ITCS 531: C1 homework solutions

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C1 Q1

Prove that if X and Y are disjoint finite sets, then the cardinal arithmetic operations agree with the usual arithmetic operations on $|X|$ and $|Y|$. In other words, prove that $|X \cup Y|$ is equal to the result of adding $|X|$ and $|Y|$ as normal, and do similar for the other two arithmetic operations we have defined.

- I'll just do X^Y.
- X^Y is the set of all functions from Y to X.
- How many functions are there?
- Such a function must map each element of Y to exactly one element of X.
- So, for each $y \in Y$ there are exactly $|X|$ choices.
- So, if $|X| = m$ and $|Y| = n$, we get m^n different functions.
- So $|X^Y| = m^n$ as required.

C1 Q2

Let X_i be countable for all $i \in \mathbb{N}$, and suppose $X_i \cap X_j = \emptyset$ for all $i \neq j \in \mathbb{N}$. Prove that $\bigcup_{i \in \mathbb{N}} X_i$ is countable.

- We will find an injective function from $\bigcup_{i \in \mathbb{N}} X_i$ to $\mathbb{N}$.
- Since $\mathbb{N} \times \mathbb{N}$ is countable, there is an injective $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.
- Since each X_i is countable, there are injective functions $g_i : X_i \to \mathbb{N}$ for all $i \in \mathbb{N}$.
- Define $g : \bigcup_{i \in \mathbb{N}} X_i \to \mathbb{N} \times \mathbb{N}$ by $g(x) = (i, g_i(x))$, where $x \in X_i$.
- This is well defined because $X_i \cap X_j = \emptyset$ for all $i \neq j$.
- Given $x_1 \in X_i$ and $x_2 \in X_j$, if $i \neq j$ then $(i, g_i(x_1)) \neq (j, g_j(x_2))$, as $i \neq j$,
- If $i = j$ but $x_1 \neq x_2$ then $g_i(x_1) \neq g_i(x_2)$, as g_i is injective.
- So g is injective.
- So $f \circ g : \bigcup_{i \in \mathbb{N}} X_i \to \mathbb{N}$ is the composition of two injective functions, and so is injective.

C1 Q3

Let X be a countable set. Prove that the set of all finite subsets of X is countable.

- Let $f : X \to \mathbb{N}$ be injective.
- Arrange the prime numbers in a list as $p_0, p_1, \ldots$.
- The set of primes is infinite, so this is a countably infinite list.
- Given $S = \{x_1, \ldots, x_n\} \subseteq X$, define $g(S) = p_{f(x_0)} \times p_{f(x_1)} \times \ldots \times p_{f(x_n)}$.
- Then g is a function from the set of all finite subsets of X to $\mathbb{N}$.
- If $S_1 \neq S_2$ then $g(S_1)$ and $g(S_2)$ will have different prime factorizations, and so by FTA are different numbers.
- So g is 1-1.
- So the set of finite subsets of X is countable.

C1 Q4

Let X be a set, let $\wp(X)$ be the powerset of X.

a) Define a simple injective function from X to $\wp(X)$.

b) Prove that there is no surjective function from X to $\wp(X)$.

c) What does this tell us about the relationship between $|X|$ and $|\wp(X)|$?

a) $f(x) = \{x\}$.

b)
- Let $f : X \to \wp(X)$ be onto.
- Let $S = \{x \in X : x \notin f(x)\}$.
- As f is onto there is $z \in X$ with $f(z) = S$.
- $z \in S \iff z \notin f(z) \iff z \notin S$.

c) $|X| < |\wp(X)|$.