0.1 Introduction

What are locales?

- They are algebraic counterparts to spaces.
- They are sites for a large, well-behaved class of Grothendieck topoi, the so-called localic topoi.
- They encode the data of generalized spaces which may not even have points. For example, the space of surjections $\mathbb{N} \to \mathbb{R}$.

0.2 Locales

Definition 0.1 (Frame)**.** The category Frame is defined as follows:

- The objects are complete Heyting algebras X, called **frames**. Equivalently X has all finite meets, arbitrary joins, and satisfies the infinite distribute law $U \vee (\bigvee_i V_i) = \bigvee_i (U \vee V_i)$.
- The morphisms are the poset homomorphisms which preserve finite meets and arbitrary joins, in articular 0 and 1. complete Heyting algebras.

Example 0.1. The function $\mathcal{O}$ sending each space X to its poset of opens $\mathcal{O}(X)$ and each continuous map $X \to Y$ to the morphism $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ is a functor $\mathsf{Top}^{\mathrm{op}} \to \mathsf{Frame}$.

Definition 0.2 (Locale)**.** A locale is an object in the category $\mathsf{Locale} = \mathsf{Frame}^{\mathrm{op}}$. We will overload $\mathcal{O}$ to mean the functor $\mathsf{Locale}^{\mathrm{op}} \to \mathsf{Frame}$ which reverses all arrows. There is an obvious covariant functor $\mathrm{Loc} : \mathsf{Top} \to \mathsf{Locale}$ sending each object to its frame of opens and then reversing arrows. Hence, every locale morphism, called a **continuous map** $f : X \to Y$ corresponds to a frame morphism $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$. We have a handy diagram:

$$\begin{tikzcd}
\mathsf{Top} \arrow[r, "\mathcal{O}"'] \arrow[rr, bend left, "\mathrm{Loc}"] & \mathsf{Frame} & \mathsf{Locale} \arrow[l, "\mathcal{O}"]
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[d, "f"] & \mathcal{O}(X) & X \arrow[d, "\mathrm{pt}(f)"] \\
Y & \mathcal{O}(Y) \arrow[u, "f^{-1}"] & Y
\end{tikzcd}$$

0.3 Points and Sober Spaces

A point in a topological space S is a continuous map $1 \to S$.

Definition 0.3 (Point)**.** A **point** of a locale X is a continuous map $1 \to X$.

What exactly is the terminal object 1 in Locale? It corresponds to the initial object in Frame, which is precisely the lattice $\{0, 1\}$ since frame morphisms must preserve finite meets and joins. The fact that $p^{-1} : \mathcal{O}(X) \to \{0, 1\}$ is frame morphism can be described by the properties of its kernel K:

$$\begin{aligned} 1 &\notin K, \\ U \wedge V \in K &\iff U \in K \text{ or } V \in K, \\ \bigvee U_i \in K &\iff U_i \in K \text{ for all } i. \end{aligned}$$

Any subset $K \subseteq \mathcal{O}(X)$ satisfying the properties above is represented by some object P in the sense that if we take $P = \bigvee_{U \in K} U$, then $U \leq P$ iff $U \in K$. K is denoted $\downarrow P$ and is called the **downward closure** of P. Similarly, the first two properties of K can be translated into properties of an object P. Similarly, we have $P \neq 1$ and $U \wedge V \leq P$ if $U \leq P$ or $V \leq P$. Any element satisfying condition 2 (and 1) is called a prime (proper prime) element. Hence,

Lemma 0.1. *The points of a locale X can be described as:*

- *Maps of locales $p : 1 \to X$,*
- *Subsets $K \subseteq \mathcal{O}(X)$ satisfying the three conditions above,*
- *Proper prime elements $P \in \mathcal{O}(X)$.*

Example 0.2. Given a point $s \in S$ in a topological space S, we get a proper prime element $S \setminus \overline{\{s\}}$ of $\mathcal{O}(S)$, or equivalently a subset $K_s = \{U \in \mathcal{O}(S) : s \notin U\}$.

Definition 0.4 (Sober Space)**.** A topological space is **sober** if for any proper prime element $P \in \mathcal{O}(S)$, there is a unique point $s \in S$ such that $P = S \setminus \overline{\{s\}}$. Alternatively: we say a closed subset U is **irreducible** if whenever we decompose $U = V \cup W$ into a union of closed sets, then $V = U$ or $W = U$. Then, S is sober iff every irreducible closed subset is the closure of a unique point.

Theorem 0.2. *We have Hausdorff $\implies$ sober $\implies$ T_0.*

Proof. The mapping $s \mapsto \overline{\{s\}}$ is a bijection iff S is sober, by definition. Suppose $s \mapsto \overline{\{s\}}$ were only injective, so $s \neq t$ implies $S \setminus \overline{\{s\}} \neq S \setminus \overline{\{t\}}$. But this means there is an open U that differentiates s and t: for example, $s \in U$ and $t \notin U$. This precisely means S is T_0. Any Hausdorff space S is sober since all irreducible nonempty closed sets are singletons: if we had distinct $x, y \in S$, then $S = (S \setminus U_x) \cup (S \setminus U_y)$ for disjoint neighbourhoods U_x, U_y. ■

0.4 Spatial Locales

Each locale X has a natural topology on its set of points $\mathrm{pt}(X)$. For each $U \in \mathcal{O}(X)$, define $\mathrm{pt}(U) = \{p \in \mathrm{pt}(X) : p^{-1}(U) = 1\}$. Since points are frame morphisms, we have

$$\begin{aligned} p^{-1}(U \wedge V) &= p^{-1}(U) \wedge p^{-1}(V), \\ p^{-1}(\bigvee_i U_i) &= \bigvee_i p^{-1}(U_i). \end{aligned}$$

Hence, pt splits over finite intersection and arbitrary union, so the sets $\mathrm{pt}(U)$ form a topology. Furthermore, each map $f : X \to Y$ of locales forms a function $\mathrm{pt}(f) : \mathrm{pt}\, X \to \mathrm{pt}\, Y$ by composition, and as just described, $\mathrm{pt}(f)$ is continuous. So $\mathrm{pt} : \mathsf{Locale} \to \mathsf{Space}$ is a functor.

Theorem 0.3 (Loc $\dashv$ pt)**.** *The functor* $\mathrm{pt} : \mathsf{Locale} \to \mathsf{Space}$ *is right adjoint to the functor* $\mathrm{Loc} : \mathsf{Space} \to \mathsf{Locales}$.

Proof. Fix the locale X and the space S. For $g : S \to \mathrm{pt}(X)$, $f : \mathrm{Loc}(S) \to X$ is given by the frame morphism $f^{-1} : \mathcal{O}(X) \to \mathcal{O}(S)$ for each V by

$$\begin{aligned} f^{-1}(V) &= \{s \in S : g(s)^{-1}(V) = 1\} \\ &= \{s \in S : g(s)(1) = (V)\} \\ &= g^{-1}(\mathrm{pt}(V)). \end{aligned}$$

f^{-1} is the composition of two frame morphisms, so it is frame itself. Now, given a continuous map $f : \mathrm{Loc}(S) \to X$, define $g : S \to \mathrm{pt}(X)$ by assigning each $s \in S$ the point $g(s) : 1 \to X$ as follows:

$$g(s)^{-1}(V) = \begin{cases} 1, & s \in f^{-1}(V),\ V \in \mathcal{O}(X), \\ 0, & \text{otherwise.} \end{cases}$$

g is continuous since

$$\begin{aligned} g^{-1}\,\mathrm{pt}(V) &= \{s \in S : g(s) \in \mathrm{pt}(V)\} \\ &= \{s \in S : g(s)^{-1}(V) = 1\} = f^{-1}(V). \end{aligned}$$

It is not hard to show these mappings are inverse. For example, given the first g, define f, g' to be the corresponding maps $\mathrm{Loc}(S) \to X$ and $S \to \mathrm{pt}(X)$. Then,

$$\begin{aligned} g'(s)^{-1}(V) = 1 &\iff s \in f^{-1}(V) = g^{-1}(\mathrm{pt}(V)) \\ &\iff g(s) \in \mathrm{pt}(V) \\ &\iff g(s)^{-1}(V) = 1. \end{aligned}$$

■

One might ask whether pt or Loc is faithful. The answer is no; we will provide a counterexample to the second, which proves that surprisingly there are nontrivial pointless locales.

Example 0.3 (Locale of surjections $\mathbb{N} \to \mathbb{R}$)**.** This example is per Johnstone. The locale X of surjections $\mathbb{N} \to \mathbb{R}$ is generated freely by the open sets $U(n, x)$, thought of as the set of surjections $f : \mathbb{N} \to \mathbb{R}$ with $f(n) = x$, modulo the following relations:

$$\bigvee_x U(n, x) = 1 \text{ for all } x \in \mathbb{R},$$

$$U(n, x) \wedge U(n, y) = 0 \text{ for } x \neq y,$$

$$\bigvee_n U(n, x) = 1 \text{ for all } x \in \mathbb{R}.$$

I give a proof of this fact: define U_n, U^x to be the sets $\{U(n,y) : y \in R\}$ and $\{U(m,x) : m \in \mathbb{N}\}$ respectively. If there existed a point $p : 1 \to X$, then for any fixed $n \in \mathbb{N}$, $p^{-1}(0)$ contains all but at most 1 open in U_n, since for distinct $U(n,x), U(n,y) \in p^{-1}(0)$, $0 = U(n,x) \wedge U(n,y) \in p^{-1}(0)$ implies $U(n,x)$ or $U(n,y)$ is in $p^{-1}(0)$. Contrarily for all $x \in \mathbb{R}$, $p^{-1}(0)$ cannot contain all opens in U_x, or else we would have $1 = \bigvee_n U(n,x) \in p^{-1}(0)$, which cannot happen since $p^{-1}(0)$ is proper. But this means that there are simultaneously countably and uncountably many $U(n,x)$ missing from $p^{-1}(0)$, which is a contradiction.

Definition 0.5 (Spatial)**.** A locale X is **spatial** or **has enough points** if $\mathrm{pt}(U) = \mathrm{pt}(V)$ implies $U = V$ for any $U, V \in \mathcal{O}(X)$.

Theorem 0.4. *The adjunction* $\mathrm{Loc} : \mathsf{Top} \rightleftarrows \mathsf{Locale} : \mathrm{pt}$ *restricts to an equivalence of categories between the full subcategories of spatial locales and sober spaces.*

Proof. This is proven by showing:

- The space S is sober $\iff$ the unit $\eta : S \to \mathrm{pt}\,\mathrm{Loc}(S)$ is a homeomorphism $\iff$ $S = \mathrm{pt}(X)$ for some locale X.
- The locale X is space $\iff$ the counit $\epsilon : \mathrm{Loc}\,\mathrm{pt}(X) \to X$ is an isomorphism $\iff$ $X = \mathrm{Loc}(S)$ for some space S.

and then applying Lemma II.6.4. ■

Example 0.4 (The Sierpinski Space)**.** Given a frame A, which we endow with the discrete topology, we have $\mathrm{pt}(A) = \mathrm{Hom}(A, 2)$, which is topologized by the compact-open topology on 2^A: recall that these were $\{p \in \mathrm{pt}(A) : p^{-1}(U) = 1\}$ for $U \in A$. We also have for a topological space X that $\mathcal{O}(X) = \mathrm{Hom}(X, 2)$, where 2 is the Sierpinski space with $\{1\}$ as its the only nontrivial open. Surprisingly, the Sierpinski space is both a locale and a topological space, with the lattice operations continuous with respect to the topology, and is thus an example of a **dualizing object**.

0.5 Geometric Morphisms

Definition 0.6 (Geometric Morphism)**.** A **geometric morphism** $f : \mathcal{F} \to \mathcal{E}$ between topoi is a pair of functors forming an adjunction $f^* : \mathcal{E} \rightleftarrows \mathcal{F} : f_*$, such that f^* is left exact. f_* and f^* are called the **direct image part** and **inverse image part** of the geometric morphism, respectively.

Definition 0.7 (Surjection)**.** A geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is **surjective** if its inverse image functor f^* is faithful.

Example 0.5. Here are some basic examples of geometric morphisms.

- Given a continuous map $f : X \to Y$, we get a geometric morphism consisting of the direct image functor $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ and the inverse image functor $f^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$.

- If we have a morphism $k : B \to A$ in a topos, we get the change-of-base functor $k^* : \mathcal{E}/A \to \mathcal{E}/B$ which has the right adjoint $k_* = \Pi_k : \mathcal{E}/B \to \mathcal{E}/A$. This forms a geometric morphism denoted $k : \mathcal{E}/B \to \mathcal{E}/A$.
- If j is a Lawvere-Tieney on a topos $\mathcal{E}$, we get a geometric $i : \mathrm{Sh}_j\,\mathcal{E} \to \mathcal{E}$, where i_* is the inclusion functor and i^* is the sheafification functor.

If S is a T_1 space, then a continuous map $f : T \to S$ is surjective iff $f^{-1} : \mathcal{O}(S) \to \mathcal{O}(T)$ is an injective frame morphism, since the former implies $ff^{-1}(U) = U$ and the latter implies $f^{-1}(1) = 1$. Similarly, if T is T_0 then $f : T \to S$ is an embedding iff f^{-1} is surjective. We promptly define

Definition 0.8. A map $f : X \to Y$ of locales is an **embedding** (resp. **surjection**) iff $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ is surjective (resp. **injective**.

Given any continuous map $f : X \to Y$ of locales, the frame morphism f^{-1} has a right adjoint $f_* : \mathcal{O}(X) \to \mathcal{O}(Y)$ given by $f_*(U) = \bigvee\{V \in B : f^{-1}(V) \leq U\}$, which satisfies $f^{-1}(V) \leq U$ iff $V \leq f_*(U)$. The (co)unit of this adjunction give

$$U \leq f_*f^{-1}(U), \text{ for } U \in \mathcal{O}(X),$$
$$f^{-1}f_*(V) \leq V, \text{ for } V \in \mathcal{O}(Y),$$

and the triangle identities give

$$f^{-1}f_*f^{-1} = f^{-1}, \qquad\qquad f_*f^{-1}f_* = f_*$$

Following easily from these identities are the following equivalences:

Theorem 0.5. *Let $f : X \to Y$ be a continuous map. TFAE:*

- *f is a surjection of locales,*
- *$f_*f^{-1} = 1 : \mathcal{O}(X) \to \mathcal{O}(X)$,*
- *$f_* : \mathcal{O}(Y) \to \mathcal{O}(X)$ is a surjection of posets.*

Furthermore, TFAE:

- *f is a surjection of locales,*
- *$f_*f^{-1} = 1 : \mathcal{O}(X) \to \mathcal{O}(X)$,*
- *$f_* : \mathcal{O}(Y) \to \mathcal{O}(X)$ is a surjection of posets.*

Now, recall that for an embedding $f : S \to T$ of a subspace, the open sets U of S are given by $T \cap V$ for some open V of T; we can suppose identify U with the largest such set $W = \cup_{f^{-1}V \subseteq U} V$. Recall that f_*U is the union of all V with $f^{-1}V \subseteq U$, so $f_*f^{-1}W$ is the union of all V with $f^{-1}V \subseteq f^{-1}W$, or just W. Hence, the opens U of S are simply the opens of W fixed by the operator $j = f_*^{-1}$, which satisfies the properties $U \leq jU$ and $j^2 \leq jU$, since j is the monad of an adjunction. Hence $j^2U = jU$, and j splits over meets since its components do. Notice that j is a modal operator on $\mathcal{O}(T)$.

Definition 0.9 (Nucleus)**.** A nucleus $j : \mathcal{O}(X) \to \mathcal{O}(X)$ on a locale X is a modal operator on $\mathcal{O}(X)$.

Theorem 0.6. *Let $j : \mathcal{O}(X) \to \mathcal{O}(X)$ be a nucleus on X. Then the poset of j-fixed points is a frame $\mathcal{O}(X_j)$, and j surjects $\mathcal{O}(X)$ onto $\mathcal{O}(X_j)$.*

The notion of **sublocale** of X is precisely captured by locales of the form X_j for some nucleus j. Here is a fact we won't prove,

Theorem 0.7. *If $f : Y \to X$ is a map of locales, then there exists a nucleus j on X such that f factors through the embedding $i : X_j \to X$. Furthermore, this surjection-embedding factorization is unique up to isomorphism.*

We conclude this part with two examples.

Example 0.6 (Closed Sublocale)**.** We will consider a specific sublocale called a closed sublocale; an open sublocale is constructed in a similar way. Define $X - U$ to be the locale given by the frame $\mathcal{O}(X - U) = \uparrow U = \{V \in \mathcal{O}(X) : U \leq V\}$. The embedding $X - U \to X$, is given by the frame map $V \mapsto V \vee U$, which may be confusing at first since $\mathcal{O}(X - U)$ consists of opens *containing* U, not *disjoint* from U. By construction, $X - U$ corresponds to the nucleus $V \mapsto V \vee U$.

Example 0.7 ($DoubleNegationNucleus$)**.** Consider the negation operator $\neg$ of a locale X. Then, $\neg\neg$ gives the **double negation nucleus**, whose corresponding sublocale consists of opens such that $\neg\neg U = U$. In other words, any locale has a Boolean sublocale $X_{\neg\neg}$.

0.6 Localic Topoi

Definition 0.10 (Localic Topos)**.** Given a locale X, endow $\mathcal{O}(X)$ with the (usual) sup topological where $\{U_i \leq U\}_{i \in I}$ covers U if $\bigvee_i U_i = U$. This is the canonical topology on $\mathcal{O}(X)$ in the sense that it is the largest Grothendieck topology in which all representable presheaves are sheaves. Then a localic topos is one of the form $\mathrm{Sh}(X)$ for some locale X.

The following theorem characterizes localic topoi.

Theorem 0.8. *Let $\mathcal{E}$ be a Grothendieck topos. TFAE:*

1. *$\mathcal{E}$ is localic,*
2. *There exists a site for $\mathcal{E}$ where the underlying category is a poset.*
3. *$\mathcal{E}$ is generated by* $\mathrm{Sub}_{\mathcal{E}}\, 1$.

Proof. $1 \implies 2$ is trivial. To show $2 \implies 3$, fix some site (P, J), and let $\mathsf{a}\,よ : P \to \mathcal{E}$ denote the operator turning any $p \in P$ into a sheafified representable presheaf $\mathsf{a}\,よ(p) \in \mathcal{E}$. But we know $よ(p) \to 1$ is a subobject of 1, and $\mathsf{a}\,よ(p) \to 1$ is monic since a is left exact. So $\mathsf{a}\,よ(p)$ are subobjects of 1, and III.6(17) follows with the remark that the objects $\mathsf{a}\,よ(p)$ generate $\mathcal{E}$. For $3 \implies 1$, recall that $\mathrm{Sub}_{\mathcal{E}}(1)$ is a cHa, so we get a locale X with $\mathcal{O}(X) = \mathrm{Sub}_{\mathcal{E}}(1)$. To show that $\mathcal{O}(X)$ is a site for $\mathcal{E}$ requires a corollary of Giraud's theorem in the appendix, which gives sufficient and necessary conditions for topos to be Grothendieck. This is in the appendix of Maclane Moerdijk. ■

Continuous maps between locales share many symmetries with geometric morphisms between localic topoi: at the basic level, continuous maps $X \to Y$ of locales correspond directly to geometric morphisms $\mathrm{Sh}(X) \to \mathrm{Sh}(Y)$, similar to the case for topological spaces. Recall that a locale X can be recovered from $\mathrm{Sh}(X)$ by $\mathcal{O}(X) \cong \mathrm{Sub}(1)$. Let $\mathcal{E}$ be a topos with small colimits. Corollary VII.9.4 states that there is a correspondence between geometric morphisms $\mathcal{E} \to \mathrm{Sh}(Y)$ with continuous left exact functors $F : \mathcal{O}(Y) \to \mathcal{E}$. F must then be a frame morphism, and since every object of $\mathcal{O}(Y)$ is a subobject of 1, F's image lies in $\mathrm{Sub}_{\mathcal{E}}(1)$. If $\mathcal{E} \cong \mathrm{Sh}(X)$ were localic, then we would have by above that $\mathrm{Sub}_{\mathcal{E}}(1) \cong X$, hence $F : \mathcal{O}(Y) \to \mathcal{O}(X)$. Even further, we have discovered the following adjunction:

Theorem 0.9. *For a cocomplete topos $\mathcal{E}$ and a locale Y, there is a natural correspondence*

$$\underline{\mathrm{Hom}}(\mathcal{E}, \mathrm{Sh}(Y)) \cong \mathrm{Hom}(\mathrm{Loc}\,\mathcal{E}, Y).$$

This fact will be important later:

Lemma 0.10. *An embedding $f : \mathcal{E} \to \mathrm{Sh}(X)$ of a topos into a localic topics forces $\mathcal{E}$ to be localic.*

Proof. Fix a generating set $\{G_i \leq \mathrm{Sub}_{\mathcal{F}}(1) : i \in I\}$. If f is an embedding, then f_* is faithful. Then if $\alpha \neq \beta : E \to E'$, $f_*\alpha \neq f_*\beta$, so there exists some fixed i and a map $u : G_i \to f_*E$ with $f_*(a)u \neq f_*(\beta)u$. Transposing along the adjunction $f^* \dashv f_*$ gives a map $\hat{u} : f^*G_i \to E$ such that $\alpha\hat{u} \neq \beta\hat{u}$, so $f^*(G_i)$ is a generating family for $\mathcal{E}$. But f^* is left exact so $f^*(G_i)$ are all subobjects of 1. ■

Also, we will refer to this theorem.

Theorem 0.11. *Let $f : X \to Y$ be a map of locales and $\tilde{f} : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ be the corresponding geometric morphism. Then f is a surjection (resp. injection) of locales iff $\tilde{f}$ is a surjection of topoi.*

Proposition 5
Corollary 6

0.7 Open Geometric Morphisms

Similar to the property of openness for maps of spaces, we wish to define a similar property for continuous maps of locales and geometric morphisms. Suppose $f : X \to Y$ is an open map. Then for any open sets $U \subseteq X$ and $V \subseteq Y$, we have $V \subseteq f^{-1}(U)$ iff $f(V) \subseteq U$. This precisely means that the functor $f_! : V \mapsto f(V)$ is a left adjoint to f^{-1}.

Definition 0.11. A geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is said to be open if for each object $E \in \mathcal{E}$, the induced subobject poset map f_E^* has a left adjoint $(f_E)_!$,

$$(f_E)_! : \mathrm{Sub}_{\mathcal{F}}(f^*E) \rightleftarrows \mathrm{Sub}_{\mathcal{E}}(E) : f_E^*$$

which is also natural in E.

The notion of open geometric morphisms are crucial to the logical aspects of localic topoi. It is mentioned that a geometric morphism f is open exactly when it preserves the interpretation of first-order logic—in particular, the interpretation of quantifers.

0.8 Open Maps of Locales

0.9 Open Maps and Sites

0.10 The Diaconescu Covering and Barr's theorem

We conclude with two interesting theorems relating localic topoi to the more Grothendieck topoi. We will need a technical theorem guaranteeing the openness of a geometric morphism.

Definition 0.12 (Cover Lifting Property)**.** Let (C, J) and (D, K) be sites. A functor $\pi : \mathsf{D} \to \mathsf{C}$ has the **cover lifting property** if for any object D of D and J-cover S of $\pi(D)$, there exists a K-cover R of D such that $\pi(R) \subseteq S$.

Theorem 0.12. *Let $\pi : \mathsf{D} \to \mathsf{C}$ be a functor with the clp. Suppose π preserves covers, i.e., for any cover T of D in D, the sieve generated by πT covers $\pi(D)$ in C, and suppose furthermore that the induced functor $\pi/D : \mathsf{D}/D \to \mathsf{C}/\pi D$ on the slice categories is surjective. Then the geometric morphism $f : \mathrm{Sh}(\mathsf{D}, K) \to Sh(\mathsf{C}, J)$ induced by π is an open geometric morphism. Lastly, if π is surjective, then so is f.*

Theorem 0.13 (The Diaconescu Covering)**.** *For every Grothendieck topos $\mathcal{E}$ there exists a locale X and an open surjective geometric morphism* $\mathrm{Sh}(X) \to \mathcal{E}$

Theorem 0.14 (Barr's Theorem)**.** *For every Grothendieck topos $\mathcal{E}$ there exists a complete Boolean algebra B and a surjective geometric morphism* $\mathrm{Sh}(B) \to \mathcal{E}$*.*

Proof of theorem . By our characterization of localic topoi it suffices to find a surjection $\mathrm{Sh}(P, K) \to \mathrm{Sh}(C, J)$ for any Grothendieck topos $\mathrm{Sh}(C, J)$, where P is a poset. We can simply construct a surjection $\pi : P \to C$ and prove that π satisfies the conditions of 8.1 and 8.3, and then we would have that π corresponds to our desired surjection of topoi. Let P be the poset category $\mathrm{Path}(C)$ whose objects are paths p of composable morphisms in C, and let $q \leq p$ if q extends p to the left:

$$\overbrace{C_{n+m} \to \cdots \to \underbrace{C_n \to \cdots \to C_0}_{p}}^{q}$$

Then we have the functor $\pi(p) = C_n$, where $\pi(q \leq p)$ is computed by evaluating the path extending q from p. Then, construct the Grothendieck topology K from J by having a sieve U cover p if the set of arrows $\pi(q \leq p) : \pi(q) \to \pi(p) \in \pi(U)$ cover $\pi(p)$. The proof that K is a Grothendieck topology is routine, since we can reward this as U covers p iff $\pi(U)$ covers p. For the conditions of the theorem above, we know that π is surjective, and it is clear that π preserves covers (it sends covers to covers), and that the induced map on slice topoi is a surjection. CLP is left as an exercise.

■

Proof of theorem . By the previous theorem, it suffices to prove that given a locale X, there exists a surjection $Y \to X$ of locales for which $\mathcal{O}(Y)$ is a complete Boolean algebra. Then, by a theorem mentioned earlier, the composite $\mathrm{Sh}(Y) \to \mathrm{Sh}(X) \to \mathcal{E}$ will be surjective. For each $U \in \mathcal{O}(X)$, let $(X - U)_{\neg\neg}$ be the Boolean sublocale of the closed sublocale $X - U$, and define

$$Y = \coprod_{U \in \mathcal{O}(X)} (X - U)_{\neg\neg},$$

where $\mathcal{O}(Y)$ will be a product in the category of frames. Hence, Y is Boolean with operations taken pointwise. Then there is a canonical map of locales $p : Y \to X$ defined on each summand $(X - U)_{\neg\neg}$ as the composite $p_U : (X - U)_{\neg\neg} \to (X - U) to U$. It remains to show that p^{-1} is injective, so that p is surjective. If we have $U, V \in \mathcal{O}(X)$ and $U < V$, then $p_U^{-1}(U) = 0$ (since $p_U(0) = 0 \cup U$) but $p_U^{-1}(V) \neq 0$. ■