ITCS 531: LA3 homework solutions

Rob Egrot

LA3 Q1(1)

Let $b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz).$$

Prove that T is linear if and only if $b = c = 0$.

- If $b = c = 0$ then $T(x, y, z) = (2x - 4y + 3z, 6x)$.
- We check the conditions of definition 3.1.
- First,

$$\begin{aligned} & T(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ =& (2(x_1 + x_2) - 4(y_1 + y_2) + 3(z_1 + z_2), 6(x_1 + x_2)) \\ =& (2x_1 - 4y_1 + 3z_1, 6x_1) + (2x_2 - 4y_2 + 3z_2, 6x_2) \\ =& T(x_1, y_1, z_1) + T(x_2, y_2, z_2). \end{aligned}$$

- Second,

$$\begin{aligned} T(\lambda x, \lambda y, \lambda z) &= (2\lambda x - 4\lambda y + 3\lambda z, 6\lambda x) \\ &= \lambda(2x - 4y + 3z, 6x) \\ &= \lambda T(x, y, z). \end{aligned}$$

LA3 Q1(2)

Let $b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz).$$

Prove that T is linear if and only if $b = c = 0$.

- Conversely, if T is linear then $2T(1,0,0) = T(2,0,0)$.
- So $2(2 + b, 6) = (4 + b, 12)$.
- I.e. $(4 + 2b, 12) = (4 + b, 12)$, and so b must be zero.
- Also, $T(1,1,1) = T(1,0,0) + T(0,1,1)$.
- So

$$(2 - 4 + 3, 6 + c) = (2, 6) + (-4 + 3, 0) = (2 - 4 + 3, 6),$$

- So $c = 0$.

LA3 Q2

Let $T \in \mathcal{L}(V, W)$. Let $v_1, \ldots, v_n \in V$ and suppose that $(T(v_1), \ldots, T(v_n))$ is linearly independent in W. Prove that $(v_1, \ldots, v_n)$ is linearly independent in V.

- Suppose $a_1 v_1 + \ldots + a_n v_n = 0$.
- Then, as T is linear, we have

  $$a_1 T(v_1) + \ldots + a_n T(v_n) = T(a_1 v_1 + \ldots + a_n v_n) = T(0) = 0.$$

- As $T(v_1), \ldots, T(v_n)$ is linearly independent, it follows that $a_1 = \ldots = a_n = 0$.
- So $(v_1, \ldots, v_n)$ is also linearly independent.

LA3 Q4(1)

Let $S \in \mathcal{L}(U, V)$ and let $T \in \mathcal{L}(V, W)$. Let $(u_1, \ldots, u_n)$, $(v_1, \ldots, v_m)$ and $(w_1, \ldots, w_p)$ be bases for U, V and W respectively. Suppose that B is the matrix of T with respect to $(v_1, \ldots, v_m)$ and $(w_1, \ldots, w_p)$, and that A is the matrix of S with respect to $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_m)$. Then BA is the matrix of TS with respect to $(u_1, \ldots, u_n)$ and $(w_1, \ldots, w_p)$.

LA3 Q4(2)

- What should TS do to the basis vector u_i of U?
- As A is the matrix of S, to find Su_i we look at what A does to the column vector that is zeroes except for 1 in the ith place.
- So the result is $a_{1i}v_1 + \ldots + a_{mi}v_m$.
- What does T do to a basis vector v_j of V?
- The matrix B tells us that $T(v_j) = b_{1j}w_1 + \ldots + b_{pj}w_p$.
- So,

$$
\begin{aligned}
&TS(u_i)\\
=&T(a_{1i}v_1 + \ldots + a_{mi}v_m)\\
=&a_{1i}T(v_1) + \ldots + a_{mi}T(v_m)\\
=&a_{1i}(b_{11}w_1 + \ldots + b_{p1}w_p) + \ldots + a_{mi}(b_{1m}w_1 + \ldots + b_{pm}w_p).
\end{aligned}
$$

LA3 Q4(3)

- We can rearrange this as

$$
\begin{aligned}
&(a_{1i}b_{11} + \ldots + a_{mi}b_{1m})w_1 \\
+&(a_{1i}b_{21} + \ldots + a_{mi}b_{2m})w_2 \\
+&\ldots \\
+&(a_{1i}b_{p1} + \ldots + a_{mi}b_{pm})w_p.
\end{aligned}
$$

- But this is the ith column of the matrix BA:

$$
\begin{bmatrix}
a_{1i}b_{11} + \ldots + a_{mi}b_{1m} \\
a_{1i}b_{21} + \ldots + a_{mi}b_{2m} \\
\vdots \\
a_{1i}b_{p1} + \ldots + a_{mi}b_{pm}
\end{bmatrix}
$$

- Since this is true for every basis vector u_i of U, the transformation TS is given by the matrix BA as claimed.

LA3 Q5

Let $T \in \mathcal{L}(V, W)$, and suppose both V and W are finite dimensional. Prove that, whatever the choice of bases for V and W, the matrix of T with respect to these bases must have at least $\dim \mathrm{ran}\, T$ entries that are not equal to 0.

- Let A be the matrix of T with respect to some pair of bases.
- If the ith column of A is all zeroes, then this means $T(v_i) = 0$, where v_i is the ith basis vector for V.
- Since $(T(v_1), \ldots, T(v_n))$ spans $\mathrm{ran}\, T$, there must be at least $\dim \mathrm{ran}\, T$ columns of A that are not all zeroes.
- This requires at least $\dim \mathrm{ran}\, T$ non-zero entries.