ITCS 531: L5 homework solutions

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L1 Q1(a)

Let ϕ be a formula where x occurs free. Write down a proof tree that shows $\forall x \phi \vdash \neg \exists x \neg \phi$.

$$\dfrac{\dfrac{[\exists x \neg \phi]_1 \qquad \dfrac{[\neg \phi[x'/x]]_2 \qquad \dfrac{\forall x \phi}{\phi[x'/x]}\,(\forall_E)}{\bot}\,(\neg_E)}{\bot}\,(\exists_E)}{\neg \exists x \neg \phi}\,(\neg_I)$$

L1 Q1(b)

Let ϕ be a formula where x occurs free. Write down a proof tree that shows $\exists x\phi \vdash \neg\forall x\neg\phi$.

$$\dfrac{\dfrac{\exists x\phi \qquad \dfrac{(\forall_E)\;\dfrac{[\forall x\neg\phi]_1}{\neg\phi[x'/x]} \qquad [\phi[x'/x]]_2}{\bot}\;(\neg_E)}{\bot}\;(\exists_E)}{\neg\forall x\neg\phi}\;(\neg_I)$$

L1 Q3

Prove that if Γ is an $\mathscr{L}$-theory then there is an $\mathscr{L}$-theory Γ' with $\Gamma \subseteq \Gamma'$ such that Γ' is *complete* (i.e. if ϕ is an $\mathscr{L}$-sentence, then either $\phi \in \Gamma'$ or $\neg\phi \in \Gamma'$).

- An $\mathscr{L}$-theory is a satisfiable set of $\mathscr{L}$-sentences.
- Since Γ is satisfiable, it must have a model.
- Let A be a model for Γ, and let

$$\Gamma' = \{\phi : \phi \text{ is an } \mathscr{L}\text{-sentence and } A \models \phi\}.$$

- Then $\Gamma \subseteq \Gamma'$, because $A \models \Gamma$.
- Γ' is complete because every $\mathscr{L}$-sentence is either true or false in A.

L1 Q4

Let Γ be an $\mathscr{L}$-theory, and let ϕ be an $\mathscr{L}$-sentence. Prove that if $\Gamma \models \phi$ then $\Delta \models \phi$ for some finite $\Delta \subseteq \Gamma$.

- Suppose $\Gamma \models \phi$.
- By completeness we have $\Gamma \vdash \phi$.
- So there is a deduction tree using Γ that proves ϕ.
- As deduction trees are finite, this tree involves only a finite number of sentences from Γ.
- Define Δ to be the set of sentences from Γ used in the proof of ϕ.
- Then $\Delta \vdash \phi$, and so $\Delta \models \phi$ by soundness.

L1 Q5

Let Γ be a set of $\mathscr{L}$-sentences. Prove that Γ has a model if and only if every finite subset of Γ has a model.

- If Γ has a model then every subset of Γ has a model.
- Conversely, suppose Γ does *not* have a model.
- Then $\Gamma \models \perp$.
- So by exercise 4 there is finite $\Delta \subseteq \Gamma$ with $\Delta \models \perp$.
- I.e. Δ does not have a model.