ITCS 532:
W7 Homework Solutions

Rob Egrot

Definitions

Definition 1 (Hamiltonian path)

A Hamiltonian path is like a Hamiltonian circuit except that there does not need to be an edge connecting v_n and v_1.

Definition 2 (HPP)

The Hamiltonian path problem (HPP) asks whether a given finite undirected simple graph contains a Hamiltonian path.

Q1

Let $G = (V, E)$ be a finite undirected simple graph. We construct a new undirected simple graph $G' = (V', E')$ as follows:

1. $V' = V \cup \{v'\}$ for some $v' \notin V$.
2. $E' = E \cup \{\{v, v'\} : v \in V\}$

a) Show that the construction of G' takes polynomial time as a function of $|V| + |E|$.

b) Prove that G has a Hamiltonian path if and only if G' has a Hamiltonian circuit.

c) Deduce that $HPP \leq_p HCP$.

Q1

a) Show that the construction of G' takes polynomial time as a function of $|V| + |E|$.

- Assume creating a vertex takes constant time.
- Then the time it takes to create the vertices of G' is a constant multiple of $|V| + 1$, and so is $O(|V|)$.
- Similarly, we must create an edge for every edge of G, and also new edges for every vertex of G.
- Again, assuming constant time for edge creation, this is $O(|E| + |V|)$.
- So total time is $O(|V|) + O(|E| + |V|)$, which is $O(|E| + |V|)$.

Q1

b) Prove that G has a Hamiltonian path if and only if G' has a Hamiltonian circuit.

- Suppose $v_0, \ldots, v_n$ is a Hamiltonian path in G.
- Then $v_0, \ldots, v_n, v'$ is a Hamiltonian circuit in G'.
- Conversely, suppose $v_0, \ldots, v_n, v_{n+1}$ is a Hamiltonian circuit in G', and suppose without loss of generality that $v_{n+1} = v'$.
- Then $v_0, \ldots, v_n$ is a Hamiltonian path in G.

c) Deduce that $HPP \leq_p HCP$.

- We have found a p-time reduction of HPP to HCP.
- So $HPP \leq_p HCP$ by definition.

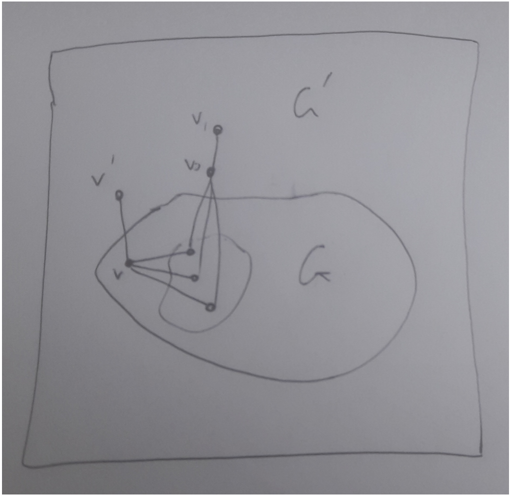
Q2

Prove $HCP \leq_p HPP$.

- We construct a new undirected simple graph $G' = (V', E')$ as follows:
  1. Pick a vertex v of G. Define new vertices v', v_0, v_1.
  2. $V' = V \cup \{v', v_0, v_1\}$.
  3. $E' = E \cup \{\{v, v'\}, \{v_0, v_1\}\} \cup \{\{v_0, u\} : \{v, u\} \in E\}$.

- I.e. We add three new vertices to G. We add edges connecting v to v', and v_0 to v_1, and we also add edges connecting v_0 to every vertex u that has edge connecting it to v.

Q2



Q2

a) Show that the construction of G' can be done in polynomial time with respect to $|V| + |E|$.

- Assume it takes constant time to choose a vertex v, and also that it takes constant time to create vertices.
- Then constructing the set of vertices of G' is obviously $O(|V|)$.
- We add one edge to G' for every edge of G, two edges $\{v, v'\}$ and $\{v_0, v_1\}$, and then an edge $\{v_0, u\}$ for every edge $\{v, u\}$ of G.
- As usual assuming constant time to create edges, this is $O(|E|) + O(|E|) = O(|E|)$.
- So the total time is $O(|V| + |E|)$.

Q2

b) Prove that G has a Hamiltonian circuit implies G' has a Hamiltonian path.

- Let $u_0, \ldots, u_n$ be a Hamiltonian circuit of G, and suppose without loss of generality that $u_0 = v$.
- Consider the sequence $v', v = u_0, \ldots, u_n, v_0, v_1$ of G'.
- There are obviously no repeated vertices in this sequence.
- There are edges $\{v', v\}$ and $\{v_0, v_1\}$ by construction of G'.
- As there is an edge $\{u_n, u_0\}$ in G, there is also an edge $\{u_n, v_0\}$.
- It follows that $v', v, u_0, \ldots, u_n, v_0, v_1$ is a Hamiltonian path.
- So G has a H. circuit implies G' has a H. path.

Q2

c) Prove that G' has a Hamiltonian path implies G has a Hamiltonian circuit.

- Conversely, suppose $u_0, u_1, \ldots, u_{m-1}, u_m$ is a Hamiltonian path in G'.
- v' and v_1 are the only possible endpoints of such a path.
- So we can suppose without loss of generality that $u_0 = v'$, $u_1 = v$, $u_{m-1} = v_0$ and $u_m = v_1$.
- It follows that $u_1, \ldots, u_{m-2}$ is a Hamiltonian path in G.
- Moreover, as there is an edge $\{u_{m-2}, u_{m-1}\}$ in G' (i.e. from u_{m-2} to v_0), there must be an edge $\{u_{m-2}, u_1\}$ in G (i.e. from u_{m-2} to $v = u_1$).
- Thus $u_1, \ldots, u_{m-2}$ is a Hamiltonian circuit as required.
- So G' has H. path implies G has H. circuit.

- That $HCP \leq_p HPP$ follows immediately.