ITCS 531: L4 homework solutions

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L4 Q1

Which of the following are formulas? In the formulas identify the free and bound variables.

a) $\forall x(f(c, f(x, d))$ – *not a formula*.

b) $R(x, y, z) \vee S(f(c, d))$ – *all free*.

c) $\exists y(R(x) \vee \forall z S(f(x, z), c))$ – *x occurs free, z occurs bound*.

d) $\exists y(R(x) \vee \forall y S(f(x, y), c))$ – *Here y occurs bound, and x occurs free. Note $\exists y$ is null*.

e) $R(x) \wedge \exists x S(x)$ – *x occurs both free and bound.*.

L4 Q2 (1)

Let $\mathscr{L} = \{0, 1, +, \times\}$ be the language of basic arithmetic. Let $\phi = \forall x(\neg(x \approx 0) \to \exists y(x \times y \approx 1))$. Let $\mathbb{N}$ and $\mathbb{R}$ have their usual meanings, and interpret $\mathscr{L}$ into these languages by giving the non-logical symbols of $\mathscr{L}$ their usual meanings.

a) Does $\mathbb{N} \models \phi$?

b) Does $\mathbb{R} \models \phi$?

- ϕ is the statement that every non-zero element has an inverse.
- This is not true in $\mathbb{N}$, but it is true in $\mathbb{R}$.

L4 Q2 (2)

c) Let $n \in \mathbb{N}$ with $n \geq 2$, and let $\mathbb{Z}_n$ be the integers mod n. For what values of n does $\mathbb{Z}_n \models \phi$?

d) Let $A = (\{a, b\}, I)$, where I interprets 0 and 1 as a and b respectively, $b \times b = b$, and $a \times b = a \times a = a$. Does $A \models \phi$?

- Remember in $\mathbb{Z}_n$, an element has an inverse iff it is coprime with n.
- So $\mathbb{Z} \models \phi$ iff every non-zero element is coprime with n. I.e. iff n is prime.

- In the structure A, the only non-zero element is b.
- Does b have an inverse? Yes, because $b \times b = b = 1$.
- So $A \models \phi$.

L4 Q2 (3)

e) Let $\psi = \exists x \forall y (\neg(y \approx 0) \to (x \times y \approx 1))$. Which of the structures in parts a)-d) is a model for ψ?

- ψ is the statement that there's an element which is an inverse to *all* non-zero elements.
- This is clearly not true in $\mathbb{N}$, $\mathbb{R}$ or $\mathbb{Z}_n$ for $n > 2$.
- $\mathbb{Z}_2 \models \psi$ as 1 is the only non-zero element of $\mathbb{Z}_2$.
- ψ is true in A, as b is the only non-zero element of A, and $b \times b = b = 1$.

L4 Q3 (1)

Let R and S be unary predicates. Let $\phi = \forall x R(x) \vee \forall x S(x)$, and let $\psi = \forall x \forall y (R(x) \vee S(y))$. Prove that $\phi \models \dashv \psi$.

- Let A be a structure of the appropriate kind.
- Suppose first that $A \models \phi$.
- Then either every element of A satisfies R, or every element of A satisfies S.
- In the former case, for every pair of elements $a, b \in A$ we must have $R(a) \vee S(b)$, because $R(a)$ must be true.
- So $A \models \forall x \forall y (R(x) \vee S(y))$. This shows $\phi \models \psi$.

L4 Q3 (2)

Let R and S be unary predicates. Let $\phi = \forall x R(x) \vee \forall x S(x)$, and let $\psi = \forall x \forall y (R(x) \vee S(y))$. Prove that $\phi \models\,\dashv \psi$.

- Conversely, suppose $A \models \psi$.
- Suppose wlog that A does not satisfy $\forall x R(x)$.
- Then there is $a \in A$ with $A \not\models R(a)$. I.e. $R(a)$ is not true.
- As $A \models \forall x \forall y (R(x) \vee S(y))$, $A \models \forall y (R(a) \vee S(y))$.
- As $R(a)$ is not true in A, it follows that $A \models \forall y S(y)$.
- By changing the variable name we have $A \models \forall x S(x)$, and so $A \models \forall x R(x) \vee \forall x S(x)$.
- This shows $\psi \models \phi$, and we are done.

L4 Q4 (1)

Let ϕ be an $\mathscr{L}$-formula, let A be an $\mathscr{L}$-structure, and let v be an assignment for $\mathscr{L}$ to A with $A, v \models \phi$. Prove that $A, u \models \phi$ for all assignments u such that $u(x) = v(x)$ for all variables x occurring free in ϕ.

- We induct on formula construction.
- It's obviously true for atomic formulas, because these have no bound variables.
- Suppose now that it's true for formulas ϕ and ψ.

L4 Q4 (2)

$\neg\phi$:

- Suppose $A, v \models \neg\phi$.
- Then $A, v \not\models \phi$.
- Suppose that u agrees with v about the free variables of ϕ.
- If $A, u \models \phi$, by the inductive hypothesis we would have $A, v \models \phi$, which would be a contradiction.
- So we must have $A, u \models \neg\phi$ as required.

$\phi \vee \psi$

- Suppose $A, v \models \phi \vee \psi$.
- Then, wlog we can assume that $A, v \models \phi$.
- Let u be an assignment agreeing with v about the free variables of $\phi \vee \psi$.
- Then u agrees with v about the free variables of ϕ.
- So $A, u \models \phi$, and thus $A, u \models \phi \vee \psi$ as required.

L4 Q4 (3)

$\forall x \phi$:

- Suppose $A, v \models \forall x \phi$, and let u be an assignment agreeing with v about the free variables of $\forall x \phi$.
- We must show that $A, u \models \forall x \phi$.
  - I.e. $A, u' \models \phi$ for all u' agreeing with u except possibly at x.
- Let u' be such an assignment, and let v' be an assignment agreeing with v except possibly at x, where we define $v'(x) = u'(x)$.
- Then, if F is the set of variables occurring free in ϕ, we have
  - v' agrees with v on $F \setminus \{x\}$.
  - u agrees with v on $F \setminus \{x\}$.
  - u' agrees with u on $F \setminus \{x\}$.
  - u' agrees with v' on F.
- So,

$$\begin{aligned} A, v \models \forall x \phi &\implies A, v' \models \phi \text{ (by definition of } \models\text{)} \\ &\implies A, u' \models \phi \text{ (by the inductive hypothesis)}, \end{aligned}$$

- And so $A, u \models \forall x \phi$ as required.