Condensed Mathematics

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1 Condensed Sets

The goal of this section is to introduce the notion of *condensed set*—a sheaf whose input ranges over the objects in category of totally disconnected *compacta*, or compact Hausdorff spaces. The sheaf condition, then—that compatible local sections can be lifted to a global section—only makes sense once we endow the category with a *Grothendieck topology*, which provides precisely the information of which families of morphisms are "open covers".

1.1 Profinite Sets

We begin by defining three categories, which are then subsequently stated to be equivalent.

Definition 1.1 (Inverse system)**.** Let a poset I be *cofiltered* if it admits finite lower bounds. That is, for every $i, j \in I$, we can find some $k \in I$ such that $k \leq i$ and $k \leq j$. Then, an *inverse system* (X_i, f_{ij}) over a category $\mathcal{C}$ is defined by the data of,

1. A family X_i of objects in $\mathcal{C}$ with $i \in I$.
2. Whenever $i \leq j$, we have a morphism $f_{ij} : X_i \to X_j$.

In short, an inverse system over $\mathcal{C}$ is a functor $I \to \mathcal{C}$ with domain a cofiltered poset I, regarded as a category in the usual way.

Definition 1.2 (Profinite sets)**.** The category ($\mathsf{Pro}(\mathsf{Fin})$ of *profinite sets* can be thought of as the formal completion of the category Fin of finite sets with by taking limits of inverse systems: let $\mathsf{Pro}(\mathsf{Fin}) \hookrightarrow \mathrm{Fun}(\mathsf{Fin}, \mathsf{Set}^{\mathrm{op}})$ be the full subcategory with objects of the form $\varprojlim よ(S_i)$, with (S_i, f_{ij}) an inverse system over Fin, and $よ : \mathsf{Fin} \hookrightarrow$ the Yoneda embedding $S \mapsto \mathrm{Hom}(-, S)$.

Remark. We can compute the set of morphisms $\mathrm{Hom}(\varprojlim_i よ(S_i), \varprojlim_j よ(T_j))$ as follows:

$$\begin{aligned}
\mathrm{Hom}(\varprojlim_i よ(S_i), \varprojlim_j よ(T_j)) &\cong \varprojlim_j \mathrm{Hom}(\varprojlim_i よ(S_i), よ(T_j)) \\
&\cong \varprojlim_j \varinjlim_i \mathrm{Hom}(よ(S_i), よ(T_j)) \\
&\cong \varprojlim_j \varinjlim_i \mathrm{Hom}(S_i, T_j).
\end{aligned}$$

using the fact that the Yoneda embedding is fully faithful.

Definition 1.3. Let $\mathsf{TCHS} \hookrightarrow \mathsf{Top}$ be the full subcategory of totally disconnected *compacta*, or compact Hausdorff spaces. This category can be constructed similar to $\mathsf{Pro}(\mathsf{Fin})$: it is the full subcategory of objects of the form $\varprojlim X_i$, where (X_i, f_{ij}) is an inverse system of discrete finite space. This time, we have a more explicit definition: $\varprojlim X_i$ is a subspace of $\prod X_i$ with the presentation $\{(x_i) : f_{ij}(x_j) = x_i\}$.

Definition 1.4. Let $\mathsf{BoolRing} \hookrightarrow \mathsf{Ring}$ be the full subcategory of Boolean (unital) rings—rings which model the axiom additional axiom $\forall x . x^2 = x$.

Theorem 1.1 (Stone Duality)**.** *There is an equivalence between the categories* $\mathsf{Pro(Set)}$*; the full subcategory* $\mathsf{TCHS} \hookrightarrow \mathsf{Top}$ *of totally disconnected compacta, or compact Hausdorff spaces; and the full subcategory* $\mathsf{BoolAlg}^{\mathrm{op}} \hookrightarrow \mathsf{Ring}^{\mathrm{op}}$*, the opposite category of Boolean algebras, via the following pair of functors:*

$$\begin{array}{ccccc} \mathsf{Pro(Set)} & \longrightarrow & \mathsf{TDCH} & \longrightarrow & \mathsf{BoolAlg}^{\mathrm{op}} \\ & & & & \\ \varprojlim \text{よ}(S_i) & \longmapsto & \varprojlim S_i & \longmapsto & \mathrm{Cont}(S, \mathbb{F}_2) \cong \varinjlim \mathbb{F}_2^{S_i} \end{array}$$

Note that $\mathbb{F}_2$ *is equipped with the discrete topology.*

Definition 1.5 (Notions of grandeur)**.** Given a profinite set $S = \varprojlim S_i$, we have the following definitions:

1. The *size* of S is $\kappa := |S|$,
2. The *weight* of S is $\lambda := |\mathrm{Cont}(S, \mathbb{F}_2)|$,
3. S is *light* if $\lambda \leq \omega$.

Example 1.1. Some immediate examples of profinite sets are:

- The one point compactification $\mathbb{N} \cup \{\infty\} = \varprojlim_n \{0, \dots, n, \infty\}$, where the transition maps $\{\dots, n+k, \infty\} \to \{\dots, n, \infty\}$ are defined by collapsing all $m > n$ to ∞, and by identity on $m < n$. Here, $\kappa = \omega = \lambda$.
- The Cantor set $\{0,1\}^{\mathbb{N}} = \varprojlim_n \{0,1\}^n$. $\kappa = 2^\omega$ and $\lambda = \omega$. The Cantor is an important light profinite set: we can always find a continuous surjection $\{0,1\}^{\mathbb{N}} \to S$ onto any light profinite S, so the Cantor set is a generating object of the category $\mathsf{Pro(Fin)}^{\mathsf{Light}}$.
- The Stone-Cêch compactification $\beta\mathbb{N}$. By the universal property, we have $\mathrm{Cont}(\beta\mathbb{N}, \mathbb{F}_2) \cong \mathrm{Cont}(\mathbb{N}, \mathbb{F}_2)$, so $\lambda = 2$; and, though it is not *a priori* clear, $\kappa = 2^{2^\omega}$.

For a multitude of reasons, we should instead restrict to the light objects in our study of sheaves on profinite sets.

Theorem 1.2 (Light Stone Duality)**.** *The following categories are equivalent:*

$$\left\{\begin{array}{c}\text{Countable projective} \\ \text{limits over } \mathsf{Fin}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{Metrizable totally} \\ \text{disconnected compacta}\end{array}\right\} \longrightarrow \{\text{Countable Boolean algebras}\}^{op}$$

Corollary 1.2.1. *The category of light profinite sets has countable limits, and sequential limits of surjections are surjective.*

Proof. The first statement follows from the fact that $\varprojlim_i \varprojlim_j S_{i,j} \cong \varprojlim_{(i,j)} S_{i,j}$, where (i,j) has the lexicographical order. To show the second statement, recall that the topological limit $X := \varprojlim_i X_i$ exists for any sequence of *stone spaces* (totally disconnected compactum) X_i, and has the underlying set $\{(x_n) \in \prod_n X_n : f_{ij}(x_j) = x_i\}$. The natural projection $X \to X_0$ is then certainly a surjection. It remains to note that X is (1) totally disconnected since it embeds into a product of totally disconnected spaces and (2) compact Hausdorff since the forgetful functor $\mathsf{CompHaus} \hookrightarrow \mathsf{Top}$ preserves limits. ∎

Theorem 1.3. *Let S be a light profinite set. Then there exists a surjection $\{0,1\}^{\mathbb{N}} \to S$ from the Cantor set.*

Proof. We can construct such a surjection inductively. Without loss of generality, let S be the limit of a diagram $\cdots \twoheadrightarrow S_2 \twoheadrightarrow S_1 \twoheadrightarrow S_0$ of finite discrete spaces S_i with maps $g_1, g_2, \dots$ all surjective. Assume inductively that there is some finite quotient $\{0,1\}^{\mathbb{N}} \twoheadrightarrow \{0,1\}^{k_n}$ of the cantor set and a cover $f_n : \{0,1\}^{k_n} \twoheadrightarrow S_n$. Pick some $k_{n+1} > k_n$ such that $2^{k_{n+1}-k_n} \geq |S_{n+1}|$. Intuitively, we need enough additional digits to give a binary encoding of each $m \in S_{n+1}$, and this encoding should be compatible with our prior choices: dropping the additional $2^{k_{n+1}-k_n}$ digits of an encoding of m automatically gives us an encoding of $g_{n+1}(m)$.

$$\begin{tikzcd}
 & \{0,1\}^{k_{n+1}} \arrow[r] & S_{n+1} \arrow[d, two heads] \\
\{0,1\}^{\mathbb{N}} \arrow[ur] \arrow[r, two heads] & \{0,1\}^{k_n} \arrow[r, two heads] & S_n
\end{tikzcd}$$

Thus, we can define the cover $\{0,1\}^{k_{n+1}} \twoheadrightarrow S_{n+1}$ as follows: for any $s \in \{0,1\}^{k_n}$, we may define f_{n+1} on the subset $\{t \in \{0,1\}^{k_{n+1}} : t = s * t'\}$ of k_{n+1} size strings with prefix s, as any arbitrarily chosen cover $2^{k_{n+1}-k_n} \twoheadrightarrow g_{n+1}^{-1}(f_n(s)) \subseteq S_{n+1}$. This concludes the inductive step. ∎

Theorem 1.4. *Let S be a light profinite set. Then S is an injective object in* $\mathsf{Pro(Fin)}$. *That is, the following commutative diagram of profinite sets can always be completed:*

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, hook] & S \\
X \arrow[ur, dashed, "\exists \tilde{f}"'] &
\end{tikzcd}$$

Proof. Let us consider the case where $S = 2$ is the set of two points, so that Z is the disjoint union of clopen sets $C \sqcup C^c$, where $C = f^{-1}(0)$. Then, constructing $\tilde{f} : X \to 2$ amounts to choosing the fibers of $\tilde{f}$ in a way that is compatible with f: we must find a clopen set $\tilde{C} \subseteq X$ such that $\tilde{C} \cap Z = C$ (and hence $\tilde{C}^c$ is a clopen set such that $\tilde{C}^c \cap Z = C^c$). First, C is open in the subspace topology, so we can find an open set $C' \subseteq X$ such that $\tilde{C} \cap X = C$. Since profinite sets admit a base of clopen sets, we can write $\tilde{C} = \bigcup_i K_i$ as a union of clopen sets. Then, $\{K_i\}$ is an open cover of the compact set C, so we may pick a finite family $\{\tilde{K}_i\}$ of clopen sets covering C. Finally, let $\tilde{C}$ be the clopen set $\bigcup \tilde{K}_i$. We verify $\tilde{C} \cap Z = C$: $\tilde{C} \cap Z \supseteq C$ (since $\tilde{C}$ covers C), and $\tilde{C} \cap Z \subseteq C' \cap Z = C$.

In the general case, consider a map $f : Z \to S := \varprojlim S_n$, and define $f_n := \pi_n f$, where $\pi_n : S \to S_n$ is the natural projection. Assume without loss of generality that all the transition maps $\varphi_n : S_{n+1} \to S_n$ are surjective—thus, φ_n can be factored as the iterated composition of maps of the form $\{1, \dots, m+1\} \to \{1, \dots, m\}$ collapsing two points at a time, so it suffices to only consider transition maps of this form. Our goal is to inductively construct maps $\tilde{f}_{m+1} : X \to S_{m+1}$ so that (1) $\varphi_m \circ \tilde{f}_{m+1} = \tilde{f}_m$ and (2) $\tilde{f}_{m+1}|_Z = f_{m+1}$. Pictorially, we need to inductively complete the following commutative diagram:

$$\begin{tikzcd}
 & & \{1, \dots, m+1\} \arrow[d, "i \mapsto \min\{i,m\}"] \\
Z \arrow[r, hook, "\iota"] \arrow[urr, "f_{m+1}"] \arrow[drr, "f_m"'] & X \arrow[ur, dashed, "\exists \tilde{f}_{m+1}"] \arrow[dr, "\tilde{f}_m"] & \\
 & & \{1 \dots m\}
\end{tikzcd}$$

Once we have such a family $\{\tilde{f}_m : X \to S_m\}$, property (1) and the universal property of S guarantees the existence of a map $\tilde{f} : X \to S$ such that

$$\begin{aligned}
\pi_m \circ \tilde{f} &= \tilde{f}_m \\
\pi_m \circ \tilde{f} \circ \iota &= \tilde{f}_m \circ \iota = f_m && \text{(for all } m\text{, by property (2))}
\end{aligned}$$

But by the universal property of S, f is the unique map satisfying $\pi_m \circ f = f_m$ for all m, which means we must have $f = \tilde{f} \circ \iota$, which is exactly what we needed from $\tilde{f}$.

Now, it remains to construct f_{m+1}. Since the fibers of f_m and f_{m+1} over $\{1, \dots, m-1\}$ are identical, we can choose the fibers of $\tilde{f}_{m+1}$ over the set $\{1, \dots, m-1\}$ to be identical with the fibers of $\tilde{f}_m$. Thus, we are left to partition the remaining elements $X' = X \setminus \tilde{f}_m^{-1}(\{1, \dots, m-1\})$ into fibers of $\tilde{f}_{m+1}$ over $\{m, m+1\}$ which are compatible with f_{m+1}, but this is proven by our first case.

$$\begin{tikzcd}
 & & \{m, m+1\} \arrow[d, dotted] \\
f_{m+1}^{-1}(m) \sqcup f_{m+1}^{-1}(m+1) \arrow[r, hook] \arrow[urr, "f_{m+1}"] \arrow[drr, dotted] & X' \arrow[ur, dashed, "\exists \tilde{f}_{m+1}"'] \arrow[dr, dotted] & \\
 & & *
\end{tikzcd}$$

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1.2 Grothendieck Topologies

What is the most natural way to extend the notion of a topology so that the objects are no longer open sets $U \hookrightarrow X$ in a topological spaces, but general morphisms $X \to Y$ in a category? An idea for a generalized topology comes from the axioms required of a sheaf: we precisely need to specify what it means for a family of morphisms $\{f_i : X_i \to Y\}$ to cover Y, so that we can formulate the sheaf condition of allowing compatible local sections to be uniquely lifted to a global section. The basic operation of this newly defined algebra will be the pullback (restriction) $\{Z \to X_i \to Y\}$ of a covering family $\{X_i \to Y\}$. Before we discuss

Grothendieck topologies, we introduce the convenient notion of a *sieve* of morphisms, which shares a relation with the notion of a family of morphisms analogous to that of topologies and topological bases. The former is easier to work with at times due to the convenience of its closure properties.

Definition 1.6 (Sieve)**.** In a category $\mathcal{C}$, a *sieve* on an object C is a *right ideal* of morphisms with codomain C with respect to composition: a sieve is a set $\mathcal{S} := \{f : D_f \to C\}$, such that if $f \in \mathcal{S}$ and $g : E \to D_f$ is any morphism, then we also have $fg \in \mathcal{S}$. We may also speak of a *sieve generated by the morphism* $f : D \to C$, which is the family $\{g : E \to D \xrightarrow{f} C\}$ of all morphisms with any domain that factor through f (as an exercise, check that this is is a sieve).

Definition 1.7. A *Grothendieck topology* on a category $\mathcal{C}$ is a functor J which assigns to each object C of C a collection $J(C)$ of sieves on C, called *covering sieves*, such that:

1. *Identity Axiom.* $J(C)$ contains the maximal sieve $\{f : D \to C : D \in \mathrm{Obj}(\mathsf{C})\}$.
2. *Stability Axiom.* If $\mathcal{S} \in J(C)$ is a covering sieve on C and $h : D \to C$ is any morphism, then the pullback $h^*(\mathcal{S}) := \{f : E \to D : h \circ f \in \mathcal{S}\} \in J(D)$ is a covering sieve on D.
3. *Transitivity Axiom.* If $\mathcal{S} \in J(C)$ is a covering sieve, and $\mathcal{R}$ is any sieve on C such that $h^*(\mathcal{R}) \in J(D)$ for all $h : D \to C \in S$, then $\mathcal{R} \in J(C)$ is also a covering sieve.

We shall call a category $\mathcal{C}$ endowed with a Grothendieck topology J, a *Grothendieck site*, or *site* for short.

Example 1.2. It would be ideal to see a demonstration that the axioms of a Grothendieck topology capture the usual notion of open covers. Let $\mathcal{S}(M)$ denote the smallest sieve containing a family M of morphisms with a common codomain X, or the *sieve generated by* M. If $\mathcal{S}(M)$ is a covering sieve, then we will say M is a cover. We will work inside the poset category $\mathcal{O}(S)$ whose morphisms are inclusions $U \subseteq V$ of opens sets U, V of a topological space S. In this context, such a family M would be a set of open subsets of U, and $\mathcal{S}(M)$ entails closure under taking open subsets $U \subseteq V$ of elements $V \in M$ (we say that $\mathcal{S}(M)$ is *downward closed* with respect to inclusion). We define the functor $J : \mathcal{O}(S) \to \mathsf{Set}$ by $J(U) = \{\mathcal{S}(\{U_i \hookrightarrow U\}) : \bigcup_i U_i = U\}$. Now, we verify that the three Grothendieck topology axioms hold.

1. Clearly $J(V)$ contains the maximal sieve $\{W \subseteq V : W \text{ open}\}$ since $\bigcup_{W \subseteq V \text{ open}} W = V$ by virtue of the family containing V itself.
2. Verifying the stability axiom amounts to noticing that if $\mathcal{S}$ is a sieve in V, and $U \subseteq V$, then $[U \subseteq V]^*(\mathcal{S}) = \{W \subseteq U : W \subseteq V \in \mathcal{S}\}$ is just the refinement of $\mathcal{S}$ to opens which are also contained in U, which by downward closure must equal $\{W \cap U : W \subseteq V \in \mathcal{S}\}$. Indeed, if $\mathcal{S}$ covers V, then $\{W \cap U : W \in \mathcal{S}\}$ must cover U.
3. Let us unravel the transitivity axiom in this context: let $\mathcal{S}$ be an open cover(ing sieve) of V and let $\mathcal{R}$ be a sieve of opens of V such that, for every open $U \subseteq V \in \mathcal{S}$, the restriction $\{W \cap U : W \in \mathcal{R}\}$ covers U. Then it should follow that $\mathcal{R}$ also covers the union $\bigcup_{\mathcal{S}} U = V$.

1.3 Light Condensed Sets

Definition 1.8. A *light condensed set* is a sheaf on the site $\mathsf{Prof(Set)}^{\mathsf{Light}}$ equipped with the coherent topology, which is the smallest topology containing the sieves generated by (1) finite disjoint unions and (2) all surjective maps. That is, the topology Coh(−) must satisfy

- $\mathcal{S}(\{S_1 \to S_1 \coprod S_2, S_2 \to S_1 \coprod S_2\}) \in \mathrm{Coh}(S_1 \sqcup S_2)$.
- $\mathcal{S}(\{T \twoheadrightarrow S\}) \in \mathrm{Coh}(S)$.

Equivalently, a light condensed set is given by a functor,

$$\mathsf{Pro}_{\mathbb{N}}(\mathsf{Fin})^{\mathsf{op}} \longrightarrow Set$$

$$S \longmapsto X(S)$$

such that,

1. $X(\emptyset) = *$,
2. $X(S_1 \sqcup S_2) \xrightarrow{\sim} X(S_1) \times X(S_2)$,
3. For all $T \twoheadrightarrow S$, the following diagram is an equalizer

$$X(S) \longrightarrow X(T) \underset{X[\pi_2]}{\overset{X[\pi_1]}{\rightrightarrows}} X(T \times_S T)$$

   where π_i are the canonical projections associated to $T \times_S T$.

Example 1.3. Any topological space A defines a light condensed set $\underline{A}$ via $\underline{A}(S) \cong \mathrm{Cont}(S, A)$. Immediately there are some distinguished values of A, namely:

- Its underlying set $\underline{A}(*) \cong \mathrm{Cont}(*, A)$,
- The set $\underline{A}(\mathbb{N} \cup \{\infty\}) \cong \mathrm{Cont}(\mathbb{N} \cup \{\infty\}, A)$ of pairs convergent sequences along with the witness of a limit point,

These notions generalize to any light condensed set X, so that we may call $X(*)$ the *underlying set of* X, or $X(\mathbb{N} \cup \{\infty\})$ the *convergent sequences in* X.

Example 1.4. Any light condensed set X is essentially determined by an abstract set $X(2^{\mathbb{N}})$ along with an action of $\mathrm{End}(2^{\mathbb{N}})$—there is a fully faithful embedding $CondSet^{Light} \hookrightarrow \mathrm{End}(2^{\mathbb{N}})\ \mathsf{Set}$. To compute $X(S)$, take a surjection $2^{\mathbb{N}} \hookrightarrow S$ and apply the sheaf condition to compute $X(S) \hookrightarrow X(2^{\mathbb{N}})$ as the equalizer of $X(S) \rightrightarrows X(\hookrightarrow 2^{\mathbb{N}} \times_S 2^{\mathbb{N}})$. We can then cover $2^{\mathbb{N}} \hookrightarrow 2^{\mathbb{N}} \times_S 2^{\mathbb{N}}$, which again computes $X(2^{\mathbb{N}} \times_S 2^{\mathbb{N}}) \overset{\iota}{\hookrightarrow} X(2^{\mathbb{N}})$ as the equalizer of some diagram. We get the composite diagram:

$$\begin{tikzcd} X(S) \arrow[r, hook] & X(2^{\mathbb{N}}) \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] \arrow[rr, bend left, "\iota\circ f_1"] \arrow[rr, bend right, "\iota\circ f_2"'] & X(2^{\mathbb{N}} \times_S 2^{\mathbb{N}}) \arrow[r, hook, "\iota"] & X(2^{\mathbb{N}}) \end{tikzcd}$$

Since ι is an injection, $X(S)$ is again the equalizer of $\iota \circ f_1$ and $\iota \circ f_2$ (recall the definition of equalizers in Set), but this is computable from the knowledge of the morphisms $\iota \circ f_1, \iota \circ f_2$ and $X(2^{\mathbb{N}})$.

Example 1.5. There is an adjunction

$$\begin{tikzcd} \mathsf{Top} \arrow[r, bend right, "A\mapsto \underline{A}"'] \arrow[r, phantom, "\perp"] & \mathsf{CondSet}^{\omega} \arrow[l, bend right, "X\mapsto X(*)_{\mathrm{top}}"'] \end{tikzcd}$$

where $X(*)$ is given the quotient topology generated by the following map:

$$\bigsqcup_{\alpha \in X(2^{\mathbb{N}})} 2^{\mathbb{N}}(*) \to X(*)..$$

(Here, $2^{\mathbb{N}}(*)$ is just the space $2^{\mathbb{N}}$, and α is naturally an element of $\mathrm{Cont}(2^{\mathbb{N}}, X)$)

It is clear that $\underline{S}(*)$ gives us back the original space S as a quotient of the Cantor set.

Theorem 1.5. *$X(*)_{\mathrm{top}}$ is metrizably compactly generated, and we have, for all metrizably compactly generated space A, $A \cong \underline{A}(*)_{\mathrm{top}}$. As a corollary, we have an embedding*

$$\{\text{Metrizably compactly generated spaces}\} \longrightarrow \mathsf{CondSet}^{\mathsf{Light}}$$

Example 1.6. Sheaves of abelian groups on any site always form a *Grothendieck abelian category*—in particular, (co)limits exist, and filtered colimits are always exact. Thus, $\mathsf{CondAb}^{\mathsf{Light}}$ is a Grothendieck abelian category. Recall that this does not hold for the category of topological abelian groups precisely for the reason that (co)kernels are not well behaved. For example, we might expect the failure of the injections $\mathbb{Q} \hookrightarrow \mathbb{R}$ and $\mathbb{R}_{\mathrm{disc}} \hookrightarrow \mathbb{R}$ to be isomorphisms, to be captured by nontrivial kernels. However, in both cases the cokernel is 0, since any continuous homomorphism $\mathbb{R} \to A$ which vanishes on $\mathbb{Q}$ or $\mathbb{R}_{\mathrm{discr}}$ must factor uniquely through 0. Contrarily, the corresponding cokernels in $\mathsf{CondAb}^{\mathsf{Light}}$ are nontrivial and have straightforward descriptions. In the case of $\underline{\mathbb{Q}}$, we have

$$(\underline{\mathbb{R}}/\underline{\mathbb{Q}})(*) = \mathbb{R}/\mathbb{Q} \cong \mathbb{R}, \qquad (\underline{\mathbb{R}}/\underline{\mathbb{Q}})(S) = \mathrm{Cont}(S, \mathbb{R})/\mathrm{Cont}_{\mathrm{loc\ const}}(S, \mathbb{Q}).$$

It is not automatically easy to take quotients: for condensed abelian groups X, Y, the quotient $(X/Y)(S)$ is computed as the sheafification $(X(S)\widehat{/}Y(S))$. The point of interest is the observation by Scholze that after sheafifiying, the resulting condensed Abelian group is a quotient of maps $S \to \mathbb{R}$ by locally constant maps $S \to \mathbb{Q}$, instead of by all maps $S \to \mathbb{Q}$.

2 Light Condensed Abelian Groups

2.1 Free Condensed Abelian Groups

The central construction considered in this section is the functor $X \mapsto \mathbb{Z}[X]$ that naturally assigns, to every light condensed set X, a light condensed abelian group $\mathbb{Z}[X]$. $\mathbb{Z}[X]$ is the smallest abelian group object one can construct from X, and one can recover topological properties of the latter from the former via a natural embedding $X \hookrightarrow \mathbb{Z}[X]$. In general, the idea is to view the object $\mathbb{Z}[X]$ as putting a topology on the actual free abelian group $\mathbb{Z}[X(*)]$.

Theorem 2.1. *The inclusion* $\mathsf{CondAb}^{\mathsf{light}} \to \mathsf{CondSet}^{\mathsf{light}}$ *has a left adjoint* $X \mapsto \mathbb{Z}[X]$*, where* $\mathbb{Z}[X]$ *is given by the sheafification of the presheaf* $S \mapsto \mathbb{Z}[X(S)]$*.*

Lemma 2.2. *If* $c : A \to C$*,* $d : B \to C$ *are maps of light profinite sets such that* $\{a, b\}$ *is a cover, then the corresponding pullback diagram in* $\mathsf{CondSet}^{Light}$ *is also a pushforward:*

$$\begin{tikzcd} \underline{A \times_C B} \arrow[r, "d"] \arrow[d, "c"] & \underline{B} \arrow[d, "b"] \\ \underline{A} \arrow[r, "a"'] & \underline{C} \end{tikzcd}$$

Proof. Let X be a light condensed set. We compute:

$$\begin{aligned} \operatorname{Hom}(\underline{C}, X) &\cong X(C) && (\text{よ}) \\ &\cong \varprojlim [X(A) \times X(B) \overset{X[a]\circ\pi_2}{\underset{X[b]\circ\pi_2}{\rightrightarrows}} X(A \times_C B)) && (\text{Sheaf definition}) \\ &\cong \{\langle f, g\rangle \in X(A) \times X(B) : X[a](f) = X[b](g)\} \\ &\cong \{\langle f, g\rangle \in \operatorname{Hom}(\underline{A}, X) \times \operatorname{Hom}(\underline{B}, X) : f_{A\times_C B}(a) = g_{A\times_C B}(b)\} \\ & \qquad (\text{Considering } a, b \text{ as elements of } \underline{A}(A \times_C B), \underline{B}(A \times_C B)) \\ &\cong \{\langle f, g\rangle \in \operatorname{Hom}(\underline{A}, X) \times \operatorname{Hom}(\underline{B}, X) : f \circ a = g \circ b\}. \end{aligned}$$

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Theorem 2.3. *The follow results hold in the subcategory* $\mathsf{CondAb}^{\mathsf{light}}$*.*

1. *Countable products are exact.*[1]

2. *Sequential limits of surjective maps are surjective. That is, the natural map* f *below is a surjection:*

$$M_\infty \cong \varprojlim(\dots \twoheadrightarrow M_2 \twoheadrightarrow M_1 \twoheadrightarrow M_0) \xrightarrow{f} M_0$$

3. $\mathbb{Z}[\underline{\mathbb{N} \cup \{\infty\}}]$ *is projective. Furthermore, it is internally projective in that* $\underline{\operatorname{Ext}}^i(\mathbb{Z}[\mathbb{N} \cup \{\infty\}], -) = 0$ *for all* $i > 0$*, where we ought to define* $\underline{\operatorname{Ext}}^i(M, N)$ *as the sheafification of the presheaf* $S \mapsto \operatorname{Ext}^i(M \otimes \mathbb{Z}[\underline{S}], N)$*.*

[1] This is axiom AB_4^* of Grothendieck's axioms.

Proof of claims 1 and 2. To show the first claim, we must show for any family of surjections $f_n : M_n \twoheadrightarrow N_n$ that $\prod f_n : \prod M_n \to \prod N_n$ is surjective. In fact, this reduces to the second claim by observing that $\prod M_n \to \prod N_n$ surfaces as the limit of the diagram:

$$\cdots \longrightarrow \prod_{n\le m} M_n \times \prod_{n>m} N_n \xrightarrow{\prod \mathrm{id} \times f_m \times \prod \mathrm{id}} \prod_{n\le m-1} M_n \times \prod_{n>m-1} N_n \longrightarrow \cdots$$

We now show claim 2. Recalling the definition of surjection in a Grothendieck topos, to say that $M_\infty \to M_0$ is surjective equates to saying for any $S_0 \in \mathsf{Pro(Fin)}^{\mathsf{Light}}$, we can find some surjection $S_\infty \twoheadrightarrow S_0$ such that the induced $M_\infty(S_\infty) \to M_0(S_0)$ is a surjection. By the Yoneda lemma, this is equivalent to saying that the following commutative diagram can be completed:

$$\begin{tikzcd}
M_\infty \arrow[r] & M_0 \\
\exists \underline{S_\infty} \arrow[u, dashed] \arrow[r, dashed, two heads] & \underline{S_0} \arrow[u]
\end{tikzcd}$$

To do this, we can apply the same surjective condition repeatedly to each surjection $M_n \to M_{n-1}$ to produce surjections $S_n \twoheadrightarrow S_{n-1}$ organized in the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & M_2 \arrow[r, two heads] & M_1 \arrow[r, two heads] & M_0 \\
\cdots \arrow[r, two heads] & \underline{S_2} \arrow[u] \arrow[r, two heads] & \underline{S_1} \arrow[u] \arrow[r, two heads] & \underline{S_0} \arrow[u]
\end{tikzcd}$$

Now, let S_∞ be the limit $\varprojlim S_n$. By Corollary 1.2.1, the natural map $S_\infty \to S_0$ is a surjection. Then, let the map $g : S_\infty \to M_\infty$ be the limit of the compositions $S_\infty \to S_i \to M_i$ of the natural projection $S_\infty \to S_i$ with the maps $S_i \to M_i$ in the diagram above. By the universal property of M_∞, the composition $S_\infty \to M_\infty \to M_0$ must equal the 0^{th} map in the diagram defining f, which is $S_\infty \to S_0 \to M_0$. Thus, we have proved claim 2.

■

Remark. Before we prove claim 3, note that $\underline{\mathbb{N} \cup \{\infty\}}$ is not projective in $\mathsf{CondSet}^{\mathsf{Light}}$. A simple example is given by the lack of lifting in the diagram below:

$$\begin{tikzcd}
 & \underline{[2\mathbb{N} \cup \{\infty\}] \sqcup [(2\mathbb{N}+1) \cup \{\infty\}]} \arrow[d, two heads] \\
\underline{\mathbb{N} \cup \{\infty\}} \arrow[ur, "\nexists"'] \arrow[r, "\sim"'] & \underline{\mathbb{N} \cup \{\infty\}}
\end{tikzcd}$$

If such a factorization existed, then one would exist on the level of spaces, by passing to $\mathsf{Prof(Set)}^{\mathsf{Light}}$ via the functor $X \mapsto X(*)$. However, such a continuous map does not exist.

The proof of claim 3 reduces to a stronger statement: that the direct summand $\mathbb{Z}[\underline{\mathbb{N} \cup \{\infty\}}]/\mathbb{Z}[\underline{\infty}]$ is internally projective. One should think of morphisms $\mathbb{Z}[\underline{\mathbb{N} \cup \{\infty\}}]/\mathbb{Z}[\underline{\infty}] \to A$ as convergent sequences in A.

Theorem 2.4. *$M := \mathbb{Z}[\underline{\mathbb{N} \cup \{\infty\}}]/\mathbb{Z}[\underline{\infty}]$ is projective in $\mathsf{Cond(Ab)}^{\mathsf{Light}}$. That is, we can always complete the follow commutative diagram:*

$$\begin{tikzcd} & \tilde{N} \arrow[d, two heads, "c"] \\ M \arrow[ur, dashed, "\exists \tilde{f}"] \arrow[r, "f"'] & N \end{tikzcd}$$

Proof. Let $d := \pi_M \circ \eta_{\underline{\mathbb{N}\cup\{\infty\}}}$ be the composite of the unit $\eta_{\underline{\mathbb{N}\cup\{\infty\}}} : \underline{\mathbb{N}\cup\{\infty\}} \to \mathbb{Z}[\underline{\mathbb{N}\cup\{\infty\}}]$ associated to the free functor $\mathbb{Z}[\cdot]$, with the quotient map $\pi_M : \mathbb{Z}[\underline{\mathbb{N}\cup\{\infty\}}] \to M$. Notably, d restricts to the map $\underline{\{\infty\}} \hookrightarrow \mathbb{Z}[\underline{\infty}] \to 0 \subseteq M$. We now use the fact that c is a surjection: this allows us to find some cover $S' \hookrightarrow \mathbb{N}\cup\{\infty\}$ with S' a light profinite set, so that there is a commutative square:

$$\begin{tikzcd} \underline{S'} \arrow[rr] \arrow[d, two heads] & & \tilde{N} \arrow[d] \\ \underline{\mathbb{N}\cup\{\infty\}} \arrow[r, "d"] & M \arrow[r, "f"] & N \end{tikzcd}$$

We may then consider a subcovering $S \twoheadrightarrow \mathbb{N}\cup\{\infty\}$ with singleton fibers over $\mathbb{N} \subseteq \mathbb{N}_\infty$, so that $S \times_{\mathbb{N}\cup\{\infty\}} \mathbb{N} \cong \mathbb{N}$—this formula also holds on the level of representable sheaves since the Yoneda embedding preserves pullbacks. This leaves us with the following commutative diagram:

$$\begin{tikzcd} \underline{S} \arrow[r, dashed] \arrow[dr, dashed, two heads, "h"'] \arrow[rrr, bend left, "g"] & \underline{S'} \arrow[rr] \arrow[d, two heads] & & \tilde{N} \arrow[d, "c"] \\ & \underline{\mathbb{N}\cup\{\infty\}} \arrow[r, "d"] & M \arrow[r, "f"] & N \end{tikzcd}$$

Define $S_\infty := S \times_{\mathbb{N}\cup\{\infty\}} \{\infty\}$ to be the fiber of S over ∞. Applying Theorem 1.6 (light profinite sets are injective) to the diagram $S \overset{\iota}{\hookleftarrow} S_\infty \overset{\mathrm{id}}{\to} S_\infty$ gives a retraction $r : S \to S_\infty$, i.e. $r \circ \iota = \mathrm{id}$. Observe then that $g - g\iota r : S \to \tilde{N}$ vanishes over S_∞ (that is, $(g - g\iota r)\iota = 0$). Expressing this vanishing as a commutative square in $\mathsf{CondSet}^{\mathsf{Light}}$ yields the commutative diagram:

$$\begin{tikzcd} \underline{S} \arrow[rr, "g-g\iota r"] & & \tilde{N} \\ \underline{S_\infty} \arrow[u, hook] \arrow[r] & \underline{\{\infty\}} \arrow[r] & 0 \arrow[u, hook] \end{tikzcd}$$

We now use Lemma 1.2: the following diagram is a pullback of light condensed sets where the bottom and right maps jointly cover $\underline{\mathbb{N}\cup\{\infty\}}$. Thus, the diagram is also a pushout.

$$\begin{tikzcd} \underline{S_\infty} \arrow[r] \arrow[d] & \underline{S} \arrow[d] \\ \underline{\infty} \arrow[r] & \underline{\mathbb{N}\cup\{\infty\}} \end{tikzcd}$$

Thus, the morphisms $\underline{S} \to \tilde{N} \leftarrow \underline{\{\infty\}}$ must factor through $\underline{\mathbb{N}\cup\{\infty\}}$

$$\begin{tikzcd}
\underline{S} \arrow[r, dashed, two heads] \arrow[rr, bend left, "g - g\iota r"] & \underline{\mathbb{N} \cup \{\infty\}} \arrow[r, dashed] & \tilde{N} \\
\underline{S_\infty} \arrow[u, hook] \arrow[r] \arrow[rr, bend right] & \{\infty\} \arrow[u, hook, dashed] \arrow[r] & 0 \arrow[u, hook]
\end{tikzcd}$$

The inner right square of the diagram above is exactly what we need. By the universal property of the free functor $\mathbb{Z}[\cdot]$, we get a corresponding commutative square with $\mathbb{Z}[\underline{\mathbb{N} \cup \{\infty\}}] \to \tilde{N}$ lifting $\mathbb{Z}[\underline{\{\infty\}}] \to 0$. The top map must then factor through a unique map $\tilde{f} : M \to \tilde{N}$, like so:

$$\begin{tikzcd}
\mathbb{Z}[\underline{S}] \arrow[r, dashed, two heads] \arrow[rr, bend left, "\mathbb{Z}[g - g\iota r]"] & \mathbb{Z}[\underline{\mathbb{N} \cup \{\infty\}}] \arrow[r] \arrow[dr, dashed, two heads] & \tilde{N} \\
& & M \arrow[u, "\tilde{f}"] \\
\mathbb{Z}[\underline{S_\infty}] \arrow[uu, hook] \arrow[r] \arrow[rr, bend right] & \mathbb{Z}[\underline{\{\infty\}}] \arrow[uu, hook] \arrow[r] & 0 \arrow[uu, hook]
\end{tikzcd}$$

Now, it remains to show that $\tilde{f}$ indeed lifts f. We can instead precompose by the surjection $\mathbb{Z}[\underline{S}] \xrightarrow{\mathbb{Z}[h]} \mathbb{Z}[\underline{\mathbb{N} \cup \{\infty\}}] \xrightarrow{\hat{d}} M$, which is the dashed morphism in the diagram above (we will let $\hat{d}$ denote the adjunct of d with respect to $\mathbb{Z}[\cdot]$), and show that the new map $\mathbb{Z}[\underline{S}] \to M \xrightarrow{\tilde{f}} \tilde{N}$ lifts $\mathbb{Z}[\underline{S}] \to M \xrightarrow{f} N$. In order words, we conclude by checking that the outer square commutes in the diagram:

$$\begin{tikzcd}
\mathbb{Z}[\underline{S}] \arrow[r, dashed, "\mathbb{Z}[g - g\iota r]"] \arrow[d, two heads, "\hat{d} \circ \mathbb{Z}[h]"'] & \tilde{N} \arrow[d, "c"] \\
M \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & N
\end{tikzcd}$$

Indeed, the top triangle commutes, and the bottom triangle commutes by the calculus of adjoints:

$$\begin{aligned}
c \circ [g \mathbin{\hat{-}} g\iota r] &= [cg \mathbin{\hat{-}} cg\iota r] \\
&= [cg - \hat{f} dh\iota r] && \text{(by commutativity of the second diagram in the proof)} \\
&= [cg - \hat{f} d(\infty)] \\
&= [cg \mathbin{\hat{-}} 0] \\
&= [f\hat{d}h] \\
&= f \circ \alpha(d) \circ \mathbb{Z}[h].
\end{aligned}$$

■

Proof of claim 3. Follows by the fact that $\mathbb{Z}[\underline{\mathbb{N}\cup\{\infty\}}] = \mathbb{Z}[\underline{\mathbb{N}\cup\{\infty\}}]/\mathbb{Z}[\underline{\infty}] \oplus \mathbb{Z}[\underline{\infty}]$, which are both projective by Theorem 1.4. ■