The Model Theory of Metric Spaces

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1 Introduction

This presentation aims to be a brief excursion into the theory of metric structures, a model-theoretic approach to the study of metric spaces. This theory is a vast generalization of ordinary model theory—in a trivial way, since every model-theoretic structure can be made into a "discrete" metric structure in the obvious way.

2 Basic Notions

Definition 2.1 (Metric Structure)**.** Let (M, d) be a complete, bounded metric space.

- A predicate on M is a uniformly continuous function from M^n to a bounded closed interval $I_p \subseteq \mathbb{R}$.
- A function or operation on M is a uniformly continuous function $M^n \to M$.
- A metric structure $\mathcal{M}$ on (M, d) consists of predicates $(R_i \mid i \in I)$, functions $(F_j \mid j \in J)$ on M, and distinguished elements $(a_k \mid k \in K)$ (there's more).

Reminder that the key restrictions are:

- M is complete and bounded (completeness can be bypassed by passing to the completion).
- Predicates and functions are uniformly continuous.

3 Examples

- A complete, bounded metric space with no additional structure.
- A structure $\mathfrak{A}$ from first-order logic, where A has the discrete metric, and predicates and relations take on values in $\{0, 1\}$.
- If (M, d) is an unbounded complete metric space with a distinguished element a, we can view (M, d) as a many-sorted metric structure $\mathcal{M}$ with sorts $\overline{B_n(a)}$ and inclusions $\overline{B_n(a)} \hookrightarrow \overline{B_m(a)}$.
- The unit ball of a Banach space X over $\mathbb{R}$ or $\mathbb{C}$: take the maps $f_{\alpha\beta}(x, y) = \alpha x + \beta y$ for each pair of scalars satisfying $|\alpha| + |\beta| \leq 1$; norm; additive identity.
- If $(X, \mathcal{M}, \mu)$ is a measure space, let M be the measure algebra of $(X, \mathcal{M}, \mu)$, elements of $\mathcal{M}$ modulo sets of measure 0, and let d be symmetric difference. As operations we can take $\cup, \cap, ^c$, and as a predicate we can take μ, with distinguished elements $0, 1$ in the lattice structure of M.

4 Uniform continuity

Definition 4.1. A modulus of uniform continuity is a function $\Delta : (0,1] \to (0,1]$. If $f : M \to M'$ is a function of metric spaces, Δ is a modulus of uniform continuity for f if for every $\varepsilon \in (0,1]$ and $x, y \in M$,

$$d(x,y) < \Delta(\varepsilon) \iff d'(f(x), f(y)) \leq \varepsilon$$

We say f is uniformly continuous if it has a modulus of uniform continuity.

Remark. These properties of Δ are crucial to term/formula construction making sense:

- If Δ is a muc for $f : M \to M'$, then it is also a muc for the unique extension $f : \overline{M} \to \overline{M'}$.
- If $f : M \to M', f' : M' \to M''$ have mucs Δ, Δ', then $\Delta' \circ r\Delta$ is a muc for $f' \circ f$, for any $r \in (0,1)$.
- If $f_n \to f$ uniformly and f_n is uniformly continuous, then so is f.

5 Quantifiers

Definition 5.1. Let $f : M \times M' \to \mathbb{R}$. Define the functions $\sup_y f : M \to \mathbb{R}$ and $\inf_y f : M \to \mathbb{R})$ by $\sup_y f(x) = \sup\{f(x,y) : y \in M'\}$, respectively for $\inf_y f$.

Theorem 5.1. *If Δ is a muc for f above, then it is a muc for $\sup_y f$ and $\inf_y f$.*

Theorem 5.2. *If $f_n : M \times M' \to M$ go to f uniformly, then $(\sup, \inf)_y f_n \to (\sup, \inf)_y f$. i.e. $\sup_y, \inf_y$ are continuous operators on the subspace of bounded continuous functions.*

6 $\star$ Prestructures

The only objects we will call structures are built complete metric spaces, but we can consider the process of structurifying weaker spaces:

$$\underbrace{\text{Pseudometric spaces} \xrightarrow{\text{mod } d(x,y)=0} \text{Metric spaces}}_{\text{Prestructures}} \xrightarrow{\text{completion}} \underbrace{\text{Complete metric spaces}}_{\text{Structures}}$$

Thus we can generalize weaken our assumptions without losing information.

Theorem 6.1. *For arbitrary functions $F, G : X \to [0,1]$, tfae:*

1. $\forall \epsilon > 0 \exists \delta > 0 \forall x \in X (F(x) \leq \delta \implies G(x) \leq \varepsilon)$.
2. *There is an increasing, continuous $\alpha : [0,1] \to [0,1]$ such that $\alpha(0) = 0$ and $(\forall x \in X)(G(x) \leq \alpha(F(x))$*

7 Syntax

Definition 7.1 (Language)**.** A language L contains logical and nonlogical symbols. The nonlogical symbols are as follows:

- Predicate symbols P^n, each with associated interval I_p and muc Δ_P (these are not symbols).
- Function symbols f^n, each with an associated muc Δ_f.
- Constant symbols a_k.
- The diameter $D_L = \max_{x,y} d(x, y)$.
- wlog we may assume $D_L = 1$ and $I_P = [0, 1]$ for all P.

L also has the logical symbols:

- The metric symbol d. This is the notion of equality in a metric structure.
- An infinite list V_L of variables.
- A connective symbol for each continuous function $u : [0, 1]^n \to [0, 1]$ with finitely many variables. It suffices to take a countable dense set (with respect to uniform norm).
- The quantifier symbols $\sup, \inf$.

Remark. Terms, formulas, and their interpretations in a metric structure are defined as usual. The set of terms/formulas respectively is closed by the facts we proved about mucs.

8 Conditions of L

Since logic is no longer 2-valued, we need to recover a notion of truth, or satisfaction.

Definition 8.1 (L-Condition)**.** An L-condition E is a formal expression of the form $\varphi = 0$, where φ is an L-formula. We say E is closed if φ is a sentence; otherwise, we can write E as $E(x_1, \ldots, x_n)$. We say $\mathcal{M} \models E[a_1, \ldots, a_n]$ if $\varphi^{\mathcal{M}}(a_1, \ldots, a_n) = 0$.

Definition 8.2 (Logical equivalence)**.** E_1 and E_2 are logically equivalent if for every L-structure $\mathcal{M}$ and every $a_1, \ldots, a_n \in M$, we have

$$\mathcal{M} \models E_1[a_1, \ldots, a_n] \iff \mathcal{M} \models E_2[a_1, \ldots, a_n].$$

Remark. It is useful to introduce the expression $\varphi = \psi$ as an abbreviation for $|\varphi - \psi| = 0$, where $t_1, t_2 \mapsto |t_1 - t_2|$ is a connective. Similarly, we can define $\varphi \leq \psi$ by $\varphi \dot{-} \psi = 0$.

Definition 8.3 (Theory)**.** A theory in L is a set of closed L-conditions. We write $\mathcal{M} \models T$ if $\mathcal{M} \models E$ for every $E \in T$

9 ≥ 2 Interesting Theorems

Definition 9.1 (Elementary Equivalence)**.** $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent, or $\mathcal{M} \equiv \mathcal{N}$, if $\sigma^{\mathcal{M}} = \sigma^{\mathcal{N}}$ for all L-sentences σ. Equivalently, $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$ (Why?).

Theorem 9.1 (Tarski-Vaught Test)**.** *Let S be a dense set of L-formulas and let $\mathcal{M} \subseteq \mathcal{N}$ be L-structures. Then $\mathcal{M} \preceq \mathcal{N}$ if and only for all $\varphi(\vec{x}, y) \in S$ and $a_1, \ldots, a_n \in M$, we have* $\inf\{\varphi^{\mathcal{N}}(a_1, \ldots, a_n, b) \mid b \in N\} = \inf\{\varphi^{\mathcal{N}}(a_1, \ldots, a_n, c) \mid c \in M\}$.

Theorem 9.2 (Compactness Theorem)**.** *Compactness theorem: Let T be an L-theory and $\mathfrak{C}$ a class of L-structures. If T is finitely satisfiable in $\mathfrak{C}$, then there exists an ultraproduct of all structures from $\mathfrak{C}$ that is a model of T. More generally, it suffices to check that $T^+ = (\varphi \leq \frac{1}{n} \mid \varphi = 0 \in T, n \geq 1)$ is finitely satisfiable.*

10 Other Model Theoretic Concepts

- Peturbation Lemma: Let $\pi : M_0 \to M$ be the quotient for $\mathcal{M}_0, \mathcal{M}$, and let $\mathcal{N}$ be $\varphi^{\mathcal{M}}$. Adjunction.
- Logically equivalent (over all L-structures), logical distance
- Condition: expression E is of the form $\varphi = 0$, E is closed if φ a sentence. "E is true of $a_1, \ldots, a_n$ in M", or $\mathcal{M} \models E[a_1, \ldots, a_n]$. We can use $\dot{-}$ and $|\cdot|$ to press $\varphi = \psi$ or $\varphi \leq \psi$.
- $M \models E_1[a_1, \ldots, a_n]$.
- Theory is a set of closed L-conditions.
- Model only applies to complete metric spaces, but if $\mathcal{M}_0$ prestructure such that each $\varphi^{\mathcal{M}_0} = 0$, then the associated structure is a model of T
- Conditions are a lot like ideals.
- Elementary equivalence: $\sigma^{\mathcal{M}} = \sigma^{\mathcal{N}}$ for all L-sentences σ. Or, $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$. Why?
- ELementary extension: $\varphi^{\mathcal{M}}(a_1, \ldots, a_n) = \varphi^{\mathcal{N}}(a_1, \ldots, a_n)$.
- Elementary map: like elementary extension but with F instead of ι.
- Elementary embedding: injective elementary extension: injective elementary extension???
- Remark: Elementary maps are isometries, closed under composition and inverse, and isomorphism are elementary embeddings.
- Tarski-Vaught Test for $\leq$: If S is a (logically) dense set of L-formulas, and suppose $\mathcal{M} \subseteq \mathcal{N}$ are L-structures. Then $\mathcal{M} \preceq \mathcal{N}$ if and only for all $\varphi(\vec{x}, y) \in S$ and $a_1, \ldots, a_n \in M$, $\inf\{\varphi^{\mathcal{N}}(a_1, \ldots, a_n, b) \mid b \in N\} = \inf\{\varphi^{\mathcal{N}}(a_1, \ldots, a_n, c) \mid c \in M\}$.

Proof. For $\Longleftarrow$, first prove (2) for all L-formulas, then prove $\psi^{\mathcal{M}}(a_1, \ldots, a_n) = \psi^{\mathcal{N}}(a_1, \ldots, a_n)$ inductively. ∎

11 Further Topics

- Fundamental theorem of ultraproducts
- Compactness theorem: Let T be an L-theory and C a class of L-structures. If T is finitely satisfiable in C, then there exists an ultraproduct of structures from C that is a model of T.
- More general: Let $T^+ = (\varphi \leq \frac{1}{n} \mid \varphi = 0 \in T, n \geq 1)$.
- Compactness theorem for formulas: let T be an be an L-theory and $\Sigma(x_j \mid j \in J)$ a set of L-conditions. Σ is said to be consistent with T if for every finite subset $F \subseteq \Sigma$, there is a model $\mathcal{M} \models T$ and a list of elements $\vec{a} \in M$ such that for each $E \in F$, $M \models E[\vec{a}]$.
- Corollary