EBERHARD KARLS UNIVERSITÄT TÜBINGEN

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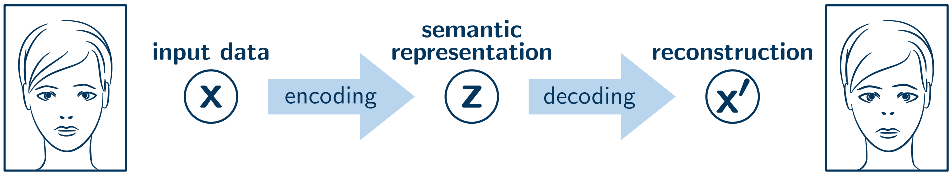
Lecture 10:

Losy Compression: From VAEs to Rate/Distortion Theory

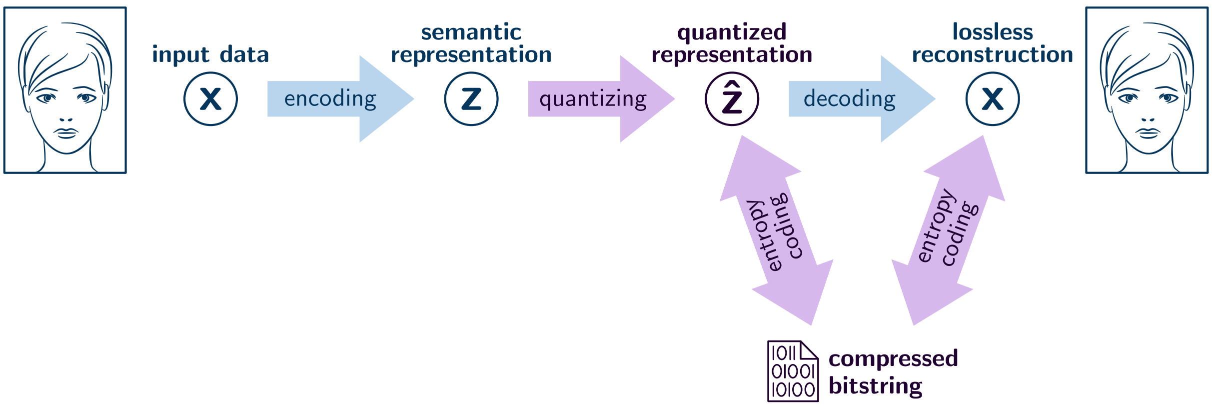
Robert Bamler · Summer Term of 2023

These slides are part of the course *"Data Compression With and Without Deep Probabilistic Models"* taught at University of Tübingen. More course materials—including video recordings, lecture notes, and problem sets with solutions—are publicly available at `https://robamler.github.io/teaching/compress23/`.

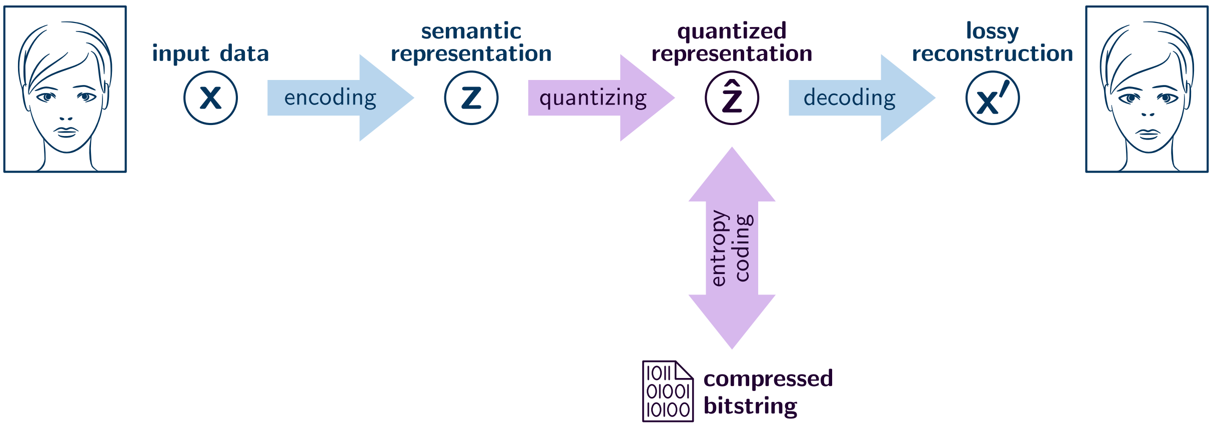
Recall: Variational Autoencoder (VAE)



Problem Set 9: Lossless Compression With a VAE



Today: *Lossy* Compression With a VAE



VAEs for Lossy (Image) Compression

- **Example:** data $\mathbf{X} = (X_{i,j,k})$ are color images.
  - $(i, j, k) =$ (x-position, y-position, red/green/blue); $X_{i,j,k} \in [0, 1]$ is a (continuous) RGB value
  - Likelihood is again parameterized by a neural network g_θ: $P_\theta(\mathbf{X} \,|\, \mathbf{Z}) = \prod_i P_\theta(X_{i,j,k} \,|\, \mathbf{Z})$
  - This time: Gaussian likelihood, i.e., density function $p_\theta(x_{i,j,k} \,|\, \mathbf{Z} = \mathbf{z}) = \mathcal{N}(x_{i,j,k}; g_\theta(\mathbf{z})_{i,j,k}, \frac{1}{\beta} I)$
- **Idea:** just use $g_\theta(\mathbf{z})$ as the reconstruction of an image.
  (Don't bother using the likelihood $P_\theta(\mathbf{X} \,|\, \mathbf{Z} = \mathbf{z})$ to encode the true image.)
- $\text{ELBO}_\beta(\theta, \phi, \mathbf{x}) =$

Quantizing Latent Space

- Latents $\mathbf{z} \in \mathbb{R}^d$ are *continuous* $\Longrightarrow$ can't be entropy coded
- Problem 9.2: for *lossless* compression with bits-back coding,
  we can simply quantize $\mathbf{z}$ to an arbitrarily fine grid.
  $D_{\text{KL}}\big(Q_\phi(\mathbf{Z} \,|\, \mathbf{X} = \mathbf{x}) \,\big\|\, P_\theta(\mathbf{Z})\big) = \mathbb{E}_{Q_\phi(\mathbf{Z} \,|\, \mathbf{X} = \mathbf{x})}\big[\log Q_\phi(\mathbf{Z} \,|\, \mathbf{X} = \mathbf{x}) - \log P_\theta(\mathbf{Z})\big]$
  - bit rate for encoding $\hat{\mathbf{z}} = \lceil \mathbf{z} \rfloor_\delta$ with quantized prior $P_\theta(\hat{\mathbf{Z}})$:
  - bit rate for decoding $\hat{\mathbf{z}}$ with quantized var. dist. $Q_\phi(\hat{\mathbf{Z}} \,|\, \mathbf{X} = \mathbf{x})$:

  $\Longrightarrow$ for $\delta \to 0$, expected net bit rate is $D_{\text{KL}}\big(Q_\phi(\mathbf{Z} \,|\, \mathbf{X} = \mathbf{x}) \,\big\|\, P_\theta(\mathbf{Z})\big)$ (independent of δ)
- **Problem:** bits-back coding does not work (out of the box) for lossy compression.
  - Receiver never recovers the exact message $\mathbf{x}$ $\Longrightarrow$ can't encode $\hat{\mathbf{z}}$ with $Q_\phi(\hat{\mathbf{Z}} \,|\, \mathbf{X} = \mathbf{x})$.
  - Thus, bit rate would depend on how fine we make the grid.

Simplest Solution: Uniform Quantization [Ballé et al., 2017]

- **Idea:** take quantization into account already during training.
  - **Goal:** model should learn to encode important information on length scales $\geq \delta$.
  - **Problem:** quantization $\mathbf{z} \mapsto \hat{\mathbf{z}} := \lfloor \mathbf{z} \rceil_\delta$ is not differentiable.
  - **Observation:** $(\hat{\mathbf{z}} - \mathbf{z}) \in \left[-\frac{\delta}{2}, \frac{\delta}{2} \right]^d$ and (empirically) approximately uniformly distributed.
- **Proposal:** at training, replace quantization by adding uniform noise $\boldsymbol{\epsilon} \sim \mathcal{U}\Big(\big[-\frac{\delta}{2}, \frac{\delta}{2}\big]^d\Big)$
  - Equivalent to using a box-shaped variational distribution with fixed width:
  - δ is fixed (i.e., data-independent)

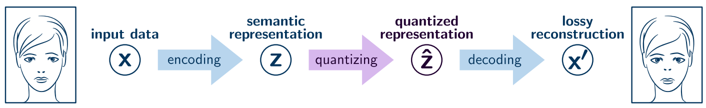
    $\Longrightarrow$ might as well set $\delta = 1$ as long as the prior $P_\theta(\mathbf{Z})$ does not impose any fixed length scale.

    $\Longrightarrow$ at deployment, encode each component $\hat{z}_i := \lfloor z_i \rceil_{\mathbb{Z}}$ with model $P_\theta(\hat{Z}_i = \hat{z}_i) =$

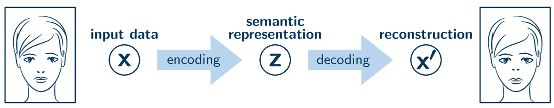
    $\Longrightarrow$ bit rate $P_\theta(\hat{\mathbf{Z}} = \hat{\mathbf{z}}) =$

Model Training: Rate/Distortion Trade-Off

- **In deployment:**



  - quantized latent representation: $\hat{\mathbf{z}} := \lfloor \mathbf{z} \rceil = \lfloor f_\phi(\mathbf{x}) \rceil$
  - bit rate: $-\log P(\hat{\mathbf{Z}} = \hat{\mathbf{z}}) = -\log \left(\int_{\mathbf{z} \in [\hat{\mathbf{z}} - \frac{1}{2}, \hat{\mathbf{z}} + \frac{1}{2}]} p(\mathbf{z}) \, \mathrm{d}\mathbf{z} \right)$
  - reconstructed message: $\mathbf{x}' = g_\theta(\hat{\mathbf{z}})$
- **At training time:**

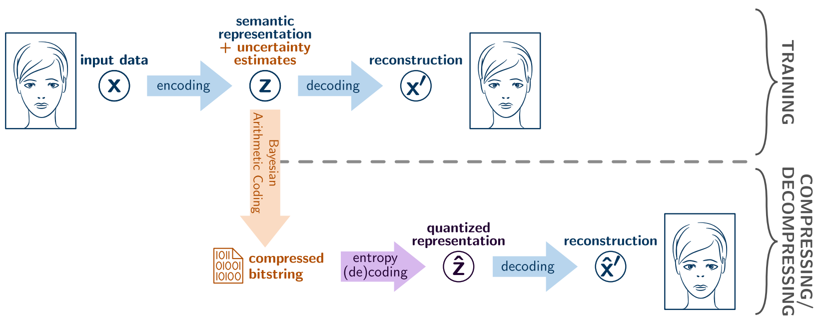


  - added uniform noise: $\mathbf{z} = f_\phi(\mathbf{x}) + \boldsymbol{\epsilon}$ where $\boldsymbol{\epsilon} \sim \mathcal{U}\big(\big[-\frac{1}{2}, \frac{1}{2}\big]^d\big)$
  - approximated bit rate: $\mathcal{R}(\theta, \phi, \mathbf{x}) := -\log \tilde{p}(\mathbf{z})$ where $\tilde{p}(\mathbf{z}) := \int_{\mathbf{z}' \in [\mathbf{z} - \frac{1}{2}, \mathbf{z} + \frac{1}{2}]} p(\mathbf{z}') \, \mathrm{d}\mathbf{z}'$
  - reconstruction error (distortion): e.g., MSE: $\mathcal{D}(\theta, \phi, \mathbf{x}) := \big\| g(\mathbf{z}) - \mathbf{x} \big\|^2$
  - loss function: rate/distortion trade off: $\mathcal{L}_\beta(\theta, \phi, \mathbf{x}) = \mathbb{E}_{\boldsymbol{\epsilon}}\big[\beta\, \mathcal{R}(\theta, \phi, \mathbf{x}) + \mathcal{D}(\theta, \phi, \mathbf{x})\big]$

    $\longrightarrow$ Problem 10.1: $\exists$ probabilistic model such that $\mathcal{L}_\beta(\theta, \phi, \mathbf{x}) \propto -\mathrm{ELBO}(\theta, \phi, \mathbf{x}) + \text{const}$

Limitations of Uniform Quantization

- quantization gap: rounding $\neq$ adding uniform noise.
  - various proposals exist for better quantization at training time (Lecture 12)
- rate/distortion trade-off β must be set *at training time*.
  - less studied in the literature.
  - Variational Inference can help: [Yang et al., 2020, Tan & Bamler, 2022]



Lecture 10:

Lower Bound on the Bit Rate of Lossy Compression

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Recall: Source Coding Theorem

- optimal expected bit rate of *lossless* compression: entropy $H[\mathbf{X}]$
- we proved that $H[\mathbf{X}]$ is both:
  - a lower bound: $\mathbb{E}[\text{bitrate}(\mathbf{X})] \geq H[\mathbf{X}] = \mathbb{E}[-\log P(\mathbf{X})] \quad \forall$lossless codes
  - achievable with negligible overhead: $\exists$ lossless code : $\text{bitrate}(\mathbf{x}) \geq -\log P(\mathbf{x}) + 1 \quad \forall \mathbf{x}$
- *Lossy* compression can have bit rates $< H[\mathbf{X}]$.
  - today and problem set: lower bound
  - next week: achievability of lower bound

Lower bound on the Bit Rate of Lossy Compression

- Encoder/decoder form a *Markov chain:*
- Problem 10.2: data processing inequality: $\forall$ Markov chains:
- Lower bound on bit rate: