ITCS 532:
W2 Homework Solutions

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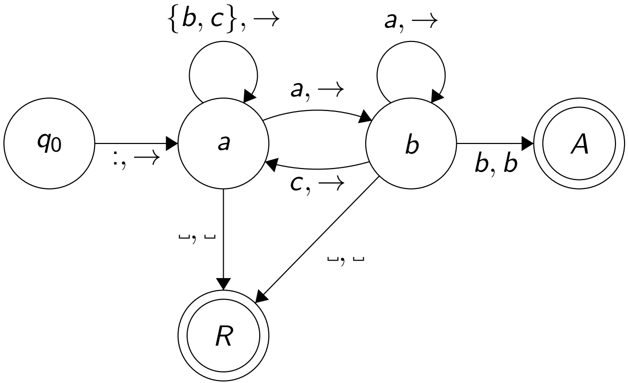
Q1

Let $\Sigma = \{a, b, c\}$. Let L be the set of all finite strings containing the substring ab. Design a Turing machine that decides L. What regular expression describes L?



Regular expression: $\cdot ab \cdot$ or $(a|b|c)^* ab(a|b|c)^*$.

Q2

Let Σ be a finite alphabet. Consider the class $\mathcal{C}$ of Turing machine variants over Σ with a countably infinite number of tape heads instead of only one. So the transition function is defined by

$$\delta : (Q \setminus H) \times \Sigma^{\omega} \to Q \times (\Sigma \setminus \{:\} \cup \{\leftarrow, \rightarrow\})^{\omega}$$

Let $L \subseteq \Sigma^*$ be any language over Σ. Show that there is a machine in $\mathcal{C}$ that decides L. What does this tell us about the relative power of $\mathcal{C}$ and the class of regular Turing machines?

Q2 - solution

If we assume the machine can tell when two or more heads are reading the same space, we can design a machine as follows:

1. Move all the tape heads to the right, so they are all reading the first symbol on the tape.
2. Move all tape heads except the first to the right, so the first is still reading the first symbol, and the rest are reading the second symbol.
3. Move all except the first and second tape head to the right.
4. Keep following this pattern till the first blank space is found.
5. Now there are n tape heads reading the n characters of the input string, and the rest of the tape heads are all reading a blank space.
6. Write the transition function so that if the string being read is in L the machine accepts, and it rejects otherwise.

This tells us this new kind of machine is much more powerful than regular Turing machines.

Q2 - better solution

Without the assumption:

- (Step 0) Move all the tape heads to the right, so they are all reading the first symbol on the tape.
- (Step 1) Move all the tape heads h_i such that $2|i$ to the right, and keep the others in place, except for h_1 which moves back to the start of the tape.
- (Step 2) Move all the tape heads h_i such that 2 and 3 divide i to the right. Keep the others in place except h_1 which moves back to square 1 (as it must do this), and also move h_3 back to the start of the tape.
- (Step 3) Move all the tape heads h_i such that i is divisible by $2, 3, 5$ to the right. Keep the others in place except h_3 which moves back to square 1, and also move h_5 back to the start of the tape.

Q2 - better solution continued

- (Step k) Suppose primes are $p_1, p_2, \ldots$. At the start of computation step $k > 2$, the tape heads with numbers divisible by $p_1 \times \ldots \times p_{k-1}$ are reading cell $k-1$, and $h_{p_{k-1}}$ is reading the : symbol.
  During step k all tape heads with numbers divisible by $p_1 \times \ldots \times p_k$ advance to the right, and head h_{p_k} goes back to read :.
- At some point there will be an infinite number of tape heads reading the blank space symbol. This tells the machine the end of the input has been reached.
- Write the transition function so that if the string being read is in L the machine accepts, and it rejects otherwise.

Q3

Let Σ be a finite alphabet. Prove that Σ^* is countably infinite.

- Σ^* is the set of all finite strings using Σ, so is infinite.
- To prove it is countable it is sufficient to find a 1-1 function $f : \Sigma^* \to \mathbb{N}$.
- Suppose $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$.
- Let $p_1, p_2, p_3, \ldots$ list the prime numbers in ascending order.
- Given a string $s = \sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k}$, define $f(s) = p_1^{i_1} \times p_2^{i_2} \times \ldots \times p_k^{i_k}$.
- E.g. if $\Sigma = \{a = \sigma_1, b = \sigma_2, c = \sigma_3\}$, the string *aacb* corresponds to $2^1 3^1 5^3 7^2$.
- Then f is 1-1 because, by the Fundamental Theorem of Arithmetic, numbers are specified uniquely by their prime factorizations (up to reordering).

Q4

Let Σ again be a finite alphabet. Is $\wp(\Sigma^*)$ countable? Justify your answer.

- No (unless Σ is empty).
- As proved in the notes for the 531 course, given a non-empty set X we always have $|X| < |\wp(X)|$.

Q5

Recall that the union of a countable number of countable sets is countable. Let $\Sigma = \{0, 1\}$ and let $X = \wp(\Sigma^*)$ be the uncountable set of all languages over Σ. Let C_1 and C_2 be countable subsets of X, and let U be an uncountable subset of X. Remember that if Y is a set we use $\bar{Y}$ to denote the complement of Y. For each of the following sets say whether it is countable, uncountable, or dependent on the choice of C_1, C_2, U. Justify your answers.

(a) $C_1 \cup C_2$

(b) $\bar{C}_1$

(c) $\bar{U}$

(d) $\bar{C}_1 \cap U$

Q5 - solutions (a),(b)

(a) $(C_1 \cup C_2)$ Union of a countable number of countable sets is countable.

(b) $(\bar{C}_1)$ Uncountable.

- $C_1 \cup \bar{C}_1 = X$, so if both C_1 and $\bar{C}_1$ were countable then X would be too.
- But X is uncountable.
- So, since C_1 is countable, $\bar{C}_1$ must be uncountable.

Q5 - solutions (c)

(c) $(\bar{U})$ This depends on the choice of U.

- If $U = X$ then $\bar{U} = \emptyset$, which is countable.
- Let U be the set of all languages containing the string s, for some arbitrary choice of s.
- Then U is in bijection with $\wp(\Sigma^* \setminus \{s\})$.
- Bijection takes $L \in U$ to $L \setminus \{s\}$, which is in $\wp(\Sigma^* \setminus \{s\})$, and conversely takes $L' \in \wp(\Sigma^* \setminus \{s\})$ to $L' \cup \{s\}$, which is in U.
- Moreover, $\bar{U} = \wp(\Sigma^* \setminus \{s\})$.
- $\wp(\Sigma^* \setminus \{s\})$ is uncountable.

Q5 - solutions (d)

(d) $(\bar{C}_1 \cap U)$ This is uncountable, for the following reason.

$$\begin{aligned} U &= X \cap U \\ &= (\bar{C}_1 \cup C_1) \cap U \\ &= (\bar{C}_1 \cap U) \cup (C_1 \cap U). \end{aligned}$$

Now, $(C_1 \cap U)$ is countable, so $(\bar{C}_1 \cap U)$ must be uncountable as U is.