ITCS 531: L3 homework solutions

Rob Egrot

Q1

Complete the proof of theorem 3.3 (don't forget the extra axiom, $\neg\neg_E$).

- Must check rules $\vee_I$, $\vee_E$, $\rightarrow_E$ and $\neg\neg_E$.
- We'll just do $\vee_E$.

$$\dfrac{\phi \vee \psi \qquad \dfrac{[\phi]}{\theta} \qquad \dfrac{[\psi]}{\theta}}{\theta}$$

- We have $\Gamma \vdash \phi \vee \psi$, $\Gamma \cup \{\phi\} \vdash \theta$, and $\Gamma \cup \{\psi\} \vdash \theta$.
- by induction, $\Gamma \models \phi \vee \psi$, $\Gamma \cup \{\phi\} \models \theta$, and $\Gamma \cup \{\psi\} \models \theta$.
- Let v be an assignment satisfying Γ. Then v satisfies $\phi \vee \psi$.
- So v satisfies one or both of ϕ and ψ.
- So v also satisfies θ.

Q2

Prove that soundness of a deduction system is equivalent to the statement "every satisfiable set of sentences is consistent".

- Soundness $= \Gamma \vdash \phi \implies \Gamma \models \phi$ (†).
- "satisfiable implies consistent" $= \Gamma \vdash \bot \implies \Gamma \models \bot$ (‡).

(†) $\implies$ (‡).

$$
\begin{aligned}
\Gamma \vdash \bot &\iff \Gamma' \cup \{\psi\} \vdash \bot \\
&\iff \Gamma' \vdash \neg\psi && \text{by lemma 3.5(2)} \\
&\implies \Gamma' \models \neg\psi && \text{by (†)} \\
&\iff \Gamma \models \bot && \text{by lemma 3.5(1).}
\end{aligned}
$$

(‡) $\implies$ (†).

$$
\begin{aligned}
\Gamma \vdash \phi &\iff \Gamma \vdash \neg\neg\phi && \text{by classical logic} \\
&\iff \Gamma \cup \{\neg\phi\} \vdash \bot && \text{by lemma 3.5(2)} \\
&\implies \Gamma \cup \{\neg\phi\} \models \bot && \text{by (‡)} \\
&\iff \Gamma \models \neg\neg\phi && \text{by lemma 3.5(1)} \\
&\iff \Gamma \models \phi.
\end{aligned}
$$

Q3

Let Γ be a set of sentences in propositional logic. Then Γ is satisfiable if and only if every finite subset of Γ is satisfiable.

- If Γ is satisfiable then every finite subset is obviously satisfiable too.
- We must now prove that if every finite subset of Γ is satisfiable then so is Γ.
- Prove contrapositive: If Γ is not satisfiable then it has a finite subset that is not satisfiable.
- Suppose Γ is not satisfiable (i.e. $\Gamma \models \bot$). Then, by completeness, $\Gamma \vdash \bot$.
- The deduction of $\bot$ from Γ can only use a finite set of axioms from Γ. Call this set Γ'.
- Then $\Gamma' \vdash \bot$, and so by soundness $\Gamma' \models \bot$.
- I.e. Γ' is not satisfiable.