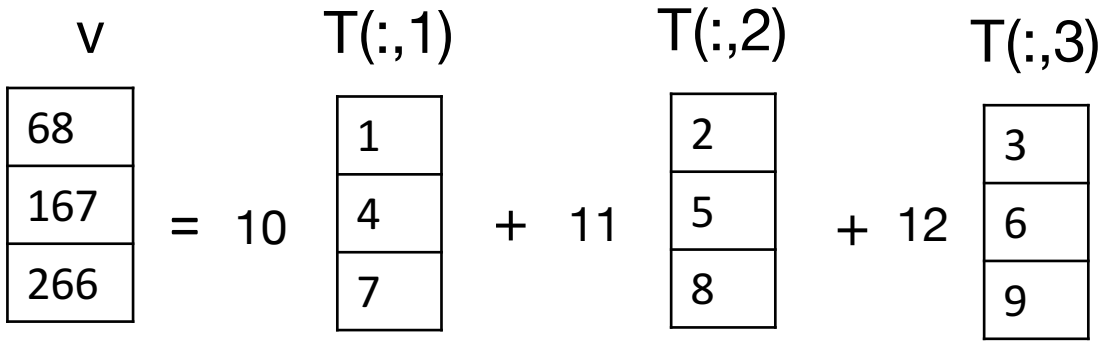


Matrix-vector products – two useful interpretations

1. Inner products per entry:



2. Weighted column sum:



Discrete convolution

$$V(x_o) = \int_{-\infty}^{\infty} U(x_i) h(x_o - x_i) dx_i$$

$$v[x_0] = \sum_{x_i=-M}^{M} u[x_i] h[x_o - x_i]$$

Discrete 1D Convolution – an example

deep imaging

Steps to follow:

Step 1	List the index 'k' covering a sufficient range
Step 2	List the input x[k]
Step 3	Obtain the reversed sequence h[-k], and align the rightmost element of h[n-k] to the leftmost element of x[k]
Step 4	Cross-multiply and sum the nonzero overlap terms to produce y[n]
Step 5	Slide h[n-k] to the right by one position
Step 6	Repeat step 4; stop if all the output values are zero or if required.

http://host.uniroma3.it/laboratori/sp4te/teaching/sp4bme/documents/LectureConvolution.pdf

Example 2: Find the convolution of the two sequences x[n] and h[n] given by,

$$x[k] = [3\ \underset{\uparrow}{1}\ 2] \qquad h[k] = [3\ \underset{\uparrow}{2}\ 1]$$

k:	-2	-1	0	1	2	3	4	5
x[k]:			3	1	2			
h[-k]:	1	2	3					
h[1-k]:		1	2	3				
h[2-k]:			1	2	3			
h[3-k]:				1	2	3		
h[4-k]:					1	2	3	
h[5-k]:						1	2	3

Hint: The value of k starts from **(– length of h + 1)** and continues till **(length of h + length of x – 1)**

Here ***k*** starts from **-3 + 1 = -2** and continues till **3 + 3 – 1 = 5**

http://host.uniroma3.it/laboratori/sp4te/teaching/sp4bme/documents/LectureConvolution.pdf

Example 2: Find the convolution of the two sequences x[n] and h[n] given by,

$$x[k] = [3\ 1\ 2] \qquad h[k] = [3\ 2\ 1]$$

(↑ under the 3 in x[k] and ↑ under the 3 in h[k])

k:	-2	-1	0	1	2	3	4	5
x[k]:			3	1	2			
h[-k]:	1	2	3					
h[1-k]:		1	2	3				
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h[4-k]:					1	2	3	
h[5-k]:						1	2	3

y: 9

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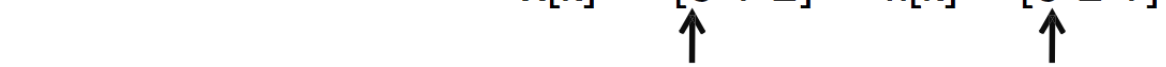

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y: 9 6+3

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h[3-k]:				1	2	3		
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h[5-k]:						1	2	3
y:			9	6+3	3+2+6			

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y: 9 6+3 3+2+6 1+4+0

y: [9 9 11 5 2 0]

Discrete convolution

$$V(x_o) = \int_{-\infty}^{\infty} U(x_i) h(x_o - x_i) dx_i$$

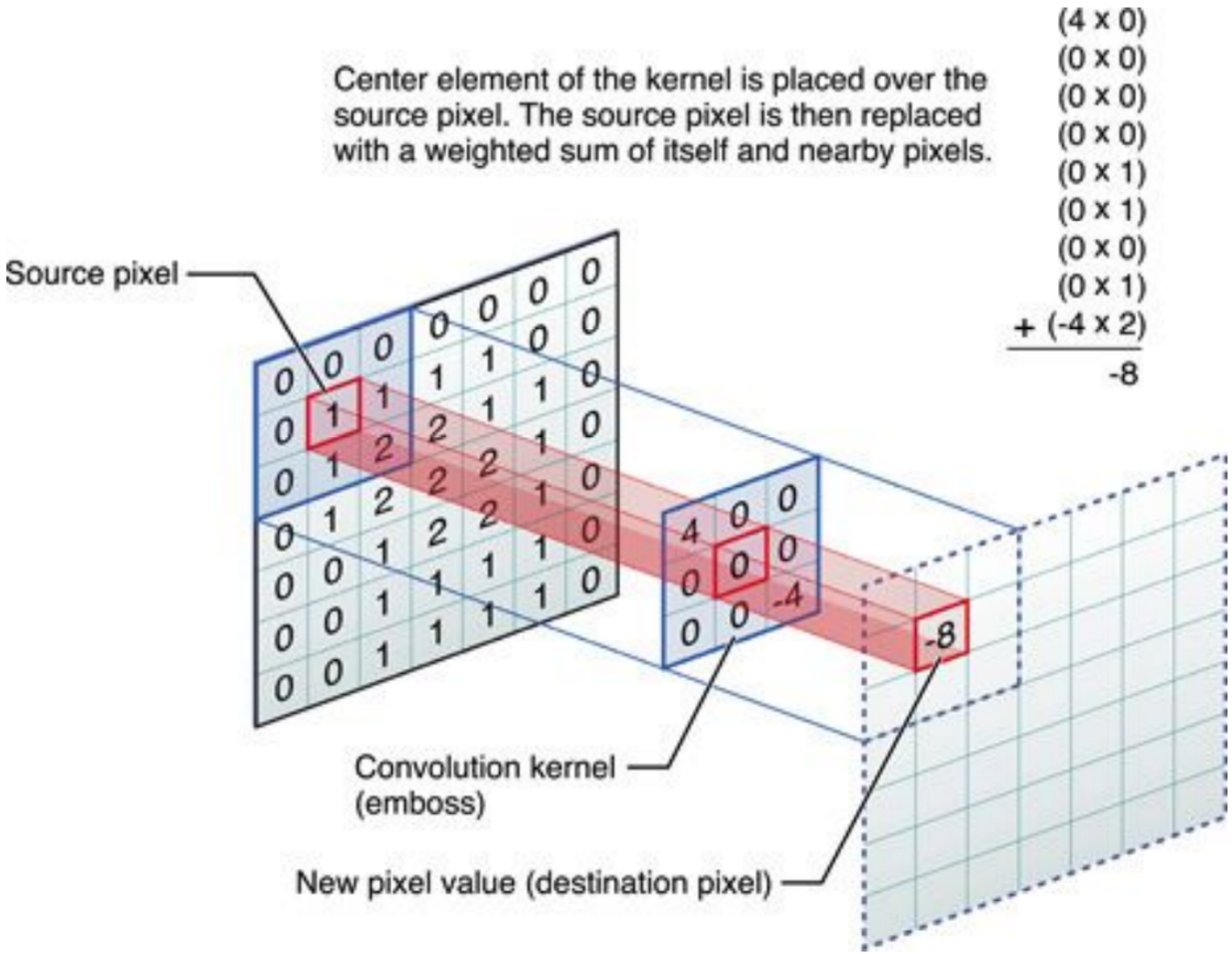
$$v[x_0] = \sum_{x_i=-M}^{M} u[x_i] h[x_o - x_i]$$

Discrete 2D convolution

$$V(x_o, y_o) = \iint_{-\infty}^{\infty} U(x_i, y_i) h(x_o - x_i, y_o - y_i) dx_i dy_i$$

$$v[x_0, y_o] = \sum_{y_i=-L}^{L} \sum_{x_i=-M}^{M} u[x_i, y_i] h[x_o - x_i, y_o - y_i]$$

Discrete 2D convolution



https://www.psi.toronto.edu/~jimmy/ece521/Tut1.pdf

Linear Algebra – notation and basics

Some basic types of matrices & terms that you should know about:

- Symmetric (Hermitian) matrix: $A=A^T$ if A is real, $A=A^H$ if A is complex
- Square, hot-dog and hamburger matrices
- Invertible matrix
- Diagonal matrix
- Toeplitz matrix
- Banded matrix

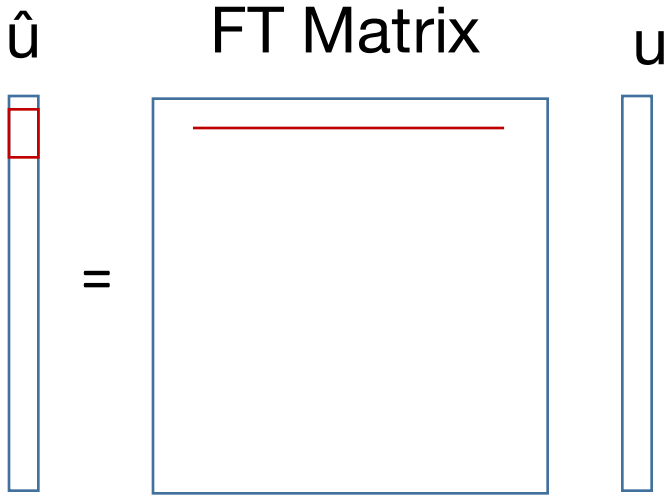
Discrete Fourier Transforms

$$\hat{U}(f_x) = \int_{-\infty}^{\infty} U(x) \exp\left(-2\pi i (f_x x)\right) dx$$

f_x=0

$$\hat{u}[f_x] = \sum_{x=0}^{M-1} u[x] \exp\left(-2\pi i f_x x / M\right)$$

Inner product of u with different complex expon.



- np.fft(u), np.fftshift(np.fft(np.ifftshift(u)))
- fft = fast Fourier transform, much more comp. efficient than matrix multiplication!

Discrete Fourier Transforms

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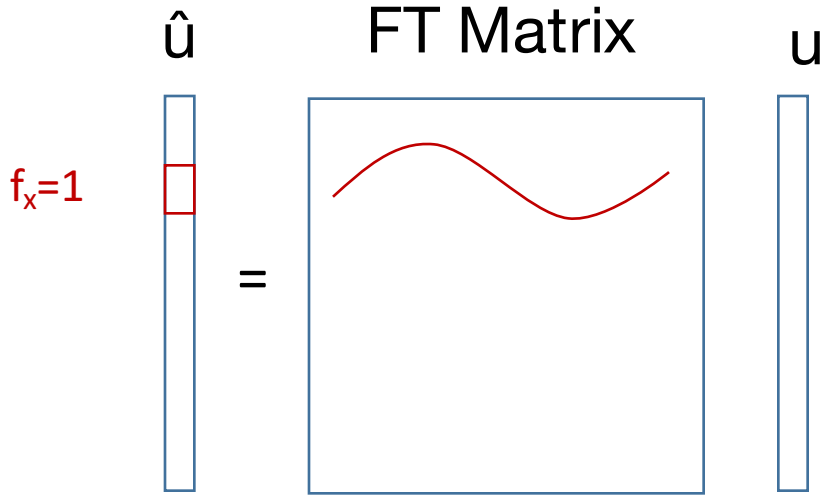

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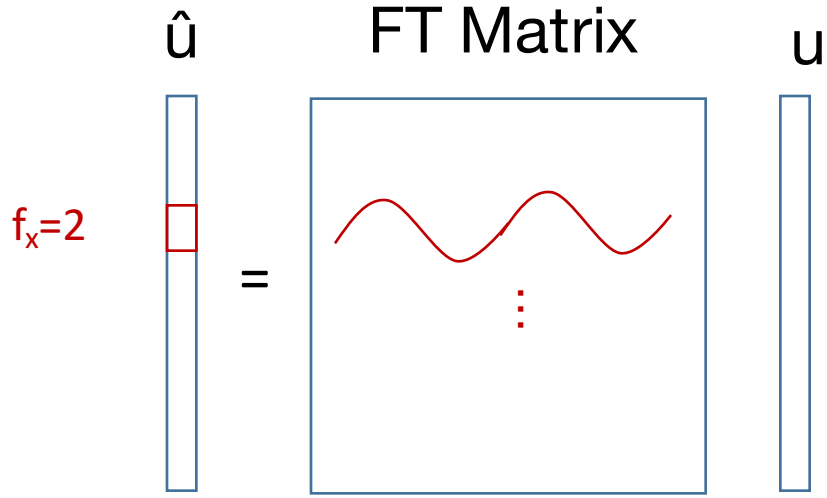

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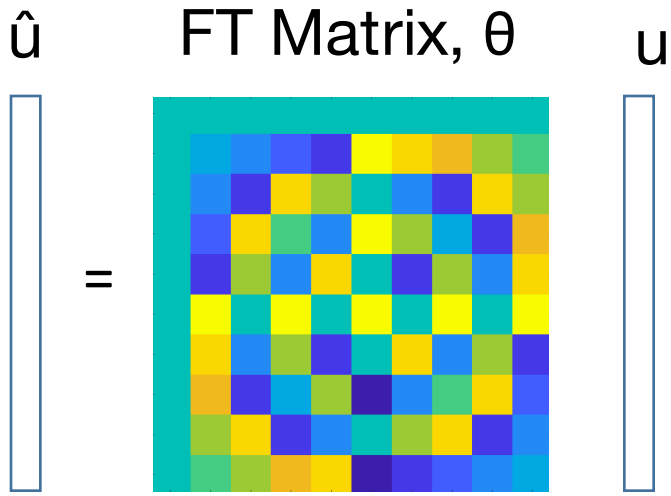

np.fft(np.eye(10))

Treats 1st entry of û as f_x=0

Discrete Fourier Transforms

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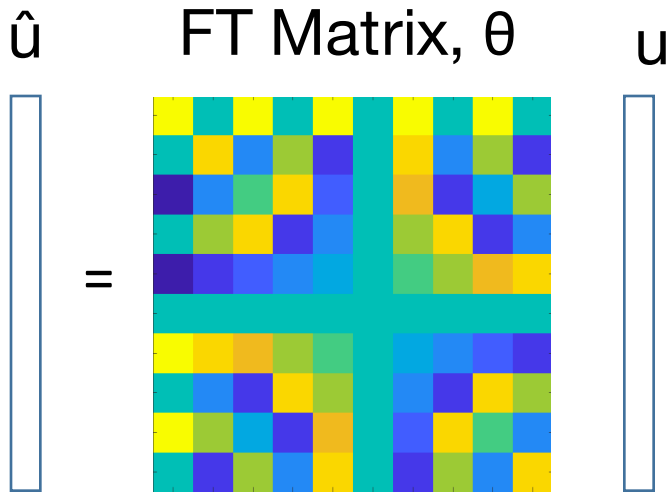


np.fftshift(np.fft(np.ifftshift(np.eye(10)))))

Treats middle entry of û as f_x=0

Discrete convolution theorem

Convolution theorem. If $\mathcal{F}\{g(x, y)) = G(f_X, f_Y)$ and $\mathcal{F}\{h(x, y)\} = H(f_X, f_Y)$, then

$$\mathcal{F}\left\{\int_{-\infty}^{\infty}\int g(\xi, \eta)\, h(x-\xi, y-\eta)\, d\xi\, d\eta\right\} = G(f_X, f_Y)\, H(f_X, f_Y). \qquad (2\text{-}15)$$

Discrete convolution theorem

Convolution theorem. If $\mathcal{F}\{g(x, y)) = G(f_X, f_Y)$ and $\mathcal{F}\{h(x, y)\} = H(f_X, f_Y)$, then

$$\mathcal{F}\left\{\int_{-\infty}^{\infty}\int g(\xi, \eta)\, h(x - \xi, y - \eta)\, d\xi\, d\eta\right\} = G(f_X, f_Y)\, H(f_X, f_Y). \qquad (2\text{-}15)$$

If $\mathcal{F}[g[x, y]] = G[f_x, f_y]$ and $\mathcal{F}[h[x, y]] = H[f_x, f_y]$, and if we know that

$$g[x, y] * h[x, y] = \sum_{l=-L}^{L} \sum_{m=-M}^{M} g[m, l] h[x - m, y - l],$$

then from the Convolution Theorem we have,

$$\mathcal{F}[g[x, y] * h[x, y]] = G[f_x, f_y] H[f_x, f_y]$$

Discrete convolution theorem example

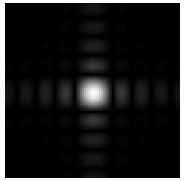
deep imaging

Input image

$U_1(x,y)$

Convolution filter h

$*$



$=$

Output image

$U_2(x,y)$

Discrete convolution theorem example

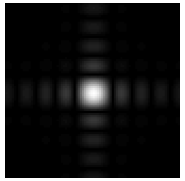
deep imaging

Input image

$U_1(x,y)$

Convolution filter h

$*$



$=$

Output image

$U_2(x,y)$

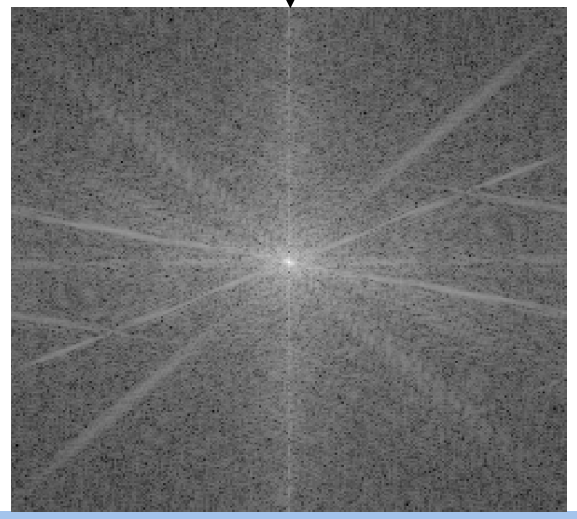
$F\,[h]$

$F\,[U_1]$

$F^{-1}[H\hat{U}_1]$

Input spectrum

$\hat{U}_1(f_x,f_y)$



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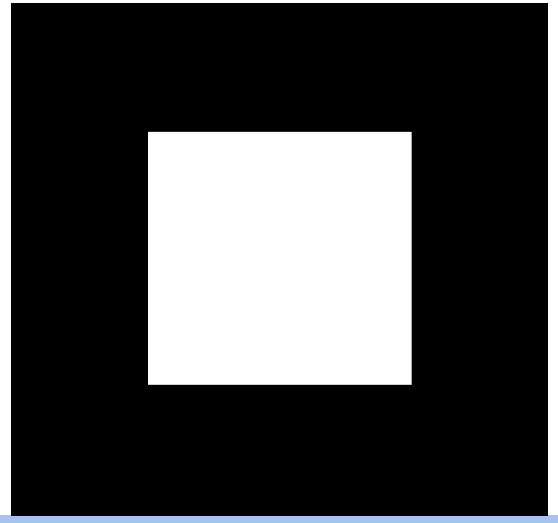

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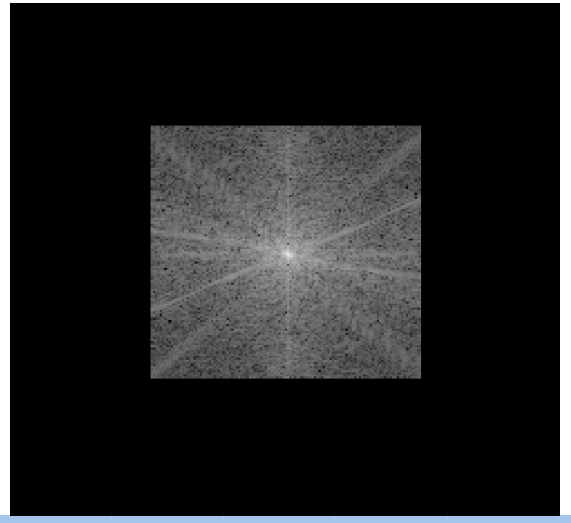


Discrete convolution theorem example

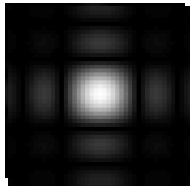
deep imaging

Input image

$U_1(x,y)$

Convolution filter h

$*$



$=$

Output image

$U_2(x,y)$

Discrete convolution theorem example

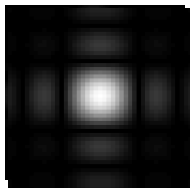
deep imaging

Input image

$U_1(x,y)$

Convolution filter h

$*$



$=$

Output image

$U_2(x,y)$

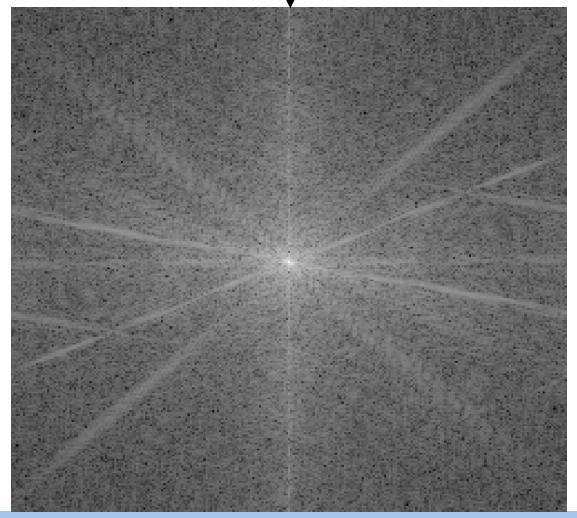
$F\,[h]$

$F\,[U_1]$

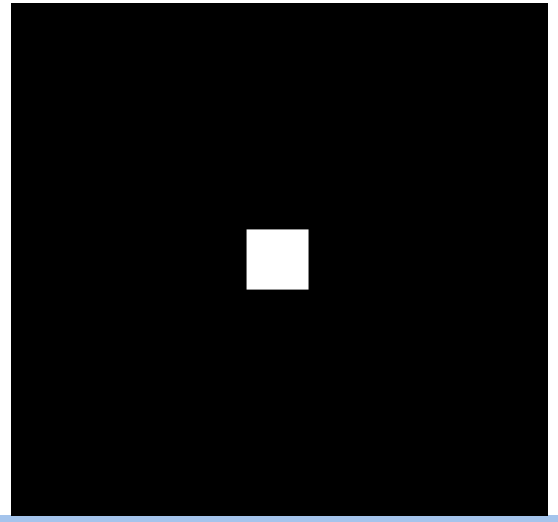
$F^{-1}[H\hat{U}_1]$

Input spectrum

$\hat{U}_1(f_x,f_y)$



$\cdot$



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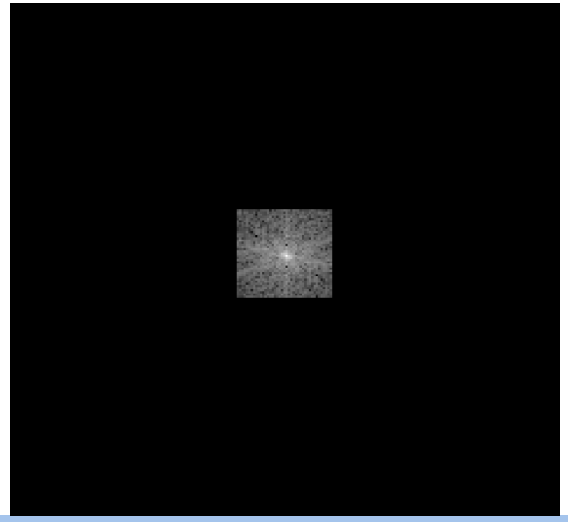
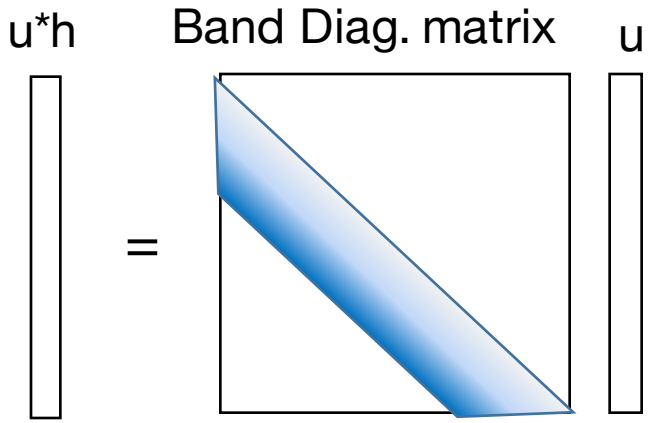


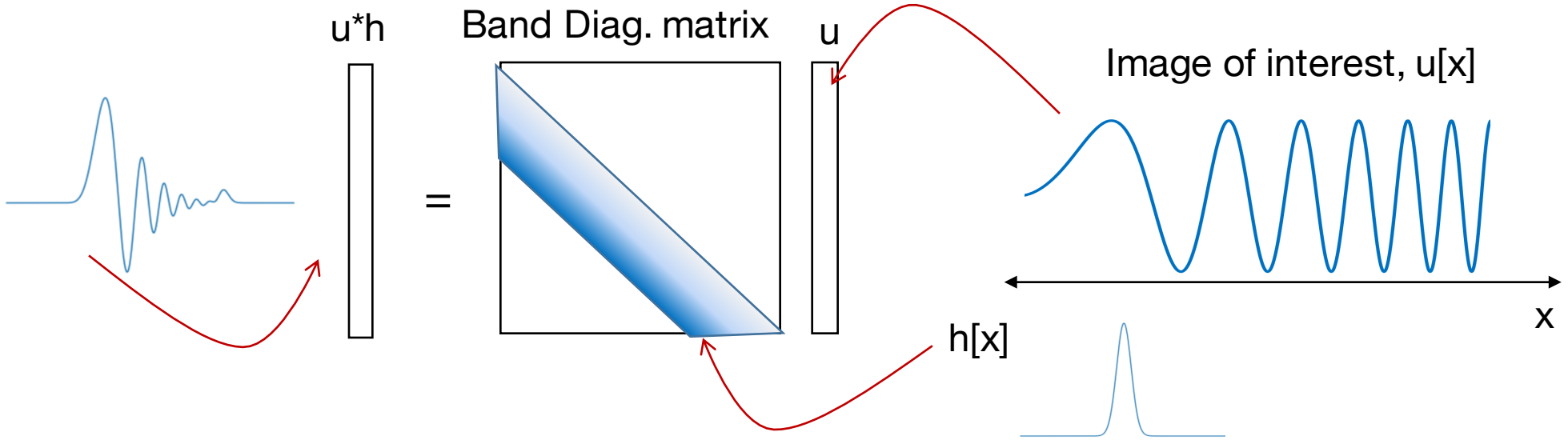
Convolutions as a big matrix multiplication

$$(u * h)[x] = \sum_{m=0}^{N+M-2} u[m]h[x-m] \longrightarrow y = u * h = \begin{bmatrix} h_1 & 0 & \dots & 0 & 0 \\ h_2 & h_1 & \dots & \vdots & \vdots \\ h_3 & h_2 & \dots & 0 & 0 \\ \vdots & h_3 & \dots & h_1 & 0 \\ h_{m-1} & \vdots & \dots & h_2 & h_1 \\ h_m & h_{m-1} & \vdots & \vdots & h_2 \\ 0 & h_m & \dots & h_{m-2} & \vdots \\ 0 & 0 & \dots & h_{m-1} & h_{m-2} \\ \vdots & \vdots & \vdots & h_m & h_{m-1} \\ 0 & 0 & 0 & \dots & h_m \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix}$$

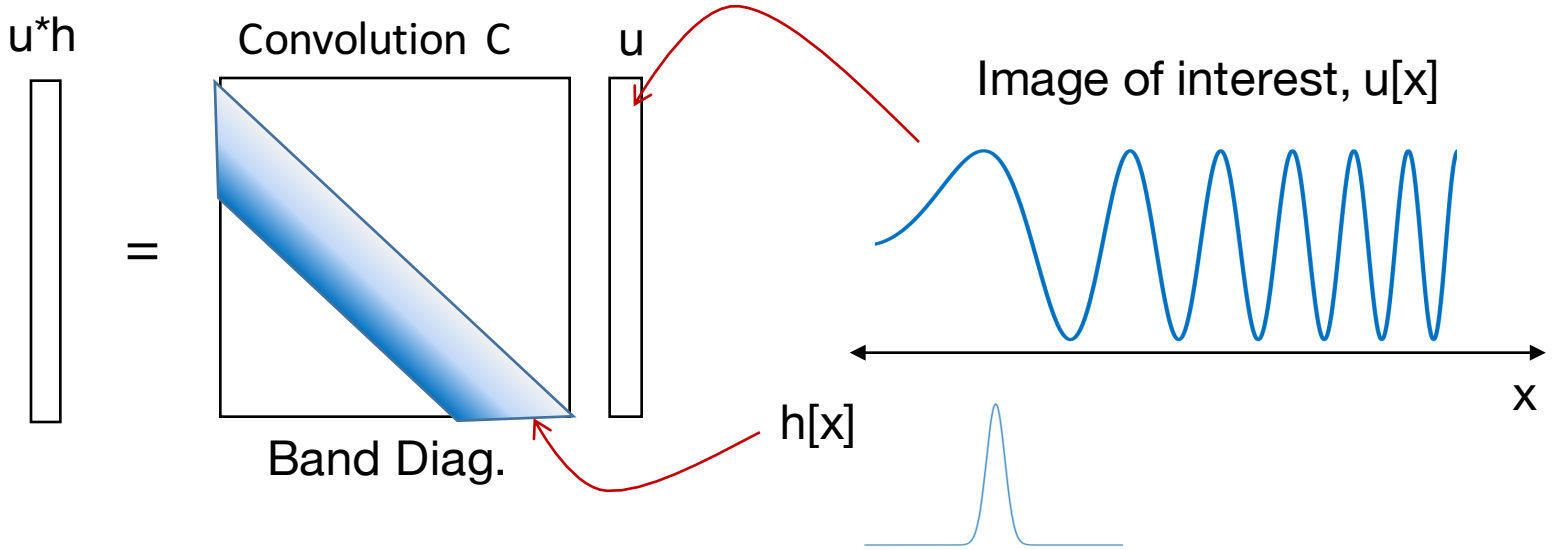
Convolutions as a big matrix multiplication



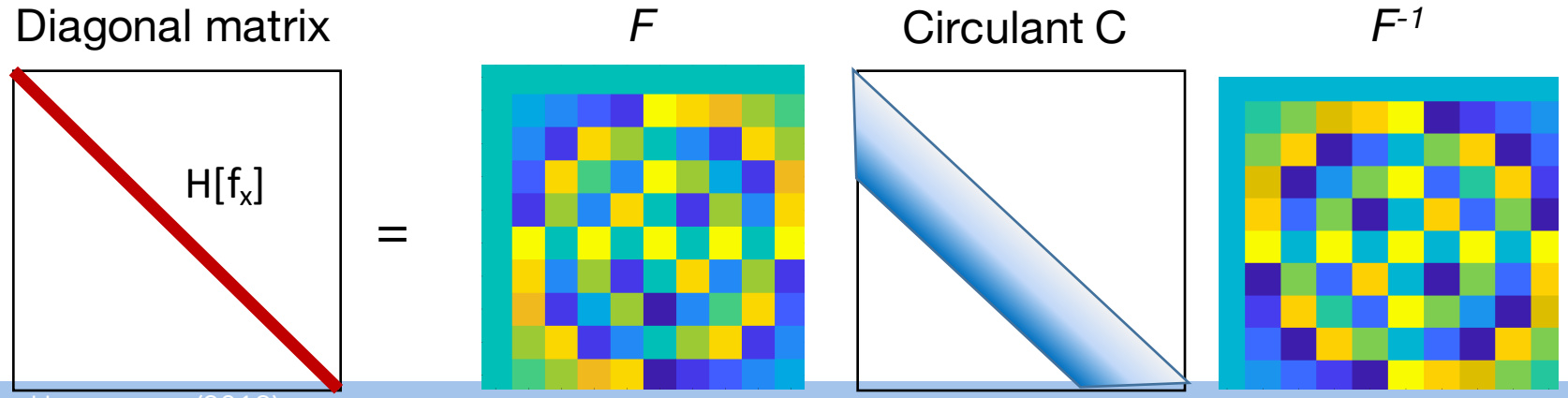
Convolutions as a big matrix multiplication



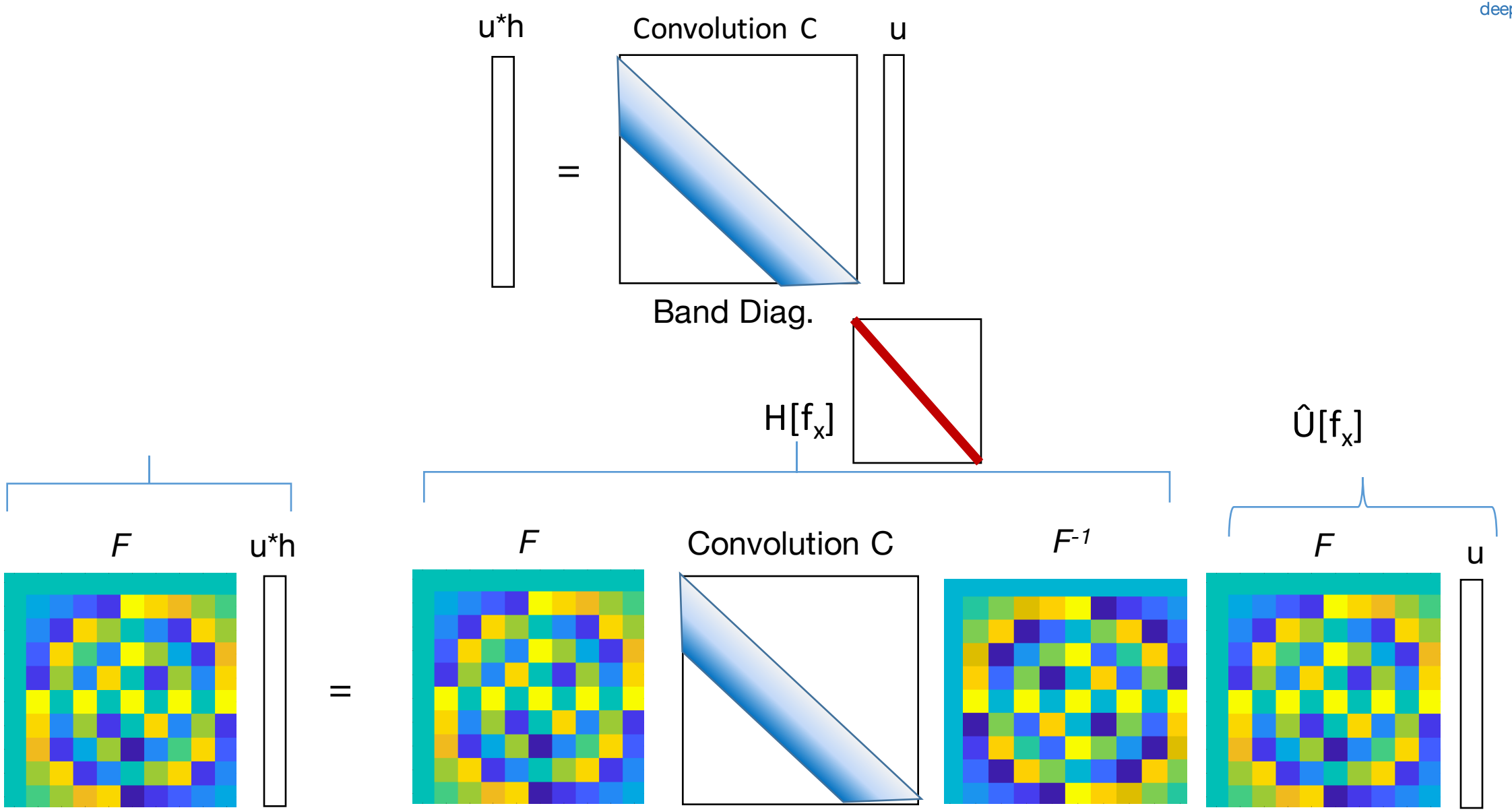
Convolutions as a big matrix multiplication



Discrete Fourier transform diagonalizes convolution matrix with $H[f_x]$ along diag.



Convolutions as a big matrix multiplication



Convolutions as a big matrix multiplication