ITCS 531: Linear Algebra - Dimension

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Bases

- Think about the Euclidean plane $\mathbb{R}^2$.
- In $\mathbb{R}^2$, every vector is defined by coordinates (x, y).
- I.e. every vector in $\mathbb{R}^2$ can be written as a sum $x(1,0) + y(0,1)$ of the vectors $(1,0)$ and $(0,1)$.
- These vectors $(1,0)$ and $(0,1)$ are special, as they form a minimal set that generates the whole space.
- We can generalize this idea.

Definition 1

If V is a vector space, then a *basis* for V is a linearly independent set that spans V.

Expressing vectors with a basis

Lemma 2

Let V be a vector space over $\mathbb{F}$. Let $v_1, \ldots, v_n \in V$. Then $(v_1, \ldots, v_n)$ is a basis for V if and only if every element u can be expressed as $a_1 v_1 + \ldots + a_n v_n$, for some unique $\{a_1, \ldots, a_n\} \subseteq \mathbb{F}$.

Proof.

- Suppose $(v_1, \ldots, v_n)$ is a basis for V.
- Given $u \in V$, we have $u = a_1 v_1 + \ldots + a_n v_n$ for some $\{a_1, \ldots, a_n\} \subseteq \mathbb{F}$ as$(v_1, \ldots, v_n)$ spans V.
- If $u = a_1 v_1 + \ldots + a_n v_n = b_1 v_1 + \ldots + b_n v_n$. Then $0 = (a_1 - b_1) v_1 + \ldots + (a_n - b_n) v_n$, so $a_i = b_i$ for all $i \in \{1, \ldots, n\}$, as $(v_1, \ldots, v_n)$ is linearly independent.
- Conversely, if $(v_1, \ldots, v_n)$ satisfies the two stated properties then it is certainly a linearly independent spanning set.

$\square$

The importance of bases

- Bases are extremely important in the study of vector spaces.
- Like the prime numbers generate the integers, a vector space is generated by a basis.
- In other words, if you have a basis, then you know the space.
- There are natural questions we can ask about bases.
- Does every vector space have one? Can a space have more than one?
- If a space has two (or more) possible bases, does it matter what basis we choose?
- We will see answers to these questions soon.

Removing redundant vectors from a spanning list

Lemma 3

Let V be a vector space over $\mathbb{F}$, and let $v_1, \dots v_n \in V$. Suppose that $(v_1, \dots, v_n)$ is linearly dependent. Then there is $j \in \{1, \dots, n\}$ such that:

1. $v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$.
2. $\operatorname{span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n) = \operatorname{span}(v_1, \dots, v_n)$.

Proof.

- Since $(v_1, \dots, v_n)$ is linearly dependent there are $a_1, \dots, a_n \in \mathbb{F}$ with $a_1 v_1 + \dots + a_n v_n = 0$ and at least one $a_i \neq 0$.
- Let j be the largest value such that $a_j \neq 0$.
- Then $a_1 v_1 + \dots + a_j v_j = 0$, and, since $a_j \neq 0$ we can rewrite this as $v_j = -\frac{a_1}{a_j} v_1 - \dots - \frac{a_{j-1}}{a_j} v_{j-1}$.
- This proves (1), and (2) follows easily from (1).

$\square$

Linearly independent lists cannot be bigger than spanning lists

Proposition 4

Let V be a vector space over $\mathbb{F}$, let $(u_1, \ldots, u_k)$ be linearly independent, and let $(v_1, \ldots, v_n)$ span V. Then $k \leq n$.

Proof.

- We will use lemma 3 multiple times.
- The idea is to replace elements of $(v_1, \ldots, v_n)$ with different elements of $(u_1, \ldots, u_k)$, till we have used all the elements of $(u_1, \ldots, u_k)$.
- Being able to do this implies that $k \leq n$.

Proof continued

- Consider the list $(u_1, v_1, \ldots, v_n)$.
- By lemma 3 there is an element w_1 of $(u_1, v_1, \ldots, v_n)$ such that w_1 is in the span of the part of the list $(u_1, v_1, \ldots, v_n)$ that precedes it.
- Obviously we can't have $w_1 = u_1$, so w_1 is in $(v_1, \ldots, v_n)$.
- Let e.g. $(v_1, \ldots, v_n) \setminus \{w_1\}$ be $(v_1, \ldots, v_n)$ with w_1 removed.
- Applying lemma 3 to $(u_2, u_1, v_1, \ldots, v_n) \setminus \{w_1\}$ we get w_2 in the span of the part of the list $(u_1, v_1, \ldots, v_n)$ that precedes it.
- w_2 can't be in (u_2, u_1) as this is linearly independent.
- Apply lemma 3 to $(u_3, u_2, u_1, v_1, \ldots, v_n) \setminus \{w_1, w_2\}$ to get w_3 and so on. These lists all span V.
- In the end we get $(u_k, \ldots, u_1, v_1, \ldots, v_n) \setminus \{w_1, \ldots, w_k\}$, and each $w_i \in (v_1, \ldots, v_n)$.
- Thus $k \leq n$ as claimed.

Finite dimensional spaces

Definition 5

A vector space V is *finite dimensional* if it contains a finite spanning list $(v_1, \ldots, v_n)$. If V is not finite dimensional then it is *infinite dimensional*.

Obtaining bases from spanning/linearly independent lists

Theorem 6

Let V be a vector space over $\mathbb{R}$. Then:

1. *If $s = (v_1, \ldots, v_n)$ spans V, then s can be reduced to a basis for V.*
2. *If V is finite dimensional, and if $t = (u_1, \ldots, u_k)$ is linearly independent in V, then t can be extended to a basis for V.*

Proof.

- For (1), we apply lemma 3 as many times as we can. The resulting list has the same span as the original, but is linearly independent as we can't apply the lemma again.
- For (2), since V is finite dimensional it has a spanning list $(w_1, \ldots, w_m)$. Now, the list $(u_1, \ldots, u_k, w_1, \ldots, w_m)$ also spans V, and so, by (1), reduces to a basis for V.
- This reduction does not remove any elements of t, as t is linearly independent.

The existence of bases

Corollary 7

Every finite dimensional vector space has a basis.

Proof.

Just reduce the finite spanning list to a basis. □

- Every *infinite* dimensional vector space also has a basis, but this proof is more difficult.
- We need an infinite choice principle.

The size of bases

Proposition 8

If V is a finite vector space then every basis for V has the same length.

Proof.

- Let s and t be bases for V.
- Then, as s is linearly independent and t spans V, by proposition 4, we must have $|s| \leq |t|$.
- But t is also linearly independent, and s also spans V, so by the same proposition we also have $|t| \leq |s|$.
- So $|s| = |t|$ as claimed.

□

Defining dimension

Definition 9

If V is a finite dimensional vector space, then we define the *dimension* of V to be the size of its bases. We use $\dim(V)$ to denote the dimension of V.

Example 10

1. The vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ provide a basis for $\mathbb{R}^3$. So $\dim(\mathbb{R}^3) = 3$.
2. The vectors $(2, 0, 1)$, $(2, 3, 0)$ and $(0, 6, -1)$ also provide a basis for $\mathbb{R}^3$.
3. The vectors $(1, 2, 3)$, $(-1, -1, 0)$, $(1, 1, 1)$ and $(3, -2, 0)$ must be linearly dependent in $\mathbb{R}^3$.
4. The vectors, $1, x, x^2, x^4, \ldots$ provide a basis for $\mathbb{R}[x]$, which is infinite dimensional.

Spanning/linearly independent lists of the right size are bases

Theorem 11

Let V be a finite dimensional vector space. Then:

1. *If s is a spanning list for V and $|s| = \dim(V)$ then s is a basis for V.*
2. *If t is a linearly independent list in V and $|t| = \dim(V)$ then t is a basis for V.*

Proof.

1. If s spans V then s can be reduced to a basis, s', for V. By proposition 8 we must have $|s'| = \dim(V) = |s|$, so s' must be equal to s.

2. If t is linearly independent then t can be extended to a basis, t'. for V. We have $|t'| = \dim(V) = |t|$, so t is a basis for V.

$\square$

Subspaces of finite dimensional spaces

Proposition 12

Every subspace of a finite dimensional vector space is finite dimensional.

Proof.

- Let V be a finite dimensional vector space and let U be a subspace of V.
- If $U = \{0\}$ then the empty list spans U.
- If $U \neq \{0\}$ then we construct a basis for U by recursion:
  - Since $U \neq \{0\}$ we can choose $v_1 \in U \setminus \{0\}$. Define $s_1 = (v_1)$.
  - Given linearly independent $s_i = (v_1, \ldots, v_i)$ in U, if s_i does not span U then there is $v_{i+1} \in U \setminus \mathrm{span}(s_i)$.
  - In this case define $s_{i+1} = (v_1, \ldots, v_i, v_{i+1})$.
- s_i is linearly independent for all i, and $|s_i| \leq \dim(U)$.
- There is k with $|s_k| = \dim(U)$. This s_k is a basis for U.

$\square$

The dimension of subspaces

Corollary 13

If V is a finite dimensional vector space and U is a subspace of V, then $\dim(U) \leq \dim(V)$.

Proof.

- Let $t = (v_1, \ldots, v_n)$ be a basis for V.
- Let $s = (u_1, \ldots, u_k)$ be a basis for U.
- Then s is linearly independent in V, and t spans V, so $|s| \leq |t|$.
- Thus $\dim(U) \leq \dim(V)$ as claimed.

□

Subspaces and direct sums

Proposition 14

Let V be a finite dimensional vector space, and let U be a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$.

Proof.

- Let $s = (u_1, \ldots, u_k)$ be a basis for U.
- Then s is linearly independent in V, so s can be extended to a basis $(u_1, \ldots, u_k, w_1, \ldots, w_m)$ for V.
- Define W to be $\mathrm{span}(w_1, \ldots, w_m)$.
- To show $V = U \oplus W$ we check $V = U + W$, and $U \cap W = \{0\}$.
- $V = U + W$ as $(u_1, \ldots, u_k, w_1, \ldots, w_m)$ spans V.
- $U \cap W = \{0\}$ because $(u_1, \ldots, u_k)$ is basis for U, $(w_1, \ldots, w_m)$ is a basis for W, and $(u_1, \ldots, u_k, w_1, \ldots, w_m)$ is linearly independent.

$\square$

The dimension of a sum

Proposition 15

Let V be a finite dimensional vector space, and let U and W be subspaces of V. Then $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$.

Proof.

- Let $(v_1, \ldots, v_n)$ be a basis for $U \cap W$.
- We can extend $(v_1, \ldots, v_n)$ to a basis $(u_1, \ldots, u_k, v_1, \ldots, v_n)$ for U, and a basis $(v_1, \ldots, v_n, w_1, \ldots, w_m)$ for W.
- We claim that

  $$s = (u_1, \ldots, u_k, v_1, \ldots, v_n, w_1, \ldots, w_m)$$

  is a basis for $U + W$.
- s clearly spans $U + W$, so we must check linear independence.

The dimension of a sum - proof continued

- Suppose that

$$a_1 u_1 + \ldots + a_k u_k + b_1 v_1 + \ldots + b_n v_n + c_1 w_1 + \ldots + c_m w_m = 0.$$

- Then

$$c_1 w_1 + \ldots + c_m w_m = -a_1 u_1 - \ldots - a_k u_k - b_1 v_1 - \ldots - b_n v_n,$$

- So $c_1 w_1 + \ldots + c_m w_m \in U \cap W$, and there are $b'_1, \ldots b'_n \in \mathbb{F}$ with $c_1 w_1 + \ldots + c_m w_m = b'_1 v_1 + \ldots + b'_n v_n$. I.e.

$$c_1 w_1 + \ldots + c_m w_m - b'_1 v_1 - \ldots - b'_n v_n = 0.$$

- But $(v_1, \ldots, v_n, w_1, \ldots, w_m)$ is linearly independent, so $c_i = 0$ for all i.
- So $a_1 u_1 + \ldots + a_k u_k + b_1 v_1 + \ldots + b_n v_n = 0$.
- As $(u_1, \ldots, u_k, v_1, \ldots, v_n)$ is linearly independent $a_i = b_j = 0$ for all i, j.
- So s is linearly independent as required.