ITCS 531: LA1 homework solutions

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LA1 Q3

Show $-1v = -v$.

- By proposition 1.7(3) we have $0v = 0$.
- So $(1-1)v = 0$, and so $v + (-1)v = 0$ by definition 1.5(8) and (6).
- So $(-1)v = -v$ by proposition 1.7(2).

LA1 Q4

Given $v \in V$, prove that $-(-v) = v$.

- We know $v + (-v) = 0$, so $-(-v) = v$, as additive inverses are unique, by proposition 1.7(2).

LA1 Q5

Given $a \in \mathbb{F}$ and $v \in V$ prove that $av = 0$ if and only if either $a = 0$ or $v = 0$.

- If $a = 0$ then $av = 0$ by proposition 1.7(3).
- Now, let $v = 0$, and suppose $a \neq 0$. Let w be any vector.
- Then $a0 + w = a0 + aa^{-1}w = a(0 + a^{-1}w) = a(a^{-1}w) = 1w = w$.
- So $a0 = 0$, as the zero of a vector space is unique (by proposition 1.7(1)).
- Conversely, suppose $av = 0$ and that $a \neq 0$.
- Then $a^{-1}av = a^{-1}0 = 0$, and so $v = 0$, as $aa^{-1} = 1$.

LA1 Q7

Let U and W be subspaces of V. Prove that if $U \cup W$ is a subspace of V, then either $U \subseteq W$ or $W \subseteq U$.

- Suppose $U \cup W$ is a subspace of V, and suppose U is not a subspace of W.
- Choose $u \in U \setminus W$, and let $w \in W$.
- Then $u + w \in U \cup W$, as $U \cup W$ is a subspace, and so is closed under $+$.
- So either $u + w \in U$ or $u + w \in W$.
- If $u + w \in W$, then $u + w - w = u$ is also in W, but this contradicts the choice of u.
- So $u + w \in U$, and so $u + w - u = w \in U$.
- This is true for all $w \in W$, so W is a subspace of U.

LA1 Q8

Let V be vector space over $\mathbb{F}$, and let $v_1, \ldots v_n \in V$ such that $(v_1, \ldots, v_n)$ is linearly independent. Let $w \in V$. Prove that $(v_1, \ldots, v_n, w)$ is linearly independent if and only if $w \notin \mathrm{span}(v_1, \ldots, v_n)$.

- "If $w \in \mathrm{span}(v_1, \ldots, v_n)$ then $(v_1, \ldots, v_n, w)$ is not linearly independent".
  - If $w \in \mathrm{span}(v_1, \ldots, v_n)$, then $w = a_1 v_1 + \ldots + a_n v_n$ for some $a_1, \ldots, a_n$.
  - So $0 = (-1)w + a_1 v_1 + \ldots + a_n v_n$.
  - So $(v_1, \ldots, v_n, w)$ is not linearly independent.
- "If $(v_1, \ldots, v_n, w)$ is not linearly independent then $w \in \mathrm{span}(v_1, \ldots, v_n)$".
  - If $(v_1, \ldots, v_n, w)$ is not linearly independent then we have $a_0 w + a_1 v_1 + \ldots + a_n v_n = 0$ for some $a_0, \ldots, a_n$ not all zero.
  - a_0 cannot be zero, as $(v_1, \ldots, v_n)$ is linearly independent.
  - So $w = \frac{-a_1}{a_0} v_1 + \ldots + \frac{-a_n}{a_0} v_n$, and is therefore in $\mathrm{span}(v_1, \ldots, v_n)$.