Sections over Vector Bundles and the Serre-Swan Theorem

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- Sections, continuous choice of a vector at each vector space. Can be added and scaled by continuous functions.
- Recurring theme: algebraic vs geometric perspective.
- The Serre-Swan provides a way to switch between these two perspectives.

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- *Remark.* the dimension $n(x) = \dim p^{-1}(y)$ of fibers around x is locally constant, so dimension is constant on connected components.
- A map between vector bundles $p_1 : E_1 \to B, p_2 : E_2 \to B$ is a continuous map $f : E_1 \to E_2$ sending $p_1^{-1}(x)$ to $p_2^1(x)$, which restricts to a linear map on each fiber.

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- The canonical line bundle $p : E \to \mathbb{R}P^n$. $\mathbb{R}P^n$ can be viewed as the space of lines in $\mathbb{R}^{n+1}$ intersecting 0; $E \subseteq \mathbb{R}P^n \times \mathbb{R}^{n+1}$ contains elements (ℓ, v), where $v \in \ell$.

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- *(Lemma 1.6)* Using a local trivialization around $x \in B$, we can pull back a basis of $B \times \mathbb{R}^n$ to sections $s_1, \ldots, s_n$ over $U \ni x$ which are a basis at every fiber of $y \in U$.

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- *(Lemma 1.7)* Sections which are linearly independent at x remain linearly independent around x.
- The set $\Gamma(E)$ of sections of $p : E \to B$ has a natural module structure over $C(B)$.

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- Take $f : [0, 1] \times \mathbb{R} \to [0, 1] \times \mathbb{R}$ defined by $(t, x) \mapsto (t, tx)$, where $p : [0, 1] \times \mathbb{R} \to [0, 1]$ is a trivial bundle.

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- *(Proposition 1.10)* The fibers of $\operatorname{im} f$ have locally constant dimension $\iff$ $\operatorname{im} f$ is a subbundle $\iff$ $\ker f$ is a subbundle $\iff$ the fibers of $\ker f$ have locally constant dimension.

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- *Question.* is a subbundle of a vector bundle always a summand?

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- Theme: vector bundles decompose and coalesce nicely if B is nice.

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- Quick computation: $\Gamma(B \times K^n) = \Gamma(\bigoplus_{i=1}^n B \times K) = \bigoplus_{i=1}^n \Gamma(B \times K) = C(B)^n$.

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- Conclusion: both perspectives study essentially the same thing.

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  3. $\Gamma(E_1)/\Gamma(E_1)_x \cong p_1^{-1}(x)$.

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  2. Projection $F : C(B)^n \to C(B)^n$ onto M is uniquely induced by endomorphism f on $B \times K^n$.
  3. Show $\mathrm{im}\, f$ is a subbundle, implying $\Gamma(f : B \times K^n \to \mathrm{im}\, f) = F : C(B)^n \to \Gamma(\mathrm{im}\, f)$.