Lecture 5, Part 1:

# Stream Codes: Encoding Into Fractional Bits

Robert Bamler · Summer Term of 2023

These slides are part of the course *"Data Compression With and Without Deep Probabilistic Models"* taught at University of Tübingen. More course materials—including video recordings, lecture notes, and problem sets with solutions—are publicly available at `https://robamler.github.io/teaching/compress23/`.

## Recall: 4 Kinds of Scalable Probabilistic Models

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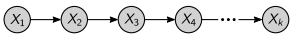
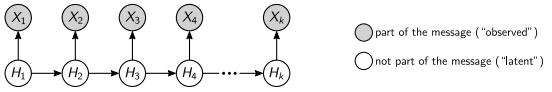
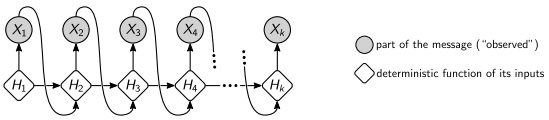
### (1) Markov Process



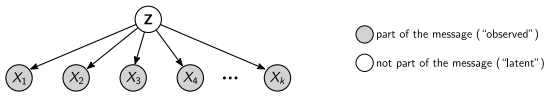

### (2) Hidden Markov Model




### (3) Autoregressive Model




### (4) Latent Variable Model






## Recall: Autoregressive Model + Huffman Coding

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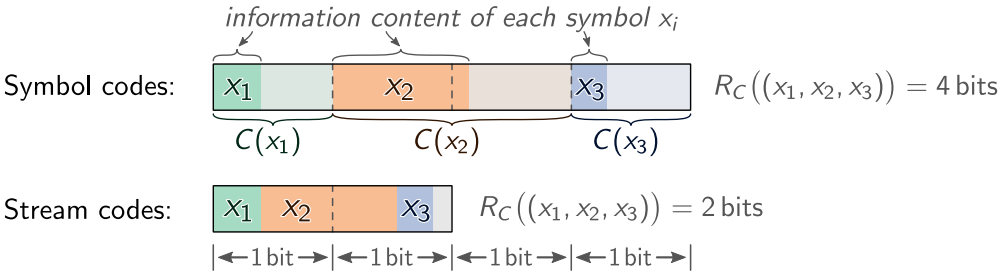
- **Problem 3.3:** (link to solutions in video description)

| | msg. len | bits per character | | | | | |
|---|---|---|---|---|---|---|---|
| | (chars) | Huffman | Shannon | inf. cont. | `gzip` | `bzip2` | `bzip2'` |
| validation set | 106,864 | **2.38** | 2.72 | 2.12 | 3.43 | 2.82 | 2.40 |
| test set | 219,561 | **2.38** | 2.73 | 2.12 | 3.33 | 2.65 | **2.38** |
| wikipedia-en | 24,618 | 4.99 | 5.67 | 5.14 | 3.22 | **2.92** | 5.14 |
| wikipedia-de | 8,426 | 6.77 | 7.70 | 7.19 | 3.96 | **3.76** | 7.22 |

  - **Observation:** Huffman coding has overhead over information content of up to 1 bit *per symbol*.
  - Can be substantial in modern ML-based compression methods:
    e.g., information content $\approx 0.3$ bits per symbol;
    but Huffman coding needs $\geq 1$ bit per symbol.
    $\Big\} \implies$ about 200 % overhead.

- **Solution:** *amortize* compressed bits over symbols $\longrightarrow$ *"stream code"*

# Stream Codes: Amortizing Bits Over Symbols




- **Intuitively:** stream codes "pack" information content as closely as possible.
  - We can no longer associate each bit in the compressed representation with any specific symbol.
- **2 important stream codes** with 2 different application domains:
  - **This lecture:** Arithmetic Coding & Range Coding [Pasco, 1976; Rissanen and Langdon, 1979]
  - **Next lecture:** Asymmetric Numeral Systems (ANS) [Duda et al., 2015]



# Arithmetic Coding: Overview [Pasco, 1976]

- **Idea:** similar to Shannon coding, but on entire *message space* $\mathcal{X}^*$ instead of alphabet $\mathcal{X}$.
  - **Thus:** overhead now only *per message* (up to 2 bit).
  - **Challenge:** computational efficiency
- **2 variants:** Arithmetic Coding & Range Coding
  - very similar to each other (Range Coding is faster on real hardware);
  - both conceptionally simple;
  - but a bit tricky in practice due to edge cases.



# Reminder: Shannon Coding (for $B = 2$)

**Input:** alphabet $\mathcal{X}$, probability measure $P$ on $\mathcal{X}$; **output:** prefix code $C_S : \mathcal{X} \to \{0,1\}^*$.

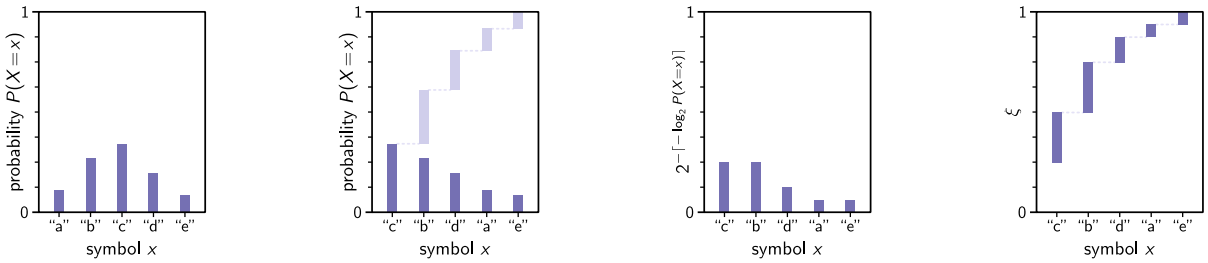
Initialize $\xi \leftarrow 1$

**for** $x \in \mathcal{X}$ *in order of increasing* $P(X=x)$ **do**

  Update $\xi \leftarrow \xi - 2^{-\lceil -\log_2 P(X=x) \rceil}$

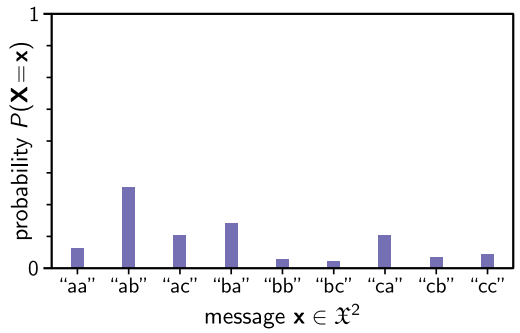
  Write out $\xi \in [0,1)$ in binary: $\xi = (0.????\ldots)_2$

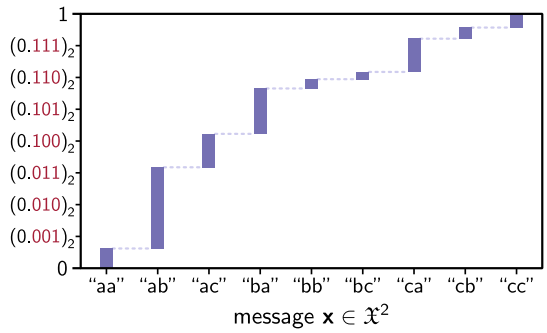
**end**



# Arithmetic Coding: General Idea

- Consider probability measure $P$ on *entire message space* $\mathcal{X}^k$ (with fixed length $k$, for now).




- **Observation:** $\forall \mathbf{x} \in \mathcal{X}^k$: interval $\big[P(\mathbf{X}<\mathbf{x}), P(\mathbf{X}\leq\mathbf{x})\big)$ contains a point $\xi_{\mathbf{x}} = (0.\underbrace{????}_{\mathcal{R}(\mathbf{x}) \text{ bits}})_2$ if:
  size of interval $\leq$ spacing between numbers of form $(0.\underbrace{????}_{\mathcal{R}(\mathbf{x}) \text{ bits}})_2$



# Arithmetic Coding: Super Naive Algorithm

- **Encoder:**
  Initialize $c \leftarrow 0$ and $p \leftarrow 1$.
  **for** $i$ **from** 1 **to** $k$ **do**
  Update $c \leftarrow c + p\,P(X_i < x_i \,|\, \mathbf{X}_{1:i-1} = \mathbf{x}_{1:i-1})$.
  Update $p \leftarrow p\,P(X_i = x_i \,|\, \mathbf{X}_{1:i-1} = \mathbf{x}_{1:i-1})$.
  **end**
  // ***Claim:*** *at this point,* $c = P(\mathbf{X}<\mathbf{x})$ *and* $p = P(\mathbf{X}=\mathbf{x})$
  Encode some $\xi \in [c, c+p)$ in binary: $\xi = (0.\underbrace{?????}_{\lceil -\log_2 p \rceil \text{ bits}})_2$
- **Decoder:** Analogous to encoder, but introduce an extra steps.



# Caveats

- **Unique Decodability:**
  - not such a big deal as it was with symbol codes
    (it's unusual to concatenate entire *messages* without deliminators);
  - can be solved with 1 extra bit: guarantee that $[\xi, \xi + 2^{-\mathcal{R}(\mathbf{x})}) \subseteq [c, c+p]$
- **Variable message length:**
  - end of bit string $\not\Longleftrightarrow$ end of message (since symbols can have information content $< 1$ bit)
  - for variable-length messages, the message length is fundamentally a *part of the message*
  - simple solution: introduce EOF symbol (→ Problem Set 7)
- **Numerical precision:**
  - e.g, if bit rate = 1 Mbit then $c$ and $p$ on last slide are 1-million-bit numbers
  - Run-time complexity for encoding $k$ symbols: $\Theta(k^2)$

# Arithmetic Coding: Naive Algorithm

- **Encoder:**
  Initialize $c \leftarrow 0$ and $p \leftarrow 1$ (fixed point numbers $\in [0,1]$ with `precision` bits);
  **for** $i$ **from** 1 **to** $k$ **do**
    Update $c \leftarrow c + p\,P(X_i < x_i \mid \mathbf{X}_{1:i-1} = \mathbf{x}_{1:i-1})$; *(rounding to fixed point precision)*
    Update $p \leftarrow p\,P(X_i = x_i \mid \mathbf{X}_{1:i-1} = \mathbf{x}_{1:i-1})$; *(rounding to fixed point precision)*
    **while** $p < \frac{1}{2}$ **do**
      Emit first bit of $c = (0.????)_2$;
      Update $p \leftarrow 2p$;
      Update $c \leftarrow$ fractional part of $2c$;
    **end**
  **end**
  Emit first bit of $c = (0.????)_2$;
- **Decoder:** exercise



# Arithmetic Coding: Actual Algorithm

Initialize $c \leftarrow 0$ and $p \leftarrow 1$ (fixed point numbers $\in [0,1]$ with `precision` bits);
Initialize `inverted` $\leftarrow 0$ (nonnegative integer);
**for** $i$ **from** 1 **to** $k$ **do**
  Update $c \leftarrow c + p\,P(X_i < x_i \mid \mathbf{X}_{1:i-1} = \mathbf{x}_{1:i-1})$;
  Update $p \leftarrow p\,P(X_i = x_i \mid \mathbf{X}_{1:i-1} = \mathbf{x}_{1:i-1})$;

  **while** $p < \frac{1}{2}$ **do**
    ~~Emit first bit of $c = (0.????)_2$;~~

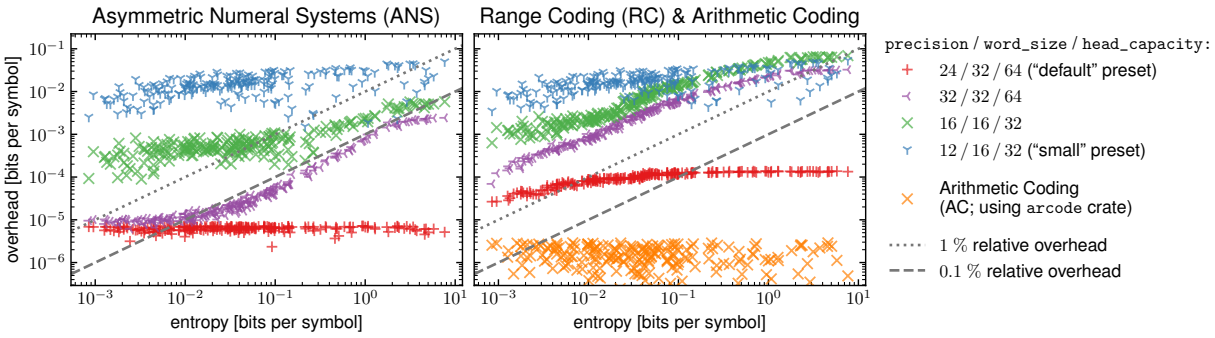
    Update $p \leftarrow 2p$;
    Update $c \leftarrow$ fractional part of $2c$;
  **end**
**end**
~~Emit first bit of $c = (0.????)_2$;~~



# Real Hardware: Range Coding

- CPUs are not optimized to operate on single bits
- *mechanical sympathy:* to best exploit the capabilities of a tool (e.g., a computer), one has to understand how the tool works.
- **Range Coding:** like arithmetic coding, but operating on `precision` bits at a time
  - accumulators $c$ and $p$ become numbers with $2\times$`precision` bits
  - individual symbol probabilities $P(X_i = x_i \mid \mathbf{X}_{1:i-1} = \mathbf{x}_{1:i-1}) \in (0,1)$ are `precision`-bit numbers
    $\Rightarrow P(X_i = x_i \mid \mathbf{X}_{1:i-1} = \mathbf{x}_{1:i-1}) \geq 2^{-\texttt{precision}}$ (smallest representable nonzero number)
  - Emit `precision` bits at once when $p < 2^{-\texttt{precision}}$
    $\Rightarrow$ always restores $p \geq 2^{-\texttt{precision}}$, thus at most 1 emission per symbol is necessary.
  - `inverted` keeps track of how many "$\underbrace{00000000}_{\texttt{precision}}$" or "$\underbrace{11111111}_{\texttt{precision}}$" blocks have accumulated.

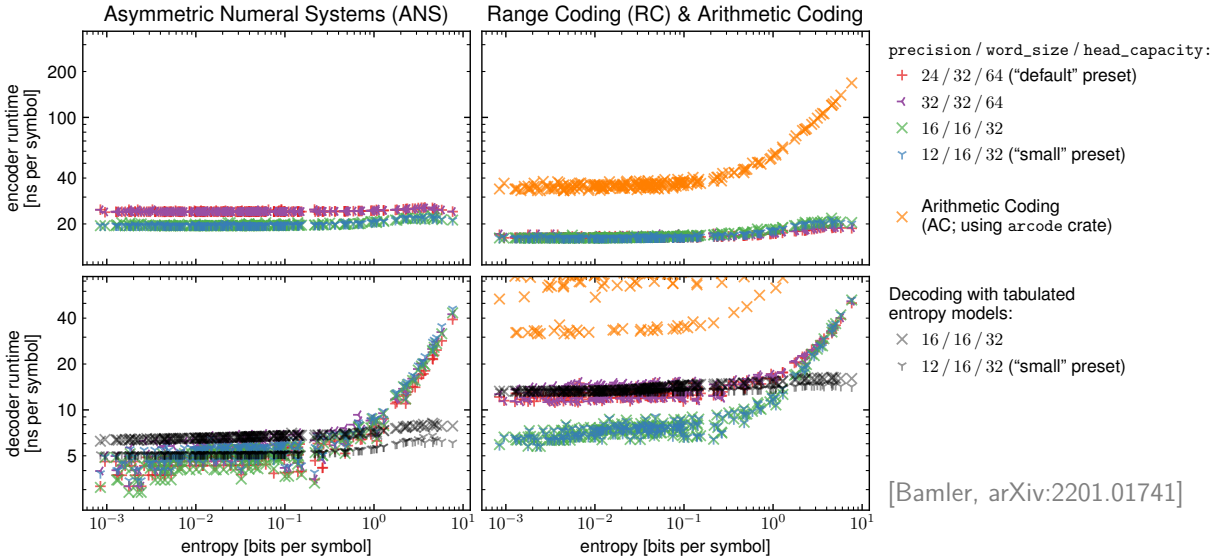
# Empirical Compression Performances: bit rates




[Bamler, arXiv:2201.01741 (2022)]



# Empirical Compression Performances: run times




[Bamler, arXiv:2201.01741]



# Outlook

- **Next week (Lecture 6):** Asymmetric Numeral Systems
  - modern stream code that operates as a *stack* ("last-in-first-out")
  - conceptionally more difficult but easier to implement in real code
  - uses "bits-back trick"
  - enables "bits-back trick" for latent variable models (→ Lecture 7)
- **Problem Set 7:** *use* range coding (from a library) for our autoregressive model of natural language from Problem Set 3.