Topos Theory

Notes from *Sheaves in Geometry and Logic*

Robbert Liu

University of Toronto

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Contents

1 Categories of Functors

1.1 Presheaves

Definition 1.1 (Presheaf). Let $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ be the category of all contravariant functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ for some small category C. This category is sometimes denoted by $\hat{\mathsf{C}}$, and its elements are called **presheaves on** C. For any $\mathscr{F} \in \hat{\mathsf{C}}$ and a morphism $f : C \to D \in \mathsf{C}$, we get an induced morphism $\mathscr{F}f$ whose action on some $x \in \mathscr{F}D$ is described by the equivalent notation

$$\mathscr{F}f(x) = x|f = x \cdot f = \text{“The } \textbf{restriction of } x \textbf{ along } f\text{.”}$$

1.2 Subobject Classifiers

Definition 1.2 (Subobject). A **subobject** of X is an equivalence class $[S \rightarrowtail X]$ of monomorphisms with codomain X. Abusing notation, we sometimes refer to the subobject by a representative $S \rightarrowtail X$, or simply by S.

Definition 1.3 (Subobject Classifer). In a category C with finite limits, a **subobject classifer** is a monomorphism $\mathrm{true} : 1 \rightarrowtail \Omega$ such that for every monomorphism $S \rightarrowtail X$, there is a unique **characteristic morphism** $\phi : X \to \Omega$ forming the pullback square:

$$\begin{tikzcd} S \arrow[r] \arrow[d, tail] & 1 \arrow[d, tail] \\ X \arrow[r, dashed, "\phi"'] & \Omega \end{tikzcd}$$

Definition 1.4 (Subobject Functor). There is a functor $\mathrm{Sub}_{\mathsf{C}} \in \hat{\mathsf{C}}$ sending $X \in \mathsf{C}$ to its set of subobjects. For each morphism $f : X \to X'$, we get a morphism $\mathrm{Sub}_{\mathsf{C}}f : \mathrm{Sub}_{\mathsf{C}}X \to \mathrm{Sub}_{\mathsf{C}}X'$ defined on each subobject $S \rightarrowtail X$ by pullback along f to a subobject $S' \rightarrowtail X'$.

Theorem 1.1. *A locally small, finitely complete category* C *has a subobject classifer iff* Sub_{C} *is representable: that is, for some object* Ω*, we have a natural isomorphism*

$$Sub_{\mathsf{C}} \cong Hom_{\mathsf{C}}(-, \Omega) \cong \text{よ}(\Omega).$$

Example 1.1 (Sieves). If a subobject classifer Ω exists in $\hat{\mathsf{C}}$, then we can characterize Ω completely by computing the subobjects of the Hom functors:

$$\mathrm{Sub}_{\hat{\mathsf{C}}}(\text{よ}(C)) \cong \mathrm{Hom}_{\mathsf{C}}(\text{よ}(C), \Omega) \overset{\text{よ}}{\cong} \Omega(C).$$

So Ω is the functor mapping C to the set of subfunctors of $\text{よ}(C)$. Incidientally, the notion of subfunctor is adjacent to that of sieve. Given an object $C \in \mathsf{C}$, a **sieve** on C is a set of morphisms with codomain C closed under precomposition with any morphism in C. Then there is a correspondence:

$$\text{Sieves on } C \leftrightarrow \text{Subfunctors of } \text{よ}(\mathrm{C}).$$

1.3 Limits and Colimits

Theorem 1.2. $\hat{\mathsf{C}}$ *admits all finite limits and colimits.*

Proof. In the case of limits—Set admits a final object 1 and, for each pair of morphisms $f, g : B, C \to D$, the pullback $B \times_D C = \{(b, c) : f(b) = g(d)\}$. Since limits in $\hat{\mathsf{C}}$ are evaluated pointwise, it too admits a final functor $1_{\hat{\mathsf{C}}} : C \mapsto 1_{\mathsf{Set}}$ and a pullback functor $\mathscr{F} \times_{\mathscr{H}} \mathscr{G} : C \mapsto \mathscr{F}C \times_{\mathscr{H}C} \mathscr{G}C$ for each pair of natural transformations $\mathscr{F}, \mathscr{G} \to \mathscr{H}$.

This is sufficient to know $\hat{\mathsf{C}}$ admits finite limits, since any fimite limit can be constructed from equalizers and a products, which can be constructed as the following pullbacks, respectively:

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & 1 \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r] \arrow[d, "e"'] & Y \arrow[d, "\Delta"] \\ X \arrow[r, "(f,g)"'] & Y \times X \end{tikzcd}$$

By the principle of duality, since Set admits an initial object 0 and all pushforwards, all finite colimits exist in $\hat{\mathsf{C}}$ as well. This proof is superfluous since both categories are (co)complete anyway. ∎

Theorem 1.3. *Let* $\mathscr{A} : \mathsf{C} \to \mathsf{E}$ *be a functor from a small category* C *to a cocomplete category* E*. Then there is an adjunction* $L_{\mathscr{A}} : \hat{\mathsf{C}} \overset{\dashv}{\rightleftarrows} \mathsf{E} : R_{\mathscr{A}}$ *defined by*

$$L_{\mathscr{A}}(\mathscr{F}) = Colim(\int \xrightarrow{\pi} \mathsf{C} \xrightarrow{\mathscr{A}} \mathsf{E})$$

$$R_{\mathscr{A}}(E) = \mathscr{E} : C \mapsto Hom_{\mathsf{E}}(\mathscr{A}(C), E)$$

Corollary 1.3.1. *If we take* E *to be* $\hat{\mathsf{C}}$ *and* $\mathscr{A}$ *to be the Yoneda embedding* よ*, then we have*

$$R_{よ}(E)(C) = Hom_{\mathsf{E}}(よ(C), E) \cong E(C),$$

which proves $L_{\mathscr{A}} \dashv R_{\mathscr{A}}$ *to be (up to isomorphism) the adjunction* $id_{\hat{\mathsf{C}}} \dashv id_{\hat{\mathsf{C}}}$.

Corollary 1.3.2. *For each such functor* $\mathscr{A} : \mathsf{C} \to \mathsf{E}$*, there exists a colimit-preserving functor* $L_{\mathscr{A}} : \hat{\mathsf{C}} \to \mathsf{E}$ *as defined above for which the following commutates:*

$$\begin{tikzcd} \hat{\mathsf{C}} \arrow[r, "L_{\mathscr{A}}"] & \hat{\mathsf{E}} \\ \mathsf{C} \arrow[u, "よ"] \arrow[ur, "\mathscr{A}"'] & \end{tikzcd}$$

1.4 Exponentials

Definition 1.5 (Exponential)**.** In a category C with finite products, if the functor $- \times X : \mathsf{C} \to \mathsf{C}$, taking any object to its product with X, has a right adjoint, written $Z \mapsto Z^X$, then C has an **exponential** for X. Then, if each object X admits an exponential, this induces a functor $\langle X, Z \rangle \mapsto Z^X : \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{C}$ called the **exponential** for C.

The notation stems from the fact that in Set, the exponential Z^X turns out to be the usual set of functions $X \to Z$.

Definition 1.6 (Evaluation)**.** The counit of the adjunction $(-)^X \dashv - \times X$ is called the **evaluation** $e : (-)^X \times X \to \mathrm{id}_{\mathsf{C}}$, which satisfies a familiar universal property:

$$\begin{tikzcd} Y \times X \arrow[d, "\varphi' \times \mathrm{id}"'] \arrow[dr, "\varphi"] & \\ Z^X \times X \arrow[r, "e"'] & Z \end{tikzcd}$$

Definition 1.7 (Cartesian Closed)**.** C is said to be **cartesian closed** if it admits a final object and an exponential. In such a category, all of the "exponential laws" hold.

$$1^X \cong 1, \qquad X^1 \cong X, \qquad (Y \times Z)^X \cong Y^X \times Z^X, \qquad X^{Y \times Z} \cong (X^Y)^Z$$

Theorem 1.4. *For any small category* C*, the category of presheaves* $\hat{\mathsf{C}}$ *is cartesian closed.*

Definition 1.8 (Topos)**.** An **elementary topos**, or simply topos, is a category with all finite limits and colimits, an exponential $(-)^{(-)}$, and a subobject classifer $1 \to \Omega$.

1.5 Propositional Calculus

Definition 1.9 ((Distributive) Lattice)**.** A **lattice** is an set endowed with distinguished "endpoints" 0 and 1 and two binary operations $\wedge$ and $\vee$ which are associative and commutative, and satisfy

$$x \wedge x = x, \qquad x \vee x = x, \qquad 1 \wedge x =, 0 \vee x = x, \qquad x \wedge (y \vee x) = x = (x \wedge y) \vee x$$

A lattice is **distributive** if the follow identity holds:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

which implies the complementary identity

$$x \vee (y \wedge) z = (x \vee y) \wedge (x \vee z).$$

Definition 1.10 (Boolean Algebra)**.** A **complement** for an element x in a lattice with 0 and 1 is an element $\neg x$ satisfying $x \wedge \neg x = 0$, $x \vee \neg x = 1$. In a distributive lattice, a complement $\neg x$, if it exists, is unique.

A **Boolean algebra** is a distributive lattice with 0 and 1 in which every x has a complement $\neg x$. Every boolean algebra satisfies the DeMorgan laws.

1.6 Heyting Algebras

Definition 1.11 (Heyting Algebra)**.** A **Heyting algebra** is a poset with finite products and coproducts (meets and joins) which is cartesian closed. The exponential is usually denoted $x \Rightarrow y$ and it is characterized by the adjunction

$$z \leq (x \to y) \text{ if and only if } z \wedge x \leq y.$$

So $x \to y$ is the least upper bound for element z with $z \wedge x \leq y$.

Example 1.2. The prime example of a Boolean algebra is powerset of some set X, whereas Heyting algebras correspond to the set of open sets of some set X. Alternatively, Heyting algebras can be thought of as models of intuitionistic logic.

Definition 1.12 (Negation in Heyting Algebras)**.** In a Heyting algebra, we define the negation of x as $\neg x = (x \to 0)$. Thus, "not x" is equivalent to "x implies absurdity".

Theorem 1.5. *A Heyting algebra H is Boolean if and only if $\neg\neg x = x$ for all $x \in H$, or, if and only if $x \vee \neg x = 1$ for all x.*

Theorem 1.6. *Given a presheaf category $\hat{\mathsf{C}}$ on a small category C. Then for any object F in C, the partially ordered set $Sub_{\hat{\mathsf{C}}}(F)$ of subobjects of F is a Heyting algebra.*

1.7 Quantifiers

Theorem 1.7. *For any function $f : Z \to Y$ between sets Z and Y, the inverse image functor $f^* : \mathcal{P}Y \to \mathcal{P}Z$ between subsets has left and right adjoints $\exists_f$ and $\forall_f$. The functor $\exists_f$ is defined by $S \mapsto f(S)$, and the functor $\forall_f$ maps each $S \subseteq Z$ to the set $T \subseteq Y$ containing all $y \in Y$ whose fiber $f^{-1}(y)$ is a subset of S.*

Theorem 1.8. *Let C be a category with pullbacks, and let B be an object of C. For each $f : B' \to B$, the change of base functor $f^* : \mathsf{C}/B \to \mathsf{C}/B'$ has a left adjoint. Moreover, if C/B is cartesian closed, then f^* has also a right adjoint.*

Theorem 1.9. *If B, B' are objects in a complete category C with pullbacks such that $\mathsf{C}, \mathsf{C}/B, \mathsf{C}/B'$ are cartesian closed, then pullback along any motphism $f : B' \to B$ preserves all colimits which exist in C/B.*

2 Sheaves of Sets

2.1 Sheaves

Definition 2.1 (Sheaf)**.** A **sheaf of sets** F on a topological space X is a functor $F : \mathcal{O}(X)^{\text{op}} \to \mathsf{Set}$, where $\mathcal{O}(X)$ is the poset category of the open sets of X, satisfying additional properties. For each inclusion $V \subseteq U$ of open subsets, we get a **restriction map** $Ff : U \to V$ where $Ff(u)$ is denoted $u|_V$, such that for each open covering $\{U_i\}$ of an open set $U \subseteq X$, we have an equalizer

$$FU \xrightarrow{e} \prod_i FU_i \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j} F(U_i \cap U_j)$$

where the map p (the map q) sends each component $f_i \in FU_i$ to the restriction $f_i|_{U_i \cap U_j}$ (the restriction $f_j|_{U_i \cap U_j}$).

Remark. The maps e, p, q are completely characterized by the commutativity of the equalizer diagram. Thus, we can define a sheaf $F : \mathcal{O}(X)^{\text{op}} \to C$ to *any* category C with small products, e.g. abelian groups, rings, R-modules/algebras.

Example 2.1 (Sheaf of maps)**.** There exists a sheaf of continuous (smooth) maps on a topological space (smooth manifold), since compatible maps on subsets U_i of a space glue together to form a unique map on the union of the subsets U_i. The restriction map is the usual restriction of a map to an open subset.

Example 2.2 (Representable sheaf)**.** The representable sheaf $\text{Hom}(-, U) = よU$ assigns V the set $\{V \to U\} \cong 1$, if $V \subseteq U$, and $\emptyset$. Restriction is usually the unique map $1 \to 1$, or the unique map $0 \to 1$.

Theorem 2.1. *Let F be a sheaf on X. A subfunctor $S \leq F$ is a subsheaf if and only if, for every open set $U \subseteq X$, $f \in FU$, and an open covering $\bigcup_i U_i = U$, we have:*

$$f \in SU \text{ if and only if } f|_{U_i} \in SU_i \text{ for each } i.$$

Proof. Idea: since S is a subfunctor of F, gluing and restriction on S is inherited from F. It suffices to necessitate that S is "closed" under gluing and restriction, for S to be a sheaf. ■

Theorem 2.2 (Sh is almost a sheaf)**.** *The functor* Sh *sending a space X to the set of sheaves over it is almost a sheaf itself:*

1. *For each continuous map $f : X \to Y$, we get an induced map $f_* : \text{Sh}(X) \to \text{Sh}(Y)$, where f_*F is defined by $f_*FU = Ff^{-1}(U)$.*

2. *Given an open subset $U \subseteq X$ of a set X, any sheaf $F \in \text{Sh}(X)$ restricts to a sheaf $F|_U \in \text{Sh}(U)$ on the subspace U.*

3. *For any open cover $\{W_k\}$ of X and a family of sheaves $F_k \in \text{Sh}(X)$ satisfying*

$$F_k|_{W_k \cap W_\ell} = F_\ell|_{W_k \cap W_\ell} \text{ for all } k, \ell,$$

   there exists a unique (up to isomorphism) sheaf $F \in \text{Sh}(X)$ such that $F|_{W_k} = F_k$.

Proof. Idea: if F exists, then for each open set $U \subseteq X$, we must have an equalizer for the covering $\bigcup_k (U \cap W_k) = U$:

$$FU \xrightarrow{e} \prod_i F_k(U \cap W_k) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{i,j} F_{k\ell}(U \cap W_k \cap W_\ell)$$

where $F_{k\ell} = F_k|_{W_k \cap W_\ell}$. We can hence take this to be the definition of FU. ■

Definition 2.2 (Sheaf on a Basis)**.** Let X be generated by a basis $\mathcal{B}$. A **sheaf on** $\mathcal{B}$ F is a presheaf on the poset category of open sets in $\mathcal{B}$ which satisfies the same conditions as a sheaf on a space.

Theorem 2.3. *The restriction functor* $\mathrm{Sh}(X) \to \mathrm{Sh}(\mathcal{B})$ *is an equivalence of categories.*

2.2 Sieves and Sheaves

Remark. We may think of the representable sheaf よU as a sieve $S_U = \{V \in \mathcal{O}(X) : V \subseteq U\}$. Each family of open sets $\{U_i\}$ generates or *spans* a sieve of open sets of X which factor through U_i. This recontextualization of sheafs using sieves lead to two central notions of this section, which will allow us to define sheaves in terms of (presheaves and) sieves: this is theoretically advantageous since both are objects of the category of presheaves.

Definition 2.3 (Principle Sieve)**.** A **principle sieve** S_U is a sieve spanned by an open set $U \subseteq X$.

Definition 2.4 (Covering Sieve)**.** A sieve S is a **covering sieve** of U if S is generated by an open cover of U.

Theorem 2.4. *A presheaf F on X is a sheaf if and only if, for every open set $U \subseteq X$ and every covering sieve S on U, the inclusion of functors $\iota_S : S \to$ よ(U) induces an isomorphism*

$$Hom(\text{よ}(U), F) \cong \mathrm{Hom}(S, F).$$

Theorem 2.5. *A subobject of a sheaf F in the category* $\mathrm{Sh}(X)$ *is isomorphic to a subsheaf of F.*

Theorem 2.6. *For any space X, there is an isomorphism* $\mathcal{O}(X) \cong \mathrm{Sub}_{\mathrm{Sh}(X)}(1)$ *of partially ordered sets.*

2.3 Sheaves and Sections

In this section we will build towards the characterization of each sheave as the sheaf of sections of some bundle.

Definition 2.5 (Germ)**.** Let F be a presheaf on X. Given $x \in X$, the **stalk** F_x **of** F **at** x is defined as the colimit

$$P_x = \varinjlim_{x \in U} FU.$$

Elements of F_x are called **germs**, and can be described as equivalence classes of pairs (U, s), where $U \subseteq X$ is an open neighbourhood of x, and $s \in FU$, such that $(U, s) \sim (V, t)$ if $s|_W = t|_W$ for some open set $W \subseteq U \cap V$. In this case, we say $s_x = t_x$ is the germ of s at x (t at x). Furthermore, each morphism of presheaves $F \to G$ induces a map $F_x \to G_x$ at each stalk, which turns the mapping $F \mapsto F_x$ into a functor.

Definition 2.6 (Bundle of Stalks)**.** Given a presheaf F on a space X, define the set $\Lambda_F = \coprod_x F_x$ and a projection map $p : \Lambda_F \to X$ sending $(U, s_x) \mapsto x$. For each $s \in FU$, we get a map $\dot{s} : U \to \Lambda_F$ defined by $\dot{s}x = s_x$, and endow Λ_F with the topology generated by open sets $\dot{s}U$, which makes each $\dot{s}$ a homeomorphism—continuity is proven by applying the definition of a germ. Thus the mapping $F \to \Lambda_F$ is a functor from presheaves to bundles over X.

Definition 2.7 (Sheaf of Sections)**.** For each bundle $p : E \to X$, let ΓE denote the corresponding sheaf of sections $X \to E$. Note that Γ is a functor from bundles to presheaves over X.

Theorem 2.7. *Given a presheaf F, let $\eta : F \to \Gamma\Lambda_F$ be the natural transformation defined by components $\eta_U : s \mapsto \dot{s}$. If F is a sheaf, then η is an isomorphism.*

Theorem 2.8. *Let F be a presheaf on X, and let $\sigma, \tau : \Gamma\Lambda_F \to G$ be two maps into a sheaf G on X. If $\sigma\eta = \tau\eta$, then $\sigma = \tau$.*

Theorem 2.9 (Sheafification)**.** *Let $\iota : \mathrm{Sh}(X) \to \hat{\mathcal{O}(X)}$ be the forgetful functor. Then, there is an adjunction $\Gamma\Lambda \dashv \iota$, with η as the unit. $\Gamma\Lambda F$ is called the* ***sheafification*** *of a presheaf F, or the free sheaf generated by F.*

2.4 Sheaves as Étale Spaces

Definition 2.8 (Étale Bundle)**.** A bundle $p : E \to X$ is **étale** if p is a local homeomorphism: for each $e \in E$, there is an open neighbourhood V of e, such that pV is open in X and $p|_V$ is a homeomorphism.

Theorem 2.10. *For any space X, there is an adjunction*

$$\mathsf{Bund}_X \underset{\Lambda}{\overset{\Gamma}{\rightleftarrows}} \hat{\mathcal{O}(X)},$$

where Γ sends a bundle to its sheaf of sections, and its left adjoint Λ sends a presheaf to its bundle of germs. Furthermore, the unit and counit restrict to natural isomorphisms on the subcategories of sheaves and étale spaces, respectively.

Theorem 2.11. *Let Etale_X denote the full subcategory of étale bundles. Then, Γ and Λ restrict to an equivalence of categories*

$$\mathrm{Sh}(X) \rightleftarrows \mathsf{Etale}_X$$

2.5 Sheaves with Algebraic Structure

Although we have been working with sheaves of sets up until this point, our work mostly applies to sheaves of algebraic structures of a certain kind, such that abelian groups. For example, a sheaf of abelian groups on X is simply an abelian group object in the category $\mathrm{Sh}(X)$. Given a ring object R in $\mathrm{Sh}(X)$, or a sheaf of rings, we may further define an R-module object, or a sheaf A of left R-modules—that is, for each open set $U \subseteq X$, AU is an RU-module.

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Theorem 2.12. *If F is a sheaf and P is a presheaf of sets on the space X, then the presheaf exponential F^P, defined by $F^P(U) = \mathrm{Hom}(よ(U) \times P, F)$, is a sheaf. Since $よ(U)(V)$ is equal to a singleton when $V \subseteq U$ and empty otherwise, it suffices to define $F^P(U)$ on open sets $V \subseteq U$. Hence,*

$$F^P(U) \cong \mathrm{Hom}(P|_U, F|_U)$$

Definition 2.9 (Internal hom)**.** If F, G are sheaves of sets on X, then $F^G \cong \mathrm{Hom}(G, F)$ is a sheaf called the internal hom from G to F, since it is a sheaf that behaves like the object of morphisms from G to F.

Theorem 2.13. *Construct the presheaf Ω on X by defining $\Omega U = \mathcal{P}(U) \cap \mathcal{O}(X)$ to be the set of open subsets of U, and defining the restriction $W|_V = W \cap V$ for any open subset $W \subseteq U$. Then, Ω is a sheaf, and is the subobject classifer of* $\mathrm{Sh}(X)$.

Theorem 2.14. $\mathrm{Sh}(X)$ *has all finite limits and colimits, exponentials, and a subobject classifier. Thus, it is an elementary topos. By Theorem 2.11, so is* Etale_X.

2.6 Inverse Image Sheaf

Definition 2.10 (Inverse Image Sheaf)**.** Given a map $f : X \to Y$, we may pull the bundle $E \to Y$ back along f to obtain a bundle $f^*E \to X$. Moreover, if $E \to Y$ is étale, then so is $f^*E \to X$. Using the equivalence of categories $\mathrm{Sh}(X) \rightleftarrows \mathsf{Etale}_X$, we can define, for any sheaf E on X, the **inverse image sheaf** f^*E, by passing to Etale_X, applying f^*, and then passing back to $\mathrm{Sh}(X)$.

Theorem 2.15. *If $f : X \to Y$ is a continuous map, then the functor f^*, sending each sheaf G on Y to its inverse image on X, is left adjoint to the direct image functor f_*.*

3 Grothendieck Topologies

3.1 Generalized Neighbourhoods

The need of a more relaxed notion of topology arose from several areas of mathematics. Around 1961, Grothendieck uncovered the surprising duality between the Galois groups of a field and the fundamental group of a space.

One one hand, a normal extension N of a field K of characteristic 0 is a monomorphism $m : K \to N$ in the category of fields. The Galois group G of N/K consists of field automorphisms of N fixing K, and the fundamental theorem of Galois theory states that the subgroups $S \leq G$ correspond to factorizations $K \leq L \leq N$, in the following way: $S \mapsto L$ if S is the subgroup of automorphisms fixing L.

On the other hand, given arcwise connected and locally arcwise connected spaces X, Y, a covering space is a particular epimorphism $\rho : Y \to X$ in a category containing X, Y. The covering group G of ρ contains deck transformations: automorphisms $\sigma : Y \to Y$ such that $\rho\sigma = \rho$. In particular, if ρ is a regular covering, then the subgroups $S \leq G$ correspond precisely to the factorizations $Y \to Y' \to X$ of ρ, where Y' is the quotient of Y given by gluing together points in the orbits of $\sigma \in S$.

Notice further Top/X admits products and coproducts, while Field can both be embedded in the larger category CRing which admits the same constructions. Continuing the analogy, the definition of covering space implies that each $x \in X$ admits a neighbourhood $U \rightarrowtail X$ such that the pullback $Y \times_X U \to U$ (the bundle product) is a coproduct of copies of U. Oppositely, the field extension $K \to L$ has a splitting field $K \to N$ with N normal, which means the tensor product (the commutative ring coproduct) $L \otimes_K N$ is a direct product of copies of N.

The analogy falls apart here: while $U \rightarrowtail X$ is a monomorphism, the map $K \to N$ is not an epimorphism. This motivates a paradigm shift in the primitive notions of topology: instead of neighbourhoods $U \rightarrowtail X$, we will consider more general maps $U \to X$ which fit into a cover of X. The following table summarizes the actors involved in the duality, and singles out the point of contention.

Galois Theory	Covering Spaces
Field K of characteristic 0	(Locally) arcwise connected space X
Normal extension $m : K \rightarrowtail N$	Covering space $\rho : Y \twoheadrightarrow X$
Galois group of N/K	Covering group of ρ
Field automorphisms σ of N/K	Deck transformations $\sigma : Y \to Y$
Factorizations $K \rightarrowtail N_\sigma \rightarrowtail N$	Factorizations $Y \twoheadrightarrow Y_\sigma \twoheadrightarrow X$
$Y \otimes_K N \cong \bigotimes_i N$, N splitting	$Y \times_X U \to U \cong \prod_i U$ for some $U \ni x$
$K \to N$	$U \rightarrowtail X$

3.2 Grothendieck Topologies

Definition 3.1 (Grothendieck Topology)**.** A **Grothendieck topology** on a category C is a functor J which assigns to each object C of C a collection $J(C)$ of sieves on C, called the *covering sieves* on C, such that:

1. $J(C)$ contains the maximal sieve $\{f : D \to C : D \in \mathrm{Obj}(\mathsf{C})\}$.

2. **Stability Axiom.** If $\mathcal{S} \in J(C)$ is a covering sieve on C and $h : D \to C$ is any morphism, then the pullback $h^*(\mathcal{S}) := \{fh : f \in \mathcal{S}\} \in J(D)$ is a covering sieve on D.

3. **Transitivity Axiom.** If $\mathcal{S} \in J(C)$ is a covering sieve, and $\mathcal{R}$ is any sieve on C such that $h^*(\mathcal{R}) \in J(D)$ for all $h : D \to C \in S$, then $\mathcal{R} \in J(C)$.

A tuple (C, J) containing a category C with a topology J on it is called a **site**.

Definition 3.2 (Grothendieck Topology (in terms of covers))**.** Given a site (C, J), we will say a sieve S **covers** C if $S \in J(C)$, that S covers $f : D \to C$ if $f^*(S)$ covers D. Then, we can reformulate the Grothendieck topology axioms:

1. If S is a sieve on C and $f \in S$, then S covers f.

2. **Stability.** If S covers a morphism $f : D \to C$, it also covers fg for any morphism $g : E \to D$.

3. **Transitivity.** If S covers a morphism $f : D \to C$, and R is a sieve on C which covers all morphisms of S, then R covers f.

Remark. Following up on the philosophy in the previous section, if we pretend the morphisms in a sieve $S \in J(C)$ are neighbourhoods $U \rightarrowtail X$, then the axioms reflect basic properties of covering sieves:

1. S covers any open subset in S.

2. If S covers an open subset $U \subseteq X$, then it also covers any open subset $V \subseteq U$.

3. If R covers any open subset in S, and S covers $U \subseteq X$, then R covers U as well.

This immediately demonstrates that any classical topology on X is a Grothendieck topology on $\mathcal{O}(X)$. A curious reader will soon obtain the answer to their question of whether ordinary open covers also have a Grothendieck doppelgänger.

Definition 3.3 (Basis for a Grothendieck Topology)**.** A **basis for a Grothendieck topology** on a category (admitting certain pullbacks) is a function K which assigns to each object C a collection $K(C)$ of of families of morphisms with codomain C, such that:

1. If $f : C' \to C$ is an isomorphism, then $\{f\} \in K(C)$.

2. If $\{f_i : C_i \to C : i \in I\} \in K(C)$, then for each morphism $g : D \to C$, the family of pullbacks $\{\pi_2 : C_i \times_C D \to D : i \in I\}$ is in $K(D)$ (these pullbacks behave like restrictions of open sets).

3. If $\{f_i : C_i \to C : i \in I\} \in K(C)$, and there exists a family $\{g_{ij} : D_{ij} \to C_i : j \in I_i\} \in K(C_i)$ for each $i \in I$, then the family of composites $\{f_i g_{ij} : D_{ij} \to C : i \in I, j \in I_i\} \in K(C)$.

K **generates** a topology J by $S \in J(C)$ if and only if $R \in K(C)$ for some $R \subseteq S$.

Theorem 3.1. *Given a topology J, if R, S cover C, then $R \cap S$ covers S. Alternatively, given a basis K, if R, S cover C, then R, S have a common refinement in $K(C)$—that is, there exists a cover T of C such that every morphism in T factors through some $f \in R$, and independently factors through some $g \in S$.*

Example 3.1.

The trivial topology, containing only the maximal sieve.

The open cover topology on a small subcategory $\mathsf{T} \leq \mathsf{Top}$, formed by gluing together the topologies of $\mathcal{O}(\mathcal{X})$.

3.3 The Zariski Site

Definition 3.4 (Zariski Site)**.** Consider the category of all affine varieties $V \subseteq \mathbb{C}^\infty$, where each morphism $\phi : V \to W$, given $V \subseteq \mathbb{C}^n, W \subseteq \mathbb{C}^m$, is an m-tuple n-input rational function mapping V into W. Define the **Zarisiki topology** on $\mathbb{C}^n$ by letting the closed sets be of the form $V(I) = \{z \in \mathbb{C}^n : I \text{ vanishes at } x\}$, for any ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$. Now, consider the site formed from the category above by endowing it with the open cover topology associated to the Zariski topology. This is called the **Zariski site**. This construction is central to algebraic geometry, and can be generalized from $\mathbb{C}$ to an arbitrary commutative ring.

3.4 Sheaves on a Site

Definition 3.5 (Sheaf)**.** Let $P : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ be a presheaf. Given a covering sieve S of C, a **matching family** for S of elements of P is a family $\{x_f \mid f : D \to C \in S\}$, such that $x_f \cdot g = x_{fg}$ for any morphism $g : E \to D$. An **amalgamation** of such a matching family is an element $x \in P(C)$ such that $x \cdot f = x_f$ for all $f \in S$. Then, P is a sheaf precisely when all matching families for any cover of any object of C has a unique amalgamation. This is equivalent to necessitating that $\mathrm{Hom}(S, P) \cong \mathrm{Hom}(よ(C), P) \cong P(C)$, or that the following diagram is always an equalizer:

$$P(C) \xrightarrow{x \mapsto \{x_f\}_f} \prod_{f \in S} P(\operatorname{dom} f) \underset{x_f \mapsto x_f \cdot g}{\overset{x_f \mapsto x_{fg}}{\rightrightarrows}} \prod_{\substack{f \in S \\ \operatorname{dom} f = \operatorname{cod} g}} P(\operatorname{dom} g)$$

Theorem 3.2. *If P is a presheaf on C, then P is a sheaf for J iff for any cover $\{f_i : C_i \to C : i \in I\}$ in the basis K, any matching family $\{x_i\}_i$ has a unique amalgamation.*

Theorem 3.3. *A presheaf P is a sheaf for the atomic topology on C iff for any morphism $f : D \to C$ and $y \in P(D)$, if $y \cdot g = y \cdot h$ for all commutative diagrams*

$$E \underset{h}{\overset{g}{\rightrightarrows}} D \xrightarrow{f} C \ ,$$

then $y = x \cdot f$ for a unique $x \in P(C)$.

Definition 3.6 (Grothendieck Topos)**.** A **Grothendieck topos** is a category which is equivalent to the category $\mathrm{Sh}(\mathsf{C}, J)$ of sheaves on some site (C, J).

Theorem 3.4 (A limit of sheaves is a sheaf)**.** *Let (C, J) be a site and let $I \to \hat{\mathsf{C}}$ be a diagram of presheaves. If all P_i are sheaves then so is $\lim_{\longleftarrow} P_i$.*

3.5 The Associated Sheaf Functor

Definition 3.7 (Associated Separated Presheaf)**.** Given a presheaf $P \in \hat{\mathsf{C}}$, we can construct a presheaf P^+ in the following manner: let $\mathrm{Match}(R, P)$ denote the set of matching families for the cover R of C, and define

$$P^+ = \varinjlim_{R \in J(C)} \mathrm{Match}(R, P).$$

To be more precise, $P^+(C)$ contains equivalence classes of matching families $\{x_f \mid f : D \to C \in R\}$—that is, $x_f \in P(D)$ and $x_f \cdot g = x_{fg}$ for all composable f, g—where two matching families $\{x_f \mid f \in R\}, \{y_f \mid f \in S\}$ are equivalent if there is a common refinement $T \subseteq R \cap S$ such that $T \in J(C)$ and $x_f = y_f$ for all $f \in T$.

Lemma 3.5. *Let $\eta : P \to P^+$ be the canonical map sending $x \in P(C)$ to the matching family $\{x \cdot f \mid f \in t_C\}$, where t_C is the maximal sieve on C. The following two properties characterize the inclusion $\eta : P \to P^+$. Then, the following two properties hold*

1. *η is a monomorphism iff P is separated.*
2. *η is an isomorphism iff P is a sheaf.*

Lemma 3.6. *If F is a sheaf and P is a presheaf, then any map $\phi : P \to F$ of presheaves factors uniquely through η as $\phi = \tilde{\phi} \circ \eta$.*

$$\begin{tikzcd}
P \arrow[r, "\eta"] \arrow[dr, "\phi"'] & P+ \arrow[d, dashed, "\tilde{\phi}"] \\
& F
\end{tikzcd}$$

Lemma 3.7. *For any presheaf P, P^+ is a separated presheaf. If P is separated, then P^+ is a sheaf.*

Theorem 3.8. *The inclusion functor $i : \mathrm{Sh}(\mathsf{C}, J) \rightarrowtail \hat{\mathsf{C}}$ has a left adjoint $a : \hat{\mathsf{C}} \to \mathrm{Sh}(\mathsf{C}, J)$ called the **associated sheaf functor**. Moreover, a commutes with finite limits.*

Corollary 3.8.1. *The composite functor $a \circ i$ is naturally isomorphic to the identity functor.*

3.6 Exponentials

Remark. Let F, G be sheaves on C. If the exponential G^F exists in $\mathrm{Sh}(\mathsf{C}, J)$, then a quick computation of isomorphisms of hom sets, natural in P, shows that

$$\begin{aligned}
P \to i(G^F) &\cong a(P) \to G^F \\
&\cong a(P) \times F \to G \\
&\cong a(P \times i(F)) \to G \\
&\cong P \times i(F) \to i(G) \\
&\cong P \to i(G)^{i(F)}.
\end{aligned}$$

Theorem 3.9. *Let P, F be presheaves on $\hat{\mathsf{C}}$. If F is a sheaf, then so is the presheaf exponential F^P.*

Remark. Given any sheaf F on C, we can regard it as a presheaf iF. Then, Yoneda lemma and the adjunction $a \dashv i$ yields

$$F(C) \cong \mathrm{Hom}(\text{よ}(C), iF) \cong \mathrm{Hom}(a\,\text{よ}(C), F).$$

Moreover, recall that the presheaf iF is isomorphic to the colimit $\lim_{\longrightarrow k} \text{よ}(C_k)$ of representable sheaves. Since left adjoints preserve colimits,

$$F \cong ai(F) \cong a \lim_{\longrightarrow k} \text{よ}(C_k) \cong \lim_{\longrightarrow k} a\,\text{よ}(C_k).$$

This mean that the set of sheafifications of representable presheafs $a\,\text{よ}(C)$ **generate** the category $\mathrm{Sh}(\mathsf{C}, J)$.

3.7 Subobjects

Definition 3.8 (Closed Sieve)**.** A sieve M on C is **closed** with respect if J if for all morphisms $f : D \to C$ in C, M covers f implies $f \in M$.

Definition 3.9. Notice that if M is a closed sieve on C, then h^*M is closed for any $h : B \to C$. Hence, we can define a presheaf Ω by assigning $\Omega(C)$ the set of closed sieves on C, with restriction defined by $M \cdot h = h^*(M)$.

Theorem 3.10. *Ω is a sheaf. Furthermore, it is the subobject classifer of* $\mathrm{Sh}(\mathsf{C}, J)$ *with the canonical morphism* $\mathrm{true} : 1 \to \Omega$ *; here, 1 is the final presheaf assigning each C the singleton 1, and* true *is defined by $C \mapsto t_C$, the maximal sieve.*

Corollary 3.10.1 (Epimorphisms are Locally Surjective)**.** *A morphis $\phi : F \to G$ of sheaves is an epimorphism in* $\mathrm{Sh}(\mathsf{C}, J)$ *iff for each object C of C and any $y \in G(C)$, there is a cover S of G such that for all $f : D \to C$ in S, the element $y \cdot f$ is in the image of $\phi_D : F(D) \to G(D)$.*

Corollary 3.10.2. *Given $\phi : PtoQ$, $a(\phi) : aP \to aQ$ is an epimorphism iff ϕ is locally surjective.*

Corollary 3.10.3. *A family $\{f_i : C_i \to C\}$ covers C iff the induced map*

$$\coprod_i a\,\text{よ}(C_i) \to a\,\text{よ}(C)$$

is an epimorphism.

3.8 Subsheaves

Theorem 3.11. *For subsheaves $A_i \leq E$, define the **meet** $\bigwedge_i A_i$ by $(\bigwedge_i A_i)(C) = \bigcap_i A_i(C)$. Then, we can express the **join** $\bigvee_i A_i$, by $(\bigvee_i A_i)(C) = \bigwedge_i \{B \leq E : A_i \leq E \text{ for all } i\}$. With these operations,* $\mathrm{Sub}(E)$ *is a complete Heyting algebra.*

Example 3.2 (Implication)**.** $\mathrm{Sub}(E)$ must admit an implication object $A \Rightarrow B$ for sheaves A, B. It turns out that $(A \Rightarrow B)$ can be described in the following manner: $e \in (A \Rightarrow B)(C)$ if for all $f : D \to C$, $e \cdot f \in A(D)$ implies $e \cdot f \in B(D)$. To check that this is the correct description, it suffices to check that it saffices the crucial property $U \subseteq (A \Rightarrow B)$ iff $U \wedge A \subseteq B$, for all $U \in \mathrm{Sub}(E)$.

Example 3.3 (Universal and Existential Quantification)**.** Any morphism $\phi : E \to F$ of sheaves induces a functor $\phi^{-1} : \mathrm{Sub}(F) \to \mathrm{Sub}(E)$ be pullback. ϕ^{-1} has familiar left and right adjoints:

1. The left adjoint $\exists_\phi : \mathrm{Sub}(E) \to \mathrm{Sub}(F)$ maps a subsheaf $A \leq E$ to the sheaf $\exists_\phi(A)$ described by
$$y \in \exists_\phi(A)(C) \quad \text{if} \quad \{f : D \to C \mid \exists a \in A(D), \phi_D(a) = y \cdot f\} \text{ covers } C,$$
for $y \in E(C)$. As usual, $\exists_\phi(A) \leq B$ iff $A \leq \phi^{-1}(B)$.

2. The right adjoint $\forall_\phi : \mathrm{Sub}(E) \to \mathrm{Sub}(F)$ maps a subsheaf $A \leq E$ to the sheaf $\forall_\phi(A)$, which has a more nuanced description than the one above. First, we might guess the description of
$$y \in \forall'_\phi(A)(C) \quad \text{if} \quad \text{for all } x \in E(C), \phi_C(x) = y \text{ implies } x \in A(C),$$
for $y \in F(C)$. The definition of $\forall'_\phi(A)$ does not produce a presheaf, but it is enough to account for the action of morphisms on y. Hence, we define
$$y \in \forall_\phi(A)(C) \quad \text{if} \quad \text{for all } f : D \to C, y \cdot f \in \forall'_\phi(A)(D).$$

4 Properties of Elementary Topoi

4.1 Definition of a Topos

Definition 4.1 ((Elementary) Topos)**.** A **topos** $\mathcal{E}$ is a category with all finite limits, a **subobject classifer** Ω, and a **power object** functor $P : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$, such that we have isomorphisms

$$\begin{aligned} \mathrm{Sub}_{\mathcal{E}}(A) &\cong \mathrm{Hom}_{\mathcal{E}}(A, \Omega), \\ \mathrm{Hom}_{\mathcal{E}}(B \times A, \Omega) &\cong \mathrm{Hom}_{\mathcal{E}}(A, PB), \end{aligned}$$

which are natural in A and B. Moreover, for any object A, PA is, more familiarly, the exponential Ω^A.

Remark. Note that the isomorphisms can be combined to form the isomorphism

$$\mathrm{Sub}_{\mathcal{E}}(B \times A) \cong \mathrm{Hom}_{\mathcal{E}}(A, PB).$$

Setting $B = 1$ gives us the first isomorphism above with $\Omega = P1$. Hence, we get the second isomorphism immediately.

Remark. Even though our initial definition of topos relied on the notion of sets (isomorphisms and hom sets), the axioms of topoi can be formulated in an *elementary* way, free of any reference to sets. It is because of this formulation that topos theory may serve as a foundation of mathematics.

Definition 4.2 (Topos (Elementary Form))**.** A topos is a category $\mathcal{E}$ with

1. A pullback for every diagram $X \to B \leftarrow Y$;

2. A terminal object 1;

3. An object Ω and a monomorphism $\mathrm{true} : 1 \rightarrowtail \Omega$ such that for any monomorphism $m : S \rightarrowtail B$, there is a unique morphism $\mathrm{char}\, S : B \to \Omega$ in $\mathcal{E}$ such that the following square is a pullback.

$$\begin{tikzcd} S \arrow[r] \arrow[d, tail, "m"'] & 1 \arrow[d, tail, "\mathrm{true}"] \\ B \arrow[r, "\mathrm{char}\, m"'] & \Omega \end{tikzcd}$$

   $\mathrm{char}\, S$, sometimes called $\mathrm{char}\, m$, is the **characteristic map of** m.

4. To each object B and object PB and a morphism $\in_B : B \times PB \to \Omega$ such that for each morphism $f : B \times A \to \Omega$ there is a unique arrow $g : A \to PB$ such that the following diagram commutes:

$$
\begin{tikzcd}
A \arrow[d, "g"'] & & B \times A \arrow[r, "f"] \arrow[d, "1\times g"'] & \Omega \arrow[d, equal] \\
PB & & B \times PB \arrow[r, "\in_B"'] & \Omega
\end{tikzcd}
$$

•

g is called the P**-transpose** of f, and $f = \hat{g}$ the P-transpose of g. Moreover, for each morphism $h : B \to C$, we get the arrow $Ph : PC \to PB$ which is the unique arrow making the following diagram commute:

$$
\begin{tikzcd}
 & C \times PC \arrow[dr, "\in_C"] & \\
B \times PC \arrow[ur, "h\times 1"] \arrow[dr, "1\times Ph"'] & & \Omega \\
 & P \times PB \arrow[ur, "\in_B"'] &
\end{tikzcd}
$$

Example 4.1. In the category Set, given some map $f : B \times A \to \Omega$, each element $a \in A$ *indexes* some subset $S \subseteq B$ defined by $S = \{b \in B : (b, a) \mapsto 0\}$. Then, $g(a)$ is precisely S.

Example 4.2 (Generalized Elements and Predicates)**.** A morphism $b : X \to B$ is a **generalized element** of B, defined over X. The elements over the terminal object 1 are the **global elements**—in Set, these are the elements in B, and in $\mathrm{Sh}(X)$, these are the global sections in B. For example, given generalized elements $a : X \to A, b : X \to B$, we get an element $(a, b) : X \to A \times B$ uniquely determined by a and b, by the universal property of $A \times B$.

A morphism $\theta : B \to \Omega$ can be thought of as a **predicate** of elements of B. For example, the predicate true_B is the unique predicate that factors through true. Thus, we can interpret the diagram in Definition 4.2 being a pullback as $b : X \to B$ factors through (is in) the subobject $S \rightarrowtail B$ if $(\mathrm{char}\, S)b = \mathrm{true}_X$. $\mathrm{char}\, S$ is thus the membership predicate of S.

Remark (The Subobject Trinity)*.* Thus far, a subobject of A has three descriptions,

$$m : S \rightarrowtail A, \qquad \phi : A \to \Omega, \qquad s : 1 \to PA,$$

as an equivalence class of monomorphisms to A, as a predicate of A, and as a global element of the power object PA. When m, ϕ, s correspond, we write

$$S = \{a \mid \phi\}, \qquad \phi = \mathrm{char}\, S \qquad s = \ulcorner \phi \urcorner,$$

where S is the **extension** of ϕ, ϕ is the **characteristic function** of S, and s is the **name** of ϕ.

Example 4.3 (The Diagonal and Singleton Morphisms)**.** Given the diagonal map $\Delta_B : B \to B \times B = (\mathrm{id}_B, \mathrm{id}_B)$, the corresponding characteristic map $\delta_B = \mathrm{char}\,\Delta_B$ is the **predicate of equality**, usually known as the Kronecker delta function in Set. Let $\{\cdot\}_B$ denote the P-transpose of δ_B: it satisfies the relation $\in_B \langle b, \{\cdot\}_B b' \rangle = \delta_B \langle b, b' \rangle$, so $\{\cdot\}_B$ decisively sends b' to the subobject of B whose only X-based element is b'. We call the monomorphism $\{\cdot\}_B$ the **singleton arrow**, since in Set it sends $b' \in B$ to the singleton $\{b'\}$; it becomes a useful fact later that $\mathrm{char}\{\cdot\}_B$ is the **singleton predicate** σ_b.

Theorem 4.1. *In a topos, every monomorphism $S \rightarrowtail B$ is an equalizer of* true_B *and* $\mathrm{char}\, m$*. Furthermore, every morphism which is monic and epic is an isomorphism.*

4.2 The Construction of Exponentials

Theorem 4.2. *Every topos has exponentials.*

Proof. Given objects B, C, the exponential C^B is defined in three steps.

$$C \times B \times P(C \times B) \xrightarrow{\in_{C \times B}} \Omega$$

$$B \times P(C \times B) \xrightarrow{v := \hat{\in}_{C \times B}} PC \xrightarrow{\sigma_B} \Omega$$

$$\begin{tikzcd} P(C \times B) \arrow[r, "u := (\sigma_B \hat{\circ} v)"] & PB \\ C^B \arrow[u, dashed, "m"] \arrow[r] & 1 \arrow[u, "\ulcorner \mathrm{true}\, B \urcorner"'] \end{tikzcd}$$

First, we take the morphism $\in_{C \times B}$, which is the P-transpose of $1 : P(C \times B) \to P(C \times B)$ and acts like the membership relation. Then, v is the P-transpose of $\in_{C \times B}$ with respect to $C \times -$; its composition with σ_C behaves like the predicate in Set which asks if $R^{-1}(b')$ is a singleton, for $b' \in B$ and $R \subseteq C \times B$. Lastly, the P-transpose u of $\sigma_C \circ v$ acts like the Set morphism sending each $R \subseteq P(C \times B)$ to the set $B' \subseteq B$ of elements such that $R^{-1}(b')$ is a singleton for all $b' \in B'$. Hence, we define C^B to be the pullback of $\ulcorner \mathrm{true}\, B \urcorner$ along u, which can be translated in Set as "the collection of objects $R \subseteq C \times B$ which is the graph of some well-defined function". The existence of an evaluation map $e : B \times C^B \to C$ is proved by fiddling with P-tranposes. ■

Example 4.4 (Internal Hom)**.** The exponential B^A is called the **internal Hom**, and admits an operation of **internal composition** $m : C^B \times B^A \to C^A$ which is defined as the transpose of the composition

$$C^B \times B^A \times A \xrightarrow{1 \times e} C^B \times B \xrightarrow{e} C.$$

Definition 4.3 (Logical Morphism)**.** A **logical morphism** is functor $T : \mathcal{E} \to \mathcal{E}'$ which preserves, up to isomorphisms, all structures required to defined a topos: T preserves finite limits, the subobject classifer, and the exponential.

4.3 Direct Image

Definition 4.4 (Direct Image for Power Objects)**.** Given a set function $k : B' \to B$, we can describe the **direct image** function $\exists_k : PB' \to PB$ mapping $S \subseteq B$ to the set $\{b \in B : \exists b' \in S',\ k(b') = b\}$. The direct image can be given for any topos (for now, we consider the case when k is monic), which is defined by first taking the monomorphism $u'_B : U \rightarrowtail B' \times PB'$ whose characteristic morphism is the predicate $\in_{B'}$, passing to the characteristic morphism e_k of $k \times 1 \circ u_{B'}$, and then taking the P-transpose $\exists_k$.

Theorem 4.3. *For monomorphisms $S \overset{m}{\rightarrowtail} B' \overset{k}{\rightarrowtail} B$ in a topos,*

$$\exists_k \ulcorner \mathrm{char}\, m \urcorner = \ulcorner \mathrm{char}\, km \urcorner : 1 \to PB.$$

Definition 4.5 (Direct Image for Subobjects)**.** Given a morphism $k : B' \to B$, there is a corresponding direct image map $k! : \mathrm{Sub}\, B' \to \mathrm{Sub}\, B$ for subobjects which pushes forward the subobject $k' : B'' \rightarrowtail\!\!\to B'$ by composing with k.

Remark. There is a nice interplay between the direct image and the morphisms (by pullback) in $\mathrm{Sub}_{\mathcal{E}}$, since a composition of pullbacks is a pullback of the composite morphism. This amounts to saying that

$$\begin{tikzcd}
\mathrm{Sub}\, B' \arrow[r, "\mathrm{Sub}\, g'"] \arrow[d, "k!"'] & \mathrm{Sub}\, C' \arrow[d, "m!"] \\
\mathrm{Sub}\, B \arrow[r, "\mathrm{Sub}\, g"'] & \mathrm{Sub}\, C
\end{tikzcd}$$

where we have morphisms $g' : C' \to B', g : C \to B$, and m' and m are equal to $(\mathrm{Sub}\, g')(k')$ and $(\mathrm{Sub}\, g)(k)$. This anticipates an adjacent result that holds *internally*—that is, for power objects rather than subobjects.

Theorem 4.4 (The Beck-Chevalley Condition for $\exists$)**.** *If the square on the left forms a pullback, then the square on the right commutes.*

$$\begin{tikzcd}
C' \arrow[r, "g'"] \arrow[d, "m"'] & B' \arrow[d, "k"] \\
C \arrow[r, "g"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
PB' \arrow[r, "Pg'"] \arrow[d, "\exists_k"'] & PC' \arrow[d, "\exists_m"] \\
PB \arrow[r, "Pg"'] & PC
\end{tikzcd}$$

Corollary 4.4.1. *If $k : B' \rightarrowtail B$ is a monomorphism, then the composite*

$$PB' \xrightarrow{\exists_k} PB \xrightarrow{Pk} PB'$$

is the identity.

4.4 Monads and Beck's Theorem

Definition 4.6 (Eilenberg-Moore Category)**.** Let C be a category with a monad (T, η, μ). The **Eilenberg-Moore category**, or **category of T-algebras**, is the category C^T whose objects are pairs $(A \in \mathsf{C}, a : TA \to A)$ such that the diagrams

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & TA \arrow[d, "a"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} T^2A \arrow[r, "\mu_a"] \arrow[d, "Ta"'] & TA \arrow[d, "a"] \\ TA \arrow[r, "a"'] & A \end{tikzcd}$$

commute in C, and morphisms are T-algebra homomorphisms: maps $f : A \to B$ in C so that the square

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"'] & TB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes.

Definition 4.7 (Kleisli Category)**.** Let C be a category with a monad (T, η, μ). The **Kleisli category** C_T has the same objects of C, and a morphism $A \rightsquigarrow B$ is a morphism $A \to TB$ in C, where

- The unit $\eta_A : A \to TA$ defines the identity morphism $A \rightsquigarrow A \in \mathsf{C}_T$.
- The composite of a morphism $f : A \rightsquigarrow B$ and $g : B \rightsquigarrow C$ is defined to be

$$A \xrightarrow{f} TB \xrightarrow{Tg} T^2C \xrightarrow{\mu_C} TC.$$

Theorem 4.5 (All Monads are Induced by Adjunctions)**.** *For any monad (T, η, μ) acting on a category C, there are adjunctions*

$$\begin{tikzcd} \mathsf{C} \arrow[r, shift right, "U^T"'] \arrow[r, phantom, "\perp"] & \mathsf{C}^T \arrow[l, shift right, "F^T"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{C} \arrow[r, shift right, "U_T"'] \arrow[r, phantom, "\perp"] & \mathsf{C}_T \arrow[l, shift right, "F_T"'] \end{tikzcd}$$

Here, U^T is the forgetful functor and F^T sends an object A to the ***free T-algebra*** *$(TA, \mu_A : T^2A \to TA)$. On the other hand, F_T is the identity on objects and acts on morphisms $f : A \to B$ by pushforward via η_B, and U_T sends and object A to TA and sends the morphism $g : A \rightsquigarrow B$ to*

$$U_T g : TA \xrightarrow{Tg} T^2B \xrightarrow{\mu_B} TB.$$

Theorem 4.6. *Let Adj_T be the category of adjunctions inducing the monad (T, η, μ). Then, the associated Kleisli and Eilenberg-Moore categories are, respectively, initial and final objects. Moreover, there is a* ***canonical comparison functor*** *$K : \mathsf{C}_T \to catC^T$, which is fully faithful and whose image consists of the free T-algebras.*

Definition 4.8 (Monadic Functor). A functor $U : \mathsf{D} \to \mathsf{C}$ is **monadic** if it has a left adjoint F, and if the canonical comparison functor $K : D \to \mathsf{C}^{UF}$ is an equivalence of categories.

Theorem 4.7. *Any monadic functor F creates limits; that is, given a diagram $K : \mathsf{C} \to \mathsf{D}$, if FK has a limit in D, then there is a limit cone over FK that can be lifted to a limit cone over K, and F reflects these limit.*

Definition 4.9 (Reflexive Pair). A reflexive pair is a pair of morphisms $s, t : A \rightrightarrows B$ along with a morphism $i : B \to A$ such that $si = ti = 1_B$.

Theorem 4.8 (Beck's Theorem). *Let $G : \mathsf{A} \to \mathsf{C}$ be a functor with a left adjoint, let T be the corresponding monad in C, and let $K : \mathsf{A} \to \mathsf{C}^T$ be the comparison functor. Then,*

1. *If A has coequalizers of* ***reflexive pairs****, K has a left adjoint L.*
2. *If, in addition, G preserves these coequiazliers, the unit of this adjunction is an isomorphism $I_{\mathsf{C}^T} \cong KL$.*
3. *If, in addition to (1) and (2), G reflects isomorphisms, then the counit of this adjunction is also an isomorphism; consequently, G is monadic in this case.*

4.5 The Construction of Colimits

Theorem 4.9. *The functor $P : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ has a left adjoint, which is $P^{\mathrm{op}} : \mathcal{E} \to \mathcal{E}^{\mathrm{op}}$*

Proof. A quick computation shows that we have the following natural isomorphisms:

$$\mathcal{E}(A, PB) \cong \mathcal{E}(B \times A, \Omega) \cong \mathcal{E}(A \times B, \Omega) \cong \mathcal{E}(A, PB) = \mathcal{E}^{\mathrm{op}}(PB, A).$$

∎

Theorem 4.10. *The power-set functor $P : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ is monadic.*

Proof. It suffices to prove P satisfies the three conditions of Beck's theorem:

1. $\mathcal{E}^{\mathrm{op}}$ has coequalizers since $\mathcal{E}$ has all finite limits, which subsumes equalizers.
2. The Beck-Chevalley condition is used to prove that P reflects coequalizers.
3. It is proven that P is faithful (using the P-tranpose), so it must reflect monomorphisms and epimorphisms, and hence isomorphisms (all epic and monic morphisms are isomorphisms in a topos).

∎

Corollary 4.10.1. *A topos $\mathcal{E}$ has all finite colimits.*

Proof. Let $T = PP^{\mathrm{op}}$ be the monad associated to $P^{\mathrm{op}} \dashv P$. The forgetful functor $\mathcal{E}^T \to \mathcal{E}$ creates limits. If J is a finite index category, $\mathcal{E}$ has J^{op}-limits, so $\mathcal{E}^T$ has all J^{op} limits. Since P is monadic, $\mathcal{E}^{\mathrm{op}}$ is equivalent to $\mathcal{E}^T$, which means $\mathcal{E}^{\mathrm{op}}$ has all J^{op} limits, which are simply J limits in $\mathcal{E}$. ∎

4.6 Factorization and Images

Definition 4.10 (Image)**.** The **image** of a morphism f, if it exists, is a monomorphism such that $f = me$ factors through m, and if f factors through some monomorphism h, then so does m.

Theorem 4.11. *In a topos, every morphism f has an image m and factors as $f = me$, with e epic.*

Theorem 4.12. *If $f = me$ and $f' = m'e'$ are two* monic $\circ$ epic *decompositions, then each commutative square $f \Rightarrow f'$ extends to unique adjacent commutative squares $m \circ e \Rightarrow m' \circ e'$.*

Theorem 4.13. *For each object A in a topos the partially ordered set $\operatorname{Sub} A$ of subobjects of A is a Heyting algebra, and for each morphism $k : A \to B$, pullback along k is a morphism $k^{-1} : \operatorname{Sub} B \to \operatorname{Sub} A$ of posets. Again, there exists a left adjoint $\exists_k$ which sends each each subobject $u : S \rightarrowtail A$ to the image $m : \exists_k S \rightarrowtail B$ of ku.*

Remark. Since k^{-1} is a right adjoint, it is a **meet-semilattice homomorphism**, which is to say k^{-1} preserves finite products: $k^{-1}(S \cap T) = k^{-1}(S) \cap k^{-1}(T)$. Alternatively, we can say that the meet $\bigcap : \operatorname{Sub} B \times \operatorname{Sub} B \to \operatorname{Sub}$ is natural in B. Moreover, under the natural isomorphism $\operatorname{Hom}(B \cap \Omega) \cong \operatorname{Sub} B$, we get an induced morphism $\bigwedge : \Omega \times \Omega \to \Omega$ through the Yoneda lemma; similarly, the natural isomorphism $\operatorname{Sub}(B \times X) \cong \operatorname{Hom}(X, PB)$ gives us a morphism $\bigwedge : PB \times PB \to PB$, called the **internal meet** on PB.

Definition 4.11 (Open Object)**.** An object U is open in $\mathcal{E}$ is **open** if $U \to 1$ is monic.

Theorem 4.14. *In a topos $\mathcal{E}$ the lattice* $\operatorname{Sub} 1$ *regarded is a category is equivalent to the full subcategory* $\operatorname{Open}(\mathcal{E})$ *of open objects $\mathcal{E}$. An object U is open in $\mathcal{E}$ if and only if there is at most one arrow $X \to U$ from any object X.*

4.7 The Slice Topos

Theorem 4.15 (Fundamental Theorem of Topos Theory)**.** *For any object B in a topos $\mathcal{E}$, the slice category $\mathcal{E}/B$ is also a topos.*

Remark. A toy version of the theorem in the topos Set follows from the fact that $\mathsf{Set}^B \cong \mathsf{Ser}/B$, where Set^B is the discrete category.

Theorem 4.16. *For any arrow $k : B \to A$ in a topos $\mathcal{E}$, we get a* ***change of base*** *functor $k^* : \varepsilon/A \to \varepsilon/B$ defined by pullback. Furthermore, k^* has a left adjoint $\sum_k$, given by composition with k, and a right adjoint $\prod_k$. Moreover, k^* preserves the subobject classifer and exponentials, and hence is a logical morphism.*

Corollary 4.16.1. *In a topos, the following important properties of morphisms hold:*

1. *The pullback of an epimorphism is epic*
2. *Any morphism $k : A \to 0$ is an isomorphism; that is, 0 is a* ***strict*** *initial object.*
3. *The morphism $0 \to B$ is monic.*

Definition 4.12 (Disjoint Subobjects)**.** We say $S, T \in \operatorname{Sub} B$ are **disjoint** if $S \cap T \cong 0$

Theorem 4.17. *If S, T are disjoint subobjects of B, then $S \cup T \cong S + T$, where $+$ denotes coproduct.*

Theorem 4.18. *Consider a family $m_i : S_i \rightarrowtail B$ of disjoint subobjects of B. If their coproduct $\coprod S_i$ exists, the induced map $m : \coprod S_i \to B$ is monic, and represents the supremum of the subobjects S_i.*

Theorem 4.19. *In a topos, if $f : X \to Y$ and $g : W \to Z$ are epimorphisms, then so is $f \times g : X \times W \to Y \times Z$. Furthermore, every epimorphism is the coequalizer of its **kernel pair**; that is, the pair $A \times_B A \rightrightarrows A$ obtained from the pullback of f along itself.*

4.8 Lattice Objects in a Topos

Definition 4.13 (Internal Lattice/Heyting Algebra)**.** A lattice (Heyting algebra) object is sometimes called an **internal lattice** (**internal Heyting algebra**).

Remark (Internal Poset). If L is an internal lattice in a category C, we have a notion of a partial order: $x \leq y$ iff $x \bigwedge y = x$. Thus, define the subobject $\leq_L$ of $L \times L$ as the equalizer

$$\leq_L \xrightarrow{\quad e \quad} L \times L \underset{\pi_1}{\overset{\wedge}{\rightrightarrows}} L$$

Then, $(L, \leq_L)$ is an **internal partial order**, which means there exist appropriate commutative diagrams witnessing the axioms of partial orders. One can further introduce a binary operation $\Rightarrow : L \times L \to L$ to make L a Heyting algebra; these two constructions of an internal Heyting algebra are equivalent.

Theorem 4.20. *For any object A in a topos $\mathcal{E}$, we have adjacent external and internal results:*

1. ***(External)*** *The poset $\operatorname{Sub} A$ of subobjects of A has the structure of a Heyting algebra. Moreover, this structure is natural in A in the sense that the pullback along any morphism $k : A \to B$ induces a map k^{-1} of Heyting algebras*

$$\begin{tikzcd} \mathrm{Sub}_{\mathcal{E}}(A) \arrow[r, "k^{-1}"] \arrow[d, hook, "\iota_B"'] & \mathrm{Sub}_{\mathcal{E}}(B) \arrow[d, hook, "\iota_A"] \\ \mathcal{E}/A \arrow[r, "k^*"'] & \mathcal{E}/B \end{tikzcd}$$

2. ***(Internal)*** *The power object PA is an internal Heyting algebra. In particular, so is $\Omega = P1$. Moreover, this structure is natural in A, in the sense that, for a morphism $k : A \to B$ in $\mathcal{E}$, the induced map $Pk : PB \to PA$ is a homomorphism of internal Heyting algebras. For each X in $\mathcal{E}$ the internal structure on PA makes $\operatorname{Hoim}(X, PA)$ an external Heyting algebra so that the canonical isomorphism*

$$\mathrm{Sub}_{\mathcal{E}}(A \times X) \cong \mathrm{Hom}_{\mathcal{E}}(X, PA),$$

is an isomorphism of external Heyting algebras.

4.9 The Beck-Chevalley Condition

Theorem 4.21 (Frobenius Identity)**.** *Given a morphism $f : A \to B$, subobjects U, V of B, and a subobject W of A in a topos $\mathcal{E}$, we have the identity*

$$\exists_f(W) \bigwedge U = \exists_f(W \cap f^{-1}(U)).$$

Theorem 4.22 (External Beck-Chevalley Condition)**.** *Give the pullback square on the left, the diagram on the right satisfies the Beck-Chevalley condition:*

$$\begin{tikzcd} C \times_A B \arrow[r, "p"] \arrow[d, "q"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Sub}(C \times_A B) \arrow[r, shift right, "\exists_p"'] \arrow[d, shift left, "\exists_q"] & \mathrm{Sub}(B) \arrow[l, shift right, "p^{-1}"'] \arrow[d, shift right, "\exists_f"'] \\ \mathrm{Sub}(B) \arrow[r, shift left, "\exists_p"] \arrow[u, shift left, "q^{-1}"] & \mathrm{Sub}(A) \arrow[l, shift left, "g^{-1}"] \arrow[u, shift right, "f^{-1}"'] \end{tikzcd}$$

That is,$g{-}1\exists_f U = \exists_q p^{-1}U$.

Theorem 4.23 (Internal Beck-Chevalley Condition)**.** *Let $f : A \to B$ be a map $\mathcal{E}$. Then $Pf : PB \to PA$ has an internal left adjoint $\exists_f : PA \to PB$. This can be expressed by saying $\langle \exists_f Pf, 1_{PA} \rangle$ factors through $\leq_{PA}$, and $\langle 1_{PA}, Pf\exists_f \rangle$ factors through $\leq_{PB}$. Moreover, the internal Frobenius identity and Beck-Chevalley conditions hold.*

Remark. The same results above can be repeated for the internal/external right adjoint $\forall_f$.

4.10 Injective Objects

5 Basic Constructions of Topoi

5.1 Lawvere-Tierney Topologies

Definition 5.1 (Lawvere-Tierney Topology)**.** Given a topos $\mathcal{E}$, a **Lawvere-Tierney Topology** is a morphism $j : \Omega \to \Omega$ satisfying the three axioms

1. $j \circ \text{true} = \text{true}$.

2. $j \circ j = j$.

3. $j \circ \wedge = \wedge \circ (j \times j)$.

By definition, j is uniquely the characteristic morphism of some subobject $J \rightarrowtail \Omega$.

Example 5.1. Here is an example of a topology for the topos $\mathcal{E} = \hat{\mathcal{O}(X)}$ of presheaves on a space X. Define $J \leq \Omega$ by $J(U) = \{S : S \text{ is a covering sieve of } U\}$. Notice that J is a subfunctor since if S covers U and $W \subseteq U$, then $S \cap W$ covers W. Recall that the map $\text{true}_U : 1 \to \Omega(U)$ picks the maximal sieve $\hat{U}$ on U Then the corresponding characteristic morphism $j : \Omega \to \Omega$ must be defined by

$$j_U(S) = \{W : W \text{ is open in } U \text{ and } S \cap W \text{ covers } W\},$$

which sends S to the principal sieve $\hat{V}$, where $V = \bigcup_{W \in S} W$. In other words, $j_U(S)$ specifies the largest open set $V \subseteq U$ covered by S, and S covers U precisely when $j_U(S) = \hat{U}$. j is a Lawvere-Tierney topology since

1. $j_U(\hat{U}) = \hat{U}$.

2. j_U is idempotent by its description.

3. $j_U(S \cap T) \subseteq j_U(S) \cap j_U(T)$ since j_U is order preserving, and if $W \in j_U(S) \cap j_U(T)$, then $W = \bigcup V_i = \bigcup V_j'$, for $V_i \in S$ and $V_j \in T$, but this means that $W = \bigcup (V_i \cap V_j') \in j_U(S \cap T)$.

Definition 5.2 (Closure Operator)**.** By the following correspondence, we get a unary **closure operator** $A \mapsto \overline{A}$ on the subobjects $A \rightarrowtail E$ of each object E:

$$\begin{tikzcd}
\mathrm{Hom}(E, \Omega) \arrow[r, "\cong"] \arrow[d, "\mathrm{Hom}(1,j)"'] & \mathrm{Sub}(E) \arrow[d, "A \mapsto \overline{A}"] \\
\mathrm{Hom}(E, \Omega) \arrow[r, "\cong"'] & \mathrm{Sub}(E)
\end{tikzcd}$$

In other words, $\mathrm{char}(\overline{A}) = j\, \mathrm{char}(A)$. Furthermore, closure is natural in E, which is to say for any morphism $f : E \to F$ and a subobject B of F, $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$. We call $\overline{A}$ the **closure** of $A \rightarrowtail E$, and we say that A is **dense** in E when $\overline{A} = E$.

Theorem 5.1. *For any topos $\mathcal{E}$, each arrow $j : \Omega \to \Omega$ determines by the diagram above an operator $A \mapsto \overline{A}$ on the subobjects of each object E, which is natural in E. Moreover, j is a Lawvere-Tierney topology if and only if for all $A, B \in \mathrm{Sub}(E)$,*

$$A \leq \overline{A}, \qquad \overline{\overline{A}} = A, \qquad \overline{A \wedge B} = \overline{A} \wedge \overline{B}.$$

Conversely, any operator satisfying all properties above arises from unique Lawvere-Tierney topology j.

Theorem 5.2. *Every Grothendieck topology J on a small category C determines a Lawvere-Tierney topology j on the presheaf topos $\hat{\mathsf{C}}$ by defining $j_C(S) = \{g : S \text{ covers } g : D \to C\}$.*

5.2 Sheaves

Definition 5.3 (Sheaf)**.** An object F of $\mathcal{E}$ is called a **sheaf** for the Lawvere-Tierney topology j if for every dense monomorphism $m : A \rightarrowtail E$, pullback by m induces an isomorphism $m^* : \mathrm{Hom}_{\mathcal{E}}(E, F) \to \mathrm{Hom}_{\mathcal{E}}(A, F)$. In other words, each map from a dense subobject of E into a sheaf can be uniquely extended to a map on E. We write $\mathrm{Sh}_j\, \mathcal{E}$ for the full subcategory of sheaves of $\mathcal{E}$.

Definition 5.4 (Separated Object)**.** An object F of $\mathcal{E}$ is called **separated** if for each dense $A \rightarrowtail E$, $\mathrm{Hom}_{\mathcal{E}}(E, F) \to \mathrm{Hom}_{\mathcal{E}}(A, F)$ is monic. We let $\mathrm{Sep}_j\, \mathcal{E}$ denote the full subcategory of separated objects of $\mathcal{E}$.

Lemma 5.3. $\mathrm{Sep}_j\, \mathcal{E}$ *and* $\mathrm{Sh}_j\, \mathcal{E}$ *are closed under all finite limits and exponentiation with an arbitrary object from $\mathcal{E}$.*

Remark. The morphisms $j, 1 : \Omega \to \Omega$ have an equalizer $\Omega_j \xrightarrow{m} \Omega$. Since j is idempotent, the universal property of equalizers implies that j factors through a unique morphism $r : \Omega \to \Omega_j$. Since $mrm = jm = m$ and m is monic, it turns out that $rm = 1$, which means Ω_j is a retract of Ω. Moreover, if a subobject A of E is characterized by a map $a : E \to \Omega$, then $\overline{A}$ is characterized by $j \circ a$. Notice then that A is closed iff $j \circ a = a$ iff a factors through $\Omega_j \rightarrowtail \Omega$. This gives us the following lemma.

Lemma 5.4. *Ω_j classifies closed subobjects: for each object E, there is a bijection natural in E*

$$\mathrm{Hom}_{\mathcal{E}}(E, \Omega_j) \to \mathrm{Cl}\,\mathrm{Sub}_{\mathcal{E}}(E),$$

where $\mathrm{Cl}\,\mathrm{Sub}_{\mathcal{E}}(E)$ is the lattice of closed subobjects of E.

Lemma 5.5. *If $m : A \rightarrowtail E$ is dense, then the inverse image morphism $m^{-1} : \mathrm{Cl}\,\mathrm{Sub}_{\mathcal{E}}(E) \to \mathrm{Cl}\,\mathrm{Sub}_{\mathcal{E}}(A)$ is an isomorphism.*

Lemma 5.6. *If $m : A \rightarrowtail E$ is a subobject of a sheaf E, then A is closed in E iff A is also a sheaf.*

Theorem 5.7. *Let $\mathcal{E}$ be a topos with a Lawvere-Tierney topology j. Then $\mathrm{Sh}_j\, \mathcal{E}$ is a topos, and the inclusion $\mathrm{Sh}_j\, \mathcal{E} \rightarrowtail \mathcal{E}$ is left exact and preserves exponentials.*

5.3 The Associated Sheaf Functor

Lemma 5.8. *If $B \rightarrowtail C$ is monic and C is separated, then so is B.*

Lemma 5.9. *For any object C in $\mathcal{E}$, the following are equivalent:*

1. *C is separated.*
2. *The diagonal $\Delta_C \rightarrowtail C \times C$ is a closed subobject of $C \times C$.*
3. *The following diagram commutes:*

$$\begin{tikzcd} C \arrow[r, "\{\cdot\}_C"] \arrow[dr, "\{\cdot\}_C"'] & \Omega^C \arrow[d, "j^C"] \\ & \Omega^c \end{tikzcd}$$

4. *For any $f : A \to C$, the graph of f is a closed subobject of $A \times C$.*

Theorem 5.10. *E is separated iff it can be embedded in an injective sheaf, the immediate candidate being Ω_j^E.*

Lemma 5.11. *For any object E, there is an epimorphism θ_E to a separated object E' such that the kernel pair of $\theta_E : E \to E'$ is precisely the closure $\overline{\Delta}$ of the diagonal $\Delta \rightarrowtail E \times E$.*

Corollary 5.11.1. *The map θ_E mentioned above is universal for maps from E into separated objects, which defines a left adjoint to the forgetful functor* $\mathrm{Set}_j\,\mathcal{E} \rightarrowtail \mathcal{E}$.

Corollary 5.11.2. *The forgetful functor* $\mathrm{Sh}_j\,\mathcal{E} \rightarrowtail \mathrm{Set}_j$ *has a left adjoint.*

Theorem 5.12. *Let j be a Lawvere-Tierney topology on a topos $\mathcal{E}$. Then, the forgetful functor has a left adjoint $\alpha : \mathcal{E} \to \mathrm{Sh}_j\,\mathcal{E}$.*

5.4 Lawvere-Tierney Subsumes Grothendieck

Theorem 5.13. *If C is a small category, the Grothendieck topologies J on C correspond exactly to Lawvere-Tierney topologies on $\hat{\mathsf{C}}$.*

Theorem 5.14. *Let C be a small category with a Lawvere-Tierney topology j on $\hat{\mathsf{C}}$ and the corresponding Grothendieck topology j on C. Then the presheafs of j coincide with the presheafs of J.*

5.5 Internal vs External

So far, we've developed in parallel two competing philosophies for examining the topos $\mathcal{E}$, which are the internal and external perspectives. The internal perspective is one where $\mathcal{E}$ is taken as a mathematical objects which satisfies all the elementary axioms of a topos. From this perspective, $\mathcal{E}$ need not rely on the notions of sets and can be treated as an independent universe of discourse. Contrarily, the external dogma treats ε as a set-theoretical structure with a set of objects and a set of arrows. Here, we gather a table comparing the respective notions of each perspective.

Notion	Internal	External
Power Object	PA	$\mathrm{Sub}(A)$
Hom Object	B^A	$\mathrm{Hom}(A, B)$
Category of Sheaves	$\mathrm{Sh}_j(\mathcal{E})$	$\mathrm{Sh}(\mathsf{C}, J)$
Beck-Chevalley	Theorem 4.23	4.22
Composition	$m : C^B \times B^A \to C^A$	$\circ : \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$

6 Topoi and Logic

6.1 The Topos of Sets

Definition 6.1 (Natural Numbers Object)**.** Any topos $\mathcal{E}$ is said to satisfy the axiom of infinity if it admits a **natural numbers object** $\mathbb{N}$ with morphisms $1 \xrightarrow{0} \mathbb{N} \xrightarrow{s} \mathbb{N}$, such that for any object X and morphisms $1 \xrightarrow{x} X \xrightarrow{f} X$, there is a unique arrow h that makes the following diagram commute:

$$\begin{tikzcd}
1 \arrow[r, "0"] \arrow[d] & \mathbb{N} \arrow[r, "s"] \arrow[d, dashed, "h"] & \mathbb{N} \arrow[d, dashed, "h"] \\
1 \arrow[r, "x"'] & X \arrow[r, "f"'] & X
\end{tikzcd}$$

It follows but he definition that $\mathbb{N}$ is unique up to isomorphism.

Remark. Suppose there is an adjunction $g^* \dashv g_*$ with $g_* : \mathcal{F} \rightleftarrows \mathcal{E} : g^*$, such that g^* preserves the terminal object, e.g. if g^*, g_* come from a **geometric morphism** $g : \mathcal{F} \to \mathcal{E}$. If $\mathcal{E}$ admits a nno $\mathbb{N}$, then

$$1 \cong g^*(1) \xrightarrow{g^*(0)} g^*(\mathbb{N}) \xrightarrow{g^*(s)} g^*(\mathbb{N})$$

is a nno for $\mathcal{F}$. In particular, any Grothendieck topos has a nno $\hat{\mathbb{N}} = a\Gamma(\mathbb{N})$.

Theorem 6.1. *For a topos $\mathcal{E}$, the following conditions are equivalent:*

1. *$\mathcal{E}$ is boolean.*
2. *The negation operator $\neg : \Omega \to \Omega$ satisfies $\neg\neg = 1$.*
3. *For every object E of $\mathcal{E}$, the Heyting algebra* $\mathrm{Sub}(E)$ *is boolean.*
4. *For every subobjects $S \rightarrowtail E$ in $\mathcal{E}$, $\neg S \vee S = E$.*
5. *The maps* $\mathrm{true} : 1 \to \Omega$ *and* $\mathrm{false} = \neg\,\mathrm{true} : 1 \to \Omega$ *induce an isomorphism* $1 + 1 \cong \Omega$.

Lemma 6.2. *Given a topos $\mathcal{E}$, a topology j, and a j-sheaf F, the following identities hold in* $\mathrm{Sub}_{\mathcal{E}_j}(F)$ *for closed subobjects S and T of F:*

1. *$1_j = 1$ and $S \wedge_j T = S \wedge T$.*
2. *$0_j = \overline{0}$ and $S \vee_j T = \overline{S \vee T}$.*
3. *$(S \Rightarrow_j T) = (S \Rightarrow T)$.*
4. *$\neg_j S = \overline{\neg S}$.*

Theorem 6.3. *In any topos $\mathcal{E}$, the operator $\neg\neg : \Omega \to \Omega$ of double negation is a Lawvere-Tierney topology, and the resulting category of $\neg\neg$-sheaves is a Boolean topos.*

Lemma 6.4. *For any subobjects $A \rightarrowtail E$ in $\hat{\mathsf{C}}$ and any objects C,*

$$\neg\neg A(C) = \{x \in E(C) : \text{ for all } f : B \to C, \text{ there exists } g : D \to B \text{ with } x \cdot f \cdot g \in A(D)\}.$$

Corollary 6.4.1. *For any presheaf topos $\hat{\mathsf{C}}$, the dense topology coincides with the double negation topology.*

Theorem 6.5. *Let $\mathcal{E}$ be a Boolean topos, and let $\mathcal{U}$ be a maximal filter of subobjects of 1 in $\mathcal{E}$. Then the filter-quotient topos $\mathcal{E}/\mathcal{U}$ is two-valued and boolean.*

Definition 6.2 (Axiom of Choice)**.** A topos $\mathcal{E}$ is said to satisfy the **axiom of choice** if every epimorphism $r : X \twoheadrightarrow I$ in $\mathcal{E}$ has a section; that is, some $s : I \to X$ such that $rs = 1$. $\mathcal{E}$ satisfies the **internal axiom of choice** if for any object E, the functor $(-)^E : \mathcal{E} \to \mathcal{E}$ preserves epimorphisms. Notice that any topos satisfying AC also satisfies IAC, since r, s induce maps $r^E : X^E \to I^E, s^E : I^E \to X^E$, such that s^E is a section for r.

Definition 6.3 (Well-pointed)**.** A family $\mathcal{G}$ of objects of a category C **generates** C if $f \neq g : A \to B$ implies $fu \neq gu$ for some $u : G \to A$ from an object G in the family $\mathcal{G}$. C is called **well-pointed** if it is generated by the terminal object 1. For example, equality of set functions can be checked on the elements of their domain.

Definition 6.4 (Nondegenerate)**.** $\mathcal{E}$ is nondegenerate if $0 \ncong 1$. Notice that a nondegenerate topos is well-pointed iff the functor $\mathrm{Hom}(1, -)$ is faithful.

Proof. A well-pointed topos is two-valued and Boolean. ■

Theorem 6.6. *Let $\mathcal{E}$ be generated by subobjects of 1, such that for each E,* $\mathrm{Sub}(E)$ *is a complete Boolean algebra. Then $\mathcal{E}$ satisfies the axiom of choice. In particular,* $\mathrm{Sh}(P, \neg\neg)$ *satisfies the axiom of choice given any poset P.*

6.2 The Cohen Topos

Theorem 6.7. *There exists a Boolean topos satisfying the axiom of choice, in which the continuum hypothesis fails.*

Sketch. To sketch the proof, we begin with a model $\mathcal{S}$ of set theory and some set B larger than $P\mathbb{N}$, we construct a new model $\mathcal{S}'$ in which there is a monomorphism $g : B \rightarrowtail P\mathbb{N}$, which will nearly guarantee $\mathbb{N} < gB < P\mathbb{N}$ in the new model. Constructing g amounts to constructing its transpose $f : B \times \mathbb{N} \to 2$. For f to be monic, it would need to admit an n for each distinct pair $b \neq b' \in B$, such that $f(b, n) \neq f(b', n)$. f does not exist in the first model, but we can construct approximations p of f, which are partial functions defined on a finite subset $F_p \subseteq B \times \mathbb{N}$:

$$p(b_i, n_i) = 0, \qquad p(c_j, m_j) = 1, \qquad i \in [k], j \in [\ell].$$

We say (F_p, p), or p, is a **condition**, and the set of conditions forms a poset P called a **notion of forcing**, with $q \leq p$ if q extends p. Define the **Cohen topos** to be the topos $\mathrm{Sh}(P, \neg\neg)$, which will play the role of our larger model of set theory. The goal then is to prove the existence of monomorphisms $\mathbb{N} \rightarrowtail K \rightarrowtail \Omega^{\mathbb{N}}$, with no epimorphisms $\Omega^N \twoheadrightarrow K$ or $K \twoheadrightarrow \mathbb{N}$. ■

Lemma 6.8. *For any p in the Cohen poset P, the representable presheaf $よ(p) \in \hat{P}$ is a sheaf for the dense topology.*

Lemma 6.9. *Let $A \in \hat{P}$ be defined by $A(p) = \{(b, n) : p(b, n) = 0\}$. Then, A is a closed subobject of $\Delta(B \times \mathbb{N})$ with respect to the dense topology, and* $\operatorname{char} A : \Delta B \times \Delta\mathbb{N} \to \Omega$ *factors through some morphism $f : \Delta B \times \Delta\mathbb{N} \to \Omega_{\neg\neg}$.*

Lemma 6.10. *The transpose of f is a monomorphism $g : \Delta B \to \Omega_{\neg\neg}^{\Delta\mathbb{N}}$.*

Corollary 6.10.1. *Let $\hat{S}$ denote the sheafification of ΔS, for a set S. Then sheafification sends the map g from above to a monormophism $m : \hat{B} \rightarrowtail \Omega_{\neg\neg}^{\hat{\mathbb{N}}} \cong P(\hat{\mathbb{N}})$. This gives us the chain*

$$\hat{\mathbb{N}} \rightarrowtail \hat{B} \rightarrowtail P(\hat{\mathbb{N}}).$$

6.3 The Preservation of Cardinal Inequalities

Definition 6.5 (im_E)**.** Define the operation $\mathrm{im}_E : \mathrm{Hom}(E, Y^X) \to \mathrm{Hom}(E, \Omega^Y)$ via the following process. Given $f : E \to Y^X$, let $\hat{f} : E \times X \to Y$ be the transpose, and let $\mathrm{Im}_E(f) \in \mathrm{Sub}(E \times Y)$ be the image of the map $(\pi_1, \hat{f}) : E \times X \to E \times Y$. Finally, define $\mathrm{im}_E(f)$ to be the transpose of $\operatorname{char} \mathrm{Im}_E(f) : E \times Y \to \Omega$.

Lemma 6.11. im_E *is natural in E. By the Yoneda lemma, we conclude that* im_E *is induced via composition by a uniquely determined map* $\mathrm{im}\, Y^X \to \Omega^Y$

Definition 6.6 (Epimorphisms Object)**.** Let $t_Y : 1 \to \Omega^Y$ be the tranpose of $1 \times Y \to 1 \xrightarrow{\text{true}} \Omega$, and define $\mathrm{Epi}(X, Y)$ as the pullback of t_Y along im.

Lemma 6.12. *For any object E of $\mathcal{E}$, a morphism $f : E \to Y^X$ factors through the subobject* $\mathrm{Epi}(X, Y) \rightarrowtail Y^X$.

Corollary 6.12.1. *In a nondegenerate topos,* $\mathrm{Epi}(X \times Y) = 0$ *implies that there is no epimorphism $X \to Y$.*

Lemma 6.13. *Let $p : Y \to Z$ be an epimorphism in mathcalE. Then the induced map $p^X : Y^X \to Z^X$ restricts to a map* $\mathrm{Epi}(X, Y) \to \mathrm{Epi}(X, Z)$.

Lemma 6.14. *In a Boolean topos, let X be an object, $m : Z \to Y$ a monomorphism and $z_0 : 1 \to Z$ a global section, If* $\mathrm{Epi}(X, Z) \cong 0$*, then* $\mathrm{Epi}(X, Y) \cong 0$.

Definition 6.7 (Souslin Property)**.** For an object X in a topos $\mathcal{E}$, X has the **Souslin property** if any family $\mathcal{A}$ of subobjects of X which is pairwise disjoint, that is, $U \wedge V = 0$ for $U \neq V$, is at most countable. A Grothendieck topos is said to have the Souslin property if it is generated by objects having the Souslin property.

Theorem 6.15. *In a Grothendieck topos $\mathcal{E}$ satisfying the Souslin property, any two infinite sets S, T satisfy the property that if* $\mathrm{Epi}(S, T) \cong 0$ *in* Set*, then* $\mathrm{Epi}(\hat{S}, \hat{T}) \cong 0$ *in $\mathcal{E}$.*

Lemma 6.16. *The Cohen topos has the Souslin property.*

Lemma 6.17. *On the Cohen poset P, any set of incompatible conditions is countable.*

Corollary 6.17.1. *Extending Corollary 6.10.1, there do not exist epimorphisms $\Omega^N \twoheadrightarrow K$ or $K \twoheadrightarrow \mathbb{N}$.*

6.4 The Axiom of Choice

Theorem 6.18. *There exists a two-valued Boolean Grothendieck topos $\mathcal{F}$ with a natural numbers object $\hat{\mathbb{N}}$ which has a sequence of objects $F_0, F_1, \ldots$ such that*

1. *For each natural number n, $F_n \to 1$ is an epimorphism.*
2. *$\prod_m F_m$ exists and is 0.*
3. *Each F_n is a subobject of $P(\hat{\mathbb{N}})$*

Remark. The fact that each F_n is 0, but their product is not, implies that AC fails in $\mathcal{F}$. More surprisingly, these conditions are enough for $\mathcal{F}$ to violate IAC. The maps $F_m \to 1$ combine to form a map $p : \coprod_{m \in \mathbb{N}} F_m \to \coprod_{n \in \mathbb{N}} 1 \cong \hat{\mathbb{N}}$. We construct the pullback

$$\begin{tikzcd} P \arrow[r, "k"] \arrow[d] & (\coprod_{n\in\mathbb{N}} F_n)^{\hat{\mathbb{N}}} \arrow[d, "p^{\hat{\mathbb{N}}}"] \\ 1 \arrow[r, "\tilde{\mathrm{id}}"] & \hat{\mathbb{N}}^{\hat{\mathbb{N}}} \end{tikzcd}$$

By a diagram chase, P is the product $\prod_n F_n$. $F_n \to 1$ is epic by (1), so we have the epimorphism $\prod_n F_n \cong \prod_m 1 \cong \hat{\mathbb{N}}$. But $p^{\hat{\mathbb{N}}}$ cannot be epic since then its pullback $0 \cong P \to 1$ would be epic, by (2), but this is false (since $\mathcal{F}$ is Boolean and thus nondegenerate). Therefore, the functor $()^{\hat{\mathbb{N}}}$ does not preserve epimorphisms, so IAC does not hold.

Lemma 6.19. *densemeet In the presheaf category $\hat{\mathsf{C}}$, a subobject $A \rightarrowtail C$ is dense in the $\neg\neg$-topology iff $B \neq 0$ implies $B \cap A \neq 0$. We say A* ***meets*** *every object which is non-zero.*

Definition 6.8 (The Freyd Topos)**.** Let A be the category with objects of all finite sets of the form $n = \{0, \ldots, n\}$, and a morphism $f : n \to m$ is a function with $n \geq m$ and $\{1, \ldots, m\}$ fixed: a retraction from n onto m. Notice that $f = g$ if the following square commutes:

$$\begin{tikzcd} p \arrow[r, "h"] \arrow[d, "k"'] & n \arrow[d, "g"] \\ n \arrow[r, "f"'] & m \end{tikzcd}$$

simply by the fact that h, k are surjective and coincide on n. Let $H_n = \mathrm{Hom}(-, n)$ be the representable functor and let F_n be the sheafification of $\cap H_n$. We will prove that $\mathcal{F} = (\mathrm{Sh}_{\neg\neg}(\mathsf{A})$ is two-valued and the objects F_n satisfy (1)-(3).

Theorem 6.20. *$\mathcal{F}$ is two valued.*

Proof. The subfunctors of 1 are the empty functor and functors U_n which are 0 until $U_n(n) = 1$, which means $U_n(m) = 1$ for $m \geq n$. Every nonempty subobject meets every nonempty subobjects since $U_n \cap U_m = U_{n+m}$, but by Lemma **??**, all nonempty subobjects are dense in 1. By V.2.4, the only dense subsheaf of 1 is 1 itself, so we are done. ∎

Lemma 6.21. *The subobjects classifer $\Omega = \hat{2}$ is an injective presheaf.*

6.5 The Mitchel-Bénabou Language

Definition 6.9 (Term)**.** Given **types** $X, Y, \ldots$ of a topos $\mathcal{E}$, we give a recursive procedure for defining terms of a particular type. Moreover, each term $\sigma \in X$ may have an **interpretation**, which will be a morphism $\sigma : U \to X$ between types of a particular kind. There will be numerous instances of overloaded notation in order to conflate a term with its interpretation.

- Each type X has countably many variables x, each of which is a term of type X. The interpretation of x is the identity $x = 1 : X \to X$.
- Terms σ, τ of types X, Y, interpreted by $sigma : U \to X$ and $\tau : V \to Y$, yield a term $\langle \sigma, \tau \rangle$ of type $X \times Y$; its interpretation is $\langle \sigma p, \tau q \rangle : W \to X \times Y$, where $p : W \to U, q : W \to V$ are the associated projections.
- Terms $\sigma, \tau : U \to X, V \to X$ yield a term $\sigma = \tau$ of type Ω, interpreted by $\delta_X \langle \sigma, \tau \rangle : W \to \Omega$. Recall that δ_X is the equality predicate.
- A morphism $f : X \to Y$ and a term $\sigma : U \to X$ of type X yield a term $f \circ \sigma$ of type Y, interpreted by $f \circ \sigma : U \to Y$.
- Terms $\theta : V \to Y^X$ and $\sigma : U \to X$ of types Y^X and X yield a term $\theta(\sigma)$ of type Y interpreted by $\theta(\sigma) : W \to Y^X \times X \xrightarrow{e} Y$.
- Similarly, terms $\sigma : U \to X$ and $\tau : V \to \Omega^X$ yield a term $\sigma \in \tau$ of types Ω, which is a special case of the term above.
- A variable x of term X and a term $\sigma : X \times U \to Z$ yield $\lambda x \sigma$, a term of type Z^X, interpreted by the transpose of σ, $\lambda x \sigma : U \to Z^X$.
- Moreover, terms of type Ω will be called **formulas** of the language; if we have formulas ϕ, ψ, then we get naturally obtain the formulas $\phi \vee \psi$, $\phi \wedge \psi, \phi \implies \psi$, and $\neg\phi$, whose interpretations are pushforwards of the interpretation of $\phi \times \psi$ (or ϕ, in the last case) by the respective operator on Ω.
- Given the unique map $p : X \to 1$, we get an induced map $P(p) : P1 \to PX$, as well as its internal adjoints $\forall_p, \exists_p : \Omega^X \to \Omega$. Then, taking the pushforward of the term $\lambda x \phi(x, y)$ gives us the terms $\forall x \phi(x, y)$ and $\exists x \phi(x, y)$.
- The interpretation of a formula $\phi(x) : X \to \Omega$ naturally characterizes a subobject of X, which we will denote as $\{x : \phi(x)\}$.

Definition 6.10 (Truth)**.** A formula $\phi(x)$ of the language of a topos is **universally valid** if its interpretation factors through true. If ϕ has no free variables, we say that ϕ is **true** in $\mathcal{E}$. It is asserted that $\phi(x)$ is universally valid iff $\forall x \phi(x)$ is true.

Example 6.1. All the objects we painstakingly derived using universal properties are easily defined using the language of a topos. For example, we have the correspondence

$$\mathrm{Epi}(X, Y) = \{f \in X^Y : \forall y \in Y \exists x \in X f(x) = y\}.$$

Also, a topos is Boolean iff $\forall p(p \vee \neg p)$ is true, and the IAC holds iff the following formula holds:

$$\forall f \in Y^X (\forall y \exists x f(x) = y \implies \exists g \in X^y \forall y f(g(y)) = y).$$

6.6 Kripke-Joyal Semantics

Definition 6.11 (Forcing)**.** Given $\alpha : U \to X$, we say U **forces** $\phi(a)$, written $U \Vdash \phi(a)$, if $\operatorname{Im}\alpha \leq \{x | \phi(x)\}$, or equivalently if α factors through $\{x|\phi(x)\}$.

Theorem 6.22 (Properties of Forcing)**.** *The following properties follow from the definition of the forcing relation:*

- ***Monotonicity.*** *If $U \Vdash \phi(\alpha)$, then, for any arrow $f : U' \to U$ in $\mathcal{E}$, $U' \Vdash \phi(\alpha \cdot f)$.*
- ***Local character.*** *If $f : U' \to U$ is epic and $U' \Vdash \phi(\alpha \circ f)$, then $U \Vdash \phi(a)$.*

Theorem 6.23. *If $\alpha : U \to X$ is a generalized element of X, and $\phi(x), \psi(x)$ are formulas with a free variable x of type X, then*

1. *$U \Vdash \phi(\alpha) \wedge \psi(\alpha)$ iff $U \Vdash \phi(\alpha)$ and $U \Vdash \psi(\alpha)$.*
2. *$U \Vdash \phi(\alpha) \vee \psi(\alpha)$ iff there are arrows $p : V \to U$ and $q : W \to U$ such that $p + q : V + W \twoheadrightarrow U$ is epic, such that $V \Vdash \phi(\alpha p)$ and $W \Vdash \psi(\alpha q)$.*
3. *$U \Vdash \phi(\alpha) \implies \psi(\alpha)$ iff for any morphism $p : V \to U$ such that $V \Vdash \phi(\alpha p)$, we also have $V \Vdash \psi(\alpha p)$.*
4. *$U \Vdash \neg\phi(\alpha)$ iff whenever $p : V \to U$ satisfies $V \Vdash \phi(\alpha p)$, then $V \cong 0$.*

Furthermore, let $\phi(x, y)$ be a formula with free variables in two types. Then,

1. *$U \Vdash \exists y \phi(\alpha, y)$ iff there exist an epimorphism $p : V \twoheadrightarrow U$ and a generalized element $\beta : V \to Y$ such that $V \Vdash \phi(\alpha p, \beta)$.*
2. *$U \Vdash \forall \phi(\alpha, y)$ iff for every object V, morphism $p : V \to U$, and generalized element $\beta : V \to Y$, we have $V \Vdash \phi(\alpha p, \beta)$.*
3. *$U \Vdash \forall y \phi(\alpha, y)$ iff $U \times Y \Vdash \phi(\alpha\pi_1, \pi_2)$.*

Extrapolating from the properties of $\forall$, we note that $\phi(x, y)$ is universally valid iff $1 \Vdash \forall x \forall y \phi(x, y)$.

Theorem 6.24. *If $\sigma(x)$ and $\tau(x)$ are terms of type Y in the free variable x of type X, while $\alpha : U \to X$ is a generalized element of type $X \in \mathcal{E}$, and σ', τ' are their interpretations, then*

$$U \Vdash \phi(\alpha) = \tau(\alpha) \quad \textit{iff} \quad \sigma'\alpha = \tau\alpha : U \to Y.$$

Theorem 6.25. *If $\sigma(x)$ and $\tau(x)$ are terms of types Y and Ω^Y respectively in a free variable x of type X and σ', τ' are their interpretations, then for any generalized element $\alpha : U \to X$,*

$$U \vdash \sigma(\alpha) \in \tau(\alpha) \quad \textit{iff} \quad \langle \sigma'(\alpha), \tau'(\alpha)\rangle : U \to Y \times \Omega^Y \textit{ factors through } M_Y \rightarrowtail Y \times \Omega^Y.$$

6.7 Sheaf Semantics

7 Completed Exercises

Exercise V.1. We iteratively construct the diagram below:

$$\begin{tikzcd}
A \arrow[rr, "!_A"] \arrow[dd, tail, "\iota_A"'] \arrow[ddr, phantom, "\lrcorner", very near start] & & 1 \arrow[dd, tail, "\mathrm{true}"] \arrow[dr, tail, "\mathrm{true}"] & \\
& J \arrow[ur, "!_J"] \arrow[dr, tail, "\iota_J"'] \arrow[r, phantom, ">"] & & \Omega \\
E \arrow[ur, "f"] \arrow[rr, dashed, "\mathrm{char}\, A"'] & & \Omega \arrow[ur, "j"'] &
\end{tikzcd}$$

First begin with $f : E \to J$, which we compose with the inclusion $\iota_J : J \rightarrowtail \Omega$, which exists since $J \rightarrowtail \Omega$ is a pullback of true along j. Hence, their composition $\iota_J \circ f$ is the characteristic function of a subobject $\iota_A : A \rightarrowtail E$, such that the square on the left is a pullback. Then, form the unique morphism $!_J : J \to 1$, and paste the skewed pullback square on the right along $!_J, \iota_J$. Finally, define the map $\varphi : \mathrm{Hom}(E, J) \to \mathrm{Den}(E)$ by $f \mapsto A$.

First, $\varphi(f)$ is dense in E, since

$$\begin{aligned}
\mathrm{char}\, \overline{\varphi(f)} &= j \circ \mathrm{char}\, \varphi(f) \\
&= j \circ \iota_J \circ f \\
&= \mathrm{true} \circ !_J \circ f \\
&= \mathrm{true}_E, i && \text{(By the pullback on the right)}
\end{aligned}$$

which uniquely characterizes $\overline{\varphi(f)} = E$.

φ is an isomorphism since if we began with a subobject $A \rightarrowtail E$ such that $\overline{A} = E$, then $\mathrm{true} \circ !_E = \mathrm{char}\, \overline{A} = j\, \mathrm{char}\, A$. By the right pullback's universal property, we get a *unique* morphism $f : E \to J$ with respect to the property that $\iota_J f = \mathrm{char}\, A$, and hence that $\varphi(f) = A$. Finally, φ is natural since it has a decomposition into morphisms natural in E

$$\mathrm{Hom}(E, J) \xrightarrow{(\iota_J)_*} \mathrm{Hom}(E, \Omega) \xrightarrow{\cong} \mathrm{Den}(E)$$

We first check that $\mathrm{Den}(E)$ is a lattice, which amounts to checking that $\mathrm{Den}(E)$ is closed under $\wedge$ and $\vee$. First, we use the purely formal theorem that closure is monotonic: if $A \leq B$, then

$$\overline{A} \cap B \leq \overline{A} \cap \overline{B} = \overline{A \cap B} = \overline{A}, \quad \text{thus } \overline{A} \leq B \leq \overline{B}.$$

Now, if $A, B \in \mathrm{Den}(E)$, then

$$\begin{aligned}
\overline{A \wedge E} &= \overline{A} \wedge \overline{B} = E \wedge E = E, \\
E \geq \overline{A \vee B} &\geq \overline{A} = E, \\
\overline{A \Rightarrow B} &= \overline{\neg A \vee B} = E.
\end{aligned}$$

(By the identity above, which only requires one argument to be dense)

Note that we can always define $\neg A = A \Rightarrow 0$ in a Heyting algebra. ∎

8 Thoughts

- Topos theory is highly related to the study of modal operators on Heyting algebras.
  - All LTTs one can put on a topos $\mathcal{E}$ already exist intrinsically.
  - One must specify an LTT to define sheaves. The pairs (presheaf topos, LTT) correspond precisely to the Grothendieck topologies on C (V.4).