ITCS 531: Number Theory 1 solutions

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Q1

Consider the following (false) theorem:

Theorem

If $a, b \in \mathbb{N}$ and $a = b$ then $a = 0$.

Proof.

$$
\begin{aligned}
a &= b \\
a^2 &= ab \\
a^2 - b^2 &= ab - b^2 \\
(a-b)(a+b) &= (a-b)b \\
a + b &= b \\
a &= 0
\end{aligned}
$$

□

What is wrong with this proof?

- Division by zero!

Q2

Use the well-ordering principle to show that

$$2 + 4 + 6 + \ldots + 2n = n(n+1).$$

- Suppose this is not true for all natural numbers.
- By well-ordering, let k be smallest number for which the identity is not true.
- k cannot be 1 as identity is true for $n = 1$.
- Identity must be true for $k - 1$. So we have

  $$2 + \ldots + 2(k-1) = (k-1)(k),$$

  and so, adding $2k$ to both sides we have

  $$2 + \ldots + 2(k-1) + 2k = (k-1)(k) + 2k.$$

  But

  $$(k-1)(k) + 2k = (k)(k+1),$$

  and so

  $$2 + \ldots + 2k = (k)(k+1).$$

- This is a contradiction because k is supposed to invalidate the identity.

Q3

Let $n \in \mathbb{N}$. If n^2 is even must n also be even? Give a proof or a counterexample.

- By the fundamental theorem of arithmetic, n^2 has a unique prime factorization.
- The same is true for n.
- Suppose $n = p_1 \dots p_k$.
- Then $n^2 = p_1 \dots p_k p_1 \dots p_k$.
- If n^2 is even then $2 | n^2$.
- Since $n^2 = p_1 \dots p_k p_1 \dots p_k$, we must have $2 | p_i$ for some $i \in \{1, \dots, k\}$.
- But this means $2 | n$.
- So n is even too.

Q4

Let $n \in \mathbb{N} \setminus \{0\}$. Then using theorem 1.2 prove that $\log_5(n)$ is either a natural number or irrational.

- If $5^{\frac{a}{b}} = n$ then $5^a = n^b$.
- n^b can be uniquely factorized into primes.
- Since $n^b = 5^a$ we know this factorization must just be $55\ldots5$ (a list of a fives).
- n must also have a unique factorization into primes.
- As $n^b = 55\ldots5$, the factorization of n must be a list of fives.
- I.e. $n = 5^k$ for some k.
- But if we take a product of b copies of this list of fives we get n^b, which is 5^a.
- I.e. $5^a = (5^k)^b$, so $a = kb$.
- This means that b must divide a.
- In other words, $\frac{a}{b}$ must be a natural number.

Q5

Is the result from exercise 4 still true if we replace 5 with 4?
Provide a proof or a counterexample.

- It's not true.
- For example, $\log_4 2 = \frac{1}{2}$.