Fast sparse period estimation

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Received signal model

Receive L noisy M-ary phase-shifted-keyed (M-PSK) symbols of the form

$$y_i = a_0 s_i + w_i, \qquad i = 1, \dots, L.$$

- $a_0 = \rho_0 e^{j\theta_0}$ is the unknown carrier phase θ_0 and amplitude ρ_0,
- $s_1, \dots, s_L$ are transmitted M-PSK symbols,
- $w_1, \dots, w_L \in \mathbb{C}$ are i.i.d. circularly symmetric complex random variables representing noise.

Interested in estimating ρ_0 and θ_0.

Least squares estimator

If all symbols $s_1, \dots, s_L$ are known

$$\hat{a}_{\text{uc}} = \arg\min_{a \in \mathbb{C}} \sum_{i=1}^{L} |y_i - a s_i|^2 = \frac{1}{L} \sum_{i=1}^{L} y_i s_i^*.$$

More interested in the practical situation where symbols are not known,

$$\hat{a} = \arg\min_{a \in \mathbb{C}} \min_{s_1,\dots,s_L} \sum_{i=1}^{L} |y_i - a s_i|^2.$$

This estimator can be computed in $O(L \log L)$ operations.

Theorem (Almost sure convergence)

Let $R_i \geqslant 0$ and $\Phi_i \in [-\pi, \pi)$ be real random variables satisfying

$$R_i e^{j\Phi_i} = 1 + \frac{w_i}{a_0 s_i},$$

and define the continuous function

$$G(x) = \mathbb{E} R_1 \cos\langle x + \Phi_1 \rangle.$$

If $G(x)$ is uniquely maximised at $x = 0$ over the interval $[-\frac{\pi}{M}, \frac{\pi}{M})$, then:

1. $\langle \hat{\theta} - \theta_0 \rangle \to 0$ almost surely as $L \to \infty$,
2. $\hat{\rho} \to \rho_0 G(0)$ almost surely as $L \to \infty$,

where $\langle \cdot \rangle$ takes its argument 'modulo $\frac{2\pi}{M}$' into $[-\frac{\pi}{M}, \frac{\pi}{M})$.

Theorem (Asymptotic normality)

Let $f(r, \phi)$ be the joint pdf of R_1 and Φ_1, and let

$$g(\phi) = \int_0^\infty r f(r, \phi) dr.$$

Put $\hat{\lambda}_L = \langle \hat{\theta} - \theta_0 \rangle$ and $\hat{m}_L = \hat{\rho} - \rho_0 G(0)$. The distribution of $(\sqrt{L}\hat{\lambda}_L, \sqrt{L}\hat{m}_L)$ converges to the bivariate normal with zero mean and covariance matrix

$$\begin{pmatrix} H^{-2}A & 0 \\ 0 & \rho_0^2 B \end{pmatrix}$$

as $L \to \infty$, where

$$H = G(0) - 2\sin\left(\tfrac{\pi}{M}\right) \sum_{k=0}^{M-1} g\left(\tfrac{2\pi}{M}k + \tfrac{\pi}{M}\right),$$

$$A = \mathbb{E} R_1^2 \sin^2\langle \Phi_1 \rangle, \quad B = \mathbb{E} R_1^2 \cos^2\langle \Phi_1 \rangle - G^2(0).$$

Simulations

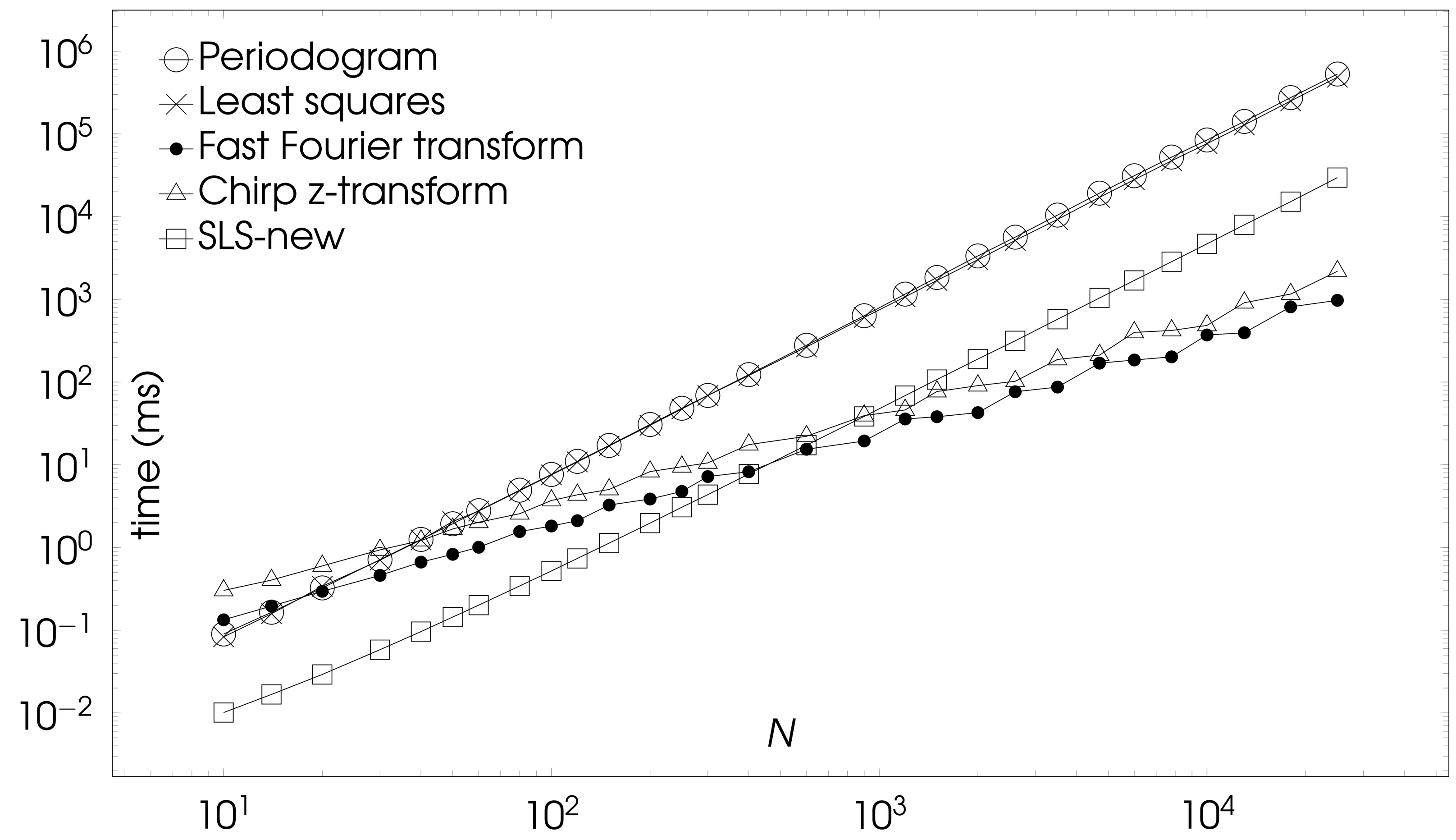

Figure : Computation time in milliseconds versus N for the periodogram, least squares, and quantized periodogram estimators computed using the chirp z-transform and a single fast Fourier transform.

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