ITCS 531: L1 homework solutions

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Q1

Let ϕ and ψ be sentences. Show that

$$\phi \leftrightarrow \psi \models\dashv (\phi \land \psi) \lor (\neg\phi \land \neg\psi)$$

- The truth tables are the same:

ϕ	ψ	$\phi \leftrightarrow \psi$	$(\phi \land \psi) \lor (\neg\phi \land \neg\psi)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

Q2

Prove that the set $\{\wedge, \neg\}$ is functionally complete.

- Must show that for every sentence ϕ there is a sentence ϕ' using only $\wedge$ and $\neg$ such that $\phi \models\dashv \phi'$.
- Know that $\{\wedge, \vee, \neg, \leftrightarrow\}$ is functionally complete.
- Assume without loss of generality that ϕ only contains connectives from $\{\wedge, \vee, \neg, \leftrightarrow\}$.
- Induction on formula construction.
  - Base case $\phi = p$. Just set $\phi' = p$.
  - Inductive step: if $\phi = \psi_1 \vee \psi_2$ then define $\phi' = \neg(\neg\psi_1' \wedge \neg\psi_2')$.
  - Use truth tables to show $\phi \models\dashv \phi'$.
  - If $\phi = \psi_1 \leftrightarrow \psi_2$, notice that $(\psi_1' \wedge \psi_2') \vee (\neg\psi_1' \wedge \neg\psi_2') \models\dashv \psi_1 \leftrightarrow \psi_2$.
  - So, using the first case we can define $\phi' = \neg\big(\neg(\psi_1' \wedge \psi_2') \wedge \neg(\neg\psi_1' \wedge \neg\psi_2')\big)$.
  - Finally, if $\phi = \neg\psi$ or $\phi = \psi_1 \wedge \psi_2$ just set $\phi' = \neg\psi'$ or $\phi' = \psi_1' \wedge \psi_2'$.

Q3

Define a binary connective $|$ using the following truth table.

ϕ	ψ	$\phi\|\psi$
T	T	F
T	F	T
F	T	T
F	F	T

Prove that $\{|\}$ is functionally complete.

- Show that if ϕ is a sentence involving only symbols from $\{\wedge, \neg\}$, there is a sentence ϕ' using only $|$ such that $\phi \models \Dashv \phi'$.
- Suppose $\phi = \neg\psi$. Observe that

ψ	$\neg\psi$	$\psi\|\psi$
T	F	F
F	T	T

- So we can define $\phi' = \psi'|\psi'$.
- Suppose $\phi = \psi_1 \wedge \psi_2$. Observe that

ψ_1	ψ_2	$\psi_1 \wedge \psi_2$	$\psi_1\|\psi_2$	$(\psi_1\|\psi_2)\|(\psi_1\|\psi_2)$
T	T	T	F	T
T	F	F	T	F
F	T	F	T	F
F	F	F	T	F

- So we can define $\phi' = (\psi_1'|\psi_2')|(\psi_1'|\psi_2')$.

Q4

Let p and q be basic propositions. How many possible distinct truth tables are there for formulas involving only the propositions p and q?

- There are 4 rows in each truth table for p and q.

p	q	ϕ
T	T	?
T	F	?
F	T	?
F	F	?

- Here each ? can be true or false.
- This gives $2^4 = 16$ distinct possibilities.

Q5

Show that every sentence is equivalent to a sentence in DNF.

- Consider this example.
- Suppose ϕ contains only the proposition symbols p, q, r, and that its truth table is as follows:

p	q	r	ϕ
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	F

- Then ϕ is obviously logically equivalent to $(p \land q \land r) \lor (\neg p \land q \land r) \lor (\neg p \land q \land \neg r)$.
- This is a DNF sentence.
- This method obviously generalizes.
- If ϕ is a contradiction then ϕ is equivalent to e.g. $p \land \neg p$.