STA261 - Module 4

Intervals and Model Checking

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Uncertainty in Point Estimates

- In Module 2, we learned how to produce the "best" point estimates of θ possible using statistics of our data
- The "best" unbiased estimator $\hat{\theta}(\mathbf{X})$ is the one that has the lowest possible variance among all unbiased estimators of θ
- But even so, suppose we observe $\mathbf{X} = \mathbf{x}$ and calculate $\hat{\theta}(\mathbf{x})$; how do we know this is close to the true θ? It may not be!
- We can't know for sure
- But we can use the data to get a range of *plausible* values of θ

Eg: UofT heights $\sim N(\mu, 1)$. Say I calculate $\hat{\mu}(\vec{x}) = \bar{X}_n = 5'6.5''$

Probably more plausible that $\mu \in [4'6.5'', 6'6.5'']$ than $\mu \in (2', 4')$

$= [\hat{\mu} - 1, \hat{\mu} + 1]$

Random Sets

- Suppose for now that $\Theta \subseteq \mathbb{R}$
- If $\hat{\theta}(\mathbf{X})$ is a continuous random variable, then $\mathbb{P}_\theta\left(\theta = \hat{\theta}(\mathbf{X})\right) = 0$

  Useless...

- But we can try to find a random set $C(\mathbf{X}) \subseteq \mathbb{R}$ based on $\mathbf{X}$ such that $\mathbb{P}_\theta\left(\theta \in C(\mathbf{X})\right) = 0.95$, for example

  a set which is a function of the random sample $\vec{X}$

  e.g. $[\bar{X}_n - 1, \bar{X}_n + 1]$

- Example 4.1: Let $X \sim \mathcal{N}(\mu, 1)$ where $\mu \in \mathbb{R}$. Show that the region $C(X) = (X + z_{0.025}, X + z_{0.975})$ satisfies $\mathbb{P}_\mu(\mu \in C(X)) = 0.95$.

$$\begin{aligned}
&\mathbb{P}_\mu(\mu \in C(X)) \\
&= \mathbb{P}_\mu(X + z_{0.025} < \mu < X + z_{0.975}) \\
&= \mathbb{P}_\mu(z_{0.025} < \mu - X < z_{0.975}) \\
&= \mathbb{P}(z_{0.025} < Z < z_{0.975}) \quad \text{where } Z \sim N(0,1) \\
&= \Phi(z_{0.975}) - \Phi(z_{0.025}) \\
&= 0.975 - 0.025 = 0.95
\end{aligned}$$

Interval Estimators and Confidence Intervals

- Definition 4.1: An **interval estimate** of a parameter $\theta \in \Theta \subseteq \mathbb{R}$ is any pair of statistics $L, U : \mathcal{X}^n \to \mathbb{R}$ such that $L(\mathbf{x}) \leq U(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}^n$. The random interval $\underbrace{(L(\mathbf{X}), U(\mathbf{X}))}_{\text{An interval with random endpoints;}}$ is called an **interval estimator**.

  Note: some authors use closed endpoints!

- Example 4.2: $N(\mu,1)$: $(X_{(1)}, X_{(n)}+5)$ Bernoulli(p): $(-\bar{X}_n+4, -\bar{X}_n+5)$

- Definition 4.2: Suppose $\alpha \in [0,1]$. An interval estimator $(L(\mathbf{X}), U(\mathbf{X}))$ is a $(\mathbf{1}-\boldsymbol{\alpha})$**-confidence interval** for θ if $\mathbb{P}_\theta(L(\mathbf{X}) < \theta < U(\mathbf{X})) \geq 1-\alpha$ for all $\theta \in \Theta$. We refer to $1-\alpha$ as the **confidence level** of the interval.

- Example 4.3: $X \sim N(\mu,1) \Rightarrow$ We showed in Ex. 4.1 that

  $(X+z_{0.025}, X+z_{0.975})$ is a 0.95-confidence interval for μ.

One-Sided Intervals

- Definition 4.3: A **lower one-sided** confidence interval is a confidence interval of the form $(L(\mathbf{X}), \infty)$. An **upper one-sided** confidence interval is a confidence interval of the form $(-\infty, U(\mathbf{X}))$.

- Example 4.4: Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, 1)$. Find a lower one-sided 0.5-confidence interval for μ.

$0.5 = \mathbb{P}(Z < 0)$ where $Z \sim N(0,1)$

$= \mathbb{P}_\mu\left(\frac{\bar{X}_n - \mu}{\sqrt{1/n}} < 0\right)$

$= \mathbb{P}_\mu(\bar{X}_n < \mu)$

$= \mathbb{P}_\mu(\mu \in (\bar{X}_n, \infty))$

So $(\bar{X}_n, \infty)$ is a lower one-sided 0.5-$\underbrace{\text{CI}}_{\text{Confidence Interval}}$ for μ

But (X_i, ∞) is another one!

So $(1-\alpha)$-CIs are not unique!

Confidence Intervals: Warmups

- The reason for the "$\geq 1-\alpha$" in the definition is that $\mathbb{P}_\theta\left(L(\mathbf{X}) \leq \theta \leq U(\mathbf{X})\right)$ may not be free of θ, depending on the choices of $L(\mathbf{X})$ and $U(\mathbf{X})$

- The lower bound means we want $1-\alpha$ confidence even in the "worst case"; equivalently,

$$\inf_{\theta \in \Theta} \mathbb{P}_\theta\left(L(\mathbf{X}) \leq \theta \leq U(\mathbf{X})\right) \geq 1-\alpha$$

$F_\theta(x) = \frac{x}{\theta}$

- Example 4.5: Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Unif}(0, \theta)$, where $\theta > 0$. Find $a \in \mathbb{R}$ such that $(aX_{(n)}, 2aX_{(n)})$ is a 95% confidence interval for θ.

$$1-\alpha = \mathbb{P}_\theta\left(\theta \in (aX_{(n)}, 2aX_{(n)})\right)$$
$$= \mathbb{P}_\theta\left(aX_{(n)} < \theta < 2aX_{(n)}\right)$$
$$= \mathbb{P}\left(\frac{\theta}{2a} < X_{(n)} < \frac{\theta}{a}\right)$$
$$= F_{X_{(n)}}\left(\frac{\theta}{a}\right) - F_{X_{(n)}}\left(\frac{\theta}{2a}\right)$$
$$= \left(\frac{\theta/a}{\theta}\right)^n - \left(\frac{\theta/(2a)}{\theta}\right)^n$$

$$\left(\frac{1}{a}\right)^n - \left(\frac{1}{2a}\right)^n$$

$$\Rightarrow \text{Choose } a = \left(\frac{1-2^{-n}}{1-\alpha}\right)^{\frac{1}{n}}, \quad \alpha = 0.05$$

Check!

Poll Time!

$$\mathbb{P}(L(\vec{X}) < \theta < U(\vec{X})) = 0.95$$

Observe $\vec{X} = \vec{x} \Rightarrow \mathbb{P}(L(\vec{x}) < \theta < U(\vec{x})) \in \{0,1\}$

$\theta \in \mathbb{R} \Rightarrow$ $3 < \theta < 5$ is either T or F

$\Rightarrow \mathbb{P}(3 < \theta < 5) \in \{0,1\}$

If 0, then it's < 0.95

If 1, then it's > 0.95

Some Confidence Intervals Are Better Than Others

- A confidence interval is only useful when it tells us something we didn't know before collecting the data

- Example 4.6: Suppose $X_1, X_2, \ldots, X_n \stackrel{iid}{\sim}$ Bernoulli (θ), where $\theta \in (0, 1)$. Find a 100% confidence interval for θ.

$(0,1)$ ... not helpful at all!

$(X_1 - 1, X_2 + 1)$ ... also not helpful!

$(X_1 - 200, \infty)$ ... REALLY not helpful!

A 100%-CI tells us nothing!

- A good confidence interval shouldn't be any longer than necessary

- We interpret the length of the interval as a measure of how accurately the data allow us to know the true value of θ

Bringing Back Hypothesis Tests

- In Module 3, we learned about test statistics and rejection regions for hypothesis tests

- Pick some arbitrary $\theta_0 \in \Theta$, and suppose we want a level-α test of $H_0 : \theta = \theta_0$ versus $H_A : \theta \neq \theta_0$ using a test statistic $T(\mathbf{X})$

- This means finding a rejection region R_{θ_0} such that

$$\mathbb{P}_{\theta_0}(T(\mathbf{X}) \in R_{\theta_0}) \leq \alpha$$

- This is equivalent to finding an *acceptance region* $A_{\theta_0} = R_{\theta_0}^c$ such that

$$\mathbb{P}_{\theta_0}(T(\mathbf{X}) \in A_{\theta_0}) \geq 1 - \alpha$$

Confidence Intervals Via Test Statistics

- If the statement $T(\mathbf{X}) \in A_{\theta_0}$ can be manipulated into an equivalent statement of the form $L(\mathbf{X}) < \theta_0 < U(\mathbf{X})$, then

$$\mathbb{P}_{\theta_0}(L(\mathbf{X}) < \theta_0 < U(\mathbf{X})) \geq 1 - \alpha$$

- But $\theta_0 \in \Theta$ was arbitrary!

- So if we did this right, we must have

$$\mathbb{P}_{\theta}\left(L(\mathbf{X}) \leq \theta \leq U(\mathbf{X})\right) \geq 1 - \alpha \quad \text{for all } \theta \in \Theta$$

- This method of finding confidence intervals is called *inverting a hypothesis test*

- You can also go the other way! Start with a $(1-\alpha)$-CI $(L(\vec{x}), U(\vec{x}))$, and "invert" it to form a level-α test of $H_0: \theta = \theta_0$ vs $H_A: \theta \neq \theta_0$. (Assignment 4)

Famous Examples: Z-Intervals

- Example 4.7: Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$ where $\mu \in \mathbb{R}$ and σ^2 is known. Find a $(1-\alpha)$-confidence interval for μ by inverting the two-sided Z-test.

Let $\mu_0 \in \mathbb{R}$. We need a level-α test of $H_0: \mu = \mu_0$ vs $H_a: \mu \neq \mu_0$.

From Ex. 3.15, $R_{\mu_0} = \left\{ \vec{x} \in \mathcal{X}^n : \left| \frac{\bar{X}_n - \mu_0}{\sqrt{\sigma^2/n}} \right| > z_{1-\alpha/2} \right\}$

$\Rightarrow A_{\mu_0} = \left\{ \vec{x} \in \mathcal{X}^n : \left| \frac{\bar{X}_n - \mu}{\sqrt{\sigma^2/n}} \right| \leq z_{1-\alpha/2} \right\}$.

Therefore
$$
\begin{aligned}
1-\alpha &= \mathbb{P}_\mu(\vec{X} \in A_\mu) \\
&= \mathbb{P}_\mu\left(-z_{1-\alpha/2} < \frac{\bar{X}_n - \mu}{\sqrt{\sigma^2/n}} < z_{1-\alpha/2}\right) \\
&= \mathbb{P}_\mu\left(-z_{1-\alpha/2} \cdot \sqrt{\sigma^2/n} < \bar{X}_n - \mu < z_{1-\alpha/2} \cdot \sqrt{\sigma^2/n}\right) \\
&= \mathbb{P}_\mu\left(-z_{1-\alpha/2} \cdot \sqrt{\sigma^2/n} - \bar{X}_n < -\mu < z_{1-\alpha/2} \cdot \sqrt{\sigma^2/n} - \bar{X}_n\right) \\
&= \mathbb{P}_\mu\left(\bar{X}_n - z_{1-\alpha/2} \cdot \sqrt{\sigma^2/n} < \mu < \bar{X}_n + z_{1-\alpha/2} \cdot \sqrt{\sigma^2/n}\right)
\end{aligned}
$$

$\Rightarrow$ Our $(1-\alpha)$-CI for μ is $\left(\bar{X}_n - z_{1-\alpha/2} \cdot \sqrt{\sigma^2/n},\ \bar{X}_n + z_{1-\alpha/2} \cdot \sqrt{\sigma^2/n}\right)$.

Famous Examples: One-Sided Z-Intervals

- Example 4.8: Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$ where $\mu \in \mathbb{R}$ and σ^2 is known. Find a lower one-sided $(1-\alpha)$-confidence interval for μ by inverting an appropriate one-sided Z-test.

Ex 3.16: $R_{\mu_0} = \left\{ \vec{x} \in \mathcal{X}^n : \frac{\bar{x}-\mu_0}{\sqrt{\sigma^2/n}} > z_{1-\alpha} \right\} \Rightarrow A_{\mu_0} = \left\{ \vec{x} \in \mathcal{X}^n : \frac{\bar{x}-\mu_0}{\sqrt{\sigma^2/n}} \leq z_{1-\alpha} \right\}$

So $1-\alpha = \mathbb{P}_\mu\left(\frac{\bar{X}_n - \mu}{\sqrt{\sigma^2/n}} \leq z_{1-\alpha} \right)$

$= \mathbb{P}_\mu\left(-\mu \leq z_{1-\alpha} \cdot \sqrt{\frac{\sigma^2}{n}} - \bar{X}_n \right)$

$= \mathbb{P}_\mu\left(\mu \geq \bar{X}_n - z_{1-\alpha} \cdot \sqrt{\frac{\sigma^2}{n}} \right)$

$\Rightarrow$ Choose $\left(\bar{X}_n - z_{1-\alpha} \sqrt{\frac{\sigma^2}{n}}, \infty \right)$

EXERCISE: upper $(1-\alpha)$-CI for μ.

Famous Examples: t-Intervals

- Example 4.9: Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mathcal{N}\left(\mu, \sigma^2\right)$ where $\mu \in \mathbb{R}$ and $\sigma^2 > 0$. Find a $(1-\alpha)$-confidence interval for μ by inverting the two-sided t-test.

Ex 3.17: $R_{\mu_0} = \left\{ \vec{x} \in \mathcal{X}^n : \left| \frac{\bar{x} - \mu_0}{\sqrt{s^2/n}} \right| > t_{n-1, 1-\alpha/2} \right\}$

$\Rightarrow A_{\mu_0} = \left\{ \vec{x} \in \mathcal{X}^n : \left| \frac{\bar{x} - \mu_0}{\sqrt{s^2/n}} \right| \leq t_{n-1, 1-\alpha/2} \right\}$

So $1 - \alpha = \mathbb{P}_\mu\left(-t_{n-1, 1-\alpha/2} < \frac{\bar{X}_n - \mu}{\sqrt{S^2/n}} < t_{n-1, 1-\alpha/2} \right\}$

$= \mathbb{P}_\mu\left(\bar{X}_n - t_{n-1, 1-\alpha/2} \cdot \sqrt{\frac{S^2}{n}} < \mu < \bar{X}_n + t_{n-1, 1-\alpha/2} \cdot \sqrt{\frac{S^2}{n}} \right)$

So choose $\left(\bar{X}_n - t_{n-1, 1-\alpha/2} \cdot \sqrt{\frac{S^2}{n}}, \bar{X}_n + t_{n-1, 1-\alpha/2} \cdot \sqrt{\frac{S^2}{n}} \right)$. "$t$-interval"

Famous Examples: One-Sided t-Intervals

- Example 4.10: Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mathcal{N}\left(\mu, \sigma^2\right)$ where $\mu \in \mathbb{R}$ and $\sigma^2 > 0$. Find an upper one-sided $(1-\alpha)$-confidence interval for μ by inverting an appropriate one-sided t-test.

EXERCISE!

An LRT-Based Interval

$F_\theta(x) = 1 - e^{-(x-\theta)}$

- Example 4.11: Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with pdf $f_\theta(x) = e^{-(x-\theta)} \cdot \mathbb{1}_{x \geq \theta}$, where $\theta \in \mathbb{R}$. Find a $(1-\alpha)$-confidence interval for θ, by inverting an LRT.

From Ex 3.21, the LRT of $H_0: \theta \leq \theta_0$ vs $H_A: \theta > \theta_0$ had a rejection region of the form

$R_{\theta_0} = \{ \vec{x} \in \mathcal{X}^n : x_{(1)} > \theta_0 - \frac{\log(c)}{n} \text{ OR } x_{(1)} < \theta_0 \}$

$\Rightarrow A_{\theta_0} = \{ \vec{x} \in \mathcal{X}^n : x_{(1)} + \frac{\log(c)}{n} < \theta_0 \text{ AND } x_{(1)} \geq \theta_0 \} = \{ \vec{x} \in \mathcal{X}^n : x_{(1)} + \frac{\log(c)}{n} < \theta_0 < x_{(1)} \}.$

So if we can choose c to make a size-α test, then $\left(X_{(1)} + \frac{\log(c)}{n}, X_{(1)} \right)$ will be a $(1-\alpha)$-CI for θ.

How? $1 - \alpha = \mathbb{P}_\theta\left(X_{(1)} \leq \theta - \frac{\log(c)}{n} \wedge \underbrace{X_{(1)} \geq \theta}_{\text{always true!}} \right)$

$= \mathbb{P}_\theta\left(X_{(1)} \leq \theta - \frac{\log(c)}{n} \right)$

$= 1 - \left(1 - F_\theta\left(\theta - \frac{\log(c)}{n} \right) \right)^n$

$F_{X_{(1)}}(t) = (1 - F_\theta(t))^n$ .. Ass. 0

$= 1 - \left(1 - 1 + \exp\left(-\left(\theta - \frac{\log(c)}{n} - \theta \right) \right) \right)^n$

$= 1 - c$

$\Rightarrow$ Choose $c = \alpha$ $\Rightarrow$ $\left(X_{(1)} + \frac{\log(\alpha)}{n}, X_{(1)} \right)$ is a $(1-\alpha)$-CI for θ.

Functions of the Data *and* the Parameter

- In constructing our confidence intervals, we've often encountered statements that look like
$$\mathbb{P}_\theta\left(a < Q(\mathbf{X}, \theta) < b\right) \geq 1 - \alpha,$$
where $Q : \mathcal{X}^n \times \Theta \to \mathbb{R}$ is a function of the data $\mathbf{X}$ *and* the parameter θ, and a, b are constants

- We were able to choose those constants a and b because we knew exactly what the distribution of $Q(\mathbf{X}, \theta)$ was

- We could then "invert" the statement $a < Q(\mathbf{X}, \theta) < b$ to produce a confidence interval for θ

$Q(\vec{X}, \mu) \sim N(0,1)$

- Example 4.12: $N(\mu, \sigma^2)$, σ^2 known. $\mathbb{P}_\mu\left(-z_{1-\alpha/2} < \frac{\bar{X}_n - \mu}{\sqrt{\sigma^2/n}} < z_{1-\alpha/2}\right) = 1 - \alpha$

- Example 4.13: $\mathrm{Unif}(0, \theta)$: $\mathbb{P}_\theta\left(\frac{1}{2a} \leq \frac{X_{(n)}}{\theta} \leq \frac{1}{a}\right) = 1 - \alpha$, where a was chosen as before

$Q(\vec{X}, \theta)$, distribution is free of θ

Pivotal Quantities

- The key in these examples was that the *distribution* of $Q(\mathbf{X}, \theta)$ is free of θ
- Definition 4.4: A random variable $Q(\mathbf{X}, \theta)$ is a **pivotal quantity** (or **pivot**) for θ if its distribution is free of θ.
- So if $\mathbf{X} \sim f_{\theta_1}$ and $\mathbf{Y} \sim f_{\theta_2}$, then $Q(\mathbf{X}, \theta_1) \overset{d}{=} Q(\mathbf{Y}, \theta_2)$
- Every ancillary statistic is a pivotal quantity

  $Q(\vec{X}, \mu) \sim N(0,1)$, free of μ
- Example 4.14: $N(\mu, \sigma^2)$, σ^2 known: $\mathbb{P}_\mu\left(-z_{1-\alpha/2} < \frac{\bar{X}_n - \mu}{\sqrt{\sigma^2/n}} < z_{1-\alpha/2}\right) = 1-\alpha$
- Example 4.15: $\text{Exp}(\lambda)$: $Q(\vec{X}, \lambda) = \frac{X_1}{\lambda} \sim \text{Exp}(1)$ is pivotal for λ

Poll Time!

We can calculate $Q(x, \theta')$.

We don't know which θ generated X, so we don't know the dist'n of X $\Rightarrow$ don't know the dist'n of $Q(X, \theta')$.

Confidence Intervals from Pivotal Quantities

- Example 4.16: Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Exp}(\lambda)$, $\lambda > 0$. Show that $Q(\mathbf{X}, \lambda) = 2\lambda \sum_{i=1}^{n} X_i$ is a pivotal quantity, and use it to find a $1 - \alpha$ confidence interval for λ.

Use mgfs! $M_{\Sigma X_i}(t) = \left(\frac{\lambda}{\lambda - t}\right)^n$, $t < \lambda$

(FYI)

$\Rightarrow M_{2\lambda \Sigma X_i}(t) = \left(\frac{\lambda}{\lambda - 2\lambda t}\right)^n = (1-2t)^{-n} \Rightarrow Q(\vec{X}, \lambda) \sim \chi^2_{(2n)}$, free of λ, hence pivotal.

So set $1 - \alpha = \mathbb{P}(a < 2\lambda \Sigma X_i < b)$ for some $a, b \in \mathbb{R}$, $a < b$,

where a and b satisfy $1 - \alpha = F_{\chi^2_{(2n)}}(b) - F_{\chi^2_{(2n)}}(a)$. Many choices!

For example, if we choose $a = 0$, we get $b = F^{-1}_{\chi^2_{(2n)}}(1-\alpha) = \chi^2_{(2n), 1-\alpha}$.

Then $1 - \alpha = \mathbb{P}\left(0 < \lambda < \frac{\chi^2_{(2n),1-\alpha}}{2 \Sigma X_i}\right)$

$\Rightarrow$ Choose $\left(0, \frac{\chi^2_{(2n),1-\alpha}}{2 \Sigma X_i}\right)$.

Finding Pivotal Quantities

- There's no all-purpose strategy to finding pivotal quantities, but there's a neat trick that sometimes lets us pull one out of the pdf of a statistic $T(\mathbf{X})$

- Theorem 4.1: Suppose that $T(\mathbf{X}) \sim f_\theta$ is univariate and continuous, such that the pdf can be expressed as

$$f_\theta(t) = g(Q(t,\theta)) \cdot \left| \frac{\partial}{\partial t} Q(t,\theta) \right|$$

  for some function $g(\cdot)$ which is free of θ and some function $Q(t,\theta)$ which is continuously differentiable and one-to-one as a function of t (i.e., with θ fixed). Then $Q(T(\mathbf{X}),\theta)$ is a pivot.

Proof.

Fix $\theta \in \Theta$ and let $h_\theta(q)$ be the pdf $Q(T(\vec{X}), \theta)$. We'll just write $Q_\theta(T(\vec{X}))$. Let $Q_\theta^{-1}(q)$ be the functional inverse of $Q_\theta(t)$.

Then

$$h_\theta(q) = f_\theta\left(Q_\theta^{-1}(q)\right) \cdot \left| \frac{\partial}{\partial q} Q_\theta^{-1}(q) \right|$$

$$= f_\theta\left(Q_\theta^{-1}(q)\right) \cdot \left| \frac{\partial}{\partial t} Q_\theta(t) \Big|_{t = Q_\theta^{-1}(q)} \right|^{-1} \quad \text{by the inverse function theorem}$$

$$= \underbrace{g\left(Q_\theta\left(Q_\theta^{-1}(q)\right)\right) \cdot \left| \frac{\partial}{\partial t} Q_\theta(t) \Big|_{t = Q_\theta^{-1}(q)} \right|}_{\text{by assumption}} \cdot \left| \frac{\partial}{\partial t} Q_\theta(t) \Big|_{t = Q_\theta^{-1}(q)} \right|^{-1}$$

$$= g(q) \quad \text{which is free of } \theta.$$

So the distribution of $Q(T(\vec{X}), \theta)$ is free of θ. $\square$

Finding Pivotal Quantities: Examples

$F_\theta(x) = \left(\frac{x}{\theta}\right)$ (Assignment 0)

$F_{X_{(n)}}(t) = [F_\theta(t)]^n$

- Example 4.17: Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim}$ Unif $(0, \theta)$ where $\theta > 0$. Find a pivotal quantity based on $T(\mathbf{X}) = X_{(n)}$, and use it to construct a $1-\alpha$ confidence interval for θ.

The pdf of $T(\vec{X})$ is $X_{(n)}$ is $n \cdot f_\theta(t) \cdot F_\theta(t)^{n-1} = n \cdot \frac{1}{\theta} \cdot \left(\frac{t}{\theta}\right)^{n-1} = \frac{n t^{n-1}}{\theta^n}$

$= \underbrace{1}_{g(\cdot)} \cdot \left| \frac{\partial}{\partial t} \underbrace{\frac{t^n}{\theta^n}}_{Q(t,\theta)} \right|$

$$f_\theta(t) = g(Q(t,\theta)) \cdot \left| \frac{\partial}{\partial t} Q(t,\theta) \right|$$

By Theorem 4.1, $Q(X_{(n)}, \theta) = \frac{X_{(n)}^n}{\theta^n}$ is a pivotal quantity.

What's its distribution? For $x \in (0,1)$,

$\mathbb{P}\left(\frac{X_{(n)}^n}{\theta^n} \le x\right)$

$= \mathbb{P}(X_{(n)} \le \theta x^{1/n})$

$= F_\theta(\theta x^{\frac{1}{n}})^n$

$= \left(\frac{\theta x^{\frac{1}{n}}}{\theta}\right)^n$

$= x \implies Q(X_{(n)}, \theta) \sim \text{Unif}(0,1)$.

$\Rightarrow$ Choose $a<b$ s.t. $1-\alpha = \mathbb{P}\left(a < \frac{X_{(n)}^n}{\theta^n} < b\right)$.

For example, take $a = \alpha/2$, $b = 1-\alpha/2$. Then

$1-\alpha = \mathbb{P}\left(\alpha/2 < \frac{X_{(n)}^n}{\theta^n} < 1-\alpha/2\right)$

$= \mathbb{P}\left(\frac{X_{(n)}^n}{1-\alpha/2} < \theta^n < \frac{X_{(n)}^n}{\alpha/2}\right)$

$\Rightarrow$ Choose $\left(\frac{X_{(n)}}{(1-\alpha/2)^{\frac{1}{n}}}, \frac{X_{(n)}}{(\alpha/2)^{\frac{1}{n}}}\right)$.

Finding Pivotal Quantities: Examples

- Example 4.18: Let $X \sim f_\theta(x) = \frac{2(\theta - x)}{\theta^2} \cdot \mathbb{1}_{0 \le x \le \theta}$, where $\theta > 0$. Find a pivotal quantity based on X, and use it to construct a $1-\alpha$ confidence interval for θ.

Observe that if $Q(x,\theta) = \frac{\theta - x}{\theta}$, then $f_\theta(x) = g(Q(x,\theta)) \cdot \left|\frac{\partial}{\partial x} Q(x,\theta)\right|$, where $g(x) = 2x$.

By Theorem 4.1, $Q(X,\theta) = \frac{\theta - X}{\theta}$ is a pivotal quantity. What's its distribution?

For $x \in (0,1)$,

$$\mathbb{P}\left(\frac{\theta - X}{\theta} \le x\right) = \mathbb{P}(X \ge (1-x)\cdot\theta) = \int_{(1-x)\cdot\theta}^{\theta} \frac{2(\theta - t)}{\theta^2}\, dt = x^2.$$

Plenty of choices to make $1-\alpha = \mathbb{P}\left(a < \frac{\theta - X}{\theta} < b\right) = b^2 - a^2$.

If $a = 0$, then $b = \sqrt{1-\alpha}$.

Then $1-\alpha = \mathbb{P}\left(0 < \frac{\theta - X}{\theta} < \sqrt{1-\alpha}\right) = \mathbb{P}\left(X < \theta < \frac{X}{1-\sqrt{1-\alpha}}\right)$

$\Rightarrow$ Choose $\left(X, \frac{X}{1-\sqrt{1-\alpha}}\right)$.

Confidence Intervals: Interpretations

- Confidence intervals are almost as widely misinterpreted as p-values

- Suppose that in a published scientific study, you see a stated 95% confidence interval such as $(0.932, 1.452)$

  CI for θ

- How do you interpret this correctly?

  $(0.932, 1.452)$ is the observed "value" of the 0.95-CI $(L(\vec{X}), U(\vec{X}))$.

  random $\vec{X}$ . . . $\vec{x}$ observed

  $(L(\vec{X}), U(\vec{X}))$ random . . . $(L(\vec{x}), U(\vec{x}))$ observed

- Should we be surprised if we try and reproduce the study and calculate a 95% confidence interval of $(0.824, 1.734)$?

- What about $(-0.232, 1.440)$?

Poll Time!

$$0.95 \le \mathbb{P}(L(X_i) < \theta < U(X_i)) \quad \forall i \in \{1, 2, \ldots, 100\}$$

$$\mathbb{E}[\# \text{ of } \theta \text{ coverages}]$$

$$= \mathbb{E}\left[\sum_{i=1}^{100} \mathbb{1}_{L(X_i) < \theta < U(X_i)}\right]$$

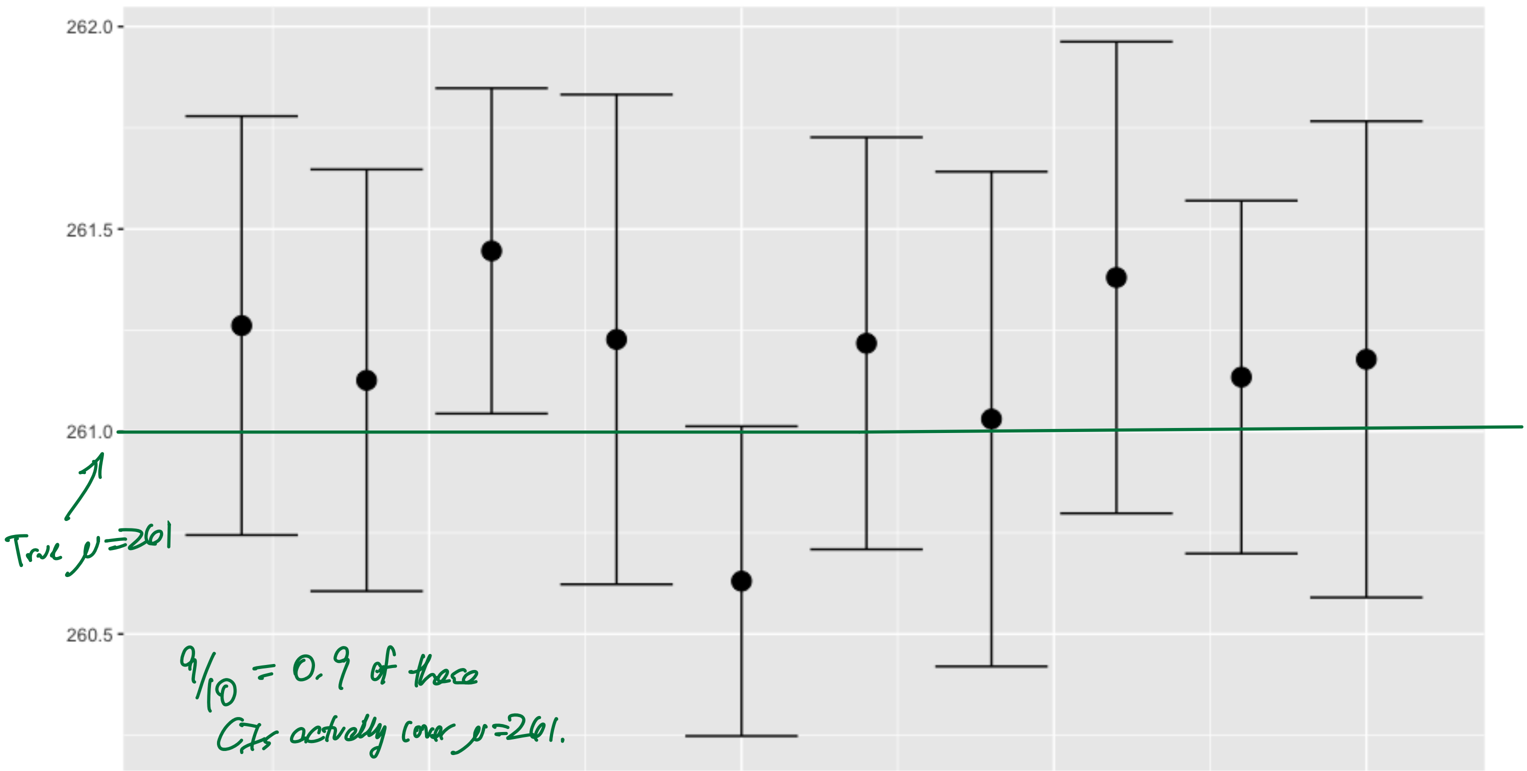
$$= \sum_{i=1}^{100} \mathbb{P}(L(X_i) < \theta < U(X_i))$$

$$\ge \sum_{i=1}^{100} 0.95$$

$$= 95$$

Confidence Intervals: Interpretations

- Here are ten observed 95% Z-intervals for μ calculated from ten random samples of size $n = 15$ from a $\mathcal{N}(\mu, 1)$ distribution:



Questioning Our Assumptions...

- All of the theory we've done up to this point has depended on the assumption of an underlying statistical model
- When we say "Suppose $X_1, X_2, \ldots, X_n \overset{iid}{\sim} f_\theta$...", we're assuming the data follows one of the distributions in the parametric family $\{f_\theta : \theta \in \Theta\}$ and only the parameter θ is unknown
- If we get the statistical model wrong, then any inferences we make about θ are likely to be completely invalid
- So it's extremely important to be able to check that statistical model assumption

Nothing Is Certain

- Of course, we can't *know* for sure that a model is correct
- Unless we generate the data ourselves... but then there's no point in doing inference
- But we can perform checks that give us confidence in our assumptions
- This is called *model checking*
- We will study two kinds of model checks: visual diagnostics and goodness-of-fit tests

Histograms: Preliminaries

- Suppose we have iid data $X_1, X_2, \ldots, X_n$, which we hypothesize are distributed according to a cdf F_θ

$h_1 < h_2 < \cdots < h_m$

- Let's group the range of the data into bins $[h_1, h_2], (h_2, h_3], \ldots, (h_{m-1}, h_m]$

- By the law of large numbers, EXERCISE!

$X \sim F_\theta$

$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{X_i \in (h_j, h_{j+1}]} \xrightarrow{p} \mathbb{P}_\theta(X \in (h_j, h_{j+1}])$$

$$= \mathbb{P}_\theta(h_j < X \leq h_{j+1}) = F_\theta(h_{j+1}) - F_\theta(h_j)$$

- So if n is large and we're correct about F_θ, then

$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{X_i \in (h_j, h_{j+1}]} \approx F_\theta(h_{j+1}) - F_\theta(h_j)$$

The Histogram Density Function

$F_\theta' = f_\theta$

- If, in addition, we believe the distribution ~~F_θ~~ is continuous with pdf f_θ, then there exists $h^* \in (h_j, h_{j+1})$ such that

$$\frac{1}{n(h_{j+1} - h_j)} \sum_{i=1}^{n} \mathbb{1}_{X_i \in (h_j, h_{j+1}]} \approx \frac{F_\theta(h_{j+1}) - F_\theta(h_j)}{h_{j+1} - h_j} = f_\theta(h^*)$$

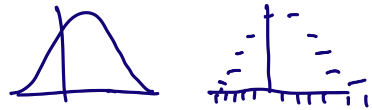
by the mean value theorem!

- Definition 4.5: Given data $X_1, \ldots, X_n$ and a partition $h_1 < h_2 < \cdots < h_m$, the **density histogram function** is defined as

$$\hat{f}_n(t) = \begin{cases} \frac{1}{n(h_{j+1} - h_j)} \sum_{i=1}^{n} \mathbb{1}_{X_i \in (h_j, h_{j+1}]}, & t \in (h_j, h_{j+1}] \\ 0, & \text{otherwise} \end{cases}$$

Histograms

- If we believe that our observed data $x_1, \ldots, x_n$ are realizations of $X_1, X_2, \ldots, X_n \overset{iid}{\sim} f_\theta$, then the observed $\hat{f}_n(t)$ should look like a "discretized" version of $f_\theta(t)$

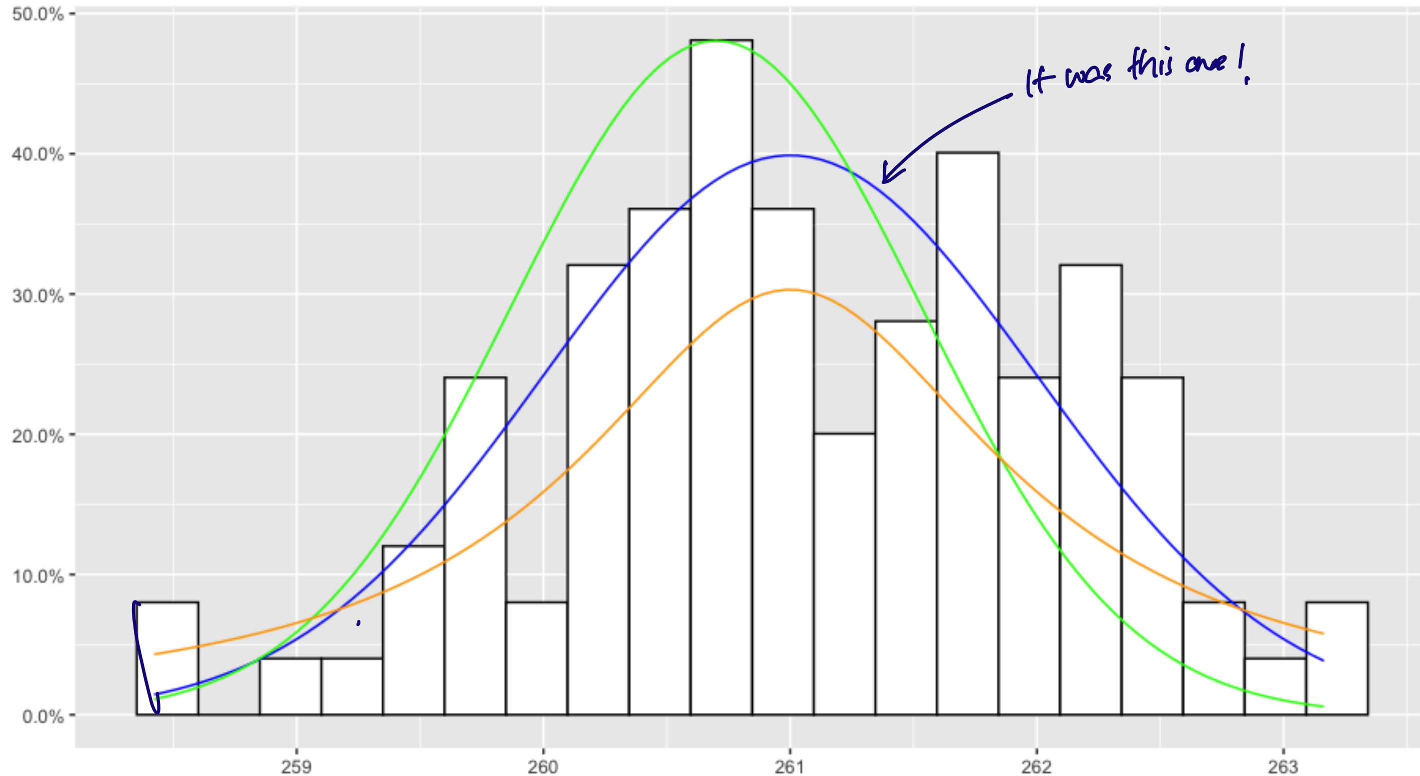


- ...and the resemblance should improve as n gets larger and each bin size $h_{j+1} - h_j$ gets smaller

- Definition 4.6: A plot of a density histogram function $\hat{f}_n(t)$ with vertical lines drawn at each h_j is called a **histogram**. A histogram where each bin width $h_{j+1} - h_j = 1$ is called a **relative frequency plot**.

Histograms: An Example

- Here's a histogram $(n = 100)$ overlaid with three hypothesized pdfs; which is more likely to have generated the data?



Poll Time!

The blue one!

Empirical CDFs

- We might prefer to deal with the cdf F_θ instead

- If we fix any $t \in \mathbb{R}$, then the law of large numbers says that

$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{X_i \le t} \xrightarrow{p} \mathbb{P}_\theta\left(X \le t\right) = F_\theta(t)$$

$X \sim F_\theta$

- So if n is large and we're correct about F_θ, then

$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{X_i \le t} \approx F_\theta(t)$$

This will approximate the true data-generating cdf (regardless of whether or not it was F_θ)

- Definition 4.7: Given a random sample ~~observations~~ $X_1, \ldots, X_n$, the **empirical distribution function (ecdf)** is defined as

$$\hat{F}_n(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{X_i \le t}$$

Empirical CDFs Are Nice

- If we believe that our observed data $x_1, \ldots, x_n$ are realizations of $X_1, X_2, \ldots, X_n \overset{iid}{\sim} F_\theta$, then $\hat{F}_n(t)$ should look like $F_\theta(t)$
- In fact, a famous result called the **Glivenko-Cantelli theorem** says that if $F_{\cancel{X}\theta}$ really is the true cdf, then $\hat{F}_n(t) \longrightarrow F_\theta(t)$ as $n \to \infty$ in a *much* stronger sense than convergence in probability

"uniformly almost surely": $\forall \varepsilon > 0$, $\mathbb{P}_\theta\left(\sup_{t \in \mathbb{R}} |\hat{F}_n(t) - F_\theta(t)| > \varepsilon\right) \xrightarrow{n \to \infty} 0$

- Theorem 4.2: For any fixed $t \in \mathbb{R}$, the ecdf $\hat{F}_n(t)$ is an unbiased estimator of $F_\theta(t)$, and it has a lower variance than $\mathbb{1}_{X_i \le t}$.

Proof.

$$\mathbb{1}_{X_i \le t} \sim \text{Bernoulli}\left(\mathbb{P}_\theta(\mathbb{1}_{X_i \le t} = 1)\right) = \text{Bernoulli}\left(\mathbb{P}_\theta(X_i \le t)\right) = \text{Bernoulli}\left(F_\theta(t)\right)$$

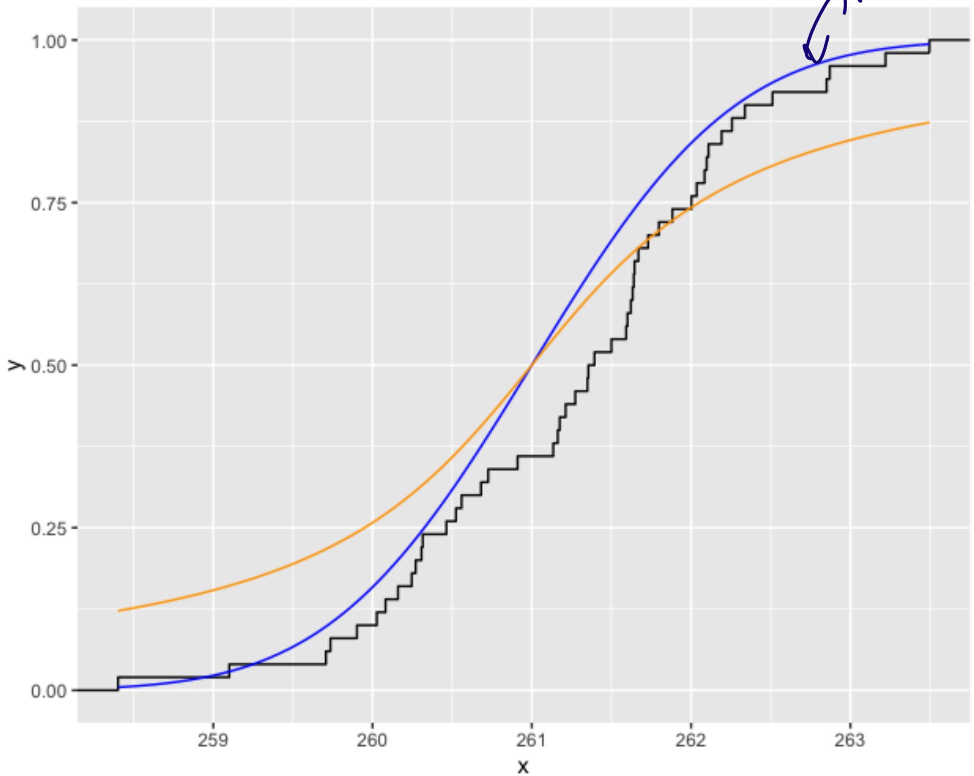
Therefore, $\mathbb{E}_\theta[\hat{F}_n(t)] = \frac{1}{n}\sum_{i=1}^n \mathbb{E}[\mathbb{1}_{X_i \le t}] = \frac{1}{n}\sum_{i=1}^n F_\theta(t) = F_\theta(t)$. So unbiased.

Also, $\text{Var}_\theta(\hat{F}_n(t)) = \frac{1}{n} \cdot \text{Var}_\theta(\mathbb{1}_{X_i \le t}) = \frac{1}{n} F_\theta(t) \cdot (1 - F_\theta(t)) \le F_\theta(t) \cdot (1 - F_\theta(t)) = \text{Var}_\theta(\mathbb{1}_{X_i \le t})$. $\square$

Empirical CDFs: An Example

- Here's an ecdf ($n = 50$) overlaid with two hypothesized cdfs; which is more likely to have generated the data?

It was this one!



Poll Time!

$X_1, \ldots, X_n \overset{iid}{\sim} N(0,1)$

$$\mathbb{E}[\hat{F}_n(0)] = F(0)$$

$$= \Phi(0)$$

$$= 1/2$$

Bringing Back Ancillarity and Sufficiency

- We know from Module 1 that if $\mathbf{X} \sim f_\theta$, the distribution of an ancillary statistic $S(\mathbf{X})$ is free of θ

- But if we've gotten the model $\{f_\theta : \theta \in \Theta\}$ wrong, $S(\mathbf{X})$ could very well depend on θ! (or some unknown parameter in the "true" model)

- So some ancillary statistics provide a model check: if our realization $S(\mathbf{x})$ is "surprising", we have evidence against the model being true

- Similarly, if $T(\mathbf{X})$ is sufficient for θ, then $\mathbf{X} \mid T(\mathbf{X}) = t$ shouldn't depend on θ

- This leads to the idea of **residual analysis**

- Loosely speaking, residuals are based on the information in the data that is left over after we have fit the model

(there's actually no formal definition of "residual!")

Residual Plots

- Example 4.19: Let $X_1, \ldots, X_n$ be a random sample from a suspected $\mathcal{N}(\mu, \sigma^2)$ distribution, where $\mu \in \mathbb{R}$ and σ^2 is known. If we're correct, then $R(\mathbf{X}) = (X_1 - \bar{X}_n, \ldots, X_n - \bar{X}_n)$ is ancillary for μ, because

$$X_i - \bar{X}_n \sim \mathcal{N}\left(0, \frac{n-1}{n}\sigma^2\right), \quad i = 1, \ldots, n$$

Assignment 1!

and therefore **standardized residuals**

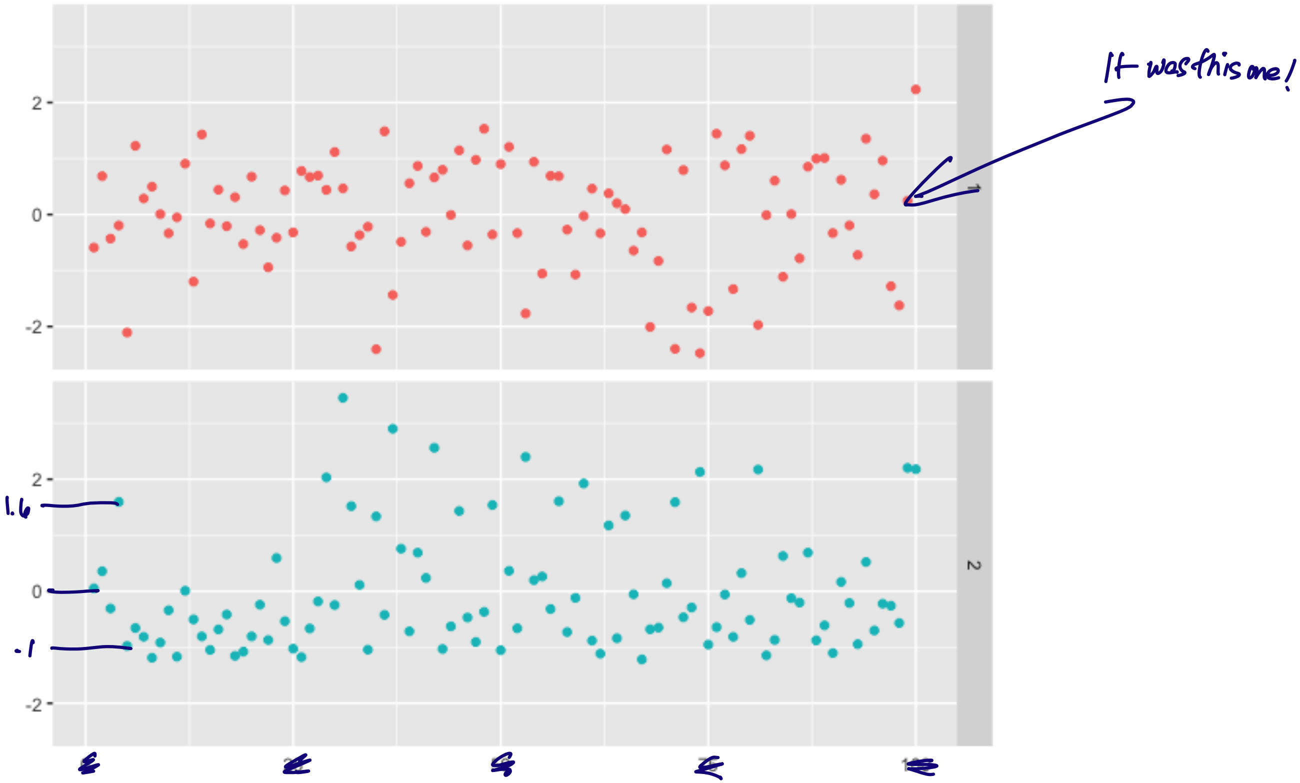
$$R_i^*(\mathbf{X}) := \frac{X_i - \bar{X}_n}{\sqrt{\frac{n-1}{n}\sigma^2}} \sim \mathcal{N}(0, 1).$$

If σ^2 is unknown, we can swap σ^2 with S^2, whence $R_i^*(\vec{X}) \sim t_{n-1}$

So if we're right about $\mathcal{N}(\mu, \sigma^2)$, then a scatterplot of the residuals shouldn't exhibit any discernable pattern, and should mostly stay within $(-3, 3)$

Residual Plots

- Example 4.20: Here are two standardized residual plots constructed from two samples ($n = 100$) with equal variances σ^2; which looks more like it came from a $\mathcal{N}(\mu, \sigma^2)$ distribution?



Probability Plots

- Probability plots extend this idea

- We need a fundamental result of probability theory first

- Theorem 4.3 (**Probability integral transform**): Let X be a continuous random variable with cdf $F_\theta(x)$, and let $U = F_\theta(X)$. Then $U \sim \text{Unif}(0,1)$.
  EXERCISE (or STA257)

- The order statistics of $U_1, \ldots, U_n \overset{iid}{\sim} \text{Unif}(0,1)$ follow a Beta distribution: $U_{(j)} \sim \text{Beta}(j, n-j+1)$, and so $\mathbb{E}\left[U_{(j)}\right] = \frac{j}{n+1}$ (Assignment 0)

- This suggests a recipe: If we hypothesize $X_1, \ldots, X_n \overset{iid}{\sim} F_\theta$, then we can plot

$\left(F_\theta(x_{(j)}), \frac{j}{n+1}\right), j = 1, \ldots, n.$

$F_\theta(X_{(j)})$
$= [F_\theta(X)]_{(j)}$ since F_θ is increasing
$= U_{(j)}$ if we're correct about F_θ

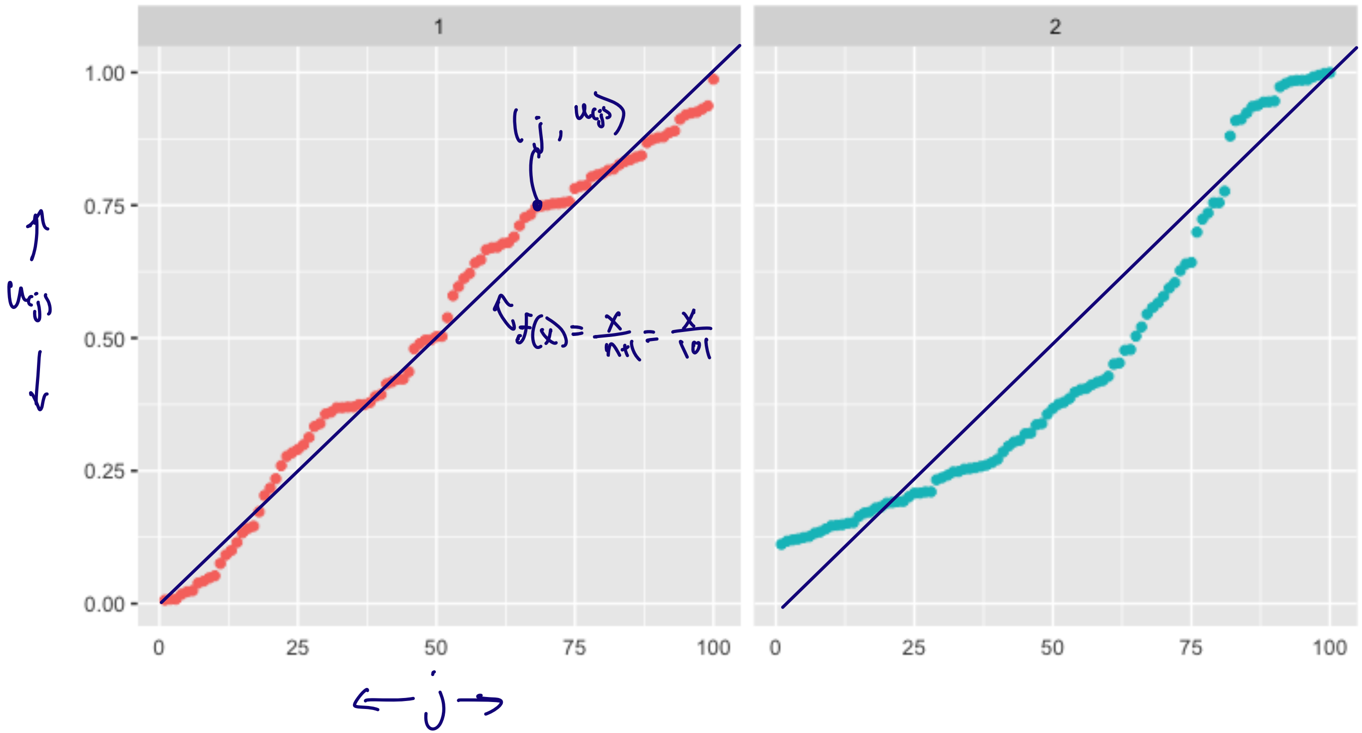
$= \mathbb{E}_\theta[F_\theta(X_{(j)})]$ if we're correct

If it doesn't look like the n points roughly lie on a straight line, we should question the assumption of F_θ.

Probability Plots

- Example 4.21: Here are two probability plots constructed from the standardized residuals as before, using $F_\theta(x) = \Phi(x)$. Which looks more like it came from a $\mathcal{N}(\mu, \sigma^2)$ distribution?

$n = 100$



Q-Q Plots

- We could also go in the other direction by looking at the quantiles

- Definition 4.8: Let X be a random variable with cdf F_θ. The **inverse cdf** (or the **quantile function**) is defined by $F_\theta^{-1}(t) = \inf\{x : F_\theta(x) \geq t\}$.

  "generalized inverse"

- When X is continuous, the inverse cdf is simply the functional inverse of F_θ

- There are plenty of software algorithms that can estimate the quantiles from a sample $x_1, \ldots, x_n$

- If we hypothesize $X_1, \ldots, X_n \sim F_\theta$ and we can compute F_θ^{-1}, then we have another recipe for model checking:

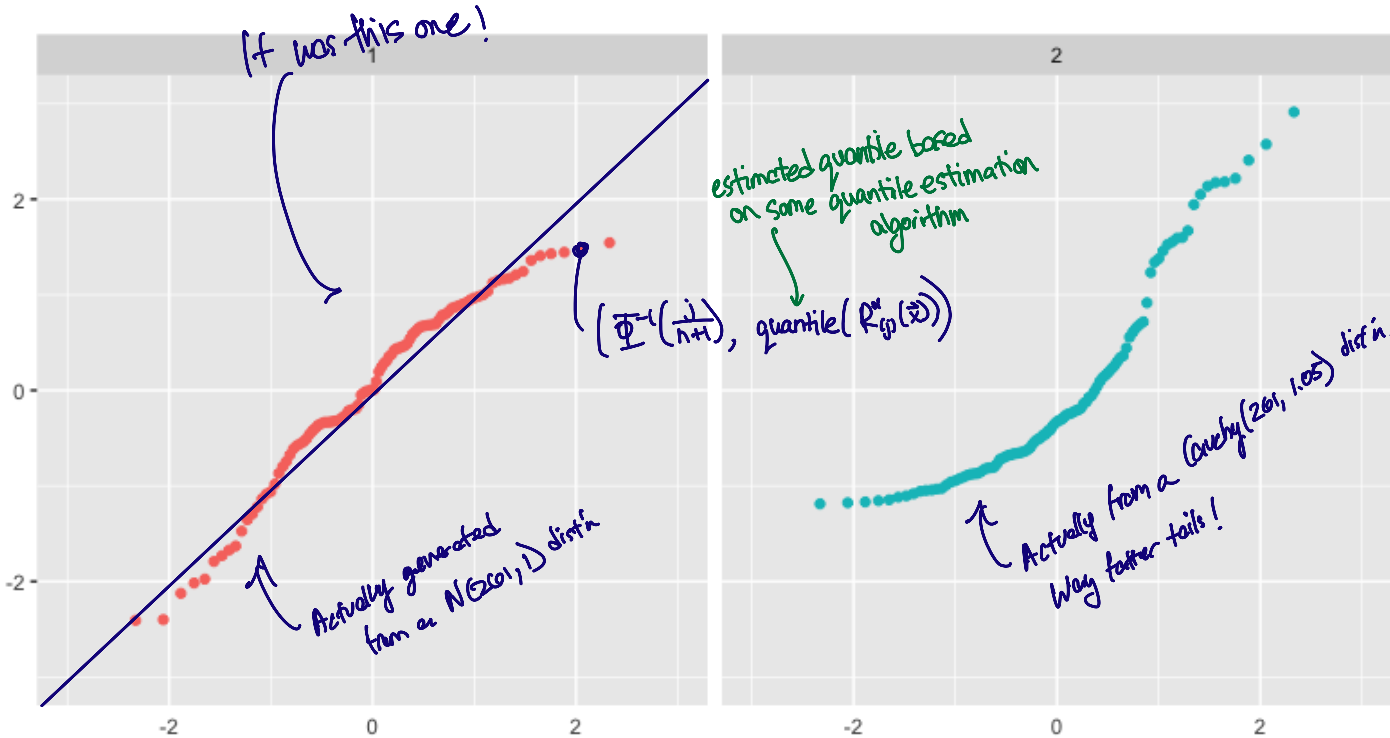
Plot the observed quantiles versus the theoretical ones! If it doesn't look (roughly) like the line $y = x$, then we should question F_θ.

Q-Q Plots

By far, the most popular use is for $F_\theta = \Phi(\cdot)$.
Use it to check if the N(0,1) does a good job of capturing the EXTREME observations (in the tails)

- Example 4.22: Here are two Q-Q plots constructed from the standardized residuals as before, using $F_\theta^{-1}(x) = \Phi^{-1}(x)$. Which looks more like it came from a $\mathcal{N}(\mu, \sigma^2)$ distribution?

$F_\theta^{-1} : [0,1] \to [-\infty, \infty]$



Q-Q Plots

- Q-Q plots are most frequently used as a test for Normality

- But technically there's no reason why we can't use them to compare *any* two distributions, observed or hypothesized

- ...provided we can actually compute (or estimate) their quantiles, of course

- Q-Q plots are particularly useful when we want to see how the "outliers" in our data compare to the extreme values predicted by the tails of a hypothesized distribution

Goodness of Fit Tests

- Instead of using visual diagnostics, we can use hypothesis tests as model checks

- This time, the null hypothesis H_0 is that the model $\{f_\theta : \theta \in \Theta\}$ for our data is "correct"

  H_0: the data are normally distributed

  H_0: the observations themselves are independent

  H_0: the two samples are independent

  etc..

- As usual, we have a test statistic $T(\mathbf{X})$ that follows some known distribution under H_0

- An observed value $T(\mathbf{x})$ which is very unlikely under H_0 (as quantified by a p-value) provides evidence that the model is wrong

- Such hypothesis tests are called **goodness of fit tests**

Towards a Foundational Test

- Suppose we observe iid random variables $W_1, W_2, \ldots, W_n$ taking values in sample space $\mathcal{X} = \{1, 2, \ldots, k\}$, which we think of as *labels* or *categories*

- We want to test whether the W_i's are distributed according to some hypothesized probability measure $\mathbb{P}_0$ on $\mathcal{X}$

- Let $X_i = \sum_{j=1}^{n} \mathbb{1}_{W_j = i}$ and let $p_i = \mathbb{P}_0(\{i\})$ so that under H_0,

$$(X_1, X_2, \ldots, X_k) \sim \text{Multinomial}(n, p_1, \ldots, p_k)$$

$X_i = \sum_{j=1}^{n} Y_j, \quad Y_j \sim \text{Bernoulli}(p_i)$

- Now define

$$R_i = \frac{X_i - \mathbb{E}[X_i]}{\sqrt{\text{Var}(X_i)}} \overset{H_0}{=} \frac{X_i - np_i}{\sqrt{np_i(1-p_i)}}$$

by the CLT

- Since $R_i \xrightarrow{d} \mathcal{N}(0,1)$ under H_0, it's tempting to think $\sum_{i=1}^{k} R_i^2 \xrightarrow{d} \chi^2_{(k)}$, but that's not true

Because the R_i's are not independent!

For the multinomial distribution, we need $\sum_{i=1}^{n} X_i = n$

Pearson's Chi-Squared Test

- Instead, we have the following result

- Theorem 4.4: If $(X_1, X_2, \ldots, X_k) \sim$ Multinomial$(n, p_1, \ldots, p_k)$, then

$$\sum_{i=1}^{k} (1-p_i) R_i^2 = \sum_{i=1}^{k} \frac{(X_i - np_i)^2}{np_i} \xrightarrow{d} \underbrace{\chi^2_{(k-1)}}_{\text{Asymptotic distribution under } H_0}.$$

- The statistic $\chi^2(\mathbf{X}) = \sum_{i=1}^{k} \frac{(X_i - np_i)^2}{np_i}$ is called a **chi-square statistic**, and it's almost always written as

O_i = # of "observed" i's

E_i = # of "expected" i's

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

- The chi-squared test is an *approximate test*, because the test statistic only has the $\chi^2_{(k-1)}$ distribution in the limit (more on this in Module 5)

A Famous Example: Fisher and Mendel's Pea Data

- Mendelian laws of inheritance establish relative frequencies of dominant and recessive phenotypes across new generations
- Gregor Mendel was known for his pioneering experiments with pea plants in the mid-1800s
- If you cross smooth, yellow male peas with wrinkled, green female peas, Mendelian inheritance predicts these relative frequencies of traits in the progeny:

	Yellow	Green
Smooth	$\frac{9}{16}$	$\frac{3}{16}$
Wrinkled	$\frac{3}{16}$	$\frac{1}{16}$

Relabel:

$1 \Leftrightarrow$ Yellow + Smooth

$2 \Leftrightarrow$ Yellow + Wrinkled

$3 \Leftrightarrow$ Green + Smooth

$4 \Leftrightarrow$ Green + Wrinkled

"$\mathbb{P}_0$": $\mathbb{P}_0(\{1\}) = 9/16$

$\mathbb{P}_0(\{2\}) = 3/16$

$\mathbb{P}_0(\{3\}) = 3/16$

$\mathbb{P}_0(\{4\}) = 1/16$

A Famous Example: Fisher and Mendel's Pea Data

- Mendel crossed 556 such pairs of peas together and recorded the following counts:

OBSERVED

	Yellow	Green
Smooth	315	108
Wrinkled	102	31

EXPECTED $= 556 \cdot \mathbb{P}_0(\{i\})$

	Yellow	Green
Smooth	312.75	104.25
Wrinkled	104.25	34.75

- Example 4.23: Do these results support the predicted frequencies?

$$\chi^2 = \frac{(315-312.75)^2}{312.75} + \frac{(108-104.25)^2}{104.23} + \frac{(102-104.25)^2}{104.25} + \frac{(31-34.75)^2}{34.75} \simeq 0.6043$$

Our p-value is $p(\vec{x}) = \mathbb{P}(\chi^2_{(3)} \geq 0.6043)$

$$= 1 - \mathbb{P}(\chi^2_{(3)} < 0.6043)$$

$$\simeq 0.895.$$

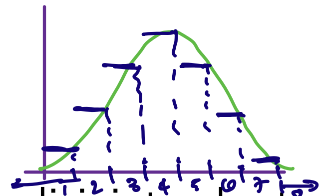
So we fail to reject H_0 at the 0.05 significance level.

Check out the "Mendelian paradox"

Extending the Chi-Squared Test

- What if our hypothesized distribution is not categorical, but quantitative?

- We can still use a chi-squared test – but how?



- The trick is to partition the sample space $\mathcal{X}$ into k disjoint subsets $\mathcal{X}_1, \ldots, \mathcal{X}_k$, and let $X_i = \sum_{j=1}^n \mathbb{1}_{W_j \in \mathcal{X}_i}$ and $p_i = \mathbb{P}_0(\mathcal{X}_i)$ $= \mathbb{P}_0(X \in \mathcal{X}_i)$

Eg: $\mathcal{X} = \mathbb{R}$. Maybe $\mathcal{X}_1 = (-\infty, -3]$, $\mathcal{X}_2 = (-3, 3]$, $\mathcal{X}_3 = (3, 27]$, $\mathcal{X}_4 = (27, \infty)$.

- The finer our partition, the better we can distinguish between distributions

- But of course, we need to increase our sample size accordingly so that each category gets sufficiently "filled" with data

Guideline: each $\mathcal{X}_i$ should contain at least 5 observations before doing this!

If you have 0 observations in $\mathcal{X}_i$, then you can't hypothesize anything other than $p_i = 0$.

* Probability plots: use the probability integral transform to make $U_1, \ldots, U_n \overset{iid}{\sim} \text{Unif}(0,1)$ under $H_0: F_\theta$. χ^2 test is basically a quantitative version of visual probability plot check.

A Famous Example: Testing for Uniformity

- There are many reasons why we might want to test whether some data $U_1, \ldots, U_n$ arises from a Unif$(0,1)$ distribution

* Probability plots: use the probability integral transform to make $U_1, \ldots, U_n \overset{iid}{\sim}$ Unif(0,1) under $H_0 : F_\theta$. The chi-squared test is basically a quantitative version of the visual check of the probability plot

* Random number generation: when simulating data from some distribution F_θ, we generally need to start with $U_1, \ldots, U_n \overset{iid}{\sim}$ Unif(0,1) data and then transform it (eg. Assignment 0 Q 13). We can't generate truly random numbers*, but we can construct deterministic sequences $u_1, u_2, u_3, \ldots$ that hopefully "look" random enough

- We can use a chi-squared test for this by binning $[0,1]$ into k equal-sized sub-intervals of length $1/k$, and letting $X_i = \sum_{j=1}^{n} \mathbb{1}_{U_j \in (\frac{i-1}{k}, \frac{i}{k}]}$ and $p_i = 1/k$
$= \mathbb{P}(U_j \in (\frac{i-1}{k}, \frac{i}{k}])$.

* Exception: numbers generated by radioactive decay ("HotBits", etc)

A Famous Example: Testing for Uniformity

- Example 4.24: How can we test whether an iid sequence $U_1, U_2, \ldots$ arises from a Unif $(0,1)$ distribution using 10 categories?

Partition $[0,1]$ into $[0, \frac{1}{10}], (\frac{1}{10}, \frac{2}{10}], \ldots, (\frac{9}{10}, 1]$.

Let $X_i = \sum_{j=1}^{n} \mathbb{1}_{U_j \in (\frac{i-1}{10}, \frac{i}{10}]}$, $i = 1, \ldots, 10$.

OR: let $V_i = \lceil 10 \cdot U_i \rceil$ so $V_1, \ldots, V_n \overset{iid}{\sim} \text{Unif}(\{1, 2, \ldots, 10\})$ under H_0

and let $X_i = \sum_{j=1}^{n} \mathbb{1}_{V_j = i}$

Then carry out a chi-squared goodness-of-fit test with $\chi^2 = \sum_{i=1}^{10} \frac{(X_i - n/10)^2}{n/10} \overset{H_0}{\underset{\text{approx}}{\sim}} \chi^2_{(9)}$

Note: this is actually a very low-powered test

There are way better randomness tests out there! The "Diehard tests" are the standard these days...

Other Goodness of Fit Tests

- Pearson's chi-squared isn't the only goodness of fit test out there; there are countless others

- Many apply to one particular parametric family specifically

Eg: for testing normality, Shapiro-Wilk, Anderson-Darling, Jarque-Bera...

- Others are completely generic and test for equality between *any* two distributions

Kolmogorov-Smirnov and Cramer-von Mises are by far the most popular

- These latter tests allow us to compare an ecdf $\hat{F}_n$ to a hypothesized cdf F_θ

Very helpful! Basically a quantitative version of the ecdf-vs-F_θ visual check

Other Goodness of Fit Tests

- In most cases, the distribution of the test statistic under H_0 is only known in the limit as $n \to \infty$
- Even then, cutoffs often can't be calculated exactly and require simulations to approximate
- When there's more than one test out there for the same thing, it's always a good idea to read up on the benefits/drawbacks of each one before deciding which to use
- One might have a lower probability of Type I error, another might higher power for lower sample sizes, another might be more robust to outliers, and so on

Very active area of research!