① a) $X \sim N(0, \sigma^2)$.

$$\mathbb{E}[X^{2k+1}] = \int_{-\infty}^{\infty} x^{2k+1} \cdot \exp(-x^2/2\sigma^2)\, dx$$

$$= \int_{-\infty}^{0} x^{2k+1} \cdot \exp(-x^2/2\sigma^2)\, dx + \int_{0}^{\infty} x^{2k+1} \cdot \exp(-x^2/2\sigma^2)\, dx$$

let $y = -x$
$-dy = dx$

$$= -\int_{0}^{\infty} y^{2k+1} \cdot \exp(-y^2/2\sigma^2)\, dy + \int_{0}^{\infty} x^{2k+1} \cdot \exp(-x^2/2\sigma^2)\, dx$$

$$= 0.$$

b) Let X be cts with pdf f_X satisfying $f_X(x) = f_X(-x)$ $\forall x \in \mathbb{R}$.

Then $\mathbb{E}[X^{2k+1}]$

$$= \int_{-\infty}^{\infty} x^{2k+1} \cdot f_X(x)\, dx$$

$$= \int_{-\infty}^{0} x^{2k+1} \cdot f_X(x)\, dx + \int_{0}^{\infty} x^{2k+1} \cdot f_X(x)\, dx$$

let $u = -x$
$-du = dx$

$$= \int_{\infty}^{0} u^{2k+1} \cdot f_X(-u)\, du + \int_{0}^{\infty} x^{2k+1} \cdot f_X(x)\, dx$$

$$= -\int_{0}^{\infty} u^{2k+1} \cdot f_X(u)\, du + \int_{0}^{\infty} x^{2k+1} \cdot f_X(x)\, dx$$

since $f_X(-u) = f_X(u)$ by assumption.

$$= 0. \qquad \square$$

2 $\mathbb{E}[\lambda \cdot h(X)]$

$$= \lambda \cdot \sum_{j=0}^{\infty} \frac{h(j) \cdot \lambda^{j} e^{-\lambda}}{j!}$$

$$= \sum_{j=0}^{\infty} \frac{h(j) \cdot \lambda^{j+1} e^{-\lambda}}{j!}$$

$$= \sum_{j=0}^{\infty} \frac{h(j) \cdot \lambda^{j+1} \cdot e^{-\lambda} \cdot (j+1)}{(j+1)!}$$

let $k = j+1$

$$= \sum_{k=1}^{\infty} \frac{h(k-1) \cdot \lambda^{k} e^{-\lambda} \cdot k}{k!}$$

$$= \sum_{k=0}^{\infty} k \cdot h(k-1) \cdot \frac{\lambda^{k} e^{-\lambda}}{k!}$$

$$= \mathbb{E}[X \cdot h(X-1)] . \qquad \square$$

3

$\mathbb{E}[g(X)\cdot(X-\mu)]$

$= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} g(x)\cdot(x-\mu)\cdot \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$

by parts:

let $u = g(x) \Rightarrow du = g'(x)\,dx$

let $dv = \frac{1}{\sqrt{2\pi}\sigma}(x-\mu)\cdot e^{-(x-\mu)^2/2\sigma^2} \Rightarrow v = \frac{-\sigma}{\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$

$= \underbrace{\frac{-\sigma}{\sqrt{2\pi}} g(x)\cdot e^{-(x-\mu)^2/2\sigma^2}\Big|_{-\infty}^{\infty}}_{=\,0-0 \text{ by assumption}} + \int_{-\infty}^{\infty}\frac{\sigma}{\sqrt{2\pi}} g'(x)\cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$

$= \sigma^2\left(\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} g'(x)\cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx\right)$

$= \sigma^2\cdot\mathbb{E}[g'(X)]$. $\square$

4) a) First, $\mathbb{P}(X_{(j)} = X_{(k)}) = 0$ for any $j \neq k$, since these random variables are continuous. Then for h small enough, each $[x, x+h]$ can only contain at most one order statistic. If $X_{(j)} \in [x, x+h]$, then one X_i must be in $[x, x+h]$ (i.e., the j'th largest one), and the lower X_i's — of which there are j — must be $< x$.
Conversely, if there are exactly $j-1$ X_i's which are $< x$ and one X_i in $X_i \in [x, x+h]$, then it must be that the j'th highest one is in $[x, x+h]$.

b) $\mathbb{P}(\text{one } X_i \in [x, x+h] \wedge \text{exactly } j-1 \text{ others are} < x)$
$= \mathbb{P}(\text{one } X_i \in [x, x+h]) \cdot \mathbb{P}(\text{exactly } j-1 \text{ others are} < x)$ by independence
$= \mathbb{P}\left(\bigcup_{i=1}^{n} \{X_i \in [x, x+h]\}\right) \cdot \mathbb{P}(\text{exactly } j-1 \text{ others are} < x)$
$= \sum_{i=1}^{n} \mathbb{P}(X_i \in [x, x+h]) \cdot \mathbb{P}(\text{exactly } j-1 \text{ others are} < x)$
$= n \cdot \mathbb{P}(X_1 \in [x, x+h]) \cdot \mathbb{P}(\text{exactly } j-1 \text{ others are} < x)$ since all X_i's have the same distribution

c) Let $Y_i = \mathbb{1}_{\{X_i < x\}} \sim \text{Bernoulli}(F_X(x))$.
Excluding X_1, the number of X_i's which are $< x$ is $\sum_{i=2}^{n} Y_i \sim \text{Bin}(n-1, F_X(x))$.
Therefore,
$$\mathbb{P}(\text{exactly } j-1 \text{ others are} < x) = \mathbb{P}\left(\sum_{i=2}^{n} Y_i = j-1\right) = \binom{n-1}{j-1} F_X(x)^{j-1} (1-F_X(x))^{n-j}.$$

d) Write $\mathbb{P}(X_1 \in [x, x+h]) = F_X(x+h) - F_X(x)$ so that $f_X(x) = \lim_{h \to 0} \frac{F_X(x+h) - F_X(x)}{h}$, and similarly for $X_{(j)}$. Then just check that $n\binom{n-1}{j-1} = \frac{n!}{(j-1)!(n-j)!}$.

5

If $X \sim Unif(0,1)$, then $f_X(x) = 1$ and $F_X(x) = x$, $x \in (0,1)$.
Using Q4, the density of $X_{(j)}$ is

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!\,(n-j)!} f_X(x) \cdot [F_X(x)]^{j-1} [1 - F_X(x)]^{n-j}$$

$$= \frac{n!}{(j-1)!\,(n-j)!} \cdot 1 \cdot x^{j-1} \cdot (1-x)^{n-j}$$

$$= \frac{\Gamma(n+1)}{\Gamma(j) \cdot \Gamma(n-j+1)} x^{j-1} (1-x)^{(n-j+1)-1} \quad , \; x \in (0,1)$$

which is the pdf of a Beta$(j, n-j+1)$ distribution.

Since the Beta(α, β) distribution has expectation $\frac{\alpha}{\alpha+\beta}$ and variance $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$, we get

$$\mathbb{E}[X_{(j)}] = \frac{j}{j+(n-j+1)} = \frac{j}{n+1} \quad \text{and}$$

$$Var(X_{(j)}) = \frac{j(n-j+1)}{(j+[n-j+1])^2(j+[n-j+1]+1)} = \frac{j(n-j+1)}{(n+1)^2(n+2)}$$

(6) We need

$$1 = \sum_{x=1}^{\infty} \frac{c}{x!} = c \sum_{x=1}^{\infty} \frac{1}{x!} = c\left(-1 + \sum_{x=0}^{\infty} \frac{1}{x!}\right) = c(-1+e),$$

so $c = \frac{1}{e-1}$.

Now, $\mathbb{P}(Z \le z)$

$$= \mathbb{P}(\min\{U_1, \ldots, U_X\} \le z)$$

$$= \sum_{x=1}^{\infty} \mathbb{P}(\min\{U_1, \ldots, U_x\} \le z \mid X = x) \cdot \mathbb{P}(X = x)$$

$$= \sum_{x=1}^{\infty} \int_0^z x \cdot f_u(t)(1 - F_u(t))^{x-1} \, dt \cdot \frac{c}{x!}$$

$$= \sum_{x=1}^{\infty} x \cdot \int_0^z (1-t)^{x-1} \, dt \cdot \frac{c}{x!}$$

$$= c \sum_{x=1}^{\infty} x \cdot \frac{1-(1-z)^x}{x} \cdot \frac{1}{x!}$$

$$= c \sum_{x=1}^{\infty} \frac{1-(1-z)^x}{x!}$$

$$= c\left(e - e^{1-z}\right)$$

$$= \frac{e - e^{1-z}}{e-1} = \frac{1 - e^{-z}}{1 - e^{-1}}.$$

(7) a) <u>Base case</u> $(k=1)$: $\quad S_1 = U_1 \sim \text{Unif}(0,1)$, so $\mathbb{P}(S_1 \le t) = t = \frac{t^1}{1!}$ ✓.

<u>Induction step</u>: assume true for $k \in \mathbb{N}$. Then

$$\mathbb{P}(S_{k+1} \le t) = \mathbb{P}(S_k + U_{k+1} \le t)$$

$$= \int_0^1 \mathbb{P}(S_k + u \le t \mid U_{k+1} = u) \cdot \underbrace{f_U(u)}_{=1}\, du$$

$$= \int_0^1 \mathbb{P}(S_k \le t-u)\, du \qquad \begin{array}{l} t-u \ge 0 \\ \Leftrightarrow \; u \le t \end{array}$$

$$= \int_0^t \mathbb{P}(S_k \le t-u)\, du$$

$$= \int_0^t \frac{(t-u)^k}{k!}\, du \quad \text{by our induction hypothesis}$$

$$= \frac{t^{k+1}}{(k+1)\cdot k!} = \frac{t^{k+1}}{(k+1)!}. \quad ✓$$

By the principle of mathematical induction, $\mathbb{P}(S_k \le t) = \frac{t^k}{k!}$.

b) $\mathbb{P}(N=n) = \mathbb{P}(\min\{k : S_k > 1\} = n)$

$$= \mathbb{P}(S_n > 1 \text{ and } S_{n-1} < 1)$$

$$= \mathbb{P}(S_{n-1} < 1 \text{ and } (S_n < 1)^c)$$

$$= \mathbb{P}(S_{n-1} < 1) - \mathbb{P}(S_n - 1).$$

c) $$\mathbb{E}[N] = \sum_{n=2}^{\infty} n\left(\frac{1}{(n-1)!} - \frac{1}{n!}\right) = \sum_{n=2}^{\infty}\left(\frac{n}{(n-1)!} - \frac{1}{(n-1)!}\right) = \sum_{n=2}^{\infty} \frac{1}{(n-2)!} = e.$$

8 First find the marginals:

$$f_X(x) = \int_0^1 f_{X,Y}(x,y)\,dy = \int_0^1 (x+y)\,dy = x + \frac{1}{2}, \quad x \in (0,1).$$

By symmetry, $f_Y(y) = y + \frac{1}{2}$, $y \in (0,1)$.

$$\mathbb{P}(X \le \sqrt{Y}) = \int_0^1 \mathbb{P}(X \le \sqrt{y} \mid Y = y) \cdot f_Y(y)\,dy$$

$$= \int_0^1 \left[\int_0^{\sqrt{y}} \left(x + \frac{1}{2}\right) dx\right] \cdot \left(y + \frac{1}{2}\right) dy$$

$$= \int_0^1 \left[\frac{x^2 + x}{2} \Big|_0^{\sqrt{y}} \right] dy$$

$$= \int_0^1 \frac{y + \sqrt{y}}{2}\,dy$$

$$= \frac{1}{2}\left(\frac{y^2}{2} + \frac{y^{1.5}}{1.5}\right)\Big|_0^1$$

$$= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3/2}\right) = 7/12.$$

So $\mathbb{P}(X > \sqrt{Y}) = 5/12$.

9. The roots of x^2+Bx+C are $\dfrac{-B \pm \sqrt{B^2-4C}}{2}$,
and these are real iff $B^2-4C \geq 0$, which has probability

$$\mathbb{P}(B^2-4C \geq 0)$$

$$= \mathbb{P}(C \leq B^2/4)$$

$$= \int_0^1 \mathbb{P}(C \leq b^2/4 \mid B=b) \cdot \underbrace{f_B(b)}_{=1} \, db$$

$$= \int_0^1 b^2/4 \, db$$

$$= b^3/12 \Big|_0^1$$

$$= 1/12.$$

(10) Let $f(x) = \mathbb{E}[(Y-x)^2]$

$$= \mathbb{E}[Y^2 - 2xY + x^2]$$

$$= \mathbb{E}[Y^2] - 2x \cdot \mathbb{E}[Y] + x^2$$

Set $0 = \frac{d f(x)}{dx} = -2 \cdot \mathbb{E}[Y] + 2x$

$\Leftrightarrow \quad x = \mathbb{E}(Y).$

Check this is a minimum: $\frac{d^2 f(x)}{dx^2} = 2 > 0.$ Yep!

So the smallest that $\mathbb{E}[(Y-x)^2]$ can get is $\mathbb{E}[(Y - \mathbb{E}[Y])^2] = Var(Y).$

(can prove using mgfs)

(11) a) Since $X+Y \sim \mathrm{Pois}(\lambda+\rho)$, we have that

$$\mathbb{P}(X=x \mid X+Y=n)$$

$$= \frac{\mathbb{P}(X=x \wedge X+Y=n)}{\mathbb{P}(X+Y=n)}$$

$$= \frac{\mathbb{P}(X=x \wedge Y=n-x)}{\mathbb{P}(X+Y=n)}$$

$$= \frac{\mathbb{P}(X=x)\cdot \mathbb{P}(Y=n-x)}{\mathbb{P}(X+Y=n)}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} \cdot \frac{\rho^{n-x} e^{-\rho}}{(n-x)!} \cdot \frac{n!}{(\lambda+\rho)^n e^{-(\lambda+\rho)}}$$

$$= \binom{n}{x}\left(\frac{\lambda}{\lambda+\rho}\right)^x \left(1-\frac{\lambda}{\lambda+\rho}\right)^{n-x} \sim \mathrm{Bin}\left(n, \frac{\lambda}{\lambda+\rho}\right).$$

(12) Let $g = g(x,y) = x+y \in (0,\infty)$ and $h = h(x,y) = \frac{x}{x+y} \in (0,1)$ so that $x = gh$ and $y = g(1-h)$. The Jacobian is

$$\begin{vmatrix} \frac{\partial x}{\partial g} & \frac{\partial x}{\partial h} \\ \frac{\partial y}{\partial g} & \frac{\partial y}{\partial h} \end{vmatrix} = \begin{vmatrix} h & g \\ 1-h & -g \end{vmatrix} = -gh - g(1-h) = g.$$

Thus the joint pdf of (G,H) is given by

$$f_{G,H}(g,h) = f_{X,Y}(gh, g(1-h)) \cdot g$$

$$= f_X(gh) \cdot f_{X,Y}(g(1-h)) \cdot g \quad \text{by independence}$$

$$= \frac{1}{\Gamma(\lambda)} (gh)^{\lambda-1} e^{-gh} \cdot \frac{1}{\Gamma(\rho)} g^{\rho-1} (1-h)^{\rho-1} e^{-g(1-h)} \cdot g$$

$$= \frac{1}{\Gamma(\lambda)} g^{\lambda+\rho-1} e^{-g} \cdot \frac{1}{\Gamma(\rho)} h^{\lambda-1} (1-h)^{\rho-1}$$

$$= \underbrace{\frac{1}{\Gamma(\lambda+\rho)} g^{\lambda+\rho-1} e^{-g}}_{\text{pdf of Gamma}(\lambda+\rho,1)} \cdot \underbrace{\frac{\Gamma(\lambda+\rho)}{\Gamma(\lambda)\Gamma(\rho)} h^{\lambda-1} (1-h)^{\rho-1}}_{\text{pdf of Beta}(\lambda,\rho)}.$$

So $G \sim \text{Gamma}(\lambda+\rho, 1)$ and $B \sim \text{Beta}(\lambda, \rho)$.
The above factorization shows that $G \perp\!\!\!\perp B$.

(13) a) Let $r = r(x,y) = \sqrt{x^2+y^2} \in (0,\infty)$ and $\theta = \arctan(y/x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Then $x = r\cdot\cos(\theta)$ and $y = r\cdot\sin(\theta)$. The Jacobian is

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\cdot\sin(\theta) \\ \sin(\theta) & r\cdot\cos(\theta) \end{vmatrix} = r\cdot\cos^2(\theta) + r\cdot\sin^2(\theta) = r.$$

The joint distribution of (R,θ) is

$$\begin{aligned} f_{(R,\theta)}(r,\theta) &= f_{X,Y}(r\cdot\cos(\theta), r\cdot\sin(\theta))\cdot r \\ &= f_X(r\cdot\cos(\theta))\cdot f_Y(r\cdot\sin(\theta))\cdot r \quad \text{by independence} \\ &= \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-r^2\cdot\cos^2(\theta)}{2}\right)\cdot\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-r^2\cdot\sin^2(\theta)}{2}\right)\cdot r \\ &= \frac{1}{2\pi}\cdot\exp\left(-r^2/2\right) \end{aligned}$$

Thus $f_{(R^2,\theta)}(r,\theta) = \underbrace{\frac{1}{2\pi}}_{\text{pdf of Unif}(0,2\pi)} \cdot \underbrace{\frac{1}{2}\exp(-r/2)}_{\text{pdf of Exp}(1/2)}$

Hence $R^2 \sim \text{Exp}(1/2)$ and $\theta \in \text{Unif}(0,2\pi)$, and $R^2 \perp\!\!\!\perp \theta$.

b) $U_i \sim \text{Unif}(0,1) \Rightarrow 2\pi U_i \sim \text{Unif}(0,2\pi) \Rightarrow 2\pi U_i \overset{d}{=} \theta$.

Also, $\mathbb{P}(\sqrt{-2\log(U_j)} \le x) = \mathbb{P}(U_j \ge e^{-x^2/2}) = 1 - e^{-x^2/2}$

$\Rightarrow \sqrt{-2\log(U_j)} \sim \text{Exp}(1/2) \Rightarrow \sqrt{-2\log(U_j)} \overset{d}{=} R$.

So $X \overset{d}{=} R\cdot\cos(\theta) \overset{d}{=} \sqrt{-2\log(U_i)}\cdot\cos(2\pi U_j)$

and $Y \overset{d}{=} R\cdot\sin(\theta) \overset{d}{=} \sqrt{-2\log(U_j)}\cdot\sin(2\pi U_i)$.

c) If I give you u_1 and u_2, you give me back

$\sigma_1\cdot\sqrt{-2\log(u_1)}\cdot\sin(2\pi u_2) + \mu_1$ and $\sigma_2\cdot\sqrt{-2\log(u_2)}\cdot\cos(2\pi u_1) + \mu_2$.

(14) $\mathrm{Cov}(X_1+X_2, X_2+X_3)$

$$= \mathbb{E}[(X_1+X_2)(X_2+X_3)] - \underbrace{\mathbb{E}[X_1+X_2]}_{=\mathbb{E}[X_1]+\mathbb{E}[X_2]=2\mu}\cdot\mathbb{E}[X_2+X_3]$$

$$= \mathbb{E}[X_1X_2 + X_1X_3 + X_2^2 + X_2X_3] - 4\mu^2$$

$$= \mathbb{E}[X_1]\cdot\mathbb{E}[X_2] + \mathbb{E}[X_1]\cdot\mathbb{E}[X_3] + \mathbb{E}[X_2^2] + \mathbb{E}[X_2]\cdot\mathbb{E}[X_3] - 4\mu^2$$

$$= \mu^2 + \mu^2 + \mathrm{Var}(X_2) + \mathbb{E}[X_1]^2 + \mu^2 - 4\mu^2$$

$$= \mathrm{Var}(X_2)$$

$$= \sigma^2$$

$\mathrm{Cov}(X_1+X_2, X_1-X_2)$

$$= \mathbb{E}[(X_1+X_2)(X_1-X_2)] - \mathbb{E}[X_1+X_2]\cdot\underbrace{\mathbb{E}[X_1-X_2]}_{=0}$$

$$= \mathbb{E}[(X_1+X_2)(X_1-X_2)]$$

$$= \mathbb{E}[X_1^2 - X_2^2]$$

$$= \mathbb{E}[X_1^2] - \mathbb{E}[X_2^2]$$

$$= 0.$$

(15) We have that

$$(n-1)S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2$$

$$= \sum_{i=1}^{n}([X_i-\mu]-[\bar{X}-\mu])^2$$

$$= \sum_{i=1}^{n}\left[(X_i-\mu)^2 - 2(X_i-\mu)(\bar{X}-\mu) + (\bar{X}-\mu)^2\right]$$

$$= \sum_{i=1}^{n}(X_i-\mu)^2 - n(\bar{X}-\mu)^2$$

So $(n-1)\mathbb{E}[S^2] = \sum_{i=1}^{n}\mathbb{E}[(X_i-\mu)^2] - n\,\mathbb{E}[(\bar{X}-\mu)^2]$

$= n\cdot \mathrm{Var}(X_i) - n\cdot \mathrm{Var}(\bar{X})$ since $\mathbb{E}[\bar{X}] = \mu$

$= n\cdot\sigma^2 - n\cdot \sigma^2/n$ since $\mathrm{Var}(\bar{x}) = \sigma^2/n$

$= (n-1)\sigma^2$.

Hence $\mathbb{E}[S^2] = \sigma^2$.

16 Let B_{ij} be the $(i,j)^{th}$ entry of B.

If $i=j$, then $B_{ii} \sim \frac{1}{2}(A_{ii}+A_{ii}) = A_i \sim N(0,1)$.

If $i \neq j$, then $B_{ij} \sim \frac{1}{2}(A_{ij}+A_{ji}) \sim N(0, \frac{1}{2})$.

So the joint distribution is given by

$$f(b_{11}, \dots, b_{nn}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}}\left(\exp\left(-\frac{b_{ii}^2}{2}\right) \cdot \prod_{j \neq i} \frac{1}{\sqrt{\pi}} \exp\left(-b_{ij}^2\right)\right]$$

$$= 2^{-\frac{n}{2}} \pi^{-\frac{n(n-1)}{4}} \cdot \exp\left(-\sum_{i=1}^{n}\left[b_{ii}^2/2 + \sum_{j \neq i} b_{ij}^2\right]\right).$$

(17) a) $M_{\Sigma X_i}(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n}\left(\frac{1}{1-t}\right) = \left(\frac{1}{1-t}\right)^n = (1-t)^{-n}$

... which is the mgf of a Gamma$(n,1)$ distribution.

b) By the CLT, $\dfrac{\bar{X}-\mathbb{E}[X_i]}{\sqrt{Var(X_i)/n}} \xrightarrow{d} N(0,1)$.

In this case, $\mathbb{E}[X_i]=1$ and $Var(X_i)=1^2=1$, so that becomes $(\bar{X}-1)/\frac{1}{\sqrt{n}} \xrightarrow{d} N(0,1)$.

Convergence in distribution means that $\mathbb{P}\left(\frac{\bar{X}-1}{1/\sqrt{n}} \le x\right) \xrightarrow{n\to\infty} \Phi(x)$,

so by taking derivatives, for large n we have $\frac{d}{dx}\mathbb{P}\left(\frac{\bar{X}-1}{1/\sqrt{n}} \le x\right) \doteq \phi(x)$.

c) $\frac{d}{dx}\mathbb{P}\left(\frac{\bar{X}-1}{1/\sqrt{n}} \le x\right) = \frac{d}{dx}\mathbb{P}\left(\sum_{i=1}^{n} X_i \le \sqrt{n}x+n\right)$

$$= \frac{d}{dx}\int_0^{\sqrt{n}x+n} \frac{1}{\Gamma(n)} t^{n-1} e^{-t}\,dt$$

let $u = \frac{t-n}{\sqrt{n}}$

$\sqrt{n}\,du = dt$

$$= \frac{d}{dx}\int_0^{x} \frac{\sqrt{n}}{\Gamma(n)} (\sqrt{n}u+n)^{n-1} e^{-\sqrt{n}u-n}\,du$$

$$= \frac{\sqrt{n}}{\Gamma(n)} (\sqrt{n}x+n)^{n-1} e^{-\sqrt{n}x-n}$$ by the FTC

d) So $\frac{\sqrt{n}}{\Gamma(n)}(\sqrt{n}x+n)^{n-1} e^{-\sqrt{n}x-n} \doteq \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

As $x\to 0$, this becomes $\frac{\sqrt{n}}{\Gamma(n)} n^{n-1} e^{-n} \doteq \frac{1}{\sqrt{2\pi}}$

$$\Rightarrow \quad \Gamma(n) \doteq \sqrt{2\pi}\, e^{-n} n^{n-\frac{1}{2}}$$

$$\Rightarrow \quad (n-1)! \doteq \sqrt{2\pi}\, e^{-n} n^{n-\frac{1}{2}}$$

$$\Rightarrow \quad n! = n\cdot(n-1)! \doteq \sqrt{2\pi}\, e^{-n} n^{n+\frac{1}{2}} \quad \square$$

e) I get 0.02523...