Calculus: Early Transcendentals, 9th Ed.
Diagnostic A: Algebra

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Problem 1

Evaluate each expression without using a calculator.

1. $(-3)^4 = 81$
2. $-3^4 = -81$
3. $3^{-4} = 1/3^4 = 1/81$
4. $5^{23}/5^{21} = 5^{23-21} = 5^2 = 25$
5. $(2/3)^{-2} = (2/3)^{-1*2} = (3/2)^2 = 9/4$
6. $16^{-3/4} = (16^{\frac{3}{4}})^{-1} = (\frac{1}{16^{\frac{3}{4}}}) = \frac{1}{(\sqrt[4]{16})^3} = \frac{1}{2^3} = \frac{1}{8}$

Problem 2

Simplify each expression. Write your answer without negative exponents.

1. $\sqrt{200} - \sqrt{32} = \sqrt{100*2} - \sqrt{16*2} = 100 * \sqrt{100*2} - 8 * \sqrt{16*2} = 10 * \sqrt{2} - 4 * sqrt2 = 6 * \sqrt{2}$
   (a) note that 32 is a power of 2; we can get a clean factor
   (b) since $100 * 2$ is also a perfect square factor of 200
   (c) many factors of 200: $1, 2, 4, 5, 10, 20, 40, 50, 100, 200$
2. $(3a^3b^3)(4ab^2)^2 = 3 * 4 * 4 * a^3 * a * a * b^3 * b^2 * b^2 = 48a^5b^7$
3. $(\frac{3x^{3/2}y^3}{x^2y^{\frac{-1}{2}}})^{-2} = \frac{(x^2y^{\frac{-1}{2}})^2}{(3x^{3/2}y^3)^2} = \frac{x^4y^{-1}}{9x^3y^6} = \frac{x}{9y^7}$

Problem 3

Expand and simplify

1. $3(x+6) + 4(2x-5) = 3x + 3*6 + 8x - 20 = 11x - 2$
2. $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$
3. $(x+3)(4x-5) = 4x^2 + 7x - 15$
4. $(2x+3)^2 = 4x^2 + 12x + 9$
5. $(x+2)^3 == (x+2)(x+2)(x+2) = x^3 + 6x^2 + 12x + 8$

Problem 4

Factor each expression.

1. $4x^2 - 25 = (2x-5)(2x+5)$
2. $2x^2 + 5x - 12 = (2x-3)(1x+4)$ - solved by factor and brute force
   (a) System of equations... hmm - how can we solve below?
   (b) $a, b, c, d : b * d = -12, a * c = 2, a * d + b * c = 5$
3. $x^3 - 3x^2 - 4x + 12$

Problem 5

Give an appropriate positive constant c such that $f(n) \leq c \cdot g(n)$ for all $n > 1$.

1. $f(n) = n^2 + n + 1$, $g(n) = 2n^3$
2. $f(n) = n\sqrt{n} + n^2$, $g(n) = n^2$
3. $f(n) = n^2 - n + 1$, $g(n) = n^2/2$

Solution
We solve each solution algebraically to determine a possible constant c.

Part One

$$\begin{aligned} n^2 + n + 1 = & \\ & \leq n^2 + n^2 + n^2 \\ & = 3n^2 \\ & \leq c \cdot 2n^3 \end{aligned}$$

Thus a valid c could be when $c = 2$.

Part Two

$$\begin{aligned} n^2 + n\sqrt{n} = & \\ & = n^2 + n^{3/2} \\ & \leq n^2 + n^{4/2} \\ & = n^2 + n^2 \\ & = 2n^2 \\ & \leq c \cdot n^2 \end{aligned}$$

Thus a valid c is $c = 2$.

Part Three

$$\begin{aligned} n^2 - n + 1 = & \\ & \leq n^2 \\ & \leq c \cdot n^2/2 \end{aligned}$$

Thus a valid c is $c = 2$.

Problem 6

Let $\Sigma = \{0, 1\}$. Construct a DFA A that recognizes the language that consists of all binary numbers that can be divided by 5.

Let the state q_k indicate the remainder of k divided by 5. For example, the remainder of 2 would correlate to state q_2 because $7 \mod 5 = 2$.

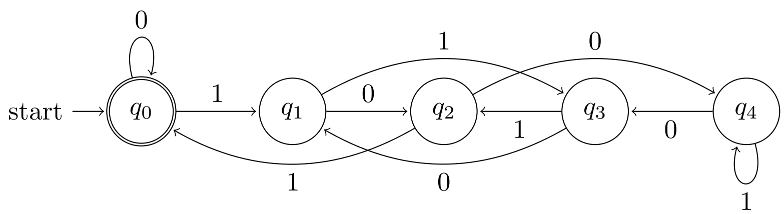

Figure 1: DFA, A, this is really beautiful, ya know?

Justification

Take a given binary number, x. Since there are only two inputs to our state machine, x can either become $x0$ or $x1$. When a 0 comes into the state machine, it is the same as taking the binary number and multiplying it by two. When a 1 comes into the machine, it is the same as multipying by two and adding one.

Using this knowledge, we can construct a transition table that tell us where to go:

	$x \mod 5 = 0$	$x \mod 5 = 1$	$x \mod 5 = 2$	$x \mod 5 = 3$	$x \mod 5 = 4$
$x0$	0	2	4	1	3
$x1$	1	3	0	2	4

Therefore on state q_0 or ($x \mod 5 = 0$), a transition line should go to state q_0 for the input 0 and a line should go to state q_1 for input 1. Continuing this gives us the Figure 1.

Problem 7

Write part of **Quick-Sort(**$list, start, end$**)**

```
function QUICK-SORT(list, start, end)
    if start ≥ end then
        return
    end if
    mid ← PARTITION(list, start, end)
    QUICK-SORT(list, start, mid − 1)
    QUICK-SORT(list, mid + 1, end)
end function
```

Algorithm 1: Start of QuickSort

Problem 8

Suppose we would like to fit a straight line through the origin, i.e., $Y_i = \beta_1 x_i + e_i$ with $i = 1, \ldots, n$, $\mathrm{E}[e_i] = 0$, and $\mathrm{Var}[e_i] = \sigma_e^2$ and $\mathrm{Cov}[e_i, e_j] = 0, \forall i \neq j$.

Part A

Find the least squares esimator for $\hat{\beta}_1$ for the slope β_1.

Solution

To find the least squares estimator, we should minimize our Residual Sum of Squares, RSS:

$$\begin{aligned} RSS &= \sum_{i=1}^{n} \left(Y_i - \hat{Y}_i\right)^2 \\ &= \sum_{i=1}^{n} \left(Y_i - \hat{\beta}_1 x_i\right)^2 \end{aligned}$$

By taking the partial derivative in respect to $\hat{\beta}_1$, we get:

$$\frac{\partial}{\partial \hat{\beta}_1}(RSS) = -2\sum_{i=1}^{n} x_i(Y_i - \hat{\beta}_1 x_i) = 0$$

This gives us:

$$\begin{aligned} \sum_{i=1}^{n} x_i(Y_i - \hat{\beta}_1 x_i) &= \sum_{i=1}^{n} x_i Y_i - \sum_{i=1}^{n} \hat{\beta}_1 x_i^2 \\ &= \sum_{i=1}^{n} x_i Y_i - \hat{\beta}_1 \sum_{i=1}^{n} x_i^2 \end{aligned}$$

Solving for $\hat{\beta}_1$ gives the final estimator for β_1:

$$\hat{\beta}_1 = \frac{\sum x_i Y_i}{\sum x_i^2}$$

Problem 8 continued on next page. . .

Part B

Calculate the bias and the variance for the estimated slope $\hat{\beta_1}$.

Solution

For the bias, we need to calculate the expected value $\mathrm{E}[\hat{\beta_1}]$:

$$\begin{aligned}
\mathrm{E}[\hat{\beta_1}] &= \mathrm{E}\left[\frac{\sum x_i Y_i}{\sum x_i^2}\right] \\
&= \frac{\sum x_i \mathrm{E}[Y_i]}{\sum x_i^2} \\
&= \frac{\sum x_i(\beta_1 x_i)}{\sum x_i^2} \\
&= \frac{\sum x_i^2 \beta_1}{\sum x_i^2} \\
&= \beta_1 \frac{\sum x_i^2 \beta_1}{\sum x_i^2} \\
&= \beta_1
\end{aligned}$$

Thus since our estimator's expected value is β_1, we can conclude that the bias of our estimator is 0.

For the variance:

$$\begin{aligned}
\mathrm{Var}[\hat{\beta_1}] &= \mathrm{Var}\left[\frac{\sum x_i Y_i}{\sum x_i^2}\right] \\
&= \frac{\sum x_i^2}{\sum x_i^2 \sum x_i^2} \mathrm{Var}[Y_i] \\
&= \frac{\sum x_i^2}{\sum x_i^2 \sum x_i^2} \mathrm{Var}[Y_i] \\
&= \frac{1}{\sum x_i^2} \mathrm{Var}[Y_i] \\
&= \frac{1}{\sum x_i^2} \sigma^2 \\
&= \frac{\sigma^2}{\sum x_i^2}
\end{aligned}$$

Problem 9

Prove a polynomial of degree k, $a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n^1 + a_0 n^0$ is a member of $\Theta(n^k)$ where $a_k \ldots a_0$ are nonnegative constants.

Proof. To prove that $a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n^1 + a_0 n^0$, we must show the following:

$$\exists c_1 \exists c_2 \forall n \geq n_0,\ c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$$

For the first inequality, it is easy to see that it holds because no matter what the constants are, $n^k \leq a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n^1 + a_0 n^0$ even if $c_1 = 1$ and $n_0 = 1$. This is because $n^k \leq c_1 \cdot a_k n^k$ for any nonnegative constant, c_1 and a_k.

Taking the second inequality, we prove it in the following way. By summation, $\sum_{i=0}^{k} a_i$ will give us a new constant, A. By taking this value of A, we can then do the following:

$$\begin{aligned} a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n^1 + a_0 n^0 = & \\ & \leq (a_k + a_{k-1} \ldots a_1 + a_0) \cdot n^k \\ & = A \cdot n^k \\ & \leq c_2 \cdot n^k \end{aligned}$$

where $n_0 = 1$ and $c_2 = A$. c_2 is just a constant. Thus the proof is complete. $\square$

Problem 18

Evaluate $\sum_{k=1}^{5} k^2$ and $\sum_{k=1}^{5}(k-1)^2$.

Problem 19

Find the derivative of $f(x) = x^4 + 3x^2 - 2$

Problem 6

Evaluate the integrals $\int_0^1 (1-x^2)\mathrm{d}x$ and $\int_1^\infty \frac{1}{x^2}\mathrm{d}x$.