# Topos Theory III: Sheaves

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# Sheaves

## Definition (Sheaf)

Let $\mathcal{O}(X)$ be the poset of open sets of a topological space. A sheaf is a presheaf $F \in \hat{\mathcal{O}(X)}$ with the following equalizer for every open cover $\{U_i\}$ of an open set $U$:

$$FU \xrightarrow{e} \prod_i FU_i \overset{\alpha,\beta}{\rightrightarrows} \prod_{i,j} F(U_i \cap U_j)$$

where $\alpha$ ($\beta$) sends each component $f_i \in FU_i$ to $f_i|_{U_i \cap U_j}$ ($f_j|_{U_i \cap U_j}$).

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- For every open set $U \subseteq X$, we get a set $FU$.
- For every inclusion $U \subseteq V$ of open sets, we get a *restriction map* $-|_U : FV \to FU$. Functorality implies $x|_V|_W = x|_W$, for $x \in U \supset V \supset W$.

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- For each family $f_i \in FU_i$ glues together to form a unique $f \in FU$ if $f_i, f_j$ coincide when restricted to $U_i \cap U_j$.

# Examples

- *The sheaf $F$ of continuous (smooth) maps on a topological space (smooth manifold) $X$.* $FU$ is the set of such maps on the neighbourhood $U \subseteq X$, and the restriction maps are the usual function restrictions.

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- *The representable sheaf* $\mathrm{Hom}_{\mathcal{O}(X)}(V, U)$ assigning $V$ the set $\{V \to U\} \cong 1$ if $V \subseteq U$, and $\emptyset$ otherwise.

## Proposition 1

Let $F$ be a sheaf on $X$. A subfunctor $S \leq F$ is a subsheaf iff for every open set $U \subseteq X$ and $f \in FU$, and an open covering $\bigcup_i U_i = U$ we have

$$f \in SU \text{ if and only if } f|_{U_i} \in SU_i \text{ for each } i. \tag{1}$$

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## Proof.

Idea: since $S$ is a subfunctor of $F$, gluing and restriction on $S$ is inherited from $F$. It suffices to add the condition that $S$ must be "closed" under gluing and restriction, for $S$ to be a sheaf. □

# The presheaf $\mathrm{Sh}(-)$

- For each continuous map $f : X \to Y$, we get an induced map $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$, where $f_*F$ is the sheaf with local sets $f_*FU = Ff^{-1}(U)$.

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- A sheaf $F \in \mathrm{Sh}(X)$ restricts to a sheaf $F|_U \in \mathrm{Sh}(U)$ for any open set $U \subseteq X$.

**Theorem 2**

Suppose $X$ has an pen cover $\{W_k\}$ with a family of sheaves $F_k \in \mathrm{Sh}(W_k)$, such that

$$F_k|_{W_k \cap W_\ell} = F_\ell|_{W_k \cap W_\ell} \text{ for all } k, \ell.$$

Then there exists a unique (up to iso.) sheaf $F \in \mathrm{Sh}(X)$ such that $F|_{W_k} = F_k$.

# The presheaf $\mathrm{Sh}(-)$

**Proof.**

Idea: if $F$ exists, then for each open set $U \subseteq X$, we must have an equalizer for the cover $\bigcup_k (U \cap W_k) = U$.

$$FU \xrightarrow{e} \prod_i F_k(U \cap W_k) \overset{\alpha,\beta}{\rightrightarrows} \prod_{i,j} F_{k\ell}(U \cap W_k \cap W_\ell)$$

where $F_{k\ell} = F_k|_{W_k \cap W_\ell}$. Thus, we take this to be the definition of $FU$. $\square$

# Sheaf on a basis

- If $\mathcal{O}(X)$ has a basis $\mathcal{B}$, then we may define a *sheaf on* $\mathcal{B}$ as a presheaf $\mathcal{B}^{\text{op}} \to Set$ with appropriate equalizers for each covering $B = \cup_i B_i$.

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**Theorem 3**

The restriction functor $r : \text{Sh}(X) \to \text{Sh}(\mathcal{B})$ is an equivalence of categories.

# Sieves and Sheaves

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- Furthermore, a sieve $S$ is a *covering sieve* of $U$ if the open sets in $S$ cover $U$.

### Proposition 1

A presheaf $F$ on $X$ is a sheaf if and only if, for every open set $U \subseteq X$ and every covering sieve $S$ on $U$, the inclusion of functors $\iota_S : S \to よ(U)$ induces an isomorphism

$$\mathrm{Hom}(よ(U), F) \cong \mathrm{Hom}(S, F).$$

## Proof

Let $F$ be a presheaf. For any open covering $\{U_i\}$ of an open set $U \subseteq X$, we can construct the following equalizer diagram:

$$E \xrightarrow{e} \prod_i FU_i \overset{\alpha,\beta}{\rightrightarrows} \prod_{i,j} F(U_i \cap U_j)$$

where $E$ contains families of elements $x_i \in FU_i$ such that $x_i|_{U_i \cap U_j} = x_j|_{U_i \cap U_j}$. Now, replace $\{U_i\}$ with the covering sieve $S$ generated by $\{U_i\}$, and define $x_V = x_i|_V$, for families $(x_i) \in \prod_i FU_i$. By assumption, $x_V$ is independent of $i$. If we let $S$ denote the functor taking $V \mapsto 1$ iff $V \in S$, then each element $x_V \in FV$ is a map $SV \to FV$. The equalizer is thus $\mathrm{Hom}(S, F)$.

Diagram:

## Proof Cont.

This lets us construct the commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(S, F) \arrow[r] & \prod_i FU_i \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & \prod_{i,j} F(U_i \cap U_j) \\
\mathrm{Hom}(\text{よ}(U), F) \arrow[u, "(\iota_s)^*"] \arrow[r, leftrightarrow, "\text{よ}"'] & FU \arrow[u, "e"']
\end{tikzcd}$$

Hence, $(\iota_s)^*$ is always an isomorphism iff $FU$ is always the equalizer of the top right pair of parallel morphisms. $\square$

## Proposition 2

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## Proof.

Let $m : H \rightarrowtail F$ be a subobject of the sheaf $F$. Recall that $m$ is monic iff $H$ is the pullback of $m$ along itself. By the last theorem, $m$ is monic in the category of presheaves: its components are consequently monic. Therefore, each $HU$ is isomorphic to some subset $SU \subseteq FU$. This allows us to construct the subfunctor $S$ of $F$ which is isomorphic to $H$, which is also a sheaf. $\square$

### Theorem

*For ant space $X$, there is an isomorphism*

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## Proof.

For an open set $U \subseteq X$, we assign the representable sheaf $\mathrm{Hom}(-, U)$. Conversely, if $S \leq 1$, then assign it the open set $\bigcup\{U \in \mathcal{O}(X) : SU = 1\}$. $\square$

# Manifolds

## Definition

A topological $n$-manifold $M$ is a second countable Hausdorff space such that each point $q$ admits an open neighbourhood $V$ homeomorphic to an open set $W \subseteq \mathbb{R}^n$ via a *chart* $\phi : V \to W$. A collection of charts $\{\phi_i : V_i \to W_i\}$ with $V_i$ covering $M$ is an *atlas*.

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- A map is continuous on $V \subseteq W$ if its precomposition with a chart $\phi^{-1} : W \to V$ is continuous on $W \subseteq \mathbb{R}^n$.
- Given an atlas $\{\phi_i\}$, define $\phi_{ij} = \phi_i|_{V_j}$. The image of $\phi_{ij}$ is some subset $W_ij \subseteq$. We obtain the *transition maps*, which are homeomorphisms ${}_{ij}\phi_{ji}^{-1} : W_{ij} \to W_{ji}$.

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- $M$ can be constructed by pasting together the sets $W_i$ on the subsets $W_{ij}$ using the transition maps: categorically, $M$ is the pushforward in the following diagram, where $\alpha$ $(\beta))$ sends $x_{ij} \in V_i \cap V_j$ to $x_i$ $(x_j)$:

$$\coprod_{i,j} V_i \cap V_j \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \coprod_i V_i \longrightarrow M$$

# Smooth Manifolds

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- Smooth manifolds can be defined using sheaves.
  - a smooth $n$-manifold $M$ is a second countable Hausdorff space with a subsheaf $S = S_M$ of the sheaf $C_M$ of continuous functions on $M$ with the property that each point $p \in M$ has an open neighbourhood $V$ such that there is a homeomorphism $\varphi : V \to W \subseteq \mathbb{R}^n$ carrying the sheaf $C^k$ in $W$ isomorphically onto $S|_V$.

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- Smooth manifolds are examples of *ringed spaces*.
  - A ringed space $X$ is a topological space with a fixed sheaf $R$ of rings called the structure sheaf, and a morphism $f : (X, R) \to (X', R')$ of ringed spaces is a continuous map $f : X \to X'$ inducing a homomorphism $\alpha : R' \to f_* R$ of sheaves.

# Bundles

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  such that $f$ is continuous.
- $p^{-1}(x)$ is called the fiber of $Y$ over $x$. A bundle is like a family of fibers continuously indexed by $X$. For an open subset $U \subseteq X$, any bundle $p$ restricts to a bundle $p_U : p^{-1}U \to U$ over $U$. $p_U$ is the pullback of $p$ along the inclusion $U \to X$ in $\mathsf{Top}$.

# Sections of bundles

- A section $s$ of the bundle $p_U$, is a continuous map $s : U \to Y$, such that $ps$ is the inclusion $\iota : U \to X$. Let $\Gamma_p U$ be the set of all sections over $U$.

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  - Any open set $U \subseteq X$ can be covered by singletons. By the glueing condition of the sheaf, given by the appropriate equalizer,

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$$FU \longrightarrow \prod_{x \in U} fu \rightrightarrows 1 \ ,$$

    we have $FU = \coprod_{x \in U} fx$.
  - Hence, we can define the *discrete bundle* $p : \coprod_{x \in X} fx \to X$ with the obvious projection. Additionally, $FU$ is the set of sections of $p$ over $U$.

# Principle $G$-Bundles

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- Let $Y \times_{Y/G} Y$ be the pullback of $p$ by itself. This space contains pairs $(y, y')$ such that $y, y'$ belong to the same orbit.
- Since there are two maps $Y \times G \to Y$, projection $\pi$ and right action $a$, we can construct the map $\theta = \pi \cdot a : Y \times G \to Y \times_{Y/G} Y$. If $\theta$ is a homeomorphism, then $Y \times G$ is called the *principle $G$-bundle*.

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- If $Y \times G$ is principle, then injectivity of $\theta$ implies that $G$ acts freely, and surjectivity implies $G$ acts transitively, on $Y$. Thus, each fiber of $p$ is homeomorphic to $G$.