

ITCS 531: L5 homework solutions

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L1 Q1(a)

Let ϕ be a formula where x occurs free. Write down a proof tree that shows $\forall x\phi \vdash \neg\exists x\neg\phi$.

$$\begin{array}{c}
 \frac{\frac{\frac{[\exists x\neg\phi]_1}{\perp} \quad \frac{\frac{[\neg\phi[x'/x]]_2}{\phi[x'/x]} \quad \frac{\forall x\phi}{\phi[x'/x]} (\forall E)}{\perp} (\neg E)}{\neg\exists x\neg\phi} (\neg I) \quad \perp \quad (\exists E)
 \end{array}$$

L1 Q1(b)

Let ϕ be a formula where x occurs free. Write down a proof tree that shows $\exists x\phi \vdash \neg\forall x\neg\phi$.

$$\begin{array}{c}
 \frac{\frac{\frac{\exists x\phi}{\perp} \quad \frac{\frac{[\forall x\neg\phi]_1}{\neg\phi[x'/x]} \quad \frac{[\phi[x'/x]]_2}{\perp} \quad (\neg_E)}{\perp} \quad (\exists_E)}{\neg\forall x\neg\phi} \quad (\neg_I)
 \end{array}$$

(The proof tree is a single line with multiple horizontal bars. The top bar is above $\neg\phi[x'/x]$ and $[\phi[x'/x]]_2$, with (\neg_E) to its right. The middle bar is above \perp , with (\exists_E) to its right. The bottom bar is above \perp , with (\neg_I) to its right. The left side of the tree is $\exists x\phi$ and the right side is $\neg\forall x\neg\phi$. The rule (\forall_E) is written in red to the left of the top bar.

L1 Q3

Prove that if Γ is an \mathcal{L} -theory then there is an \mathcal{L} -theory Γ' with $\Gamma \subseteq \Gamma'$ such that Γ' is *complete* (i.e. if ϕ is an \mathcal{L} -sentence, then either $\phi \in \Gamma'$ or $\neg\phi \in \Gamma'$).

- ▶ An \mathcal{L} -theory is a satisfiable set of \mathcal{L} -sentences.
- ▶ Since Γ is satisfiable, it must have a model.
- ▶ Let A be a model for Γ , and let

$$\Gamma' = \{\phi : \phi \text{ is an } \mathcal{L}\text{-sentence and } A \models \phi\}.$$

- ▶ Then $\Gamma \subseteq \Gamma'$, because $A \models \Gamma$.
- ▶ Γ' is complete because every \mathcal{L} -sentence is either true or false in A .

L1 Q4

Let Γ be an \mathcal{L} -theory, and let ϕ be an \mathcal{L} -sentence. Prove that if $\Gamma \models \phi$ then $\Delta \models \phi$ for some finite $\Delta \subseteq \Gamma$.

- ▶ Suppose $\Gamma \models \phi$.
- ▶ By completeness we have $\Gamma \vdash \phi$.
- ▶ So there is a deduction tree using Γ that proves ϕ .
- ▶ As deduction trees are finite, this tree involves only a finite number of sentences from Γ .
- ▶ Define Δ to be the set of sentences from Γ used in the proof of ϕ .
- ▶ Then $\Delta \vdash \phi$, and so $\Delta \models \phi$ by soundness.

L1 Q5

Let Γ be a set of \mathcal{L} -sentences. Prove that Γ has a model if and only if every finite subset of Γ has a model.

- ▶ If Γ has a model then every subset of Γ has a model.
- ▶ Conversely, suppose Γ does *not* have a model.
- ▶ Then $\Gamma \models \perp$.
- ▶ So by exercise 4 there is finite $\Delta \subseteq \Gamma$ with $\Delta \models \perp$.
- ▶ I.e. Δ does not have a model.