ITCS 531: Logic 2 - Deduction rules for propositional logic

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Semantic proof

- Last week we saw how formulas and sets of formulas can imply other formulas according to truth tables.
- This allows us to make deductions about when a formula must be true assuming that certain other formulas are true.
- This method of deduction is *semantic*.
- I.e. it is based on an idea of *true* and *false*.
- In other words, propositions have a meaning in a world where they are true or false.

Syntactic proof

- A different approach to logical deduction is to forget concepts like ‘true’, ‘false’ and ‘meaning’.
- I.e. just look at the structure of the formulas involved.
- This is known as *syntax*.
- We will develop a syntactical approach to deduction here.

Formal proofs in propositional logic

- A formal proof begins with a (possibly empty) set of sentences, Γ, (considered to be axioms).
- In addition we have a collection of **deduction rules** (also called *inference rules*).
- We use these to generate new sentences from combinations of ones previously generated.
- During this process the intended meaning of the sentences are irrelevant.
- The only important thing is their syntactic form.
- The set of sentences provable from Γ is the set of sentences that can be obtained from Γ using a finite number of applications of the inference rules.

Natural deduction

- There are many ways we can define deduction rules for propositional logic that are equivalent in a technical sense.
- We use a system called *natural deduction*.
- The advantage is that it is relatively human readable.
- Natural deduction proofs resemble human argument.
- The disadvantage is that its proofs more difficult to formally reason about.
- This matters to proof theorists because they want to be able to prove theorems about the deductive power of formal systems.
- We're not worried about that though.

Introduction rules

$$\top_I: \quad \frac{}{\top}$$

$$\wedge_I: \quad \frac{\phi \qquad \psi}{\phi \wedge \psi}$$

$$\vee_{I_l}: \quad \frac{\phi}{\phi \vee \psi}$$

$$\vee_{I_r}: \quad \frac{\psi}{\phi \vee \psi}$$

$$\neg_I: \quad \frac{\dfrac{[\phi]}{\perp}}{\neg\phi}$$

$$\to_I: \quad \frac{\dfrac{[\phi]}{\psi}}{\phi \to \psi}$$

Elimination rules

$$\bot_E: \quad \frac{\bot}{\phi}$$

$$\wedge_{E_l}: \quad \frac{\phi \wedge \psi}{\phi}$$

$$\wedge_{E_r}: \quad \frac{\phi \wedge \psi}{\psi}$$

$$\vee_E: \quad \frac{\phi \vee \psi \qquad \begin{array}{c}[\phi] \\ \hline\hline \theta\end{array} \qquad \begin{array}{c}[\psi] \\ \hline\hline \theta\end{array}}{\theta}$$

$$\neg_E: \quad \frac{\phi \qquad \neg\phi}{\bot}$$

$$\rightarrow_E: \quad \frac{\phi \rightarrow \psi \qquad \phi}{\psi}$$

Intuitionistic propositional logic

- These rules define *intuitionistic propositional logic*.
- This is like classical propositional logic except that here $\neg\neg\phi$ does not imply ϕ
  - The converse is still true though! (see example 3 later).
- To get classical propositional logic we need one extra rule (double negation elimination).

$$\neg\neg_E: \quad \frac{\neg\neg\phi}{\phi}$$

Using the deduction rules

- Roughly speaking, introduction rules create new sentences by combining old ones with a logical connective.
- Elimination rules create new sentences by eliminating logical connectives from old ones.
  - There are some rules that don't fit this pattern in an obvious way.
- Derivations go from top to bottom.
- We can introduce sentences based on our axioms, then use the inference rules to derive new ones.
- Derived sentences go below the line.

Assumptions and subderivations

- Sentences in square brackets, e.g. $[\phi]$, are *assumptions*.
- When we make an assumption we have to discharge it later using one of the inferences rules $\neg_I$, $\rightarrow_I$, or $\vee_E$.
- We often use a subscript when making an assumption, e.g. $[\phi]_1$, so we can keep track of when we discharge it.
- We will discharge assumptions using 'last in first out'. So, in a derivation, the last assumption made is the first to be discharged.
- Double lines (e.g. in $\vee_E$) represent a subderivation.
- That is, it stands for some arbitrary derivation beginning with the thing on the top and ending with the thing on the bottom.

Introduction rules again

$$\top_I: \quad \frac{}{\top}$$

$$\wedge_I: \quad \frac{\phi \qquad \psi}{\phi \wedge \psi}$$

$$\vee_{I_l}: \quad \frac{\phi}{\phi \vee \psi}$$

$$\vee_{I_r}: \quad \frac{\psi}{\phi \vee \psi}$$

$$\neg_I: \quad \cfrac{\cfrac{[\phi]}{\bot}}{\neg\phi}$$

$$\to_I: \quad \cfrac{\cfrac{[\phi]}{\psi}}{\phi \to \psi}$$

Elimination rules again

$$\bot_E: \quad \frac{\bot}{\phi}$$

$$\wedge_{E_l}: \quad \frac{\phi \wedge \psi}{\phi}$$

$$\wedge_{E_r}: \quad \frac{\phi \wedge \psi}{\psi}$$

$$\vee_E: \quad \frac{\phi \vee \psi \qquad \begin{array}{c}[\phi] \\ \hline\hline \theta\end{array} \qquad \begin{array}{c}[\psi] \\ \hline\hline \theta\end{array}}{\theta}$$

$$\neg_E: \quad \frac{\phi \qquad \neg\phi}{\bot}$$

$$\rightarrow_E: \quad \frac{\phi \rightarrow \psi \qquad \phi}{\psi}$$

Some conventions

- We implicitly assume we can deduce any formula from itself or an assumption of itself:

$$\frac{\phi}{\phi}$$

$$\frac{[\phi]}{\phi}$$

- When we make deductions we can usually freely switch the order of sentences.
    - E.g. ϕ and $\neg\phi$ could be switched when applying rule $\neg_E$.

Example: $\phi \to \phi$

$$\to_I: \quad \dfrac{\dfrac{[\phi]}{\psi}}{\phi \to \psi}$$

Example 1

We can deduce $\phi \to \phi$ from an empty set of axioms.

$$\dfrac{\dfrac{[\phi]_1}{\phi}}{\phi \to \phi} \quad (\to_I)_1$$

Example: $\phi \vee \psi$ implies $\psi \vee \phi$

$\vee_{I_l}$: $\dfrac{\phi}{\phi \vee \psi}$

$\vee_E$: $$\dfrac{\phi \vee \psi \qquad \dfrac{[\phi]}{\theta} \qquad \dfrac{[\psi]}{\theta}}{\theta}$$

Example 2

If $\phi \vee \psi$ is an axiom then we can deduce $\psi \vee \phi$.

$$\dfrac{\phi \vee \psi \qquad \dfrac{\dfrac{[\phi]_1}{\phi}}{\psi \vee \phi}\,(\vee_{I_r}) \qquad \dfrac{\dfrac{[\psi]_1}{\psi}}{\psi \vee \phi}\,(\vee_{I_l})}{\psi \vee \phi}\,(\vee_E)_1$$

Example: $\phi \to \neg\neg\phi$

$\neg_E$: $\dfrac{\phi \qquad \neg\phi}{\perp}$

$\neg_I$: $\dfrac{\begin{array}{c}[\phi] \\ \hline\hline \perp\end{array}}{\neg\phi}$

$\to_I$: $\dfrac{\begin{array}{c}[\phi] \\ \hline\hline \psi\end{array}}{\phi \to \psi}$

Example 3

For all sentences ϕ, we can derive $\phi \to \neg\neg\phi$ from an empty set of axioms, without using the rule $\neg\neg_E$.

$$\dfrac{\dfrac{\dfrac{\dfrac{[\phi]_1}{\phi} \qquad \dfrac{[\neg\phi]_2}{\neg\phi}}{\perp}\ (\neg_E)}{\neg\neg\phi}\ (\neg_I)_2}{\phi \to \neg\neg\phi}\ (\to_I)_1$$

Example: De Morgan's laws part 1

$$\wedge_{E_l}: \quad \frac{\phi \wedge \psi}{\phi} \qquad \neg_E: \quad \frac{\phi \qquad \neg\phi}{\bot} \qquad \neg_I: \quad \frac{\genfrac{}{}{0pt}{}{[\phi]}{=\!=\!=}\atop \bot}{\neg\phi}$$

$$\vee_E: \quad \frac{\phi \vee \psi \qquad \genfrac{}{}{0pt}{}{[\phi]}{\overline{\overline{\theta}}} \qquad \genfrac{}{}{0pt}{}{[\psi]}{\overline{\overline{\theta}}}}{\theta}$$

Example 4

From $\phi \vee \psi$ we can deduce $\neg(\neg\phi \wedge \neg\psi)$.

$$\dfrac{(\phi \vee \psi) \qquad \dfrac{\dfrac{\dfrac{[\phi]_1}{\phi} \qquad \dfrac{[\neg\phi \wedge \neg\psi]_2}{\neg\phi}\,(\wedge_{E_l})}{\bot}\,(\neg_E)}{\neg(\neg\phi \wedge \neg\psi)}\,(\neg_I)_2 \qquad \dfrac{\dfrac{\dfrac{[\psi]_1}{\psi} \qquad \dfrac{[\neg\phi \wedge \neg\psi]_3}{\neg\psi}\,(\wedge_{E_r})}{\bot}\,(\neg_E)}{\neg(\neg\phi \wedge \neg\psi)}\,(\neg_I)_3}{\neg(\neg\phi \wedge \neg\psi)}\,(\vee_E)_1$$

Example: De Morgan's laws part 2

$$\vee_{I_l}: \quad \frac{\phi}{\phi \vee \psi} \qquad\qquad \neg_E: \quad \frac{\phi \qquad \neg\phi}{\bot} \qquad\qquad \neg_I: \quad \dfrac{\dfrac{[\phi]}{\bot}}{\neg\phi}$$

$$\wedge_I: \quad \frac{\phi \qquad \psi}{\phi \wedge \psi} \qquad\qquad \neg\neg_E: \quad \frac{\neg\neg\phi}{\phi}$$

Example 5

From $\neg(\neg\phi \wedge \neg\psi)$ we can deduce $\phi \vee \psi$.

$$\dfrac{\dfrac{\neg(\neg\phi \wedge \neg\psi) \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\phi]_2}{\phi \vee \psi}\,(\vee_{I_l}) \qquad \dfrac{[\neg(\phi \vee \psi)]_1}{\neg(\phi \vee \psi)}}{\bot}\,(\neg_E)}{\neg\phi}\,(\neg_I)_2 \qquad \dfrac{[\psi]_3 \qquad [\neg(\phi \vee \psi)]_1}{\neg\psi}}{\neg\phi \wedge \neg\psi}\,(\wedge_I)}{\dfrac{\bot}{\neg\neg(\phi \vee \psi)}\,(\neg_I)_1}\,(\neg_E)}{\phi \vee \psi}\,(\neg\neg_E)$$

Example: $\phi \vee \neg\phi$

$$\vee_{I_l}: \quad \frac{\phi}{\phi \vee \psi} \qquad\qquad \neg_E: \quad \frac{\phi \qquad \neg\phi}{\perp} \qquad\qquad \neg_I: \quad \frac{\begin{array}{c}[\phi]\\ \hline\hline \perp\end{array}}{\neg\phi}$$

$$\neg\neg_E: \quad \frac{\neg\neg\phi}{\phi}$$

Example 6

$\phi \vee \neg\phi$ is a theorem of classical propositional logic (i.e. it can be deduced from an empty set of axioms).

$$\dfrac{\dfrac{\dfrac{\neg(\neg\phi \vee \phi) \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\neg(\neg\phi \vee \phi)]_1}{\neg(\neg\phi \vee \phi)} \qquad \dfrac{\dfrac{[\phi]_2}{\phi}}{\neg\phi \vee \phi}\,(\vee_{I_r})}{\perp}\,(\neg_E)}{\neg\phi}\,(\neg_I)_2}{\neg\phi \vee \phi}\,(\vee_{I_l})}{\perp}\,(\neg_E)}{\neg\neg(\neg\phi \vee \phi)}\,(\neg_I)_1}{\neg\phi \vee \phi}\,(\neg\neg_E)$$

Example: ψ from $\phi \vee \psi$ and $\neg\phi$

$$\perp_E: \quad \frac{\perp}{\phi} \qquad\qquad \neg_E: \quad \frac{\phi \qquad \neg\phi}{\perp}$$

$$\vee_E: \quad \frac{\phi \vee \psi \qquad \begin{array}{c}[\phi]\\ \hline\hline \theta\end{array} \qquad \begin{array}{c}[\psi]\\ \hline\hline \theta\end{array}}{\theta}$$

Example 7

If $\phi \vee \psi$ and $\neg\phi$ are axioms then we can deduce ψ.

$$\dfrac{\phi \vee \psi \qquad \dfrac{\dfrac{\neg\phi \qquad \dfrac{[\phi]_1}{\phi}}{\perp}\ (\neg_E)}{\psi}\ (\perp_E) \qquad \dfrac{[\psi]_1}{\psi}}{\psi}\ (\vee_E)_1$$