Algebraic Topology

Robbert Liu
MATD94H3: Readings in Mathematics
University of Toronto

April 8, 2022

Contents

1 Some Underlying Geometric Notions

All maps referenced in these notes are assumed to be continuous unless stated otherwise.

1.1 Homotopy and Homotopy Type

The notion of homotopy relates spaces that are not necessarily homeomorphic, but which have the same fundamental shape. We say two maps $f_0, f_1 : X \to Y$ are **homotopic**—denoted as $f_0 \simeq f_1$—if there exists a family of maps $\{f_t : X \to Y\}_{t \in I}$ such that the map $F : X \times I \to Y$ defined by $F(x,t) = f_t(x)$ is continuous. A natural example of a homotopy arises when $X = Y$, $f_0 = \mathbb{1}$ and f_1 is a retraction of X onto a subset $A \subseteq X$: that is, $f_1|_A = \mathbb{1}$ and $f_1(X) = A$. In this case, $\{f_t\}$ is given the suggestive name of **deformation retract**. More generally, $\{f_t\}$ is a **homotopy relative to** A when for some subspace $A \subseteq X$, each map $f_t|_A$ is identity.

Two spaces X, Y are said to be **homotopy equivalent** or to have the same **homotopy type**—again denoted $X \simeq Y$—if there exist maps $f : X \to Y, g : Y \to X$ such that $fg \simeq \mathbb{1} \simeq gf$. Notice that if $\{f_t : X \to X\}$ is a deformation retract onto a subspace A and $i : A \to X$ is the inclusion map, then $f_1 i = \nVdash$ and $i f_1 \simeq \nVdash$. Hence, $X \simeq A$ in this case. Moreover, if A is a singleton—that is, X has the homotopy type of a point—then we say X is **contractible**, and note that $\mathbb{1}$ **nullhomotopic**, or homotopic to a constant map.

1.2 Cell Complexes

The structure of a multigraph roughly arises from gluing 0-dimensional vertices to the boundary points of 1-dimensional edges and discarding non-topological information such as distance or curvature. This process is succinctly generalized to higher dimensions by the notion of **cell complex**. Let X_0 be a discrete set. For $n \in \mathbb{N}^+$, a family $\{D^n_\alpha\}_{\alpha \in A_n}$ of closed n-disks, and a family $\{\varphi_\alpha : D^n_\alpha \to X^{n-1}\}_{a \in A_n}$ of attaching maps, define X_n to be the quotient space

$$X^{n-1} \amalg_\alpha D^n_\alpha / \sim, \text{ where } x \sim \varphi_a(x) \text{ for all } x \in \partial D^n_\alpha,\ \alpha \in A_n.$$

Then, $X = \cup_n X^n$ is called a **cell complex**; X^n is called the n-**skeleton**; and $i(\text{int } D^\alpha_n)$ is the n-**cell** e^n_α. If all but finitely many A_n are non-empty, $X = X^n$ for some n, in which case we call X **finite-dimensional** or of **dimension** n. Finally, each attaching map φ_α can be extended to the **characteristic map** $\Phi_\alpha : D^n_\alpha \to X$, such that $\Phi_\alpha|_{\text{int } D^\alpha_n}$ is a homeomorphism onto e^n_α. A **subcomplex** A of a cell complex X is a closed subspace which is a union of cells; (X, A) is called a **CW pair**.

1.3 Operations on Spaces

The definition of cell complex gives rise to a rich variety of operations one can perform on these spaces. The product $X \times Y$ of two cell complexes is a cell complex, whose cells are the products $e^m_\alpha \times e^n_\beta$ of a cell in X and a cell in Y. If (X, A) is a **CW pair**, then the quotient space X/A is a cell complex, whose cells are the cells of $X \setminus A$ plus an additional 0-cell corresponding to the image of A in X/A.

We remark on four interesting classes of quotient spaces: the **suspension** $SX := \frac{X\times I/X\times\{0\}}{X\times\{1\}}$ of X; The **join** $X * Y := X \times Y \times I/\sim$ of X and Y, where $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$; the **wedge sum** $X \vee Y := X \amalg Y/\{x_0, y_0\}$ of X and Y, given the points $x_0 \in X, y_0 \in Y$; and the **smash product** $X \wedge Y := X \times Y/X \vee Y$ of X and Y.

We introduce two common methods of proving homotopy equivalence between cell complexes. Let (X, A) be a CW pair. First, if A is contractible, then the quotient map $X \to X/A$ is a homotopy equivalence. Next, given two spaces X_0, X_1 and a map $f : A \to X_0$ whose domain is a subspace $A \subseteq X_1$, define $X_0 \sqcup_f X_1$ to be the space constructed from $X_0 \amalg X_1$ by identifying each $a \in A$ with $f(a) \in X_0$. We may think of $X_0 \amalg X_1$ as the space X_0 with X_1 **attached along** A **via** f. Then, if (X_1, A) is a CW pair and the attaching maps $f, g : A \to X_0$ are homotopic, then $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$.

1.4 The Homotopy Extension Property

We conclude with a theorem which "attaches" the notion of cell complex and homotopy and provides a way to compute homotopy equivalences between cell complexes with ease. Let (X, A) be a pair. If, for any map $f_0 : X \to Y$ whose restriction $f_0|A$ has a homotopy $f_t : A \to Y$, we can extend always f_t to a homotopy $f_t : X \to Y$, then we say (X, A) has the **homotopy extension property**.

Theorem 1.1. *If (X, A) is a CW pair, then (X, A) has the homotopy extension property.*

2 Homology

Although the fundamental group is useful for studying low dimensional spaces, it cannot distinguish higher-dimensional features of spaces. On the other hand, higher homotopy groups are comparatively more laborious to compute. We thus switch our focus to a different invariant, called homology.

2.1 Simplicial and Singular Homology

2.1.1 Δ-complexes

The 2-dimensional triangle can be generalized to n dimensions as the smallest convex set D in $\mathbb{R}^m$ containing $n+1$ ordered **vertices** $v_0, \dots, v_n$, such that $v_1 - v_0, \dots, v_n - v_0$ are linearly independent. D is called an **n-simplex**, and is homeomorphic to the standard n-simplex Δ^n, whose vertices are $0, e_1, \dots, e_n$, via a map that preserves the order of vertices. We may then refer to U by its ordered vertices $[v_0, \dots, v_n]$, which induces an ordering on the vertices of its **faces** $[v_0, v_n, \hat{v}_i, \dots, v_n]$.

A **Δ-complex** structure on a face X is a collection C of maps $\sigma_\alpha : \Delta^{n_\alpha} \to X$ such that the restriction $\sigma_\alpha \mid \operatorname{int} \Delta^{n_\alpha}$ is injective, and each point of X is in the image of exactly one such restriction; C is closed under restriction of $\sigma_\alpha \in C$ to a face of Δ^{n_α}; and $A \subseteq X$ is open iff $\Delta_\alpha^{-1}(a)$ is open in Δ^{n_α} for all Δ_α. This definition allows for Δ-complexes to be constructed from disjoint simplices by identifying pairs of n-subsimplices while preserving orientation.

2.1.2 Simplicial Homology

Let $\Delta_n(X)$ be the free abelian group of **n-chains** in X, which is spanned by the open n-simplices e_α^n of X. Define the **boundary homomrphism** $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ on its basis by $[v_0, \dots, v_n] \mapsto \sum_i (-1)^i \sigma_\alpha \mid [v_0, \dots, \hat{v}_i, \dots, v_n]$. We may refer to ∂_n as ∂ when it is unambiguous. Since the set $\operatorname{img} \partial_{n+1}$ of **cycles** is a normal subgroup of the set $\ker \partial_n$ of **boundaries**, we may take the quotient $H_n^\Delta(X) := \ker \partial_n / \operatorname{img} \partial_{n+1}$, which we call the n^{th} **simplicial homology group of** X. In general, given a **chain complex**

$$\cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

where $\partial_n \partial n - 1 = 0$ for each n, the same quotient is called the n^{th} **homology group** of the chain complex.

2.1.3 Singular Homology

A **singular n-simplex** in a space X is a map $\sigma : \Delta^n \to X$. Elements of $C_n(X)$ are finite formal sums $\sum_i : \Delta^n \to X$ over $\mathbb{Z}$ called **n-chains**. Boundary maps $\partial_n : C_n(X) \to C_{n-1}(X)$ and **singular homology groups** $H_n(X)$ are defined identically to their simplicial counterparts. Although it is clear that homeomorphic spaces have isomorphic singular homology groups, it is unclear whether or not $H_n(X)$ is finitely generated or $H_n(X)$ is zero for $n \geq \dim X$. Singular homology turns out to be a special case of simplicial homology, by creating the **singular**

complex $S(X)$ by attaching an n-simplex Δ_σ^n to X for each **singular** n-simplex. A result we can derive fairly quickly is that $H_0(X) \cong \mathbb{Z}^n$, where n is the number of path-components of X.

2.1.4 Homotopy Invariance

Given spaces X, Y, a map $f : X \to Y$ induces a homomorphism $f_\sharp : H_n(X) \to H_n(Y)$, which is defined on the basis of $H_n(X)$ by $\sigma \mapsto f\sigma$. $f_\sharp$ satisfies the important property $\partial f_\sharp = f_\sharp \partial$, implying that f takes cycles to cycles and boundaries to boundaries. Thus, $f_\sharp$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$. We discover that any two homotopic maps $f, g : X \to Y$ induces the same homomorphism $f_* = g_*$, which means that f_* is an isomorphism if f is a homotopy equivalence.

2.1.5 Exact Sequences

A chain complex

$$\cdots \longrightarrow A_n{+}1 \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \dots$$

is said to be **exact** if $\ker \alpha = \operatorname{img} \alpha_{n+1}$ for all n. That is to say the homology groups of the chain complex are trivial. A **short exact sequence** has the form $0 \to A \to B \to C \to 0$.

We eventually prove that if X is a space and A is a closed empty subspace that is a deformation retract of a neighbourhood in X ((X, A) is a "**good pair**"), then there exists a **long exact sequence**

$$\cdots \longrightarrow \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_n(A) \longrightarrow \cdots \longrightarrow \tilde{H}_0(X/A) \to 0,$$

where i, j are the inclusion and quotient maps respectively. We can use this theorem to compute the homology of spheres: $\tilde{H}_n S^n \cong \mathbb{Z}$ and $\tilde{H}_i(S^n) = 0$ for $i \neq n$. We also get Brouwer's Fixed Point Theorem as a quick corollary.

We can prove the existence of the exact sequence by proving a more general result involving the **relative homology groups** $H_n(X, A)$ of a pair (X, A), which are precisely the homology groups of the chain complex whose entries are groups $C_n(X, A) := C_n(X)/C_n(A)$. The boundary map $\partial : C_n(X, A) \to C_{n-1}(X, A)$ induced by the boundary map on $C_n(X)$ is well defined, since the latter takes $C_n(A) \to C_{n-1}(A)$.

Intuitively, the relative homology groups measure the homology of X while ignoring the homology of A. The cycles and boundaries of the new chain complex are aptly named **relative cycles** and **relative boundaries**. The relative homology groups fit into a symmetric long exact sequence:

$$\cdots \longrightarrow \tilde{H}_n(A) \longrightarrow \tilde{H}_n(X) \longrightarrow \tilde{H}_n(X, A) \xrightarrow{\partial} \tilde{H}_n(A) \longrightarrow \cdots \longrightarrow \tilde{H}_0(X/A) \to 0.$$

Exactness and the existence of ∂ are proven by **diagram chasing** a **short exact sequence of chain complexes**.

The Excision Theorem gives conditions under which removing, or **excising**, a subset $Z \subseteq A$ fixes relative homology group: given a chain $Z \subseteq \overline{Z} \subseteq \operatorname{int} A \subseteq A \subseteq X$ of spaces the inclusion $(X-Z, A-Z) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X-Z, A-Z) \to (X, A)$. This is proven via Proposition 2.21: given an open cover $\mathcal{U}$, define the subgroup $C_n^{\mathcal{U}}(X) \leq C_n(X)$ of chains $\sum_i n_i\sigma_i$ such that each $\operatorname{img}\sigma_i$ is contained by the interior of some $U \in \mathcal{U}$; then the inclusion $C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ is a chain homology equivalence. The idea is that we can perform **barycentric subdivision** on each singular n-simplex σ until the simplices of the iterated subdivision have radii smaller than $\mathcal{U}$'s Lebesgue number.

Excision Theorem is used to prove Theorem 2.13 from Theorem 2.16 by showing for good pairs (X, A) that $H_n(X, A) \cong \tilde{H}_n(X/A)$ for each n. Furthermore, Excision Theorem proves that inclusions $i_\alpha : X\alpha \to \vee_\alpha X_\alpha$ induce an isomorphism $\oplus_\alpha i\alpha_* : \oplus_\alpha \tilde{H}_n(X_\alpha) \to \tilde{H}_n(\vee_\alpha X_\alpha)$, as well as the classical result Invariance of Dimension: nonempty open sets $U \subseteq \mathbb{R}^m, V \subseteq \mathbb{R}^n$ are homeomorphic only if $m = n$.

Finally, we remark that the long exact sequence constructed in Theorem 2.16 is **natural**: that is, any map $f : (X, A) \to (Y, B)$ induces a chain map f_* between the long exact sequence.

2.2 Computations and Applications

2.2.1 Degree

If $n > 0$, then the induced map $f* : H_n(S^n) \to H_n(S^n)$ of a map $f : S^n \to S^n$ is a homomorphism $\mathbb{Z} \to \mathbb{Z}$ and thus be multiplication by a constant d_f. We call d the **degree** of f. Using the degree of elementary functions on S^n, we can prove several nontrivial applications. We can use the properties $\deg \mathbb{1} = 1$, $\deg(-\mathbb{1}) = (-1)^{n+1}$, and the invariance of degree under homotopy to prove that S^n has a continuous field of nonzero tangent vectors iff n is odd. Furthermore, maps with no fixed points are homotopic to the antipodal map and thus have degree $(-1)^{n+1}$, which leads to the result that $\mathbb{Z}/2\mathbb{Z}$ is the only nontrivial group that acts freely on S^n if n is even.

Suppose the fibre $f^{-1}(y)$ of some point $y \in S^n$ contains finitely many points $x_1, \dots, x_m$ with corresponding disjoint neighbourhoods U_i. The **local degree** $\deg f|x_i$ of f at x_i is the number d_f induced by the map

$$f_* : H_n(S^n) \cong H_n(U_i, U_i - x_i) \to H_n(S^n) \cong H_n(f(U_i), f(U_i) - y),$$

and can be thought of as the degree of f restricted to a small neighbourhood of x_i. It turns out that for any y with a finite fibre, $\deg f = \sum_i \deg f|x_i$. For example, on S^1, $\deg z^m|0 = m$, since z^m maps the disjoint open arcs containing, respectively, m distinct points, homeomorphically onto an open arc containing 1.

2.2.2 Cellular Homology

We prove three results on CW complexes: first, if X is a CW Complex, then $H_k(X^n, X^{n-1})$ is trivial when $k \neq n$, and is free with generators in bijection with X's n-cells when $k = n$; secondly, $H_k(X^n) = 0$ when $k > n$, and $H_k(X) = 0$ for $k > \dim X$ if X is finite dimension; and lastly, the inclusion $X^n \hookrightarrow X$ induces a map $H_k(X^n) \to H_k(X)$, which is an isomorphism for $k < n$ and surjective for $k = n$. This allows us to construct the **cellular chain complex**

$$\cdots \longrightarrow H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \longrightarrow \dots.$$

Surprisingly, the **cellular homology groups** $H_n^{CW}(X)$ of the chain complex are isomorphic to $H_n(x)$. It remains to compute the **cellular boundary maps** d_n, which are given by the Cellular Boundary Formula,

$$d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1},$$

where $d_{\alpha\beta}$ is the degree of attaching map $S_\alpha^{n-1} \to X^{n-1}$ of e_α^n with the map $X^{n-1} \to S_\beta^{n-1}$ that collapses $X^{n-1} - e_\beta^{n-1}$ to a point.

The **Euler characteristic** $\chi(X)$ of a finite CW complex X is the sum $\sum_n (-1)^n c_n$, where c_n is the number of n-cells of X. However, Euler Characteristic may be expressed solely in terms of the homology of X via the formula $\chi(X) = \sum_n (-1)^n \operatorname{rank} H_n(X)$, where the **rank** of a finitely generated abelian group is its number of $\mathbb{Z}$ summands.

Suppose there is a retraction $X \to A$ and let $i : A \hookrightarrow X$ be the inclusion map. Since $ri_* = \mathbb{1}$, i is injective, the long exact sequence for (X, A) decomposes into short exact sequences

$$0 \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \to 0 \tag{*}$$

Then, by the **Splitting lemma**, there is an isomorphism $H_n(X) \cong H_n(A) \oplus H_n(X, A)$ and homomorphisms $H_n(A) \to H_n(A) \oplus H_n(X, A) \to H_n(X, A)$ such that the entire diagram commutes—in that case, (*) is said to **split**.

2.2.3 Mayer-Vietoris Sequences

Give spaces $A, B \subseteq X$, if X is the union of the interiors of A and B, then there exists a long exact sequence, called a **Mayer-Vietoris sequence**, of the following form

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots \to H_n(X) \to 0$$

Similar to the Mayer-Vietoris sequences is a long exact sequence involving two maps $f, g : X \to Y$. Let Z be the quotient of the space $(X \times I) \amalg Y$ under the identifications $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$. For example, if f is the identity, then Z is the **mapping torus** of g. Then, we have the following exact sequence

$$\cdots \longrightarrow H_n(X) \xrightarrow{f_* - g_*} H_n(Y) \xrightarrow{i_*} H_n(Z) \longrightarrow H_{n-1}(X) \to \dots$$

where i is the inclusion $Y \hookrightarrow Z$.

There the Mayer-Vietoris sequences also have relative counterparts. Suppose we have a pair $(X, Y) = (A \cup B, C \cup D)$ with $C \subseteq A$ and $D \subseteq B$, such that X (resp. Y) is the union of the interiors of A and B (resp. C and D). Then, there is a relative Mayer-Vietoris sequence

$$\cdots \longrightarrow H_n(A \cap B, C \cap D) \longrightarrow H_n(A, C) \oplus H_n(B, D) \longrightarrow H_n(X, Y) \xrightarrow{\partial} \dots$$

2.2.4 Homology with Coefficients

We can generalize our theory of homology of $\mathbb{Z}$-modules to modules over any group G, by defining the chain groups $C_n(X; G)$ over G and extracting the homology groups $H_n(X; G)$, otherwise known as **homology groups with coefficients in** G. Even though these homology groups are generally different than those with coefficients in $\mathbb{Z}$, our prior tools and theorems carry over to the new homology groups—for example, there exist the relative and reduced counterparts $H_n(X, A; G), \tilde{H}_n(X; G)$, and if $f : S^k \to S^k$ has degree m, then the induced map $f_* : H_k(S^k; G) \to H_k(S^k; G)$ is also multiplication by m.

2.3 The Formal Viewpoint

2.3.1 Axioms for Homology

Up until this point, we have discussed several ways to define associate spaces with some notion of (un)reduced homology. However, we may abstract these specific instances into a general set of axioms which are desirable in *any* theory of homology. We restrict the case to CW complexes for now. A reduced **homology theory** associates with each nonempty CW complex X a sequence of abelian groups $\tilde{h}_n(X)$ and with each map $f : X \to Y$ a chain map $f_* : \tilde{h}_n(X) \to \tilde{h}_n(Y)$ such that $(fg)* = f * g*$ and $\mathbb{1}_* = \mathbb{1}$, and

1. If $f \simeq g$, then $f_* = g_*$.

2. For each CW pair (X, A) there exists natural boundary homomorphisms $\partial : \tilde{h}_n(\frac{X}{A}) \to \tilde{h}_{n-1}(A)$ which make exact the sequence

$$\dots \xrightarrow{\partial} \tilde{h}_n(A) \xrightarrow{i_*} \tilde{h}_n(X) \xrightarrow{q_*} \tilde{h}_n(X/A) \xrightarrow{\partial} \tilde{h}_{n-1}(A) \xrightarrow{i_*} \dots$$

   where i, q are the inclusion and boundary maps respectively.

3. Given $X = \vee_\alpha X_\alpha$ and inclusion maps $i_\alpha : X_\alpha \hookrightarrow X$, the direct sum maps $\oplus_\alpha i_{\alpha *} : \oplus_\alpha \tilde{h}_n(X_\alpha) \to \tilde{h}_n(X)$ are isomorphisms.

Conveniently, existence of Mayer-Vietoris sequences follow from the axioms. One thing to note is that axiom 3 is redundant when only considering finite wedge products.

We may also produce a list of axioms for unreduced homology theories, which are constructed by first defining relative homology groups $h_n(X, A)$, and then defining $h_n(X) := h_n(X, \emptyset)$. Then, axiom 1 is simply changed to refer to unreduced homology groups; axiom 2 consists of the same exact sequence except involving relative groups, and an additional condition that introduces some form of excision, for example $h_n(X, A) \cong h_n(X/A, A/A$; and in axiom 3, the wedge sum is exchanged for a disjoint sum. One may consider a fourth **dimension axiom** stating that $h_n(\text{point})$ is trivial for $n \neq 0$; an example of a homology theory where this axiom fails is **bordism**. However, unreduced and reduced homology are essentially equivalent since an (un)reduced homology theory can be easily converted into the other.

2.3.2 Categories, Functors, Natural Transformations

A **category** $\mathcal{C}$ is a collection of **objects** $\text{Ob}(\mathcal{C})$ and, for each pair X, Y of objects, a set of **morphisms** $\text{Mor}(X, Y)$, such that:

1. $\text{Mor}(X, X)$ contains a distinguished morphism $\mathbb{1}_X$.

2. There exists a morphism composition operation $\circ : \text{Mor}(X, Y) \times \text{Mor}(Y, Z) \to \text{Mor}(X, Z)$ for any bjects X, Y, Z, which is associative and admits an identity element $\mathbb{1}_Y$.

An **isomorphism** is a morphism $i : X \to Y$ that has an inverse $i^{-1} : Y \to X$ such that $i^{-1}i = \mathbb{1}_X$ and $ii^{-1} = \mathbb{1}_Y$. A **functor** $F : \mathcal{C} \to \mathcal{D}$ is a morphism between categories: it assigns each object $c \in \mathcal{C}$ an object $Fc \in \mathcal{D}$, and each morphism $f : c \to c'$ a morphism $Ff : Fc \to Fc'$, such that $Fg \circ Ff = F(g \circ f)$ and $F(\mathbb{1}_c) = \mathbb{1}_{Fc}$. A **natural transformation** $\alpha : F \to G$ is a morphism between functors $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$: it assigns for each object $c \in \mathcal{C}$ a morphism $\alpha_c : F \Rightarrow D$ such that for any morphism $f : c \to c'$ in $\mathcal{C}$, $Gf\alpha_c = \alpha_{c'} Ff$.

2.4 Additional Topics

2.4.1 Classical Applications

Let $g : D^k \to S^n, h : S^k \to S^n$ be embeddings. Then $S^n - g(D^k)$ has trivial reduced homology, and $\tilde{H}_i(S^n - h(S^k))$ equals $\mathbb{Z}$ when $i = n - k - 1$ and is trivial otherwise. The idea behind the proof is to argue inductively using the right Mayer-Vietoris sequence: let $A = S^n - X_1$ and $B = S^n - X_2$, such that $X_1 \cup X_2$ is S^k or D^k, and $X_1 \cap X_2$ is S^{k-1} or D^{k-1}. A direct consequence of this result is that there is no continuous injection $\mathbb{R}^m \to \mathbb{R}^n$, since one would restrict to an embedding $S^n \hookrightarrow \mathbb{R}^n$.

We obtain as further corollaries the Jordan Curve Theorem, as well as the result that any continuous injection $U \to \mathbb{R}^n$ of an open set $U \subseteq \mathbb{R}^n$ is open. The latter proves that any embedding of a compact n-manifolds in a connected n-manifold must be a homeomorphism. This idea can be further developed to show that $\mathbb{R}$ and $\mathbb{C}$ are the only commutative finite-dimension division algebras with identity over $\mathbb{R}$, by using such a structure on $\mathbb{R}^n$ to construct an injective map $f : \mathbb{R}P^{n-1} \to S^{n-1}$. If $n > 1$, S^n is connected, so f must be a homeomorphism by the result above, implying that $n = 2$ as $\mathbb{R}P^{n-1}$ and S^{n-1} have differing homology for other values of n.

The usual embedding $S^2 \hookrightarrow \mathbb{R}^3$ separates $\mathbb{R}^3$ into two simply-connected components, but this is not true for all embeddings. The Alexander Horned Sphere is one such embedding that produces a non simply-connected unbounded component, yet with trivial homology in dimension 1. The embedding is created by inductively towering increasingly smaller horns onto a sphere such that the aggregate of the horns approaches a closed handle.

To prove the Borsuk-Ulam theorem in full generality, it is easier to first prove the lemma that all odd maps $S^n \to S^n$ have odd degree. This is proven algebraically using the long exact sequence associated with the short exact sequence of chain complexes consisting of the sequences

$$0 \longrightarrow C_n(S^n; \mathbb{Z}/2\mathbb{Z}) \longrightarrow C_n(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z}) \longrightarrow C_n(S^n; \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

which are exact since any singular simplex into S^n always lifts to $\mathbb{R}P^n$.

2.4.2 Simplicial Approximation

The added structure of simplicial complexes has some advantages over that of CW complexes. For instance, we can consider **simplicial maps** $f : K \to L$ between simplicial complexes K, L, which are uniquely defined by its values on vertices in K, sending them to vertices in L. The usefulness of simplicial maps are highlighted by the following theorem.

Theorem 2.1 (Simplicial Approximation Theorem)**.** *If $f : K \to L$ maps a finite simplicial complex K to an arbitrary simplicial complex L, then f is homotopic to a map that is simplicial with respect to an iterated barycentric subdivision of K.*

The proof uses the notions of the **star** $\operatorname{St}\sigma$ of a simplex σ, which is the subcomplex containing all simplices that contain σ, and **open star** $\operatorname{st}\sigma$, which contains the interiors of all simplices containing σ. The idea is to find a fine enough subdivision K' so that $f(\operatorname{St} v) \subseteq \operatorname{st} u$ for some vertex $u \in L$, given each vertex $v \in K'$. We then correspondingly set $g(v) = u$.

Simplicial maps can be used to directly generalize Brouwer's fixed point theorem. Define the **trace** of a homomorphism $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$ to be the trace of its matrix $[a_{ij}]$. More generally, the trace of a map $\varphi : A \to A$ of a finitely generated abelian group is the trace of its free component. Define the **Lefschetz number** $\tau(f)$ of a map $f : X \to X$ to be the number $\sum_n (-1)^n \operatorname{tr}(f_* : H_n(X) \to H_n(X))$, given that X is CW complex with finitely many nontrivial homology groups, all of which are finitely generated.

Theorem 2.2 (Lefschetz Fixed Point Theorem)**.** *If X is a retract of a finite simplicial complex map, then any map $f : X \to X$ with $\tau(f) \neq 0$ has a fixed point.*

This proving using simplicial approximation: we can subdivide $X \to L \to K$ such that L, K are fine enough for $\operatorname{St}\sigma \cap \operatorname{st} f(\sigma)$ to be empty for each simplex σ, and $f : K \to L$ is homotopic to a simplicial map g. Hence, we can use the equation

$$\tau(f) = \tau(g) = \sum_n (-1)^n \operatorname{tr}(g_* : H_n(K_n, K_{n-1}) \to H_n(K_n, K_{n-1})) = 0.$$

It is not unfair to doubt the utility of these results should CW complexes be irreparably dissimilar to simplicial complexes in general. This concern is rectified by the following theorem:

Theorem 2.3. *Every CW complex X is homotopy equivalent to a simplicial complex $S(X)$, such that $\dim S(X) = X$ and $S(X)$ is finite (resp. countable) if X is as well.*

2.4.3 Universal Coefficients for Homology

We require a robust method to compute homology with arbitrary coefficients in terms of homology with $\mathbb{Z}$ coefficients. This is done by reformulating $H_n(X;G)$ in terms of the tensor product. Since we have a natural isomorphism $C_n(X,A;G) \to C_n(X,A) \otimes G$, it is natural to ask, given a chain complex C, if there is a way to compute the homology of the chain complex $C \otimes G$ with the boundary map $\partial \otimes \mathbb{1}$. By taking the tensor product of each term in the short exact sequence of chain complexes $0 \to Z \to C \to B \to 0$ with G, where Z, C, B denote the cycle, chain, and boundary chain complexes respectively, we then extract the associated long exact sequence, which breaks up into exact sequences

$$0 \longrightarrow H_n(C) \otimes G \longrightarrow H_n(C;G) \longrightarrow \mathrm{Tor}(H_{n-1}(C), G) \longrightarrow 0,$$

where $\mathrm{Tor}(H_{n-1}(C), G)$ denotes the homology group at the term $B_n \otimes G$ of the sequence

$$0 \longrightarrow \ker(i_n \otimes \mathbb{1}) \longrightarrow B_n \otimes G \longrightarrow Z_n \otimes G \longrightarrow H_n(C) \otimes G \longrightarrow 0.$$

The first family consists of natural, splitting, exact sequences, whose existence and properties are encapsulated in the **universal coefficient theorem for homology**. Indeed, this is exactly what we set out to find. The relative variant of this theorem also exists: its statement is exactly as imagined.

Furthermore, we have also constructed the group $\mathrm{Tor}(A, B)$, whose definition depends only on A, B; in a sense, $\mathrm{Tor}(A, B)$ measures the common torsion between A and B. Tor satisfies many nice properties: it is reflexive and bilinear with respect to direct summation; it equals 0 if either argument is torsionfree; the argument A can be exchanged with the torsion subgroup of A; $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, A) \cong \ker(A \xrightarrow{n} A)$; and for exact sequences $0 \to B \to C \to D \to 0$, there is a naturally associated exact sequence

$$0 \longrightarrow \mathrm{Tor}(A,B) \longrightarrow \mathrm{Tor}(A,C) \longrightarrow \mathrm{Tor}(A,D) \to A \otimes B \to A \otimes C \to A \otimes D \to 0.$$

If the group of coefficients is a field, usually $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$, then the respective homology is easier to compute and still retains the most essential information. For example, we have $H_n(X;\mathbb{Q}) \cong H_n(X;\mathbb{Z}) \otimes \mathbb{Q}$; and if $H_n(X;\mathbb{Z}), H_{n-1}(X;\mathbb{Z})$ are finitely generated, then for each prime p, $H_n(X;\mathbb{Z}/p\mathbb{Z})$ is equal to a direct sum of $\mathbb{Z}/p\mathbb{Z}$ summands for each $\mathbb{Z}$ summand of $H_n(X;\mathbb{Z})$, and each $\mathbb{Z}/p^k\mathbb{Z}$ summand in $H_n(X;\mathbb{Z})$ and $H_{n-1}(X;\mathbb{Z})$.

2.4.4 Tensor Products and Adjoint Functors

There is an isomorphism $\mathrm{Hom}_R(A \otimes_R B, C) \to \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C))$ given by $\phi \mapsto (a \mapsto (b \mapsto \phi(a \otimes b)))$. Moreover, this isomorphism is natural in A, B, C. This is essentially a reformulating of the universal property of tensor products, and leads into the notion of adjoint functor. Covariant functors $F : C \to D, G : D \to C$ form an adjoint pair if $\mathrm{Mor}_D(Fc, d) \cong \mathrm{Mor}_C(c, Gd)$, for all $c \in \mathrm{Ob}\, C, d \in \mathrm{Ob}\, D$, which is natural in c and d. Here, F is the left adjoint of G and G is the right adjoint of F. Thus, the adjoint property of tensor products says that the functors $- \otimes_R B : R - \mathrm{MOD} \to R - \mathrm{MOD}$ and $\mathrm{Hom}_R(B, -) : R - \mathrm{MOD} \to R - \mathrm{MOD}$ form an adjoint pair.

3 Cohomology

3.1 Cohomology Groups

3.1.1 The Universal Coefficient Theorem

The cohomology groups of a chain complex $\cdots \to C_n \to C_{n-1} \to \dots$ are constructed by first dualizing the complex—replacing each chain group C_n by the dual **cochain group** $\mathrm{Hom}(C_n, G)$, for some fixed group G, and each boundary map ∂_n with the dual **coboundary map** $\delta_n : C^*_{n-1} \to C_n$—and finally forming the **cohomology group** $H^n(C;G) := \ker \delta_{n+1}/\operatorname{img} \delta_n$. A coset $\overline{\varphi} \in H^n(C;G)$ is represented by a map $\varphi : C_n \to G$ such that $\delta\varphi = \varphi\partial = 0$; that is, φ vanishes on the boundary group B_n. It turns out that $H^n(C;G)$ is determined algebraically by G and the homology groups $H_n(C)$.

This relationship is given by the **universal coefficient theorem for cohomology**

Theorem 3.1. *If C is a chain complex of free abelian groups, then its homology groups $H_n(C)$ determine the cohomology groups $H^n(C;G)$ of the cochain complex $Hom(C_n, G)$ via the split exact sequences*

$$0 \longrightarrow Ext(H_{n-1}(C), G) \longrightarrow H^n(C;G) \xrightarrow{h} Hom(H_n(C), G) \to 0,$$

where h is a surjection that sends the coset represented by the map $\varphi : C_n \to G$ to the quotient map $\overline{\varphi|_{Z_n}} : Z_n/B_n \to G$ induced by the restriction $\varphi|_{Z_n}$; and given a free completion F of H, $Ext(H,G)$ is the 1^{st} cohomology group of the cochain complex $Hom(F,G)$, which is independent of the choice of F.

As an aside, $\mathrm{Ext}(H, G)$ satisfies properties similar to its dual, $\mathrm{Tor}(H, G)$. Using the universal coefficient theorem, we get a nice isomorphism directly relating $H^n(C)$ and $H^n(C;\mathbb{Z})$. Fix a chain complex C of finitely generated free abelian groups, and let $T_n \subseteq H_n$ be the torsion subgroup. Then, $H^n(C;\mathbb{Z}) \cong (H_n/T_n) \oplus T_{n-1}$.

3.2 Cohomology of Spaces

We hence define the group $C^n(X;G)$ of **singular n-cochains with coefficients in G** to be the dual group $\mathrm{Hom}(C_n(X), G)$ of the singular chain group $C_n(X)$; an arbitrary n-cochain φ extends some assignment of each n-simplex σ to a value $\varphi(\sigma) \in G$. Moreover, we define the **cohomology group $H^n(X;G)$ with coefficients in G** to be the cohomology groups of the respective cochain complex. Notably, $H^1(X;G) \cong \mathrm{Hom}(H_1(X), G)$.

Cohomology shares the same plethora of tools available to homology. Reduced and relative cohomology arise from a similar dualization process, which also shows the existence of a dual long exact sequence associated with a pair (X, A). Each map $f : (X, A) \to (Y, B)$ contravariantly induces a cochain map $f^{\#} : H^n((Y, B); G) \to H^n((X, A); G)$, which is invariant under homotopy equivalence. More notions that carry over to the realm of cohomology include excision, simplicial and cellular cohomology, and Mayer-Vietoris sequences.

Finally, we can consider axioms which determine generalized **cohomology theories** on CW complexes, which include a homotopy invariance axiom, an axiom guaranteeing the existence of a long exact sequence associated with a pair, and an axiom stating $\prod_\alpha i^*_\alpha$:

$\tilde{h}^n(\vee_\alpha X_\alpha) \cong \prod_\alpha \tilde{h}^n(X_\alpha)$. The product noticeably replaces the direct sum in the respective homology axiom due to the natural isomorphism $\mathrm{Hom}(\oplus_\alpha A_\alpha, G) \cong \prod_\alpha \mathrm{Hom}(A_\alpha, G)$.

3.3 Cup Product

A surprising aspect of cohomology is that its construction gives rise to a more complicated algebraic structure not shared by homology: a product which expands the set of all cohomology classes into a ring. For cochains, $\varphi \in C^k(X;R), \psi \in C^\ell(X;R)$, the **cup product** $\varphi \smile \psi \in C^{k+\ell}(X;R)$ has the formula $(\varphi \smile \psi)(\sigma) = \varphi(\sigma|[v_0,\dots,v_k])\psi(\sigma|[v_k,\dots,v_{k+\ell}])$.

The cup product satisfies $\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^k\varphi \smile \delta\psi$, which implies that a product of two cocycles is a cocycle, and a product of a cocycle and coboundary is a coboundary. Thus, we have an induced cup product on the cohomology groups which is associative and distributive, and has the identity $[\sigma \mapsto 1] \in H^0(X;\mathbb{R})$. Furthermore, for any map $f : X \to Y$, the induced a map $f^* : H^n(Y;R) \to H^n(X;R)$ is multiplicative, i.e. $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$; and we have a skew symmetric rule $\alpha \smile \beta = (-1)^{k\ell}\beta \smile \alpha$ always holds. Also, the cup product has a relative form: if either input is a cohomology class of X relative to A, then the output is also relative to A.

3.3.1 The Cohomology Ring

Define $H^*(X;R)$ to be the direct sums of the cohomology groups $H^n(X;R)$. Under addition and the cup product as multiplication, we obtain not just a ring, but an R-algebra, on the cohomology classes of X. This sort of construction is generally known as a **graded ring**: a ring $A = \sum_{k\geq 0} A_k$ of additive subgroups A_k augmented with a multiplication $A_k \times A_\ell \to A_{k+\ell}$. Let the dimension $|\alpha| = k$ of α index the subgroup A_k containing α.

We provide some examples of graded rings which are cohomological. $H^*(\mathbb{R}P^n;\mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2[\alpha]/(\alpha^{n+1})$, where α is a generator of H^1, and $H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[\beta]/(\beta^{n+1})$, where β is a generator of H^2; if $\alpha_1,\dots,\alpha_n$ have odd dimension, then $\vee_n S^{|\alpha|}$ is isomorphic to the exterior algebra $\Lambda_R[\alpha_1,\dots,\alpha_n]$.

The isomorphism $H^*(\coprod_\alpha X_\alpha;R) \to \prod_\alpha H^*(X_\alpha;R)$ induced by the inclusions $X_\alpha \hookrightarrow \coprod_\alpha X_\alpha$ is a ring isomorphism if define multiplication to be coordinatewise. A similar ring isomorphism holds for reduced cohomology One might ask the question of which graded commutative R-algebras occur as a cup product algebra of some space. This true for “essentially every” graded commutative $\mathbb{Q}$-algebra, not well known for $\mathbb{Z}_p$-algebras with p prime, and barely known for $\mathbb{Z}$-algebras. Results such as this that compute the homology or cohomology of a product space are called **Künneth formulas**.

3.3.2 A Künneth Formula

It is possible to relate the product of cohomology rings of spaces X, Y to the cohomology ring of $X \times Y$ via the cup product. We begin by defining $\times : H^*(X;R) \times H^*(Y;R) \to H^*(X \times Y;R)$ by $a \times b = \pi_X^*(a) \smile \pi_Y^*(b)$. However, $\times$ is usually not a homomorphism, so we instead take the domain to be the tensor product $H^*(X;R) \otimes_R H^*(Y;R)$. Then, if X and Y are CW complexes and $H^k(Y;R)$ is always finitely generated free, then the cross product $H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y;R)$ is a ring isomorphism.

3.3.3 Spaces with Polynomial Cohomology

Using the cohomology rings structure of $\mathbb{R}P^n$, we can prove that division algebra structures on $\mathbb{R}^n$ over $\mathbb{R}$ exist only when n is a power of 2.

4 Exercises

Q0.9

Suppose there exists a retraction $r : X \to A$ for spaces $A \subseteq X$. Let $1 := \{q\}$ denote an arbitrary singleton, and assume there exist maps $f : X \to 1, g : 1 \to X$ such that we can find a homotopy h_t between $h_0 = gf$ and $h_1 = \mathbb{1}$ with the associated continuous map $h : (t, x) \mapsto h_t(x)$. Finally, let $i : A \to X$ denote the inclusion map. This information is captured by the following diagram:

$$\begin{tikzcd} A \arrow[r, "i"] & X \arrow[d, "h_t"] \arrow[dr, bend left, "f"] & \\ & X \arrow[ul, bend left, "r"] & 1 \arrow[l, "g"] \end{tikzcd}$$

It suffices to show that $fi \circ rg \simeq \mathbb{1}|_1$ and $rg \circ fi \simeq \mathbb{1}_A$.

Since $fi \circ rg$ necessarily maps q to q, $fi \circ rg$ is both constant and identity, so we can define the homotopy $F_t : 1 \to 1$ by $F_t(q) = q$. We verify immediately that the associated map $F : (t, q) \mapsto F_t(q)$, as well each F_t, is constant and thus continuous, and $fi \circ rg = F_0 = F_1 = \mathbb{1}$.

Next, define the map $G_t : A \to A$ simply to be rh_ti, which is continuous for any fixed t as a composition of continuous maps. Furthermore, by assumption we have $G_0 = rg \circ fi$ and $G_1 = r\mathbb{1}i = \mathbb{t}$, since $r|_A = \mathbb{1}$ when viewing A as a subspace of X. Finally, to verify that the map $G : (t, a) \mapsto G_t(a)$ is continuous, we notice that it can be factored as

$$(t, a) \overset{i}{\longmapsto} (t, a)_{I \times X} \overset{h}{\longmapsto} h_t(a_X) = h_ti(a) \overset{r}{\longmapsto} rh_ti(a),$$

which is a composition of continuous maps.

Therefore, A is contractible.

Q2.1.8

We begin by describing the n-simplices of X. The n vertices comprising the outer cycle are collapsed to a single 0-simplex b, as is the central two vertices to a 0-simplex a: informally, in the process of gluing T_n to T_{n+1}, we "translate" the outer vertices of T_n to the right by one, and the inner vertices up by one.

The edges connecting two outer vertices or two inner vertices are collapsed to either of the loops $r : a \circlearrowleft a$ and $s : b \circlearrowleft b$. Furthermore, the top right edge of T_i (adjacent to T_{i+1}) is identified with the bottom left edge of T_i; we will refer to this quotient edge by t_i.

There are n horizontal 2-simplices E_i which are the quotients of the bottom face of each T_i, and n vertical simplices F_i which are the quotients of the right face of each T_i. Finally, we can identify each 3-simplex with exactly one T_i.

Now, we will compute the simplicial homotopy groups of X:

- Since X is non-empty and path-connected, $H_0^\Delta(X) \cong \mathbb{Z}$ (Prop 2.7).

- Next, we will compute $H_1^\Delta(x)$. $\operatorname{img} \partial_2$ is spanned by $\{\partial E_i, \partial F_i\}_i = \{t_{i+1} - t_i + s, t_{i+1} - t_i + r\}_i$, where the index i is reduced mod n, which suggests the basis $\{r - s\} \cup \{t_{i+1} - t_i + s\}_{1 \leq i \leq n}$. Furthermore, fixing $\sum_i n_i t_i + n_r r + n_s s \in \ker \partial_1$ implies $\sum_i n_i(a - b) + n_r(a - a) + n_s(b - b) = 0$, which means $\ker \partial_1$ has the basis $\{r - s, s\} \cup \{t_{i+1} - t_i\}_i$. We chose these bases so that each element in $\ker \partial_1 / \operatorname{img} \partial_2$, except possibly the cosets of $s, 2s, \ldots$, are 0. Incidentally, the only multiples of s in $\operatorname{img} \partial_2$ are in the span of $\sum_i [t_{i+1} - t + i + s] = ns$. Thus, $\ker \partial_1 / \operatorname{img} \partial_2 \cong \mathbb{Z}/n\mathbb{Z}$.

- Onto $H_2^\Delta(X)$. $\operatorname{img} \partial_3$ is spanned by $\{\partial T_i\}_i = \{F_{i+1} - F_i + E_i - E_{i+1}\}_i$, and if $\sum_i [n_i E_i + m_i F_i] \in \ker \partial_2$, then

$$\begin{aligned} 0 &= \sum_i [n_i(t_{i+1} - t_i + s) + m_i(r - t_i + t_{i+1})] \\ &= \sum_i [(n_i + m_i)(t_{i+1} - t_i) + n_i s + m_i r], \end{aligned}$$

  which implies $\ker \partial_2$ is spanned by the same set. So $H_2^\Delta(X) \cong 0$.

- Since $\delta_4(X)$ is trivial, $H_3^\Delta(X) \cong \ker \partial_3$. Suppose $\partial \sum_i n_i T_i = 0$. This implies that $\sum_i n_i(F_{i+1} - F_i + E_i - E_{i+1}) = 0$, which occurs when n_i is constant. Hence, $\ker \partial_3$ has the basis $\{\sum_i T_i\}$, which means $H_3^\Delta(X) \cong \mathbb{Z}$.

- $H_n^\Delta(X)$ is trivial for $n \geq 4$ since X only contains n-simplices, $n \leq 3$.

Q2.1.22

Proof of (a). The base case for $n = 0$ follows from Prop 2.7 and Prop 2.8. Now, let $n > 0$ and assume the IH on $n - 1$. For $i \geq n$, consider the following sequence of maps:

$$H_i(X) \cong \tilde{H}_i(X) \xrightarrow{f} \tilde{H}_i(X/X^{n-1}) \cong \tilde{H}_i(\vee_{\alpha\in A} S^n) \cong \oplus_{\alpha \in A} \tilde{H}_i(S^n).$$

Here, A indexes the n-cells. The $1^{\text{st}} \cong$ comes from $n > 0$. f is an injection since the following sequence is exact by Thm 2.13, recalling the IH and the fact that (X, X^{n-1}) is a good pair:

$$0 \cong \tilde{H}_i(X^{n-1}) \to \tilde{H}_i(X) \to \tilde{H}_i(X/X^{n-1}). \tag{*}$$

The $2^{\text{nd}} \cong$ comes from homotopy invariance, since the space X/X^{n-1} is the wedge of the quotients of the n-cells of X by their respective boundaries, at the point x_0 identified with X^{n-1}. The last $\cong$ is by Cor 2.25. Finally, since α is non-empty by the dimension of X, Cor 2.14 tells us that $\tilde{H}_i(S^n)$ is $\mathbb{Z}$ when $i = n$, in which case $H_i(X)$ is free, and 0 when $i > n$, in which case $H_i(X) = 0$. This finishes the inductive step. ■

Proof of (b). Suppose X has dimension m. By Thm 2.16, the following sequence is exact for all $i > n$:

$$H_n(X^i) \xrightarrow{f_i} H_n(X^{i+1}) \to H_n(X^{i+1}, X^i).$$

By Prop 2.22 and (a), when $i < m$,

$$H_n(X^{i+1}, X^i) \cong \tilde{H}_n(X^{i+1}/X^i) \cong 0,$$

so exactness forces f_i to be an isomorphism. It follows that

$$H_n(X^{n+1}) \cong \ldots \cong H_n(X^m) = H_n(X).$$

Now, we prove the statement of (b), assuming X has no cells of dimension $n-1, n+1$. If $n > m$, then (b) holds by (a). If $n = m$, then the map $f : \tilde{H}_i(X) \to \tilde{H}_i(X/X^{n-1})$ defined in (a) is an isomorphism, since the following sequence is exact:

$$0 \cong \tilde{H}_n(X^{n-1}) \to \tilde{H}_i(X) \xrightarrow{f} \tilde{H}_n(X/X^{n-1}) \to H_{n-1}(X^{n-1}) \cong H_{n-1}(X^{n-2}) \cong 0.$$

This completes the chain (*) of isomorphisms in (a) when $i = n$, which implies that $H_n(X)$ is the free product with n-cells as generators. If $n < m$, then

$$H_n(X) \cong H_n(X^{n+1}) \cong H_n(X^n),$$

which reduces to the second case. ■

Proof of (c). Let $m = \dim X$. If $n \geq m$, this follows from the previous parts, so let $n < m$. Recall from (a) that $H_n(X^n) \leq \mathbb{Z}^k$, so $H_n(X^n)$ has at most k generators. By exactness of the following sequence,

$$H_n(X^n) \to H_n(X^{n+1}) \to H_n(X_{n+1}, X_n) \cong 0,$$

$H_n(X^n)$ surjects onto $H_n(X^{n+1}) \cong H_n(X)$, which proves the statement. ■

Q2.2.1

Proof. Let $F : S^n \to S^n$ be the map that sends the northern and southern hemispheres N and S of S^n to S via f. F is continuous since its restriction to either hemisphere is continuous, and the two values of F coincide on the compact intersection S^{n-1} of N and S. Since F is not surjective, $\deg F = 0$. Thus, F must have a fixed point, since otherwise it would have non-zero degree. Because img $F = F(S)$, the restriction $F|S = f$, where S is identified with D^n, must admit a fixed point. ■

Q2.2.9

Proof. Let $\lambda_1, \ldots, \lambda_k$ be the roots of f, and let $U_1, \ldots, U_k$ be their respective disjoint open neighbourhoods on S^2. Let m_i be the multiplicity of λ_i, and let $g(z)$ satisfy $f(z) = h(z)g(z)$, where $h(z) = (z - \lambda_i)^{m_i}$. Since the neighbourhoods U_j are disjoint, $\hat{g}$ is nonzero on U_i, so locally $\hat{g}$ behaves like a nonzero constant. Since multiplication by a nonzero complex number rotates S^2 about the $\{0, \infty\}$ axis and latitudinally stretches the sphere by some nonzero magnitude, $\hat{f}(z)$ is homotopic to $\hat{h}(z)$, which implies $\deg \hat{f}|\lambda_i = \deg \hat{h}|\lambda_i$. We also know $\hat{h}^{-1}(0) = \lambda_i$ and hence $\hat{h}|\lambda_i = \deg \hat{h} = m_i$, since, for any $z \neq \lambda_i$, $\hat{h}$ maps m_i disjoint open neighbourhoods homeomorphically to an open neighbourhood of z. This proves that the local degree of $\hat{f}$ at each λ_i is the corresponding multiplicity, and by the Degree Sum Formula and Fundamental Theorem of Algebra, $\deg \hat{f} = \deg f$. ■

Q2.2.31

Proof. Note that the subspaces $A := X \vee V$ and $B := Y \vee U$ of $X \vee Y$ are open sets which cover $X \vee Y$. For $n > 0$, we extract from the corresponding Mayer-Vietoris sequence the following short exact sequence:

$$\begin{tikzcd}
\tilde{H}_n(A \cap B) \arrow[r] \arrow[d, "\cong"] & \tilde{H}_n(A) \oplus \tilde{H}_n(B) \arrow[r, "\varphi"] \arrow[d, "\cong"] & \tilde{H}_n(X \vee Y) \arrow[r] \arrow[d, equal] & \tilde{H}_{n-1}(A \cap B) \arrow[d, "\cong"] \\
0 \arrow[r] & \tilde{H}_n(X) \oplus \tilde{H}_n(Y) \arrow[r, "\varphi"] & \tilde{H}_n(X \vee Y) \arrow[r] & 0
\end{tikzcd}$$

The first isomorphism is induced by $A \cap B = U \vee V \simeq x_0 \vee x_0$, which has trivial reduced homology. The second isomorphism is induced by $X \vee V \simeq X \vee x_0 \cong X$ and $Y \vee U \simeq Y \vee x_0 \cong Y$. The last isomorphism exists for the same reason as the first. By exactness, φ is an isomorphism. ■

Q2.2.32

Proof. Define $A, B \subseteq SX$ to be the images of $[0, \frac{3}{4}) \times X, (\frac{1}{4}, 1] \times X$ under the quotient map $X \times I \to SX$. Note that $\{A, B\}$ forms an open cover of SX, the corresponding Mayer-Vietoris sequence grants us the following short exact sequence for each $n > 0$:

$$\begin{tikzcd}
\tilde{H}_n(A) \oplus \tilde{H}_n(B) \arrow[r] \arrow[d, "\cong"] & \tilde{H}_n(SX) \arrow[r, "\varphi"] \arrow[d, equal] & \tilde{H}_{n-1}(A \cap B) \arrow[r] \arrow[d, "\cong"] & \tilde{H}_{n-1}(A) \oplus \tilde{H}_{n-1}(B) \arrow[d, "\cong"] \\
0 \arrow[r] & \tilde{H}_n(SX) \arrow[r, "\varphi"] & \tilde{H}_{n-1}(X) \arrow[r] & 0
\end{tikzcd}$$

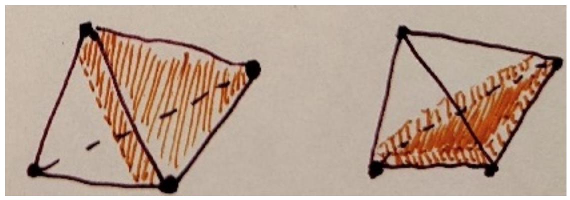
The first and last isomorphisms come from the fact that $[0, \frac{3}{4}) \times X \cong (\frac{1}{4}, 1] \times X \cong CX$, which is contractible. The second isomorphism comes from the fact that $(\frac{1}{4}, \frac{3}{4}) \times X$ deformation retracts onto X. By exactness, φ is an isomorphism. ■

Q2.2.33

Proof. We prove the first claim by induction on n. For $n = 1$, if A_1 is either empty or has trivial reduced homology, then $\tilde{H}_k(A_1) = 0$ for all $k \geq 0$ by assumption. Now, assume the IH for $n \geq 1$, and let $X := X' \cup A_{n+1}$, where $X' := \cup_{i=1}^n A_i$ and A_i satisfy the required properties. This implies that X' and A_{n+1} have trivial homology; by consequence $X' \cap A_i = \cup_{i=1}^n (A_i \cap A_{n+1})$ does as well, since each $A_i \cap A_{n+1}$ is open and $\cap_{j=1}^m (A_{i_j} \cap A_{n+1}) = \cap_{j \in \{i_1, \ldots, i_m, n+1\}} (A_{i_j})$ is always empty or has trivial reduced homology. Hence, the following exact sequence obtained from the Mayer-Vietoris sequence guarantees that $\tilde{H}_k(X) = 0$ for $k \geq n$:

$$\begin{tikzcd}
\tilde{H}_k(X') \oplus \tilde{H}_k(A_{n+1}) \arrow[r] \arrow[d, "\cong"] & \tilde{H}_k(X) \arrow[r, "\varphi"] \arrow[d, "\cong"] & \tilde{H}_{k-1}(X' \cap A_{i+1}) \arrow[d, "\cong"] \\
0 \arrow[r] & \tilde{H}_k(X) \arrow[r] & 0
\end{tikzcd}$$

We must now show that the bound is tight. For $n \geq 2$, consider the space $\partial\Delta^{n-1} = \cup_{i=1}^{n} U(\Delta_i^{n-2})$, where Δ_i^{n-2} are the faces of Δ^{n-1}, and $U(\Delta_i^{n-2})$ is a "thickening" of Δ_i^{n-2}, such that the elements of any subset of $\{U(\Delta_i^{n-2})\}_i$ intersect at an open set V which deformation retracts to the proper face containing exactly the vertices of Δ^{n-1} contained by V. For example, here are two such $U(\Delta_i^2) \subseteq \partial\Delta^3$.



By construction, $\cap_{j=1}^{m} U(\Delta_{i_j}^{n-2})$ is always empty or contractible, but $H_{n-2}(\partial\Delta^{n-1})$ is non-trivial: to see this, notice that in the chain complex

$$0 \cong \Delta_{n-1}(\partial\Delta^{n-1}) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(\partial\Delta^{n-1}) \xrightarrow{\partial_{n-2}} \Delta_{n-2}(\partial\Delta^{n-1}),$$

∂_{n-1} has trivial image whereas ∂_{n-2} does not have trivial kernel, which is one of the earliest example of homology we computed. ■

Q2.3.3

Proof. Via Axiom 3,

$$\tilde{h}_n(\text{point}) = \tilde{h}_n(\vee_\emptyset \text{ point}) \cong \oplus_\emptyset \tilde{h}_n(\text{point}) = 1,$$

or

$$\tilde{h}_n(\text{point}) \cong \frac{\tilde{h}_n(\vee_2 \text{ point})}{\tilde{h}_n(\text{point})} = \frac{\tilde{h}_n(\text{point})}{\tilde{h}_n(\text{point})} = 1.$$

■

Q1.1.ii

Proof. Since the subcategory C' inherits associativity of morphism composition, it suffices to show that the identity morphism 1_X is an isomorphism for each X, which is true since its inverse is itself, and that the composition fg of isomorphisms is itself an isomorphism, which holds since $fgg^{-1}f^{-1} = 1$ implies $g^{-1}f^{-1} = (fg)^{-1}$. ■

Q1.4.i

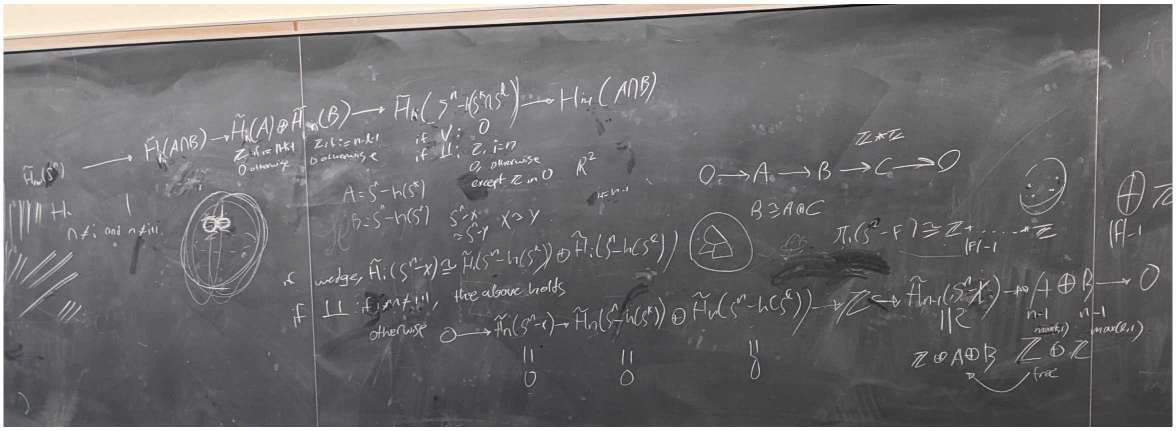
Proof. We can turn the statement expressing commutativity of the square diagram containing a pair of components a_c, a'_c of α into commutativity of the opposite square diagram containing $a_c^{-1}, a_c'^{-1}$:

$$\begin{aligned} Gfa_c &= a'_c Ff \\ Gf &= a'_c Ffa_c^{-1} \\ a_{c'}^{-1} Gf &= Ffa_c^{-1}. \end{aligned}$$

■

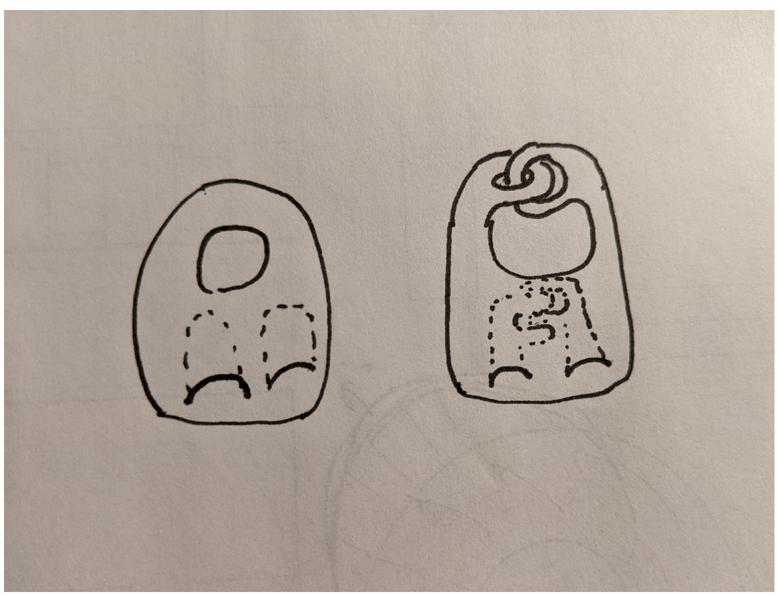
Q2.B.1

I have solved this but I did not have the time to properly present the solution. For what it's worth, here are my computations which I worked on with help from Marcus.



Q2.B.6

We can mimic the construction of the Alexander Horned Sphere given in Hatcher by taking of boundary ∂X of the intersection X of a countable family $X_0 \supseteq X_1 \supseteq \ldots$, except in addition to the process of transforming each short handle connecting a pair of horns in X_i into a pair of linked horns-and-short-handles, we indent cavities into X_i such that the union C of the cavities of X has a negative space which resembles the union of all horns in X: more formally, the complement of C is homeomorphic to the union H of the handles of X. The first two iterations are illustrated below:



To see that ∂X is homeomorphic to S^2, note that it decomposes into the union of the two "hemispheres" H^+ and H^- which intersect at the "equator". It is clear that $H^+ \cong H^-$, and $H^+ \cong D^2$ since it can be identified with the upper hemisphere of the Alexander Horned

Sphere. Thus, X has the standard CW structure on S^2 consisting of two 2-cells attached along a closed loop.

Furthermore, the bounded component induced by ∂X is non simply-connected for the same reason that the unbounded component induced by the Alexander Horned Sphere is non simply-connected.

Q2.B.8

Proof. For $a \in \mathbb{R}^{2n-1}$, let $M(a) \in \mathbb{R}^{2(2n+1)}$ be the matrix associated with the transformation $x \mapsto ax$. The map $f : \mathbb{R}^{2n+1} \to \mathbb{R}$ defined by $a \mapsto M(a) \mapsto \det M(a)$ is continuous since each individual mapping is k-linear for some k. Then if $p : I \to \mathbb{R}^{2n+1} - \{0\}$ is a path joining any two antipodal points, we have

$$\begin{aligned} f(p(0)) &= \det M(p(0)) = p(0)^{2n+1}, \\ f(p(1)) &= \det M(-p(0)) = -p(0)^{2n+1}. \end{aligned}$$

By the intermediate value theorem, $f(p(t)) = 0$ for some $t \in I$, but this means that $M(p(t))$ is invertible, so $x \mapsto p(t)x$ is not an injective mapping. ∎

Q2.C.2

Proof. For $f : S^n \to S^n$

$$\begin{aligned}\tau(f) &= \sum_k (-1)^k \operatorname{tr} f_*|_{H_k(S^n)} \\ &= \operatorname{tr} f_*|_{H_0(S^n)} + (-1)^k \operatorname{tr} f_*|_{H_n(S^n)} \quad \text{(trivial homomorphisms induce “empty” matrices)} \\ &= 1 + (-1)^n \deg f. \qquad (f \text{ acts on the generator via multiplication by } \deg f)\end{aligned}$$

Thus, $\tau(f) = 0$ iff $\deg f = (-1)^{n+1} = \deg(x \mapsto -x)$. ∎

Q2.C.8

Proof. Let $f : X \to S(X), g : Y \to S(Y)$ be homotopy equivalences from X, Y to the simplicial complexes $S(X), S(Y)$, where $S(X)$ is finite and $S(Y)$ is countable. Let f', g' be the respective homotopy inverses. Then, for maps $u, v : X \to Y$,

$$u \simeq v \implies guf' \simeq gvf' \implies g'guf'f \simeq g'gvf'f \implies u \simeq v,$$

implying that $u \simeq v$ if and only if the maps $guf', gvf' : S(x) \to S(Y)$ are homotopic. Hence, we've reduced the problem to proving that there are at most countably many maps from a finite simplicial complex K to a countable simplicial complex L.

By simplicial approximation, any map $K \to L$ is homotopic to a map which is simplicial with respect to some barycentric subdivision K'. There are at most $\aleph_0^m \cdot \aleph_0 = \aleph_0$ such maps, since $\aleph_0^{|\{\text{vertices of K'}\}|}$ bounds the number of simplicial maps $K' \to L$, and $\aleph_0$ is the number of barycentric subdivisions of K. Thus, there are at most $\aleph_0$ homotopy types $S(X) \to S(Y)$. ∎

Q2.C.9

Proof. Recall that each finite CW complex is homotopy equivalent to some finite simplicial complex, so it suffices to consider only simplicial complexes. A simplicial complex C is uniquely determined by a finite sequence $c_1 \ldots c_n$ determining the number of k-simplices of C for $k \leq n = \dim C$, as well as a rule that attaches each k-simplex K to C by identifying the vertices of K with the vertices of previously attached m-simplices according to some restrictions. There are countably many such sequences, and each sequence admits finitely many attaching rules, since there are finitely many maps $f : \{\text{vertices of } K\} \to \{\text{vertices of } C\}$. Therefore, there are at most countably many simplicial complexes. ∎

Q3.A.2

The short exact sequence $0 \to \mathbb{Q} \to \mathbb{Z} \to \mathbb{Z}/\mathbb{Q} \to 0$, along with property (6) of the Tor functor, gives us

$$\operatorname{Tor}(\operatorname{Tor}(A), \mathbb{Q}) \longrightarrow \operatorname{Tor}(\operatorname{Tor}(A), \mathbb{Q}/\mathbb{Z}) \longrightarrow \operatorname{Tor}(A) \otimes \mathbb{Z} \longrightarrow \operatorname{Tor}(A) \otimes \mathbb{Q}.$$

By property (2), the first term equals $\operatorname{Tor}(A, \mathbb{Q})$, which reduces to 0 by property (3), and the second term equals $\operatorname{Tor}(A, \mathbb{Q}/\mathbb{Z})$; as easy algebraic facts, the third term is $\operatorname{Tor}(A)$ and the last term is 0 (tensoring by $\mathbb{Q}$ kills torsion). Thus, we get an isomorphism $\operatorname{Tor}(A, \mathbb{Q}/\mathbb{Z}) \to \operatorname{Tor}(A)$. For the remaining part, ($\Longrightarrow$) follows from property (3), and the hypothesis of ($\Longleftarrow$) forces $\operatorname{Tor}(A) = \operatorname{Tor}(A, \mathbb{Q}/\mathbb{Z}) = 0$.

Q3.1.1

Given a map $\alpha : H \to H'$, $\mathrm{Ext}(-, G) : \mathrm{Grp} \to \mathrm{Grp}$ induces a map $\alpha^* : \mathrm{Ext}(H', G) \to \mathrm{Ext}(H, G)$ in the following way: we can construct a free resolution $0 \to F_1 \to F_0 \to H \to 0$, by letting F_0 be the free group with a generator set of H as its basis, and letting F_1 be the free kernel of the usual surjection $F_0 \to H$. Similarly, construct a free resolution for H'. α hence induces the chain map

$$\begin{tikzcd}
0 \arrow[r] & F_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & F_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"] & H \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & F_1' \arrow[r, "f_1'"] & F_0 \arrow[r, "f_0'"] & H' \arrow[r] & 0
\end{tikzcd}$$

whose construction is given in Hatcher 194—the idea is to inductively construct α_n on $x \in F_n$ by pulling $\alpha_{n-1} f_n(x)$ back to some $x' \in F_n'$ by exactness, and setting $\alpha_n(x) = x'$. Dualizing this diagram via $\mathrm{Hom}(-, G)$, we get a chain map $\alpha_1^* : F_1'^* \to F_1^*$, which induces a map $\alpha^* : \mathrm{Ext}(H', G) \to \mathrm{Ext}(H, G)$ between cohomology groups. The induced map $\mathbb{1}^*$ equals $\mathbb{1}$ since $\mathbb{1}^* \varphi = \varphi$ for any map $\varphi : A \to G$, and the composition $\alpha\beta$ of maps $\alpha : F' \to F''$, $\beta : F \to F'$ induces a map $(\alpha\beta)^* = \beta^* \alpha^* : F'' \to F$, since $(\alpha\beta)_n$ can constructed to coincide with $\alpha_n \beta_n$. Thus, $\mathrm{Ext}(-, G)$ is a contravariant functor.

Given another map $\gamma : F \to G$, $\mathrm{Ext}(H, -) : \mathrm{Grp} \to \mathrm{Grp}$ induces a map $\gamma_* : \mathrm{Ext}(H, F) \to \mathrm{Ext}(H, G)$ which pushes a coset $\overline{\varphi} = \varphi + K \in \mathrm{Ext}(H, F)$ forward to $\gamma\overline{\varphi} \in \mathrm{Ext}(H, G)$: given $\varphi : F_1 \to F$, we have $\gamma_* \varphi : F_1 \to G$, and for each

$$\psi \in K = \mathrm{img}(f_1^* : \mathrm{Hom}(F_0, F) \to \mathrm{Hom}(F_1, F)),$$

we have $\gamma_* \psi : F_1 \to G \in (f_1^* : \mathrm{Hom}(F_0, G) \to \mathrm{Hom}(F_1, G))$, since $\gamma_* \psi$ equals the composition

$$F_1 \xrightarrow{f_1} F_0 \xrightarrow{\psi' \in (f_1^*)^{-1}(\psi)} F \xrightarrow{\gamma} G.$$

This shows that γ_* is well-defined. Clearly, $\mathbb{1}_* = \mathbb{1}$ and $(\alpha\beta)_* = \alpha_* \beta_*$, which shows that $\mathrm{Ext}(H, -)$ is a covariant functor.

3.2.3

Let α be a generator of $H^1(\mathbb{R}P^m;\mathbb{Z}_2)$, which equals $\mathbb{Z}_2$ (Theorem 3.19). If f^* is not trivial on H^1, then $f^*(\alpha)$ equals another generator $\beta \in H^1(\mathbb{R}P^m;\mathbb{Z}_2)$. By the equivalence $H^*(\mathbb{R}P^m;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$, and likewise for $\mathbb{R}P^n$ with β instead of α, we have

$$\begin{aligned}
0 &= f^*(0) \\
&= f^*(\alpha^{m+1}) \\
&= f^*(\alpha)^{m+1} && (f^* \text{ is a ring homomorphism}) \\
&= \beta^{m+1} \\
&\neq 0. && (\text{since } n \geq m+1)
\end{aligned}$$

which is a contradiction. Similarly, $f : \mathbb{C}P^n \to \mathbb{C}P^m$ always induces a trivial map $f^* : H^2 \to H^2$ for $n > m$, and the proof follows from Theorem 3.19 almost identically: the only difference is that, assuming nontriviality guarantees that f maps $\alpha \mapsto n\beta$, where α, β are generators of H^2, and $f^*(\alpha)^{m+1} = n^{m+1}\beta^{m+1} \neq 0$.

3.2.7

Example 3.14 gives us an isomorphism $\tilde{H}^*(\mathbb{R}P^2 \vee S^3;\mathbb{Z}_2) \cong \tilde{H}^*(\mathbb{R}P^2;\mathbb{Z}_2) \oplus \tilde{H}^*(S^3;\mathbb{Z}_2)$. Taking a generator $\alpha \in H^1(\mathbb{R}P^3;\mathbb{Z}_2)$ and $\alpha^2 \in H^2$, we compute $\alpha \smile \alpha^2 = \alpha^3 \neq 0$ by the isomorphism $H^*(\mathbb{R}P^3;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^3)$. However, examples 3.14, 3.16 provides a formula for the cohomology of $\mathbb{R}P^2$, S^3 with coefficients in $\mathbb{Z}_2$:

$$\begin{aligned}
\tilde{H}^1(\mathbb{R}P^2;\mathbb{Z}_2) \oplus \tilde{H}^1(S^3;\mathbb{Z}_2) &\cong \mathbb{Z}_2 \oplus 0 \\
\tilde{H}^2(\mathbb{R}P^2;\mathbb{Z}_2) \oplus \tilde{H}^2(S^3;\mathbb{Z}_2) &\cong \mathbb{Z}_2 \oplus 0 \\
\tilde{H}^3(\mathbb{R}P^2;\mathbb{Z}_2) \oplus \tilde{H}^3(S^3;\mathbb{Z}_2) &\cong 0 \oplus \mathbb{Z}_2.
\end{aligned}$$

Since ring multiplication is componentwise, $\gamma \smile \eta = 0$ for any $\gamma \in \tilde{H}^1(\mathbb{R}P^2;\mathbb{Z}_2), \eta \in \tilde{H}^2(\mathbb{R}P^2;\mathbb{Z}_2)$. Thus, these spaces cannot be homotopy equivalent since they have different cohomology ring structures.