For the reaction

$$A + B \underset{}{\overset{k_a}{\rightleftharpoons}} AB \tag{1}$$

we can calculate k_d as follows:

$$\frac{[A][B]}{[AB]} = \frac{1}{k_a} = k_d \tag{2}$$

Using

$$[A_0] = [A] + [AB] \tag{3}$$
$$[B_0] = [B] + [AB] \tag{4}$$

we rewrite eqn 2 so that it can be solved with the square (ABC) formula:

$$\frac{([A_0] - [AB])([B_0] - [AB])}{[AB]} = k_d \tag{5}$$
$$[A_0][B_0] - [AB]([A_0] + [B_0]) + [AB]^2 = k_d[AB] \tag{6}$$
$$[AB]^2 - [AB]([A_0] + [B_0] + k_d) + [A_0][B_0] = 0 \tag{7}$$

Percentage bound is:

$$\frac{[AB]}{[A_0]} \cdot 100\% \tag{8}$$

Determine k_a

According to paper Indyk (1998).

Protein, M_T is now P_0. Ligand, L_T is now L_0. The binding constant k_b is now k_a.

Reaction

$$P + L \overset{k_a}{\rightleftharpoons} PL \tag{9}$$

with binding constant k_a

$$\frac{[PL]}{[P][L]} = \frac{1}{k_d} = k_a \tag{10}$$

$$[P_T] = [P] + [PL] \tag{11}$$
$$[L_T] = [L] + [PL] \tag{12}$$
$$[PL] = [P][L]k_a \tag{13}$$
$$[P_T] = [P] + [P]k_a[L] = [P](1 + k_a[L]) \tag{14}$$
$$[L_T] = [L] + [L]k_a[P] \tag{15}$$
$$\frac{[P_T]}{1 + k_a[L]} = [P] = \frac{[L_T] - [L]}{k_a[L]} \tag{16}$$
$$k_a[P_T][L] = ([L_T] - [L])(1 + k_a[L]) \tag{17}$$

This yields

$$k_a[P_T][L] - [L_T] - [L_T]k_a[L] + [L] + k_a[L]^2 = 0 \tag{18}$$
$$k_a[L]^2 + k_a[P_T][L] - [L_T]k_a[L] + [L] - [L_T] = 0 \tag{19}$$

Following the paper

Am I stupid? In the paper, from eq 9 to 10 doesn't work. Anyway, to continue with eq. 10.

$$[L]^2 + [L]([P_T] - [L_T] + 1/k_a) - [L_T] = 0 \tag{20}$$

We fill this in the quadratic equation:

$$a = 1 \tag{21}$$
$$b = [P_T] - [L_T] + 1/k_a \tag{22}$$
$$c = -[L_T] \tag{23}$$
$$\tag{24}$$

Huh?

$$[L] = \frac{-([L_T] - [P_T] - 1/k_a) \pm \sqrt{([P_T] - [L_T] - 1/k_a)^2 + 4[L_T])}}{2} \tag{25}$$

We divide by $[P_T]$ and define $X_r = [L_T]/[P_T]$ and $r = 1/(k_a[P_T])$

$$\frac{[L]}{[P_T]} = \frac{\frac{[L_T]}{[P_T]} - \frac{[P_T]}{[P_T]} - \frac{1}{k_a[P_T]} \pm \sqrt{(\frac{[P_T]}{[P_T]} - \frac{[L_T]}{[P_T]} + \frac{1}{k_a[P_T]})^2 + 4\frac{[L_T]}{[P_T]}}}{2} \tag{26}$$
$$\frac{[L]}{[P_T]} = \frac{X_r - 1 - r \pm \sqrt{(1 - X_r + r)^2 + 4X_r}}{2} \tag{27}$$

This again deviates from the paper, where the signs of X_r are the other way around.

The calorimeter measures $dq/d[L_T]$, where q is heat.

Since

$$\frac{d[L]}{d[L_T]} = \frac{d[L]}{dX_r}\frac{dX_r}{d[L_T]} \tag{28}$$

$$\frac{dX_r}{d[L_T]} = \frac{1}{[P_T]} \tag{29}$$

the derivative is

$$\frac{d[L]}{dX_r} = \left(\frac{1}{2} + \frac{X_r + r - 1}{\sqrt{(X_r + r + 1)^2 - 4X_r}}\right)[P_T] \tag{30}$$

$$\frac{d[L]}{d[L_T]} = \left(\frac{1}{2} + \frac{X_r + r - 1}{\sqrt{(X_r + r + 1)^2 - 4X_r}}\right) \tag{31}$$

My own derivation

$$[L]^2 + [L]([P_T] - [L_T] + 1/k_a) - [L_T]/k_a = 0 \tag{32}$$

We fill this in the quadratic equation:

$$a = 1 \tag{33}$$

$$b = [P_T] - [L_T] + 1/k_a \tag{34}$$

$$c = -[L_T]/k_a \tag{35}$$

$$[L] = \frac{-\left([P_T] - [L_T] + 1/k_a\right) \pm \sqrt{([P_T] - [L_T] + 1/k_a)^2 + 4[L_T]/k_a)}}{2} \tag{36}$$

We divide by $[P_T]$ and define $X_r = [L_T]/[P_T]$ and $r = 1/(k_a[P_T])$

$$\frac{[L]}{[P_T]} = \frac{-\frac{[P_T]}{[P_T]} + \frac{[L_T]}{[P_T]} - \frac{1}{k_a[P_T]} \pm \sqrt{\left(\frac{[P_T]}{[P_T]} - \frac{[L_T]}{[P_T]} + \frac{1}{k_a[P_T]}\right)^2 + 4\frac{[L_T]}{k_a[P_T]}}}{2} \tag{37}$$

$$\frac{[L]}{[P_T]} = \frac{-1 + X_r - r \pm \sqrt{(1 - X_r + r)^2 + 4X_r/k_a}}{2} \tag{38}$$

the derivative is

$$\frac{d[L]}{dX_r} = \left(\frac{1}{2} + \frac{k_a(X_r - r - 1) + 2}{2k_a\sqrt{(-X_r + r + 1)^2 + 4X_r/k_a}}\right)[P_T] \tag{39}$$

$$\frac{d[L]}{d[L_T]} = \left(\frac{1}{2} + \frac{k_a(X_r - r - 1) + 2}{2k_a\sqrt{(-X_r + r + 1)^2 + 4X_r/k_a}}\right) \tag{40}$$

For one reason or the other, they took $X_r = -X_r$.

Calculating the heat

The expression for the total heat that has been released is

$$q = [PL]\Delta HV \tag{41}$$

The machine measures $dq/d[L_T]$:

$$\frac{dq}{d[L_T]} = \Delta HV \frac{d[PL]}{d[L_T]} \tag{42}$$

$$\frac{d[PL]}{d[L_T]} = \frac{d[L_T]}{d[L_T]} - \frac{d[L]}{d[L_T]} \tag{43}$$

Which is what we calculated above.

We should take the positive part of the quadratic function.

Plugging things into each other:

$$\frac{dq}{d[L_T]} = \Delta HV \left(\frac{1}{2} + \frac{k_a(-X_r - r - 1) + 2}{2k_a\sqrt{(X_r + r + 1)^2 - 4X_r/k_a}} \right) \tag{44}$$