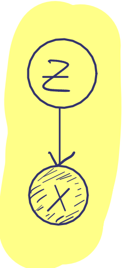
Variational Autoencoders & Lossy Neural Compression

Lecture 8 (23 June 2022); lecturer: Robert Bamler
more course materials online at https://robamler.github.io/teaching/compress22/

Recap from last lecture: Variational Inference (VI)



- latent variable model: $P(Z, X) = P(Z)\, P(X|Z)$
- goal: approximate the posterior: $P(Z|X{=}x) = \dfrac{P(Z)\, P(X{=}x|Z)}{\int P(Z{=}z)\, P(X{=}x|Z{=}z)\, dz}$

→ VI turns the inference problem into an optimization problem

→ variational distribution: $\boxed{Q_\phi(Z|X{=}x)}$ ← variational parameters

Evidence lower bound (ELBO):

- $\mathrm{ELBO}(\phi) = -\mathbb{E}_{Q_\phi(Z|X=x)}\big[\tilde{R}^{(z)}_{\mathrm{net}}(x)\big]$ ← how we motivated it

Problem Set 8

$$= \mathbb{E}_{Q_\phi(Z|X=x)}\Big[\log P(Z, X{=}x) \underbrace{- \log Q_\phi(Z|X{=}x)}_{\to H[Q_\phi(Z|X=x)]}\Big]$$ ← most explicit formulation ("regularized MAP")

$$= \mathbb{E}_{Q_\phi(Z|X=x)}\big[\log P(X{=}x|Z)\big] - D_{KL}\big(Q_\phi(Z|X{=}x)\,\|\,P(Z)\big)$$ ← "regularized maximum likelihood"

$$= \log P(X{=}x) - D_{KL}\big(Q_\phi(Z|X{=}x)\,\|\,P(Z|X{=}x)\big)$$ ← connects VI to Bayesian inference

How to maximize the ELBO with stochastic gradient optimization:

- reparameterization gradients: $\nabla_\phi \mathbb{E}_{Q_\phi(Z|X=x)}\big[\ell(Z, \phi)\big] = \nabla_\phi \mathbb{E}_{\epsilon\sim Q_0}\big[\ell(g(\epsilon,\phi), \phi)\big]$

  $Z = g(\epsilon, \phi)$ where $\epsilon \sim Q_0$ ← fixed distribution

- score function gradients: $\nabla_\phi \mathbb{E}_{Q_\phi(Z|X=x)}\big[\ell(Z,\phi)\big] = \mathbb{E}_{Q_\phi(Z|X=x)}\big[(\nabla_\phi \log Q_\phi(Z|X{=}x)) \times \ell(Z,\phi)\big]$
  (= REINFORCE method)

Limitations so far:

TODAY: overcoming these limitations

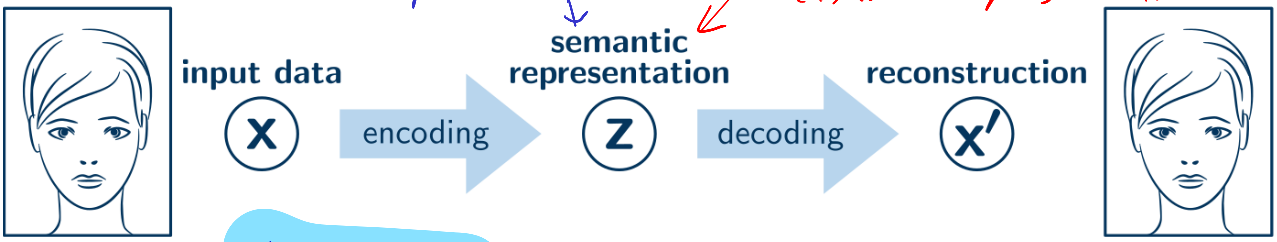
(i) the generative model P(Z, X) is fixed -- and therefore limited to simple models that we can come up with manually; and → Variational Expectation Maximization (Variational EM) → "learn the prob. gen. model P from training data"

(ii) for every concrete message x that we want to compress, we have to run an expensive optimization procedure to find the optimal variational parameters ϕ^*. → Amortized variational inference "learn how to do inference"

Spoiler: Variational Autoencoders (VAEs) = amortized variational expectation maximization

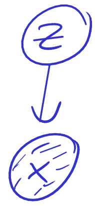
prior $P(Z)$

decorrelated representation ⇒ easier to compress than x



approximate Bayesian inference (VI)

probabilistic generative model



Variational Expectation Maximization: learning a latent variable model

[Beal & Ghahramani, Bayesian statistics, 2003]

Introduce free parameters into the probabilistic model P(Z, X):

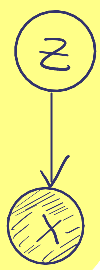
$$P_\theta(Z, X) = P_\theta(Z)\, P_\theta(X \mid Z)$$

model parameters θ

$$\mathbb{E}_{x\sim\text{train set}}\left[\log P_\theta(X=x)\right]$$

$$= \mathbb{E}_{x\sim\text{train set}}\left[\log\left(\sum_z P_\theta(Z=z, X=x)\right)\right]$$

prohibitively expensive

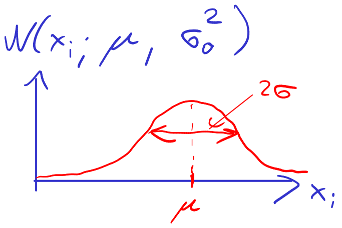


For example, the likelihood could be parameterized by a neural network with weights θ:

$$\underline{P_\theta(X \mid Z=z)} = \underbrace{\mathcal{N}\big(\overbrace{g_\theta(z)}^{\mu},\ \sigma_0^2 I\big)}$$

"a normal distribution with mean $g_\theta(z)$ and variance σ_0^2 in each coordinate direction"

where g_θ : latent space (z) → data space (x) is a neural network with trainable weights θ



Thus, the ELBO now depends both on the variational parameters ϕ and on the model parameters θ:

$$\mathrm{ELBO}(\theta, \phi, x) = -\mathbb{E} \quad R_{net}$$

$$= \mathbb{E}\quad\left[\log P - \log Q\right]$$

$$= \underbrace{\log P_\theta(X=x)}_{-\text{optimal bitrate with model } P_\theta} \underbrace{- D_{KL}}_{\text{overhead due to VI}}$$

- maximize over both of these jointly (at training time)
- at compression time: keep θ fixed & maximize only over ϕ

minimize the expected net bitrate of modified bits-back

$$\Rightarrow \text{maximize: } \mathrm{ELBO}(\phi, \vartheta) = -\,\mathbb{E}_{Q_\phi(z|x=x)}\left[\tilde{R}^{(z,\vartheta)}_{net}(x)\right]$$

ϕ: variational params — ϑ: model params

Alternative:

- store ϕ_x $\forall x \in$ train set on disk
- training loop:

for training_step in $\{1, 2, 3, \ldots, n\}$:

sample $x \sim$ train set

look up ϕ_x on disk

calculate $g_\vartheta = \nabla_\vartheta \, \mathrm{ELBO}(\vartheta, \phi_x, x)$

$g_\phi = \nabla_{\phi_x} \, \mathrm{ELBO}(\vartheta, \phi_x, x)$

update $\vartheta \leftarrow \vartheta + \rho\, g_\vartheta$

$\phi \leftarrow \phi + \rho\, g_\phi$

store ϕ_x back to disk

Data compression with learnt latent variable models (try 1: without amortization):

1) When designing the compression method:
   - collect large (unlabeled) data set of training samples (e.g., a large collection of images)
   - come up with a model architecture for the generative model P_θ (that still has free parameters)
   - train the model by maximizing the ELBO jointly over both θ and ϕ.

     in detail: $\theta^*, \{\phi_x^*\} := \arg\max_{\theta, \{\phi_x\}} \mathbb{E}_{x \sim \text{trainset}}\left[\mathrm{ELBO}(\theta, \phi_x, x)\right]$

     ϕ_x are the variational params for training point x

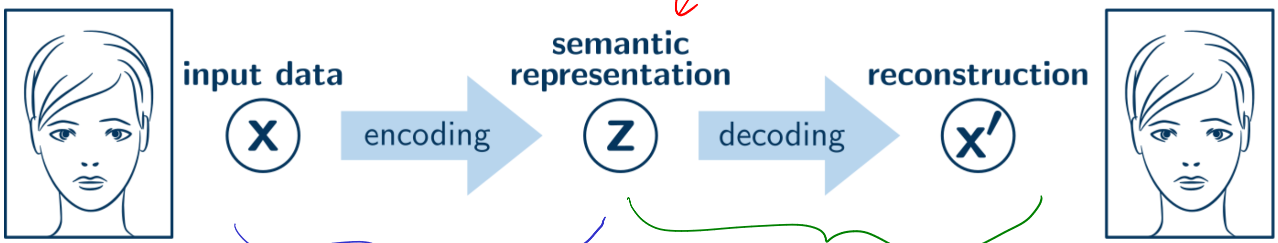
     for training step $\in \{1, 2, \ldots, N\}$:
     find $\phi_x^* := \arg\max \mathrm{ELBO}(\theta, \phi_x)$
     set $\hat{g} := \nabla_\theta \mathrm{ELBO}(\ldots)$
     update $\theta \leftarrow \theta + \rho \hat{g}$

   - throw away $\{\phi_x^*\}$ and share θ* between sender and receiver

2) When compressing some given data x (i.e., on the sender side):
   - perform variational inference, i.e., maximize ELBO(θ*, ϕ, x) over ϕ but keep θ* fixed at the agreed-upon values.
   - use probabilistic generative model P_{θ^*}(Z, X) and the resulting variational distribution Q_{ϕ^*}(Z | X=x) to compress x.

3) When decompressing data (i.e., on the receiver side):
   - needs the exact same probabilistic generative model P_{θ^*}(Z, X) that the sender used for comprestion.
   - if the data was compressed with bits-back coding, then the receiver also needs to perform variational inference once it has reconstructed x (i.e., maximize ELBO(θ, ϕ) over ϕ but keep θ^*.

compress z using prior $P_{\theta^*}(Z)$



$Q_{\phi^*}(Z|X=x)$

where $\phi^* := \underbrace{\arg\max_\phi}_{\text{expensive operation}} \mathrm{ELBO}(\phi, \theta^*, x)$

$P_{\theta^*}(X|Z)$

where θ^* is known at (de-)compression time

Amortized Variational Inference: learn how to do inference

Variational inference maps data x to a variational distribution Q_{ϕ^*}(Z | X=x):

Idea: learn this mapping from x to Q_{ϕ^*}(Z | X=x):

   - rename the parameteres of Q_λ(Z) from ϕ to λ

     for example: $Q_\lambda(Z) = \mathcal{N}(\mu, \mathrm{diag}(\sigma^2))$
     $\lambda = (\mu, \sigma^2)$

   - rather than optimizing over λ, learn a function f that maps x to λ (and that is parameterized by some neural network weights ϕ:

$\lambda = f_\phi(x)$ where f_ϕ is a neural network with weights ϕ

$\boxed{Q_\phi(Z|X=x) := Q_\lambda(Z) \text{ where } \lambda = f_\phi(x)}$

⇒ variational parameters ϕ are now shared between all data points x

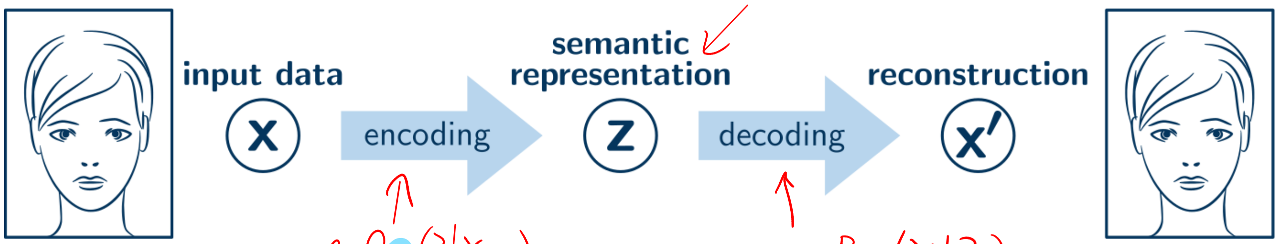
$$\mathrm{ELBO}(\phi, \theta, x) = \mathbb{E}_{Q_\phi(Z|X=x)}\left[\log P(Z, X=x) - \log Q_\phi(Z|X=x)\right]$$

Combining amortized inference with variational expectation maximization results in:

Variational Autoencoers (VAEs):

prior $P_{\theta^*}(Z)$

[Kingma and Welling, 2015]



use $Q_{\phi^*}(Z|X=x)$

use $P_{\theta^*}(X|Z)$

both ϕ^* & θ^* are fixed once the model is deployed

training objective: E[ELBO] = ...

Problem Set: implement a variational autoencoder for simple images (MNIST)

Note: Variational expectation maximization (EM) is not limited to VAEs. Even without amortized inference, variational EM is a very useful algorithm that is very simple and allows you to treat some model parameters (Z) probabilistically while using point estimates for others (θ).