Specifying models – fixed and random effects

Rob Davies

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Aims for the class

1. Understand the use of *Information Criteria* and *Likelihood Ratio Tests* for evaluating models
2. Practise running linear mixed-effects models with varying fixed or random effects structures
3. Practise evaluating models

Linear mixed-effects models – Model comparison approach

We focus on building a series of models from the simplest to the most complex model supported by the data

- A minimal model – observed responses can be predicted only by the average response – the intercept
- Specific response values are then random deviations from this average
- Will our our capacity to predict responses be improved by adding other terms?
- What about the experimental factors we manipulate? Or variables carrying information about test conditions, particants or stimuli?

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We can expect to deal with a trade-off between too much and too little simplicity in model specification

- Models with too many parameters may tend to identify effects that are spurious
- Effects may be unintuitive and hard to explain *and* not reproduced in future samples
- Contrastingly, if we omit key variables we will likely mis-estimate the ‘true’ population value of the effects we seek to identify

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Understood within an approach: Information-theoretic methods– for linear models we can compare the F, R^2 and information criteria indices

- Information-theoretic methods are grounded in the insight that the distance between a model and observed outcomes corresponds to the 'information lost' when you use a model to approximate reality
- Information criteria – AIC or BIC – estimates of information loss
- The process of model selection aims to minimise information loss

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Akaike showed you could estimate information loss in terms of the likelihood of the model given the data

$$AIC = -2ln(l) + 2k \tag{1}$$

- $-2ln(l)$ -2 times the log of the likelihood of the model given the data
- (l) – likelihood
  - Is proportional to the probability of observed data conditional on some hypothesis being true

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- You can link models that are more likely – closer to reality – with models with smaller residuals
- Linear models with smaller residuals would have larger R^2 – more of observed outcome (response) variance is explained by better models

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Schwartz proposed an alternative estimate

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Model evaluation: using AIC and BIC

- Example – compare models varying in fixed effects: model 1, just main effects; model 2, main effects plus interactions
- Example – compare models varying in random effects: model 1, just random effect of subjects on intercepts; model 2, random effects of subjects and items on intercepts
- If the more complex model better approximates reality then it will be more likely given the data
  - BIC or AIC will be closer to negative infinity: $-2ln(l)$ will be larger
  - e.g. 10 is better than 1000, -1000 better than -10

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Reporting standards; Using AIC and BIC

So far, I tend to see such analyses reported more in ecology, little in psychology

- Report briefly the model comparisons: "Compared a simpler model: model 1, just main effects; model 2, main effects plus interactions"
- Report the AIC or BIC for the different models
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Likelihood ratio test comparison

Likelihood ratio test comparisons appear increasingly wide-spread in psychology and are recommended for evaluating the relative utility of fixed or random effects

Baayen et al., 2008; Barr et al., 2013; Bates (LME developer list); glmm wiki dot; but see Bolker et al., 2009; Pinheiro & Bates, 2000

- The test statistic is the comparison of the likelihood of the simpler model with the more complex model
- Comparison by division $2log\frac{likelihood-complex}{likelihood-simple}$
- The likelihood ratio is compared to the χ^2 distribution for a significance test
- Assuming the null hypothesis that the simpler model is adequate
- With df equal to the difference in the number of parameters of the models being compared
- Alternatively, we can say: $deviance = deviance_{simple} - deviance_{complex}$ where $deviance = -2log(likelihood_{model})$

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Model comparisons among mixed-effects models – use *anova()* function

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anova(full.lmer0, full.lmer1)
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- `anova(...,....)` compare pairs of models named in brackets
- `full.lmer0` including a simpler model – fewer parameters (predictors)
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Running the *anova*(,) comparison will deliver AIC, BIC, and likelihood comparisons for varying models

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> anova(full.lmer1, full.lmer2)
Data: subjects.behaviour.items.nomissing
Models:
full.lmer1: logrt ~ cAge + cTOWRE_wordacc + cTOWRE_nonwordacc + cART_HRminusFR +
full.lmer1:     (1 | subjectID) + (1 | item_name)
full.lmer2: logrt ~ cAge + cTOWRE_wordacc + cTOWRE_nonwordacc + cART_HRminusFR +
full.lmer2:     item_type + cLength + cOrtho_N + cBG_Mean + (1 | subjectID) +
full.lmer2:     (1 | item_name)
           Df    AIC    BIC logLik deviance  Chisq Chi Df Pr(>Chisq)
full.lmer1  8 -17983 -17925 8999.4   -17999
full.lmer2 12 -18319 -18232 9171.3   -18343 343.81      4  < 2.2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
>
```

Figure : Comparison of model with subject attribute predictors and model also with item effects

Model comparison

- AIC, BIC and LRT comparisons should be consistent in their indications – which model to prefer
- Can be tricky where dealing with complex sets of predictors – indicators may diverge
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A more realistic repeated measures model of word frequency and reading ability effects

Taking into account random effect of subjects and items

$$RT = \beta_0 + \beta_{ability} + \beta_{frequency} + \beta_{ability*frequency} + \beta_{subject} + \beta_{item} + \beta_{subject*frequency} + \beta_{item*ability} + \epsilon \quad (4)$$

- What about effect of random variation between participants?
  - Allow intercept to vary – random effect of subject – some have slower average some have faster average than average overall
  - Allow effect of frequency to vary – random effect of subject – subjects can be affected by word frequency in different ways

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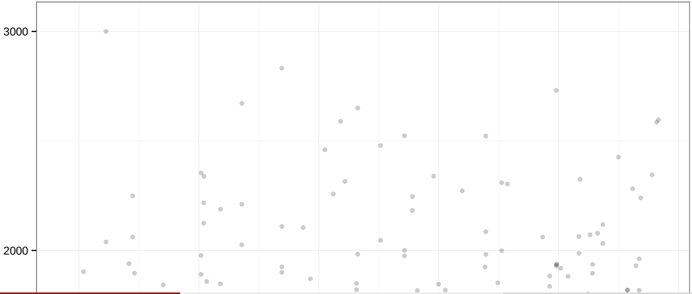
Random effect of subjects on intercepts

Differences in individuals' average response speed compared to the group average

$$Y_{ij} = \beta_0 + \beta_1 X_i + U_{0,j} + \epsilon_{ij} \tag{6}$$

- β_0 common intercept, average outcome given the other effects
- $U_{0,j}$ adjustments to the intercept required to explain differences between common average and average for each j individual



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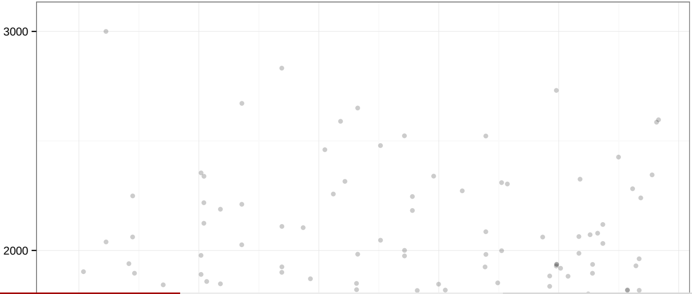
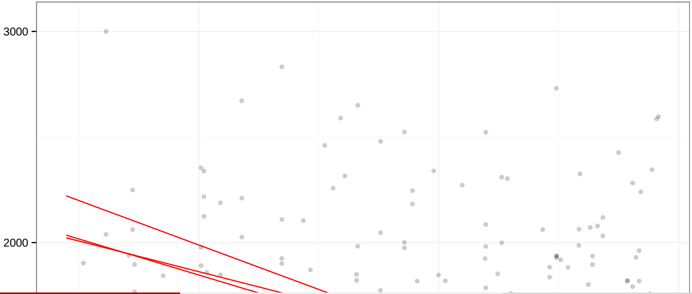


Random effect of subjects on slopes of fixed effects

Accounting for variation in *within-subjects* frequency effect on RTs

$$Y_{ij} = \beta_0 + \beta_1 X_i + U_{0,j} + U_{1,j} + \epsilon_{ij} \quad (7)$$

- $\beta_1 X_i$ the group average frequency effect
- $U_{1,j}$ adjustments required to model differences between group average frequency effect and frequency effect for each j individual

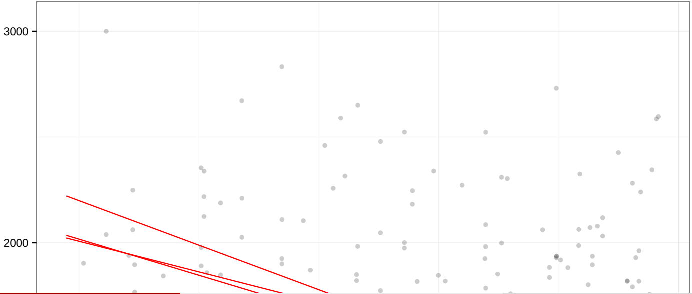


Random effect of subjects on slopes of fixed effects

Accounting for variation in *within-subjects* frequency effect on RTs

$$Y_{ij} = \beta_0 + \beta_1 X_i + U_{0,j} + U_{1,j} + \epsilon_{ij} \tag{7}$$

- $\beta_1 X_i$ the group average frequency effect
- $U_{1,j}$ adjustments required to model differences between group average frequency effect and frequency effect for each j individual



Can allow random effects of both subjects and items

Solving the 'language-as-fixed-effect' problem in one model

$$Y_{ij} = \beta_0 + \beta_1 X_i + U_{0,j} + U_{1,j} + W_{0,i} + W_{1,i} + \epsilon_{ij} \tag{8}$$

- $W_{0,i}$ random effect of items on intercepts – where W terms correspond to random effects of items
- $W_{1,i}$ may require a random effect of items on slopes of within-items effects e.g. effect of ability differences between participants

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Mixed effects models – both fixed effects and random effects

$$Y_{ij} = \beta_0 + \beta_1 X_i + U_{0,j} + U_{1,j} + W_{0,i} + W_{1,i} + \epsilon_{ij} \quad (9)$$

- $\beta_0 + \beta_1 X_i$ fixed effect of predictors – fixed because replicable by manipulation or selection
- $U_{0,j} + U_{1,j} + +W_{0,i} + W_{1,i} + \epsilon_{ij}$ random effects of group on intercepts and slopes – random because differences due to sampling

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We could model random differences between participants items as fixed effects but we do not

We incorporate random effects terms in models to capture the spread, *the variance*, associated with random differences in intercepts or slopes

$$Y_{ij} = \beta_0 + \beta_1 X_i + U_{0,j} + U_{1,j} + W_{0,i} + W_{1,i} + \epsilon_{ij} \tag{10}$$

- $\sigma^2_{U_{0,j}} + \sigma^2_{U_{1,j}} + \sigma^2_{U_{0,j}U_{1,j}}$ variances and covariance of random effects
- $\sigma^2_{W_{0,j}} + \sigma^2_{W_{1,j}} + \sigma^2_{W_{0,j}W_{1,j}}$ variances and covariance of random effects
- $\sigma^2_{\epsilon_{ij}}$ residuals – error variance left over
- Covariances are included because random intercepts and slopes may correlate e.g. if slow subjects are more vulnerable to difference in conditions

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Estimation methods

An intuitive account of estimation in mixed-effects models

- If we knew the random effects, we could find the fixed effects estimates by minimising differences between predicted and observed outcomes – like linear modelling
- If we knew the fixed effects – the regression coefficients – we could work out the residuals and the random effects

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- At the start, we know neither, but we can move between partial estimation of fixed and random effect in an iterative approach
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Restricted maximum likelihood ($REML = TRUE$) and maximum likelihood ($REML = FALSE$) methods

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- Therefore not recommended to compare the likelihood of models varying in fixed effects fitted using REML (Pinheiro & Bates, 2000)
- REML method recommended for comparing the likelihood of models with *the same fixed effects* but *different random effects*
- ML method `REML= FALSE` recommended for comparing models with *different fixed effects* but *the same random effects*

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Model comparisons among mixed-effects models fitted using maximum likelihood ($REML = FALSE$) method

```
full.lmer.0 <- lmer(logrt ~

(1|subjectID) + (1|item_name),

data = ML.all.correct, REML = F)
```

- Maximum likelihood ($REML = FALSE$)
  - ML estimation methods can be used to fit models with varying fixed effects but the same random effects

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Examine the fixed effects then the random effects

Start by examining models varying in the *fixed effects* but constant in the *random effects*

- Compare maximum likelihood ($REML = FALSE$) models varying in fixed effects
  - Think about simpler models as simplifications or subsets of more complex models
- Add effects of interest in blocks or sets of predictors

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Compare maximum likelihood models with varying fixed effects but the same random effects

```
full.lmer.1 <- lmer(logrt ~

  zAge + zTOWRE_wordacc + zTOWRE_nonwordacc +

  (1|subjectID) + (1|item_name),

  data = ML.all.correct, REML = FALSE)

summary(full.lmer.1)
```

- `zAge + zTOWRE_wordacc + zTOWRE_nonwordacc +` add fixed effects
- `(1|subjectID) + (1|item_name)` to random effects of subjects and items on intercepts

Compare maximum likelihood models with varying fixed effects but the same random effects

```
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- `zAge + zTOWRE_wordacc + zTOWRE_nonwordacc +` add fixed effects
- `(1|subjectID) + (1|item_name)` to random effects of subjects and items on intercepts

Compare the model 1 with just subject effects to the model 2 also with item effects

```
full.lmer.2 <- lmer(logrt ~

zAge + zTOWRE_wordacc + zTOWRE_nonwordacc +

item_type + zLength + zOrtho_N +

(1|subjectID) + (1|item_name),

data = ML.all.correct, REML = F)

summary(full.lmer.2)
```

- `item_type + zLength + zOrtho_N` add item effects
- Everything else stays the same

Compare the model 1 with just subject effects to the model 2 also with item effects

```
full.lmer.2 <- lmer(logrt ~

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data = ML.all.correct, REML = F)

summary(full.lmer.2)
```

- `item_type + zLength + zOrtho_N` add item effects
- Everything else stays the same

Compare model 2 with subject and item effects to model 3 specifying interactions between subject and item effects

```
full.lmer.3  <- lmer(logrt ~

    (zAge + zTOWRE_wordacc + zTOWRE_nonwordacc)*

    (item_type + zLength + zOrtho_N) +

    (1|subjectID) + (1|item_name),

    data = ML.all.correct, REML = FALSE)
summary(full.lmer.3)
```

- Notice that the `(something)*(something)` get you interactions and main effects for all possible pairs of variables in the first set `(zAge + zTOWRE_wordacc + zTOWRE_nonwordacc)` and the second set `(item_type + zLength + zOrtho_N)` of predictors

How do we know if increasing *model complexity* by adding predictors actually helps us to account for variation in outcome values?

We can use the *anova()* function to do the comparison

```
anova(full.lmer.0, full.lmer.1, full.lmer.2, full.lmer.3)
```

- `anova(full.lmer.0, full.lmer.1, full.lmer.2, full.lmer.3)` compare the named models in pairs

See that *anova()* results present information criteria statistics for each model plus *likelihood ratio test* comparisons

Each step increase in complexity appears warranted by improved model fit to data

```
> anova(full.lmer.0, full.lmer.1, full.lmer.2, full.lmer.3)
Data: ML.all.correct
Models:
full.lmer.0: logrt ~ (1 | subjectID) + (1 | item_name)
full.lmer.1: logrt ~ zAge + zTOWRE_wordacc + zTOWRE_nonwordacc + (1 | subjectID) +
full.lmer.1:     (1 | item_name)
full.lmer.2: logrt ~ zAge + zTOWRE_wordacc + zTOWRE_nonwordacc + item_type +
full.lmer.2:     zLength + zOrtho_N + (1 | subjectID) + (1 | item_name)
full.lmer.3: logrt ~ (zAge + zTOWRE_wordacc + zTOWRE_nonwordacc) * (item_type +
full.lmer.3:     zLength + zOrtho_N) + (1 | subjectID) + (1 | item_name)
            Df    AIC    BIC logLik deviance    Chisq Chi Df Pr(>Chisq)
full.lmer.0  4 -17981 -17952 8994.4   -17989
full.lmer.1  7 -17982 -17931 8998.0   -17996   7.1782      3    0.06643 .
full.lmer.2 10 -18320 -18247 9169.7   -18340 343.5400      3    < 2e-16 ***
full.lmer.3 19 -18416 -18278 9227.0   -18454 114.5129      9    < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
> 
```

Figure : Model comparisons

The distinction between exploratory and confirmatory studies may be useful here

Cumming (2014): between *pre specified* and *exploratory* or *question answering* and *question formulating* studies

- Confirmatory – pre specified – models test the effects of the variables or conditions you manipulated
- Exploratory – question formulating – models may be developed by building up complexity, with initially no clear idea about predictions

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Examine the random effects

When the goal of a confirmatory analysis is to test hypotheses about one or more critical fixed effects, what random-effects structure should one use?

- Current recommendations (Barr et al., 2013; JML): Maximal random effects structure
- If you are testing effects manipulated according to a prespecified – confirmatory study – design
  - Test random intercepts – subjects and items
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Model comparison and selection

Remember: Simplicity and parsimony

- Trade-off between too much and too little simplicity in model selection
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- Estimation procedures may run into convergence problems where there is too much model complexity and not enough data

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