EE Circuit Theory

The Use of Bode Plots, and Laplace and Fourier Transforms in Circuit Analysis

Hello and welcome to this session of Electrical Engineering Circuit Theory.

Today's presentation begins with an introduction to the topic of circuit analysis in the time and frequency domains. It is a prelude to the following in depth study of the mathematical concepts and processes used in the design and analysis of circuits with various input source characteristics. Bode plots and the use of Laplace and Fourier transforms rests on the premise that if it is possible to express the mathematical characteristics of the circuit input signal in terms of a set of orthogonal components, then, for a linear circuit, the characteristics of the circuit output signal can be obtained in terms of these components. In fact this concept has application in many areas of engineering and as we study the application of these tools in the context of linear circuit analysis, you should recognize, at an intellectual level, the extent that it applies not only to this subject but also in other engineering areas as well.

Consider for example, an input source provided by human speech or the sound from a musical instrument to an audio amplifier designed to convert the input signal into an electrical signal for recording or transmission purposes. This lesson's focus is to provide an overview of the mathematical concepts typically used to model the electrical input and output signals given a mathematical representation of a circuit's input – output transfer function. Previously, we defined a circuit transfer function $H(\cdot)$ as the ratio of the output signal (e.g. V_{out}) to an input signal (e.g. V_{in}). We begin our discussion assuming knowledge of the input signal characteristics and the circuit's physical composition and layout expressed in terms of the independent variables ω (where $\omega = 2\pi f$, f is frequency in cycles per second and ω is frequency in radians per second) and t, the time. Consider the following circuit with component values shown in phasor notation:

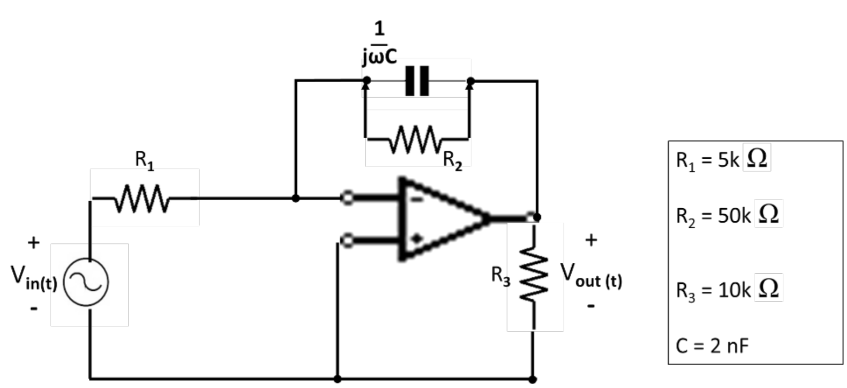


Figure 1

If the input signal, for a given frequency ω, is sinusoidal in the form:

$$V_{in}(t) = A\cos(\omega t)$$

Then, for a linear circuit, the steady state output voltage will also be sinusoidal in nature and at the same frequency ω as the input, with the general form:

$$V_{out}(t) = B\cos(\omega t + \theta).$$

Where A and B represent amplitudes of the input and output signals. The phase angle of the input is often taken as the zero reference and the output signal phase is shifted θ° relative to the input.

We define the gain of the circuit (G) to be the ratio of the output to input amplitude, expressed as

G = B/A

And the circuit phase shift, defined as the difference between the angular phase of the output and input signals, expressed, for this example, as

Phase shift = $\theta° - 0^{\circ} = \theta°$

The Circuit Transfer Function H(·)

We begin the analysis by determining the circuit transfer function defined as the ratio of the circuit output signal to its input signal. Please refer to the separately attached materials for the algebraic details that obtain the following calculations.

1. Express the input and output voltage functions in phasor notation:

    a. $V_{in}(t) = A\cos(\omega t) \rightarrow V_{in}(\omega t) = \mathrm{A}\angle 0°$
    b. $V_{out}(t) = B\cos(\omega t + \Theta) \rightarrow V_{out}(\omega t) = V_{out}(\omega t) = \mathrm{B}\angle\theta°$

2. In the steady state, assuming an ideal model for the Op Amp & expressing the circuit impedances in complex notation, the Kirchoff nodal equation at the inverting input node of the Op Amp is given by the following equation:

$$\frac{V_{in}(\omega)}{R_1} + \frac{V_{out}(\omega)}{R_2} + j\omega C V_{out}(\omega) = 0$$

3. Rearranging this equation to provide the ratio $V_{out}(j\omega)/V_{in}(j\omega)$ provides the transfer function:

$$H(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{-R_2}{R_1 + j\omega C R_1 R_2}$$

Note that the independent time variable does not appear explicitly in this formulization and the independent variable, expressed as jω, indicates the equation H(jω) is a complex function of the signal frequency ω in radians. Additionally, also note that the shape and amplitude of the time based sinusoidal input and output functions are determined by the frequency variable. That is, once the frequency is set, a time based sinusoid signal, such as cos (ωt), is completely determined on the interval $-\infty \le \omega t \le \infty$.

4. For this circuit, the complex transfer function is obtained as

$$H(j\omega) = \frac{R_2/R_1}{\sqrt{1+j\omega^2 C^2 R_2^2}}\,(-1 + j\omega C R_2)$$

5. Expressing equation 4 in phasor notation results in:

$$H\,(j\omega) = \frac{R_2/R_1}{\sqrt{1+\omega^2 C^2 R_2^2}} \angle(180^\circ - tan^{-1}(\omega C R_2))$$

   Where the magnitude of the transfer function (i.e. the circuit Gain) is:

$$G = |H\,(j\omega)| = \frac{R_2/R_1}{\sqrt{1+\omega^2 C^2 R_2^2}}$$

   And the Transfer Function phase shift is:

$$\text{Phase shift} = \angle H(j\,\omega) = \left(180^\circ - tan^{-1}(\omega C R_2)\right)$$

This equation set shows that the gain of the circuit is a determined by the input signal's frequency ω and the passive circuit component values and, additionally, that the gain is an inverse function of ω. Similarly, the phase shift of the output is determined by the input signal's frequency ω and the passive circuit component values and, additionally, that H(j ω) is located in the upper left quadrant of the complex plain.

6. Substituting the passive circuit component values given in Figure 1, obtains for this circuit the following:

$$H(j\omega) = \frac{-10}{1 + j\omega * 10^{-4}}$$

   Therefore, the output gain G equals

$$G = |H(j\omega)| = \frac{10}{\sqrt{1 + (\omega^2 * 10^{-8})}}$$

   And the output phase shift equals

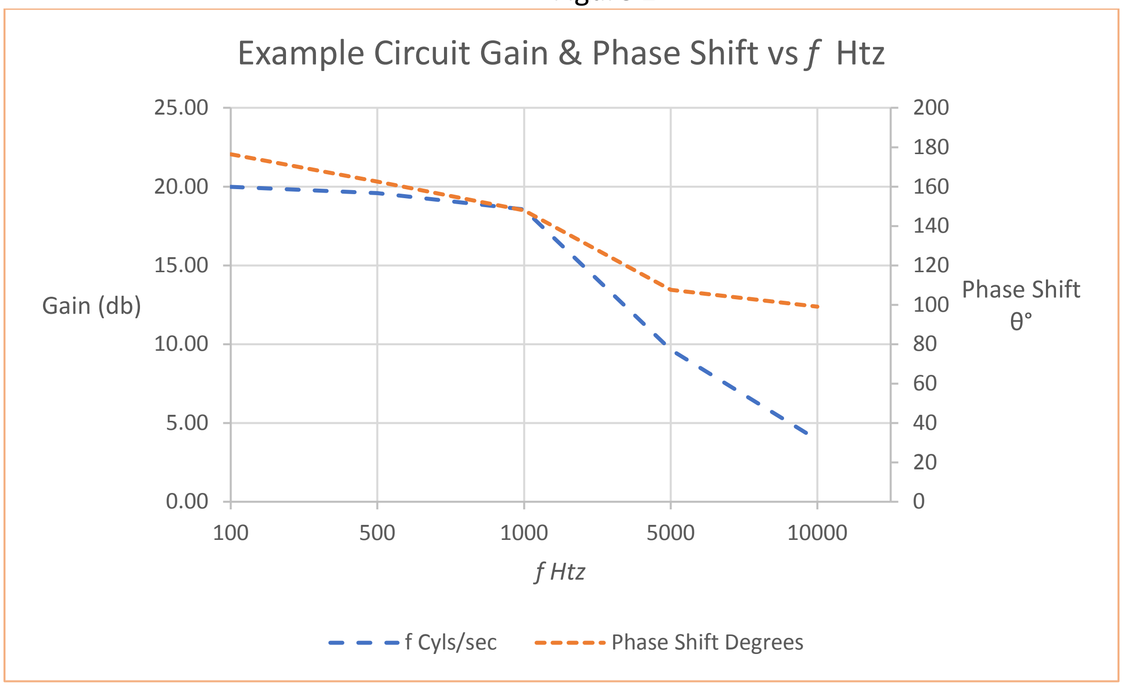
$$\angle H(j\omega) = 180^\circ - (\tan^{-1}(\omega * 10^{-4}))^\circ$$

Using the transfer function for the circuit shown in Figure 1 the following table and graphic show the circuit gain and phase shifts for input signal frequencies in a range from 100 to 10,000 Htz.

Table 1

f *Cyls/sec*	ω rad/sec	Gain Ratio	Gain db	Phase Shift Degrees	Phase Shift Radians
100	628.3	9.98	19.98	176	3.0788
500	3141.6	9.54	19.59	163	2.8372
1000	6283.2	8.47	18.55	148	2.5806
5000	31415.9	3.03	9.64	108	1.8790
10000	62831.9	1.57	3.93	99	1.7286

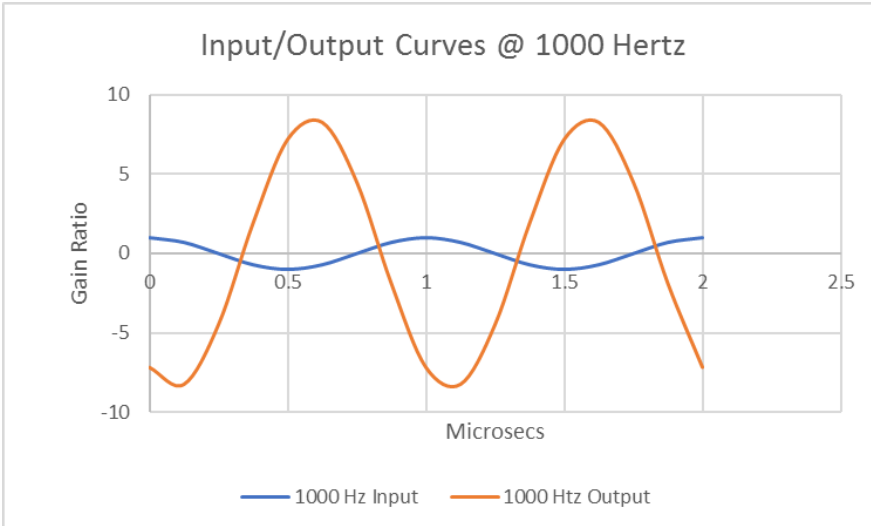
Figure 2

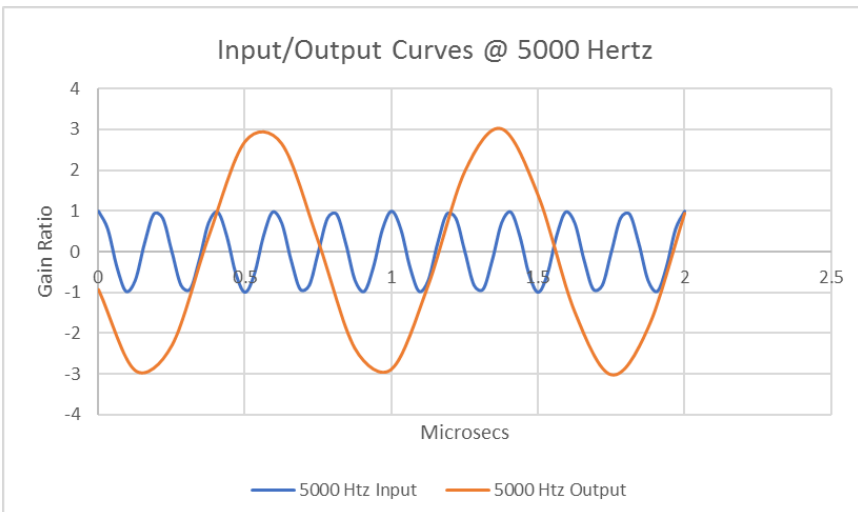


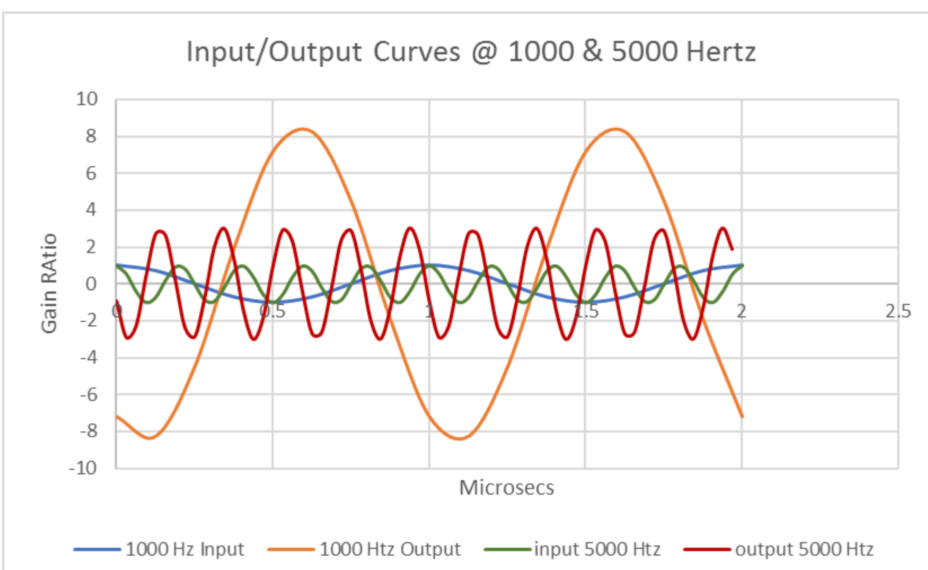
Where, the frequency axis is logarithmic, the gain is shown on the left vertical axis in db [20 Log_{10} (Gain Ratio)], and the phase shift is shown in degrees on the right vertical axis.

As is obvious from Table 1 and Figure 2, the output gain and phase shift are inversely proportional to the input frequency, with distinct break points at 1000 and 5000 Htz.

One could of course obtain the same information provided in Table 1 and Figure 2 from the time based input and output equations $V_{in}(\omega t)$ and $V_{out}(\omega t)$. Typically, this approach requires constructing and plotting a family of curves (Gain and Phase Shift) at multiple input signal frequencies over 1 or more signal periods and illustrated as follows:


Input/Output Curves @ 1000 Hertz
Gain Ratio
Microsecs
1000 Hz Input
1000 Htz Output


Input/Output Curves @ 5000 Hertz
Gain Ratio
Microsecs
5000 Htz Input
5000 Htz Output


Input/Output Curves @ 1000 & 5000 Hertz
Gain RAtio
Microsecs
1000 Hz Input
1000 Htz Output
input 5000 Htz
output 5000 Htz

The foregoing illustrates the extremely significant and important fact that transforming the model to the complex domain allows a simpler and more intuitive representation of the results of the circuit analysis. For example, the breakpoints in the gain and phase shift output response that are observed in Figure 2 are immediately obvious to the analyst.

In fact, the depictions shown in Figure 2 are called Bode Plots for the gain and phase shift frequency response of the circuit. As we will see in the following discussion on Bode Plots, further circuit characteristics and circuit performance measures are available from these plots. Finally, in the following sessions, we will investigate the application of the Laplace Transform, Fourier Series and Fourier Transform mathematics to construct periodic representations of cyclic input signals, that meet certain mathematical criteria, in terms of orthogonal sinusoidal components. If this is possible then, applying the principle of linear superposition, the transfer function and frequency response of a linear circuit can be obtained and analyzed as the sum of these orthogonal component responses.

Transfer Function H(·) Calculation Details

By Kirchhoff's Current Law (KCL), the node equation at the OP Amp inverting (–) input terminal is given as:

$$\frac{V_{in}(\omega)}{R_1} + \frac{V_{out}(\omega)}{R_2} + j\omega C V_{out}(\omega) = 0$$

$$\frac{-V_{in}(\omega)}{R_1} = V_{out}(\omega)\left(\frac{1}{R_2} + j\omega C\right) = V_{out}(\omega) * \left(\frac{1 + j\omega C R_2}{R_2}\right)$$

$$H(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{-R_2}{R_1 + j\omega C R_1 R_2}$$

Where the change in the independent variable, expressed as jω, indicates the equation H(jω) is a complex function of the signal frequency ω in radians.

Referring to Figure 1, the KCL node equation given above follows directly from the decision to model the Operational Amplifier as an ideal Op Amp. This implies that in steady state operation the magnitude of the current flow into the OP Amp's inverting and noninverting terminals each equal 0. Therefore, since the noninverting terminal is clamped at ground potential, the inverting terminal also appears to be at ground potential. Applying KCL at the node connection with the inverting Op amp input, with the assigned voltage levels $V_{in}(\omega)$ and $V_{out}(\omega)$ as given, one obtains the net in-flow of the currents through the impedance R_1 and through the parallel combination of R_2 and the capacitor C must equal the out-flow current, which is 0 amps by the ideal Op Amp model.

To express H(jω) in phasor notation one proceeds as follows:

Multiply the above complex function H(jω) by the value 1 in form of

$$\frac{R_1 - j\omega C R_1 R_2}{R_1 - j\omega C R_1 R_2}, \quad \text{to obtain}$$

$$H(j\omega) = \frac{-R_2}{R_1^2} * \frac{R_1 - j\omega C R_1 R_2}{1 + \omega^2 C^2 R_2^2} = \frac{R_1/R_2}{1 + j\omega^2 C^2 R_2^2} * (-1 + j\omega C R_2)$$

and, noting that the magnitude of the term (-1+j ωCR₂) equals $\sqrt{1 + \omega^2 C^2 R_2^2}$, convert this expression for H(j ω) from rectangular to polar coordinates to obtain the final result:

$$H(j\omega) = \frac{R_1/R_2}{1+j\omega^2C^2R_2^2} * \left[\left(\sqrt{1+\omega^2C^2R_2^2}\right)\angle(180° - tan^{-1}(\omega CR_2))\right]$$

$$H(j\omega) = \frac{R_1/R_2}{\sqrt{1+\omega^2C^2R_2^2}}\ \angle(180° - tan^{-1}(\omega CR_2))$$

Note here that the determination of the transfer function phase angle recognizes that the complex point $(-1 + j\omega CR_2)$ is located in the upper left quadrant in the complex plain. Therefore, the phase angle must be expressed relative to the phase shift 0° reference which is the real axis in the upper right quadrant of the complex plane as shown in Figure 3.

Figure 3