Topos Theory I: Presheaf Categories and Related Constructions

Robbert Liu

September 18, 2022

Contents

Some text.

Contents

Some text.

Contents

Some text.

- Presheaf Categories
- Pullbacks
- Subobject Classifers

Contents

Some text.

- Presheaf Categories
- Pullbacks
- Subobject Classifers
- Colimits

Contents

Contents

Contents

Contents

- Functor Categories
- Pullbacks
- Subobject Classifers
- Colimits

Functor Categories at a Distance

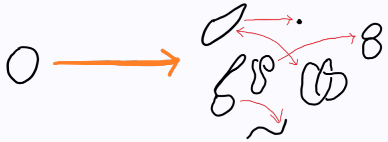
- Functor $\leftrightarrow$ continuous map.

Functor Categories at a Distance

- Functor $\leftrightarrow$ continuous map.
- Natural transformation $\leftrightarrow$ homotopy.

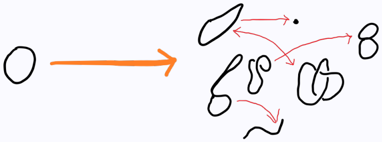
Functor Categories at a Distance

- Functor $\leftrightarrow$ continuous map.
- Natural transformation $\leftrightarrow$ homotopy.
- Consider the space of continuous maps $X \to Y$ and homotopies between them:



Functor Categories at a Distance

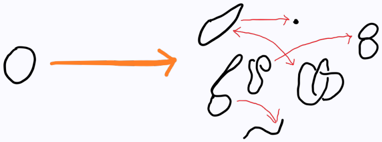
- Functor $\leftrightarrow$ continuous map.
- Natural transformation $\leftrightarrow$ homotopy.
- Consider the space of continuous maps $X \to Y$ and homotopies between them:



- *Analogously...* a functor $F : \mathbf{J} \to \mathbf{C}$ is a "drawing" of $\mathbf{J}$ in $\mathbf{C}$.

Functor Categories at a Distance

- Functor $\leftrightarrow$ continuous map.
- Natural transformation $\leftrightarrow$ homotopy.
- Consider the space of continuous maps $X \to Y$ and homotopies between them:



- *Analogously...* a functor $F : \mathbf{J} \to \mathbf{C}$ is a "drawing" of $\mathbf{J}$ in $\mathbf{C}$.
- A functor category $\mathbf{C}^{\mathbf{J}}$ considers all such "drawings" along with compatible maps between them.

Category of Presheaves

Definition

The **category of presheaves** on $\mathbf{C}$ is a functor category $\hat{\mathbf{C}} := \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$.

- Examples:

Category of Presheaves

Definition

The **category of presheaves** on $\mathbf{C}$ is a functor category $\hat{\mathbf{C}} := \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$.

- Examples:
  - $\mathbf{Set} = \hat{\mathbf{1}}$

$$
\begin{tikzcd}
F(1) \arrow[r] \arrow[d, equal] & G(1) \arrow[d, equal] \\
F(1) \arrow[r] & G(1)
\end{tikzcd}
$$

Category of Presheaves

Definition

The **category of presheaves** on $\mathbf{C}$ is a functor category $\hat{\mathbf{C}} := \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$.

- Examples:
  - $\mathbf{Set} = \hat{\mathbf{1}}$

$$\begin{tikzcd} F(1) \arrow[r] \arrow[d, equal] & G(1) \arrow[d, equal] \\ F(1) \arrow[r] & G(1) \end{tikzcd}$$

  - $\mathbf{Set}^n = \hat{\mathbf{C}}$, where $\mathbf{C}$ is the discrete category of size n.

Category of Presheaves

Definition

The **category of presheaves** on $\mathbf{C}$ is a functor category $\hat{\mathbf{C}} := \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$.

- Examples:
  - $\mathbf{Set} = \hat{\mathbf{1}}$

$$\begin{tikzcd} F(1) \arrow[r] \arrow[d, equal] & G(1) \arrow[d, equal] \\ F(1) \arrow[r] & G(1) \end{tikzcd}$$

  - $\mathbf{Set}^n = \hat{\mathbf{C}}$, where $\mathbf{C}$ is the discrete category of size n.
  - $\mathbf{Set}^{\mathbf{N}} = \widehat{\mathbf{N}^{\mathrm{op}}}$, where $\mathbf{N}$ is the total order category of $\mathbb{N}$.

$$\begin{tikzcd} Pn \arrow[r] \arrow[d] & Pm \arrow[d] \\ Qn \arrow[r] & Qm \end{tikzcd} \qquad \begin{tikzcd} P0 \arrow[r] \arrow[d] & P1 \arrow[r] \arrow[d] & \cdots \\ Q0 \arrow[r] & Q1 \arrow[r] & \cdots \end{tikzcd}$$

Pullbacks, briefly

- Pullbacks and final objects exist in **Set**...

Pullbacks, briefly

- Pullbacks and final objects exist in **Set**. . .
- . . . and in $\hat{\mathbf{C}}$, since limits are computed pointwise.

$$(X \times_B Y)(C) \cong X(C) \times_{B(C)} Y(C)$$
$$1(C) \cong 1$$

Pullbacks, briefly

- Pullbacks and final objects exist in **Set**...
- ...and in $\hat{\mathbf{C}}$, since limits are computed pointwise.

$$(X \times_B Y)(C) \cong X(C) \times_{B(C)} Y(C)$$
$$1(C) \cong 1$$

- All finite limits can be constructed from equalizers and products:

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & 1 \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "f|_E"] \arrow[d, hook] & Y \arrow[d, "\Delta"] \\ X \arrow[r, "{(f,g)}"'] & Y \times Y \end{tikzcd}$$

Characteristic Functions of Subobjects

- A subset $S \subseteq X$ can be thought of as either...

Characteristic Functions of Subobjects

- A subset $S \subseteq X$ can be thought of as either...
  - a monomorphism $S \rightarrowtail X$ given by inclusion, or

Characteristic Functions of Subobjects

- A subset $S \subseteq X$ can be thought of as either...
  - a monomorphism $S \rightarrowtail X$ given by inclusion, or
  - a characteristic function $\phi_S : X \to 2$ defined by $\phi_S(x) = 0 \iff x \in S$.

Characteristic Functions of Subobjects

- A subset $S \subseteq X$ can be thought of as either...
  - a monomorphism $S \rightarrowtail X$ given by inclusion, or
  - a characteristic function $\phi_S : X \to 2$ defined by $\phi_S(x) = 0 \iff x \in S$.
- If we define $\mathrm{true} : 1 \hookrightarrow 2$, then S can be recovered from ϕ_S by pulling back along true.

$$\begin{tikzcd}
X \times_2 1 \arrow[r] \arrow[d] & 1 \arrow[d] \\
X \arrow[r, "\phi_S"] & 2
\end{tikzcd}$$

Characteristic Functions of Subobjects

- A subset $S \subseteq X$ can be thought of as either...
  - a monomorphism $S \rightarrowtail X$ given by inclusion, or
  - a characteristic function $\phi_S : X \to 2$ defined by $\phi_S(x) = 0 \iff x \in S$.
- If we define $\mathrm{true} : 1 \hookrightarrow 2$, then S can be recovered from ϕ_S by pulling back along true.

$$\begin{tikzcd} X \times_2 1 \arrow[r] \arrow[d] & 1 \arrow[d] \\ X \arrow[r, "\phi_S"] & 2 \end{tikzcd}$$

- *Idea:* generalize the set 2 to an object $\Omega \in \mathbf{C}$ of "truth values" for a general category $\mathbf{C}$.

Subobject Classifers

Definition

In a category $\mathbf{C}$ with finite limits, a **subobject classifer** is a monomorphism $true : 1 \to \Omega$ such that for every monomorphism $S \rightarrowtail X$, we get a unique morphism ϕ forming a pullback square

$$\begin{tikzcd} S \arrow[r] \arrow[d, tail] & 1 \arrow[d, tail, "true"] \\ X \arrow[r, dashed, "\phi"'] & \Omega \end{tikzcd}$$

Subobject Classifers

Definition

In a category $\mathbf{C}$ with finite limits, a **subobject classifer** is a monomorphism $\mathit{true} : 1 \to \Omega$ such that for every monomorphism $S \rightarrowtail X$, we get a unique morphism ϕ forming a pullback square

$$\begin{tikzcd} S \arrow[r] \arrow[d, tail] & 1 \arrow[d, tail, "true"] \\ X \arrow[r, dashed, "\phi"'] & \Omega \end{tikzcd}$$

- *Remark:* there is a functor $\mathrm{Sub}_{\mathbf{C}} : \mathbf{C}^{\mathrm{op}} \to \mathrm{Set}$:

Subobject Classifers

Definition

In a category $\mathbf{C}$ with finite limits, a **subobject classifer** is a monomorphism $true : 1 \to \Omega$ such that for every monomorphism $S \rightarrowtail X$, we get a unique morphism ϕ forming a pullback square

$$\begin{tikzcd} S \arrow[r] \arrow[d, tail] & 1 \arrow[d, tail, "true"] \\ X \arrow[r, dashed, "\phi"'] & \Omega \end{tikzcd}$$

- *Remark:* there is a functor $\mathrm{Sub}_{\mathbf{C}} : \mathbf{C}^{\mathrm{op}} \to \mathsf{Set}$:
    - $\mathrm{Sub}_{\mathbf{C}}(X)$ is the set of subobjects $S \rightarrowtail X$.

Subobject Classifers

Definition

In a category $\mathbf{C}$ with finite limits, a **subobject classifer** is a monomorphism $true : 1 \to \Omega$ such that for every monomorphism $S \rightarrowtail X$, we get a unique morphism ϕ forming a pullback square

$$\begin{tikzcd} S \arrow[r] \arrow[d, tail] & 1 \arrow[d, tail, "true"] \\ X \arrow[r, dashed, "\phi"'] & \Omega \end{tikzcd}$$

- *Remark:* there is a functor $\mathrm{Sub}_{\mathbf{C}} : \mathbf{C}^{\mathrm{op}} \to \mathrm{Set}$:
  - $\mathrm{Sub}_{\mathbf{C}}(X)$ is the set of subobjects $S \rightarrowtail X$.
  - For $f : S' \to S$, $\mathrm{Sub}_{\mathbf{C}}(f)(S \rightarrowtail X)$ is the (monic) pullback $S' \rightarrowtail X'$ of $S \rightarrowtail X$ along f.

Subobject Classifers

Proposition

A locally small category $\mathbf{C}$ with finite limits has a subobject classifer iff $\mathrm{Sub}_{\mathbf{C}}$ is representable: there exists an object Σ and a natural isomorphism:

$$\theta_X : \mathrm{Sub}_{\mathbf{C}}(X) \cong \mathrm{Hom}_{\mathbf{C}}(X, \Omega).$$

If this holds, then $\mathbf{C}$ is well-powered: $\mathrm{Sub}_{\mathbf{C}}(X)$ is isomorphic to a small set for all X.