ITCS 531: Logic 3 - soundness, completeness and compactness

Rob Egrot

Two concepts of consequence

- We have two operations $\models$ and $\vdash$.
- $\Gamma \models \phi$ when ϕ is a logical consequence of Γ according to truth tables.
- $\Gamma \vdash \phi$ when ϕ is deducible from Γ according to the deduction rules.
- $\models$ captures a notion of truth derived from the 'meaning' of formulas.
- $\vdash$ captures a notion of truth based on the 'structure' of the formulas.

Soundness and completeness

Definition 1 (Sound)

A formal deduction system for propositional logic is *sound* if whenever $\Gamma \vdash \phi$, we also have $\Gamma \models \phi$.

Definition 2 (Complete)

A formal deduction system for propositional logic is *complete* if whenever $\Gamma \models \phi$, we also have $\Gamma \vdash \phi$.

- The main purpose of this class is to prove the natural deduction system with the $\neg\neg_E$ rule is sound and complete.
- In other words, $\models$ and $\vdash$ are different ways of talking about the same thing.

Some notation

- We will often write things like $\Gamma \models \bot$.
- This means *there is no assignment that makes every sentence in* Γ *true*.
- In other words, every assignment of true or false to the basic propositions makes at least one sentence in Γ false according to the truth table.

Soundness

Theorem 3

The natural deduction system for propositional logic is sound.

Proof.

- We are trying to prove that for all Γ, χ we have $\Gamma \vdash \chi$ implies $\Gamma \models \chi$.
- I.e. if χ is deducible from Γ, then every assignment that makes Γ true also makes χ true.
- We will use a form of induction on the 'length' of the proof of χ from Γ.
- 'Length' here means number of uses of deduction rules.
- The base case is easy - if we only use one deduction rule then χ is either in Γ or is $\top$. In either case every assignment making Γ true also makes χ true.

$\square$

Soundness - inductive steps 1

We think about the last deduction rule used in the deduction of χ.

$\top_I$: In this case $\chi = \top$, which is true for every assignment.

$\bot_E$:
- The last step is deriving χ from $\bot$, where $\bot$ has first been derived from Γ. I.e. $\Gamma \vdash \bot$.
- By inductive hypothesis, $\Gamma \models \bot$. In other words, there is no assignment satisfying Γ.
- So $\Gamma \models \chi$, because there are no assignments satisfying Γ to worry about!

Soundness - inductive steps 2

$\wedge_I$:

- Here we deduce $\chi = \phi \wedge \psi$ from ϕ and ψ, with $\Gamma \vdash \phi$ and $\Gamma \vdash \psi$.
- By the inductive hypothesis we have $\Gamma \models \phi$ and $\Gamma \models \psi$.
- I.e. any assignment that satisfies Γ will satisfy both ϕ and ψ.
- But then the truth table says it will also satisfy $\phi \wedge \psi$.

$\wedge_{E_l}$:

- Here $\chi = \phi$, which we deduce from $\phi \wedge \psi$, and $\Gamma \vdash \phi \wedge \psi$.
- Again, by the inductive hypothesis we have $\Gamma \models \phi \wedge \psi$.
- So any assignment that satisfies Γ also satisfies $\phi \wedge \psi$.
- But then it must also satisfy ϕ.

Remaining cases in the notes and exercises.

Completeness

Theorem 4

The natural deduction system for propositional logic is complete.

- We must prove that for all Γ, χ we have $\Gamma \models \chi$ implies $\Gamma \vdash \chi$.
- This is harder than soundness.
- Before we can start the proof properly we will need to develop some more theory.

Dealing with $\bot$

Lemma 5

Let Γ be a set of sentences, then:

1. $\Gamma \models \neg\phi \iff \Gamma \cup \{\phi\} \models \bot$, *and*
2. $\Gamma \vdash \neg\phi \iff \Gamma \cup \{\phi\} \vdash \bot$.

Proof.

- If $\Gamma \models \neg\phi$ then every assignment satisfying Γ satisfies $\neg\phi$.
- So no assignment satisfies $\Gamma \cup \{\phi\}$ i.e. $\Gamma \cup \{\phi\} \models \bot$.
- Conversely, if no assignment satisfies $\Gamma \cup \{\phi\}$, then every assignment that satisfies Γ must satisfy $\neg\phi$ i.e. $\Gamma \models \neg\phi$.
- For part 2, suppose $\Gamma \vdash \neg\phi$. Then we can derive $\bot$ from $\Gamma \cup \{\phi\}$ using rule $\neg_E$.
- Conversely, if $\Gamma \cup \{\phi\} \vdash \bot$, then starting with Γ we can apply rule $\neg_I$ with assumption ϕ to derive $\neg\phi$, and so $\Gamma \vdash \neg\phi$.

$\square$

An equivalent statement of completeness

Definition 6 (Consistent)

A set of sentences Γ is *consistent* if $\Gamma \nvdash \bot$. I.e. if we cannot deduce a contradiction from it.

Lemma 7

Completeness of the natural deduction system with $\neg\neg_E$ is equivalent to the statement:

$$\textit{Every consistent set of sentences is satisfiable.} \qquad (\dagger)$$

An equivalent statement of completeness - proof 1

- Completeness can be stated as $\Gamma \models \phi \implies \Gamma \vdash \phi$ for all sets of sentences Γ
- $(\dagger)$ translates as $\Gamma \models \bot \implies \Gamma \vdash \bot$ for all sets of sentences Γ.

Now, assuming $(\dagger)$, we have:

$$
\begin{aligned}
\Gamma \models \phi &\iff \Gamma \models \neg\neg\phi \\
&\iff \Gamma \cup \{\neg\phi\} \models \bot \\
&\implies \Gamma \cup \{\neg\phi\} \vdash \bot \\
&\iff \Gamma \vdash \neg\neg\phi \\
&\iff \Gamma \vdash \phi
\end{aligned}
$$

So $\Gamma \models \phi \implies \Gamma \vdash \phi$, which is the statement of completeness.

An equivalent statement of completeness - proof 2

- Conversely, assume completeness, and suppose $\Gamma \models \bot$.
- Then Γ cannot be empty, so let $\phi \in \Gamma$.
- We have:

$$\begin{aligned}
\Gamma \models \bot &\iff \Gamma \setminus \{\phi\} \cup \{\phi\} \models \bot \\
&\iff \Gamma \setminus \{\phi\} \models \neg\phi \\
&\implies \Gamma \setminus \{\phi\} \vdash \neg\phi \\
&\iff \Gamma \setminus \{\phi\} \cup \{\phi\} \vdash \bot \\
&\iff \Gamma \vdash \bot
\end{aligned}$$

So $\Gamma \models \bot \implies \Gamma \vdash \bot$, which is (†).

Maximal consistent theories

Definition 8 (maximal consistent)

A consistent set of sentences Γ is maximal consistent if for every sentence ϕ, either $\phi \in \Gamma$ or $\neg\phi \in \Gamma$.

Note that if Γ is maximal consistent, then a sentence is deducible from Γ if and only if it is actually in Γ. I.e. for all sentences ϕ we have $\Gamma \vdash \phi \iff \phi \in \Gamma$.

Maximal consistent theories

Lemma 9

For every consistent Γ there is a maximal consistent Γ' with $\Gamma \subseteq \Gamma'$.

- We will extend Γ to a maximal consistent theory using a recursive construction.
- Let $\phi_0, \phi_1, \phi_2, \ldots$ list all the possible formulas.
- Define $\Gamma_0 = \Gamma$, given Γ_n define

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\phi_n\} \text{ if consistent} \\ \Gamma_n \cup \{\neg\phi_n\} \text{ otherwise.} \end{cases}$$

- Γ_0 is consistent, and if Γ_n is consistent then so is Γ_{n+1}.
  - Remember Γ_{n+1} is $\Gamma_n \cup \{\phi_n\}$ or $\Gamma_n \cup \{\neg\phi_n\}$.
  - So if $\Gamma_{n+1} \vdash \bot$, then either $\Gamma_n \vdash \neg\phi_n$ or $\Gamma_n \vdash \phi_n$ (by lemma 5).
  - If $\Gamma_n \vdash \phi_n$ then $\Gamma_{n+1} = \Gamma_n \cup \{\phi_n\}$ is consistent as Γ_n is.
  - If $\Gamma_n \vdash \neg\phi_n$ then $\Gamma_{n+1} = \Gamma_n \cup \{\neg\phi_n\}$ is consistent as Γ_n is.

Maximal consistent theories - continued

- Define $\Gamma' = \bigcup_{n \in \mathbb{N}} \Gamma_n$.
- Suppose $\Gamma' \vdash \bot$. Then $\Gamma_n \vdash \bot$ for some n.
- I.e. Γ_n is not consistent. But we proved that Γ_n must be consistent.
- So we see that Γ' must be consistent too.
- Γ' is obviously maximal consistent too.
- Clearly $\Gamma \subseteq \Gamma'$, so we have proved the lemma.

Proving completeness 1

- We want to prove natural deduction with $\neg\neg_E$ is complete (theorem 4).
- By lemma 6 it is sufficient to prove that if Γ is consistent, then it is satisfiable.
- So let Γ be consistent.
- We will construct an assignment v that makes every formula in Γ true.
- By lemma 8 there is maximal consistent Γ' with $\Gamma \subseteq \Gamma'$.
- We will build v that satisfies Γ' - this v will obviously satisfy Γ too.

Proving completeness 2

- Define v so that for each basic proposition p we have

$$v(p) = \begin{cases} T \text{ if } p \in \Gamma' \\ F \text{ if } p \notin \Gamma' \end{cases}$$

- v is an assignment because Γ' is maximal consistent.
- Now let $\phi \in \Gamma$ - we must show that $v(\phi) = T$.
- We use induction on formula construction - the base case where $\phi = p$ is automatic.
- Let $\phi = \neg\psi$ and $v(\psi) = T \iff \psi \in \Gamma'$.
  - Then $v(\phi) = T \iff v(\psi) = F \iff \psi \notin \Gamma' \iff \phi \in \Gamma'$
- Let $\phi = \psi_1 \vee \psi_2$ and $v(\psi_i) = T \iff \psi_i \in \Gamma'$ for $i = 1, 2$.
  - Then

$$\begin{aligned} v(\phi) = T &\iff v(\psi_1) = T \text{ and/or } v(\psi_2) = T \\ &\iff \psi_1 \in \Gamma' \text{ and/or } \psi_2 \in \Gamma' \\ &\iff \phi \in \Gamma'. \end{aligned}$$

Proving completeness 3

- Remember $\{\neg, \vee\}$ is functionally complete.
- So every formula ϕ is logically equivalent to some ϕ' using only connectives $\neg$ and $\vee$.
- Let $\phi \in \Gamma'$, and suppose $v(\phi) = F$.
- Then $v(\phi') = F$ as $\phi \models \dashv \phi'$.
- So $\phi' \notin \Gamma'$, by previous slide.
- So $\neg\phi' \in \Gamma'$, i.e. $\Gamma' \vdash \neg\phi'$, by maximality.
- So $\Gamma' \models \neg\phi'$, by soundness (theorem 3).
- So $\Gamma' \models \neg\phi$.
- Contradiction as $\Gamma' \models \neg\phi$ and $\Gamma' \models \phi$.
- So $v(\phi) = T$ after all - this completes the proof.

Compactness

- Soundness and completeness are the big results uniting $\vdash$ and $\models$.
- There's another important result known as **compactness**.
- Roughly speaking, compactness translates statements about infinite structures into statements about finite ones.
- Since humans are poorly equipped to understand infinity, this can be very useful.
- You will see a precise statement of compactness for propositional logic in the exercises.