VECTOR BUNDLES, SECTIONS, AND THE SERRE-SWAN THEOREM

ROBBERT LIU

Abstract. In 1962, Swan provided a topological analogue to a theorem in algebraic geometry proven by Serre. Swan's theorem intimately relates two adjacent but seemingly distinct constructions, one geometric and the other algebraic in essence. A K-vector bundle is a continuously varying family of finite dimensional K-vector spaces parametrized by some base space B, which can be locally flattened to the product of an open set with a vector space. A section of a vector bundle is a continuously varying choice of vector in each fiber over B, and the set of sections of a vector bundle has a module structure over the ring $C(B)$ of continuous functions from B to K. The Serre-Swan theorem asserts that the functor taking a vector bundle to its module of sections is an equivalence from the category of vector bundles over a fixed compact Hausdorff space B to the category of finitely generated projective $C(B)$-modules. In this paper, we discuss various constructions related to vector bundles and sections, explore the structure of the categories Vect_B and $\mathsf{Mod}_{C(B)}$, and prove the Serre-Swan theorem. We discuss the implications of the theorem and a direct application.

1. Vector Bundles

Let K denotes either the real or complex numbers. Note that [Swa62] generalizes the presented proofs to the case where K is the quaternions, with minute reconsiderations. A topological K-vector bundle, or simply a vector bundle, is a continuous surjective map $p : E \to B$ between topological spaces E, called the total space, and B, the base space, such that the following two properties hold for each $x \in B$:

(1) The fiber $p^{-1}(x)$ has the structure of a finite dimensional vector space.
(2) There exists an open neighbourhood $U \ni x$ and a homeomorphism $h : p^{-1}(U) \to U \times K^{n(x)}$, called a local trivialization of $p : E \to B$ around x, which is a linear isomorphism on each fiber $p^{-1}(y)$ over $y \in U$.

Vector bundles formalize the notion of a continuous family of vector spaces parametrized by some other space. If $n(x) = n$ is constant over B, then we call it the dimension of the vector bundle; otherwise, we may still refer to the local dimension $n(x)$ around x.

Remark 1.1. The union E^k of all fibers of dimension k is an open subspace of E, since for each $y \in E_k$, a local trivialization $h : p^{-1}(U) \to U \times K^k$, exists around $p(y)$, so $y \in p^{-1}(U) \subseteq E_k$. Indubitably, if E is connected, then $p : E \to B$ has a (constant) dimension.

A map between vector bundles $p_1 : E_1 \to B$, $p_2 : E_2 \to B$ is a continuous map $f : E_1 \to E_2$ such that the following commutes,

Date: June 1, 2025.

$$\begin{tikzcd} E_1 \arrow[rr, "h"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

and moreover we require that f is linear on each fiber. Commutativity essentially says that f takes the fiber of a basepoint to the fiber of the same point. Moreover, an isomorphism of vector bundles is simply an invertible bundle map. Several examples of vector bundles are provided by [Hat17].

Example 1.2 (The trivial bundle). The simplest example of a vector bundle is one which admits a local trivialization whose domain covers the total space. This is known as the trivial bundle $p : B \times K^n \to B$, where p simply projects $(x, y) \mapsto x$.

Example 1.3 (The normal and tangent bundle of a sphere). Viewing an n-dimensional S^n by its standard embedding in $\mathbb{R}^{n+1}$, we can consider the tangent bundle $p : \tau^n \to S^n$, whose fibers $p^{-1}(x)$ are the tangent spaces $\{(x, v) \in S^n \times \mathbb{R}^{n+1} : x \perp v\}$ of S^n. The map p is again defined by projection onto the basepoint, and a local trivialization of τ^n around $x \in S^n$ is defined on the open hemisphere H^n_x centered at x by projecting the tangent spaces of $y \in H^n_x$ to the plane orthogonal to x. Alternatively, we can consider the normal bundle $p : \nu^1 \to S^n$ containing fibers $p^{-1}(x)$ of the form $\{(x, tx) \in S^n \times \mathbb{R}^{n+1}\}$. It turns out that ν is trivial, that is, isomorphic to the trivial bundle, since we can define the homeomorphism $(x, tx) \mapsto (x, t)$, which is linear in each fiber.

Example 1.4 (The annulus and the Möbius strip). We can define two 1-dimensional real vector bundle structures on the circle—the trivial bundle $p : S^1 \times \mathbb{R} \to S^1$, otherwise known as the annulus, and the Möbius strip $p : \mu \to S^1$, where μ is the quotient of $[0, 1] \times \mathbb{R}$ obtained by identifying $(1, x) \sim (0, -x)$. These are not isomorphic as vector bundles since there is no homeomorphism $S^1 \times \mathbb{R} \to \mu$. For instance, the former is orientable whereas the latter is not. These turn out to be the *only* 1-dimensional vector bundles over S^1 [Hat17].

Example 1.5 (Canonical line bundle over $\mathbb{R}P^n$). Recall that the real projective space $\mathbb{R}P^n$ is the space of lines intersecting the origin in $\mathbb{R}^{n+1}$. We define the canonical line bundle $p : E \to \mathbb{R}P^n$ as a 1-dimensional vector bundle whose fiber at $\ell \in \mathbb{R}P^n$ is ℓ itself: points in the fiber at ℓ are of the form $(\ell, v) \in \mathbb{R}P^K \times \ell$. p again acts as projection onto ℓ, and a local trivialization is defined via orthogonal projection.

A section of a vector bundle $p : E \to B$ over $U \subseteq B$ is a continuous choice map $s : U \to E$, associating some $y \in p^{-1}(x)$ to each $x \in U$. We refer to a section over B simply as a section.

Example 1.6. A section s of the tangent bundle τ^n is a continuous choice of a tangent vector at each basepoint in S^n. In more familiar terms, s is a continuous vector field on S^n. Assuming for the moment that τ^n is trivial, we can pull back the constant non-zero section (x, v) of $B \times K^n$ via an isomorphism h to a section $s : \tau^n \to S^n$. But since h acts as a linear bijection on each fiber, s is a nonzero, continuous vector field on S^n. Therefore, n must be odd [Hat17].

We introduce two elementary results on sections of vector bundles. Most of the presented proofs in this essay follow [Swa62] with minor differences.

Lemma 1.7 (Existence of a local basis)**.** *Given a bundle $p : E \to B$, each $x \in B$ admits sections $s_1, \ldots, s_n : U \to E$ over some neighbourhood $U \ni x$ which form a basis of $p^{-1}(y)$ for each $y \in U$. Moreover, any section $s : U \to E$ has a unique representation $s(y) = \sum_n a_i(y)s_i(y)$ for continuous a_i.*

Proof. For existence, let $h : p^{-1}(U) \to U \times K^n$ be a local trivialization around x, and let $s'_1, \ldots, s'_n$ be any basis of K^n. Then, we obtain the continuous map $s_i = h^{-1}\iota_i : U \to E$, where $\iota_i : U \to U \times K^n$ is defined by $x \mapsto (x, s'_i)$. Since h^{-1} is a linear isomorphism on fibers, $s_1(y), \ldots, s_n(y)$ are a local basis around x.

Onto the second statement: given such a section s and for each $y \in U$, we may expand $s(y) = \sum_n a_i(y)s_i(y)$ for unique coefficients $a_i(y)$ since $\{s_i\}_i$ is a local basis, which defines unique maps $a_i : U \to E$. If $\{s_i\}_i$ is the standard basis $\{e_i\}$, then continuity follows immediately: since h is linear on each fiber, a_i decomposes as

$$y \mapsto s(y) = \sum_n a_i(y)e_i(y) \overset{h}{\mapsto} (y, \sum_n a_i(y)e_i(y)) \overset{\pi_{e_i}}{\longmapsto} a_i(y),$$

where π_{e_i} is the canonical projection map. In the general case, we can write $s_i(y) = \sum_n a_{ij}(y)e_j(y)$ for continuous a_{ij}, so $[a_{ij}(y)]_{ij}$ is a continuous map $B \to GL(n, K)$, since $[a_{ij}(y)]_{ij}$ is an change of basis. Hence, $[a_{ij}(y)]_{ij}^{-1}$ is also continuous, so by expressing $s(y) = \sum_n \alpha_i(y)e_i(y)$, it follows that each component of $(a_1(x), \ldots, a_n(x)) = [a_{ij}(y)]^{-1}(\alpha_1(x), \ldots, \alpha_n(x))$ is continuous. □

Lemma 1.8. *If the sections $s_1, \ldots, s_k : U \to E$ are linearly independent at $x \in U$, then they remain linearly independent within a neighbourhood of x.*

Proof. By Lemma 1.7, let $t_1, \ldots, t_n : W \to E$ be a local basis around x and write $s_i(y) = \sum_{j=1}^n a_{ij}(y)t_j(y)$ on $U \cap W$. $[a_{ij}]_{ij}$ is a continuous map $V \to M_{k\times n}(K)$, which has a nonsingular $k \times k$ submatrix at x. By continuity of the determinant, the submatrix remains nonsingular within a neighbourhood of x, which finishes the proof. □

If $p : E \to B$ is a vector bundle and $E' \subseteq E$, then $p : E' \to B$ is a subbundle of $p : E \to B$ if the former is itself a vector bundle. Define the kernel and image of a bundle map $f : E_1 \to E_2$ by $\ker f := \{x \in E_1 : f(x) = 0_{p_1(x)}\}$ and $\operatorname{im} f := f(E_1)$ respectively. In well-behaved cases, these two spaces induce subbundles of $p_1 : E_1 \to B, p_2 : E_2 \to B$ respectively in the obvious way, since $\ker f|_{p_1^{-1}(x)}, \operatorname{im} f|_{p_1^{-1}(x)}$ are subspaces of $p_1^{-1}(x), p_2^{-1}(x)$ respectively. Woefully, this is not the general case, and one such counterexample is provided in [Swa62]: let $p : [0,1] \times \mathbb{R} \to [0,1]$ be the trivial bundle, and let f be the endomorphism $(x, y) \mapsto (x, xy)$. f fixes fibers and is continuous and linear in y. However, $\operatorname{im} f$ does not have a local trivialization around 0 since $p|_{\operatorname{im} f}^{-1}(0) = \{0\}$ and $p|_{\operatorname{im} f}^{-1}(x) = \mathbb{R}$ for $x > 0$. A similar problem arises for $\ker f$. However, if we require that fibers have locally constant dimension, our situation is ameliorated.

Proposition 1.9. *Let $f : E_1 \to E_2$ be a bundle map between vector bundles $p_1 : E_1 \to B, p_2 : E_2 \to B$. Then, the following are equivalent:*

(1) $\operatorname{im} f$ *is a subbundle of* $p_2 : E_2 \to B$.
(2) $\ker f$ *is a subbundle of* $p_1 : E_1 \to B$.
(3) The fibers of $\operatorname{im} f$ *have locally constant dimensions.*
(4) The fibers of $\ker f$ *have locally constant dimensions.*

Proof. Clearly, (1) $\implies$ (3) $\iff$ (4) $\impliedby$ (2), so it suffices to show that (3) implies (1) and (2). As hinted at earlier, $\ker f, \operatorname{im} f$ satisfy almost all of the vector bundle axioms except for the existence of local trivializations, so we will show that (3) guarantees this property. By Lemma 1.7, $s_1, \ldots, s_n$ be a local basis for fibers $p_1^{-1}(y)$ around $x \in p_2(\operatorname{im} f)$. Since $fs_1, \ldots, fs_n$ span $\operatorname{im} f$, we can extract a reordered basis $fs_1, \ldots, fs_k$ for $\operatorname{im} f$ at x, which is locally linearly independent by Lemma 1.8.

From here, assuming (3) implies that $fs_1, \ldots, fs_k$ are a local basis around U, hence the mapping $s_i(y) \to (y, t_i)$ for a basis $t_1, \ldots, t_k$ of K^n extends to a local trivialization around x. This grants us (1). If we instead assume (4), we can write, for $i > k$, $fs_i(x) = \sum_{i=1}^{k} a_{ij}(x) fs_j(x)$, where the maps a_{ij}'s are continuous by Lemma 1.8, using the fact that $fs_1, \ldots fs_k$ are a local basis. Then, define $u_i(y) = s_i - \sum_{i=1}^{k} a_{ij}(y) s_j(y)$ for $i > k$, which are locally linearly independent since $s_1, \ldots, s_n$ were. By linearity on fibers, we compute $fu_i(x) = fs_i - \sum_{i=1}^{k} a_{ij}(x) fs_j(x) = 0$, which means $u_k(x), \ldots, u_n(x) \in \ker f|_{p_1^{-1}(x)}$. Furthermore, rank-nullity forces $u_k, \ldots, u_n$ to be a basis at x, and (4) implies that this remains true around x. $\square$

While subbundles allow us to construct a smaller vector bundle from a larger vector bundle, there exists an operation, called the direct sum, which allows us to form a larger vector bundle from smaller ones. The direct sum $p : E_1 \oplus E_2 \to B$ of vector bundles $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ is another vector bundle with fibers $p^{-1}(x) = p_1^{-1}(x) \oplus p_2^{-1}(x)$ and local trivializations $\phi_1 \times \phi_2 : p^{-1}(U) \to U \times K^{m+n}$, where $\phi_1 : p_1^{-1}(U) \to U \times K^m, \phi_2 : p_1^{-1}(U) \to U \times K^n$ are local trivializations of E_1, E_2 respectively around x. $E_1 \oplus E_2$ is constructed by first taking the product $E_1 \times E_2$, which induces the product of the projections maps $p_1 \times p_2$ as well as products of local trivializations, and defining $E_1 \oplus E_2 = \{(x, y) \in E_1 \times E_2 : p_1(x) = p_2(y)$; that is, $E_1 \oplus E_2$ is the diagonal of $E_1 \times E_2$ with respect to the basepoint.

An inner product on a vector bundle $p : E \to B$ is a continuous functional $\langle \cdot, \cdot \rangle : E \oplus E \to K$ which restricts to an inner product on each fiber. To guarantee existence of an inner product, it suffices for B to be paracompact. A space X is said to be paracompact if it is Hausdorff and each open cover $\{U_\alpha\}$ of X admits a partition of unity $\{\eta_\beta : X \to [0, 1]\}$ subordinate to $\{U_\alpha\}$—that is to say, $\{\eta_\beta\}$ is a family of continuous maps whose supports are each contained by some U_α, $\sum_\beta \eta_\beta$ is constantly 1, and every point in X has an open neighbourhood that intersects the supports of only finitely many η_β.

Lemma 1.10. *If B is paracompact, then the vector bundle $p : E \to B$ has an inner product.*

Proof. Take a cover $\{U_\alpha\}$ of B such that each U_α admits a local trivialization $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times K^n$, and let $\{\eta_\beta\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Each $U_\alpha \times K^n$ admits an inner product $\langle \cdot, \cdot \rangle_\alpha$ which is constant with respect to with respect to U_α. Finally, define $\langle x, y \rangle = \sum_\beta \eta_\beta p(x) \langle h_\alpha(x), h_\alpha(y) \rangle_{\alpha(\beta)}$, where $\alpha(\beta)$ is such that η_β has support contained inside $U_{\alpha(\beta)}$. Immediately, $\langle \cdot, \cdot \rangle$ is a continuous map which restricts to an inner product on each fiber. $\square$

The following proposition lends weight to the notion that vector bundles decompose and coalesce nicely, given the right condition on B. With this result, we gain a satisfying relation between the two directions, upward and downward, in which we can construct new vector bundles from old ones.

Proposition 1.11. *If B is paracompact, any subbundle $p : F \to B$ of vector bundle $p : E \to B$ is a direct summand.*

Proof. Lemma 1.10 grants us an inner product on E, which defines a projection $f_x : p^{-1}(x) \to p^{-1}|_F(x)$ for each $x \in B$. Then $E = F \oplus \ker f$, where $\ker f$ is a subbundle by Proposition 1.9. □

2. Products and Coproducts

Before we address the algebraic structure of sections on a vector bundle, we must first take a detour into category theory. Fixing a category $\mathcal{C}$ and a family $\{X_\alpha\}$ of objects, the product $X_\prod := \prod_{\alpha \in A} X_\alpha$, if it exists, is an object which admits morphisms $\pi_\alpha : X_\prod \to X_\alpha$ satisfying the following universal property: given a family of morphisms $f_\alpha : Y \to X_\alpha$ for a fixed Y, there exists a unique morphism $\varphi : Y \to X_\prod$ such that each f_α factors through $X_\prod$ via $\varphi\pi_\alpha$. Equivalently, the following diagram commutes for each α:

$$\begin{tikzcd} & X_\alpha \\ Y \arrow[ur, "f_\alpha"] \arrow[r, dotted, "\exists!\varphi"'] & X_\prod \arrow[u, "\pi_\alpha"'] \end{tikzcd}$$

As a reminder, $\exists!$ denotes unique existence. We can also speak of the coproduct $X_\coprod := \coprod_\alpha X_\alpha$, whose construction involves reversing all the morphisms above. Hence, $X_\coprod$, if it exists, admits morphisms $\iota_\alpha : X_\alpha \to X_\coprod$ such that any object Y and family of morphisms $g_\alpha : X_\alpha \to Y$ gives rise to a unique morphism $\psi : X_\coprod \to Y$ makes the following diagram commute for each α:

$$\begin{tikzcd} X_\alpha \arrow[d, "\iota_\alpha"'] \arrow[dr, "g_\alpha"] & \\ X_\coprod \arrow[r, dotted, "\exists!\psi"'] & Y \end{tikzcd}$$

If it exists, the product (resp. coproduct) of a family is unique up to isomorphism, which is easily proven using the uniqueness of the maps φ, ψ. Thus, it is moral to speak of *the* product (resp. coproduct) of some family.

Example 2.1 (Cartesian product and disjoint union)*.* In the category Set of sets and set maps, products and coproducts coincide with cartesian products and disjoint unions respectively, which exist for arbitrary families of maps. Indeed, φ is uniquely realized as the map $(\ldots, f_\alpha, \ldots)$ which evaluates at $y \in Y$ to $f_\alpha(y)$ in the α^{th} component, and π_α is the canonical projection. Similarly, ι_α is the canonical inclusion $x \mapsto (\alpha, x)$ into $\bigsqcup_\alpha X_\alpha$, and ψ is the evaluation map $(\alpha, x) \mapsto g_\alpha(x)$.

Example 2.2 (Meet and join)*.* A partially ordered set $(U, \leqslant)$ forms a category with objects $x \in X$ and a unique morphism $x \to y$ whenever $x \leqslant y$. Given a family $V \subseteq U$, the product $V_\prod := \prod_{v \in V} v$, if it exists, is the infimum of V. Indeed, there exist comparisons $V_\prod \leqslant v$ for all $v \in V$, and if we have some element $a \in U$ such that $a \leqslant v$ for all $v \in V$, then these comparisons necessarily factor through $V_\prod$. Dually, the coproduct $V_\coprod$ is the supremum of V, which the reader should verify on their own.

Example 2.3 (Direct sum of abelian groups)*.* In the category Ab of abelian groups and group homomorphisms, finite products and coproducts coincide with the direct sum

$\bigoplus_\alpha X_\alpha$, where $A \ni \alpha$ is finite. ι_α, π_α are the canonical inclusions and projections, respectively, and given Y and families $\{f_\alpha : Y \to X_\alpha\}, \{g_\alpha : X_\alpha \to Y\}$, φ is defined by $y \mapsto \sum_\alpha f_\alpha(y) \in \bigoplus_\alpha X_\alpha$; and given $x_\alpha \in \iota(X_\alpha)$, ψ is defined by $\sum_\alpha x_\alpha \mapsto \sum_\alpha g_\alpha(x_\alpha)$. Uniqueness should again be checked by the reader. Ab is the prime example of an additive category, one that admits all finite biproducts—objects which are simultaneously the product and coproduct of some family $\{X_1, \dots, X_n\}$. This will not be our last encounter with additive categories.

Example 2.4 (Initial and final objects). It is natural to ask, if the question makes any sense, what an empty product or coproduct looks like. If we trace our established definition, we find that the empty product F, if it exists, provides precisely the data of a unique morphism $Y \to F$ for any object Y. Similarly, the empty coproduct I admits a unique morphism $I \to Y$ for any object Y. The objects F, I are suggestively called the final and initial objects, respectively, of a category. Before returning to the subject of vector bundles, we mention that in an additive category, the final object coincides with the initial object, which are collectively called the zero object.

3. Modules of Sections

Let $C(B)$ be the ring of continuous K-valued functions on B, and let $\Gamma_p(E, B)$, or $\Gamma(E)$ when unambiguous, denote the set of sections of a vector bundle $p : E \to B$. It is easy to put a $C(B)$-module structure on $\Gamma(E)$—with pointwise addition $(s_1 + s_2)(x) = s_1(x) + s_2(x)$ and scaling $(as)(x) = a(x)s(x)$.

Let $\textsc{Vect}_B$ denote the category of K-vector bundles over B and bundle maps, and let $\textsc{Mod}_{C(B)}$ denote the category of $C(B)$-modules and $C(B)$-module homomorphisms. Then $\Gamma : \textsc{Vect}_B \to \textsc{Mod}_{C(B)}$ is a functor which takes the bundle map $f : E_1 \to E_2$ to the map $\Gamma f : \Gamma(E_1) \to \Gamma(E_2)$ defined by $s \mapsto fs$, which is a module homomorphism since f is linear in each fiber. Specifically, $\textsc{Vect}_B, \textsc{Mod}_{C(B)}$ are examples of additive categories, with the direct sum of vector bundles (resp. modules) acting as the biproduct of $\textsc{Vect}_B$ (resp. $\textsc{Mod}_{C(B)}$). .

Proposition 3.1. *Let X, Y, and E_α for $\alpha \in A$ be vector bundles over B, and let $\pi_\alpha : \bigoplus_\alpha E_\alpha \to E_\alpha, \iota_\alpha : E_\alpha \to \bigoplus_\alpha E_\alpha$ be the canonical projections and inclusions respectively. Then for bundle map families $\{f_\alpha : X \to E_\alpha\}_\alpha, \{g_\alpha : E_\alpha \to Y\}_\alpha$, there exist unique homomorphisms $\varphi : X \to \bigoplus_\alpha E_\alpha, \psi : \bigoplus_\alpha E_\alpha \to Y$ that make the following diagram commute:*

$$\begin{tikzcd}
& E \arrow[d, shift right, "\iota_\alpha"'] \arrow[dr, "g_\alpha"] \arrow[dd, bend right] & \\
X \arrow[ur, "f_\alpha"] \arrow[r, dotted, "\exists!\varphi"] \arrow[dr] & \bigoplus_\alpha E_\alpha \arrow[u, shift right, "\pi_\alpha"'] \arrow[r, dotted, "\exists!\psi"] \arrow[d] & Y \arrow[dl] \\
& B &
\end{tikzcd}$$

Proof. Commutativity of the left side forces $\varphi(x) = \iota_\alpha f_\alpha(x)$ for $x \in X$, which completely defines $\varphi(x) = \sum_\alpha f_\alpha(x) \in \bigoplus_\alpha E_\alpha$ since φ is a homomorphism. Similarly, commutativity of the right side forces $\psi(\iota_\alpha(e_\alpha)) = g_\alpha(e_\alpha)$, for $e_\alpha \in E_\alpha$, so we must have $\psi(\sum_\alpha e_\alpha) = \sum_\alpha g_\alpha(e_\alpha)$ since ψ is a homomorphism. $\square$

Remark 3.2. Γ turns out to be an additive functor, in that it translates the additive structure of Vect_B into that of $\mathsf{Mod}_{C(B)}$. Indeed, for a finite family $\{E_\alpha\}$ of vector bundles over B, $\Gamma(\bigoplus_\alpha E_\alpha) \cong \bigoplus_\alpha \Gamma(E_\alpha)$ since each section $s \in \Gamma(\bigoplus_\alpha E_\alpha)$ can be decomposed as $\prod_\alpha s_\alpha$, where s_α is a section of E_α.

Example 3.3. It is clear that $\Gamma(B \times K) = C(B)$. By Remark 3.2, we can compute

$$\Gamma(B \times K^n) = \Gamma\left(\bigoplus_{i=1}^{n} B \times K\right) \cong \bigoplus_{i=1}^{n} \Gamma(B \times K) = C(B)^n.$$

For the remainder of this section, when we speak of a vector bundle $p : E \to B$, we shall assume that B is normal. The goal is to then show that Γ is an isomorphism $\mathrm{mor}(E_1, E_2) \cong \mathrm{mor}(\Gamma(E_1), \Gamma(E_2))$.

Lemma 3.4. *Let $p : E \to B$ be a vector bundle. Any section over an open neighbourhood $U \subseteq E$ around x has a restriction to some open $V \subseteq U$ containing x, which can be extended to B.*

Proof. Let $V, W \ni x$ be open such that $\overline{W} \subseteq V \subseteq \overline{V} \subseteq U$, and by Urysohn's lemma (a proof of which can be found in [Mun14]), let $\omega : B \to [0,1]$ be 0 on $B - V$ and 1 on $\overline{W}$. Then $s|_W$ can be extended to a section $s' : B \to E$ by $s' = \omega s$ on U and $s' = 0$ on U^c. Continuity follows immediately. □

Corollary 3.5. *If $p : E \to B$ is a vector bundle, any $x \in B$ admits sections $s_1, \dots, s_n \in \Gamma(E)$ which are a local basis.*

Proof. Follows from Lemma 1.7 and Lemma 3.4. □

Corollary 3.6 (Γ is injective on morphisms sets)**.** *If $f, g : E_1 \to E_2$ are maps between vector bundles $p_1 : E_1 \to B, p_2 : E_2 \to B$ and $\Gamma(f) = \Gamma(g)$, then $f = g$.*

Proof. Given $y \in E$, we can construct a section s over some open $U \ni x := p(y)$ such that $s(x) = y$, via a local basis, and the restriction of s to some open $V \ni x$ extends to a section $s' \in \Gamma(E)$. Hence,

$$f(y) = \Gamma(f)s'(x) = \Gamma(g)s'(x) = g(y).$$

□

Lemma 3.7. *Let $p : E \to B$ be a vector bundle and let $\Gamma(E)_x$ be the submodule of sections that vanish at $x \in B$. Then $\Gamma(E)/\Gamma(E)_x \cong p^{-1}(x)$ via the isomorphism $e_x : \overline{s} \mapsto s(x)$.*

Proof. It is clear why the evaluation map e_x is a well-defined homomorphism. The map is surjective by the proof of Corollary 3.6, so it suffices to show injectivity. If $s_1(x) = s_2(x)$ for $s_1, s_2 \in \Gamma(E)$, then $t := s_1 - s_2 \in \Gamma(E)_x$, so $\overline{s_1} = \overline{s_2}$. □

Corollary 3.8 (Γ is surjective on morphism sets)**.** *Let $p_1 : E_1 \to B, p_2 : E_2 \to B$ be vector bundles. Given a homomorphism $F : \Gamma(E_1) \to \Gamma(E_2)$, there is a bundle map $f : E_1 \to E_2$ such that $F = \Gamma(f)$.*

Proof. For each $x \in B$, since $F(\Gamma(E_1)_x) \subseteq \Gamma(E_2)_x$, we have an induced linear map $f_x : \Gamma(E_1)/\Gamma(E_1)_x \cong p_1^{-1}(x) \to \Gamma(E_2)/\Gamma(E_2)_x \cong p_2^{-1}(x)$. Then, define f to coincide with f_x on each fiber $p^{-1}(x)$, which implies $fs = f_x s = F(s)$ for all $x \in B$.

We now show continuity of f. Let $s_1, \dots, s_n \in \Gamma(E_1)$ be a local basis around $x \in B$. For y near x, we can represent $y = \sum_{i=1}^n a_i(y) s_i(p(y))$ with a_i continuous, such that $f(y) = \sum_{i=1} a_i(y) f s_i(p(y))$. Therefore, f is locally a composition of continuous maps at every point in E_1. □

Theorem 3.9. *If E_1, E_2 are vector bundles over a normal space B, then for each homomorphism $F : \Gamma(E_1) \to \Gamma(E_2)$, there exists a unique vector bundle map $f : E_1 \to E_2$ such that $F = \Gamma(f)$.*

4. Projective Modules

Now that we know Γ is an isomorphism on morphism sets, we would like to characterize the $C(B)$-modules mapped to by Γ. A $C(B)$-module is projective if it is a direct summand of a free module. Projective modules share a similar property to that of subbundles: by Proposition 1.11, any subbundle of a vector bundle of $p : E \to B$ is a direct summand of E. Since free modules shares a correspondence with trivial bundles, as seen in Remark 3.2, we guess that a statement of the form "any vector bundle over a nice enough base space is a direct summand of a trivial bundle" holds. The exact condition we require is for B to be compact Hausdorff. Keep in mind that a compact Hausdorff space is normal [Mun14] and paracompact [Hat17].

Lemma 4.1. *Let $p : E \to B$ be a vector bundle such that B is compact Hausdorff. Then there is a trivial bundle $p' : B \times K^n \to B$ that surjects onto E.*

Proof. By Corollary 3.5, we can construct an open cover $\{U_\alpha\}$ of B such that each set U_α admits sections $s_1^\alpha, \ldots, s_{k_\alpha}^\alpha \in \Gamma(E)$ which are a basis at each fiber over U_α. By compactness, we can extract a finite subcover $\{U_\beta\}$ and finitely many sections $s_1, \ldots, s_n$ which are a spanning set at every fiber over B. Then, take the homomorphism $F : \Gamma(B \times K^n) \to \Gamma(E)$ extending the mapping $e_i \mapsto s_i$. By Theorem 3.9, F is induced by a map $f : B \times K^n \to E$, which can be shown to be surjective using the sections $s_1, \ldots, s_n$. $\square$

Corollary 4.2. *Any vector bundle $p : E \to B$ with compact Hausdorff B is a direct summand of a trivial vector bundle. Furthermore, $\Gamma(E)$ is a finitely generated projective module.*

Proof. By Lemma 4.1, let $f : B \times K^n \to E$ be a surjective bundle map from the trivial bundle $p' : B \times K^n \to B$. Since the fibers of $\operatorname{im} f$ have locally constant dimension, $\ker f$ is a subbundle of $B \times K^n$ by Proposition 1.9. Then, Proposition 1.11 implies that $\ker f$ is a direct summand of $B \times K^n$ and has the complement $\operatorname{im} f = E$. By f's construction, $\Gamma(E)$ is finitely generated, and it is projective by Remark 3.2. $\square$

Theorem 4.3. *If B is compact Hausdorff, then a $C(B)$-module M is finitely generated projective iff $M \cong \Gamma(E)$ for some vector bundle $p : E \to B$.*

Proof. $\Longleftarrow$ follows from Corollary 4.2. Now, assume M is finitely generated and that $M \oplus N = F := C(B)^n$ for some $C(B)$-module N. Let $g : F \to F$ be the projection onto M. By Theorem 3.9, g is induced by some surjective map $f : B \times K^n \to \operatorname{im} f \subseteq B \times K^n$. It suffices to show that $\operatorname{im} f$ is a subbundle, since by functorality we would have $\Gamma(\operatorname{im} f) = \operatorname{im} g = M$.

It remains to prove that the fibers of $\operatorname{im} f$ have locally constant dimension. Since f is idempotent, $\ker f = \operatorname{im}(\mathbb{1} - f)$, where $\mathbb{1}$ is the identity, so each fiber $p^{-1}(x)$ of $B \times K^n$ is a direct sum $p|_{\operatorname{im} f}^{-1}(x) \oplus p|_{\ker f}^{-1}(x)$. But in the proof of Proposition 1.9, we constructed a set of maps $s_1, \ldots, s_n$ which are linearly independent on each fiber of $\ker f$ (resp. $\operatorname{im} f$) near x, implying that the fibers of $\ker f$ (resp. $\operatorname{im} f$) near x are bounded below by $\dim p|_{\ker f}^{-1}(x)$ (resp. $\dim p|_{\operatorname{im} f}^{-1}(x)$). The statement thus follows from the rank-nullity theorem. $\square$

5. Discussion

The Serre-Swan theorem asserts that the category Vect_B is equivalent to the subcategory of $\mathsf{Mod}_{C(B)}$ of finitely generated projective modules, via Γ (recall that a functor is an equivalence of categories if it is bijective on sets of morphisms and bijective on objects up to isomorphism [Rie16]). We conclude that it is equally valid to understand vector bundles over a compact Hausdorff space B, which are inherently structures of a geometric flavor, instead as finitely generated projective $C(B)$-modules, which are decidedly algebraic, and that this change in perspective does not result in a loss of information. One the other hand, Serre-Swan gives us an easy way to import tools and intuition from one perspective over to the other. The following is an example given by [Swa62] where the geometric properties of vector bundles are used to shed light on an algebraic problem.

Example 5.1*.* The summands of free modules are generally not as well-behaved as we would like. At first glance, one might expect a cancellation law of sorts to hold for projective and free modules which are finitely generated: if $M \times C(B)^n \cong C(B)^k$, then perhaps $M \cong C(B)^{n-k}$. This intuition is erroneous, and we provide an example of a projective module whose complement. Let τ^n, ν^1 denote the tangent and normal bundles of the n-sphere S^n. We've seen that $v_1 \cong S^n \times \mathbb{R}$ is a trivial vector bundle, and it can be shown that $\tau^n \oplus \nu^1 \cong S^n \times \mathbb{R}^{n+1}$ is also trivial [MS74]. However, τ^n is nontrivial for even n, as seen in Example 1.6. In this case, $\Gamma(\tau^n)$ is a counterexample to the statement above.

References

[Hat17] Allen Hatcher, *Vector bundles and k-theory*, 2017. 2, 8
[MS74] John W. Milnor and James D. Stasheff, *Characteristic classes*, 1974, p. 30. 9
[Mun14] James R. Munkres, *Topology*, 2nd ed., Pearson Education Limited, 2014. 7, 8
[Rie16] Emily Riehl, *Category theory in context*, Dover Publications, 2016. 9
[Swa62] Richard G. Swan, *Vector bundles and project modules*, 1962, pp. 264–277. 1, 2, 3, 9