Contact lens case flips: #H = 15
#T = 8

STA261 - Module 6

Bayesian Statistics

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The Bayesian Model

- So θ is now treated as a *random variable* with its own distribution expressing our beliefs

- The Bayesian framework for inference contains the statistical model $\{f_\theta : \theta \in \Theta\}$ and adds a **prior probability measure** $\Pi : \Theta \to [0, 1]$ describing our beliefs about θ *before* we observe the data like "$\mathbb{P}$", but the prior version

- We usually refer to the prior by its pdf/pmf, which we denote generically as $\pi(\cdot)$

eg: $\pi(p) = \mathbb{1}_{p \in [0,1]}$ → Unif(0,1) prior on p

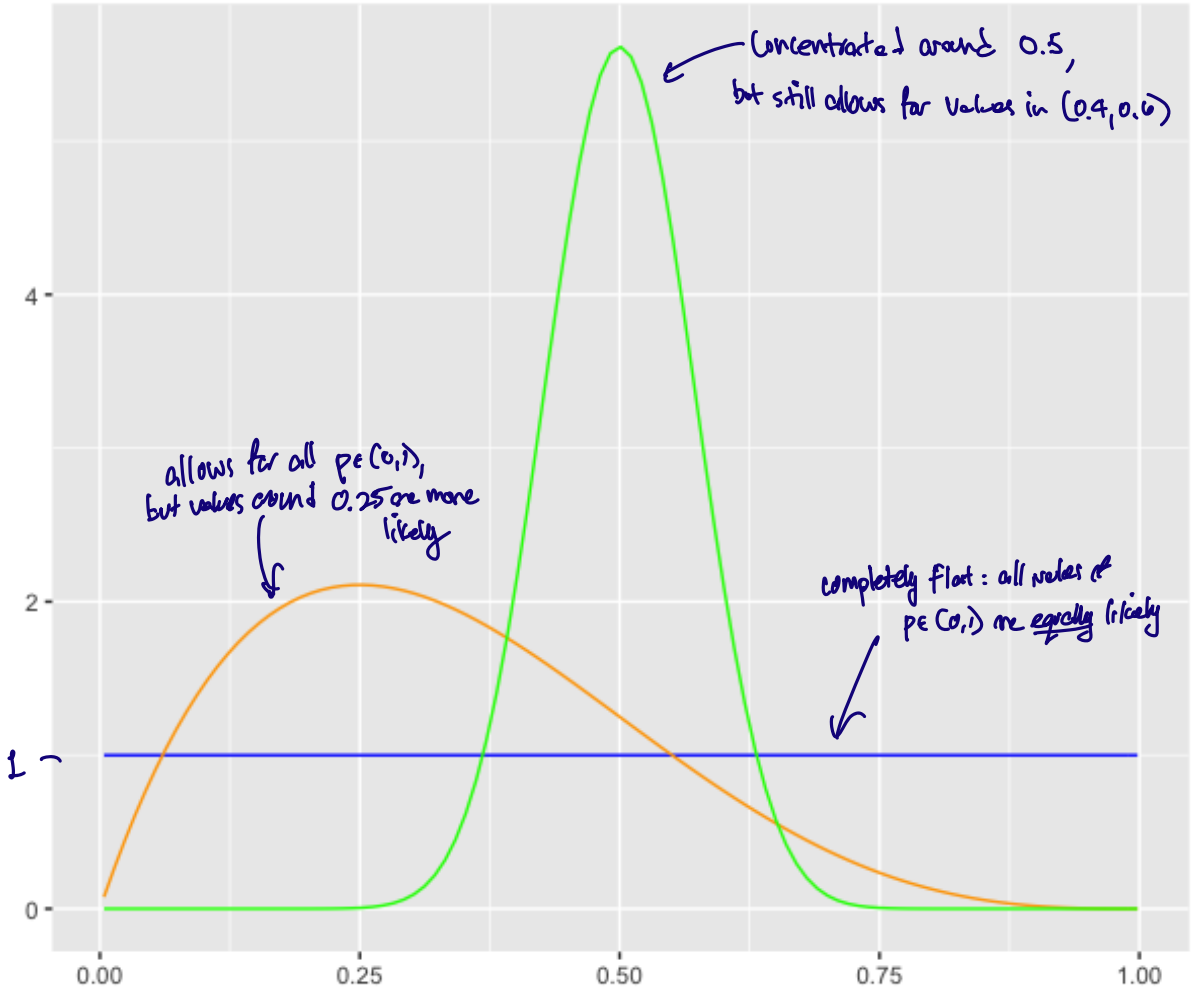
$\pi(p) = 3e^{-3p}$ → Exp(3) prior on p

$\pi(\theta) = \frac{1}{\sqrt{2\pi}} \exp(-\theta^2/2)$ → $N(0,1)$ prior on θ

A Simple Example of a Prior

- Suppose we're shown a coin, and we are told to infer whether it's biased or not, just from looking at it (before flipping it)
- If $X = \mathbb{1}_{\text{heads}}$, then we want to make inferences about the random variable p, where $X \mid p \sim \text{Bernoulli}\,(p)$
- What should our prior on $\Theta = [0, 1]$ look like?
- It depends on what we know (or don't know) about the coin
- Here are three of many possible choices

Prior Distributions for the Coin Example



The Prior Predictive Distribution

- What if we were asked to predict the likelihood of the coin coming up heads at this point?

- It's reasonable to take a weighted average of all possible Bernoulli (p) distributions, each one weighted by our prior confidence $\pi(p)$, which is

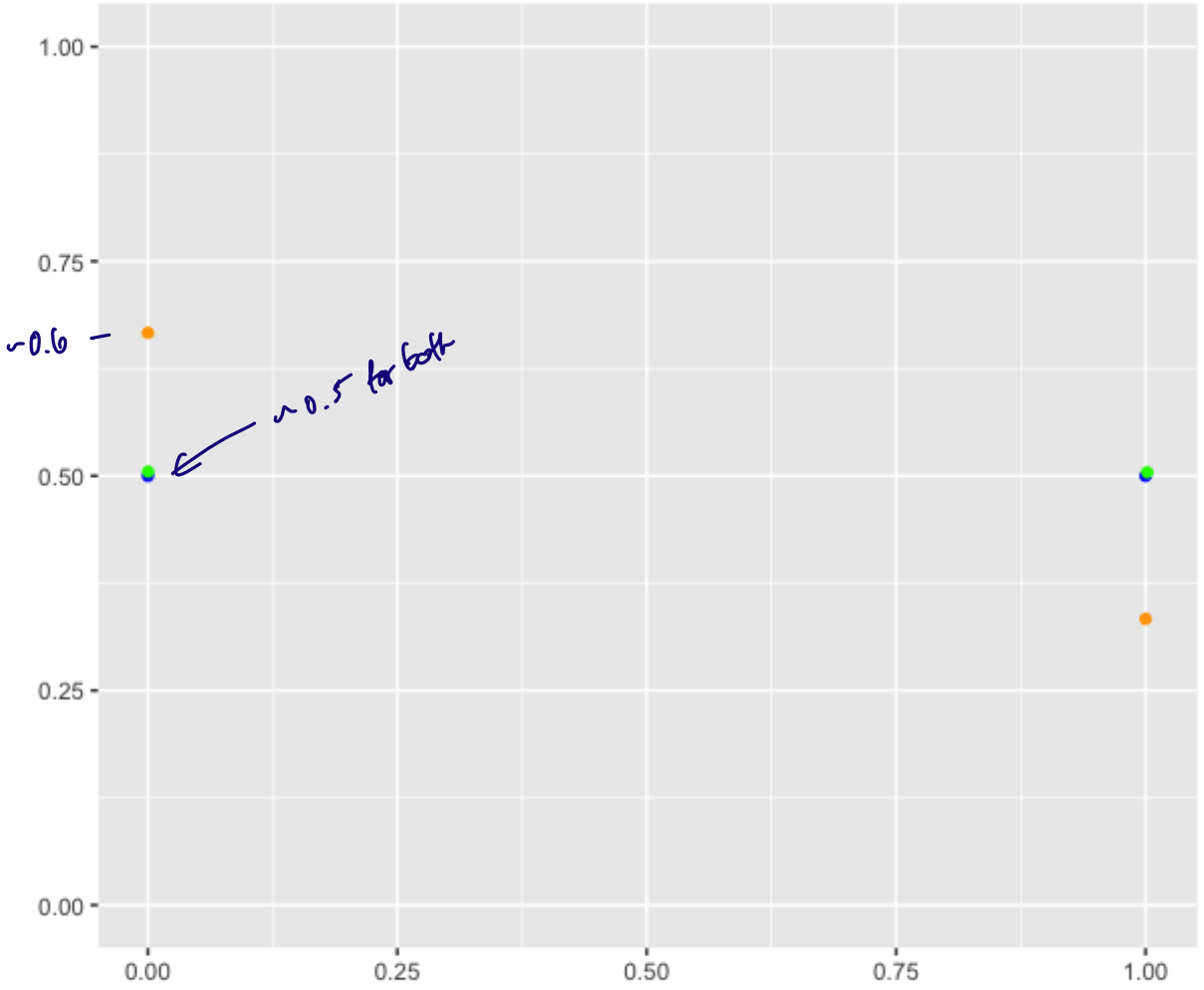
$$\int_\Theta \mathbb{P}_p(X=1)\cdot\pi(p)\,\mathrm{d}p = \int_0^1 p\cdot\pi(p)\,\mathrm{d}p$$

- There's a name for this

- Definition 6.1: Given a pdf f_θ and a prior distribution π on θ, the **prior predictive distribution** of the data $\mathbf{x}$ is given by the pdf

$$f(\mathbf{x}) = \int_\Theta f_\theta(\mathbf{x})\cdot\pi(\theta)\,\mathrm{d}\theta.$$

Prior Predictive Distributions for the Coin Example



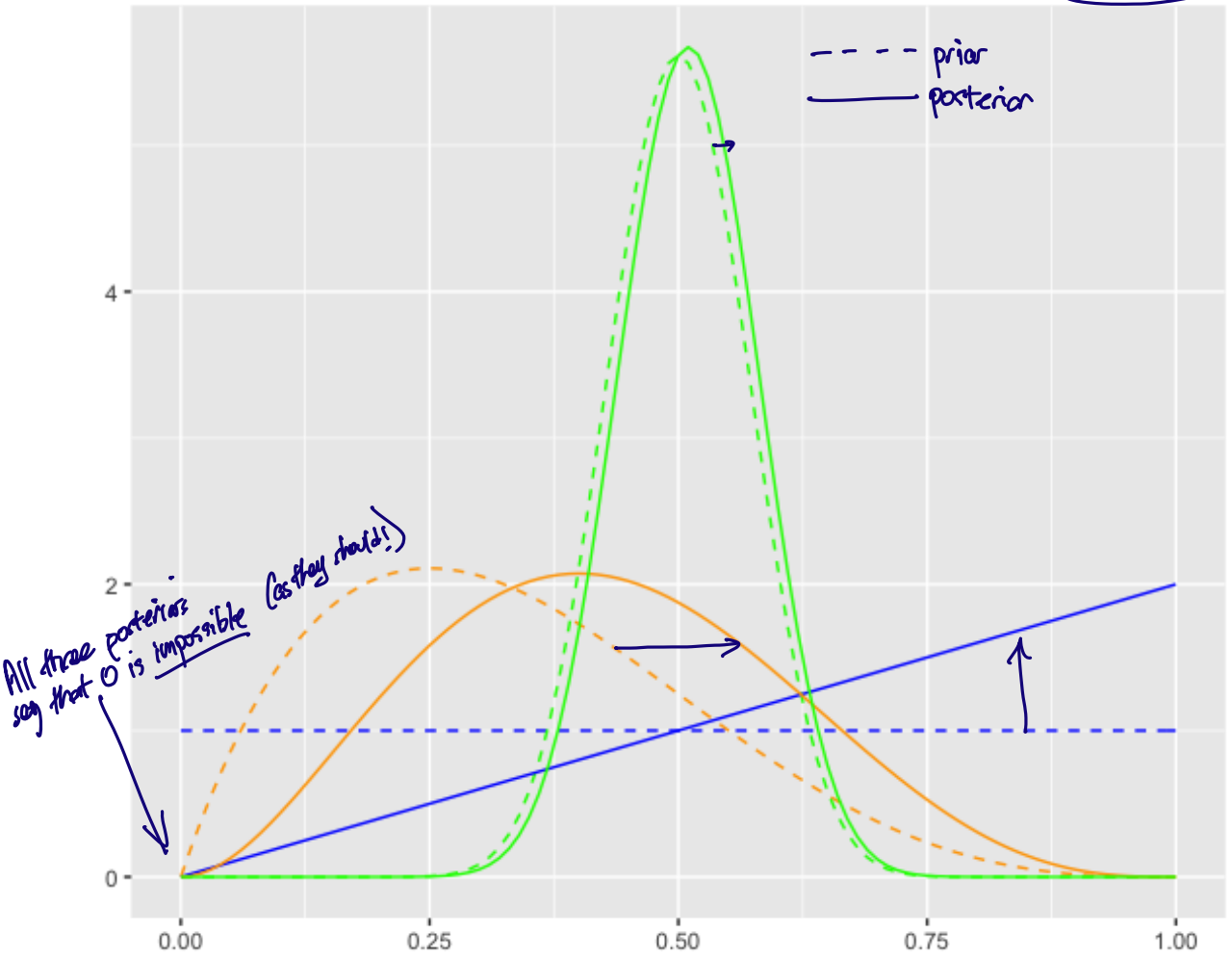
The Posterior Distribution - A Motivation

- Now, suppose we actually flip the coin once and observe $X = 1$
- If we were asked what the likelihood of some $p' \in [0, 1]$ is now, we could take our prior probability $\pi(p')$ and weigh it down by the likelihood of observing $X = 1$ if the "true" parameter really were p'
- That is, it's reasonable to answer with $\mathbb{P}_{p'}(X = 1) \cdot \pi(p')$, since data in support of p' will make this relatively high, while data in support of some p'' far away from p' will make it relatively low
- To put everything on the same scale, may as well normalize those quantities over all possible $p \in [0, 1]$ and answer instead with

$$\frac{\mathbb{P}_{p'}(X = 1) \cdot \pi(p')}{\int_0^1 \mathbb{P}_p(X = 1) \cdot \pi(p)\, \mathrm{d}p} = \frac{p' \cdot \pi(p')}{\int_0^1 p \cdot \pi(p)\, \mathrm{d}p}$$

EXERCISE: prove this is a valid pdf (as a function of p')

Posterior Distributions for the Coin Example $(X = 1)$



The Posterior Distribution - A Derivation

- In general, $f_\theta(\mathbf{x}) \cdot \pi(\theta)$ is the joint pdf of $(\mathbf{X}, \theta)$
- From Bayes' rule, the conditional pdf of $\theta \mid \mathbf{X}$ is given by

$$\frac{f_\theta(\mathbf{x}) \cdot \pi(\theta)}{f(\mathbf{x})}$$

$f(\mathbf{x})$ ← prior predictive distribution:

$= \int_\Theta f_\theta(\vec{x}) \cdot \pi(\theta)\, d\theta$

- There's also a name for this
- Definition 6.2: The **posterior distribution of θ** is the conditional distribution of $\theta \mid (\mathbf{X} = \mathbf{x})$, given by the pdf

$$\pi(\theta \mid \mathbf{x}) = \frac{f_\theta(\mathbf{x}) \cdot \pi(\theta)}{\int_\Theta f_\theta(\mathbf{x}) \cdot \pi(\theta)\, \mathrm{d}\theta}.$$

$\pi(\theta)$ is the prior pdf

$\pi(\theta|\vec{x})$ is the posterior pdf

Poll Time!

$$f_\theta(\vec{x}) = P_\theta(\vec{X} = \vec{x})$$
$$= P(\vec{X} = \vec{x} \mid \theta)$$

More on the Posterior

"proportional to" $f(x) \propto g(x) \iff$ there exists some $c \neq 0$ which is free of x such that $f(x) = c \cdot g(x)$

- The posterior $\pi(\theta \mid \mathbf{x})$ is a function of θ, and the data $\mathbf{x}$ is *observed* (we treat $\vec{x}$ as constant)
- So we could write $\pi(\theta \mid \mathbf{x}) \propto f_\theta(\mathbf{x}) \cdot \pi(\theta)$ because $\pi(\theta|\vec{x}) = \frac{f_\theta(\vec{x}) \cdot \pi(\theta)}{\int_\Theta f_\theta(\vec{x}) \cdot \pi(\theta)\, d\theta}$, constant wrt θ
- Thus, $[\int_\Theta f_\theta(\mathbf{x}) \cdot \pi(\theta)\, \mathrm{d}\theta]^{-1}$ plays the role of normalizing constant for the unnormalized pdf $f_\theta(\mathbf{x}) \cdot \pi(\theta)$
- If the functional form of $f_\theta(\mathbf{x}) \cdot \pi(\theta)$ looks familiar, then we'll know what $(\int_\Theta f_\theta(\mathbf{x}) \cdot \pi(\theta)\, \mathrm{d}\theta)^{-1}$ must be, and we can get $\pi(\theta \mid \mathbf{x})$ for free
- Example 6.1: Suppose we calculate $f_\theta(x) \cdot \pi(\theta) \propto \theta^{x+1}(1-\theta)^{2-x}$ for $\theta \in (0, 1)$. What is $\pi(\theta \mid x)$?

Integration exercise: check $\int_0^1 \theta^{x+1}(1-\theta)^{2-x}\, d\theta = \frac{\Gamma(x+2)\cdot\Gamma(3-x)}{\Gamma(5)}$

It's a Beta! What are the parameters? If $Z \sim \text{Beta}(a, b)$, then it has pdf $f_Z(z) \propto z^{a-1}(1-z)^{b-1}$

So $\theta|x \sim \text{Beta}(x+2, 3-x)$

Therefore, $\pi(\theta|x) = \frac{\Gamma(5)}{\Gamma(x+2)\cdot\Gamma(3-x)} \cdot \theta^{x+1}(1-\theta)^{2-x}$

More on the Posterior

- The observed data dictates how much the posterior distribution differs from the prior

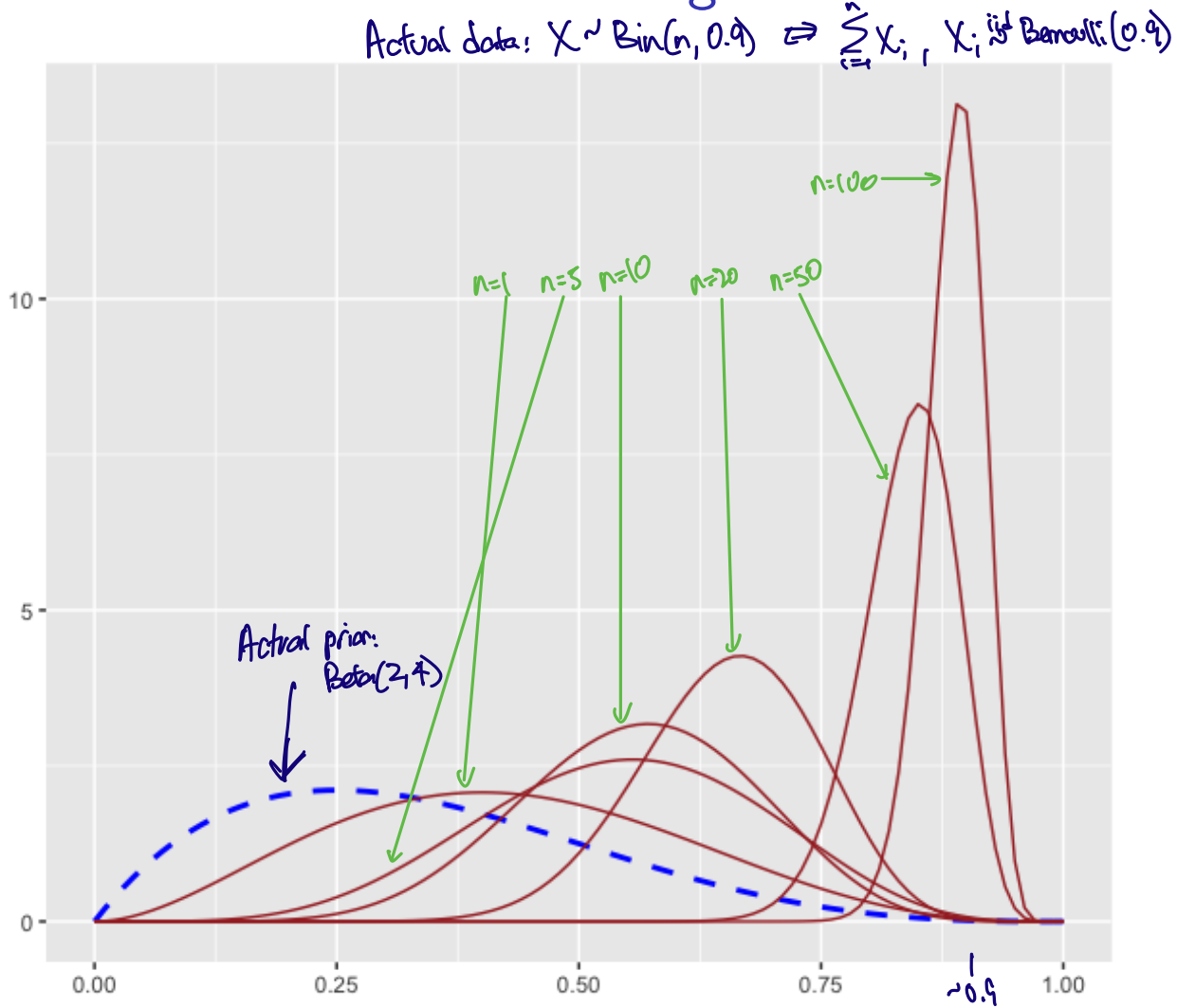
- Consider three different priors:
  - π_1 is highly concentrated at $\theta_1 \in \Theta$
  - π_2 is highly concentrated at $\theta_2 \in \Theta$
  - π_3 is Unif (Θ)

- Now we observe $\mathbf{x}$; suppose the likelihood $L(\theta \mid \mathbf{x}) = f_\theta(\mathbf{x})$ "supports" θ_2 in the frequentist sense

- What do the posteriors look like?
  - $\pi_1(\cdot \mid \mathbf{x})$ less concentrated at Θ_1
  - $\pi_2(\cdot \mid \mathbf{x})$ even more concentrated at Θ_2
  - $\pi_3(\cdot \mid \mathbf{x})$ somewhat concentrated at Θ_2

- Even if the prior is strong, the likelihood will eventually "overpower" it as the sample size n grows

When the Prior and the Data Disagree



Computing Posteriors: Examples

- Example 6.2: Suppose that $\pi(p) = \text{Beta}(\alpha, \beta)$ and $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Bernoulli}(p)$. Find the posterior $\pi(p \mid \mathbf{x})$.

$\pi(p|\vec{x}) \propto f_p(\vec{x}) \cdot \pi(p) = L(p|\vec{x}) \cdot \pi(p)$ also fine

$= \left(\prod_{i=1}^{n} p^{x_i} (1-p)^{(1-x_i)} \right) \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} p^{\alpha-1} (1-p)^{\beta-1}$

$\propto p^{\Sigma x_i} (1-p)^{n-\Sigma x_i} \cdot p^{\alpha-1} (1-p)^{\beta-1}$

$= p^{\Sigma x_i + \alpha - 1} (1-p)^{n-\Sigma x_i + \beta - 1}$

this is an un-normalized Beta(α', β') pdf where $\alpha' = \Sigma x_i + \alpha$ and $\beta' = n - \Sigma x_i + \beta$

$\propto \frac{\Gamma(\Sigma x_i + \alpha + n - \Sigma x_i + \beta)}{\Gamma(\Sigma x_i + \alpha)\cdot\Gamma(n - \Sigma x_i + \beta)} p^{\Sigma x_i + \alpha - 1} (1-p)^{n-\Sigma x_i + \beta - 1}$

So $p|\vec{x} \sim \text{Beta}(\Sigma x_i + \alpha, n - \Sigma x_i + \beta)$

Same parametric family as $\pi(p)$, but with the original parameters α and β "updated" in light of $\vec{X} = \vec{x}$

Computing Posteriors: Examples

- Example 6.3: Suppose that $\pi(\lambda) = \text{Gamma}(\alpha, \beta)$ and $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$. Find the posterior $\pi(\lambda \mid \mathbf{x})$.

$$\pi(\lambda|\vec{x}) \propto f_\lambda(\vec{x}) \cdot \pi(\lambda)$$

$$= \left(\prod_{i=1}^{n} \frac{x_i^{x_i} e^{-\lambda}}{x_i!} \right) \lambda^{\alpha-1} e^{-\beta\lambda}$$

$\lambda^{\alpha-1} e^{-\beta\lambda}$: unnormalized Gamma$(\alpha, \beta)$ pdf

$$\propto \lambda^{\sum x_i} e^{-n\lambda} \lambda^{\alpha-1} e^{-\beta\lambda}$$

$$= \lambda^{\sum x_i + \alpha - 1} e^{-[n+\beta]\cdot\lambda}$$

$$\Longrightarrow \quad \lambda|\vec{x} \sim \text{Gamma}(\textstyle\sum x_i + \alpha,\ n+\beta)$$

The Return of Sufficiency

- What if instead of observing $\mathbf{x}$, we only have access to a sufficient statistic $T(\mathbf{x})$?

- Sufficiency kind of carries over to the Bayesian setting, in the following sense

- Theorem 6.1: Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} f_\theta$ and let $\pi(\theta)$ be a prior on θ. If $T(\mathbf{X})$ is a sufficient statistic for θ (in the frequentist sense), then $\pi(\theta \mid \mathbf{x}) = \pi(\theta \mid T(\mathbf{x}))$.

posterior given $\pi(\theta)$ and $\vec{X} = \vec{x}$

posterior given $\pi(\theta)$ and $T(\vec{X}) = T(\vec{x})$

Different functions!

Proof: EXERCISE!

Computing Posteriors: Examples

- Example 6.4: Suppose that $\pi(p) = \text{Beta}(\alpha, \beta)$ and $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Bernoulli}(p)$. Find the posterior $\pi(p \mid \sum_{i=1}^n x_i)$.

Sufficient for p

$$\pi(p \mid \textstyle\sum x_i) \propto f_p(\textstyle\sum x_i) \cdot \pi(p)$$

$$\propto \left(\binom{n}{\sum x_i} \cdot p^{\sum x_i} (1-p)^{n - \sum x_i} \right) \cdot p^{\alpha-1} (1-p)^{\beta-1}$$

$$\propto p^{\sum x_i + \alpha - 1} (1-p)^{n - \sum x_i + \beta - 1}$$

So $p \mid \sum x_i \sim \text{Beta}(\sum x_i + \alpha, n - \sum x_i + \beta)$. Same as before!

Hyperparameters

- In the previous example, the prior $\pi(\theta) = \text{Gamma}(\alpha, \beta)$ had its own set of parameters: α, β
- Definition 6.3: The parameters λ of a prior distribution $\pi_\lambda(\cdot)$ in a parametric family $\{\pi_\lambda : \lambda \in \Lambda\}$ are called **hyperparameters**.

generic parameter (like θ in old stuff) could be a vector, eg $\lambda = (\alpha, \beta)$

- Sometimes the hyperparameter λ is a given constant (either known from prior experience or chosen based on the situation)
- Other times, we go meta and assign a prior distribution to λ itself (called a **hyperprior**, possibly with its own **hyperhyperparameters**)
- Models of this sort are called **hierarchical Bayesian models**

An actual word!

- We could keep going and assign a hyperhyperprior to the hyperhyperparameters, and a hyperhyperhyperprior to the hyperhyperhyperparameters, and... but we've gotta stop somewhere!

Poll Time!

Choosing Priors

- How do we choose an appropriate prior (both for the parameter associated with the data, as well as any hyperparameters)?
- There's no single answer to this question Bayesians blog about this a lot
- One of a Bayesian statistician's key roles is arguing with other statisticians about prior selection Almost every paper that uses Bayesian statistics will justify choices & priors. It's important!
- Some priors are simply not sensible given the parametric family for the data
- Example 6.5: $X_1, \dots, X_n$ iid Bernoulli(p), $\pi(p) = \text{Unif}(-1, 0)$ makes no sense! $\pi(p)$ $N(-10, 20)$ makes no sense either!

$X_1, \dots, X_n$ iid $N(\mu, \sigma^2)$, $\pi(\sigma^2) = \text{Unif}\{5, 8\}$ probably not that sensible...

- We'll discuss several commonly used methods of prior selection, but these certainly aren't the only ones (nor are they mutually exclusive)

Objectivity Versus Subjectivity

- One can very roughly classify Bayesians into two groups: *objective Bayesians* and *subjective Bayesians*
- Subjective Bayesians prefer to integrate personal beliefs about the world – or lack thereof – into their inferences, and they would choose priors that reflect their beliefs (to the extent possible)
- Of course, these would influence the posterior, so two subjective Bayesians might come up with different posteriors (even if they both agree on a model for the data itself); these reflect their differing opinions
- Objective Bayesians prefer to let the data speak for itself, and they would choose priors that do not reflect any personal biases
- To an objective Bayesian, there should be a fixed procedure for choosing a prior, and therefore everyone should agree on the same posterior

Conjugate Priors

- In the previous examples, the posterior distribution was in the same parametric family as the prior (albeit with "updated" parameters)

- This doesn't always happen – most of the time, the posterior will be an unfamiliar distribution – but when it does happen, there's a special name for it

- Definition 6.4: A family of priors $\{\pi_\lambda : \lambda \in \Lambda\}$ for the parameter θ of the model $\mathcal{F} = \{f_\theta : \theta \in \Theta\}$ is called **conjugate for $\mathcal{F}$** if, for all data $\mathbf{x} \in \mathcal{X}^n$ and all $\lambda \in \Lambda$, the posterior $\pi(\cdot \mid \mathbf{x}) \in \{\pi_\lambda : \lambda \in \Lambda\}$

- Example 6.6: Beta(α, β) is conjugate for Bernoulli(p) (and Bin(n,p))

- Example 6.7: Gamma(α, β) is conjugate for Poisson(λ)

Conjugate Priors

- Example 6.8: Suppose that $\pi(\mu) = \mathcal{N}\left(\theta, \tau^2\right)$ and $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mathcal{N}\left(\mu, \sigma^2\right)$ where σ^2 is known. Find the posterior $\pi(\mu \mid \mathbf{x})$.

Knew that $\bar{X}_n$ is sufficient for μ, so

$$\pi(\mu \mid \vec{x}) = \pi(\mu \mid \bar{x})$$

$$\propto \exp\left(-\frac{(\theta-\mu)^2}{2\tau^2}\right) \cdot \exp\left(\frac{-(\bar{x}-\mu)^2}{2\sigma^2/n}\right)$$

$$= \exp\left(\frac{-\theta^2+2\mu\theta-\mu^2}{2\tau^2} + \frac{-\bar{x}^2+2\bar{x}\mu-\mu^2}{2\sigma^2/n}\right)$$

$$\propto \exp\left(\frac{2\theta\mu-\mu^2}{2\tau^2} + \frac{2\bar{x}\mu-\mu^2}{2\sigma^2/n}\right)$$

looks like $\exp\left(\frac{-(\mu-a)^2}{2b^2}\right)$ for some a, b...

$$\frac{2\theta\mu - \mu^2}{2\tau^2} + \frac{2\bar{x}\mu - \mu^2}{2\sigma^2/n}$$

$$= \mu^2\left[-\frac{1}{2\tau^2} - \frac{n}{2\sigma^2}\right] + \mu\left[\frac{\theta}{\tau^2} + \frac{n\bar{x}}{\sigma^2}\right]$$

$$= -\left[\frac{1}{2\tau^2} + \frac{n}{2\sigma^2}\right]\left(\mu^2 - 2\mu\left[\frac{\frac{\theta}{\tau^2} + \frac{n\bar{x}}{\sigma^2}}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}}\right]\right)$$

$$= -\left[\frac{1}{2\tau^2} + \frac{n}{2\sigma^2}\right]\left(\mu^2 - 2\mu\left[\frac{\frac{\theta}{\tau^2} + \frac{n\bar{x}}{\sigma^2}}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}}\right] + \left[\frac{\frac{\theta}{\tau^2} + \frac{n\bar{x}}{\sigma^2}}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}}\right]^2\right) + C, \text{ where } C \text{ is free of } \mu$$

$$= -\left[\frac{1}{2\tau^2} + \frac{n}{2\sigma^2}\right]\left(\mu - \left[\frac{\frac{\theta}{\tau^2} + \frac{n\bar{x}}{\sigma^2}}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}}\right]\right)^2 + C$$

$$= -\frac{\left(\mu - \left[\frac{\frac{\theta}{\tau^2} + \frac{n\bar{x}}{\sigma^2}}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}}\right]\right)^2}{2\left[\frac{1}{\tau^2} + \frac{n}{\sigma^2}\right]^{-1}} + C$$

Think about what happens when τ^2 is really close to 0 here...

$$\Rightarrow \pi(\mu|\bar{x}) \propto \exp\left(\frac{-\left(\mu - \left[\frac{\frac{\theta}{\tau^2} + \frac{n\bar{x}}{\sigma^2}}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}}\right]\right)^2}{2\left[\frac{1}{\tau^2} + \frac{n}{\sigma^2}\right]^{-1}}\right) \Rightarrow \mu|\bar{x} \sim N\left(\frac{\frac{\theta}{\tau^2} + \frac{n\bar{x}}{\sigma^2}}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}}, \frac{1}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}}\right)$$

Conjugate Priors

- In those examples, it was no coincidence that both prior and likelihood were in exponential families

- Theorem 6.2: Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} f_\theta$ where f_θ is in an exponential family:

$$f_\theta(x) = h(x) \cdot g(\theta) \cdot \exp\left(\sum_{j=1}^{k} w_j(\theta) \cdot T_j(x) \right).$$

If we choose an exponential family prior of the form

$$\pi(\theta) \propto g(\theta)^\nu \cdot \exp\left(\sum_{j=1}^{k} w_j(\theta) \cdot \eta_j \right)$$

where ν and $\eta_1, \ldots, \eta_k$ are hyperparameters, then $\pi(\theta)$ is a conjugate prior for f_θ.

Proof: EXERCISE! Identify the "updated" parameters, too!

Why Conjugate Priors?

- Conjugacy is very mathematically convenient
- But is a conjugate family actually *relevant* to whatever the statistical situation is?
- It's widely acknowledged that most conjugate families are rich enough to express a wide spectrum of prior beliefs
- Example 6.9: Normal prior for μ in the $N(\mu, \sigma^2)$ model: if we're okay encoding "symmetric" prior knowledge on μ, this accomodates a lot

Beta prior for p in the Bernoulli(p) model: can handle uniform prior beliefs, any "mode" in $(0,1)$, etc...

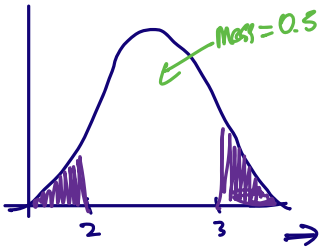
Elicitation

- Even if we do have a particular parametric family $\{\pi_\lambda : \lambda \in \Lambda\}$ selected for our prior, how do we actually set the hyperparameters?
- Ideally, we'll have some experts in the field (possibly ourselves) available to give us their thoughts on what they believe is plausible, based on their own past experiences
- We can't expect them to just tell us raw numbers for λ, but with enough information, we can try and work out the best match
- Translating those thoughts into a choice of hyperprior is called **prior elicitation**

Poll Time!

Elicitation: Examples

- Example 6.10: Suppose we're sampling from an $\mathcal{N}\left(\mu, \sigma^2\right)$ distribution with μ unknown and σ^2 known, and we restrict attention to the family $\{\mathcal{N}\left(\mu_0, \tau^2\right) : \mu_0 \in \mathbb{R}, \tau^2 > 0\}$. If an expert tells us they're 50% certain that μ lies between 2 and 3, how can we elicit our prior?



Gotta choose $\mu_0 = 2.5$. What about τ^2?

Say $\mu \sim N(2.5, \tau^2)$. Then

$$\frac{1}{2} = P(\mu \in (2,3))$$

$$\Rightarrow 1 - \frac{1}{2} = \frac{1}{2} = P(\mu < 2) + P(\mu > 3)$$

$$= P\left(\frac{\mu - 2.5}{\tau} < \frac{2 - 2.5}{\tau}\right) + P\left(\frac{\mu - 2.5}{\tau} > \frac{3 - 2.5}{\tau}\right)$$

$$= P\left(Z < -\frac{0.5}{\tau}\right) + P\left(Z > \frac{0.5}{\tau}\right) \quad \text{where } Z \sim N(0,1)$$

$$= P\left(Z < -\frac{0.5}{\tau}\right) + 1 - P\left(Z < \frac{0.5}{\tau}\right)$$

$$= 2 \cdot P\left(Z < -\frac{0.5}{\tau}\right)$$

$$\Rightarrow \tau = \frac{-0.5}{\Phi^{-1}(0.25)}$$

So we should choose $\pi(\mu) = N\left(2.5, \left(\frac{-0.5}{\Phi^{-1}(0.25)}\right)^2\right)$.

Expressing Ignorance

- What if the experts are keeping quiet and we have nothing to work with?
- Or maybe we're objective Bayesians and "expert advice" is irrelevant to us
- How do we choose a prior that expresses *complete* ignorance about θ?
- In the coin example, choosing $\pi(p) = \text{Unif}(0, 1)$ would work
- What about a completely objective prior on μ in the $\mathcal{N}(\mu, \sigma^2)$ model? There's no uniform distribution on $\mathbb{R}$ $\int_{-\infty}^{\infty} 1 \, dx$ does not exist!
- And yet, if we take $\pi(\mu) = 1$,

$$\pi(\mu|\vec{x}) \propto 1 \cdot \exp\left(\frac{-(\bar{x}-\mu)^2}{2\sigma^2/n}\right) = \exp\left(\frac{-(\mu-\bar{x})^2}{2\sigma^2/n}\right)$$

$$\Rightarrow \mu|\vec{x} \sim N(\bar{x}, \sigma^2/n)$$

That's a completely legitimate posterior!
Clearly letting the data do all of the work here...

Uninformative Priors

- Definition 6.5: A function $\pi(\theta)$ used in place of a true prior distribution that does not relect any prior beliefs about θ is called an **uninformative** (or **noninformative** or **default** or **reference**) **prior**.

$\pi(\mu) = 1$ in $N(\mu, \sigma^2)$ model, $\mu \in \mathbb{R}$

- Example 6.11: $\pi(\theta) = 1$ in Unif$(0, \theta)$ model, $\theta > 0$

$\pi(p) = 1$ in Bernoulli(p) model, $p \in (0,1)$ (An uninformative prior can still have a true prior distn!)

- We have a special name for choices like $\pi(\mu) = 1$ above

- Definition 6.6: If an uninformative prior $\pi(\theta)$ is not a true distribution (i.e., $\int_\Theta \pi(\theta)\, \mathrm{d}\theta$ is divergent), then it is called an **improper prior**.

- Improper priors are controversial, and they're difficult to interpret probabilistically

$\pi(\theta)$ is improper iff $c \cdot \pi(\theta)$ is improper for any $c > 0$

- Moreover, if chosen haphazardly they can lead to improper posteriors (which are truly meaningless)

Problems With Uninformative Priors

- Example 6.12: Suppose that $X \sim \text{Bernoulli}(p)$. What is the posterior $\pi(p \mid x)$ based on the **Haldane prior** $\pi(p) = \frac{1}{p(1-p)}$?

This is improper! $\int_0^1 \frac{1}{p(1-p)}\,dp = \infty$.

$$\pi(p|x) \propto \frac{1}{p(1-p)} \cdot p^x (1-p)^{1-x}$$

$$= p^{x-1}(1-p)^{-x}$$

Is this a pdf? $\int_0^1 \pi(p|x)\,dp = \int_0^1 p^{x-1}(1-p)^{-x}\,dp$

$= \pi \cdot \csc(\pi x)$ Calculus exercise!

$= \pm\infty$ when $x \in \mathbb{Z}$ ... which is the case here!

Not a pdf! This is an "improper posterior" — useless!

This can never happen if we choose a proper prior...

Problems With Uninformative Priors

- Example 6.13: Suppose that $X \sim \text{Bernoulli}(p)$ and we choose $\pi(p) = \text{Unif}(0,1)$. What prior does this correspond to for the log-odds $\tau = \log\left(\frac{p}{1-p}\right)$?

Original function: $\tau(p) = \log\left(\frac{p}{1-p}\right) \in \mathbb{R}$ "logit function"

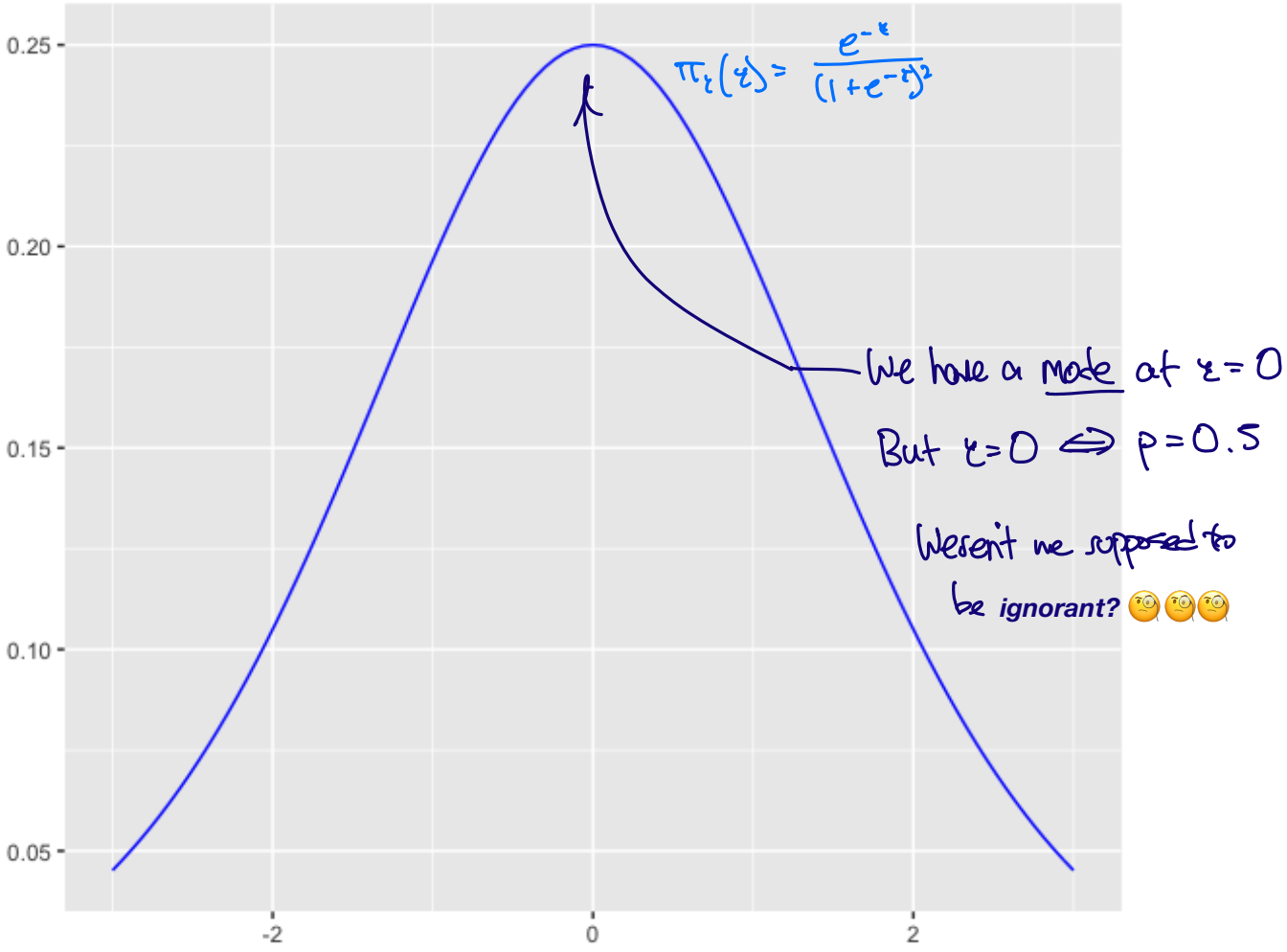
Inverse function: $p(\tau) = \frac{1}{1+e^{-\tau}} \in (0,1)$ "expit function"

$$\pi_{\tau(p)}(\tau) = \pi_p(p(\tau)) \cdot \left|\frac{d}{d\tau} p(\tau)\right|$$

$$= 1 \cdot \left|\frac{e^{-\tau}}{(1+e^{-\tau})^2}\right|$$

$$= \frac{e^{-\tau}}{(1+e^{-\tau})^2}$$

Oh No


0.25
0.20
0.15
0.10
0.05
-2
0
2
$\pi_z(z) = \frac{e^{-z}}{(1+e^{-z})^2}$
We have a mode at $z=0$
But $z=0 \Leftrightarrow p=0.5$
Weren't we supposed to be ignorant? 🧐🧐🧐

Ignorance From All Perspectives

- The previous example shows that ignorance about θ does not necessarily translate to the same ignorance about $\tau(\theta)$
- In other words, if π_θ is a prior for the model parameterized by θ and π_τ is a prior for the model parameterized by $\tau = \tau(\theta)$,

$$\pi_\tau(t) \neq \pi_\theta(\tau^{-1}(t)) \cdot \left| \frac{\mathrm{d}}{\mathrm{d}t} \tau^{-1}(t) \right|$$

  in general
- What if we insisted on "equivalent" ignorance for all monotone re-parametrizations of θ?
- It turns out there's a way to make this happen using the Fisher information

Jeffreys' Prior

- Definition 6.7: Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} f_\theta$ where θ is univariate. **Jeffreys' prior** for θ is given by $\pi_\theta^J(\theta) \propto \sqrt{I_1(\theta)}$.

- Notice that this prior *depends only the model* – there's no room for any subjectivity beyond the choice of model

- Jeffreys felt that invariance under monotone transformations is a suitably uninformative property for a prior

- Theorem 6.3: Under the regularity conditions of the Cramér-Rao Lower Bound, Jeffreys' prior is invariant under monotone transformations, in the sense that

$$\pi_\tau^J(t) = \pi_\theta^J(\tau^{-1}(t)) \left| \frac{\mathrm{d}}{\mathrm{d}t} \tau^{-1}(t) \right|$$

  if $\tau : \Theta \to \mathbb{R}$ is monotone and differentiable.

Proof. Let $f_\theta(x)$ be the original pdf, and let $g_\xi(x)$ be the pdf under the $\xi(\theta)$ parameterization.

Let $I_\theta(\theta)$ and $I_\xi(\xi)$ be the Fisher information under the two parameters.

Then $I_\theta(\theta) = \mathbb{E}\left[\left(\frac{d}{d\theta}\log(f_\theta(X))\right)^2\right]$ by definition

$= \mathbb{E}\left[\left(\frac{d}{d\theta}\log(g_{\xi(\theta)}(X))\right)^2\right]$ because $f_\theta(x) = g_\xi(x)$; reparameterization doesn't change the likelihood

$= \mathbb{E}\left[\left(\frac{d\xi}{d\theta}\cdot\frac{d}{d\xi}\log(g_\xi(X))\right)^2\right]$ by the chain rule

$= \left(\frac{d\xi}{d\theta}\right)^2 \cdot \mathbb{E}\left[\left(\frac{d}{d\xi}\log(g_\xi(X))\right)^2\right]$

$= \left(\frac{d\xi}{d\theta}\right)^2 \cdot I_\xi(\xi)$ $\quad \xi(\theta) = t$, $\theta = \xi^{-1}(t)$

Thus $\pi_\xi^J(\xi(\theta)) \propto \sqrt{I_\xi(\xi)}$ by defn of Jeffreys' prior

$= \sqrt{I_\theta(\theta)} \cdot \left|\frac{d\xi}{d\theta}\right|^{-1}$

$= \sqrt{I_\theta(\theta)} \cdot \left|\frac{d\theta}{d\xi}\right|$

The result follows by letting $t = \xi(\theta)$. □

Jeffreys' Prior: Examples

- Example 6.14: Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Bernoulli}(p)$. Determine Jeffreys' prior for this model, and determine the posterior $\pi(p \mid \mathbf{x})$ based on it.

Know $I_1(p) = \frac{1}{p(1-p)}$ from old stuff. $\pi^J(p) \propto \sqrt{\frac{1}{p(1-p)}} = p^{-\frac{1}{2}}(1-p)^{-\frac{1}{2}}$...

actually a $\text{Beta}(\frac{1}{2}, \frac{1}{2})$ distribution!

Posterior: $\pi(p(\vec{x}) \propto \pi^J(p) \cdot f_p(\vec{x})$

$$\propto \sqrt{\frac{1}{p(1-p)}}\, p^{\sum x_i}(1-p)^{n-\sum x_i}$$

$$= p^{-\frac{1}{2}+\sum x_i}(1-p)^{n-\sum x_i-\frac{1}{2}}$$

$$\Rightarrow p|\vec{x} \sim \text{Beta}\left(\textstyle\sum x_i + \frac{1}{2},\ n - \sum x_i + \frac{1}{2}\right).$$

$p \in (0,1)$
$\Rightarrow \sqrt{p} \in (0,1)$
$\Rightarrow \arcsin(\sqrt{p}) \in (0, \pi/2)$
$\Rightarrow \tau \in (0, \pi/2)$

What if $\tau(p) = \arcsin(\sqrt{p})$ $\Rightarrow$ $p(\tau) = \sin^2(\tau)$

$$\Rightarrow \pi^J_\tau(\tau) \propto \pi^J_p(p(\tau)) \cdot \left|\frac{d}{d\tau} p(\tau)\right|$$

$$= \sqrt{\frac{1}{\sin^2(\tau)(1-\sin^2(\tau))}} \cdot |2\sin(\tau)\cdot\cos(\tau)|$$

$$= 2 \quad \Rightarrow \pi^J_\tau(\tau) \propto 2 \Rightarrow \pi^J_\tau(\tau) = \text{Unif}(0, \pi/2)$$

Jeffreys' Prior: Examples

- Example 6.15: Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$ with σ^2 known. Determine Jeffreys' prior for this model, and determine the posterior $\pi(\mu \mid \mathbf{x})$ based on it.

From many examples past, $I_1(\mu) = 1/\sigma^2 \propto 1$. So $\pi^J(\mu) \propto 1$.

Improper!

The posterior is $\pi(\mu|\vec{x}) \propto \pi^J(\mu) \cdot f_\mu(\vec{x})$

$$\propto 1 \cdot \exp\left(-\frac{(\bar{x}-\mu)^2}{2\sigma^2/n}\right)$$

$$\Rightarrow \mu|\vec{x} \sim N(\bar{x}, \sigma^2/n)$$

Inferences Based On the Posterior

- If we're satisfied with a choice of prior and we've computed (or estimated) the posterior, what do we actually do with this distribution?
- The inferential techniques of Modules 2-4 (point estimation, hypothesis testing, and confidence intervals) can't be directly applied here, since $\theta \mid \mathbf{x}$ is not a fixed constant
- Our goal is to find Bayesian analogues of these techniques

There are LOTS of Bayesian analogues of frequentist concepts!
Nothing is fully agreed upon by all Bayesians.—

Bayesian Point Estimation

- If $\mathbf{X} \sim f_\theta$, how do we "estimate" either θ itself or some quantity $\tau = \tau(\theta)$ in the Bayesian context?
- We have a posterior distribution $\pi(\theta \mid \mathbf{x})$ to work with
- What quantities can we extract from it that can meaningfully take the place of our frequentist estimates?
- If we use some characteristic $\hat{\theta}$ of $\pi(\theta \mid \mathbf{x})$, then it must be a function of the data $\mathbf{x}$ and we can write $\hat{\theta} = \hat{\theta}(\mathbf{x})$

eg: the mean, the median, the mode, a quantile, etc...

- That makes $\hat{\theta}(\mathbf{X})$ a genuine point estimator, which we can compare to our favourite frequentist estimators like the MLE
- To keep the notation simple, we'll work with θ itself, but everything carries over to $\tau(\theta)$

MAP Estimators

- One reasonable approach is to choose the value that the posterior says is most probable – that is, the mode of the posterior

- Definition 6.8: Given a posterior distribution $\pi(\theta \mid \mathbf{x})$, a **maximum *a posteriori* (MAP) estimator** of θ is given by the conditional mode of the posterior:

$$\hat{\theta}_{\text{MAP}}(\mathbf{X}) = \underset{\theta \in \Theta}{\text{argmax}}\, \pi(\theta \mid \mathbf{X}).$$

If the posterior is unimodal!

- If we want the MAP estimator of $\tau = \tau(\theta)$, we'll need to maximize $\pi(\tau \mid \mathbf{x})$

- But that's the same as maximizing $f(\mathbf{x}) \cdot \pi(\tau \mid \mathbf{x}) = \pi(\tau) \cdot f_\tau(\mathbf{x})$, so we don't need to bother with the normalizing constant $f(\mathbf{x})$, which is usually a nasty integral

Posterior Means

- We might prefer to take a weighted average of all $\theta' \in \Theta$, each weighed down by how probable the posterior says it is – that is, the expectation of the posterior

- Definition 6.9: Given a posterior distribution $\pi(\theta \mid \mathbf{x})$, the **posterior mean estimator** – if it exists – is given by the conditional expectation of the posterior:

$$\hat{\theta}_{\mathsf{B}}(\mathbf{X}) = \mathbb{E}\left[\theta \mid \mathbf{X}\right] = \int_{\Theta} \theta \cdot \pi(\theta \mid \mathbf{x}) \, \mathrm{d}\theta.$$

- The posterior mean estimator is nice because it minimizes the *expected MSE* under the posterior:

$$\hat{\theta}_{\mathsf{B}}(\cdot) = \underset{T(\cdot)}{\operatorname{argmin}} \, \mathbb{E}\left[\mathsf{MSE}_{\theta}\left(T(\mathbf{X})\right)\right]$$

taken with respect to $\pi(\theta|\mathbf{x})$

Bayesian Point Estimation: Examples

- Example 6.16: Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Bernoulli}(p)$, and suppose we place a $\text{Beta}(\alpha, \beta)$ prior on p. Find the MAP estimator and the posterior mean estimator for p, and describe how they compare to the MLE.

From Example 6.2, $\pi(p|\vec{x}) = \text{Beta}(\alpha + \sum x_i, \beta + n - \sum x_i)$

MAP: gotta maximize a Beta pdf $f_{\alpha,\beta}(\theta)$ in θ. That's the same as maximizing $\log(f_{\alpha,\beta}(\theta))$:

For a general Beta(α,β) pdf:

$$\frac{d}{d\theta}\log(f_{\alpha,\beta}(\theta)) = \frac{d}{d\theta}\left((\alpha-1)\cdot\log(\theta) + (\beta-1)\cdot\log(1-\theta)\right) = \frac{\alpha-1}{\theta} - \frac{\beta-1}{1-\theta} \overset{\text{set}}{=} 0$$

$$\Rightarrow \theta = \frac{\alpha-1}{\alpha+\beta-2} \quad \text{provided that } \alpha,\beta > 1 \quad \text{Check!}$$

So $$\hat{p}_{MAP}(\vec{X}) = \frac{\sum X_i + \alpha - 1}{\sum X_i + \alpha + \beta + n - \sum X_i - 2} = \frac{\sum X_i + \alpha - 1}{\alpha + \beta + n - 2}$$

Posterior mean: the mean of a Beta(α,β) is $\frac{\alpha}{\alpha+\beta}$

So $$\hat{p}_B(\vec{X}) = \frac{\sum X_i + \alpha}{\sum X_i + \alpha + \beta - \sum X_i + n} = \frac{\sum X_i + \alpha}{\alpha + \beta + n}$$

MLE: $\hat{p}_{MLE}(\vec{X}) = \bar{X}_n = \frac{\sum X_i}{n}$

All three are pretty similar... the posterior mean and MAP estimators reflect prior information (i.e., choices of α and β) in different ways. But when n is large, the differences become negligible.

EXERCISE: what (if anything) do they converge to as $n \to \infty$?

Bayesian Point Estimation: Examples

- Example 6.17: Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mathcal{N}\left(\mu, \sigma^2\right)$ with σ^2 known, and suppose we place a $\mathcal{N}\left(\theta, \tau^2\right)$ prior on μ. Find the MAP estimator and the posterior mean estimator for μ, and describe how they compare to the MLE.

Example 6.8: $\mu|\vec{X} \sim N\left(\frac{\frac{\theta}{\tau^2}+\frac{n\bar{x}}{\sigma^2}}{\frac{1}{\tau^2}+\frac{n}{\sigma^2}}, \frac{1}{\frac{1}{\tau^2}+\frac{n}{\sigma^2}}\right)$

$$\hat{\mu}_B(\vec{X}) = \frac{\frac{\theta}{\tau^2}+\frac{n\bar{X}_n}{\sigma^2}}{\frac{1}{\tau^2}+\frac{n}{\sigma^2}}$$

$$\hat{\mu}_{MAP}(\vec{X}) = \frac{\frac{\theta}{\tau^2}+\frac{n\bar{X}_n}{\sigma^2}}{\frac{1}{\tau^2}+\frac{n}{\sigma^2}}$$

) same thing! $= \frac{\frac{\theta}{n\tau^2}+\frac{\bar{X}_n}{\sigma^2}}{\frac{1}{n\tau^2}+\frac{1}{\sigma^2}}$

$\hat{\mu}_{MLE}(\vec{X}) = \bar{X}_n$

As n gets large, $\hat{\mu}_{MAP}(\vec{X}) = \hat{\mu}_B(\vec{X}) \approx \hat{\mu}_{MLE}(\vec{X})$

Poll Time!

$$\mathbb{E}[\text{Unif}(x, x+2)] = \frac{x+x+2}{2} = x+1$$

$$\Rightarrow \quad \hat{\theta}_B(X) = X+1$$

Bayesian Hypothesis Testing

- What about Bayesian hypothesis testing?
- We might think to test every hypothesis by simply computing probability under $\pi(\theta \mid \mathbf{x})$, we'd quickly run into problems
- For example, if the posterior is continuous, then we'd reject every simple hypothesis $H : \theta = \theta_0$
- We might try to get around this by computing a **Bayesian p-value** $\Pi(\{\theta : \pi(\theta \mid \mathbf{x}) \leq \pi(\theta_0 \mid \mathbf{x})\} \mid \mathbf{x})$, but there can be problems with that as well

$\Pi(\cdot|\vec{x})$ ↑ posterior probability measure

↑ Interpretation: reject $H_0: \theta = \theta_0$ if θ_0 is in a region of low posterior probability (ie, where $\pi(\cdot|\vec{x})$ is small)

Bayesian p-Values Aren't Great

- Example 6.18: Suppose $\pi(\theta \mid \mathbf{x}) = \text{Beta}(2, 1)$. Compute Bayesian p-values for $H_0 : \theta = \frac{3}{4}$ under the posterior of $\theta \mid \mathbf{x}$ and the posterior of $\theta^2 \mid \mathbf{x}$.

$\pi(\theta|\vec{x}) = 2\theta$ for $\theta \in (0,1)$. Now $\pi(\theta|\vec{x}) \leq \pi(3/4|\vec{x})$

$\Leftrightarrow \quad 2\theta \leq 2 \cdot 3/4$

$\Leftrightarrow \quad \theta \leq 3/4$

So the Bayesian p-value is $\Pi(\{\theta : \theta \leq 3/4\}|\vec{x}) = \Pi((0, 3/4)|\vec{x}) = \int_0^{3/4} 2\theta \, d\theta = 9/16.$

What about under $\pi(\theta^2|\vec{x})$? Then we're testing $H_0 : \theta^2 = (3/4)^2 = 9/16$.

Check that $\theta^2|\vec{x} \sim \text{Beta}(1,1) = \text{Unif}(0,1)$ so $\pi(\theta^2|\vec{x}) = 1 \; \forall \theta^2 \in (0,1)$.

Check!

But $\pi(\theta^2|\vec{x}) \leq \pi(9/16|\vec{x})$

$\Leftrightarrow \quad 1 \leq 1$. Always true! So $\Pi(\{\theta : 1 \leq 1\}|\vec{x}) = 1$. So the Bayesian p-value here is always 1 (regardless of $\vec{x}$). So there can never be any evidence against H_0! Not so great....

Tweaking the Prior

- These issues happen when the prior $\pi(\theta)$ assigns zero probability to H_0, and can be avoided by tweaking the prior in such a way to fix this

- This isn't unreasonable; if we have reason to test $H : \theta \in A$, then we suspect it *could* be true, which would be contradicted if $\Pi(\theta \in A) = 0$

- If we start with a continuous prior π_2, we can create a new one using

$$\pi(\theta) = \alpha \cdot \pi_1(\theta) + (1 - \alpha) \cdot \pi_2(\theta),$$

where π_1 is degenerate at θ_0 and $\alpha \in (0, 1)$

General form of a "mixture distribution" pdf/pmf is $f(x) = \sum_{j=1}^{m} \alpha_j \cdot f_j(x)$, where $\alpha_j > 0$ and $\sum_{j=1}^{m} \alpha_j = 1$ and each $f_j(\cdot)$ is a pdf/pmf EXERCISE! Show that $f(x) = \sum_{j=1}^{m} \alpha_j \cdot f_j(x)$ is a valid pdf/pmf!

- This gives

$$\Pi(\{\theta_0\} \mid \mathbf{x}) = \frac{\alpha f_1(\mathbf{x})}{\alpha f_1(\mathbf{x}) + (1 - \alpha) f_2(\mathbf{x})},$$

where $f_i(\mathbf{x})$ is the prior predictive distribution under the prior π_i

Bayes Factors

Odds of an event A : $\frac{P(A)}{1-P(A)}$

- There's a popular approach to Bayesian hypothesis testing involves the odds

- Definition 6.10: Let $\pi(\theta)$ be a prior, let $\mathbf{X} \sim f_\theta(\mathbf{x})$, and let $\pi(\theta \mid \mathbf{x})$ be the posterior for the model. Suppose that $H_0 : \theta \in \Theta_0$ and $H_A : \theta \in \Theta_0^c$ are two competing hypotheses about plausible values of θ.

  $\Pi(\cdot)$ prior measure
  $\Pi(\cdot \mid \vec{x})$ posterior measure

  The **prior odds** in favour of H_0 is the ratio $\dfrac{\Pi(\Theta_0)}{\Pi(\Theta_0^c)} = \dfrac{\Pi(\Theta_0)}{1 - \Pi(\Theta_0)}$.

  The **posterior odds** in favour of H_0 is the ratio $\dfrac{\Pi(\Theta_0 \mid \mathbf{x})}{\Pi(\Theta_0^c \mid \mathbf{x})} = \dfrac{\Pi(\Theta_0 \mid \mathbf{x})}{1 - \Pi(\Theta_0 \mid \mathbf{x})}$.

  Provided that $\Pi(\Theta_0) > 0$, the **Bayes factor** in favour of H_0 is given by the ratio of the posterior odds to the prior odds:

$$BF_{H_0} = \frac{\Pi(\Theta_0 \mid \mathbf{x})}{1 - \Pi(\Theta_0 \mid \mathbf{x})} \Bigg/ \frac{\Pi(\Theta_0)}{1 - \Pi(\Theta_0)}.$$

Bayes Factors

- What's the point of Bayes factors?
- For one, if we let r be the prior odds, then

$$\Pi(\Theta_0 \mid \mathbf{x}) = \frac{r \cdot BF_{H_0}}{1 + r \cdot BF_{H_0}}$$

Show!

- So a small/large Bayes factor means a small/large posterior probability of H_0
- Moreover, Bayes factors have a surprising connection to likelihood ratios
- Theorem 6.4: If we want to test $H_0 : \theta \in \Theta_0$ and we choose a prior mixture $\pi(\theta) = \alpha \cdot \pi_1(\theta) + (1 - \alpha) \cdot \pi_2(\theta)$ such that $\Pi_1(\Theta_0) = \Pi_2(\Theta_0^c) = 1$, then

$$BF_{H_0} = \frac{f_1(\mathbf{x})}{f_2(\mathbf{x})}.$$

Free $\propto$ (:)

where $f_i(\vec{x})$ is the prior predictive distribution under the prior π_i

Bayes Factors: Examples

- Example 6.19: Suppose that $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Bernoulli}(\theta)$ and we place a $\text{Unif}(0,1)$ prior on θ. Compute the Bayes factor in favour of $H_0 : \theta = \theta_0$.

Let π_1 be degenerate at θ_0 so $\Pi_1(\{\theta_0\}) = 1$

Let $\pi_2 = \text{Unif}(0,1)$ so $\Pi_2(\{\theta_0\}) = 0$

Then by Theorem 6.4, $BF_{10} = \dfrac{f_1(\vec{x})}{f_2(\vec{x})}$ $\quad \delta_{\theta_0}(x) := \begin{cases} 1, & x = \theta_0 \\ 0, & \text{otherwise} \end{cases}$

Prior predictive under π_1: $\int \delta_{\theta_0}(\theta) \cdot \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}\, d\theta = \theta_0^{\sum x_i}(1-\theta_0)^{n-\sum x_i}$

Prior predictive under π_2: $\int_0^1 1 \cdot \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}\, d\theta = \dfrac{\Gamma(\sum x_i + 1)\cdot\Gamma(n-\sum x_i+1)}{\Gamma(n+2)}$

So $BF_{10} = \dfrac{\theta_0^{\sum x_i}(1-\theta_0)^{n-\sum x_i}}{\Gamma(\sum x_i+1)\cdot\Gamma(n-\sum x_i+1)/\Gamma(n+2)}$

Credible Intervals

- Assuming that $\Theta \subseteq \mathbb{R}$, what's a reasonable Bayesian analogue of confidence intervals?

- Now, it's perfectly reasonable to ask what the probability is that $l \leq \theta \leq u$ for $l, u \in \Theta$

- Definition 6.11: Let $\pi(\theta \mid \mathbf{x})$ be a posterior distribution on θ. A **$(1-\alpha)$-credible interval** for θ is an interval $[L(\mathbf{x}), U(\mathbf{x})] \subseteq \Theta$ such that

$$\Pi(L(\mathbf{x}) \leq \theta \leq U(\mathbf{x}) \mid \mathbf{x}) = \int_{L(\mathbf{x})}^{U(\mathbf{x})} \pi(\theta \mid \mathbf{x}) \, \mathrm{d}\theta \geq 1 - \alpha.$$

- As with confidence intervals, there are usually plenty of credible intervals available for a given posterior, so we look for some desirable properties

Two Types of Credible Intervals

i.e., $\pi(\theta|\mathbf{x})$ has one mode

- Definition 6.12: If $\pi(\theta \mid \mathbf{x})$ is unimodal, the $(1-\alpha)$-credible interval $[L(\mathbf{x}), U(\mathbf{x})]$ such that the length $U(\mathbf{x}) - L(\mathbf{x})$ is minimized is called the **$(1-\alpha)$-highest posterior density (HPD) interval** for θ

- An HPD interval really does capture the most likely values in Θ, since any region outside of it will be assigned a lower posterior probability

- Definition 6.13: The $(1-\alpha)$-credible interval $[L(\mathbf{x}), U(\mathbf{x})]$ which satisfies

$$\Pi((-\infty, L(\mathbf{x})] \mid \mathbf{x}) = \Pi([U(\mathbf{x}), \infty) \mid \mathbf{x}) = \alpha/2$$

  is called the **$(1-\alpha)$-equal tailed interval (ETI)** for θ

- An ETI exists for any continuous posterior, unimodal or otherwise

- One can show that if $\pi(\theta \mid \mathbf{x})$ is symmetric, unimodal, and continuous, then the HPD interval and the ETI will be equal

Credible Intervals: Examples

- Example 6.20: Suppose that $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mathcal{N}\left(\mu, \sigma^2\right)$ where σ^2 is known, and we place a $\mathcal{N}\left(\theta, \tau^2\right)$ prior on μ. What do 95% HPD intervals and ETIs for μ look like? What happens as $\tau^2 \to \infty$? $(1-\alpha)$-HPD

Posterior $\pi(\mu|\vec{x})$ is Normal: continuous, unimodal, and symmetric, so the ETI and HPD intervals will be the same!

From Example 6.8, $\mu|\vec{x} \sim N\left(\dfrac{\frac{\theta}{\tau^2}+\frac{n\bar{x}}{\sigma^2}}{\frac{1}{\tau^2}+\frac{n}{\sigma^2}}, \dfrac{1}{\frac{1}{\tau^2}+\frac{n}{\sigma^2}}\right)$.

We need $1-\alpha = \Pr\left(-z_{1-\alpha/2} < \left(\mu - \dfrac{\frac{\theta}{\tau^2}+\frac{n\bar{x}}{\sigma^2}}{\frac{1}{\tau^2}+\frac{n}{\sigma^2}}\right) \Big/ \left(\frac{1}{\tau^2}+\frac{n}{\sigma^2}\right)^{-1/2} < z_{1-\alpha/2} \,\middle|\, \vec{x}\right)$

$= \Pr\left(\dfrac{\frac{\theta}{\tau^2}+\frac{n\bar{x}}{\sigma^2}}{\frac{1}{\tau^2}+\frac{n}{\sigma^2}} - z_{1-\alpha/2}\cdot\left(\frac{1}{\tau^2}+\frac{n}{\sigma^2}\right)^{-\frac{1}{2}} < \mu < \dfrac{\frac{\theta}{\tau^2}+\frac{n\bar{x}}{\sigma^2}}{\frac{1}{\tau^2}+\frac{n}{\sigma^2}} + z_{1-\alpha/2}\cdot\left(\frac{1}{\tau^2}+\frac{n}{\sigma^2}\right)^{-\frac{1}{2}} \,\middle|\, \vec{x}\right)$

So our $(1-\alpha)$-credible intervals are both $\left(\dfrac{\frac{\theta}{\tau^2}+\frac{n\bar{x}}{\sigma^2}}{\frac{1}{\tau^2}+\frac{n}{\sigma^2}} - z_{1-\alpha/2}\cdot\left(\frac{1}{\tau^2}+\frac{n}{\sigma^2}\right)^{-\frac{1}{2}}, \dfrac{\frac{\theta}{\tau^2}+\frac{n\bar{x}}{\sigma^2}}{\frac{1}{\tau^2}+\frac{n}{\sigma^2}} + z_{1-\alpha/2}\cdot\left(\frac{1}{\tau^2}+\frac{n}{\sigma^2}\right)^{-\frac{1}{2}}\right)$.

What happens as $\tau^2 \to \infty$ (ie, as the prior $\pi(\mu)$ becomes improper)?

The $(1-\alpha)$-credible interval becomes $\left(\bar{x} - \sqrt{\frac{\sigma^2}{n}}\, z_{1-\alpha/2},\ \bar{x} + \sqrt{\frac{\sigma^2}{n}}\cdot z_{1-\alpha/2}\right)$ ... that's our z-interval!

Credible Intervals: Examples

- Example 6.21: Suppose that $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\lambda)$ and we place a Gamma (α, β) prior on λ. What do 95% HPD intervals and ETIs for λ look like?

From Example 6., $\lambda|\vec{x} \sim \text{Gamma}(\alpha + \sum x_i, \beta + n)$. Let $G(\cdot)$ be the cdf of that thing.

ETI: need $\underbrace{\alpha/2 = \Pi((-\infty, L(\vec{x})]|\vec{x})}_{\Rightarrow \alpha/2 = G(L(\vec{x}))} = \underbrace{\Pi([U(\vec{x}), \infty)|\vec{x})}_{G(U(\vec{x})) = 1-\alpha/2} = \alpha/2$

$\Rightarrow L(\vec{x}) = G^{-1}(\alpha/2)$

$\Rightarrow U(\vec{x}) = G^{-1}(1-\alpha/2)$

HPD: impossible to do by hand! Gotta use a statistical software package... many available.

ETIs are Invariant

- We've seen that posterior distributions can do unexpected things when we're interested in inferences of $\tau(\theta)$
- In general, a credible interval for θ may tell us nothing about a credible interval (or credible region) for $\tau(\theta)$
- But ETIs have a special property that bypasses this issue
- Theorem 6.5: ETIs are invariant under monotone transformations of θ, in the sense that if $(L(\mathbf{x}), U(\mathbf{x}))$ is a $(1-\alpha)$-ETI for θ and $\tau : \Theta \to \mathbb{R}$ is monotone increasing, then $(\tau(L(\mathbf{x})), \tau(U(\mathbf{x})))$ is a $(1-\alpha)$-ETI for $\tau(\theta)$.

Proof. If $\Pi(\{\theta : \theta \le L(\mathbf{x})\} \mid \vec{x}) = \Pi(\{\theta : \theta \ge U(\mathbf{x})\} \mid \vec{x}) = \alpha/2$, then

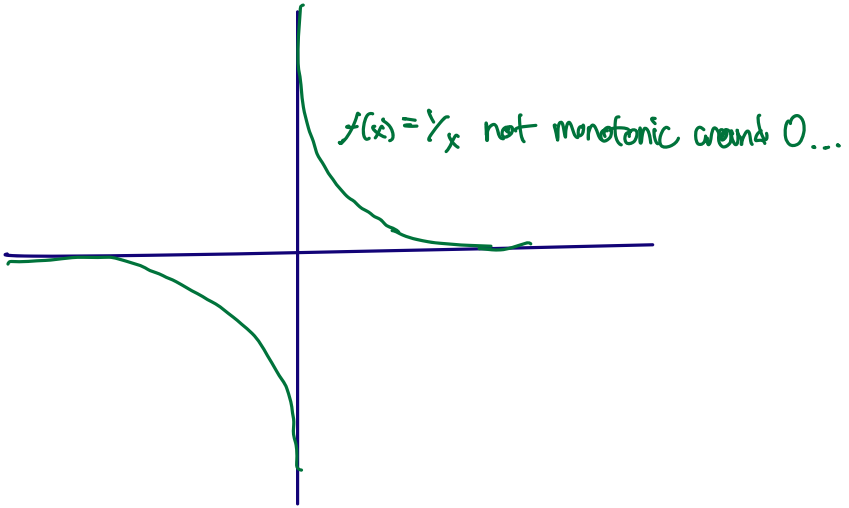
$\Pi(\{\theta : \tau(\theta) \le \tau(L(\mathbf{x}))\} \mid \vec{x}) = \Pi(\{\theta : \tau(\theta) \ge \tau(U(\mathbf{x}))\} \mid \vec{x}) = \alpha/2 \Rightarrow (\tau(L(\mathbf{x})), \tau(U(\mathbf{x})))$ is a $(1-\alpha)$-ETI for $\tau(\theta)$. □

If τ is monotone decreasing, everything flips!

- Example 6.22:

For the $N(\mu, \sigma^2)$ model, a $(1-\alpha)$-ETI for μ^3 is given by $\left(\left[\frac{\frac{\theta}{\tau^2} + \frac{n\bar{x}}{\sigma^2}}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}} - z_{1-\alpha/2} \cdot \left(\frac{1}{\tau^2} + \frac{n}{\sigma^2} \right)^{-1/2} \right]^3, \left[\frac{\frac{\theta}{\tau^2} + \frac{n\bar{x}}{\sigma^2}}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}} + z_{1-\alpha/2} \cdot \left(\frac{1}{\tau^2} + \frac{n}{\sigma^2} \right)^{-1/2} \right]^3 \right)$

Poll Time!



The Bernstein-von Mises Theorem

- Bayesian and frequentist inferences unite in this monumental result

- Theorem 6.6 (**Bernstein-von Mises**): Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} f_{\theta_0}$, let $\pi(\theta)$ be a prior distribution on θ, and let $\theta_n \sim \pi(\theta \mid \mathbf{x}_n)$. Under suitable regularity conditions,

$$\sqrt{n}\left(\theta_n - \hat{\theta}_{\text{MLE}}(\mathbf{x}_n)\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{I_1(\theta_0)}\right).$$

- This statement is a *vast* simplification of the actual Bernstein-von Mises theorem, but it preserves the essence

- The takeaway is that as the sample size of our data n gets larger, the choice of $\pi(\theta)$ matters less and the likelihood dominates

- Roughly speaking, the posterior $\pi(\theta \mid \mathbf{x}_n)$ converges to a degenerate distribution on θ_0, for *any* well-behaved prior (!)