How an Enumerative Combinatorialist Might Solve a PDE

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G([n]) \arrow[r, dashed, "|\cdot|"'] & G(n)
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Note on Notation:

- $[n] = \{1, \ldots, n\}$.
- $F(X)$ class of structures on a set X.
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Proofs are more elegant and insightful.

Counting monotonic paths, the first way

Consider

$$4^n = \sum_{k=0}^{n} \binom{2k}{k} \binom{2(n-k)}{n-k}$$

Remember: $\binom{n}{k} = |\{k - \text{subsets of } [n]\}|$.

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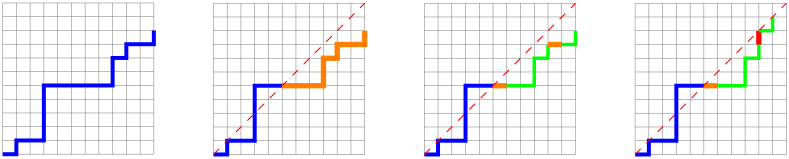
Example

$$2^{20} = \sum_{k=0}^{10} \binom{2k}{k}\binom{2(n-k)}{n-k}, \qquad k = 5 : \binom{10}{5}\binom{10}{5}$$



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Squaring both sides (RHS is a convolution)

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Enumerators destroyed!

Higher enumerable discrete structures

Wrong scheme:

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F([n]) \arrow[r, dashed, "|\cdot|"] \arrow[d, dashed, leftrightarrow, "\cong"'] & 4^n \arrow[d, equal] \\
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Right scheme:

$$\begin{tikzcd}
F[X] \arrow[rr, dashed, "|\cdot|"] \arrow[d, dashed, leftrightarrow, "\cong"'] & & \frac{1}{\sqrt{1-4x}} \arrow[d, equal] \\
G \cdot G'[X] \arrow[rr, dashed, "|\cdot|"'] & & \sum_n \left[\sum_k \binom{2k}{k}\binom{2(n-k)}{(n-k)}\right] x^k
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Overview, higher enumerable discrete structures (Cont.)

Goal: Find a model $\mathbb{S}(X)$ for the algebra of all $F[X]$.

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- Hence: $\mathbb{S}(X)$ behaves like a ring of smooth functions.
- i.e. Leibniz identity and chain rule.
- Generalize to multiple variables, i.e. $\mathbb{S}(X, Y, Z)$.

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Definition 2

A species is a functor $F[X]$: Core(FinSet) $\to$ Core(FinSet).

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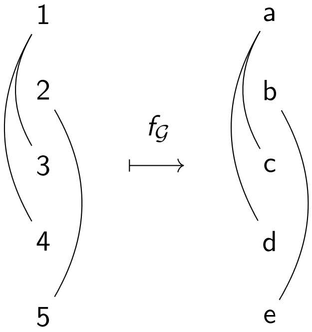
Consider $f : \{1, 2, 3, 4, 5\} \to \{a, b, c, d, e\}$.

$$f_{\mathcal{S}} : \mathcal{S}[1, \ldots, 5] \to \mathcal{S}[a, \ldots, b]$$
$$(13)(452) \mapsto (ac)(deb)$$

$$f_{\mathcal{L}} : \mathcal{L}[1, \ldots, 5] \to \mathcal{L}[a, \ldots, b]$$
$$34152 \mapsto cdaeb$$



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Immediately, $E(x) = e^x$, $X^n(x) = \frac{x^n}{n!}$, $1(x) = 1$, $0(x) = 0$.

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- $\mathcal{L}[X]$ and $\mathcal{S}[X]$ are NOT isomorphic.
- Transport functions of $\sigma : U \to U$ have different automorphism types, fixed points:

$$\sigma := (12) \in S_5$$
$$\sigma_{\mathcal{S}}(12)(345) = (12)(345).$$
$$\sigma_{\mathcal{S}}(21453) = (12453) \neq (21453).$$

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- Notice that

$$
\begin{aligned}
(F \cdot G)(x) &= \sum_n \sum_{[n] \cong [k] \sqcup [k']} |F[k]| \cdot |G[k']| \frac{x^n}{n!} \\
&= \sum_n \sum_{k=0}^{n} \binom{n}{k} |F[k]| \cdot |G[n-k]| \frac{x^n}{n!} \\
&= \sum_n \sum_{k=0}^{n} \frac{|F[k]|}{k!} \cdot \frac{|G[n-k]|}{(n-k)!} x^n = F(x) \cdot G(x)
\end{aligned}
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- (Sketch of Leibniz identity)

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The following formula holds:

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- (Example)

Composition

- If $G(0) = 0$, then $F(G(x)) := \sum F_n \frac{G(x)^n}{n!}$ is well-defined.

Theorem

The following formula holds:

$$F(G(x)) = \sum_n \sum_{\sum\{d_1,\dots,d_k\}=n} \binom{n}{d_1,\dots,d_k} F_k \prod_{i=1}^{k} G_{d_i}.$$

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Connected species

$$F = E(F^{\mathrm{c}})$$

Binary rooted trees (Lawvere, Baez)

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- For every bijection f, a transport function $F[U_1, \dots, U_n] \to F[V_1, \dots, V_n]$.

One final example

Consider the following PDE with $F(X, Y)$ a 2-sort species.

$$\frac{\partial}{\partial X} F = Y\left(1 + \frac{\partial}{\partial Y}\right) F$$

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We get a corresponding generating function $e^{(e^x - 1)y}$ for the Stirling numbers.

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Proposition

The system of differential equations:

$$\frac{\partial Y_i}{\partial X} = R_i(Y_1, \dots, Y_k), \quad Y_i(0) = Z_i, \quad i = 1, \dots, k,$$

has a computable, unique up to isomorphism, solution… for $\mathbb{L}$-species.

Fin