Lecture 2, Part 1:

Theoretical Bounds for Lossless Compression

Robert Bamler · Summer Term of 2023

These slides are part of the course *"Data Compression With and Without Deep Probabilistic Models"* taught at University of Tübingen. More course materials—including video recordings, lecture notes, and problem sets with solutions—are publicly available at `https://robamler.github.io/teaching/compress23/`.

Admin Stuff

- **Important:** next lecture *only on zoom*, not in classroom
  - Sign up to course using (new) Ilias link to get zoom link by email
    (link will also be on website ~30 minutes before next week's lecture starts)
- You'll have to sign up for *exam* on Alma starting 5 June (independently of whether you signed up to the *course* on Ilias)
  - More details will follow.

Recap: Symbol Codes

i.e., finite or countably infinite

- *alphabet* $\mathfrak{X}$ (discrete set) with *probabilities* $p(x)$ for all *symbols* $x \in \mathfrak{X}$
- *message* $\mathbf{x} = (x_1, x_2, \ldots, x_{k(\mathbf{x})}) \in \mathfrak{X}^*$
- *code book* C maps any $x \in \mathfrak{X}$ to its *code word* $C(x) \in \{0, \ldots, B-1\}^*$ (usually: $B = 2$)
  - induces a *symbol code* $C^*\colon \mathfrak{X}^* \to \{0, \ldots, B-1\}^*$ by concatenation (without delimiters):
    $C^*(\mathbf{x}) := C(x_1) \,\|\, C(x_2) \,\|\, \ldots \,\|\, C(x_{k(\mathbf{x})})$
- properties of symbol codes:
  - *unique decodability:* C^* is injective
  - *prefix code:* no code word $C(x)$ is a prefix of another code word $C(x')$ with $x' \neq x$
  - C is a prefix code $\Rightarrow$ C is uniquely decodable (but reverse is in general not true) → Problem 0.2 (d)
- *expected code word length* $L_C := \sum_{x \in \mathfrak{X}} p(x)\,|C(x)|$ ← (simple) model of the data source (crucial for source coding)
- → Problem Set 1
- *Huffman coding* generates an optimal symbol code (that minimizes L_C) for a given p

Theoretical Bounds for Lossless Compression

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- **Goal of this lecture:** Source Coding Theorem [Shannon, 1948]
  - Relates L_C to the so-called *entropy* $H_B[p]$ (which we'll define later today).
  - **The Bad News:** a uniquely decodable B-ary symbol code C *cannot* have $L_C < H_B[p]$.
  - **The Good News:** $\forall p$, one can make L_C close to $H_B[p]$ with less than 1 bit per symbol overhead.
- **Step 1:** proof bound on code word lengths, *independent of* p (KM-Theorem)
- **Step 2:** proof bound on *expected* code word length for a given model p
- **Credits:** Our proof follows:
  `https://www.youtube.com/watch?v=yHw1ka-4g0s&list=PLE125425EC837021F&index=14`

The Kraft-McMillan Theorem [Kraft, 1949; McMillan, 1956]

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(a) $\forall$ B-ary uniquely decodable symbol codes over some discrete alphabet $\mathfrak{X}$:

$$\sum_{x\in\mathfrak{X}} \frac{1}{B^{|C(x)|}} \leq 1 \qquad \text{(“Kraft inequality”)}. \tag{1}$$

Interpretation: we have a finite budget of "shortness" for code words:
- interpret $\frac{1}{B^{|C(x)|}}$ as the "shortness" of code word $C(x)$;
- the sum of all "shortnesses" must not exceed 1;
- if we shorten one code word then we may have to make another code word longer so that we don't exceed our "shortness budget".

Assume you are given a uniquely decodable symbol code and want to shorten the code word for some specific symbol $x \in \mathfrak{X}$.
$\Rightarrow |C(x)|$ shrinks
$\Rightarrow$ "shortness" $B^{-|C(x)|}$ grows
$\Rightarrow$ to avoid exceeding your "shortness budget", you must reduce the shortness $B^{-|C(x')|}$ of some $x' \neq x$
$\Rightarrow |C(x')|$ grows

(b) $\forall$ functions $\ell : \mathfrak{X} \to \mathbb{N}$ that satisfy the Kraft inequality (i.e., $\sum_{x\in\mathfrak{X}} \frac{1}{B^{\ell(x)}} \leq 1$):

$$\exists\ B\text{-ary prefix code } C_\ell \text{ with } |C_\ell(x)| = \ell(x)\ \ \forall x\in\mathfrak{X}.$$

(combining parts (a) and (b))

Corollary: $\forall$ uniquely decodable B-ary symbol codes C:
$\exists$ a B-ary *prefix* code C' with same code word lengths (i.e., $|C'(x)| = |C(x)|\ \forall x\in\mathfrak{X}$)

Lemma

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- let: $\begin{cases} C \text{ be a } B\text{-ary uniquely decodable symbol code over } \mathfrak{X}; \\ s \in \mathbb{N}_0; \\ Y_s := \left\{\mathbf{x} \in \mathfrak{X}^* \text{ with } |C^*(\mathbf{x})| = s\right\}. \end{cases}$
- then: $|Y_s| \leq B^s$.

Proof: Let $S_s := \{ \underbrace{C^*(\underline{x})}_{\in \{0,\dots,B-1\}^s} : \underline{x} \in Y_s \} \subseteq \underbrace{\{0,\dots,B-1\}^s}_{\text{size: } B^s} \Rightarrow |S_s| \leq B^s$

Assume $|Y_s| > B^s \Rightarrow |Y_s| > |S_s|$

$\Rightarrow \exists\, \underline{x}, \underline{x}' \in Y_s$ with $\underline{x} \neq \underline{x}'$ but $C^*(\underline{x}) = C^*(\underline{x}')$

$\Rightarrow C^*$ not injective, i.e., C not uniquely decodable

Proof of Part (a) of KM Theorem

Claim (reminder): C is uniquely decodable $\implies \underbrace{\sum_{x\in\mathfrak{X}} \frac{1}{B^{|C(x)|}}}_{=:r} \le 1.$

Proof: Let $k \in \mathbb{N}$.

$$r^k = \Big(\sum_{x\in\mathfrak{X}} B^{-|C(x)|}\Big)^k = \Big(\sum_{x_1\in\mathfrak{X}} B^{-|C(x_1)|}\Big)\Big(\sum_{x_2\in\mathfrak{X}} B^{-|C(x_2)|}\Big)\cdots\Big(\sum_{x_k\in\mathfrak{X}} B^{-|C(x_k)|}\Big) = \sum_{\underline{x}\in\mathfrak{X}^k} B^{-|C^*(\underline{x})|}$$

(i) if $\mathfrak{X}$ is finite: $\Rightarrow \gamma := \max_{x\in\mathfrak{X}} |C(x)| < \infty$ is well-defined & finite;

$$r^k = \sum_{\underline{x}\in\mathfrak{X}^k} B^{-|C^*(\underline{x})|} = \sum_{s=0}^{\gamma k} \sum_{\underline{x}\in Y_s} B^{-s} = \sum_{s=0}^{\gamma k} \underbrace{|Y_s|\, B^{-s}}_{\le B^{-s}\text{ (Lemma)}} \le \gamma k + 1 \Rightarrow \underbrace{\frac{r^k - 1}{k}}_{\substack{\to\infty \\ \text{for } k\to\infty \\ \text{if } r>1}} < \underbrace{\gamma}_{\substack{\text{indep.} \\ \text{of } k}}$$

$\Rightarrow r \le 1$

(ii) if $\mathfrak{X}$ is countably infinite: without restriction, assume $\mathfrak{X} = \mathbb{N}$;

$$\Rightarrow r = \sum_{x\in\mathbb{N}} B^{-|C(x)|} = \sum_{x=1}^{\infty} B^{-|C(x)|} = \lim_{n\to\infty} \underbrace{\sum_{x=1}^{n} B^{-|C(x)|}}_{\text{alphabet of size } n<\infty \Rightarrow \text{case (i) applies}} \le 1$$

all terms ≥ 0 $\Rightarrow$ absolutely convergent if convergent at all $\Rightarrow$ may reorder terms arbitrarily

Proof of Part (b) of KM Theorem

Claim (reminder): $\sum_{x\in\mathfrak{X}} \frac{1}{B^{\ell(x)}} \le 1 \implies \exists$ B-ary prefix code C_ℓ with $|C_\ell(x)| = \ell(x)\ \forall x\in\mathfrak{X}$.

input of algorithm below — output of algorithm below

Constructive proof: we show existence of C by showing how it can be obtained.

Algorithm: sort symbols in $\mathfrak{X} = \{x, x', x'', \ldots\}$ s.t. $\ell(x) \ge \ell(x') \ge \ell(x'') \ge \ldots$;

initialize $\xi \leftarrow 1$;

for each $x \in \mathfrak{X}$ in above order:

- update $\xi \leftarrow \xi - B^{-\ell(x)}$;
- write $\xi \in [0,1)$ in B-ary: $\xi = (0.\underbrace{????}\ldots)_B$;
- set $C(x)$ to first $\ell(x)$ bits here (pad with trailing zeros if necessary)

Claim: The resulting code book C_ℓ is prefix free (proof: Problem 2.1). also discusses case of (countably) infinite $\mathfrak{X}$

Example: Simplified Game of Monopoly (SGoM)

x	$\ell(x)$	$C_\ell(x)$	order after sorting by descending $\ell(x)$
2	3	111	1
3	2	10	3
4	2	01	4
5	2	00	5
6	3	110	2

$\xi \leftarrow 1$ (initialization)

$\xi \leftarrow 1 - 2^{-3} = (1.000)_2 - (0.001)_2 = (0.111)_2$

$\xi \leftarrow (0.110)_2 - (0.01)_2 = (0.10)_2$

$\xi \leftarrow (0.10)_2 - (0.01)_2 = (0.01)_2$

$\xi \leftarrow (0.01)_2 - (0.01)_2 = (0.00)_2$

$\xi \leftarrow (0.111)_2 - (0.001)_2 = (0.110)_2$

exercise: execute algorithm without sorting $\mathfrak{X}$ by descending $\ell(x)$ and verify that it fails.

- Check Kraft inequality for $B = 2$: $\sum_{x\in\mathfrak{X}} 2^{-\ell(x)} = 2\times 3^{-2} + 3\times 2^{-2} = 1 \le 1$ ✓
- **Question:** how should we choose $\ell : \mathfrak{X} \to \mathbb{N}$ for a given probabilistic model p?
  - **optimally:** via Huffman coding
  - **near-optimally:** via *information content* (next part).

Outlook

- **Problem Set 2:**
  - complete proof of part (b) of KM-Theorem
  - implement Huffman *decoder* in Python
- **Next part:**
  - theoretical bounds on the expected code word length L_C ("The Bad News" & "The Good News")
  - theoretical bounds *beyond symbol codes:* Source Coding Theorem

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Lecture 2, Part 2:

The Source Coding Theorem

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Recap: Kraft-McMillan (KM) Theorem

(a) $\forall$ B-ary uniquely decodable symbol codes over some discrete alphabet $\mathfrak{X}$:

$$\sum_{x\in\mathfrak{X}} \frac{1}{B^{|C(x)|}} \leq 1 \qquad \text{("Kraft inequality")}. \tag{1}$$

(b) $\forall$ functions $\ell : \mathfrak{X} \to \mathbb{N}$ that satisfy the Kraft inequality (i.e., $\sum_{x\in\mathfrak{X}} \frac{1}{B^{\ell(x)}} \leq 1$):

$$\exists\ B\text{-ary prefix code } C_\ell \text{ with } |C_\ell(x)| = \ell(x)\ \forall x \in \mathfrak{X}.$$

- **Question:** how should we choose $\ell : \mathfrak{X} \to \mathbb{N}$ for a given probabilistic model p?
  - **optimally:** via Huffman coding (problem: no closed-form solution)
  - **near-optimally (this part):** via *information content*
    spoiler: $\ell_S(x) := \lceil -\log_B p(x) \rceil$

Optimal Choice of ℓ

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- **Constrained optimization problem:** $(\star)$
  - minimize: $L_{C_\ell} = \sum_{x\in\mathfrak{X}} p(x)\,|C_\ell(x)| = \sum_{x\in\mathfrak{X}} p(x)\,\ell(x)$
  - constraints: (i) $\sum_{x\in\mathfrak{X}} \frac{1}{B^{\ell(x)}} \leq 1$; (ii) $\ell(x) \in \mathbb{N}\ \forall x \in \mathfrak{X}$.
- **Idea:** relax constraint (ii): $(\square)$
  - minimize: $L_\ell := \sum_{x\in\mathfrak{X}} p(x)\,\ell(x)$
  - constraints: (i) $\sum_{x\in\mathfrak{X}} \frac{1}{B^{\ell(x)}} \leq 1$; (ii') $\ell(x) \in \mathbb{R}_{>0}\ \forall x \in \mathfrak{X}$.

  $\Rightarrow$ yields *lower bound*: solution L_ℓ of $(\square)$ $\leq$ solution L_{C_ℓ} of $(\star)$
- **Observation:** solution of $(\square)$ satisfies: (i') $\sum_{x\in\mathfrak{X}} \frac{1}{B^{\ell(x)}} = 1$.
  - Enforce via Lagrange multiplier λ:
    find stationary point of $\mathcal{L}_{\ell,\lambda} := L_\ell + \lambda\big(\sum_{x\in\mathfrak{X}} \frac{1}{B^{\ell(x)}} - 1\big)$ w.r.t. $\lambda \in \mathbb{R}$ and all $\ell(x) \in \mathbb{R}_{\geq 0}\ \forall x \in \mathfrak{X}$.

- $0 = \frac{\partial \mathcal{L}_{\ell,\lambda}}{\partial \lambda} = \sum_{x\in\mathfrak{X}} B^{-\ell(x)} - 1 \Leftrightarrow$ constraint (ii')
- $\forall x \in \mathfrak{X}$: $0 = \frac{\partial \mathcal{L}_{\ell,\lambda}}{\partial \ell(x)} = \frac{\partial L_\ell}{\partial \ell(x)} + \lambda \frac{\partial}{\partial \ell(x)} B^{-\ell(x)}$

$$= p(x) + \lambda \frac{\partial}{\partial \ell(x)} e^{\ln(B^{-\ell(x)})}$$
$$= p(x) + \lambda \frac{\partial}{\partial \ell(x)} e^{-\ell(x) \ln B}$$
$$= p(x) - \lambda \ln B\, e^{-\ell(x)\ln B}$$
$$= p(x) - \lambda \ln B\, B^{-\ell(x)}$$

- solve for $\ell(x)$:

$$\ell(x) = -\log_B\left(\frac{p(x)}{\lambda \ln B}\right)$$
$$= -\log_B p(x) + \alpha \qquad \alpha = \log_B(\lambda \ln B)$$

- obtain λ from constraint:

$$1 = \sum_{x\in\mathfrak{X}} B^{-\ell(x)} = B^{-\alpha} \underbrace{\sum_{x\in\mathfrak{X}} p(x)}_{=1} \Rightarrow \alpha = 0$$

Lower Bound on Expected Code Word Length L_C

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- **Solution of relaxed optimization problem $(\square)$:** $\ell(x) = \underbrace{-\log_B p(x)}_{\text{"information content of the symbol } x\text{" (under model } p \text{ and to base } B)}$
  - $L_\ell = \sum_{x\in\mathfrak{X}} p(x)\,\ell(x) = \underbrace{-\sum_{x\in\mathfrak{X}} p(x) \log_B p(x)}_{=: H_B[p] \text{ ("entropy")}}$
- Let's now restore the constraints from $(\star)$, i.e., $\ell : \mathfrak{X} \to \mathbb{N}$ must be *integer valued*.
  - **Recall:** solution L_{C_ℓ} of $(\star)$ $\geq$ solution L_ℓ of $(\square)$
  - Thus, for all *integer valued* ℓ that satisfy Kraft inequality: $L_{C_\ell} \geq H_B[p]$
- By part (a) of the KM-Theorem:

lower bound on the expected code word length L_C of any uniquely decodable B-ary symbol code C:
$$L_C \geq H_B[p]$$

Shannon Coding [Shannon, 1948]

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- **Last slide:**
  - Lower bound for uniquely decodable B-ary symbol code: $L_C \geq H_B[p] = -\sum_{x\in\mathfrak{X}} p(x) \log_B p(x)$ "the bad news"
  - We would achieve equality ($L_C = H_B[p]$) if we were able to set $\ell(x) = \underbrace{-\log_B p(x)}_{\notin \mathbb{N} \text{ (in general)}}\ \forall x \in \mathfrak{X}$.
- **Question:** How closely can we approach this bound?
- **Idea:** choose $\ell_S : \mathfrak{X} \to \mathbb{N}$ as follows: $\ell_S(x) = \lceil -\log_B p(x) \rceil$ — $\lceil \cdots \rceil$ denotes rounding up to nearest integer
  - Satisfies Kraft inequality: $\sum_{x\in\mathfrak{X}} B^{-\ell_S(x)} = \sum_{x\in\mathfrak{X}} B^{-\lceil -\log_B p(x) \rceil} \leq \sum_{x\in\mathfrak{X}} B^{\log_B p(x)} = \sum_{x\in\mathfrak{X}} p(x) = 1$
- **By part (b) of KM-Theorem:** $\exists$ B-ary prefix code C_S with $|C_S(x)| = \ell_S(x)\ \forall x \in \mathfrak{X}$.
  - $L_{C_S} = \sum_{x\in\mathfrak{X}} p(x)\,\ell_S(x) = \sum_{x\in\mathfrak{X}} p(x) \lceil -\log_B p(x) \rceil < \sum_{x\in\mathfrak{X}} p(x)\big(-\log_B p(x) + 1\big) = H_B[p] + 1$
  - in short: $L_{C_S} < H_B[p] + 1$ "the good news"

Symmary: Theoretical Bounds for symbol codes

- **The Bad News:** no (uniquely decodable B-ary) symbol code can have an expected code word length smaller than the entropy $H_B[p]$ of a symbol.
- **The Good News:** one can always approach this lower bound with less than 1 bit of overhead *per symbol* (e.g., by using the *Shannon code* C_S).
- Thus, the *optimal* code C_{opt} (that minimizes L_C) satisfies:

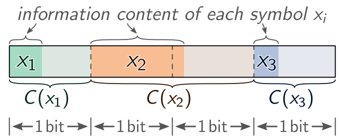
$$\boxed{H_B[p] \leq L_{C_{\text{opt}}} < H_B[p] + 1}$$

(but this requires that $|C(x')| > -\log_B p(x')$ for some $x' \neq x$, see discussion of KM-theorem)

- **Note:** The above bounds are *in expectation over all symbols $x \in \mathfrak{X}$*.
  - For any *specific* symbol $x \in \mathfrak{X}$, a code C can "violate the lower bound": $|C(x)| < -\log_B p(x)$.
  - But: *Shannon code* satisfies $-\log_B p(x) \leq |C_S(x)| < -\log_B p(x) + 1$ *for each individual $x \in \mathfrak{X}$*.

The Source Coding Theorem [Shannon, 1948]

- **So far:** theoretical bounds for *symbol codes*: $\boxed{H_B[p] \leq L_{C_{\text{opt}}} < H_B[p] + 1}$
- **Symbol codes are suboptimal.**
  - Always generate an *integer* number of bits per symbol.
  - Thus, overhead of up to 1 bit applies *per symbol*.



- **Practical solution:** stream codes (Lectures 5 and 6)
- **For theoretical analysis:** consider entire message $\mathbf{x} \in \mathfrak{X}^*$ as a single symbol.
  - New alphabet $\mathfrak{X}^*$ is still *countable*, thus theorems still apply.
  - Probability distribution p^* on $\mathfrak{X}^*$ can be complicated, but we'll assume it has a finite entropy $H_B[p^*] = -\sum_{\mathbf{x} \in \mathfrak{X}^*} p^*(\mathbf{x}) \log_B p^*(\mathbf{x})$.

infinitely large; and even if we set a maximum message length k then $|\mathfrak{X}^k| = |\mathfrak{X}|^k$ is still exponentially large ⇒ Huffman coding directly on this space would be astronomically expensive.

⇒ The optimal uniq. dec. code C_{opt} on $\mathfrak{X}^*$ (typically *not* a symbol code on $\mathfrak{X}$) satisfies: (⇒ need stream codes)

$$\boxed{H_B[p^*] \leq \text{expected bit rate of } C_{\text{opt}} < H_B[p^*] + 1}$$

Outlook

- **Problem Set 2:**
  - simple examples of Shannon coding
  - entropy and information content
- **Next week (on zoom!):**
  - proof of optimality of Huffman coding
  - machine-learning models for lossless compression (continued in Lectures 4 and 7-9)
- **Lectures 5 & 6:** beyond symbol codes: stream codes
- **Lecture 11:** theoretical bounds for *lossy* compression ("Rate/Distortion Theory")