ITCS 531: L2 homework solutions

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Deduction rules

$$\top_I: \quad \frac{}{\top}$$

$$\wedge_I: \quad \frac{\phi \qquad \psi}{\phi \wedge \psi}$$

$$\vee_{I_l}: \quad \frac{\phi}{\phi \vee \psi}$$

$$\vee_{I_r}: \quad \frac{\psi}{\phi \vee \psi}$$

$$\neg_I: \quad \frac{\begin{array}{c}\underline{\underline{[\phi]}} \\ \bot\end{array}}{\neg\phi}$$

$$\to_I: \quad \frac{\begin{array}{c}\underline{\underline{[\phi]}} \\ \psi\end{array}}{\phi \to \psi}$$

$$\bot_E: \quad \frac{\bot}{\phi}$$

$$\wedge_{E_l}: \quad \frac{\phi \wedge \psi}{\phi}$$

$$\wedge_{E_r}: \quad \frac{\phi \wedge \psi}{\psi}$$

$$\vee_E: \quad \frac{\phi \vee \psi \qquad \begin{array}{c}\underline{\underline{[\phi]}} \\ \theta\end{array} \qquad \begin{array}{c}\underline{\underline{[\psi]}} \\ \theta\end{array}}{\theta}$$

$$\neg_E: \quad \frac{\phi \qquad \neg\phi}{\bot}$$

$$\to_E: \quad \frac{\phi \to \psi \qquad \phi}{\psi}$$

$$\neg\neg_E: \quad \frac{\neg\neg\phi}{\phi}$$

Q1

The following deduction tree proves that $\phi \to \psi$ can be deduced from $\neg\phi \vee \psi$ in intuitionistic propositional logic. Add labels indicating the rules used at each stage.

$$
(\vee_E)\ \dfrac{\neg\phi \vee \psi \qquad (\to_I)\ \dfrac{(\bot_E)\ \dfrac{(\neg_E)\ \dfrac{[\neg\phi]_1 \qquad [\phi]_2}{\bot}}{\psi}}{\phi \to \psi} \qquad \dfrac{\dfrac{[\psi]_1}{\psi} \qquad [\phi]_3}{\phi \to \psi}\ (\to_I)}{\phi \to \psi}
$$

Q2

What is being proved in the following deduction tree? Add labels indicating the rules at each stage.

$$
\begin{prooftree}
\AxiomC{$[\neg(\neg\phi \vee \psi)]_1$}
\AxiomC{$\phi \to \psi$}
\AxiomC{$[\phi]_2$}
\RightLabel{$(\to_E)$}
\BinaryInfC{ψ}
\RightLabel{$(\vee_{I_r})$}
\UnaryInfC{$\neg\phi \vee \psi$}
\LeftLabel{$(\neg_E)$}
\BinaryInfC{$\bot$}
\LeftLabel{$(\neg_I)$}
\UnaryInfC{$\neg\phi$}
\LeftLabel{$(\vee_{I_l})$}
\UnaryInfC{$\neg\phi \vee \psi$}
\AxiomC{$[\neg(\neg\phi \vee \psi)]_1$}
\UnaryInfC{$\neg(\neg\phi \vee \psi)$}
\RightLabel{$(\neg_E)$}
\BinaryInfC{$\bot$}
\RightLabel{$(\neg_I)$}
\UnaryInfC{$\neg\neg(\neg\phi \vee \psi)$}
\RightLabel{$(\neg\neg_E)$}
\UnaryInfC{$\neg\phi \vee \psi$}
\end{prooftree}
$$

This tree proves that $\neg\phi \vee \psi$ can be deduced from $\phi \to \psi$ in *classical* propositional logic.

Q3

Show that $(\phi \wedge \psi) \to (\psi \wedge \phi)$ can be deduced from an empty set of axioms.

$$
\begin{prooftree}
\AxiomC{$[\phi \wedge \psi]_1$}
\UnaryInfC{$\phi \wedge \psi$}
\LeftLabel{$(\wedge_{E_r})$}
\UnaryInfC{ψ}
\AxiomC{$[\phi \wedge \psi]_1$}
\UnaryInfC{$\phi \wedge \psi$}
\RightLabel{$(\wedge_{E_l})$}
\UnaryInfC{ϕ}
\LeftLabel{$(\wedge_I)$}
\BinaryInfC{$\psi \wedge \phi$}
\LeftLabel{$(\to_I)$}
\UnaryInfC{$(\phi \wedge \psi) \to (\psi \wedge \phi)$}
\end{prooftree}
$$

Q4

Show that we can deduce $\phi \wedge (\psi \vee \chi)$ if we start with $(\phi \wedge \psi) \vee (\phi \wedge \chi)$.

To save space let $\theta = (\phi \wedge \psi) \vee (\phi \wedge \chi)$.

$$
(\wedge_I)\ \dfrac{(\vee_E)\ \dfrac{\theta \qquad (\wedge_{E_l})\ \dfrac{[\phi \wedge \psi]}{\phi} \qquad \dfrac{[\phi \wedge \chi]}{\phi}\ (\wedge_{E_l})}{\phi} \qquad \dfrac{\theta \qquad (\vee_{I_l})\ \dfrac{(\wedge_{E_r})\ \dfrac{[\phi \wedge \psi]}{\psi}}{\psi \vee \chi} \qquad \dfrac{\dfrac{[\phi \wedge \chi]}{\chi}\ (\wedge_{E_r})}{\psi \vee \chi}\ (\vee_{I_r})}{\psi \vee \chi}\ (\vee_E)}{\phi \wedge (\psi \vee \chi)}
$$

Q5

Show that we can deduce $(\phi \wedge \psi) \vee (\phi \wedge \chi)$ if we start with $\phi \wedge (\psi \vee \chi)$.

$$
(\vee_E)\,\dfrac{
(\wedge_{E_r})\,\dfrac{\phi \wedge (\psi \vee \chi)}{\psi \vee \chi}
\qquad
(\vee_{I_l})\,\dfrac{(\wedge_I)\,\dfrac{(\wedge_{E_l})\,\dfrac{\phi \wedge (\psi \vee \chi)}{\phi} \qquad \dfrac{[\psi]_1}{\psi}}{\phi \wedge \psi}}{(\phi \wedge \psi) \vee (\phi \wedge \chi)}
\qquad
\dfrac{\dfrac{\dfrac{\phi \wedge (\psi \vee \chi)}{\phi}\,(\wedge_{E_l}) \qquad \dfrac{[\chi]_1}{\chi}}{\phi \wedge \chi}\,(\wedge_I)}{(\phi \wedge \psi) \vee (\phi \wedge \chi)}\,(\vee_{I_r})
}{(\phi \wedge \psi) \vee (\phi \wedge \chi)}
$$