EE Circuit Theory

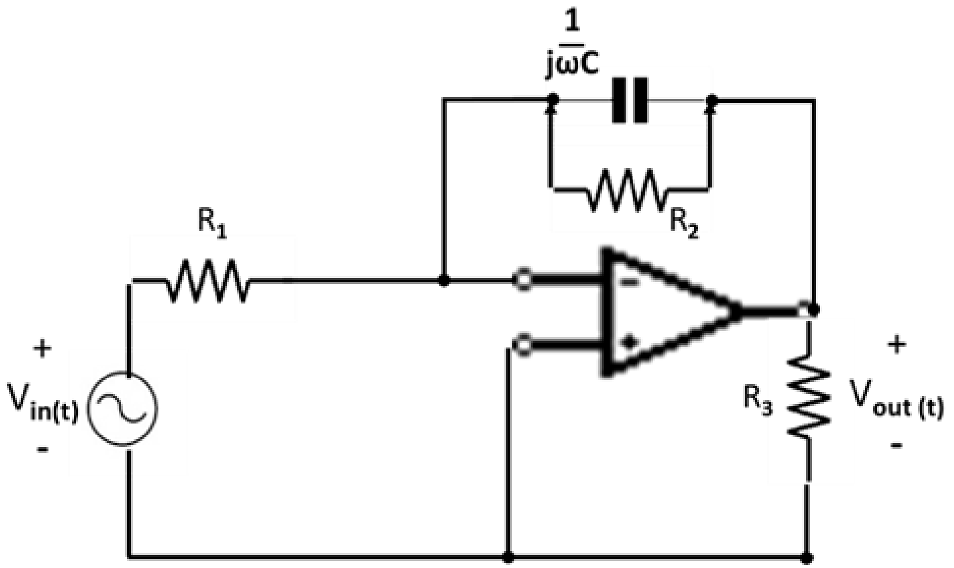
Bode, Laplace and Fourier Circuit Analysis Methods

Bode, Laplace and Fourier Transform Circuit Analysis

- Bode Plots and Laplace and Fourier Transforms are mathematical constructs used to model the response of linear circuits to time varying input signals including, for example:
    - Sinusoidal, square and triangular wave forms.
  - The utility of these tools rests on the premise that:
    - if it is possible to express the mathematical characteristics of the circuit input signal in terms of a set of orthogonal components, then
    - the characteristics of the circuit output signal can be obtained in terms of these components.
  - As prelude to the following in depth study of these tools this session will cover:
    - a general overview of a circuit analysis in the frequency domain, and
    - a demonstration, using a simple amplifier circuit, of such an analysis.
  - The text for this presentation, along with supplemental materials, is available at the following URL: https://github.com/rob76012/CircuitThry-DS

Bode, Laplace and Fourier Circuit Analysis Tools

- During this session, we will consider the following audio amplifier designed to convert an input signal into an electrical signal for recording or transmission purposes:



R_1 = 5k Ω

R_2 = 50k Ω

R_3 = 10k Ω

C = 2 nF

Previously, we defined a circuit transfer function H(·) as the ratio of the output signal (e.g. V_{out}) to an input signal (e.g. V_{in}). We begin our discussion assuming knowledge of the input signal characteristics and the circuit's physical composition and layout expressed in terms of the independent frequency variable ω radians per second (where $\omega = 2\pi f$, f is frequency in cycles per second) and the time, t.

Bode, Laplace and Fourier Circuit Analysis Tools

- If the input signal in the time domain, for a given frequency ω, is sinusoidal in the form:
  - $V_{in}(t) = A\cos(\omega t)$
- Then, for a linear circuit, the steady state output voltage will also be sinusoidal in nature and at the same frequency ω as the input, with the general form:
  - $V_{out}(t) = B\cos(\omega t + \theta)$.
  - Where A and B represent amplitudes of the input and output signals,
  - The phase angle of the input is taken as the zero reference, and the output signal phase is shifted θ° relative to the input.
- We define the gain of the circuit (G) to be the ratio of the output to input amplitudes, expressed as
  - G = B/A
- And the circuit phase shift, defined as the difference between the angular phase of the output and input signals, expressed as
  - Phase shift = $\theta° - 0^{°} = \theta°$

The Circuit Transfer Function H(·)

- We begin the analysis by determining the circuit transfer function. Please refer to the additional materials for the algebraic details that obtain the calculations shown on the following slides.
- Expressing the input and output voltage functions in phasor notation obtains:
  - $V_{in}(t) = A\cos(\omega t)$ → $V_{in}(\omega t) = \mathrm{A}\ \angle 0°$
  - $V_{out}(t) = B\cos(\omega t + \ \Theta)$ → $V_{out}(\omega t) = V_{out}(\omega t) = \mathrm{B}\angle\theta°$
- In the steady state
  - assuming an ideal model for the Op Amp, and
  - expressing the circuit impedances in complex notation,
  - the Kirchoff nodal equation at the inverting input node of the Op Amp is given by the following equation:
  - (Eq 5.1) $\dfrac{V_{in}(\omega)}{R_1} + \dfrac{V_{out}(\omega)}{R_2} + j\omega C V_{out}(\omega) = 0$

The Circuit Transfer Function H(·)

- Rearranging Equation 5.1 to provide the ratio $V_{out}(j\omega)/V_{in}(j\omega)$ obtains the transfer function:
  - (Eq 6.1) $H(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{-R_2}{R_1 + j\omega C R_1 R_2}$
- Note the following:
  - the independent time variable does not appear explicitly in this formulization;
  - the independent frequency variable is expressed as jω, indicating that equation 6.1 is a complex function of the signal frequency ω in radians;
  - the shape and amplitude of the time based sinusoidal input and output functions (LH side of Eq 6.1) are determined by the values of the circuit components and the frequency variable ω, or
  - alternately, once the frequency is set, a time based sinusoid signal, such as cos (ωt), is completely determined on the interval $-\infty \leq \omega t \leq \infty$.

The Circuit Transfer Function H(·)

- As shown in the supplemental materials, rationalizing the transfer function to place the complex term in the numerator obtains
  - (Eq 7.1) $H(j\omega) = \frac{R_2/R_1}{\sqrt{1+j\omega^2C^2R_2^2}}\ (-1 + j\omega CR_2)$
- Expressing Equation 7.1 in phasor notation (a complex amplitude and phase angle) results in:
  - (Eq 7.2) $H\ (\omega) = \frac{R_2/_{R_1}}{\sqrt{1+\omega^2C^2R_2^2}}\angle(180° - tan^{-1}(\omega CR_2))$
- Where the magnitude of the transfer function (i.e. the circuit Gain) is given by:
  - (Eq 7.3) $G = |H\ (j\omega)| = \frac{R_2/_{R_1}}{\sqrt{1+\omega^2C^2R_2^2}}$
- And the Transfer Function phase shift is given by:
  - (Eq 7.4) Phase shift = ∠H(j ω) = $\left(180° - tan^{-1}(\omega CR_2)\right)$

The Circuit Transfer Function H(·)

- Note that the transfer function Gain $G = |H\ (j\omega)| = \frac{R_2/R_1}{\sqrt{1+\omega^2 C^2 R_2^2}}$ and its Phase shift = $\angle$H(j ω) = $\left(180° - tan^{-1}(\omega C R_2)\right)$ show the following:

1. The gain of the circuit is a determined by ω, the input signal frequency, and the passive circuit component values, and, additionally, it is an inverse function of ω.
2. The phase shift of the output is determined by the input signal frequency ω and the passive circuit component values, and
3. H(j ω) is located in the upper left quadrant of the complex plain, which is indicated by the expression of the phase shift relative to 180°.

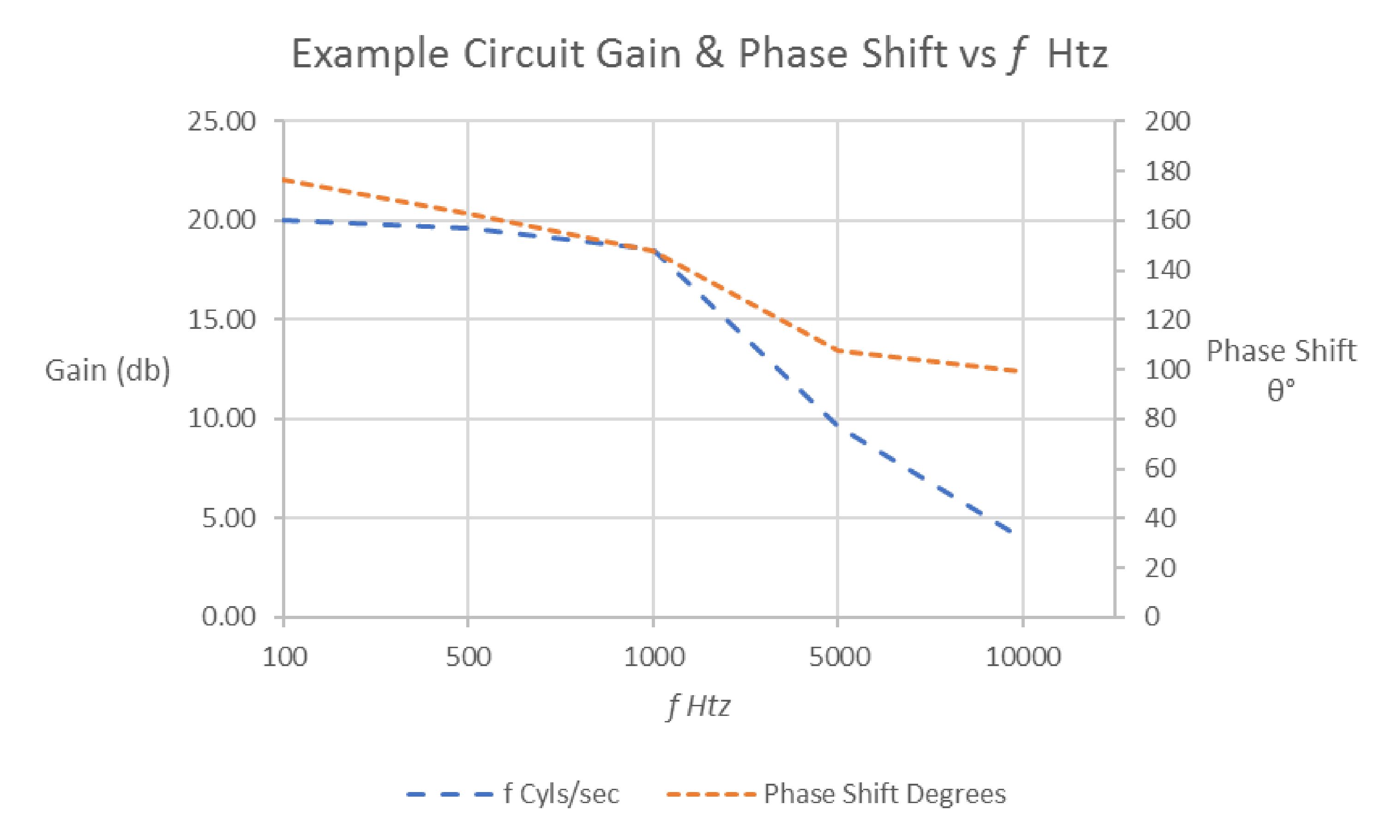
The Circuit Transfer Function H(·)

- Substituting the passive circuit component values obtains the following transfer function for the circuit shown on slide 3:
- (Eq 9.1) $H(j\omega) = \frac{-10}{1+j\omega*10^{-4}}$
- With the output gain G equal to
- (Eq 9.2) $G = |H(j\omega)| = \frac{10}{\sqrt{1+(\omega^2*10^{-8})}}$
- And the output phase shift equal to
- (Eq 9.3) $\angle H(j\omega) = 180° - (\tan^{-1}(\omega * 10^{-4}))°$

Bode Plots

The graph to the right is called a Bode Plot. It shows the relationship between the input signal frequency and the output signal gain and phase shift as a function of frequency, in this case over a range from 100 to 10,000 Htz. as shown in the table below.

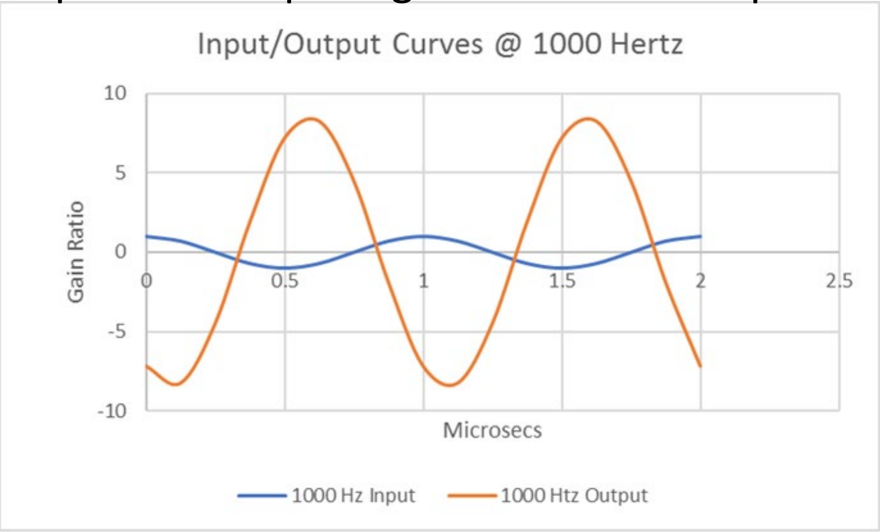
For the circuit transfer function Eq 9.1, the Bode Plots show the inverse relationship of the output gain and phase shift to the input frequency and, clearly emphasize distinct break points at 1000 and 5000 Htz.

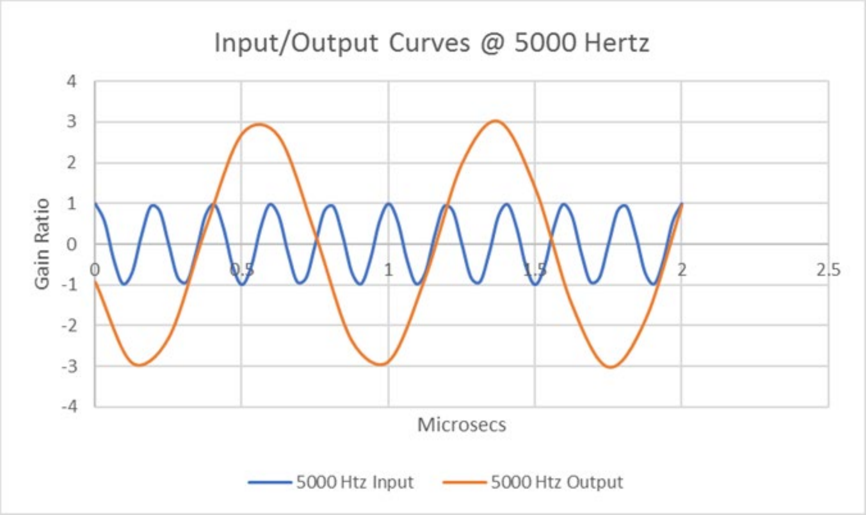


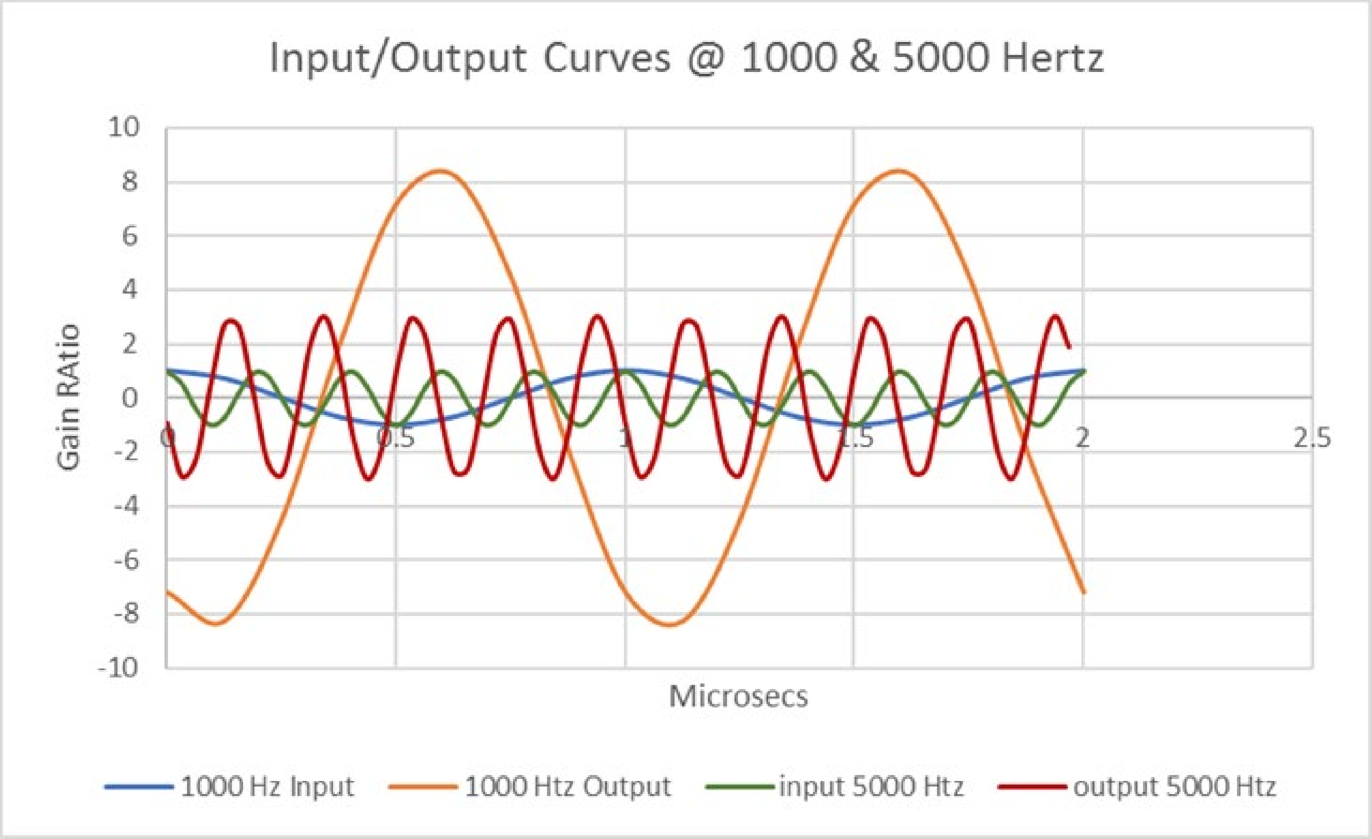
f Cyls/sec	ω rad/sec	Gain Ratio	Gain db	Phase Shift Degrees	Phase Shift Radians
100	628.3	9.98	19.98	176	3.0788
500	3141.6	9.54	19.59	163	2.8372
1000	6283.2	8.47	18.55	148	2.5806
5000	31415.9	3.03	9.64	108	1.8790
10000	62831.9	1.57	3.93	99	1.7286

Note the frequency axis is logarithmic, the gain, shown on the left vertical axis is in units of db [20 Log_{10} (Gain Ratio)], and the phase shift is shown in degrees on the right vertical axis.

One could of course obtain the same information provided by Bode Plots from the time-based input and output equations $V_{in}(\omega t)$ and $V_{out}(\omega t)$. Typically, this approach requires constructing and plotting a family of curves (Gain and Phase Shift) over a signal frequency range of interest as illustrated, in part, below. These plots show the time-based input and output signals at the break points for 1000 and 5000 Htz., in separate and composite graphic views.







Advantages of Analysis in the Complex Domain

- The foregoing illustrates the important fact that transforming the model to the complex domain allows a simpler and more intuitive representation of the results of the circuit analysis. For example, the breakpoints in the gain and phase shift output response that are observed in the Bode Plots are immediately obvious to the analyst.
- As will be seen during the following discussion on Bode Plots, further circuit characteristics and circuit performance measures are available from these plots.
- Finally, in the sessions that follow we will investigate the application of the Laplace Transform, Fourier Series and Fourier Transform:
  - to construct mathematical representations of cyclic input signals, that meet certain mathematical criteria, in terms of orthogonal sinusoidal components, and
  - to analyze the transfer function and frequency response of a linear circuit in terms of the sum of these orthogonal components.

Thank you for your attention and we will begin our detailed discussion next.

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