Topos Theory III: Sheaves

Robbert Liu

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• Sheaves

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- \bullet The Representable Sheaf and Sieves

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- Bundles

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Definition (Sheaf)

Let $\mathcal{O}(X)$ be the poset of open sets of a topological space. A sheaf is a presheaf $F \in \mathcal{O}(X)$ with the following equalizer for every open cover $\{U_i\}$ of an open set U:

$$FU \stackrel{e}{\longrightarrow} \prod_{i} FU_{i} \stackrel{\alpha,\beta}{\Longrightarrow} \prod_{i,j} F(U_{i} \cap U_{j})$$

where α (β) sends each component $f_i \in FU_i$ to $f_i|_{U_i \cap U_j}$ ($f_j|_{U_i \cap U_j}$).



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- For each family $f_i \in FU_i$ glues together to form a unique $f \in FU$ if f_i, f_j coincide when restricted to $U_i \cap U_j$.



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- The representable sheaf $\operatorname{Hom}_{\mathcal{O}(X)}(V,U)$ assigning V the set $\{V \to U\} \cong 1$ if $V \subseteq U$, and \emptyset otherwise.

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Let F be a sheaf on X. A subfunctor $S \leq F$ is a subsheaf iff for every open set $U \subseteq X$ and $f \in FU$, and an open covering $\bigcup_i U_i = U$ we have

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Proof.

Idea: since S is a subfunctor of F, gluing and restriction on S is inherited from F. It suffices to add the condition that S must be "closed" under gluing and restriction, for S to be a sheaf.

• For each continuous map $f: X \to Y$, we get an induced map $f_*: \operatorname{Sh}(X) \to \operatorname{Sh}(Y)$, where f_*F is the sheaf with local sets $f_*FU = Ff^{-1}(U)$.

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Theorem 2

Suppose X has an pen cover $\{W_k\}$ with a family of sheaves $F_k \in Sh(W_k)$, such that

$$F_k|_{W_k\cap W_\ell} = F_\ell|_{W_k\cap W_\ell}$$
 for all k, ℓ .

Then there exists a unique (up to iso.) sheaf $F \in Sh(X)$ such that $F|_{W_k} = F_k$.

Proof.

Idea: if F exists, then for each open set $U \subseteq X$, we must have an equalizer for the cover $\bigcup_k (U \cap W_k) = U$.

$$FU \xrightarrow{e} \prod_{i} F_k(U \cap W_k) \xrightarrow{\alpha, \beta} \prod_{i,j} F_{k\ell}(U \cap W_k \cap W_\ell)$$

where $F_{k\ell} = F_k|_{W_k \cap W_\ell}$. Thus, we take this to be the definition of FU.



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Sheaf on a basis

• If $\mathcal{O}(X)$ has a basis \mathcal{B} , then we may define a *sheaf on* \mathcal{B} as a presheaf $\mathcal{B}^{\mathrm{op}} \to Set$ with appropriate equalizers for each covering $B = \cup_i B_i$.

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Theorem 3

The restriction functor $r: \operatorname{Sh}(X) \to \operatorname{Sh}(\mathcal{B})$ is an equivalence of categories.

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Proposition 1

A presheaf F on X is a sheaf if and only if, for every open set $U \subseteq X$ and every covering sieve S on U, the inclusion of functors $\iota_S : S \to \pounds(U)$ induces an isomorphism

$$\operatorname{Hom}(\sharp(U), F) \cong \operatorname{Hom}(S, F).$$



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Proof

Let F be a presheaf. For any open covering $\{U_i\}$ of an open set $U \subseteq X$, we can construct the following equalizer diagram:

$$E \stackrel{e}{\longrightarrow} \prod_{i} FU_{i} \stackrel{\alpha,\beta}{\Longrightarrow} \prod_{i,j} F(U_{i} \cap U_{j})$$

where E contains families of elements $x_i \in FU_i$ such that $x_i|_{U_i \cap U_j} = x_j|_{U_i \cap U_j}$. Now, replace $\{U_i\}$ with the covering sieve S generated by $\{U_i\}$, and define $x_V = x_i|_V$, for families $(x_i) \in \prod_i FU_i$. By assumption, x_V is independent of i. If we let S denote the functor taking $V \mapsto 1$ iff $V \in S$, then each element $x_V \in FV$ is a map $SV \to FV$. The equalizer is thus Hom(S, F).

Diagram:



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Proof Cont.

This lets us construct the commutative diagram:

Hence, $(\iota_s)^*$ is always an isomorphism iff FU is always the equalizer of the top right pair of parallel morphisms.

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Proof.

Let $m: H \to F$ be a subobject of the sheaf F. Recall that m is monic iff H is the pullback of m along itself. By the last theorem, m is monic in the category of presheaves: its components are consequently monic. Therefore, each HU is isomorphic to some subset $SU \subseteq FU$. This allows us to construct the subfunctor S of F which is isomorphic to H, which is also a sheaf.

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Proof.

For an open set $U \subseteq X$, we assign the representable sheaf $\operatorname{Hom}(-,U)$. Conversely, if $S \leq 1$, then assign it the open set $\bigcup \{U \in \mathcal{O}(X) : SU = 1\}$.

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Definition

A topological n-manifold M is a second countable Hausdorff space such that each point q admits an open neighbourhood V homeomorphic to an open set $W \subseteq \mathbb{R}^n$ via a $chart \ \phi: V \to W$. A collection of charts $\{\phi_i: V_i \to W_i\}$ with V_i covering M is an atlas.

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- Given an atlas $\{\phi_i\}$, define $\phi_{ij} = \phi_i|_{V_j}$. The image of ϕ_{ij} is some subset $W_ij \subseteq$. We obtain the transition maps, which are homeomorphisms $_{ij}\phi_{ii}^{-1}:W_{ij}\to W_{ji}$.

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- M can be constructed by pasting together the sets W_i on the subsets W_{ij} using the transition maps: categorically, M is the pushforward in the following diagram, where $\alpha(\beta)$ sends $x_{ij} \in V_i \cap V_j$ to $x_i(x_j)$:

$$\coprod_{i,j} V_i \cap V_j \xrightarrow{\alpha \atop \beta} \coprod_i V_i \longrightarrow M$$



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- Smooth manifolds can be defined using sheaves.
 - ▶ a smooth n-manifold M is a second countable Hausdorff space with a subsheaf $S = S_M$ of the sheaf C_M of continuous functions on M with the property that each point $p \in M$ has an open neighbourhood V such that there is a homeomorphism $\varphi: V \to W \subseteq \mathbb{R}^n$ carrying the sheaf C^k in W isomorphically onto $S|_V$.

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- Smooth manifolds are examples of ringed spaces.
 - ▶ A ringed space X is a topological space with a fixed sheaf R of rings called the structure sheaf, and a morphism $f:(X,R)\to (X',R')$ of ringed spaces is a continuous map $f:X\to X'$ inducing a homomorphism $\alpha:R'\to f_*R$ of sheaves.

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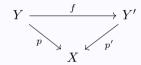
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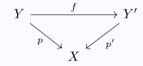


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such that f is continuous.

• $p^{-1}(x)$ is called the fiber of Y over x. A bundle is like a family of fibers continuously indexed by X. For an open subset $U \subseteq X$, any bundle p restricts to a bundle $p_U : p^{-1}U \to U$ over U. p_U is the pullback of p along the inclusion $U \to X$ in Top.



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we have $FU = \coprod_{x \in U} fx$.

▶ Hence, we can define the discrete bundle $p: \coprod_{x \in X} fx \to X$ with the obvious projection. Additionally, FU is the set of sections of p over U.



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- Since there are two maps $Y \times G \to Y$, projection π and right action a, we can construct the map $\theta = \pi \cdot a : Y \times G \to Y \times_{Y/G} Y$. If θ is a homeomorphism, then $Y \times G$ is called the *principle G-bundle*.

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- If $Y \times G$ is principle, then injectivity of θ implies that G acts freely, and surjectivity implies G acts transitively, on Y. Thus, each fiber of p is homeomorphic to G.



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