ITCS 531: LA2 homework solutions

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LA2 Q3 (1)

Let U and W be subspaces of V and suppose that $V = U \oplus W$. Let $(u_1, \ldots, u_k)$ be a basis for U, and let $(w_1, \ldots, w_m)$ be a basis for W. Prove that $(u_1, \ldots, u_k, w_1, \ldots, w_m)$ is a basis for V.

- We will first show that $(u_1, \ldots, u_k, w_1, \ldots, w_m)$ is linearly independent.
- Suppose $0 = a_1 u_1 + \ldots + a_k u_k + b_1 w_1 + \ldots + b_m w_m$.
- Since $U \oplus W$ is a direct sum, by definition there is a unique $u \in U$ and $w \in W$ with $u + w = 0$, and this must be $u = 0$ and $w = 0$.
- So $b_1 w_1 + \ldots + b_m w_m = 0$ and $a_1 u_1 + \ldots + a_k u_k = 0$.
- But $(u_1, \ldots, u_k)$ is a basis for U and $(w_1, \ldots, w_m)$ is a basis for W, so $a_1 = \ldots = a_k = b_1 = \ldots = b_m = 0$.
- But this means $(u_1, \ldots, u_k, w_1, \ldots, w_m)$ is linearly independent as claimed.

LA2 Q3 (2)

Let U and W be subspaces of V and suppose that $V = U \oplus W$. Let $(u_1, \ldots, u_k)$ be a basis for U, and let $(w_1, \ldots, w_m)$ be a basis for W. Prove that $(u_1, \ldots, u_k, w_1, \ldots, w_m)$ is a basis for V.

- Now we show the list spans V.
- Let $v \in V$.
- Then there is $u \in U$ and $w \in W$ with $v = u + w$.
- So, as $(u_1, \ldots, u_k)$ and $(w_1, \ldots, w_m)$ span U and W respectively, we have $b_1 w_1 + \ldots + b_m w_m = w$ and $a_1 u_1 + \ldots + a_k u_k = u$, for some choice of coefficients.
- But this means $v = a_1 u_1 + \ldots + a_k u_k + b_1 w_1 + \ldots + b_m w_m$, so the list does span v.

LA2 Q4

Let U and W be subspaces of $\mathbb{R}^8$, and suppose $\dim(U) = 5$, $\dim(W) = 3$, and $U \cap W = \{0\}$. Prove that $\mathbb{R}^8 = U \oplus W$.

- Let $(u_1, \ldots, u_5)$ be a basis for U, and let (w_1, w_2, w_3) be a basis for W.
- By lemma 1.14, $U + W$ is a direct sum, so $(u_1, \ldots, u_5, w_1, w_2, w_3)$ is linearly independent.
- It also has 8 elements, which is the dimension of $\mathbb{R}^8$.
- So, by theorem 2.11, $(u_1, \ldots, u_5, w_1, w_2, w_3)$ is a basis for $\mathbb{R}^8$, and so $U \oplus W = \mathbb{R}^8$.

LA2 Q5

Let V be a finite dimensional vector space and $\dim(V) = n > 0$. Show that $V = U_1 \oplus \ldots \oplus U_n$, for some set $\{U_1, \ldots, U_n\}$ of one-dimensional subspaces.

- Let $(v_1, \ldots, v_n)$ be a basis for V.
- For each $i \in \{1 \ldots, n\}$, let $U_i = \mathrm{span}(v_i)$.
- Then $V = U_1 + \ldots + U_n$, as a basis spans V, by definition.
- Also, if $0 = a_1 v_n + \ldots + a_n v_n$ then $a_1 = \ldots = a_n = 0$, by linear independence, so the sum is direct, by lemma 1.13.

L4 Q2 (1)

Let $\mathscr{L} = \{0, 1, +, \times\}$ be the language of basic arithmetic. Let $\phi = \forall x(\neg(x \approx 0) \to \exists y(x \times y \approx 1))$. Let $\mathbb{N}$ and $\mathbb{R}$ have their usual meanings, and interpret $\mathscr{L}$ into these languages by giving the non-logical symbols of $\mathscr{L}$ their usual meanings.

a) Does $\mathbb{N} \models \phi$?
b) Does $\mathbb{R} \models \phi$?

- ϕ is the statement that every non-zero element has an inverse.
- This is not true in $\mathbb{N}$, but it is true in $\mathbb{R}$.

L4 Q2 (2)

c) Let $n \in \mathbb{N}$ with $n \geq 2$, and let $\mathbb{Z}_n$ be the integers mod n. For what values of n does $\mathbb{Z}_n \models \phi$?

d) Let $A = (\{a, b\}, I)$, where I interprets 0 and 1 as a and b respectively, $b \times b = b$, and $a \times b = a \times a = a$. Does $A \models \phi$?

- Remember in $\mathbb{Z}_n$, an element has an inverse iff it is coprime with n.
- So $\mathbb{Z} \models \phi$ iff every non-zero element is coprime with n. I.e. iff n is prime.

- In the structure A, the only non-zero element is b.
- Does b have an inverse? Yes, because $b \times b = b = 1$.
- So $A \models \phi$.

L4 Q2 (3)

e) Let $\psi = \exists x \forall y(\neg(y \approx 0) \to (x \times y \approx 1))$. Which of the structures in parts a)-d) is a model for ψ?

- ψ is the statement that there's an element which is an inverse to *all* non-zero elements.
- This is clearly not true in $\mathbb{N}$, $\mathbb{R}$ or $\mathbb{Z}_n$ for $n > 2$.
- $\mathbb{Z}_2 \models \psi$ as 1 is the only non-zero element of $\mathbb{Z}_2$.
- ψ is true in A, as b is the only non-zero element of A, and $b \times b = b = 1$.

L4 Q3 (1)

Let R and S be unary predicates. Let $\phi = \forall x R(x) \vee \forall x S(x)$, and let $\psi = \forall x \forall y (R(x) \vee S(y))$. Prove that $\phi \models \dashv \psi$.

- Let A be a structure of the appropriate kind.
- Suppose first that $A \models \phi$.
- Then either every element of A satisfies R, or every element of A satisfies S.
- In the former case, for every pair of elements $a, b \in A$ we must have $R(a) \vee S(b)$, because $R(a)$ must be true.
- So $A \models \forall x \forall y (R(x) \vee S(y))$. This shows $\phi \models \psi$.

L4 Q3 (2)

Let R and S be unary predicates. Let $\phi = \forall x R(x) \vee \forall x S(x)$, and let $\psi = \forall x \forall y (R(x) \vee S(y))$. Prove that $\phi \models\!\dashv \psi$.

- Conversely, suppose $A \models \psi$.
- Suppose wlog that A does not satisfy $\forall x R(x)$.
- Then there is $a \in A$ with $A \not\models R(a)$. I.e. $R(a)$ is not true.
- As $A \models \forall x \forall y (R(x) \vee S(y))$, $A \models \forall y (R(a) \vee S(y))$.
- As $R(a)$ is not true in A, it follows that $A \models \forall y S(y)$.
- By changing the variable name we have $A \models \forall x S(x)$, and so $A \models \forall x R(x) \vee \forall x S(x)$.
- This shows $\psi \models \phi$, and we are done.

L4 Q4 (1)

Let ϕ be an $\mathscr{L}$-formula, let A be an $\mathscr{L}$-structure, and let v be an assignment for $\mathscr{L}$ to A with $A, v \models \phi$. Prove that $A, u \models \phi$ for all assignments u such that $u(x) = v(x)$ for all variables x occurring free in ϕ.

- We induct on formula construction.
- It's obviously true for atomic formulas, because these have no bound variables.
- Suppose now that it's true for formulas ϕ and ψ.

L4 Q4 (2)

$\neg\phi$:

- Suppose $A, v \models \neg\phi$.
- Then $A, v \not\models \phi$.
- Suppose that u agrees with v about the free variables of ϕ.
- If $A, u \models \phi$, by the inductive hypothesis we would have $A, v \models \phi$, which would be a contradiction.
- So we must have $A, u \models \neg\phi$ as required.

$\phi \vee \psi$

- Suppose $A, v \models \phi \vee \psi$.
- Then, wlog we can assume that $A, v \models \phi$.
- Let u be an assignment agreeing with v about the free variables of $\phi \vee \psi$.
- Then u agrees with v about the free variables of ϕ.
- So $A, u \models \phi$, and thus $A, u \models \phi \vee \psi$ as required.

L4 Q4 (3)

$\forall x \phi$:

- Suppose $A, v \models \forall x \phi$, and let u be an assignment agreeing with v about the free variables of $\forall x \phi$.
- We must show that $A, u \models \forall x \phi$.
- I.e. that $A, u' \models \phi$ for all u' agreeing with u except possibly at x.
- Let u' be such an assignment, and let v' be an assignment agreeing with v except possibly at x, where we define $v'(x) = u'(x)$.
- Then, if F is the set of variables occurring free in ϕ, we have
  - v' agrees with v on $F \setminus \{x\}$.
  - u agrees with v on $F \setminus \{x\}$.
  - u' agrees with u on $F \setminus \{x\}$.
  - u' agrees with v' on F.
- So,

$$\begin{aligned} A, v \models \forall x \phi &\implies A, v' \models \phi \text{ (by definition of } \models) \\ &\implies A, u' \models \phi \text{ (by the inductive hypothesis),} \end{aligned}$$

- And so $A, u \models \forall x \phi$ as required.