Topos Theory V: Sheaves

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October 19, 2022

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Field K of characteristic 0	(Locally) arcwise connected space X
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- *Grothendieck.* We need a more general notion of topology where the primitive notions are not open sets $U \rightarrowtail X$, but more general maps.

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- A tuple (C, J), C small, is called a **(Grothendieck) site**.

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Preliminary Consequences

- *Consequence of Transitivity.* If S covers C, and if we have a cover R_f of each $f : D_f \to C$ in S, then the set of composites fg with $f \in S$ and $g \in R_f$ covers C.

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- If R and S cover $g : D \to C$, then $R \cap S$ covers g.
- *Proof.* If $f : D \to C \in R$, then $f^*(R \cap S) = f^*(S)$, since any morphism in $f^*(S)$, when composed with $f \in R$, is also an element of R. Hence, $f^*(R \cap S)$ covers D.

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- From K, we get a unique topology J, such that $S \in J(C)$ iff S contains some $R \in K(C)$.

Examples of Topologies

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- A finer category on T is generated by the basis K assigning to X the set of all families $\{f_i : Y_i \to X \mid i \in I\}$ such that $f : \coprod_i Y_i \to X$ is an open surjection.

More Examples

- *Sup Topology.* A complete Heyting algebra is a HA admitting sups and infs over any family. The **sup topology** on a cHa is generated by the basis K, defined by $\{a_i : i \in I\} \in K(c)$ if $\bigwedge_{i \in I} a_i = c$; this generalizes the open set topology to any complete Heyting algebra. Stability follows from

$$\bigwedge_{i \in I} (b \wedge a_i) = b \wedge \bigwedge_{i \in I} a_i$$

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- *Dense Topology.* Define the **dense topology** J on a category C by $S \in J(C)$ if for any $f : D \to C$, there is a $g : E \to D$ such that $fg \in S$. When the category is a poset P, $J(p)$ is simply the set of dense subsets below $p \in P$.

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- *Atomic Topology.* The **atomic category** is defined by $S \in J(C)$ iff S is a nonempty sieve.

The Zariski Site

- Given an ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$, we define $V(I) = \{x \in \mathbb{C}^n : I \text{ vanishes on } x\}$. $V(I)$ is called a **complex affine variety**.

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- The general construction can be done for a commutative ring k.

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- The last step is to define a suitable structure sheaf on $(k - \mathrm{Alg})_{fp}$ which acts like "the ring of functions on each neighbourhood".

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- This is true if we have a family of equalizers:

$$P(C) \xrightarrow{x \mapsto \{x_f\}_f} \prod_{f \in S} P(\mathrm{dom}\, f) \underset{x_f \mapsto x_f \cdot g}{\overset{x_f \mapsto x_{fg}}{\rightrightarrows}} \prod_{\substack{f \in S \\ \mathrm{dom}\, f = \mathrm{cod}\, g}} P(\mathrm{dom}\, g)$$

- Again, it suffices to describe a sheaf on a basis K. If $R = \{f_i : C_i \to C \mid i \in I\}$ is a K-cover of C, a matching family is one such that $x_i \cdot \pi^1_{ij} = x_j \cdot \pi^2_{ij}$ always holds. An amalgamation for $\{x_i\}$ is a compatible element x.

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- Definition. A **Grothendieck topos** is a category which is equivalent to the category $\mathrm{Sh}(\mathsf{C}, J)$ of sheaves on some site (C, J).