ITCS 531: Linear Algebra - Inner products on real vector spaces

Rob Egrot

What is an inner product?

- We will work with vector spaces over $\mathbb{R}$.
- Everything can be adapted for $\mathbb{C}$, but the definitions are more complicated.
- An inner product is a generalization of the dot product.
- E.g. in $\mathbb{R}^3$, we have $(a, b, c) \cdot (d, e, f) = ad + be + cf$.
- Many geometric ideas for Euclidean spaces can be described using dot products.
- If a vector space has an inner product, then our geometric intuitions apply to it in some sense.

The definition of an inner product

Definition 1

Let V be a vector space over $\mathbb{R}$. An *inner product* for V is a function that takes a pair $(u, v) \in V^2$ to a value $\langle u, v \rangle \in \mathbb{R}$, satisfying the following properties:

1. $\langle v, v \rangle \geq 0$ for all $v \in V$ (positivity).
2. $\langle v, v \rangle = 0 \iff v = 0$ (definiteness).
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$ (additivity in first slot).
4. $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$ for all $\lambda \in \mathbb{R}$ and for all $u, v \in V$ (homogeneity in first slot).
5. $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$ (symmetry).

Definition 2

A vector space with an inner product is an *inner product space*.

Examples of inner products

Example 3

1. It's easy to check that the dot product as it is usually defined is indeed an inner product.

2. It can be shown that the set of continuous real valued functions on the interval $[-1, 1]$ is a vector space over $\mathbb{R}$. We can define an inner product on this space using $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$.

Basic properties of inner products

Proposition 4

The following properties hold in all real inner product spaces:

1. *Given* $v \in V$, *we can define a linear map* $\langle -, v \rangle : V \to \mathbb{R}$ *by defining* $\langle -, v \rangle(u) = \langle u, v \rangle$ *for all* $u \in V$.

2. $\langle v, 0 \rangle = \langle 0, v \rangle = 0$ *for all* $v \in V$.

3. $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$ *for all* $u, v, w \in V$.

4. $\langle u, \lambda v \rangle = \lambda \langle u, v \rangle$ *for all* $\lambda \in \mathbb{R}$ *and for all* $u, v \in V$

Basic properties of inner products - proof

Proof.

1. "$\langle -, v\rangle(u) = \langle u, v\rangle$ *is a linear map*".
   - Given $u_1, u_2 \in V$ we have $\langle u_1 + u_2, v\rangle = \langle u_1, v\rangle + \langle u_2, v\rangle$, by additivity in the first slot.
   - We also have $\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$ by homogeneity in the first slot.
2. "$\langle v, 0\rangle = \langle 0, v\rangle = 0$"
   - That $\langle 0, v\rangle = 0$ follows from part (1) and the fact that $T(0) = 0$ for all linear maps.
   - We then have $\langle v, 0\rangle = 0$ by symmetry.
3. "$\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$".
   - $\langle u, v + w\rangle = \langle v + w, u\rangle$ by symmetry.
   - The result follows from additivity and symmetry again.
4. "$\langle u, \lambda v\rangle = \lambda\langle u, v\rangle$".
   - Symmetry and homogeneity in the first slot.

$\square$

Norms

- In every real inner product space V we can calculate the value of $\langle v, v \rangle$.
- This must be non-negative.
- This inspires the following definition:

Definition 5

If V is an inner product space, then given $v \in V$, the *norm* of v, $||v||$, is defined by

$$\|v\| = \sqrt{\langle} v, v \rangle.$$

Example 6

In $\mathbb{R}^2$ with the usual dot product, the norm of a vector (a, b) is $\sqrt{(}a^2 + b^2)$. I.e., it is the Euclidean distance of the point (a, b) from the origin.

Basic properties of norms

Proposition 7

The following hold for all real inner product spaces V, and for all $v \in V$:

1. $\|v\| = 0 \iff v = 0$.
2. $\|\lambda v\| = |\lambda| \|v\|$ *for all* $\lambda \in \mathbb{R}$.

Proof.

- (1) follows immediately from definiteness of the inner product.
- (2) follows from homogeneity in the first slot and proposition 4(4).

$\square$

Geometric interpretation of dot product

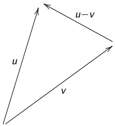
Proposition 8

Given $u, v \in \mathbb{R}^2 \setminus \{0\}$*, we have*

$$\langle u, v \rangle = \|u\| \|v\| \cos\theta,$$

where θ *is the angle between* u *and* v *when these are thought of as arrows beginning at the origin.*

Geometric interpretation of dot product - proof



- In $\mathbb{R}^2$ the norm of a vector is its length.
- Law of cosines: $\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos\theta$.
- Now, $\|u - v\|^2 = \langle u - v, u - v\rangle$, by definition, and

$$\begin{aligned}\langle u - v, u - v\rangle &= \langle u, u - v\rangle - \langle v, u - v\rangle \\ &= \langle u, u\rangle - \langle u, v\rangle - \langle v, u\rangle + \langle v, v\rangle \\ &= \langle u, u\rangle + \langle v, v\rangle - 2\langle u, v\rangle \\ &= \|u\|^2 + \|v\|^2 - 2\langle u, v\rangle.\end{aligned}$$

- So $\langle u, v\rangle = \|u\|\|v\| \cos\theta$ as required.

Orthogonality

Definition 9

If u and v are vectors in an inner product space, then we say u and v are *orthogonal* if $\langle u, v \rangle = 0$.

- By proposition 8, two non-zero vectors in $\mathbb{R}^2$ are orthogonal if and only if the cosine of the angle between them is 0.
- I.e. if and only if they are perpendicular.
- You can think of 'being orthogonal' as a generalization of the concept of 'being perpendicular'.

Orthogonality of zero

Lemma 10

1. 0 *is orthogonal to everything.*
2. 0 *is the only thing that is orthogonal to itself.*

Proof.
These follows from proposition 4(2) and the definiteness of inner products, respectively. □

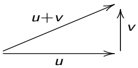
Pythagoras for inner products

Proposition 11

If u and v are vectors in a real inner product space, then

$$\|u\|^2 + \|v\|^2 = \|u + v\|^2 \iff u \text{ and } v \text{ are orthogonal.}$$

Proof.

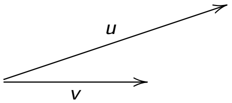


$$\begin{aligned}
\|u + v\|^2 &= \langle u + v, u + v\rangle \\
&= \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle \\
&= \|u\|^2 + \|v\|^2 + 2\langle u, v\rangle.
\end{aligned}$$

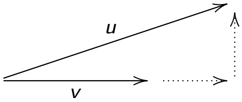
So $\|u\|^2 + \|v\|^2 = \|u + v\|^2$ if and only if $\langle u, v\rangle = 0$. $\square$

Some geometric intuition 1

- We can think of vectors as arrows. E.g:



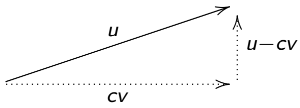
- Geometric intuition says we should be able to turn this into a right angled triangle by drawing some lines. I.e:



- In the picture above we have essentially extended v as far as we need, then added a third line.
- 'Extending' v is multiplying by some scalar c to get cv.

Some geometric intuition 2

- The associated vector equation is $u = cv + (u - cv)$.



- In an inner product space, the triangle being 'right angled' corresponds to the vectors v and $(u - cv)$ being orthogonal.
- I.e. $\langle v, u - cv \rangle = 0$.
- We should always be able to find a scalar value c such that this is true (so long as u and v are non-zero).
- From the properties of the inner product we have

$$\langle v, u - cv \rangle = 0 \iff \langle v, u \rangle - c\|v\|^2 = 0.$$

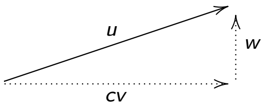
- So we can take

$$c = \frac{\langle v, u \rangle}{\|v\|^2}.$$

Some geometric intuition 3

Lemma 12

Let V be a real inner product space, let $u, v \in V$ and suppose $v \neq 0$. Then there is $w \in V$ such that $\langle v, w \rangle = 0$, and $u = cv + w$ for some $c \in \mathbb{R}$.



Proof.

Set $c = \frac{\langle v,u \rangle}{\|v\|^2}$ and $w = u - cv$. $\square$

The Cauchy-Schwarz inequality

Theorem 13 (Cauchy-Schwarz)

Let V be an inner product space, and let $u, v \in V$. Then

$$|\langle u, v\rangle| \leq \|u\|\|v\|.$$

Moreover, we have equality if and only if u is a scalar multiple of v or vice versa.

The Cauchy-Schwarz inequality - proof

- If v is zero, then everything is zero, and there is nothing to do.
- Let $v \neq 0$ and write $u = cv + w$ where $\langle v, w \rangle = 0$.
- By Pythagoras we have $\|u\|^2 = c^2\|v\|^2 + \|w\|^2$.
- We have $c = \frac{\langle v,u \rangle}{\|v\|^2}$, so

$$\|u\|^2 = \frac{\langle v, u \rangle^2}{\|v\|^4}\|v\|^2 + \|w\|^2.$$

- As $\|w\|^2 \geq 0$ this implies

$$\|u\|^2 \geq \frac{\langle v, u \rangle^2}{\|v\|^4}\|v\|^2.$$

- So $\|u\|\|v\| \geq |\langle u, v \rangle|$.
- Note that $|\langle u, v \rangle| = \|u\|\|v\|$ if and only if $\|w\| = 0$, which happens if and only if $w = 0$. I.e. if $u = cv$.

Applications of Cauchy-Schwarz

The Cauchy-Schwarz inequality is extremely useful. Here's a simple application, and we will see more soon.

Example 14

Let $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$. Then, using Cauchy-Schwarz we have

$$|x_1 y_1 + \ldots + x_n y_n|^2 \leq (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2).$$

The triangle inequality

- It is a basic fact of Euclidean geometry that the length of a side of a triangle is less than the sum of the lengths of the other two sides.
- This generalizes to inner product spaces.

Proposition 15

Let V be a real inner product space, and let $u, v \in V$. Then

$$\|u + v\| \leq \|u\| + \|v\|.$$

Moreover, we have equality if and only if u is a scalar multiple of v or vice versa.

The triangle inequality - proof

- Appealing to Cauchy-Schwarz for the inequality marked $*$ we have

$$\begin{aligned}
\|u+v\|^2 &= \langle u+v, u+v\rangle \\
&= \langle u,u\rangle + \langle v,u\rangle + \langle u,v\rangle + \langle v,v\rangle \\
&= \|u\|^2 + \|v\|^2 + 2\langle u,v\rangle \\
* &\leq \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| \\
&= (\|u\| + \|v\|)^2.
\end{aligned}$$

- So $\|u+v\| \leq \|u\| + \|v\|$ as claimed.
- We have equality if and only if $\|u\|\|v\| = \langle u, v\rangle$.
- By Cauchy-Schwarz this happens if and only if one of u or v is a scalar multiple of the other.

The parallelogram equality

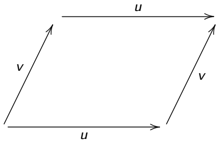
Now lets use what we have proved about inner product spaces to prove a less obvious fact about plain geometry.

Proposition 16

In a parallelogram, the sum of the squares of the lengths of the diagonals equals the sum of the squares of the sides.

The parallelogram equality - proof

- Expressed in terms of vectors, a parallelogram has form



The diagonals are given by $u - v$ and $u + v$. Now

$$
\begin{aligned}
&\|u+v\|^2 + \|u-v\|^2 \\
=&\langle u+v, u+v\rangle + \langle u-v, u-v\rangle \\
=&\|u\|^2 + \|v\|^2 + 2\langle u, v\rangle + \|u\|^2 + \|v\|^2 - 2\langle u, v\rangle \\
=&2(\|u\|^2 + \|v\|^2).
\end{aligned}
$$

The identity $\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$ is called the *parallelogram equality*.