Signals and Systems

Robby McKilliam

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Chapter 1

Signals and systems

A **signal** is a function mapping an input variable to some output variable. For example

$$\sin(\pi t), \qquad \frac{1}{2}t^3, \qquad e^{-t^2}$$

all represent **signals** with real input variable $t \in \mathbb{R}$ and real output variable. These signals are plotted in Figure 1.1. If x is a signal and t an input variable we write $x(t)$ for the output variable corresponding with t. Signals can be multidimensional. This page is an example of a 2-dimensional signal, the independent variables are the horizontal and vertical position on the page, and the signal maps this position to a colour, in this case either black or white. A moving image such as seen on your television or computer monitor is an example of a 3-dimensional signal, the three independent variables being vertical and horizontal screen position and time. The signal maps each position and time to a colour on the screen. In these notes we focus exclusively on 1-dimensional signals such as those in Figure 1.1 and we will only consider signals where the output variable is real or complex valued. Many of the results presented here can be extended to deal with multidimensional signals.

1.1 Properties of signals

A signal x is **bounded** if there exists a real number M such that

$$|x(t)| < M \qquad \text{for all } t \in \mathbb{R}$$

where $|\cdot|$ denotes the (complex) magnitude. Both $\sin(\pi t)$ and e^{-t^2} are examples of bounded signals because $|\sin(\pi t)| \leq 1$ and $|e^{-t^2}| \leq 1$ for all $t \in \mathbb{R}$. However, $\frac{1}{2}t^3$ is not bounded because its magnitude grows indefinitely as t moves away from the origin.

A signal x is **periodic** if there exists a positive real number T such that

$$x(t) = x(t + kT) \qquad \text{for all } k \in \mathbb{Z} \text{ and } t \in \mathbb{R}.$$

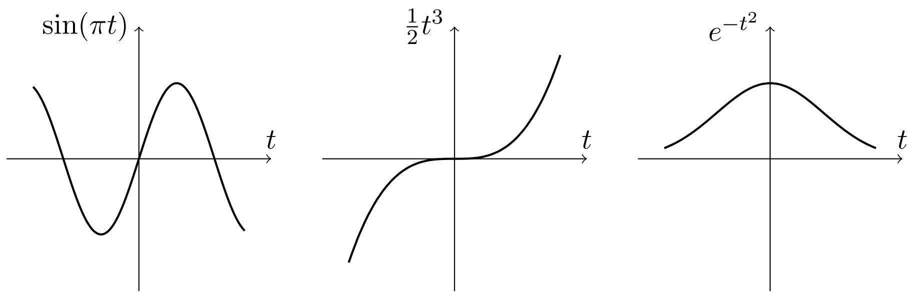

Figure 1.1: 1-dimensional signals

If there exists a smallest such positive T it is called the **fundamental period** or simply the **period**. For example, the signal $\sin(\pi t)$ is periodic with period $T = 2$. Neither $\frac{1}{2}t^3$ or e^{-t^2} are periodic.

A signal x is **right sided** if there exists a $T \in \mathbb{R}$ such that $x(t) = 0$ for all $t < T$. Correspondingly x is **left sided** if $x(t) = 0$ for all $T > t$. For example, the **step function**

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{1.1.1}$$

is right-sided. Its reflection in time $u(-t)$ is left sided (Figure 1.2). A signal x is said to be **finite** if it is both left and right sided, that is, if there exits a $T \in \mathbb{R}$ such that $x(t) = x(-t) = 0$ for all $t > T$. The signals $\sin(\pi t)$ and e^{-t^2} are not finite, but the **rectangular pulse**

$$\Pi(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{1.1.2}$$

is finite.

A signal x is **even** (or **symmetric**) if

$$x(t) = x(-t) \qquad \text{for all } t \in \mathbb{R}$$

and **odd** (or **antisymmetric**) if

$$x(t) = -x(-t) \qquad \text{for all } t \in \mathbb{R}.$$

For example, $\sin(\pi t)$ and $\frac{1}{2}t^3$ are odd and e^{-t^2} is even.

A signal x is **locally integrable** if

$$\int_a^b |x(t)|\, dt < \infty$$

for all finite constants a and b, where by $< \infty$ we mean that the integral evaluates to a finite number. An example of a signal that is not locally

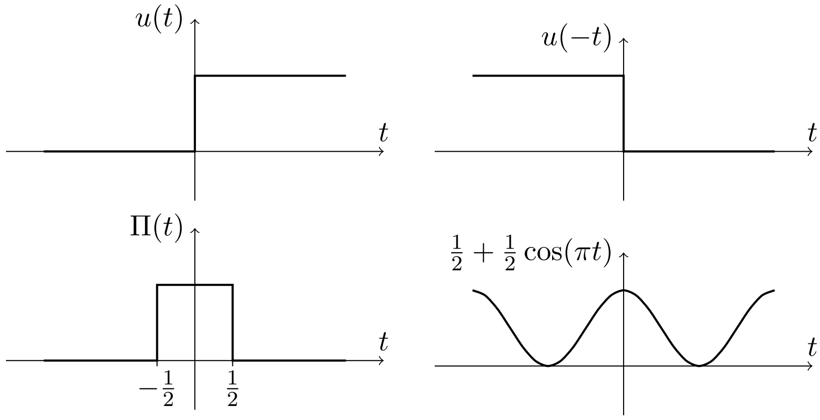

Figure 1.2: The right sided step function $u(t)$, its left sided reflection $u(-t)$, the finite rectangular pulse $\Pi(t)$ and the signal $\frac{1}{2}+\frac{1}{2}\cos(x)$ that is not finite.

integrable is $x(t) = \frac{1}{t}$ (Exercise 1.2). A signal x is **absolutely integrable** if

$$\|x\|_1 = \int_{-\infty}^{\infty} |x(t)|\, dt < \infty. \tag{1.1.3}$$

Here we introduce the notation $\|x\|_1$ called the **L^1-norm** of x. For example $\sin(\pi t)$ and $\frac{1}{2}t^3$ are not absolutely integrable, but e^{-t^2} is because [Nicholas and Yates, 1950]

$$\int_{-\infty}^{\infty} |e^{-t^2}| dt = \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}. \tag{1.1.4}$$

It is common to denote the set of absolutely integrable signals by L^1 or $L^1(\mathbb{R})$. So, $e^{-t^2} \in L^1$ and $\frac{1}{2}t^3 \notin L^1$. A signal x is called is **square integrable** if

$$\|x\|_2^2 = \int_{-\infty}^{\infty} |x(t)|^2\, dt < \infty.$$

The real number $\|x\|_2$ is called the **L^2-norm** of x. Square integrable signals are also called **energy signals**, and the squared L^2-norm $\|x\|_2^2$ is called the **energy** of x. For example $\sin(\pi t)$ and $\frac{1}{2}t^3$ are not energy signals, but e^{-t^2} is (Exercise 1.5). The set of square integrable signals is often denoted by L^2 or $L^2(\mathbb{R})$.

We write $x = y$ to indicate that two signals x and y are **equal pointwise**, that is, $x(t) = y(t)$ for all $t \in \mathbb{R}$. This definition of equality is often stronger than we desire. For example, the step function u and the signal

$$z(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}$$

are not equal pointwise because they are not equal at $t = 0$, that is, $u(0) = 1$ and $z(0) = 0$. It is useful to identify signals that differ only at isolated points and for this we use a weaker definition of equality. We say that two signals x and y are equal **almost everywhere** if

$$\int_a^b |x(t) - y(t)|\, dt = 0$$

for all finite constants a and b. So, in the previous example, while $u \neq z$ pointwise we do have $u = z$ almost everywhere. Typically the term almost everywhere is abbreviated to a.e. and one writes

$$x = y \;\text{ a.e.} \qquad \text{or} \qquad x(t) = y(t) \;\text{ a.e.}$$

to indicate that the signals x and y are equal almost everywhere.

1.2 Systems (functions of signals)

A **system** is a function that maps a signal to another signal. For example

$$x(t) + 3x(t-1), \qquad \int_0^1 x(t-\tau)d\tau, \qquad \frac{1}{x(t)}, \qquad \frac{d}{dt}x(t)$$

represent systems, each mapping the signal x to another signal. Consider the electric circuit in Figure 1.3 called a **voltage divider**. If the voltage at time t is $x(t)$ then, by Ohm's law, the current at time t satisfies

$$i(t) = \frac{1}{R_1 + R_2} x(t),$$

and the voltage over the resistor R_2 is

$$y(t) = R_2 i(t) = \frac{R_2}{R_1 + R_2} x(t). \tag{1.2.1}$$

The circuit can be considered as a system mapping the signal x representing the voltage to the signal $i = \frac{1}{R_1+R_2}x$ representing the current, or a system mapping x to the signal $y = \frac{R_2}{R_1+R_2}x$ representing the voltage over resistor R_2.

We denote systems with capital letters such as H and G. A system H is a function that maps a signal x to another signal denoted $H(x)$. We call x the **input signal** and $H(x)$ the **output signal** or the **response** of system H to signal x. The value of the signal $H(x)$ at t is denoted by $H(x,t)$ or $H(x,t)$ and we do not distinguish between these notations. It is sometimes useful to depict systems with a block diagram. Figure 1.4 is a simple block diagram showing the input and output signals of a system H.

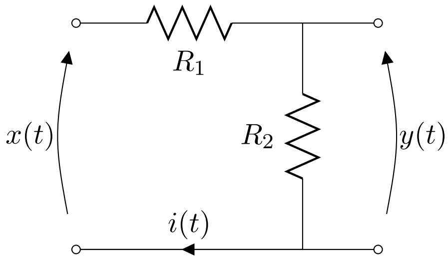

Figure 1.3: A **voltage divider** circuit.

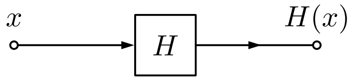

Figure 1.4: System block diagram with input signal x and output signal $H(x)$.

The electric circuit in Figure 1.3 corresponds with the system

$$H(x) = \frac{R_2}{R_1 + R_2} x = y.$$

This system multiplies the input signal x by $\frac{R_2}{R_1+R_2}$. This brings us to our first practical test.

Test 1 (**Voltage divider**) In this test we construct the voltage divider from Figure 1.3 on a breadboard with resistors $R_1 \approx 100\Omega$ and $R_2 \approx 470\Omega$ with values accurate to within 5%. Using a computer soundcard (an approximation of) the voltage signal

$$x(t) = \sin(2\pi f_1 t) \qquad \text{with} \qquad f_1 = 100$$

is passed through the circuit. The approximation is generated by sampling $x(t)$ at rate $F = \frac{1}{P} = 44100\text{Hz}$ to generate samples

$$x(nP) \qquad n = 0, \dots, 2F$$

corresponding to approximately 2 seconds of signal. These samples are passed to the soundcard which starts playback. The voltage over resistor R_2 is recorded (also using the soundcard) that returns a list of samples $y_1, \dots, y_L$ taken at rate F. The voltage over R_2 can be (approximately) reconstructed from these samples as

$$\tilde{y}(t) = \sum_{\ell=1}^{L} y_\ell \operatorname{sinc}(Ft - \ell) \tag{1.2.2}$$

where

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \tag{1.2.3}$$

is the called the **sinc function** and is plotted in Figure 5.1. We will justify this reconstruction in Section 5.4. Simultaneously the (stereo) soundcard is used to record the input voltage x producing samples $x_1, \dots, x_L$ taken at rate F. An approximation of the input signal is

$$\tilde{x}(t) = \sum_{\ell=1}^{L} x_\ell \operatorname{sinc}(Ft - \ell). \tag{1.2.4}$$

In view of (1.2.1) we would expect the approximate relationship

$$\tilde{y} \approx \frac{R_2}{R_1 + R_2}\tilde{x} = \frac{47}{57}\tilde{x}.$$

A plot of $\tilde{y}$, $\tilde{x}$ and $\frac{47}{57}\tilde{x}$ over a 20ms period from 1s to 1.02s is given in Figure 1.5. The hypothesised output signal $\frac{47}{57}\tilde{x}$ does not match the observed output signal $\tilde{y}$. A primary reason is that the circuitry inside the soundcard itself cannot be ignored. When deriving the equation for the voltage divider we implicitly assumed that current flows through the output of the soundcard without resistance (a short circuit), and that no current flows through the input device of the soundcard (an open circuit). These assumptions are not realistic. Modelling the circuitry in the sound card wont be attempted here. In Section 2.2 we will construct circuits that contain external sources of power (active circuits). These are less sensitive to the circuitry inside the soundcard.

Not all signals can be input to all systems. For example, the system

$$H(x,t) = \frac{1}{x(t)}$$

is not defined at those t where $x(t) = 0$ because we cannot divide by zero. Another example is the system

$$I_\infty(x,t) = \int_{-\infty}^{t} x(\tau)d\tau, \tag{1.2.5}$$

called an **integrator**. The signal $x(t) = 1$ cannot be input to the integrator because the integral $\int_{-\infty}^{t} dt$ is not finite for any t.

When specifying a system it is necessary to also specify a set of signals that can be input. This is called a **domain** for the system. We are free to choose the domain at our convenience. For example, a domain for the system $H(x,t) = \frac{1}{x(t)}$ is a the set of signals $x(t)$ which are not zero for any t. An example of a domain for the integrator I_∞ is the set L^1 of absolutely integrable signals because, if x is absolutely integrable, then

$$|I_\infty(x,t)| \leq \left| \int_{-\infty}^{t} x(\tau)d\tau \right| \leq \int_{-\infty}^{t} |x(\tau)|\, d\tau < \int_{-\infty}^{\infty} |x(\tau)|\, d\tau = \|x\|_1 < \infty$$

and so, $I_\infty(x,t)$ is finite for all t. In this text, the domain used for a given system will usually be obvious from the context in which the system is defined. For this reason we will not usually state the domain explicitly. We will only do so if there is chance for confusion.

1.3 Some important systems

The system

$$T_\tau(x,t) = x(t-\tau)$$

is called a **time-shifter**. This system shifts the input signal along the t axis ('time' axis) by τ. When τ is positive T_τ delays the input signal by τ. The time-shifter will appear so regularly in this course that we use the special notation T_τ to represent it. Figure 1.6 depicts the action of time-shifters $T_{1.5}$ and T_{-3} on the signal $x(t) = e^{-t^2}$. When $\tau = 0$ the time-shifter is the **identity system**

$$T_0(x) = x$$

that maps the signal x to itself.

Another important system is the **time-scaler** that has the form

$$H(x,t) = x(\alpha t), \qquad \alpha \in \mathbb{R}.$$

Figure 1.7 depicts the action of a time-scaler with a number of values for α. When $\alpha = -1$ the time-scaler reflects the input signal in the time axis. When $\alpha = 1$ the time-scaler is the identity system T_0.

Another system we regularly encounter is the **differentiator**

$$D(x,t) = \frac{d}{dt}x(t),$$

that returns the derivative of the input signal. We also define a kth differentiator

$$D^k(x,t) = \frac{d^k}{dt^k}x(t)$$

that returns the kth derivative of the input signal.

A related system is the **integrator**

$$I_a(x,t) = \int_{-a}^{t} x(\tau)d\tau.$$

The parameter a describes the lower bound of the integral. In this course it will often be that $a = \infty$. For example, the response of the integrator I_∞ to the signal $tu(t)$ where u is the step function (1.1.1) is

$$\int_{-\infty}^{t} \tau u(\tau)d\tau = \begin{cases} \int_0^t \tau d\tau = \frac{t^2}{2} & t > 0 \\ 0 & t \leq 0. \end{cases}$$

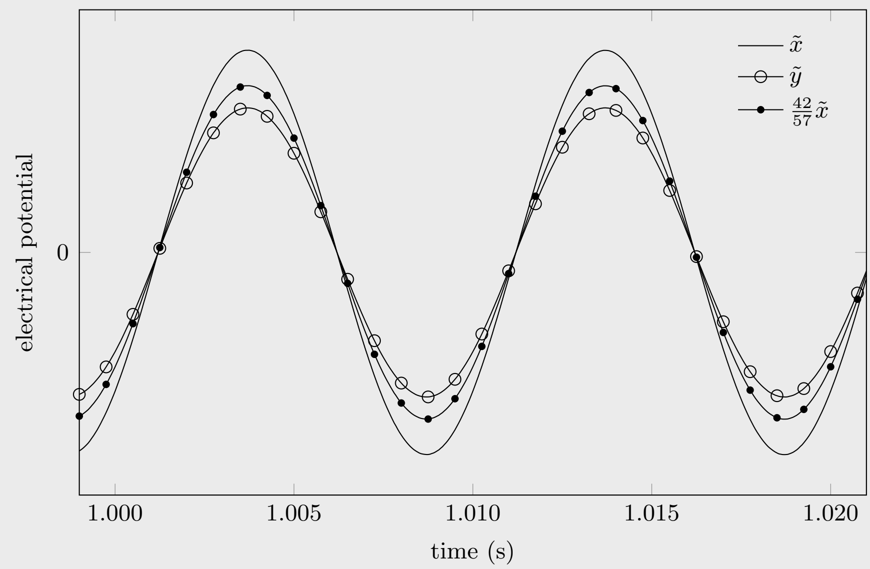

Figure 1.5: Plot of reconstructed input signal $\tilde{x}$ (solid line), output signal $\tilde{y}$ (solid line with circle) and hypothesised output signal $\frac{47}{57}\tilde{x}$ (solid line with dot) for the voltage divider circuit in Figure 1.3. The hypothesised signal does not match $\tilde{y}$. One reason is that the model does not take account of the circuitry inside the soundcard.

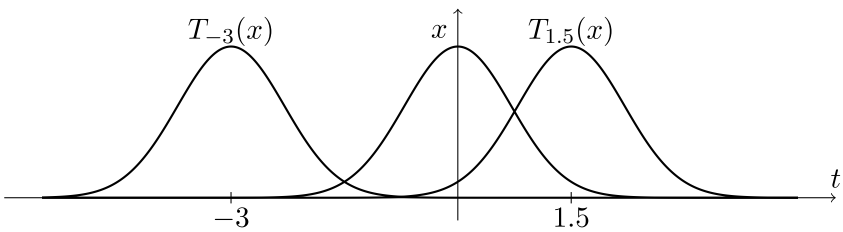

Figure 1.6: Time-shifter system $T_{1.5}(x,t) = x(t-1.5)$ and $T_{-3}(x,t) = x(t+3)$ acting on the signal $x(t) = e^{-t^2}$.

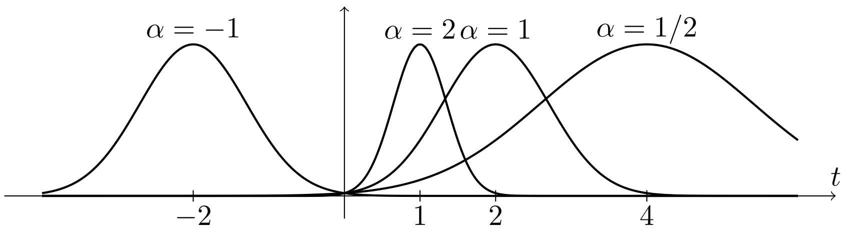

Figure 1.7: Time-scaler system $H(x,t) = x(\alpha t)$ for $\alpha = -1, \frac{1}{2}, 1$ and 2 acting on the signal $x(t) = e^{-(t-2)^2}$.

Observe that the integrator I_∞ cannot be applied to the signal $x(t) = t$ because $\int_{-\infty}^{t} \tau d\tau$ is not finite for any t. A domain for I_∞ would not contain the signal $x(t) = t$.

1.4 Properties of systems

In this section we define a number of important properties that systems can possess. In what follows H will be a system and the phrase "for all signals" will mean for all signals inside some domain for H.

A system H is called **memoryless** if the output signal $H(x)$ at time t depends only on the input signal x at time t. For example $\frac{1}{x(t)}$ and the identity system T_0 are memoryless, but

$$x(t) + 3x(t-1) \qquad \text{and} \qquad \int_0^1 x(t-\tau)d\tau$$

are not. A time-shifter T_τ with $\tau \neq 0$ is not memoryless.

A system H is **causal** if the output signal $H(x)$ at time t depends on the input signal only at times less than or equal to t. Memoryless systems such as $\frac{1}{x(t)}$ and T_0 are also causal. Time-shifters T_τ are causal when $\tau \geq 0$, but are not causal when $\tau < 0$. The systems

$$x(t) + 3x(t-1) \qquad \text{and} \qquad \int_0^1 x(t-\tau)d\tau$$

are causal, but the systems

$$x(t) + 3x(t+1) \qquad \text{and} \qquad \int_0^1 x(t+\tau)d\tau$$

are not causal.

A system H is called **bounded-input-bounded-output (BIBO) stable** or just **stable** if the output signal $H(x)$ is bounded whenever the input signal x is bounded. That is, H is stable if for every positive real number M there exists a positive real number K such that for all signals x satisfying

$$|x(t)| < M \qquad \text{for all } t \in \mathbb{R},$$

it also holds that

$$|H(x,t)| < K \qquad \text{for all } t \in \mathbb{R}.$$

For example, the system $x(t) + 3x(t-1)$ is stable with $K = 4M$ since if $|x(t)| < M$ then

$$|x(t) + 3x(t-1)| \leq |x(t)| + 3\,|x(t-1)| < 4M = K.$$

The integrator I_a for any $a \in \mathbb{R}$ and differentiator D are not stable (Exercises 1.6 and 1.7).

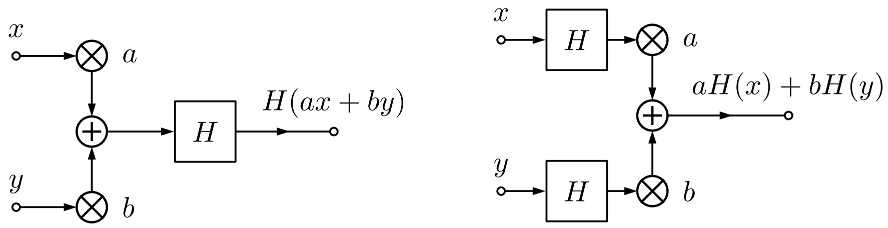

Figure 1.8: If H is a linear system the outputs of these two diagrams are the same signal, i.e. $H(ax + by) = aH(x) + bH(y)$.

A system H is **linear** if

$$H(ax + by) = aH(x) + bH(y)$$

for all signals x and y and all complex numbers a and b. That is, a linear system has the property: If the input consists of a weighted sum of signals, then the output consists of the same weighted sum of the responses of the system to those signals. Figure 1.8 indicates the linearity property using a block diagram. For example, the differentiator is linear because

$$\begin{aligned} D(ax + by, t) &= \frac{d}{dt}\big(ax(t) + by(t)\big) \\ &= a\frac{d}{dt}x(t) + b\frac{d}{dt}y(t) \\ &= aD(x, t) + bD(y, t) \end{aligned}$$

whenever both x and y are differentiable. However, the system $H(x, t) = \frac{1}{x(t)}$ is not linear because

$$H(ax + by, t) = \frac{1}{ax(t) + by(t)} \neq \frac{a}{x(t)} + \frac{b}{y(t)} = aH(x, t) + bH(y, t)$$

in general.

The property of linearity trivially generalises to more than two signals. For example, if $x_1, \dots, x_k$ are signals and $a_1, \dots, a_k$ are complex numbers for some finite k, then

$$H(a_1x_1 + \cdots + a_kx_k) = a_1H(x_1) + \cdots + a_kH(x_k).$$

A system H is **time invariant** if

$$H\big(T_\tau(x), t\big) = H(x, t - \tau)$$

for all signals x and all time-shifts $\tau \in \mathbb{R}$. That is, a system is time-invariant if time shifting the input signal results in the same time-shift of the output signal. Equivalently, H is time-invariant if it commutes with the time-shifter T_τ, that is, if

$$H\big(T_\tau(x)\big) = T_\tau\big(H(x)\big)$$

for all $\tau \in \mathbb{R}$ and all signals x. Figure 1.9 represents the property of time-invariance with a block diagram.

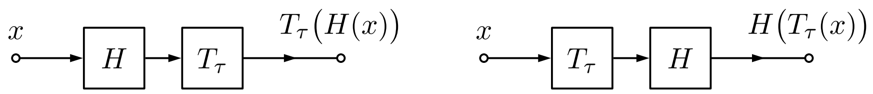

Figure 1.9: If H is a time-invariant system the outputs of these two diagrams are the same signal, i.e. $H\big(T_\tau(x)\big) = T_\tau\big(H(x)\big)$.

1.5 Exercises

1.1. State whether the step function $u(t)$ is bounded, periodic, absolutely integrable, an energy signal.

1.2. Show that the signal t^2 is locally integrable, but that the signal $\frac{1}{t^2}$ is not.

1.3. Plot the signal
$$x(t) = \begin{cases} \frac{1}{t+1} & t > 0 \\ \frac{1}{t-1} & t \leq 0. \end{cases}$$
State whether it is: bounded, locally integrable, absolutely integrable, square integrable.

1.4. Plot the signal
$$x(t) = \begin{cases} \frac{1}{\sqrt{t}} & 0 < t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$
Show that x is absolutely integrable, but not square integrable.

1.5. Compute the energy of the signal $e^{-\alpha^2 t^2}$ (Hint: use equation (1.1.4) on page 3 and a change of variables).

1.6. Show that the integrator I_a for any $a \in \mathbb{R}$ is not stable.

1.7. Show that the differentiator system D is not stable.

1.8. Show that the time-shifter is linear and time-invariant and that the time-scaler is linear, but not time invariant

1.9. Show that the integrator I_c with c finite is linear, but not time-invariant.

1.10. Show that the integrator I_∞ is linear and time invariant.

1.11. State whether the system $H(x) = x+1$ is linear, time-invariant, stable.

1.12. State whether the system $H(x) = 0$ is linear, time-invariant, stable.

1.13. State whether the system $H(x) = 1$ is linear, time-invariant, stable.

1.14. Let x be a signal with period T that is not equal to zero almost everywhere. Show that x is not absolutely integrable.

Chapter 2

Systems modelled by differential equations

Systems of particular interest in this text are those where the input signal x and output signal y are related by a linear differential equation with constant coefficients, that is, an equation of the form

$$\sum_{\ell=0}^{m} a_\ell \frac{d^\ell}{dt^\ell} x(t) = \sum_{\ell=0}^{k} b_\ell \frac{d^\ell}{dt^\ell} y(t),$$

where $a_0, \dots, a_m$ and $b_0, \dots, b_k$ are real or complex numbers. In what follows we use the differentiator system D rather than the notation $\frac{d}{dt}$ to represent differentiation. To represent the ℓth derivative we write D^ℓ instead of $\frac{d^\ell}{dt^\ell}$. Using this notation the differential equation above is

$$\sum_{\ell=0}^{m} a_\ell D^\ell(x) = \sum_{\ell=0}^{k} b_\ell D^\ell(y). \tag{2.0.1}$$

Equations of this form can be used to model a large number of electrical, mechanical and other real world devices. For example, consider the resistor and capacitor (RC) circuit in Figure 2.1. Let the signal v_R represent the voltage over the resistor and i the current through both resistor and capacitor. The voltage signals satisfy

$$x = y + v_R,$$

and the current satisfies both

$$v_R = Ri \qquad \text{and} \qquad i = CD(y).$$

Combining these equations,

$$x = y + RCD(y) \tag{2.0.2}$$

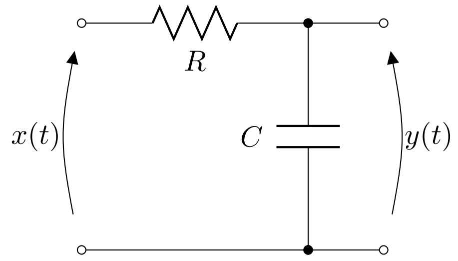

Figure 2.1: An electrical circuit with resistor and capacitor in series, otherwise known as an **RC circuit**.

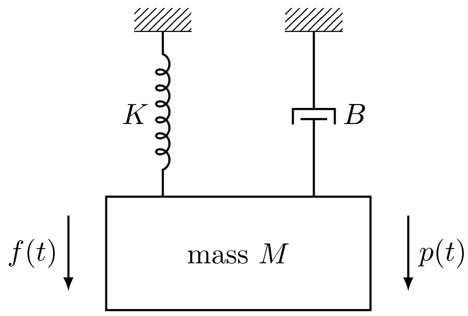

Figure 2.2: A mechanical mass-spring-damper system

that is in the form of (2.0.1).

As another example, consider the mass-spring-damper in Figure 2.2. A force represented by the signal f is externally applied to the mass, and the position of the mass is represented by the signal p. The spring exerts force $-Kp$ that is proportional to the position of the mass, and the damper exerts force $-BD(p)$ that is proportional to the velocity of the mass. The cumulative force exerted on the mass is

$$f_m = f - Kp - BD(p)$$

and by Newton's law the acceleration of the mass $D^2(p)$ satisfies

$$MD^2(p) = f_m = f - Kp - BD(p).$$

We obtain the differential equation

$$f = Kp + BD(p) + MD^2(p) \tag{2.0.3}$$

that is in the form of (2.0.1) if we put $x = f$ and $y = p$. Given p we can readily solve for the corresponding force f. As a concrete example, let the spring constant, damping constant and mass be $K = B = M = 1$. If the position satisfies $p(t) = e^{-t^2}$, then the corresponding force satisfies

$$f(t) = e^{-t^2}(4t^2 - 2t - 1).$$

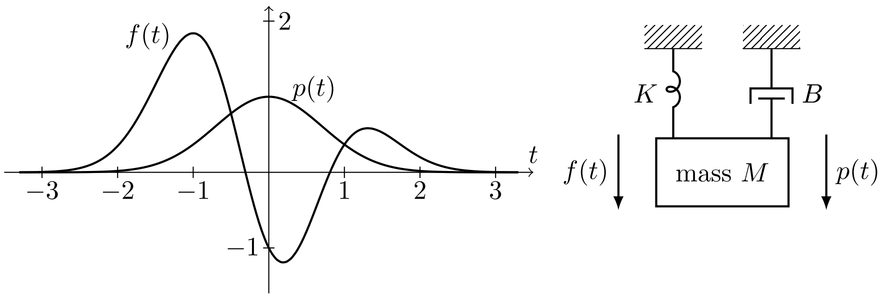

Figure 2.3: A solution to the mass-spring-damper system with $K = B = M = 1$. The position is $p(t) = e^{-t^2}$ with corresponding force $f(t) = e^{-t^2}(4t^2 - 2t - 1)$.

Figure 2.3 depicts these signals.

What happens if a particular force signal f is applied to the mass? For example, say we apply the force

$$f(t) = \Pi(t - \tfrac{1}{2}) = \begin{cases} 1 & 0 < t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the corresponding position signal p? We are not yet ready to answer this question, but will be later (Exercise 4.12).

In both the mechanical mass-spring-damper system in Figure 2.2 and the electrical RC circuit in Figure 2.1 we obtain a differential equation relating the input signal x with the output signal y. The equations do not specify the output signal y explicitly in terms of the input signal x, that is, they do not explicitly define a system H such $y = H(x)$. As they are, the differential equations do not provide as much information about the behaviour of the system as we would like. For example, is the system stable? We will be able to obtain much more information about these systems when the **Laplace transform** is introduced in Chapter 4. The remainder of this chapter details the construction of differential equations that model various mechanical, electrical, and electro-mechanical systems. We will use the systems constructed here as examples throughout the course.

2.1 Passive electrical circuits

Passive electrical circuits require no sources of power other than the input signal itself. For example, the voltage divider in Figure 1.3 and the RC circuit in Figure 2.1 are passive circuits. Another common passive electrical circuit is the resistor, capacitor and inductor (RLC) circuit depicted in Figure 2.4. In this circuit we let the output signal y be the voltage over the resistor. Let v_C represent the voltage over the capacitor and v_L the voltage over the inductor and let i be the current. We have

$$y = Ri, \qquad i = CD(v_C), \qquad v_L = LD(i),$$

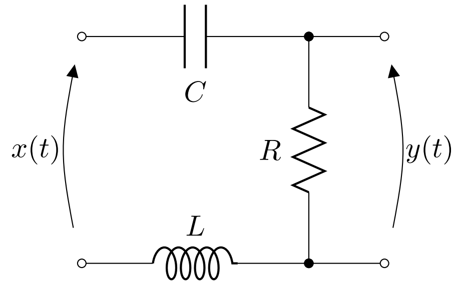

Figure 2.4: An electrical circuit with resistor, capacitor and inductor in series, otherwise known as an **RLC circuit**.

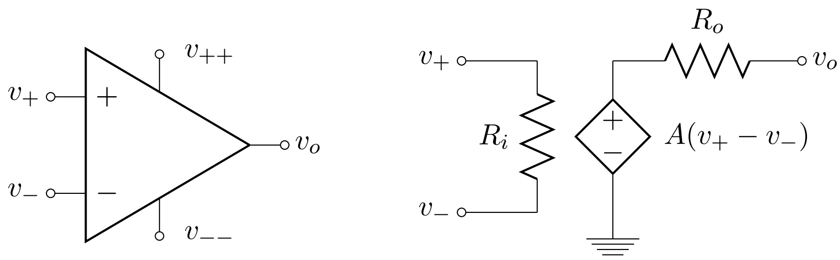

Figure 2.5: Left: triangular component diagram of an **operational amplifier**. The v_{++} and v_{--} connectors indicate where an external voltage source can be connected to the amplifier. These connectors will usually be omitted. Right: model for an operational amplifier including input resistance R_i, output resistance R_o, and open loop gain A. The diamond shaped component is a dependent voltage source. This model is usually only useful when the operational amplifier is in a negative feedback circuit.

leading to the following relationships between y, v_C and v_L,

$$y = RCD(v_C), \qquad Rv_L = LD(y).$$

Kirchhoff's voltage law gives $x = y + v_C + v_L$ and by differentiating both sides

$$D(x) = D(y) + D(v_C) + D(v_L).$$

Substituting the equations relating y, v_C and v_L leads to

$$RCD(x) = y + RCD(y) + LCD^2(y). \tag{2.1.1}$$

We can similarly find equations relating the input voltage with v_C and v_L.

2.2 Active electrical circuits

Unlike passive electrical circuits, an **active electrical circuit** requires a source of power external to the input signal. Active circuits can be modelled and constructed using **operational amplifiers** as depicted in Figure 2.5. The left hand side of Figure 2.5 shows a triangular circuit diagram for an operational amplifier, and the right hand side of Figure 2.5 shows a circuit that can be used to model the behaviour of the amplifier. The v_{++} and v_{--} connectors indicate where an external voltage source can be connected to the amplifier. These connectors will usually be omitted. The diamond shaped component is a dependent voltage source with voltage $A(v_+ - v_-)$ that depends on the difference between the voltage at the **non-inverting input** v_+ and the voltage at the **inverting input** v_-. The dimensionless constant A is called the **open loop gain**. Most operational amplifiers have large open loop gain A, large **input resistance** R_i and small **output resistance** R_o. As we will see, it can be convenient to consider the behaviour as $A \to \infty$, $R_i \to \infty$ and $R_0 \to 0$, resulting in an **ideal operational amplifier**.

As an example, an operational amplifier configured as a **multiplier** is depicted in Figure 2.6. This circuit is an example of an operational amplifier configured with **negative feedback**, meaning that the output of the amplifier is connected (in this case by a resistor) to the inverting input v_-. The horizontal wire at the bottom of the plot is considered to be ground (zero volts) and is connected to the negative terminal of the dependent voltage source of the operational amplifier depicted in Figure 2.5. An equivalent circuit for the multiplier using the model in Figure 2.5 is shown in Figure 2.7. Solving this circuit (Exercise 2.1) yields the following relationship between the input voltage signal x and the output voltage signal y,

$$y = \frac{R_i(R_o - AR_2)}{R_i(R_2 + R_o) + R_1(R_2 + R_i + AR_i + R_o)} x. \tag{2.2.1}$$

For an ideal operational amplifier we let $A \to \infty$, $R_i \to \infty$ and $R_o \to 0$. In this case terms involving the product AR_i dominate and we are left with the simpler equation

$$y = -\frac{R_2}{R_1} x. \tag{2.2.2}$$

Thus, assuming an ideal operational amplifier, the circuit acts as a multiplier with constant $-\frac{R_2}{R_1}$.

The equation relating x and y is much simpler for the ideal operational amplifier. Fortunately this equation can be obtained directly using the following two rules:

1. the voltage at the inverting and non-inverting inputs are equal,

2. no current flows through the inverting and non-inverting inputs.

These rules are only useful for analysing circuits with negative feedback. Let us now rederive (2.2.2) using these rules. Because the non-inverting input is connected to ground, the voltage at the inverting input is zero. So, the voltage over resistor R_2 is $y = R_2 i$. Because no current flows through the inverting input the current through R_1 is also i and $x = -R_1 i$. Combining these results, the input voltage x and the output voltage y are related by

$$y = -\frac{R_2}{R_1}x.$$

In Test 2 the inverting amplifier circuit is constructed and the relationship above is tested using a computer soundcard.

We now consider another circuit consisting of an operational amplifier, two resistors and two capacitors depicted in Figure 2.8. Assuming an ideal operational amplifier, the voltage at the inverting terminal is zero because the non-inverting terminal is connected to ground. Thus, the voltage over capacitor C_2 and resistor R_2 is equal to y and, by Kirchoff's current law,

$$i = \frac{y}{R_2} + C_2 D(y).$$

Similarly, since no current flows through the inverting terminal,

$$i = -\frac{x}{R_1} - C_1 D(x).$$

Combining these equations yields

$$-\frac{x}{R_1} - C_1 D(x) = \frac{y}{R_2} + C_2 D(y). \tag{2.2.3}$$

Observe the similarity between this equation and that for the passive RC circuit (2.0.2) when $R_1 = R_2$ and $C_1 = 0$ (an open circuit). In this case

$$x = -y - R_1 C_2 D(y). \tag{2.2.4}$$

We call this the **active RC circuit**. This circuit is tested in Test 3.

Consider the circuit in Figure 2.9. Assuming an ideal operational amplifier, the input voltage x satisfies

$$-i = \frac{x}{R_1} + C_1 D(x).$$

The voltage over the capacitor C_2 is $y - R_2 i$ and so the current satisfies

$$i = C_2 D(y - R_2 i).$$

Combining these equations gives

$$-\frac{x}{R_1} - C_1 D(x) = C_2 D(y) + \frac{R_2 C_2}{R_1} D(x) + R_2 C_2 C_1 D^2(x),$$

and after rearranging,

$$D(y) = -\frac{1}{R_1 C_2}x - \left(\frac{R_2}{R_1} + \frac{C_1}{C_2}\right) D(x) - R_2 C_1 D^2(x).$$

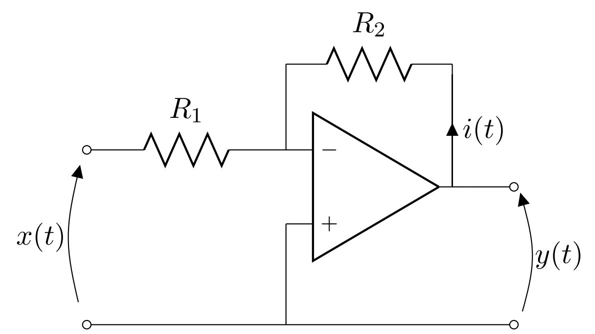

Figure 2.6: Inverting amplifier

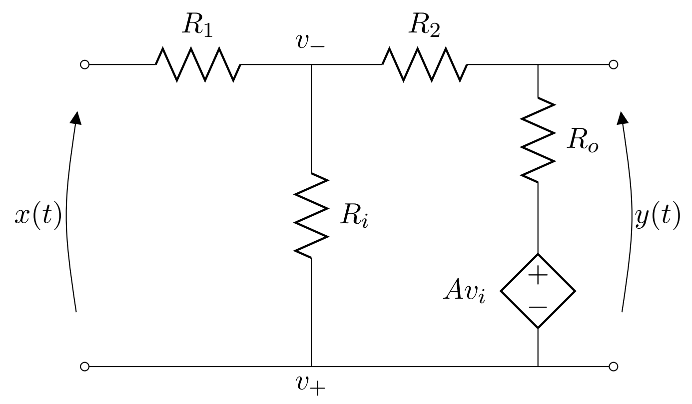

Figure 2.7: An equivalent circuit for the inverting amplifier from Figure 2.6 using the model for an operational amplifier in Figure 2.5. The symbol $v_i = v_+ - v_-$ is the voltage over resistor R_i.

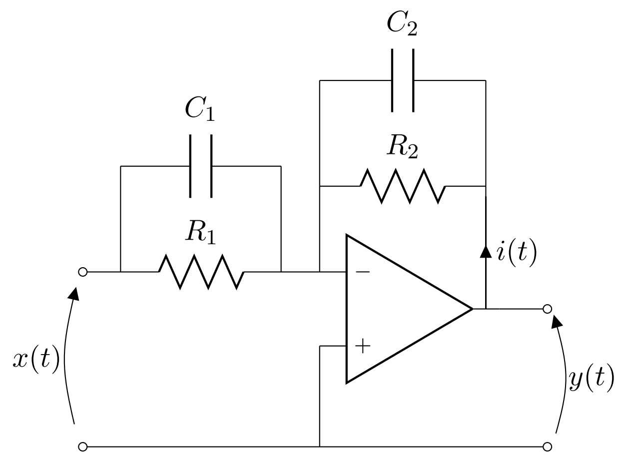

Figure 2.8: Operational amplifier configured with two capacitors and two resistors.

Test 2 (**Inverting amplifier**) In this test we construct the inverting amplifier circuit from Figure 2.6 with $R_2 \approx 22\text{k}\Omega$ and $R_1 \approx 12\text{k}\Omega$ that are accurate to within 5% of these values. The operational amplifier used is the Texas Instruments LM358P. Using a computer soundcard (an approximation of) the voltage signal

$$x(t) = \tfrac{1}{3}\sin(2\pi f_1 t) + \tfrac{1}{3}\sin(2\pi f_2 t)$$

with $f_1 = 100$ and $f_2 = 233$ is passed through the circuit. As in previous tests, the soundcard is used to sample the input signal x and the output signal y. Approximate reconstructions of the input signal $\tilde{x}$ and output signal $\tilde{y}$ are given according to (1.2.4) and (1.2.2). According to (2.1.1) we expect the approximate relationship

$$\tilde{y} \approx -\frac{R_2}{R_1}\tilde{x} = -\frac{11}{6}\tilde{x}.$$

Each of $\tilde{y}$, $\tilde{x}$ and $-\frac{11}{6}\tilde{x}$ are plotted in Figure 2.9.

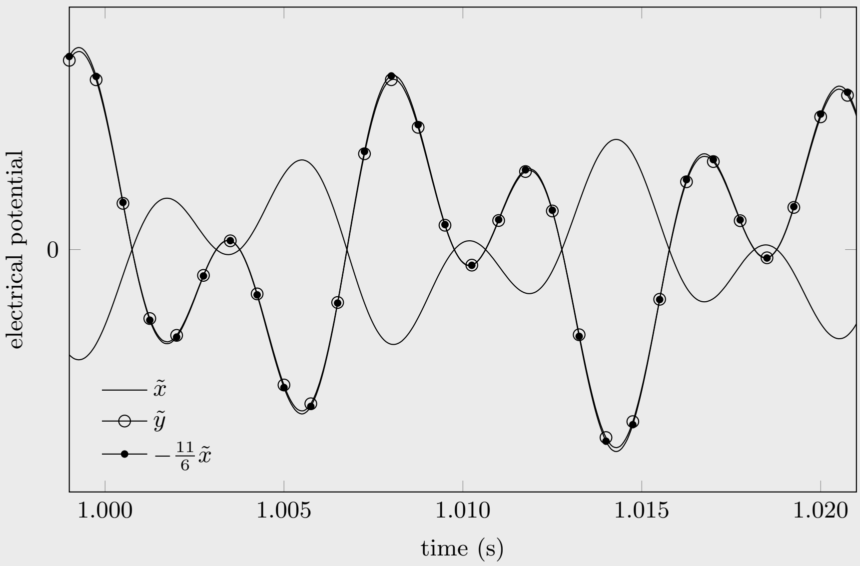

Figure 2.9: Plot of reconstructed input signal $\tilde{x}$ (solid line), output signal $\tilde{y}$ (solid line with circle) and hypothesised output signal $-\frac{11}{6}\tilde{x}$ (solid line with dot).

Test 3 (**Active RC circuit**) In this test we construct the circuit from Figure 2.8 with $R_1 \approx R_2 \approx 27\text{k}\Omega$ and $C_2 \approx 10\text{nF}$ accurate to within 5% of these values and $C_1 = 0$ (an open circuit). The operational amplifier used is a Texas Instruments LM358P. Using a computer soundcard (an approximation of) the voltage signal

$$x(t) = \tfrac{1}{3}\sin(2\pi f_1 t) + \tfrac{1}{3}\sin(2\pi f_2 t)$$

with $f_1 = 500$ and $f_2 = 1333$ is passed through the circuit. As in previous tests, the soundcard is used to sample the input signal x and the output signal y and approximate reconstructions $\tilde{x}$ and $\tilde{y}$ are given according to (1.2.4) and (1.2.2). According to (2.2.4) we expect the approximate relationship

$$\tilde{x} \approx -\frac{R_1}{R_2}\tilde{y} - R_1 C D(\tilde{y}) = -\tilde{y} - \frac{27}{10^5}D(\tilde{y}).$$

The derivative of the sinc function is

$$D(\text{sinc}, t) = \frac{1}{\pi t^2}\big(\pi t\cos(\pi t) - \sin(\pi t)\big), \tag{2.2.5}$$

and so,

$$D(\tilde{y}, t) = \frac{d}{dt}\left(\sum_{\ell=1}^{L} y_\ell \,\text{sinc}(Ft - \ell)\right) = F\sum_{\ell=1}^{L} y_\ell D(\text{sinc}, Ft - \ell). \tag{2.2.6}$$

Each of $\tilde{y}$, $\tilde{x}$ and $-\tilde{y} - \frac{27}{10^5}D(\tilde{y})$ are plotted in Figure 2.9.

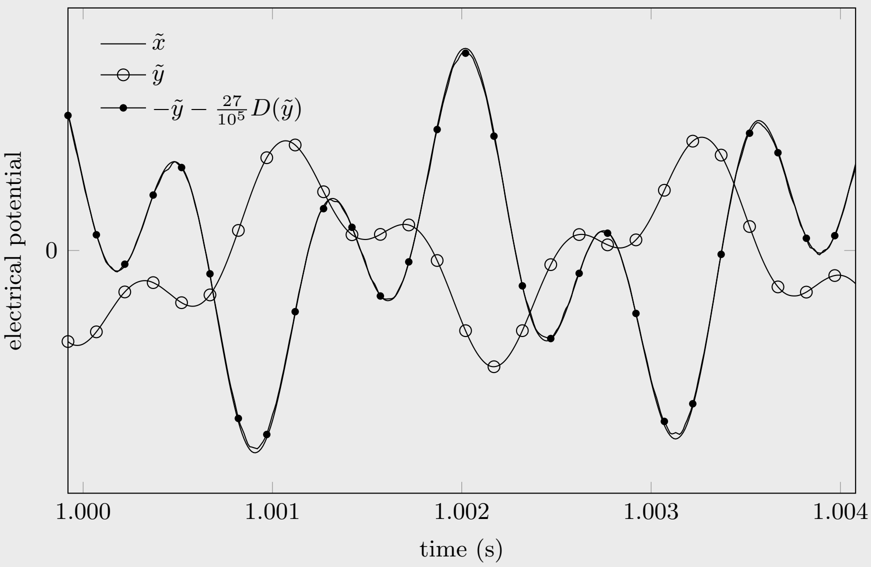

Figure 2.9: Plot of reconstructed input signal $\tilde{x}$ (solid line with circle), output signal $\tilde{y}$ (solid line), and hypothesised input signal $-\tilde{y} - \frac{27}{10^5}D(\tilde{y})$ (solid line with dot).

Put

$$K_i = \frac{1}{R_1 C_2}, \qquad K_p = \frac{R_2}{R_1} + \frac{C_1}{C_2}, \qquad K_d = R_2 C_1$$

and now

$$D(y) = -K_i x - K_p D(x) - K_d D^2(x). \tag{2.2.7}$$

This equation models what is called a **proportional-integral-derivative controller** or **PID controller**. The coefficients K_i, K_p and K_d are called the **integral gain**, **proportional gain**, and **derivative gain**.

The final active circuit we consider is called a **Sallen-Key** [Sallen and Key, 1955] and is depicted in Figure 2.10. Observe that the output of the amplifier is connected directly to the inverting input and is also connected to the noninverting input by a capacitor and resistor. This circuit has both negative *and* positive feedback. It is not immediately apparent that we can use the simplifying assumptions for an ideal operational amplifier with negative feedback. However, we will do so, and will find that it works in this case.

Let v_{R1}, v_{R2}, v_{C1}, and v_{C2} be the voltages over the components R_1, R_2, C_1, and C_2. Kirchoff's voltage law leads to the equations

$$x = v_{R1} + v_{R2} + v_{C2}, \qquad y = v_{C1} + v_{R2} + v_{C2}.$$

The voltage at the inverting and noninverting terminals is y and so the voltage over the capacitor C_2 is y, that is, $y = v_{C2}$. Using this, the equations above simplify to

$$x = v_{R1} + v_{R2} + y, \qquad v_{C1} = -v_{R2}.$$

The current i_2 through capacitor C_2 satisfies $i_2 = C_2 D(v_{C2}) = C_2 D(y)$. Because no current flows into the inverting terminal of the amplifier the current through R_2 is also i_2 and so $v_{R2} = R_2 i_2 = R_2 C_2 D(y)$. Substituting this into the equations above gives

$$x = v_{R1} + R_2 C_2 D(y) + y, \qquad v_{C1} = -R_2 C_2 D(y). \tag{2.2.8}$$

Kirchoff's current law asserts that $i + i_1 = i_2$. The current i through capacitor C_1 satisfies $i = C_1 D(v_{C1}) = -R_2 C_1 C_2 D^2(y)$ and the current through resistor R_1 satisfies

$$v_{R1} = R_1 i_1 = R_1(i_2 - i) = R_1 C_2 D(y) + R_1 R_2 C_1 C_2 D^2(y).$$

Substituting this into the equation on the left of (2.2.8) gives

$$x = y + C_2(R_1 + R_2) D(y) + R_1 R_2 C_1 C_2 D^2(y). \tag{2.2.9}$$

The Sallen-Key will be useful when we consider the design of analogue electrical filters in Section 5.2.

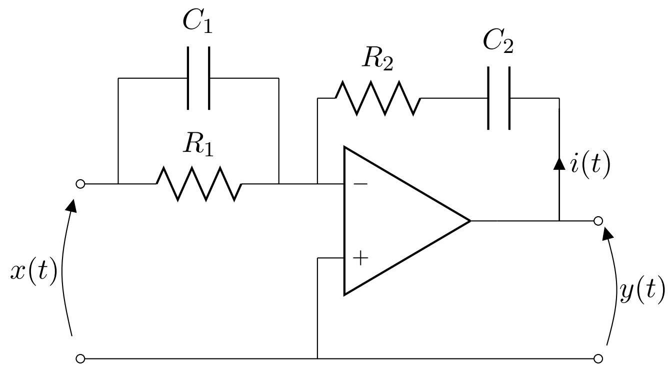

Figure 2.9: Operational amplifier implementing a **proportional-integral-derivative controller**.

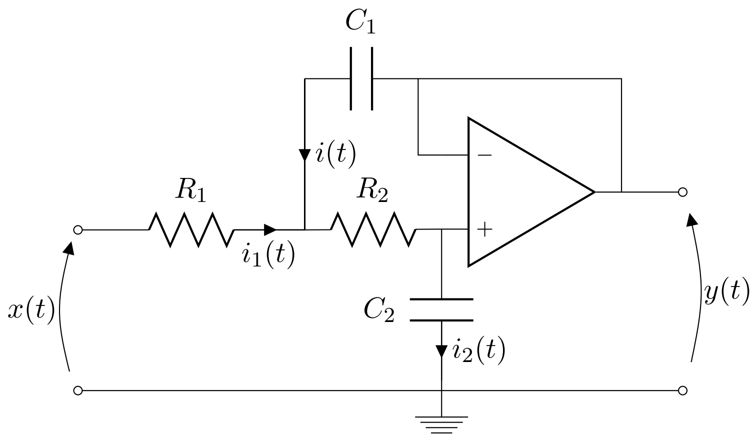

Figure 2.10: Operational amplifier implementing a **Sallen-Key**.

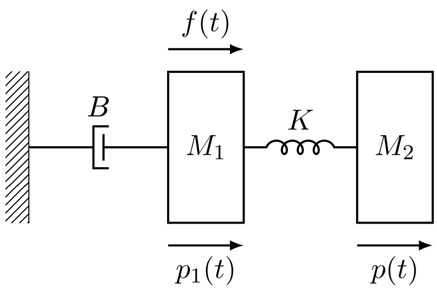

Figure 2.11: Two masses, a spring and a damper

2.3 Masses, springs, and dampers

A mechanical mass-spring-damper system was described in Section 2 and Figure 2.2. We now consider another mechanical system involving a different configuration of masses, a spring and a damper depicted in Figure 2.11. A mass M_1 is connected to a wall by a damper with constant B, and to another mass M_2 by a spring with constant K. A force represented by the signal f is applied to the first mass. We will derive a differential equation relating f with the position p of the second mass. Assume that the spring applies no force (is in equilibrium) when the masses are distance d apart. The forces due to the spring satisfy

$$f_{s1} = -f_{s2} = K(p - p_1 - d)$$

where f_{s1} and f_{s2} are signals representing the force due to the spring on mass M_1 and M_2 respectively. It is convenient to define the signal $g(t) = p_1(t) + d$ so that forces due to spring satisfy the simpler equation

$$f_{s1} = -f_{s2} = K(p - g).$$

The only force applied to M_2 is by the spring and so, by Newton's law, the acceleration of M_2 satisfies

$$M_2 D^2(p) = f_{s2}.$$

Substituting this into the previous equation gives a differential equation relating g and p,

$$Kg = Kp + M_2 D^2(p). \tag{2.3.1}$$

The force applied by the damper on mass M_1 is given by the signal

$$f_d = -BD(p_1) = -BD(g)$$

where the replacement of p_1 by g is justified because differentiation will remove the constant d. The cumulative force on M_1 is given by the signal

$$\begin{aligned} f_1 &= f + f_d + f_{s1} \\ &= f - Kg + Kp - BD(g), \end{aligned} \tag{2.3.2}$$

and by Newton's law the acceleration of M_1 satisfies

$$M_1 D^2(p_1) = M_1 D^2(g) = f_1.$$

Substituting this into (2.3.2) and using (2.3.1) we obtain a fourth order differential equation relating p and f,

$$f = BD(p) + (M_1 + M_2)D^2(p) + \frac{BM_2}{K}D^3(p) + \frac{M_1 M_2}{K}D^4(p). \tag{2.3.3}$$

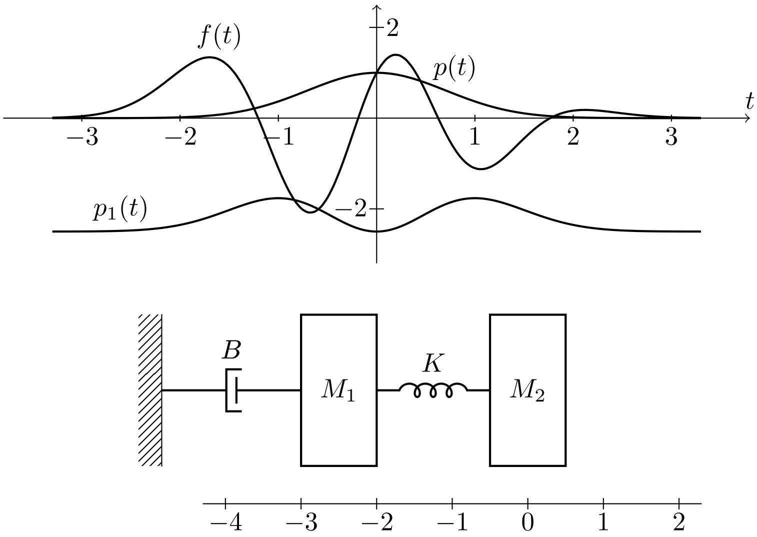

Figure 2.12: Solution of the system describing two masses with a spring and damper where $B = K = 1$ and $M_1 = M_2 = \frac{1}{2}$ and the position of the second mass is $p(t) = e^{-t^2}$.

Given the position of the second mass p we can readily solve for the corresponding force f and position of the first mass p. For example, if the constants $B = K = 1$ and $M_1 = M_2 = \frac{1}{2}$ and $d = \frac{5}{2}$, and if the position of the second mass satisfies

$$p(t) = e^{-t^2}$$

then, by application of (2.3.3) and (2.3.1),

$$f(t) = e^{-t^2}(1 + 4t - 8t^2 - 4t^3 + 4t^4), \qquad \text{and} \qquad p_1(t) = 2e^{-t^2}t^2 - \tfrac{5}{2}.$$

This solution is plotted in Figure 2.12.

2.4 Direct current motors

Direct current (DC) motors convert electrical energy, in the form of a voltage, into rotary kinetic energy [Nise, 2007, page 76]. We derive a differential equation relating the input voltage v to the angular position of the motor θ. Figure 2.13 depicts the components of a DC motor.

The voltages over the resistor and inductor satisfy

$$v_R = Ri, \qquad v_L = LD(i),$$

and the motion of the motor induces a voltage called the **back electromotive force** (EMF),

$$v_b = K_b D(\theta)$$

that we model as being proportional to the angular velocity of the motor. The input voltage now satisfies

$$v = v_R + v_L + v_b = Ri + LD(i) + K_b D(\theta).$$

The torque τ applied by the motor is modelled as being proportional to the current i,

$$\tau = K_\tau i.$$

A load with inertia J is attached to the motor. Two forces are assumed to act on the load, the torque τ applied by the current, and a torque $\tau_d = -BD(\theta)$ modelling a damper that acts proportionally against the angular velocity of the motor. By Newton's law, the angular acceleration of the load satisfies

$$JD^2(\theta) = \tau + \tau_d = K_\tau i - BD(\theta).$$

Combining these equations we obtain the 3rd order differential equation

$$v = \left(\frac{RB}{K_\tau} + K_b\right) D(\theta) + \frac{RJ + LB}{K_\tau} D^2(\theta) + \frac{LJ}{K_\tau} D^3(\theta)$$

relating voltage and motor position. In many DC motors the inductance L is small and can be ignored, leaving the simpler second order equation

$$v = \left(\frac{RB}{K_\tau} + K_b\right) D(\theta) + \frac{RJ}{K_\tau} D^2(\theta). \tag{2.4.1}$$

Given the position signal θ we can find the corresponding voltage signal v. For example, put the constants $K_b = K_\tau = B = R = J = 1$ and assume that

$$\theta(t) = 2\pi(1 + \operatorname{erf}(t))$$

where $\operatorname{erf}(t) = \frac{2}{\pi}\int_{-\infty}^{t} e^{-\tau^2} d\tau$ is the **error function**. The corresponding angular velocity $D(\theta)$ and voltage v satisfy

$$D(\theta, t) = 4\sqrt{\pi} e^{-t^2}, \qquad v(t) = 8\sqrt{\pi} e^{-t^2}(1 - t).$$

These signals are depicted in Figure 2.14. This voltage signal is sufficient to make the motor perform two revolutions and then come to rest.

2.5 Exercises

2.1. Analyse the inverting amplifier circuit in Figure 2.7 to obtain the relationship between input voltage x and output voltage y given by (2.2.1). You may wish to use a symbolic programming language (for example Sage, Mathematica, or Maple).

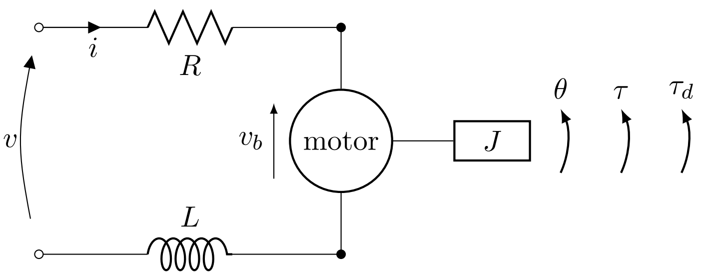

Figure 2.13: Diagram for a rotary direct current (DC) motor

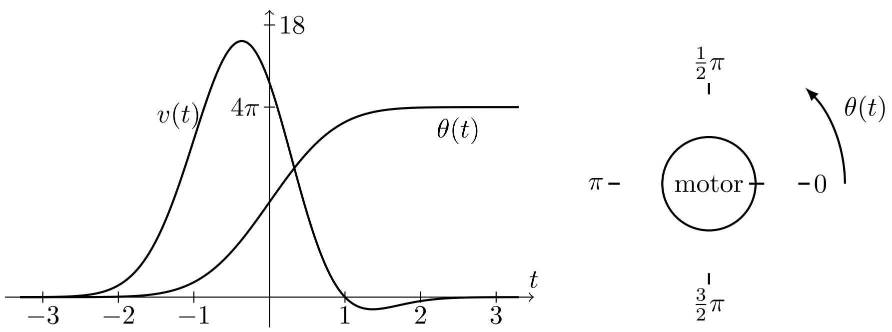

Figure 2.14: Voltage and corresponding angle for a DC motor with constants $K_b = K_\tau = B = R = J = 1$.

Chapter 3

Linear time invariant systems

In the previous section we derived differential equations that model mechanical, electrical, and electro-mechanical systems. The equations themselves often do not provide as much information about these system as we require. For example, we were able to find a signal p representing the position of the mass-spring-damper in Figure 2.2 given a particular force signal f is applied to the mass. However, it is not immediately obvious how to find the force signal f given a particular position signal p. We will be able to solve this problem and, more generally, to describe properties of systems modelled by linear differential equations with constant coefficient, if we make the added assumptions that the systems are **linear** and **time invariant**. We study linear time invariant systems in this chapter. Throughout this chapter H will denote a linear time invariant system.

3.1 Convolution, regular systems and the delta "function"

A large number of linear time invariant systems can be represented by a signal called the **impulse response**. The impulse response of a system H is a signal h such that

$$H(x,t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau,$$

that is, the response of H to input signal x can be represented as an integral equation involving x and the impulse response h. The integral is called a **convolution** and appears so often that a special notation is used for it. We write $h * x$ to indicate the signal that results from convolution of signals h and x, that is, $h * x$ is the signal satisfying

$$h * x = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau.$$

Those systems that have an impulse response we call **regular systems**[1]. Observe that regular systems are linear because

$$\begin{aligned} H(ax+by) &= h*(ax+by) \\ &= \int_{-\infty}^{\infty} h(\tau)\big(ax(t-\tau)+by(t-\tau)\big)d\tau \\ &= a\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau + b\int_{-\infty}^{\infty} h(\tau)y(t-\tau)d\tau \\ &= a(h*x)+b(h*y) \\ &= aH(x)+bH(y). \end{aligned} \tag{3.1.1}$$

The above equations show that convolution commutes with scalar multiplication and distributes with addition, that is,

$$h*(ax+by) = a(h*x)+b(h*y).$$

Regular systems are also time invariant because

$$\begin{aligned} T_\kappa\big(H(x)\big) &= T_\kappa(h*x) \\ &= \int_{-\infty}^{\infty} h(\tau)x(t-\kappa-\tau)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)T_\kappa(x,t-\tau)d\tau \\ &= h*T_\kappa(x) \\ &= H\big(T_\kappa(x)\big). \end{aligned}$$

We can define the impulse response of a regular system H in the following way. First define the signal

$$p_\gamma(t) = \begin{cases} \gamma, & 0<t\leq \frac{1}{\gamma} \\ 0, & \text{otherwise,} \end{cases}$$

that is, a rectangular shaped pulse of height γ and width $\frac{1}{\gamma}$. The signal p_γ is plotted in Figure 3.1 for $\gamma = \frac{1}{2}, 1, 2, 5$. As γ increases the pulse gets thinner and higher so as to keep the area under p_γ equal to one. Consider the response of the regular system H to the signal p_γ,

$$\begin{aligned} H(p_\gamma) &= h*p_\gamma \\ &= \int_{-\infty}^{\infty} h(\tau)p_\gamma(t-\tau)d\tau \\ &= \gamma\int_{t-1/\gamma}^{t} h(\tau)d\tau. \end{aligned}$$

[1] The name **regular system** is motivated by the term **regular distribution** [Zemanian, 1965]

Taking limits as $\gamma \to \infty$,

$$\lim_{\gamma\to\infty} H(p_\gamma) = \lim_{\gamma\to\infty} \gamma \int_{t-1/\gamma}^{t} h(\tau)d\tau = h(t) \text{ a.e.}$$

Thus, we define the impulse response of a regular system H as the limit

$$h = \lim_{\gamma\to\infty} H(p_\gamma). \tag{3.1.2}$$

The limit exists when H is regular. If this limit does not exist, the system is not regular and does not have an impulse response.

As an example, consider the integrator system

$$I_\infty(x) = \int_{-\infty}^{t} x(\tau)d\tau \tag{3.1.3}$$

described in Section 1.3. This systems response to p_γ is

$$I_\infty(p_\gamma, t) = \int_{-\infty}^{t} p_\gamma(\tau)d\tau = \begin{cases} 0, & t \leq 0 \\ \gamma t, & 0 < t \leq \frac{1}{\gamma} \\ 1, & t > \frac{1}{\gamma} \end{cases}$$

The response is plotted in Figure 3.1. Taking the limit as $\gamma \to \infty$ we find that the impulse response of the integrator is the step function

$$u(t) = \lim_{\gamma\to\infty} H(p_\gamma) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0. \end{cases} \quad \text{a.e.} \tag{3.1.4}$$

Some important systems do not have an impulse response. For example, the identity system T_0 does not because

$$\lim_{\gamma\to\infty} T_0(p_\gamma) = \lim_{\gamma\to\infty} p_\gamma$$

does not exist. Similarly, all the time shifters T_τ do not have impulse responses. However, it can be notationally useful to pretend that T_0 *does* have an impulse response and we denote it by the symbol δ called the **delta function**. The idea is to assign δ the property

$$\int_{-\infty}^{\infty} x(t)\delta(t)dt = x(0)$$

so that convolution of x and δ satisfies

$$\delta * x = \int_{-\infty}^{\infty} \delta(\tau)x(t-\tau)d\tau = x(t) = T_0(x).$$

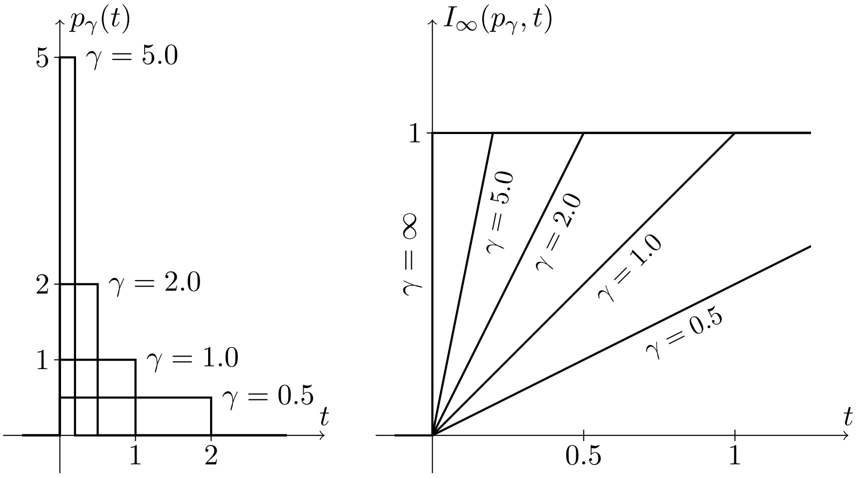

Figure 3.1: The rectangular shaped pulse p_γ for $\gamma = 0.5, 1, 2, 5$ and the response of the integrator (3.1.3) to p_γ for $\gamma = 0.5, 1, 2, 5, \infty$.

We now treat δ as if it were a signal. So $\delta(t-\tau)$ will represent the impulse response of the time shifter T_τ because

$$\begin{aligned} T_\tau(x) &= \delta(t-\tau) * x \\ &= \int_{-\infty}^{\infty} \delta(\kappa-\tau)x(t-\kappa)d\kappa \\ &= \int_{-\infty}^{\infty} \delta(k)x(t-\tau-k)dk \qquad \text{(change variable } k = \kappa - \tau) \\ &= x(t-\tau). \end{aligned}$$

For $a \in \mathbb{R}$ it is common to plot $a\delta(t-\tau)$ using an arrow of height a at $t = \tau$ as indicated in Figure 3.2. It is important to realise that δ is not actually a signal. It is not a function. However, it can be convenient to treat δ as if it were a function. The manipulations in the last set of equations, such as the change of variables, are not formally justified, but they do lead to the desired result $T_\tau(x) = x(t-\tau)$ in this case. In general, there is no guarantee that mechanical mathematical manipulations involving δ will lead to sensible results.

The only other non regular systems that we have use of are differentiators D^k, and it is convenient to define a similar notation for pretending that these systems have an impulse response. In this case we use the symbol δ^k and assign it the property

$$\int_{-\infty}^{\infty} x(t)\delta^k(t)dt = D^k(x, 0),$$

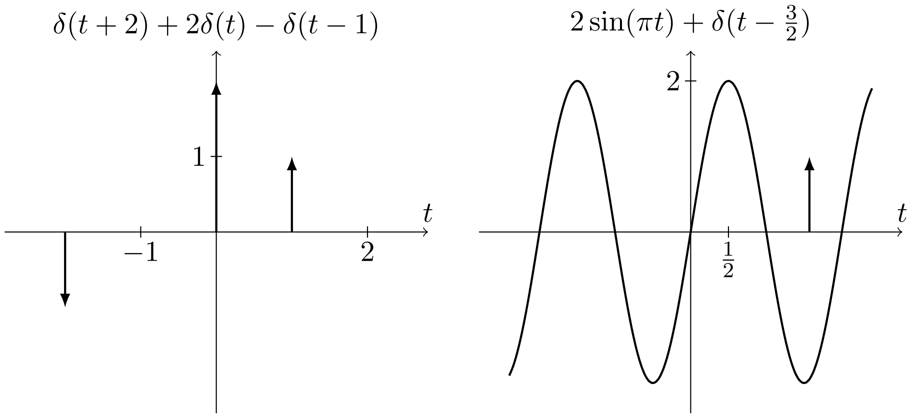

Figure 3.2: Plot of the "signal" $\delta(t+2)+2\delta(t)-\delta(t-1)$ (left) and the "signal" $2\sin(\pi t)+\delta(t-\frac{3}{2})$ (right).

so that convolution of x and δ is

$$\delta^k * x = \int_{-\infty}^{\infty} \delta^k(\tau)x(t-\tau)d\tau = D^k(x,t).$$

As with the delta function the symbol δ^k must be treated with care. This notation can be useful, but purely formal manipulations with δ^k may not lead to sensible results in general.

The impulse response h immediately yields some properties of the corresponding system H. For example, if $h(t) = 0$ for all $t < 0$, then H is causal because

$$H(x) = h * x = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$$

only depends on values of the input signal x at times less than or equal to t, i.e., only times $t-\tau$ with $\tau > 0$. The system H is stable if and only if h is absolutely integrable (Exercise 3.3).

Another important signal is the **step response** of a system that is defined as the response of the system to the step function $u(t)$. For example, the step response of the time shifter T_τ is the time shifted step function $T_\tau(u,t) = u(t-\tau)$. The step response of the integrator I_∞ is

$$I_\infty(u) = \int_{-\infty}^{t} u(\tau)d\tau = \begin{cases} \int_0^t d\tau = t & t > 0 \\ 0 & t \leq 0. \end{cases}$$

This signal is often called the **ramp function**. Not all systems have a step response. For example, the system with impulse response $u(-t)$ does not because the convolution of the step $u(t)$ and its reflection $u(-t)$ does not exist. If a system H has both an impulse response h and a step response

$H(u)$, then these two signals are related. To see this, observe that the step response is

$$H(u) = h * u = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau = I_\infty(h). \qquad (3.1.5)$$

Thus, the step response can be obtained by applying the integrator I_∞ to the impulse response.

3.2 Properties of convolution

The convolution $x * y$ of two signals x and y does not always exist. For example, if $x(t) = u(t)$ and $y(t) = 1$, then

$$x * y = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau = \int_{-\infty}^{\infty} u(\tau)d\tau = \int_{0}^{\infty} d\tau$$

is not finite for any t. We cannot convolve the step function u and the signal that is equal to 1 for all time. On the other hand, if $x(t) = y(t) = u(t)$, then

$$x * y = \int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau = \begin{cases} \int_0^t d\tau = \tau & t > 0 \\ 0 & t \leq 0, \end{cases}$$

if finite for all t.

We have already shown in (3.1.1) that convolution commutes with scalar multiplication and is distributive with addition, that is, for signals x, y, w and complex numbers a, b,

$$a(x * w) + b(y * w) = (ax + by) * w.$$

Convolution is commutative, that is, $x * y = y * x$ whenever these convolutions exist. To see this, write

$$\begin{aligned} x * y &= \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} x(t-\kappa)y(\kappa)d\kappa \qquad \text{(change variable } \kappa = t - \tau) \\ &= y * x. \end{aligned}$$

Convolution is also associative, that is, for signals x, y, z,

$$(x * y) * z = x * (y * z). \qquad \text{(see Exercise 3.2)}$$

By combining the associative and commutative properties we find that the order in which the convolutions in $x * y * z$ are performed does not mater, that is

$$x * y * z = y * z * x = z * x * y = y * x * z = x * z * y = z * y * x$$

provided that all the convolutions involved exist. More generally, the order in which any sequence of convolutions is performed does not change the final result.

3.3 Linear combining and composition

Let H_1 and H_2 be linear time invariant systems and let H be the system

$$H(x) = cH_1(x) + dH_2(x), \qquad c, d \in \mathbb{C}$$

formed by a linear combination of H_1 and H_2. The system H is linear because for signals x, y and complex numbers a, b,

$$\begin{aligned} H(ax + by) &= cH_1(ax + by) + dH_2(ax + by) \\ &= acH_1(x) + bcH_1(y) + adH_2(x) + bdH_2(y) \qquad \text{(linearity } H_1, H_2\text{)} \\ &= a\big(cH_1(x) + dH_2(x)\big) + b\big(cH_1(y) + dH_2(y)\big) \\ &= aH(x) + bH(y). \end{aligned}$$

The system is also time invariant because

$$\begin{aligned} H\big(T_\tau(x)\big) &= cH_1\big(T_\tau(x)\big) + dH_2\big(T_\tau(x)\big) \\ &= cT_\tau\big(H_1(x)\big) + dT_\tau\big(H_2(x)\big) \qquad \text{(time-invariance } H_1, H_2\text{)} \\ &= T_\tau\big(cH_1(x) + dH_2(x)\big) \qquad \text{(linearity } T_\tau\text{)} \\ &= T_\tau\big(H(x)\big). \end{aligned}$$

So, we can construct linear time invariant systems by **linearly combining** (adding and multiplying by constants) other linear time invariant systems. If H_1 and H_2 are regular systems this linear combining property can be expressed using their impulse responses h_1 and h_2. We have

$$\begin{aligned} H(x) &= aH_1(x) + bH_2(x) \\ &= ah_1 * x + bh_2 * x \\ &= (ah_1 + bh_2) * x \qquad \text{(distributivity of convolution)} \\ &= h * x, \end{aligned}$$

and so, H is a regular system with impulse response $h = ah_1 + bh_2$.

Another way to construct linear time invariant systems is by **composition**. Let H_1 and H_2 be linear time invariant systems and let

$$H(x) = H_2\big(H_1(x)\big),$$

that is, H first applies the system H_1 and then applies the system H_2. The composition $H_2\big(H_1(x)\big)$ only applies to those signals x in the domain of H_1 and such that the signal $H_1(x)$ is in the domain of H_2. The system H is linear because, for signals x, y and complex numbers a, b,

$$\begin{aligned} H(ax + by) &= H_2\big(H_1(ax + by)\big) \\ &= H_2\big(aH_1(x) + bH_1(y)\big) \qquad \text{(linearity } H_1\text{)} \\ &= aH_2\big(H_1(x)) + bH_2\big(H_1(y))\big) \qquad \text{(linearity } H_2\text{)} \\ &= aH(x) + bH(y). \end{aligned}$$

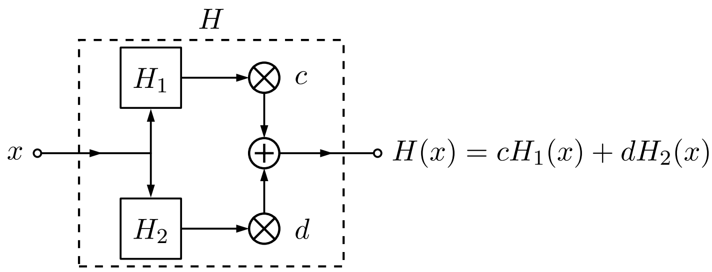

Figure 3.3: Block diagram depicting the linear combining property of linear time invariant systems. The system $cH_1(x) + dH_2(x)$ can be expressed as a single linear time invariant system $H(x)$.

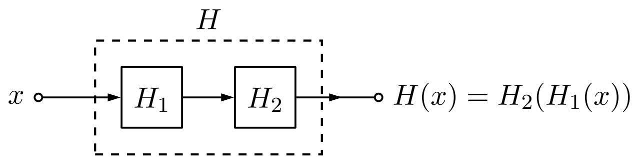

Figure 3.4: Block diagram depicting the composition property of linear time invariant systems. The system $H_2(H_1(x))$ can be expressed as a single linear time invariant system $H(x)$.

The system is also time invariant because

$$\begin{aligned} H\big(T_\tau(x)\big) &= H_2\big(H_1(T_\tau(x)\big) \\ &= H_2\big(T_\tau\big(H_1(x)\big)\big) && \text{(time-invariance } H_1) \\ &= T_\tau\big(H_2\big(H_1(x)\big)\big) && \text{(time-invariance } H_2) \\ &= T_\tau\big(H(x)\big). \end{aligned}$$

If H_1 and H_2 are regular systems the composition property can be expressed using their impulse responses h_1 and h_2. It follows that

$$\begin{aligned} H(x) &= H_2(H_1(x)) \\ &= h_2 * (h_1 * x) \\ &= (h_2 * h_1) * x && \text{(associativity of convolution)} \\ &= h * x, \end{aligned}$$

and so, H is a regular system with impulse response $h = h_2 * h_1$.

A wide variety of linear time invariant systems can now be constructed by linearly combining and composing simpler systems.

3.4 Eigenfunctions and the transfer function

Let $s = \sigma + j\omega \in \mathbb{C}$. Complex exponential signals of the form

$$e^{st} = e^{\sigma t}e^{j\omega t} = e^{\sigma t}\big(\cos(\omega t) + j\sin(\omega t)\big)$$

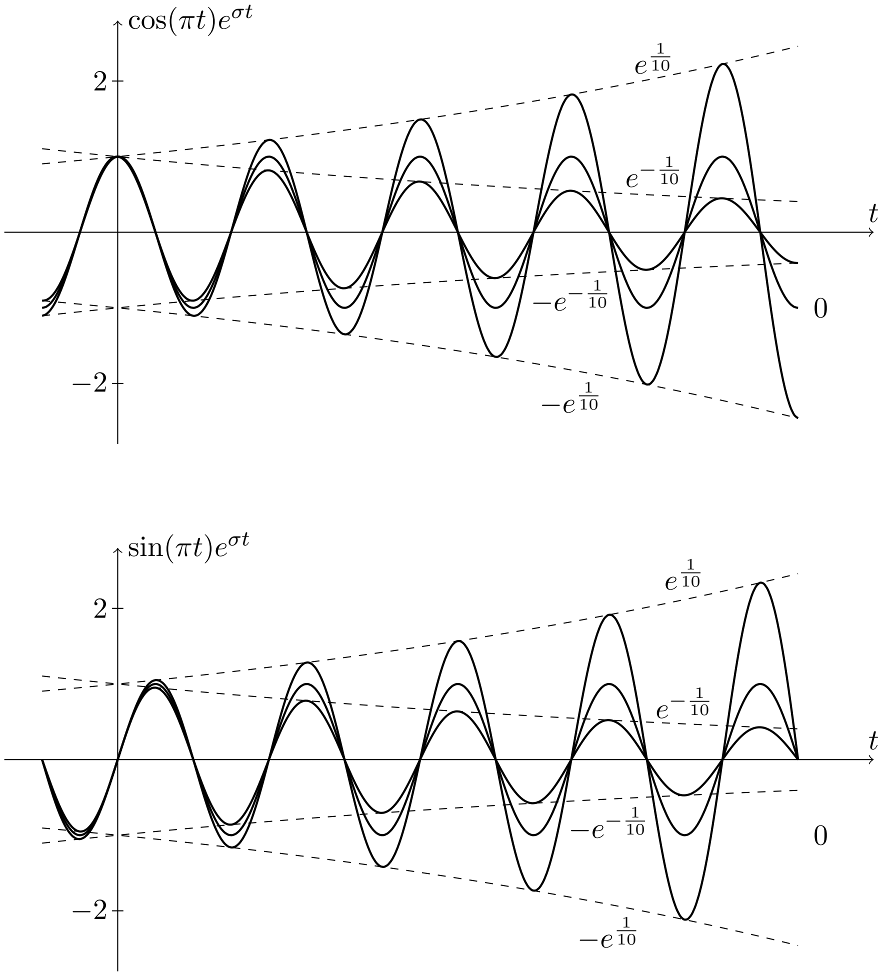

Figure 3.5: The function $\cos(\pi t)e^{\sigma t}$ (top) and $\sin(\pi t)e^{\sigma t}$ (bottom) for $\sigma = -\frac{1}{10}, 0, \frac{1}{10}$.

play an important role in the study of linear time invariant systems. The real and imaginary parts of the signal $e^{(\sigma+j\pi)t}$ with $\sigma = -\frac{1}{10}, 0, \frac{1}{10}$ are plotted in Figure 3.5. The signal is oscillatory when $\omega \neq 0$. The signal converges to zero as $t \to \infty$ when $\sigma < 0$ and diverges as $t \to \infty$ when $\sigma > 0$.

Let H be a linear time invariant system and let $y = H(e^{st})$ be the response of H to the exponential signal e^{st}. Consider the response of H to the time-shifted signal $e^{s(t+\tau)}$ for $\tau \in \mathbb{R}$. By time-invariance

$$H(e^{s(t+\tau)}, t) = H(e^{st}, t+\tau) = y(t+\tau) \qquad \text{for all } t, \tau \in \mathbb{R},$$

and by linearity

$$H(e^{s(t+\tau)}, t) = e^{s\tau} H(e^{st}, t) = e^{s\tau} y(t) \qquad \text{for all } t, \tau \in \mathbb{R}.$$

Combining these equations we obtain

$$y(t+\tau) = e^{s\tau} y(t) \qquad \text{for all } t, \tau \in \mathbb{R}.$$

This equation is satisfied by signals of the form $y(t) = \lambda e^{st}$ where λ is a complex number. That is, the response of a linear time invariant system H to an exponential signal e^{st} is the same signal e^{st} multiplied by some constant complex number λ. Due to this property exponential signals are called **eigenfunctions** of linear time invariant systems. The constant λ does not depend on t, but it does usually depend on the complex number s and the system H. To highlight this dependence on H and s we write $\lambda(H)(s)$ or $\lambda(H, s)$. Considered as a function of s, $\lambda(H, s)$ is called the **transfer function** of the system H. Thus, the transfer function satisfies

$$H(e^{st}) = \lambda(H, s) e^{s\tau}. \tag{3.4.1}$$

We can use these eigenfunctions to better understand the properties of systems modelled by differential equations, such as those in Section 2. As an example, consider the active electrical circuit from Figure 2.8. In the case that the resistors $R_1 = R_2$, and the capacitor $C_1 = 0$ (an open circuit) the differential equation relating the input voltage x and output voltage y is

$$x = -y - R_1 C_2 D(y).$$

We called this the **active RC** circuit. To simplify notation put $R = R_1$ and $C = C_2$ so that $x = -y - RCD(y)$. Observe what occurs when $y = ce^{st}$ is a complex exponential signal with $c \in \mathbb{C}$. We have

$$x = -ce^{st} - cRCse^{st} = -(1 + RCs)ce^{st} = -(1 + RCs)y,$$

and so, x is also a complex exponential signal. We immediately obtain the relationship

$$y = -\frac{1}{1 + RCs} x,$$

that holds whenever y (or equivalently x) is of the form ce^{st} with $c \in \mathbb{C}$. Let H be a system that maps the input voltage x to the output voltage y, i.e., H is a system that describes the active RC circuit. Putting $x = e^{st}$ in the equation above, we find that

$$y = H(x) = H(e^{st}) = -\frac{1}{1+RCs}e^{st},$$

and so, the transfer function of the system H describing the active RC circuit is

$$\lambda(H,s) = -\frac{1}{1+RCs}. \tag{3.4.2}$$

3.5 The spectrum

It is often of interest to focus on the transfer function when s is purely imaginary, that is, when $s = j\omega$. In this case the complex exponential signal takes the form

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t).$$

This signal is oscillatory when $\omega \neq 0$ and does not decay or explode as $|t| \to \infty$. The function

$$\Lambda(H,f) = \lambda\big(H, j2\pi f\big)$$

is called the **spectrum** of the system H. It follows from (3.4.1) that the response of the system to the complex exponential signal $e^{j2\pi ft}$ satisfies

$$H(e^{j2\pi ft}) = \lambda(H, j2\pi f)e^{j2\pi ft} = \Lambda(H,f)e^{j2\pi ft}, \qquad f \in \mathbb{R}.$$

It is of interest to consider the **magnitude spectrum** $|\Lambda(H)|$ and the **phase spectrum** $\angle\Lambda(H)$ separately. The notation $\angle$ denotes the **argument** (or **phase**) of a complex number. We have,

$$\Lambda(H,f) = |\Lambda(H,f)|\, e^{j\angle\Lambda(H,f)}$$

and correspondingly

$$H(e^{j2\pi ft}) = |\Lambda(H,f)|\, e^{j(2\pi ft + \angle\Lambda(H,f))}.$$

By taking real and imaginary parts we obtain the pair of real valued solutions

$$H\big(\cos(2\pi ft)\big) = |\Lambda(H,f)| \cos\big(2\pi ft + \angle\Lambda(H,f)\big),$$

$$H\big(\sin(2\pi ft)\big) = |\Lambda(H,f)| \sin\big(2\pi ft + \angle\Lambda(H,f)\big). \tag{3.5.1}$$

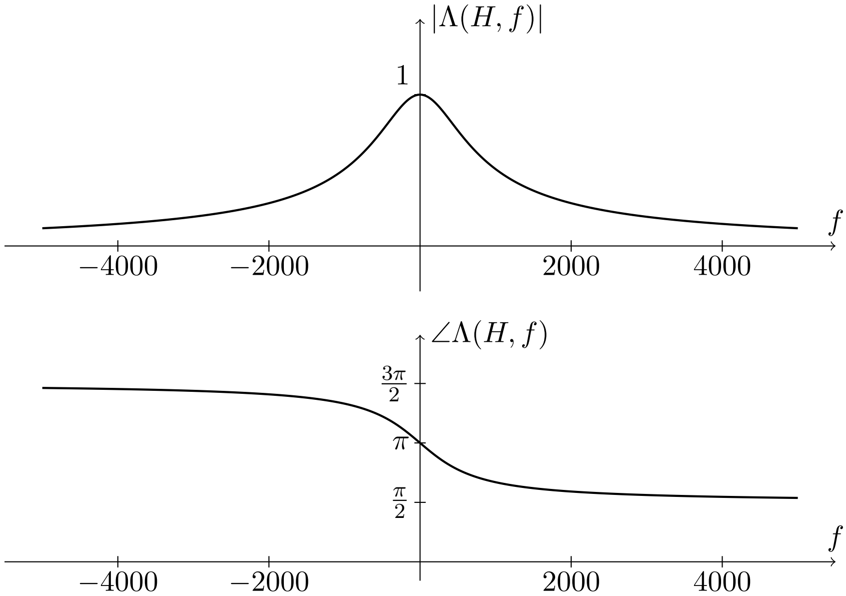

Figure 3.6: Magnitude spectrum (top) and phase spectrum (bottom) of the active RC circuit with $R = 27 \times 10^3$ and $C = 10 \times 10^{-9}$.

Consider again the active RC circuit with H the system mapping the input voltage x to the output voltage y. According to (3.4.2) the spectrum of H is

$$\Lambda(H,f) = -\frac{1}{1 + 2\pi RCfj}. \tag{3.5.2}$$

The magnitude and phase spectrum is

$$|\Lambda(H,f)| = \big(1 + 4\pi^2 R^2 C^2 f^2\big)^{-\frac{1}{2}}, \qquad \angle\Lambda(H,f) = \pi - \operatorname{atan}(2\pi RCf).$$

The magnitude and phase spectrum are plotted in Figure 3.6 when $R = 27 \times 10^3$ and $C = 10 \times 10^{-9}$. Observe from the plot of the magnitude spectrum that a low frequency sinusoidal signal, say 100Hz or less, input to the active RC circuit results in a sinusoidal output signal with the same frequency and approximately the same amplitude. However, a high frequency sinusoidal signal, say greater than 1000Hz, input to the circuit results in a sinusoidal output signal with the same frequency, but smaller amplitude. For this reason RC circuits are called **low pass filters**.

Test 4 (**Spectrum of the active RC circuit**) We test the hypothesis that the active RC circuit satisfies (3.5.1). To do this sinusoidal signals at varying frequencies of the form

$$x_k(t) = \sin(2\pi f_k t), \qquad f_k = \left\lceil 110 \times 2^{k/2} \right\rfloor, \quad k = 0, 1, \dots, 12$$

are input to the active RC circuit constructed as in Test 3 with $R = R_1 = 27\text{k}\Omega$ and $C = C_2 = 10\text{nF}$. The notation $\lceil \cdot \rfloor$ denotes rounding to the nearest integer with half integers rounded up. In view of (3.5.1) the expected output signals are of the form

$$y_k(t) = |\Lambda(H, f_k)| \sin\big(2\pi f_k t + \angle\Lambda(H, f_k)\big), \qquad k = 0, 1, \dots, 12.$$

This equality can also be shown directly using the differential equation for the active RC circuit.

Using the soundcard each signal x_k is played for a period of approximately 1 second and approximately $F = 44100$ samples are obtained. On the soundcard hardware used for this test samples near the beginning and end of playback are distorted. This appears to be an unavoidable feature of the soundcard. To alleviate this we discard the first 10^4 samples and use only the $L = 8820$ samples that follow (corresponding to 200ms of signal). After this process we have samples $x_{k,1}, \dots, x_{k,L}$ and $y_{k,1}, \dots, y_{k,L}$ of the input and output signals corresponding with the kth signal x_k. The samples are expected to take the form

$$x_{k,\ell} \approx x_k(P\ell - \tau) = \rho \sin(2\pi f_k P\ell - \theta)$$

and

$$y_{k,\ell} \approx y_k(\ell P - \tau) = |\Lambda(H, f_k)|\, \rho \sin\big(2\pi f_k P\ell - \theta + \angle\Lambda(H, f_k)\big)$$

where $P = \frac{1}{F}$ is the sample period, the positive real number ρ corresponds with the gain on the input and output of the soundcard, and $\theta = 2\pi f_k \tau$ corresponds with delays caused by discarding the first 10^4 samples and also unavoidable delays that occur when starting soundcard playback and recording.

We will not measure the gain ρ nor the delay θ, but will be able to test the properties of the circuit without knowledge of these. To simplify notation put $\gamma = 2\pi f_k P$. From the samples of the input signal $x_{k,1}, \dots, x_{k,L}$ compute the complex number

$$\begin{aligned} A &= \frac{2j}{L} \sum_{\ell=1}^{L} x_{k,\ell} e^{-j\gamma\ell} \\ &\approx \frac{2j}{L} \sum_{\ell=1}^{L} \rho \sin(\gamma\ell - \theta) e^{-j\gamma\ell} \\ &= \alpha + \alpha^* C \end{aligned}$$

where $\alpha = \rho e^{-j\theta}$ and α^* denotes the complex conjugate of α and

$$C = e^{-\gamma(L+1)} \frac{\sin(\gamma L)}{L \sin(\gamma)} \qquad \text{(Excersize 3.6).}$$

Similarly, from the samples of the output signal $y_{k,1}, \dots, y_{k,L}$ we compute the complex number

$$B = \frac{2j}{L}\sum_{\ell=1}^{L} y_{k,\ell} e^{-j\gamma\ell} \approx \beta + \beta^* C$$

where $\beta = \rho e^{-j\theta}\Lambda(H, f_k) = \alpha\Lambda(H, f_k)$. Now compute the quotient

$$Q_k = \frac{B - B^* C}{A - A^* C} \approx \frac{\beta(1 + |C|^2)}{\alpha(1 + |C|^2)} = \frac{\beta}{\alpha} = \Lambda(H, f_k).$$

Thus, we expect the quotient Q_k to be close to the spectrum of the active RC circuit evaluated at frequency f_k. We will test this hypothesis by observing the magnitude and phase of Q_k individually, that is, we will test the expected relationships

$$|Q_k| \approx |\Lambda(H, f_k)| = \sqrt{\frac{1}{1 + 4\pi^2 R^2 C^2 f_k^2}}$$

and

$$\angle Q_k \approx \angle\Lambda(H, f_k) = \pi - \operatorname{atan}(2\pi RC f_k)$$

for each $k = 0, \dots, 12$. Figure 3.7 plots the hypothesised magnitude and phase spectrum alongside the measurements Q_k for $k = 0, \dots, 12$.

3.6 Exercises

3.1. Show that convolution distributes with addition and commutes with scalar multiplication, that is, show that $a(x*w)+b(y*w) = (ax+by)*w$.

3.2. Show that convolution is associative. That is, if x, y, z are signals then $x * (y * z) = (x * y) * z$.

3.3. Show that a regular system is stable if and only if its impulse response is absolutely integrable.

3.4. Show that the system $H(x) = \int_{-1}^{1} \sin(\pi\tau)x(t + \tau)d\tau$ is linear time invariant and regular. Find and sketch the impulse response.

3.5. Show that $\sum_{\ell=1}^{L} e^{\beta\ell} = \frac{e^{\beta(L+1)} - e^{\beta}}{e^{\beta} - 1}$ (Hint: sum a geometric progression).

3.6. Show that

$$\frac{2j}{L}\sum_{\ell=1}^{L} \sin(\gamma\ell - \theta)e^{-j\gamma\ell} = \alpha + \alpha^* C$$

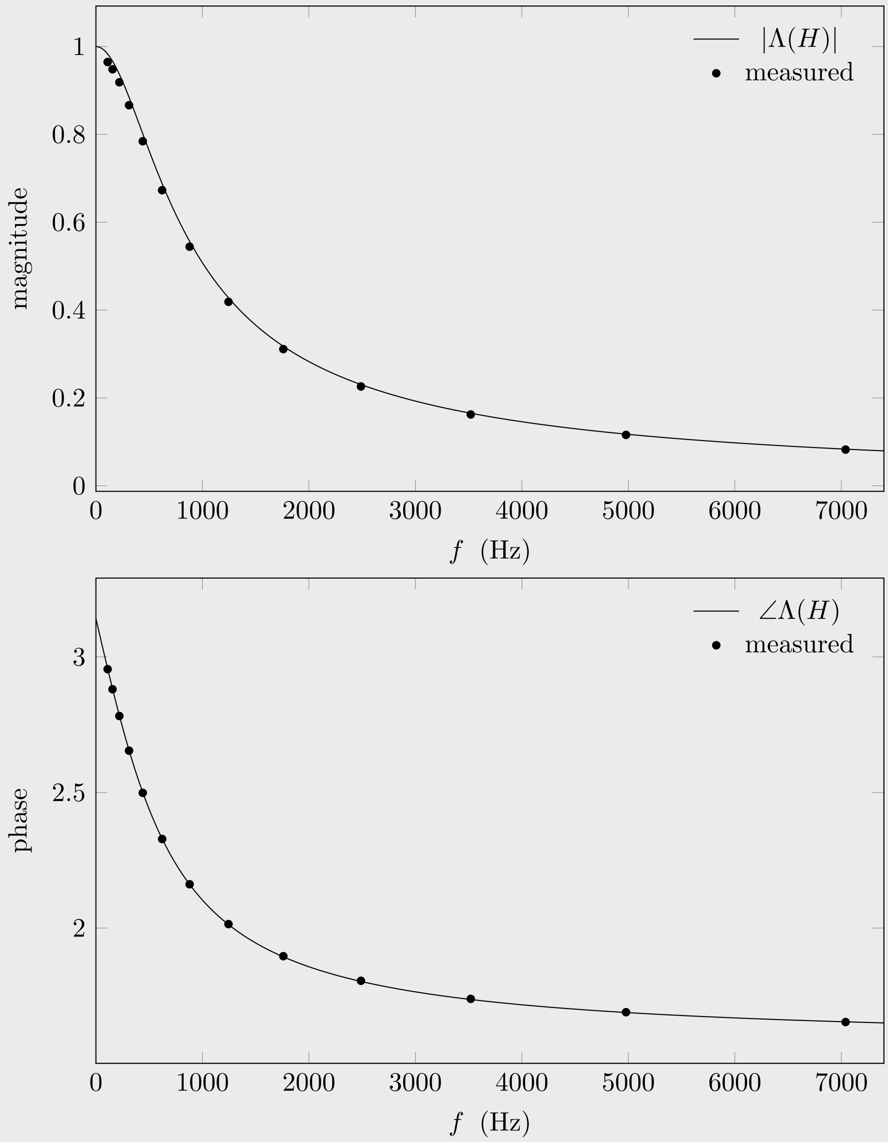

Figure 3.7: Hypothesised magnitude spectrum $|\Lambda(H, f)|$ (top) and phase spectrum $\angle\Lambda(H, f)$ (bottom) and the measured magnitude and phase spectrum $|Q_k|$ and $\angle Q_k$ for $k = 0, \dots, 12$ (dots).

where $\alpha = e^{-j\theta}$ and $C = e^{-j\gamma(L+1)} \frac{\sin(\gamma L)}{L \sin(\gamma)}$. (Hint: solve Exercise 3.5 first and then use the formula $2j \sin(x) = e^{jx} - e^{-jx}$).

Chapter 4

The Laplace transform

Let $x\colon \mathbb{R} \to \mathbb{C}$ be a complex valued function of the real line (a signal). The function

$$\mathcal{L}(x) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{4.0.1}$$

is called the **Laplace transform** of x. The Laplace transform is a function of the complex parameter s and if we need to indicate this we write $\mathcal{L}(x)(s)$ or $\mathcal{L}(x, s)$. The Laplace transform $\mathcal{L}(x)$ is not necessarily defined for all values of $s \in \mathbb{C}$. Let R be the set of real numbers such that $x(t)e^{-\sigma t}$ is absolutely integrable if and only if $\sigma \in R$, that is

$$\int_{-\infty}^{\infty} |x(t)|\, e^{-\sigma t}dt < \infty \qquad \text{if and only if } \sigma \in R.$$

In this case, $\mathcal{L}(x, s)$ is finite for all s with real part satisfying $\operatorname{Re}(s) \in R$ because

$$|\mathcal{L}(x, s)| = \left|\int_{-\infty}^{\infty} x(t)e^{-st}dt\right| \leq \int_{-\infty}^{\infty} |x(t)|\, e^{-\operatorname{Re}(s)t}dt < \infty.$$

The subset of the complex plane with real part from R is called the **region of convergence** (ROC) of the signal x.

For example, the Laplace transform of the right sided signal $e^{\alpha t}u(t)$ is

$$\begin{aligned}\mathcal{L}(e^{\alpha t}u(t)) &= \int_{-\infty}^{\infty} e^{\alpha t}e^{-st}u(t)dt \\ &= \int_{0}^{\infty} e^{(\alpha - s)t}dt \\ &= \lim_{t\to\infty} \frac{e^{(\alpha-s)t}}{\alpha - s} - \frac{1}{\alpha - s}.\end{aligned}$$

The limit converges for all s with $\operatorname{Re}(\alpha - s) < 0$. Thus, the Laplace transform of $e^{\alpha t}u(t)$ is

$$\mathcal{L}(e^{\alpha t}u(t)) = \frac{1}{s-\alpha} \qquad \operatorname{Re}(s) > \operatorname{Re}(\alpha)$$

The region of convergence of $e^{\alpha t}u(t)$ is the subset of the complex plane with real part greater than $\mathrm{Re}(\alpha)$. Figure 4.1 shows the region of convergence when $\mathrm{Re}(\alpha) = -2$. Now consider the left sided signal $e^{\beta t}u(-t)$ with Laplace transform

$$\mathcal{L}(e^{\beta t}u(-t)) = \lim_{t\to-\infty} \frac{e^{(\beta-s)t}}{\beta - s} + \frac{1}{\beta - s}.$$

The limit converges when $\mathrm{Re}(\beta - s) > 0$, and so,

$$\mathcal{L}(e^{\beta t}u(-t)) = \frac{1}{\beta - s} \qquad \mathrm{Re}(s) < \mathrm{Re}(\beta).$$

The region of convergence of $e^{\beta t}u(-t)$ is those $s \in \mathbb{C}$ such that $\mathrm{Re}(s) < \mathrm{Re}(\beta)$. The signal $ae^{\alpha t}u(t) + be^{\beta t}u(-t)$ has Laplace transform

$$\begin{aligned}\mathcal{L}\big(ae^{\alpha t}u(t) + be^{\beta t}u(-t)\big) &= \int_{-\infty}^{\infty} \big(ae^{\alpha t}u(t) + be^{\beta t}u(-t)\big)e^{-st}dt \\ &= a\int_{-\infty}^{\infty} e^{\alpha t}u(t)e^{-st}dt + b\int_{-\infty}^{\infty} e^{\beta t}u(-t)e^{-st}dt \\ &= a\mathcal{L}(e^{\alpha t}u(t)) + b\mathcal{L}(e^{\beta t}u(-t))\end{aligned}$$

that is finite only when $\mathrm{Re}(\alpha) < \mathrm{Re}(s) < \mathrm{Re}(\beta)$. The corresponding ROC is shown in Figure 4.1 when $\mathrm{Re}(\alpha) = -2$ and $\mathrm{Re}(\beta) = 3$. In the previous equation we have discovered that the Laplace transform is **linear**, that is, for signals x and y and constants a and b,

$$\mathcal{L}(ax + by) = a\mathcal{L}(x) + b\mathcal{L}(y). \tag{4.0.2}$$

In words: the Laplace transform of a linear combination of signals is the same linear combination of the Laplace transforms of those signals.

In the previous example the Laplace transform is guaranteed to be finite for any s if $\mathrm{Re}(\alpha) \geq \mathrm{Re}(\beta)$, and the region of convergence is correspondingly the empty set. Other signals also have this property. For example, the signal $x(t) = 1$ because

$$\mathcal{L}(1) = \int_{\infty}^{\infty} e^{-st}dt = \lim_{t\to-\infty} \frac{e^{-st}}{s} - \lim_{t\to\infty} \frac{e^{-st}}{s}$$

and the limit as $t \to -\infty$ converges only when $\mathrm{Re}(s) < 0$ while the limit as $t \to \infty$ converges only when $\mathrm{Re}(s) > 0$.

As a final example, consider the rectangular pulse

$$\Pi(t) = \begin{cases} 1 & -\frac{1}{2} < t \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Its Laplace transform is

$$\mathcal{L}(\Pi) = \int_{-\infty}^{\infty} \Pi(t)e^{-st}dt = \int_{-1/2}^{1/2} e^{-st}dt = \frac{e^{s/2} - e^{-s/2}}{s} \tag{4.0.3}$$

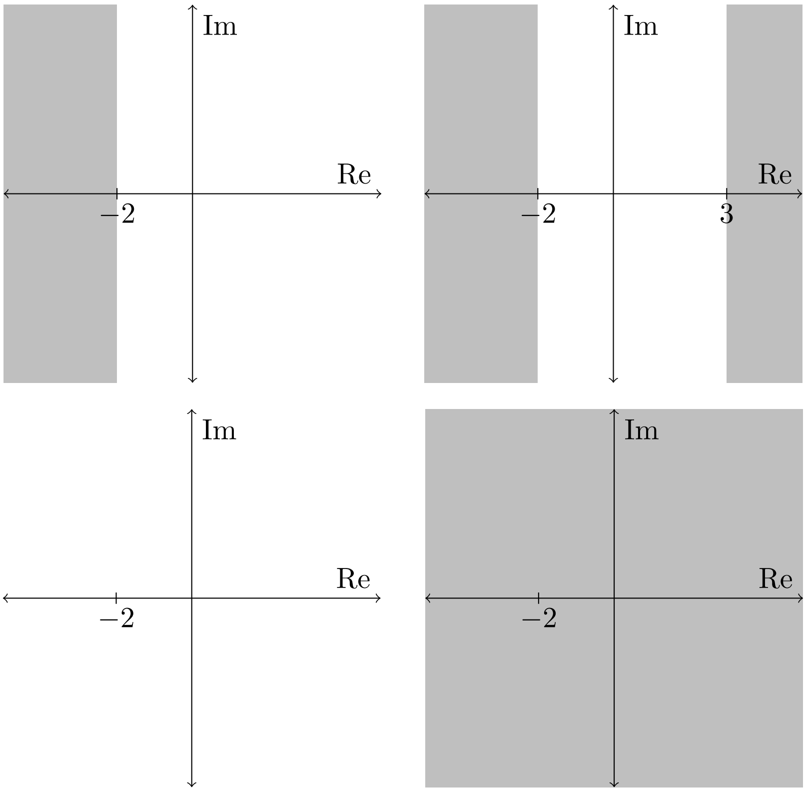

Figure 4.1: Regions of convergence (unshaded) for the signal $e^{-2t}u(t)$ (top left), the signal $e^{-2t}u(t) + e^{3t}u(-t)$ (top right), the rectangular pulse Π (bottom left), and the constant signal $x(t) = 1$ (bottom right).

and is finite for all $s \in \mathbb{C}$. The region of convergence of the rectangular pulse Π is the entire complex plane. The examples just given exhibit all the possible types of regions of convergence. The region of convergence is either the entire complex plane, a left or right half plane, a vertical strip, or the empty set.

Given the Laplace transform $\mathcal{L}(x)$ the signal x can be recovered by the **inverse Laplace transform**

$$x(t) = \frac{1}{2\pi j} \lim_{\omega \to \infty} \int_{\sigma - j\omega}^{\sigma - j\omega} \mathcal{L}(x, s) e^{st} ds,$$

where σ is a real number that is inside the region of convergence of x. Solving the integral above typically requires a special type of integration called **contour integration** that we will not consider here [Stewart and Tall, 2004]. For our purposes, and for many engineering purposes, it suffices to remember only the following Laplace transform pair

$$\mathcal{L}\big(t^n u(t)\big) = \frac{n!}{s^{n+1}} \qquad \operatorname{Re}(s) > 0, \tag{4.0.4}$$

where $n \geq 0$ is an integer (Exercise 4.2). Let $x(t)$ be a signal with region of convergence R_x. The Laplace transforms of the signal $x(t)$ and the signal $e^{\alpha t}x(t)$ are related. To see this write

$$\begin{aligned}
\mathcal{L}\big(e^{\alpha t}x(t), s\big) &= \int_{-\infty}^{\infty} e^{\alpha t}x(t)e^{-st}dt \\
&= \int_{-\infty}^{\infty} x(t)e^{-(s-\alpha)t}dt \\
&= \mathcal{L}(x, s-\alpha) \qquad s-\alpha \in R_x.
\end{aligned} \tag{4.0.5}$$

This is called the **frequency shift rule**. Combining the frequency shift rule with (4.0.4) we obtain the transform pair

$$\mathcal{L}\big(t^n e^{\alpha t}u(t)\big) = \mathcal{L}\big(t^n u(t), s-\alpha\big) = \frac{n!}{(s-\alpha)^{n+1}} \qquad \mathrm{Re}(s) > \mathrm{Re}(\alpha), \tag{4.0.6}$$

where $n \geq 0$ is an integer. This is the only Laplace transform pair we require here.

A useful relationship exists between the Laplace transform of a signal x and its time scaled version $x(\alpha t)$ where $\alpha \neq 0$. If x is a signal with region of convergence R then the time scaled signal $x(\alpha t)$ with $\alpha \neq 0$ has Laplace transform

$$\mathcal{L}\big(x(\alpha t), s\big) = \frac{1}{|\alpha|}\mathcal{L}(x, s/\alpha), \qquad \mathrm{Re}(s/\alpha) \in R. \tag{4.0.7}$$

This is called the **time scaling property** (Excercise 4.10).

4.1 The transfer function and the Laplace transform

Recall from Section 3.4 that exponential signals are **eigenfunctions** of linear time invariant systems. That is, if $s \in \mathbb{C}$ such that the complex exponential signal e^{st} is in the domain of H, then response of H to e^{st} is λe^{st} where $\lambda \in \mathbb{C}$ is a constant that does not depend on t, but may depend on s and the system H. To highlight this dependence on H and s we write $\lambda(H, s)$ or $\lambda(H)(s)$ and do not distinguish between these notations. Considered as a function of s, $\lambda(H, s)$ is called the **transfer function** of the system H. For a given system H, we would like to understand how $\lambda(H, s)$ behaves as s changes. In what follows we regularly drop the argument "(s)" and simply write $\lambda(H)$ as the transfer function of H.

Assume that H is a regular system with impulse response h. In this case,

$$\begin{aligned}
H(e^{st}) = e^{st}\lambda(H,s) &= h * e^{st} \\
&= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau \\
&= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \\
&= e^{st}\mathcal{L}(h,s),
\end{aligned}$$

and so $\lambda(H) = \mathcal{L}(h)$. That is, the transfer function of a regular system is precisely the Laplace transform of its impulse response. The region of convergence of the impulse response describes a set of complex exponential signals e^{st} in the domain of the system and we refer to this as the region of convergence of the *system*. In this way, both signals and systems have regions of convergence.

The transfer functions of the time-shifter and differentiator can be obtained by inspection. For the time-shifter

$$T_\tau(e^{st}) = e^{s(t-\tau)} = e^{-s\tau}e^{st} \qquad \text{and so} \qquad \lambda(T_\tau) = e^{-s\tau}. \tag{4.1.1}$$

The region of convergence is the whole complex plane $s \in \mathbb{C}$. For the special case of the identity system T_0 we obtain $\lambda(T_0) = 1$. For the differentiator

$$D(e^{st}) = \frac{d}{dt}e^{st} = se^{st} \qquad \text{and so} \qquad \lambda(D) = s.$$

The region of convergence is the whole complex plane $s \in \mathbb{C}$. More generally, for the kth differentiator

$$D^k(e^{st}) = \frac{d^k}{dt^k}e^{st} = s^k e^{st} \qquad \text{and so} \qquad \lambda(D^k) = s^k. \tag{4.1.2}$$

The region of convergence is again the whole complex plane. These results motivate assigning the following Laplace transforms to the delta "function" and its derivatives

$$\mathcal{L}(\delta) = 1, \qquad \mathcal{L}(\delta^k) = s^k.$$

These conventions are common in the literature [Oppenheim et al., 1996].

Let H_1 and H_2 be linear time invariant systems with regions of convergenence $R_1 \subseteq \mathbb{C}$ and $R_2 \subseteq \mathbb{C}$. Let $H = aH_1 + bH_2$ be a linear combination of H_1 and H_2. The response of H to the complex exponential signal e^{st} is

$$\begin{aligned}
H(e^{st}) &= aH_1(e^{st}) + bH_2(e^{st}) \\
&= a\lambda(H_1)e^{st} + b\lambda(H_2)e^{st} && s \in R_1 \cap R_2, \\
&= \big(a\lambda(H_1) + b\lambda(H_2)\big)e^{st} && s \in R_1 \cap R_2, \\
&= \lambda(H)e^{st} && s \in R_1 \cap R_2,
\end{aligned}$$

and so,

$$\lambda(H) = a\lambda(H_1) + b\lambda(H_2) \qquad s \in R_1 \cap R_2.$$

That is, the transfer function of a linear combination of systems is the same linear combination of the transfer functions. The region of convergence of the linear combination is the intersection of the regions of convergence of the systems being combined.

Now let H be the system given by the composition of H_1 and H_2, that is, $H(x) = H_1\big(H_2(x)\big)$. The response of H to the signal e^{st} is

$$\begin{aligned} H(e^{st}) &= H_1\big(H_2(e^{st})\big) \\ &= H_1\big(\lambda(H_2)e^{st}\big) && s \in R_2 \\ &= \lambda(H_2)H_1(e^{st}) && s \in R_2 \\ &= \lambda(H_2)\lambda(H_1)e^{st} && s \in R_1 \cap R_2 \\ &= \lambda(H)e^{st} && s \in R_1 \cap R_2, \end{aligned}$$

and so,

$$\lambda(H) = \lambda(H_1)\lambda(H_2) \qquad s \in R_1 \cap R_2. \tag{4.1.3}$$

That is, the transfer function of a composition of linear time invariant systems is the multiplication of the transfer functions of those systems. The region of convergence of the composition is the intersection of the regions of convergence of the systems being composed.

We showed in Section 3.3 that if H_1 and H_2 are regular systems with impulse responses h_1 and h_2, then the impulse of the system $H(x) = H_1\big(H_2(x)\big)$ is given by the convolution $h = h_1 * h_2$. Because,

$$\lambda(H) = \mathcal{L}(h) \qquad \lambda(H_1) = \mathcal{L}(h_1) \qquad \lambda(H_2) = \mathcal{L}(h_2),$$

and using (4.1.3), we obtain,

$$\mathcal{L}(h_1 * h_2) = \mathcal{L}(h) = \lambda(H) = \lambda(H_1)\lambda(H_2) = \mathcal{L}(h_1)\mathcal{L}(h_2), \qquad s \in R_1 \cap R_2.$$

Putting $x = h_1$, $y = h_2$, $R_x = R_1$, and $R_y = R_2$ we obtain the **convolution theorem**,

$$\mathcal{L}(x * y) = \mathcal{L}(x)\mathcal{L}(y), \qquad s \in R_x \cap R_y. \tag{4.1.4}$$

In words: the Laplace transform of a convolution of signals is the multiplication of their Laplace transforms.

Let $y = H(x)$ be the response of the system H to input signal x. Suppose that x has region of convergence R_x and that y has region of convergence R_y. In the case that H is regular with impulse response h we have $y = h * x$ and the convolution theorem asserts that

$$\mathcal{L}(y) = \mathcal{L}(h)\mathcal{L}(x) = \lambda(H)\mathcal{L}(x), \qquad s \in R_x \cap R_y \tag{4.1.5}$$

where, in this case, $R_y = R_h \cap R_x$ where R_h is the region of convergence of the impulse response h. Thus, the Laplace transform of the output signal $y = H(x)$ is the transfer function of the system H multiplied by the Laplace transform of the input signal x. This result also holds when H is a time shifter or a differentiator (Exercise 4.4).

4.2 Solving differential equations

Assume we have a system modelled by a differential equation of the form

$$\sum_{\ell=0}^{m} a_\ell D^\ell(x) = \sum_{\ell=0}^{k} b_\ell D^\ell(y), \tag{4.2.1}$$

where x and y are signals. Taking Laplace transforms of both sides of this equation,

$$\begin{aligned}
\mathcal{L}\left(\sum_{\ell=0}^{m} a_\ell D^\ell(x)\right) &= \mathcal{L}\left(\sum_{\ell=0}^{k} b_\ell D^\ell(y)\right) \\
\sum_{\ell=0}^{m} a_\ell \mathcal{L}\big(D^\ell(x)\big) &= \sum_{\ell=0}^{k} b_\ell \mathcal{L}\big(D^\ell(y)\big) && \text{(linearity (4.0.2))} \\
\sum_{\ell=0}^{m} a_\ell \lambda(D^\ell)\mathcal{L}(x) &= \sum_{\ell=0}^{k} b_\ell \lambda(D^\ell)\mathcal{L}(y) && \text{(using (4.1.5))} \\
\sum_{\ell=0}^{m} a_\ell s^\ell \mathcal{L}(x) &= \sum_{\ell=0}^{k} b_\ell s^\ell \mathcal{L}(y). && \text{(since } \lambda(D^\ell) = s^\ell \text{ by (4.1.2))}
\end{aligned}$$

We have obtained an equation relating the Laplace transforms of x and y,

$$\mathcal{L}(x)(a_0 + a_1 s + \dots a_m s^m) = \mathcal{L}(y)(b_0 + b_1 s + \dots b_k s^k).$$

Rearranging this equation we obtain

$$\mathcal{L}(y) = \frac{a_0 + a_1 s + \dots a_m s^m}{b_0 + b_1 s + \dots b_k s^k}\mathcal{L}(x).$$

Let H be a linear time invariant system such that $y = H(x)$ whenever x and y satisfy the differential equation (4.2.1). According to (4.1.5) the transfer function of H is

$$\lambda(H) = \frac{\mathcal{L}(y)}{\mathcal{L}(x)} = \frac{a_0 + a_1 s + \dots a_m s^m}{b_0 + b_1 s + \dots b_k s^k}.$$

Properties of H can be obtained by inspecting this transfer function. For example, the impulse response of H (if it exists) can be obtained by applying the inverse Laplace transform.

We now apply these results to the differential equations that model the RC electrical circuit from Figure 2.1 and the mass spring damper from Figure 2.2. The RC circuit is an example of what is called a **first order system** and the mass spring damper is an example of what is called a **second order system**.

4.3 First order systems

Recall the passive electrical RC circuit from Figure 2.1. The differential equation modelling this circuit is (2.0.1),

$$x = y + RCD(y)$$

where x is the input voltage signal, y is the voltage over the capacitor, and R and C are the resistance and capacitance. The RC circuit is an example of a **first order system**. Let H be a system mapping the input voltage signal x to the output voltage signal y. We will discover the impulse response of H. Taking the Laplace transform on both sides of the differential equation gives

$$\mathcal{L}(x) = (1 + RCs)\mathcal{L}(y)$$

and it follows that the transfer function of H is

$$\lambda(H) = \frac{\mathcal{L}(y)}{\mathcal{L}(x)} = \frac{1}{1 + RCs} = \frac{r}{r + s}$$

where $r = \frac{1}{RC}$. The value $\frac{1}{r} = RC$ is called the **time constant**. The impulse response of H is given by the inverse of this Laplace transform. There are two signals with Laplace transform $\frac{r}{r+s}$: the right sided signal $re^{-rt}u(t)$ with region of convergence $\mathrm{Re}(s) > -r$, and the left sided signal $-re^{-rt}u(-t)$ with region of convergence $\mathrm{Re}(s) < -r$. The RC circuit (and in fact all physically realisable systems) are expected to be causal. For this reason, the left sided signal $-re^{-rt}u(-t)$ cannot be the impulse response of H. The impulse response is the right sided signal

$$h(t) = re^{-rt}u(t).$$

Given an input voltage signal x we can now find the corresponding output signal $y = H(x)$ by convolving x with the impulse response h. That is,

$$y = H(x) = h * x = \int_{-\infty}^{\infty} re^{-r\tau}u(\tau)x(t - \tau)d\tau = r\int_{0}^{\infty} e^{-r\tau}x(t - \tau)d\tau.$$

If $r \geq 0$ the impulse response is absolutely integrable, that is,

$$\begin{aligned}\|h\|_1 &= \int_{-\infty}^{\infty} \left| re^{-rt}u(t) \right| dt \\ &= r\int_0^{\infty} e^{-rt} dt \\ &= 1 - \lim_{t\to\infty} e^{-rt} = 1,\end{aligned}$$

and the system is stable (Exercise 3.3). However, if $r < 0$ the impulse response is not absolutely integrable and the system is not stable. Figure 4.3 shows the impulse response when $r = -\frac{1}{5}, -\frac{1}{3}, -\frac{1}{2}, 1, 2$. In a passive electrical RC circuit the resistance R and capacitance C are always positive and $r = \frac{1}{RC}$ is positive. For this reason, passive electrical RC circuits are always stable.

From (3.1.5), the step response $H(u)$ is given by applying the integrator I_∞ to the impulse response, that is,

$$H(u) = I_\infty(h) = \int_{-\infty}^{t} re^{-r\tau}u(\tau)d\tau = \begin{cases} r\int_0^t e^{-r\tau}d\tau & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

or more simply

$$H(u) = \big(1 - e^{-rt}\big)u(t). \tag{4.3.1}$$

This step response in plotted in Figure 4.3.

Test 5 (**The impulse response of the active RC circuit**) In this test we again use the active RC circuit from Test 3 with resistors $R = R_1 = R_2 = 27\mathrm{k}\Omega$ and capacitor $C = C_2 = 10\mathrm{nF}$. In Test 3 we applied the differential equation (2.2.4) to the reconstructed output signal $\tilde{y}$ and asserted that the resulting signal was close to the reconstructed input signal $\tilde{x}$. In this test we instead convolve the input signal $\tilde{x}$ with the impulse response

$$h(t) = -\tfrac{1}{RC}e^{-t/RC}u(t) = -re^{-rt}u(t), \qquad r = \tfrac{1}{RC} = \frac{10^5}{27}$$

and assert that the resulting signal is close to the output signal $\tilde{y}$. That is, we test the expected relationship

$$\tilde{y} \approx h * \tilde{x} = \int_{-\infty}^{\infty} h(\tau)\tilde{x}(t-\tau)d\tau.$$

From (1.2.4),

$$\begin{aligned}
\tilde{y}(t) &\approx \int_{-\infty}^{\infty} h(\tau) \sum_{\ell=1}^{L} x_\ell \operatorname{sinc}(Ft - F\tau - \ell) d\tau \\
&= \sum_{\ell=1}^{L} x_\ell \int_{-\infty}^{\infty} h(\tau) \operatorname{sinc}(Ft - F\tau - \ell) d\tau \\
&= \sum_{\ell=1}^{L} x_\ell g(Ft - \ell)
\end{aligned}$$

where the function

$$g(t) = \int_{-\infty}^{\infty} h(\tau) \operatorname{sinc}(t - F\tau) d\tau = -r \int_{0}^{\infty} e^{-r\tau} \operatorname{sinc}(t - F\tau) d\tau.$$

An approximation of $g(t)$ is made using the trapezoidal sum

$$f(t) \approx \frac{K}{2N} \left(p(0) + p(K) + 2 \sum_{n=1}^{N-1} p(\Delta n) \right),$$

where $p(\tau) = h(\tau) \operatorname{sinc}(t - F\tau)$ and

$$K = -RC \log\left(10^{-3}\right), \qquad N = \lceil 10FK \rceil, \qquad \Delta = K/N.$$

Figure 4.2 plots the input signal $\tilde{x}$, output signal $\tilde{y}$, and hypothesised output signal $h * \tilde{x}$ over a 4ms window.

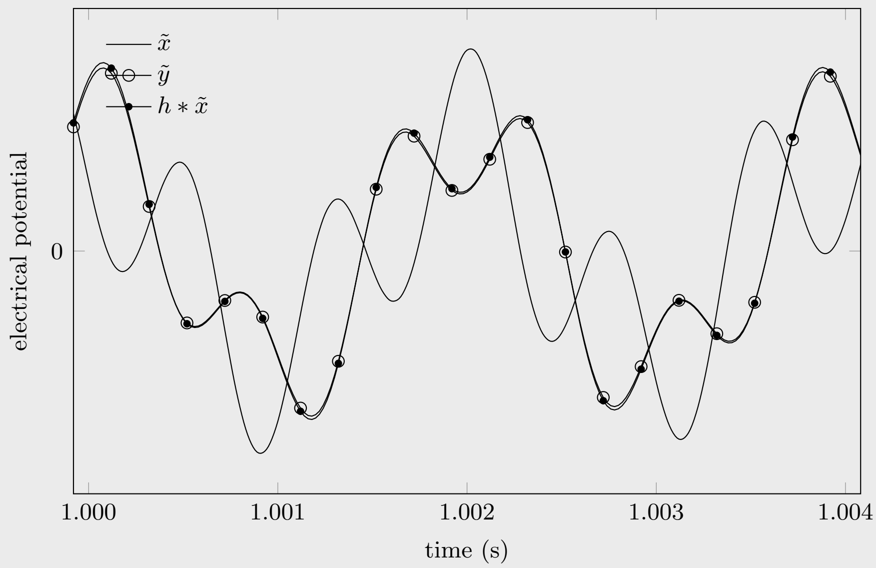

Figure 4.2: Plot of reconstructed input signal $\tilde{x}$ (solid line), output signal $\tilde{y}$ (solid line with circle), and hypothesised output signal $h * \tilde{x}$ (solid line with dot).

4.4 Second order systems

Consider the mass spring damper system from Figure 2.2 that is described by the equation

$$f = Kp + BD(p) + MD^2(p) \tag{4.4.1}$$

where f is the force applied to the mass M and p is the position of the mass and K and B are the spring and damping coefficients. The mass spring damper is an example of a **second order system**. Another example of a second order system is the Sallen-Key active electrical circuit depicted in Figure 2.10. In Section 2 we were able to find the force f corresponding with a given position signal p. Suppose that H is a linear time invariant system mapping f to p, that is, such that $p = H(f)$. We will find the impulse response of H. Taking Laplace transforms on both sides of the differential equation gives

$$\mathcal{L}(f) = (K + Bs + Ms^2)\mathcal{L}(p).$$

Rearranging gives the transfer function of H,

$$\lambda(H) = \frac{\mathcal{L}(p)}{\mathcal{L}(f)} = \frac{1}{K + Bs + Ms^2}.$$

We can invert this Laplace transform to obtain the impulse response. There are three cases to consider depending on whether the quadratic $K+Bs+Ms^2$ has two distinct real roots, is irreducible (does not have real roots), or has two identical real roots.

Case 1: (Distinct real roots) In this case, the roots are

$$\beta - \alpha, \qquad -\beta - \alpha,$$

where

$$\alpha = \frac{B}{2M}, \qquad \beta = \frac{\sqrt{B^2 - 4KM}}{2M}$$

and $B^2 - 4KM > 0$. By a partial fraction expansion (Exercise 4.7),

$$\begin{aligned}\lambda(H) &= \frac{1}{M(s-\beta+\alpha)(s+\beta+\alpha)} \\ &= \frac{1}{2\beta M}\left(\frac{1}{s-\beta+\alpha} - \frac{1}{s+\beta+\alpha}\right).\end{aligned}$$

From (4.0.6) we obtain the transform pairs

$$\mathcal{L}(e^{(\beta-\alpha)t}u(t)) = \frac{1}{s-\beta+\alpha}, \qquad \mathcal{L}(e^{-(\beta+\alpha)t}u(t)) = \frac{1}{s+\beta+\alpha}.$$

As in Section 4.3, other signals with these Laplace transforms are discarded because they do not lead to an impulse response that is zero for $t < 0$. That

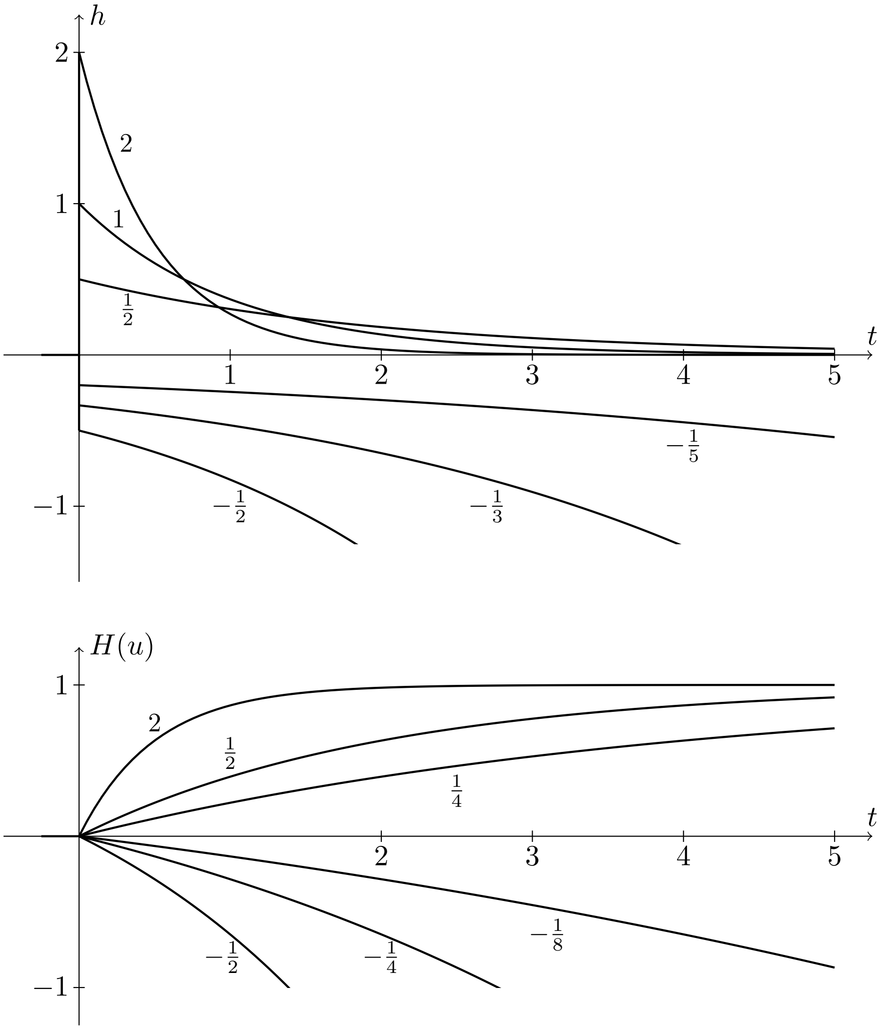

Figure 4.3: Top: impulse response of a first order system with $r = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{5}, \frac{1}{2}$, 1, 2. Bottom: step response of a first order system with $r = -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, \frac{1}{4}, \frac{1}{2}$, 2.

is, they do not lead to a causal system H. The impulse response of H is thus

$$h(t) = \frac{1}{2\beta M} u(t) e^{-\alpha t} \big(e^{\beta t} - e^{-\beta t} \big).$$

This is a sum of the impulse responses of two first order systems.

Case 2: (Distinct imaginary roots) The solution is as in the previous case, but now $4KM - B^2 > 0$ and β is imaginary. Put $\theta = \beta/j$ so that

$$e^{\beta t} - e^{-\beta t} = e^{j\theta t} - e^{-j\theta t} = 2j \sin(\theta t).$$

The impulse response of H is

$$h(t) = \frac{1}{\theta M} u(t) e^{-\alpha t} \sin(\theta t).$$

Case 3: (Identical roots) In this case, the two roots are equal to $-\alpha$ and

$$\lambda(H) = \frac{1}{M(s+\alpha)^2}.$$

From (4.0.6) we obtain the transform pair

$$\mathcal{L}\big(te^{-\alpha t} u(t)\big) = \frac{1}{(s+\alpha)^2}$$

and this is the only signal with this Laplace transform that leads to a causal impulse response. The impulse response of H is thus

$$h(t) = \frac{1}{M} t e^{-\alpha t} u(t).$$

A second order system is called **overdamped** when there are two distinct real roots, **underdamped** when their are two distinct imaginary roots, and **critically damped** when the roots are identical. The different types of impulse responses for are plotted in Figure 4.4.

With no damping (i.e. damping coefficient $B = 0$) the roots are of the form $\pm\beta$ and have no real part. In this case, the impulse response is

$$h(t) = \frac{1}{\theta M} u(t) \sin(\theta t),$$

where $\theta = \beta/j = \sqrt{KM}$ is called the **natural frequency** of the second order system. This impulse response oscillates for all $t > 0$ without decay or explosion. Two identical roots occur when the damping coefficient $B = \sqrt{4KM}$ and this is sometimes called the **critical damping coefficient**.

The impulse response of a second order system is absolutely integrable when $\alpha = \frac{B}{2M} > 0$, but not when $\alpha \leq 0$. Thus, the system is stable when

$\alpha > 0$ and not stable when $\alpha \leq 0$. For the mass spring damper both the mass M and damping coefficient B are positive and so mass spring dampers are always stable.

From (3.1.5) the step response $H(u)$ is given by applying the integrator I_∞ to the impulse response. There are three cases to consider depending on whether the system is overdamped, underdamped, or critically damped. When the system is overdamped the step response is

$$\begin{aligned}
H(u) = I_\infty(h) &= \frac{1}{2\beta M}\int_{-\infty}^{t} e^{-\alpha\tau}\big(e^{\beta\tau} - e^{-\beta\tau}\big)u(\tau)d\tau \\
&= \frac{1}{2\beta M}\int_{0}^{t} e^{-\alpha\tau}\big(e^{\beta\tau} - e^{-\beta\tau}\big)d\tau \\
&= \frac{1}{2\beta M}u(t)\left(\frac{e^{(\beta-\alpha)t} - 1}{\beta - \alpha} + \frac{e^{-(\beta+\alpha)t} - 1}{\beta + \alpha}\right).
\end{aligned}$$

When the system is underdamped the step response is

$$\begin{aligned}
H(u) = I_\infty(h) &= \frac{1}{\theta M}\int_{0}^{t} e^{-\alpha\tau}\sin(\theta\tau)dt \\
&= u(t)\left(\frac{\theta - e^{-t\alpha}\big(\theta\cos(t\theta) + \alpha\sin(t\theta)\big)}{M\theta(\alpha^2 + \theta^2)}\right).
\end{aligned}$$

When the system is critically damped the step response is

$$\begin{aligned}
H(u) = I_\infty(h) &= \frac{1}{\theta M}\int_{0}^{t} \frac{1}{M}te^{-\alpha t}dt \\
&= \frac{1}{M\alpha^2}u(t)\big(1 - e^{-t\alpha s}(1 + t\alpha)\big).
\end{aligned}$$

These step responses are plotted in Figure 4.5.

4.5 Poles, zeros, and stability

As discussed in Section 4.2 the transfer function of a system described by a linear differential equation with constant coefficients is of the form

$$\lambda(H) = \frac{\mathcal{L}(y)}{\mathcal{L}(x)} = \frac{a_0 + a_1 s + \dots a_m s^m}{b_0 + b_1 s + \dots b_k s^k}.$$

Factorising the polynomials on the numerator and denominator we obtain

$$\lambda(H) = C\frac{(s - \alpha_0)(s - \alpha_1)\cdots(s - \alpha_m)}{(s - \beta_0)(s - \beta_1)\cdots(s - \beta_k)},$$

where $\alpha_0, \dots, \alpha_m$ are the roots of the numerator polynomial $a_0 + a_1 s + \cdots + a_m s^m$, and $\beta_0, \dots, \beta_k$ are the roots of the denominator polynomial

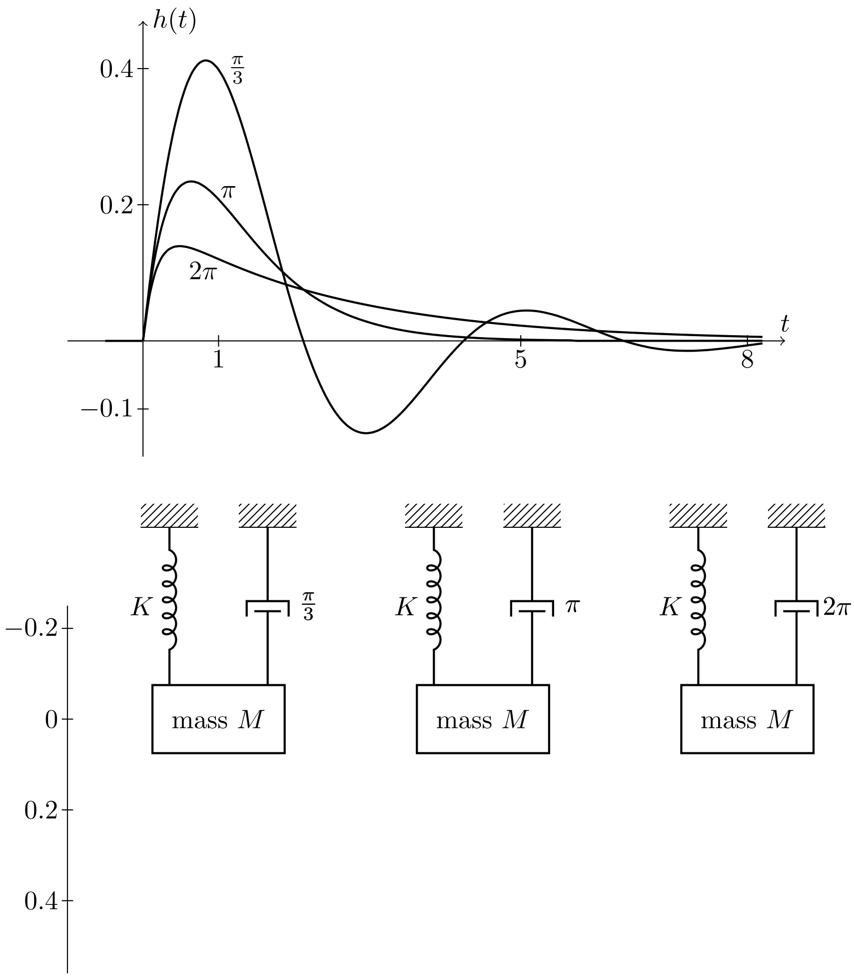

Figure 4.4: Impulse response of the mass spring damper with $M = 1$, $K = \frac{\pi^2}{4}$ and damping constant $B = \frac{\pi}{3}$ (underdamped), $B = \sqrt{4KM} = \pi$ (critically damped), and $B = 2\pi$ (overdamped).

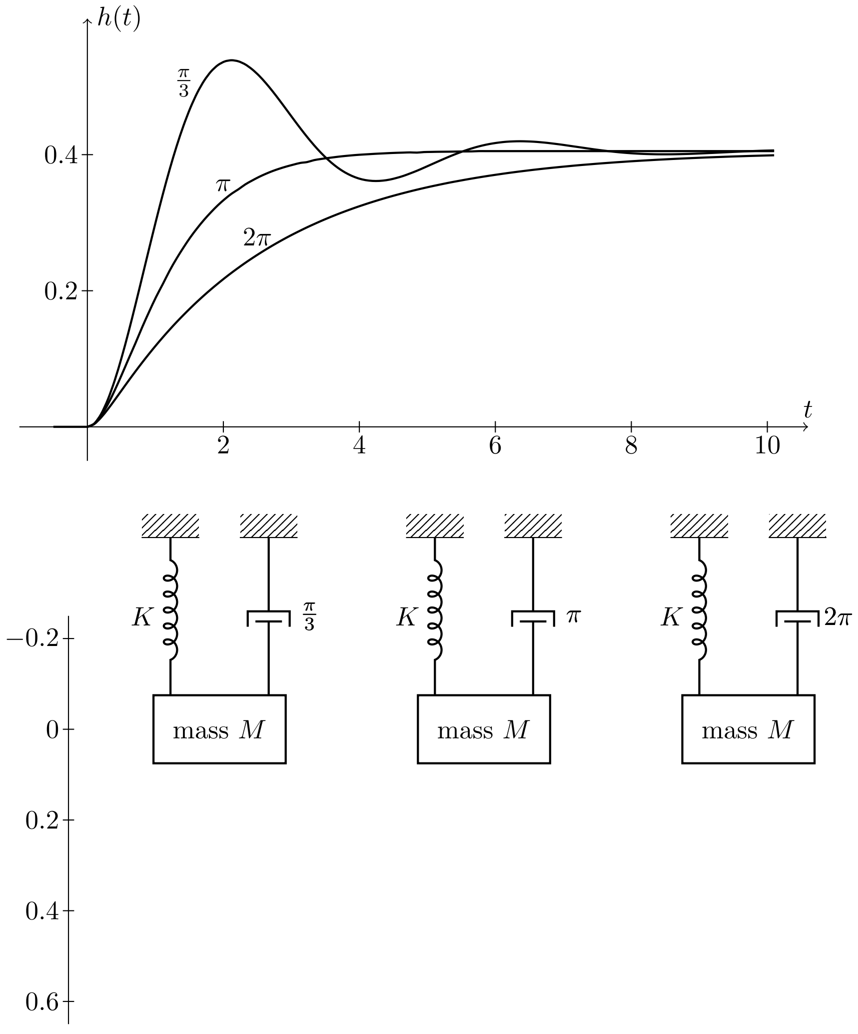

Figure 4.5: Step response of the mass spring damper with $M = 1$, $K = \frac{\pi^2}{4}$ and damping constant $B = \frac{\pi}{3}$ (underdamped), $B = \sqrt{4KM} = \pi$ (critically damped), and $B = 2\pi$ (overdamped).

$b_0+b_1s+\cdots+b_ks^k$, and $C = \frac{a_m}{b_m}$. That such a factorisation is always possible is called the **fundamental theorem of algebra** [Fine and Rosenberger, 1997]. If the numerator and denominator polynomials share one or more roots, then these roots cancel leaving the simpler expression

$$\lambda(H) = C\frac{(s-\alpha_d)(s-\alpha_{d+1})\cdots(s-\alpha_m)}{(s-\beta_d)(s-\beta_{d+1})\cdots(s-\beta_k)}, \qquad (4.5.1)$$

where d is the number of shared roots, these shared roots being

$$\alpha_0 = \beta_0, \quad \alpha_1 = \beta_1, \quad \ldots, \quad \alpha_{d-1} = \beta_{d-1}.$$

The roots from the numerator $\alpha_d, \ldots, \alpha_m$ are called the **zeros** and the roots from the denominator $\beta_d, \ldots, \beta_m$ are called the **poles**. A **pole-zero plot** is constructed by marking the complex plane with a cross at the location of each pole and a circle at the location of each zero. Pole-zero plots for the first order system from Section 4.3, the second order system from Section 4.4, and the system describing the PID controller (2.2.7) are shown in Figure 4.6.

It is always possible to apply partial fractions and write (4.5.1) in the form

$$\lambda(H) = p(s) + \sum_{\ell\in K}\frac{A_\ell}{(s-\beta_\ell)^{r_\ell}},$$

where r_ℓ are positive integers, A_ℓ are complex constants, K is a subset of the indices from $\{d, d+1, \ldots, k\}$, and $p(s)$ is a polynomial of degree $m-k$. If $k > m$ then $p(s) = 0$. The integer r_ℓ is called the **multiplicity** of the pole β_ℓ. We see that the transfer function contains the summation of two parts: the polynomial $p(s)$, and a sum of terms of the form $\frac{A}{(s-\beta)^r}$. Let $p(s) = \gamma_0 + \gamma_1 s + \cdots + \gamma_{m-k}s^{m-k}$. This polynomial is the transfer function of the nonregular system

$$H_1 = \gamma_0 T_0 + \gamma_1 D + \gamma_2 D^2 + \cdots + \gamma_{m-k}D^{m-k}.$$

This system is a linear combination of the identity system T_0 and differentiators of order at most $m-k$. From (4.0.6),

$$\mathcal{L}\left(\frac{A}{r!}t^{r-1}e^{\beta t}u(t)\right) = \frac{A}{(s-\beta)^r} \qquad \mathrm{Re}(s) > \mathrm{Re}(\beta)$$

and so the terms of the form $\frac{A}{(s-\beta)^r}$ correspond with the transfer function of a regular system with impulse response $\frac{A}{r!}t^{r-1}e^{\beta t}u(t)$. Other signals with Laplace transform $\frac{A}{(s-\beta)^r}$ are discarded because they do not correspond with the impulse response of a causal system. Thus, $\sum_{\ell\in K}\frac{A_\ell}{(s-\beta_\ell)^{r_\ell}}$ is the transfer function of the regular system H_2 with impulse response

$$h_2(t) = u(t)\sum_{\ell\in K}\frac{A_\ell}{r_\ell!}t^{r_\ell-1}e^{\beta_\ell t}. \qquad (4.5.2)$$

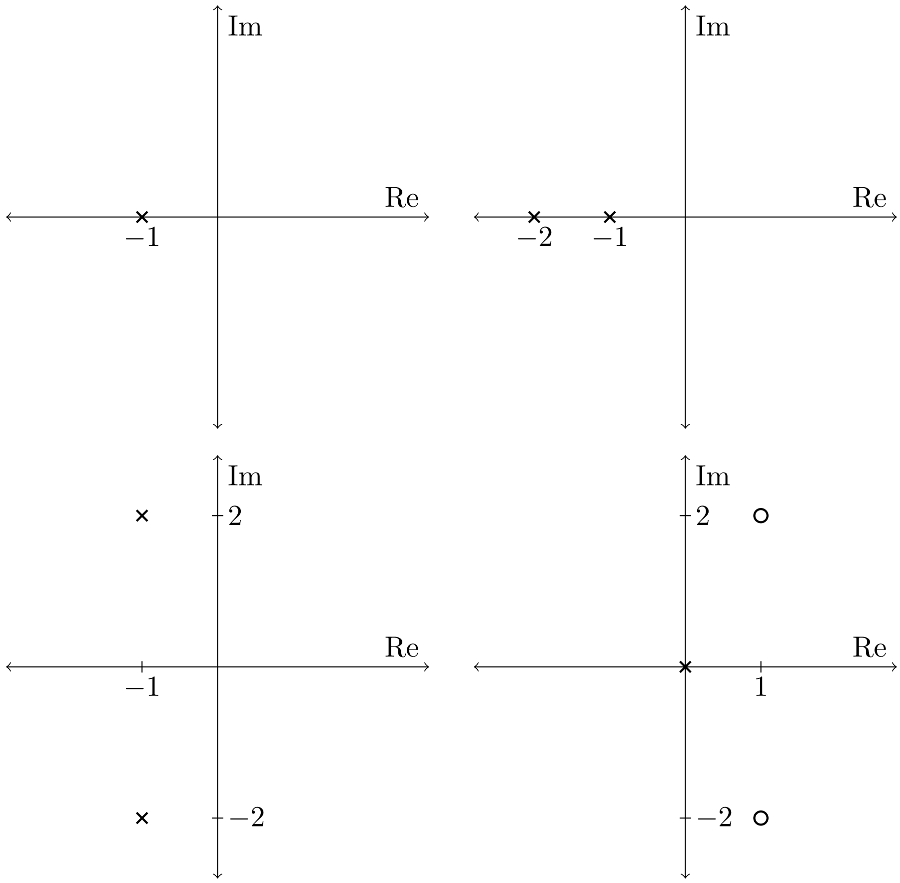

Figure 4.6: Top left: pole zero plot for the first order system $x = y + D(y)$. There is a single pole at -1. Top right: pole zero plot for the overdamped second order system $x = 2y + 3D(y) + D^2(y)$ that has two real poles at -1 and -2. Bottom left: pole zero plot for the underdamped second order system $x = 5y + 2D(y) + D^2(y)$ that has two imaginary poles at $-1 + 2j$ and $-1 - 2j$. The poles form a conjugate pair. Bottom right: pole zero plot for the equation $D(y) = 5x - 2D(x) + D^2(x)$ that models a PID controller (2.2.7). The system has a single pole at the origin and two zeros at $1 + 2j$ and $1 - 2j$.

The system H mapping x to y is the sum of the regular system H_2 and nonregular system H_1, that is,

$$y = H(x) = H_1(x) + H_2(x).$$

Observe that H is regular only if the system $H_1 = 0$, that is, only if H_1 maps all input signals to the signal $x(t) = 0$ for all $t \in \mathbb{R}$. This occurs only when the polynomial $p(s) = 0$, that is, only when the number of poles exceeds the number of zeros. The system H will be stable if both H_1 and H_2 are stable. Because the differentiator D^ℓ is not stable (Exercise 1.7) the system H_1 is stable if only if the order of the polynomial $p(s)$ is zero, that is, if $p(s) = \gamma_0$ is a constant (potentially $\gamma_0 = 0$). In this case $H_1(x) = \gamma_0 T_0(x)$ is the identity system multiplied by a constant. The polynomial $p(s)$ is a constant only when the order of the denominator polynomial is greater than or equal to the order of the numerator polynomial, that is, when the number of poles is greater than or equal to the number of zeros. The regular system H_2 is stable if and only if its impulse response h_2 is absolutely integrable. This occurs only when the terms $e^{\beta_\ell t}$ inside the sum (4.5.2) are decreasing as $t \to \infty$, that is, only if the real part of the poles $\operatorname{Re} \beta_\ell$ are negative. Thus, the system H_2 is stable if and only if the real part of the poles are strictly negative.

The stability of the system H can be immediately determined from its pole-zero plot. The system is stable if and only if:

1. the number of poles is greater than or equal to the number of zeros (there are at least as many crosses on the pole-zero plot as circles),
2. No poles (crosses) line on the imaginary axis or in the right half of the complex plane.

The pole-zero plots in Figure 4.6 all represent stable systems with the exception of the plot on the bottom right (a PID controller). This system has two zeros and only one pole. The single pole is contained on the imaginary axis.

4.5.1 Two masses, a spring, and a damper

Consider the system involving two masses, a spring, and a damper in Figure 2.11. From (2.3.3), the equation relating the force applied to the first mass f and the position of the second mass p is

$$f = BD(p) + (M_1 + M_2)D^2(p) + \frac{BM_2}{K} D^3(p) + \frac{M_1 M_2}{K} D^4(p),$$

where B is the damping coefficient, K is the spring constant, and M_1 and M_2 are the masses. Taking Laplace transforms

$$\mathcal{L}(f) = s\left(B + (M_1 + M_2)s + \frac{BM_2}{K}s^2 + \frac{M_1 M_2}{K}s^3\right)\mathcal{L}(p),$$

from which, we obtain the transfer function of a system H that maps f to p,

$$\lambda(H) = \frac{\mathcal{L}(p)}{\mathcal{L}(f)} = \frac{1}{s\left(B + (M_1 + M_2)s + \frac{BM_2}{K}s^2 + \frac{M_1M_2}{K}s^3\right)}.$$

The system has no zeros and 4 poles. One of these poles always exists at the origin. The system is not stable because this pole is not strictly in the left half of the complex plane.

Consider the specific case when $B = K = M_1 = M_2 = 1$. Factorising the denominator polynomial gives

$$\lambda(H) = \frac{1}{s(s-\beta_1)(s-\beta_2)(s-\beta_2^*)},$$

where

$$\beta_1 = \frac{1}{3}\left(\gamma - \frac{5}{\gamma} - 1\right) \approx -0.56984,$$

$$\beta_2 = \frac{1}{6}\left(\frac{5(1+j\sqrt{3})}{\gamma} - (1-j\sqrt{3})\gamma - \frac{1}{2}\right) \approx -0.21508 + 1.30714j,$$

and $\gamma = \left(\frac{3\sqrt{69}-11}{2}\right)^{1/3}$. Applying partial fractions (Exercise 4.8) gives

$$\lambda(H) = \frac{1}{s(s-\beta_1)(s-\beta_2)(s-\beta_2^*)} = \frac{A_0}{s} + \frac{A_1}{s-\beta_1} + \frac{A_2}{s-\beta_2} + \frac{A_2^*}{s-\beta_2^*},$$

where

$$A_0 = -\frac{1}{\beta_1|\beta_2|^2} = 1, \qquad A_1 = \frac{1}{\beta_1|\beta_1 - \beta_2|^2} \approx -0.956611,$$

$$A_2 = \frac{1}{\beta_2(\beta_2-\beta_1)(\beta_2-\beta_2^*)} \approx -0.0216944 + 0.212084j.$$

From (4.5.2), the impulse response of H is

$$h(t) = u(t)\big(A_0 + A_1 e^{\beta_1 t} + 2\,|A_2|\, e^{\operatorname{Re}\beta_2 t}\cos(\operatorname{Im}\beta_2 t + \angle A_2)\big).$$

This impulse response is plotted in Figure 4.7. Observe that h is not absolutely integrable and the system is not stable. The impulse response $h(t)$ does not converge to zero as $t \to \infty$ and correspondingly the mass M_2 does not come come to rest at position zero in Figure 4.7. In the figure it is assumed that the spring is at equilibrium when the two masses are $d = 1$ apart. From (2.3.1), the position of mass M_1 is given by the signal $p_1 = g - d$ where $g = h + M_2 D^2(h)$.

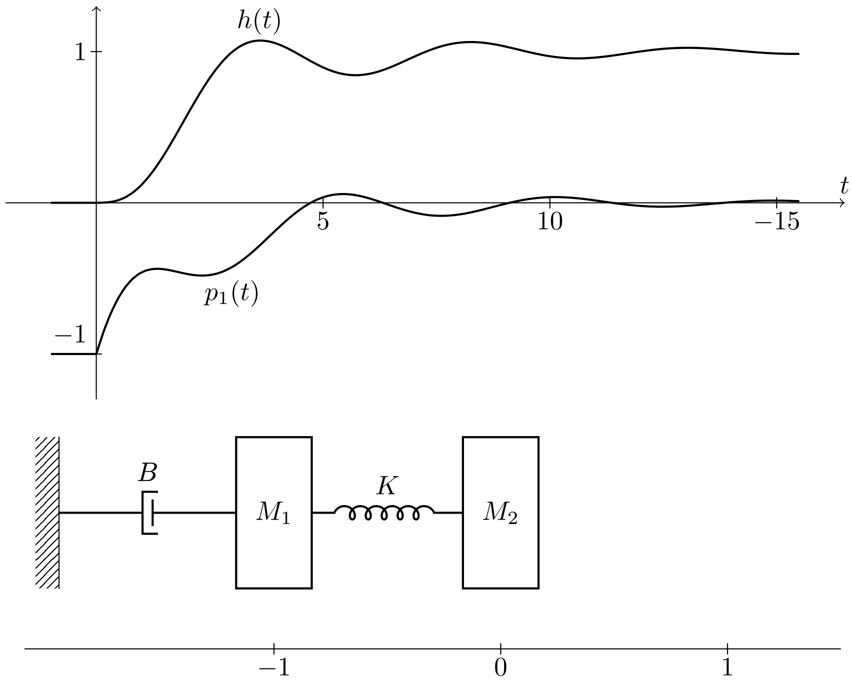

Figure 4.7: Impulse response of the system with two masses, a spring, and a damper, where $B = K = M_1 = M_2 = 1$.

4.5.2 Direct current motors

Recall the direct current (DC) motor from Figure 2.13 described by the differential equation from (2.4.1),

$$v = \left(\frac{RB}{K_\tau} + K_b\right) D(\theta) + \frac{RJ}{K_\tau} D^2(\theta),$$

where v is the input voltage signal and θ is a signal representing the angle of the motor. The constants R, B, K_τ, K_b, and J are related to components of the motor as described in Section 2.4. To simplify the differential equation put $a = \frac{RB}{K_\tau} + K_b$ and $b = \frac{RJ}{K_\tau}$ and the equation becomes

$$v = aD(\theta) + bD^2(\theta).$$

Taking Laplace transforms on both sides of this equation gives the transfer function of a system H that maps input voltage v to motor angle θ,

$$\lambda(H) = \frac{1}{s(a + bs)}.$$

This system has no zeros and two poles. One pole is at $-\frac{a}{b}$ and the other is at the origin. The system is not stable because the pole at the origin is not strictly in the left half of the complex plane.

Applying partial fractions we find that

$$\lambda(H) = \frac{1}{as} - \frac{1}{a(s-\beta)}, \tag{4.5.3}$$

where $\beta = -\frac{a}{b}$. Using (4.0.6), the impulse response of H is

$$h(t) = \frac{1}{a}u(t)\big(1 - e^{\beta t}\big). \tag{4.5.4}$$

Other signals with Laplace transform (4.5.3) are discarded because they do not lead to a causal system. The step response $H(u)$ is obtained by applying the integrator system I_∞ to the impulse response, that is

$$H(u) = I_\infty(h) = \frac{1}{a\beta}u(t)\big(\beta t + e^{\beta t} - 1\big).$$

The impulse response and step response are plotted in Figure 4.8 when $K_b = \frac{1}{8}$, $K_\tau = 8$ and $B = R = 1$ and $J = 2$ so that $a = \frac{1}{4}$, $b = \frac{1}{4}$ and $\beta = -1$.

4.6 Exercises

4.1. Sketch the signal

$$x(t) = e^{-2t}u(t) + e^{t}u(-t)$$

where $u(t)$ is the step function. Find the Laplace transform of $x(t)$ and the corresponding region of convergence (ROC). Sketch the region of convergence on the complex plane.

4.2. Find the Laplace transform of the signal $t^n u(t)$ where $n \geq 0$ is an integer.

4.3. Let $n \geq 0$ be an integer. Show that the Laplace transform of the signal $-t^n u(-t)$ is the same as the Laplace transform of the signal $t^n u(t)$, but with a different region of convergence.

4.4. Show that equation (4.1.5) on page 50 holds when the system H is the differentiator D^k or the time shifter T_τ.

4.5. What is the transfer function of the integrator system I_∞ and what is its region of convergence?

4.6. By partial fractions, or otherwise, assert that

$$\frac{as}{s+b} = a - \frac{ab}{s+b}$$

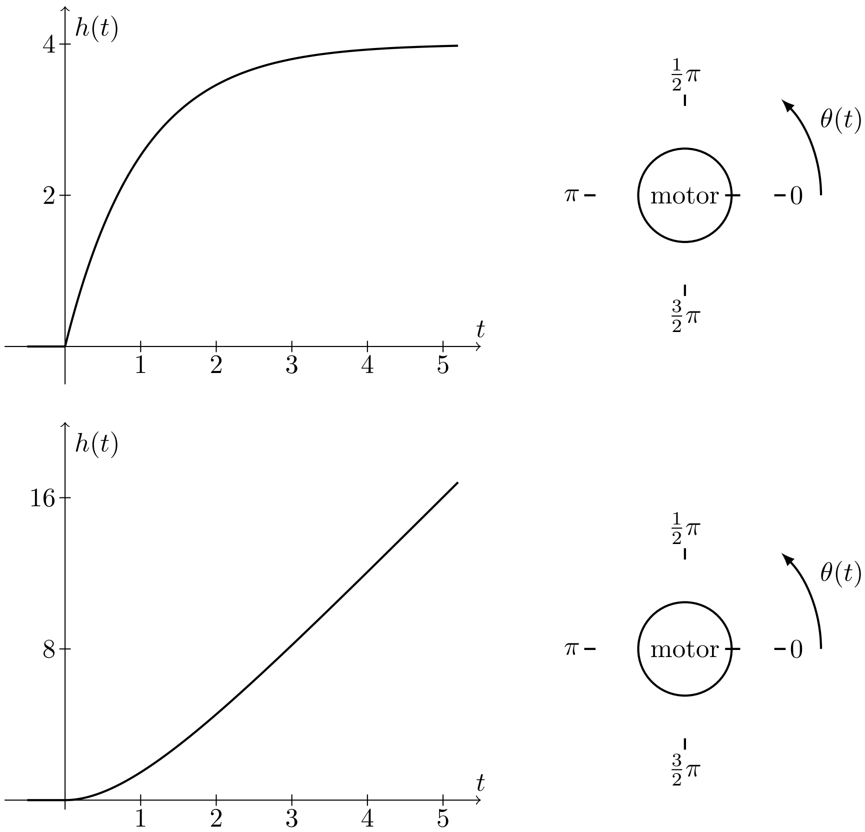

Figure 4.8: Impulse response (top) and step response (bottom) of a DC motor with constants $K_b = \frac{1}{4}$, $K_\tau = 8$ and $B = R = J = 1$.

4.7. By partial fractions, or otherwise, assert that

$$\frac{s+c}{(s+a)(s+b)} = \frac{a-c}{(a-b)(s+a)} + \frac{c-b}{(a-b)(s+b)}$$

4.8. By partial fractions, or otherwise, assert that

$$\frac{1}{s(s-a)(s-b)(s-b^*)} = \frac{A_0}{s} + \frac{A_1}{s-a} + \frac{A_2}{s-b} + \frac{A_2^*}{s-b^*}$$

where $a \in \mathbb{R}$ and $b \in \mathbb{C}$ and $\operatorname{Im}(b) \neq 0$ and

$$A_0 = -\frac{1}{a|b|^2}, \qquad A_1 = \frac{1}{a|a-b|^2}, \qquad A_2 = \frac{1}{b(b-a)(b-b^*)}.$$

You might wish to check your solution using a symbolic programming language (for example Sage, Mathematica, or Maple).

4.9. Let

$$\mathcal{L}(y) = \frac{2s+1}{s^2+s-2}$$

be the Laplace transform of a signal y. By partial fractions, or otherwise, find all possible signals y and their regions of convergence.

4.10. Let x be a signal with region of convergence R. Show that the time scaled signal $x(\alpha t)$ with $\alpha \neq 0$ satisfies equation (4.0.7) on page 48.

4.11. Consider the active electrical circuit from Figure 2.8 described by the differential equation from (2.2.3). Derive the transfer function of this system. Find an explicit system H that maps the input voltage x to the output voltage y. State whether this system is stable and/or regular.

4.12. Given the mass spring damper system described by (4.4.1), find the position signal p given that the force signal

$$f(t) = \Pi\big(t - \tfrac{1}{2}\big) = \begin{cases} 1 & 0 < t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

is the rectangular function time shifted by $\frac{1}{2}$. Consider three cases:

(a) $M = 1$, $K = \frac{\pi^2}{4}$ and $B = \frac{\pi}{3}$,

(b) $M = 1$, $K = \frac{\pi^2}{4}$ and $B = \pi$,

(c) $M = 1$, $K = \frac{\pi^2}{4}$ and $B = 2\pi$,

Plot the solution in each case, and comment on whether the system is underdamped, overdamped, or critically damped.

4.13. Plot the signal $x(t) = \sin(te^t)u(t)$ and find and plot its derivative $D(x)$. Show that the region of convergence of x contains those complex numbers s with $\mathrm{Re}(s) > 0$ and that the region of convergence of $D(x)$ contains those with $\mathrm{Re}(s) > 1$.

Chapter 5

The Fourier transform

The **Fourier transform** of an absolutely integrable signal x is defined as

$$\mathcal{F}(x) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt. \tag{5.0.1}$$

The Fourier transform is a function of the real number f. We indicate its value at f by $\mathcal{F}(x)(f)$ or $\mathcal{F}(x, f)$. For example, the rectangular pulse $\Pi(t)$ from (1.1.2) is absolutely integrable and has Fourier transform

$$\begin{aligned}\mathcal{F}(\Pi) &= \int_{-\infty}^{\infty} \Pi(t)e^{-j2\pi ft}dt \\ &= \int_{-1/2}^{1/2} e^{-j2\pi ft}dt \\ &= \frac{e^{j\pi f} - e^{-j\pi f}}{j2\pi f} = \frac{\sin(\pi f)}{\pi f} = \operatorname{sinc}(f). \end{aligned} \tag{5.0.2}$$

The sinc function is plotted in Figure 5.1.

The Fourier transform is closely related to the Laplace transform because

$$\mathcal{F}(x, f) = \mathcal{L}(x, j2\pi f)$$

for those signals x with region of convergence containing the imaginary axis, that is, for absolutely integrable x. The Fourier transform inherits the properties of the Laplace transform that were described in Section 4.1. For example, if H is a stable regular system with absolutely integrable impulse response h having Fourier transform $\mathcal{F}(h)$, then the spectrum of H satisfies

$$\Lambda(H, f) = \lambda(H, j2\pi f) = \mathcal{L}(h, j2\pi f) = \mathcal{F}(h, f),$$

that is, the spectrum of a stable regular system is given by the Fourier transform of its impulse response. Like the Laplace transform, the Fourier transform obeys the **convolution theorem** (4.1.4), that is,

$$\mathcal{F}(x * y) = \mathcal{F}(x)\mathcal{F}(y) \tag{5.0.3}$$

when each of the signals x, y, and $x * y$ are absolutely integrable. In words: the Fourier transform of a convolution of signals is given by the multiplication of the Fourier transforms of those signals.

It follows from (4.1.5) that if H is a regular system with spectrum $\Lambda(H)$ and if x is a signal with Fourier transform $\mathcal{F}(x)$, then the signal $y = H(x)$ has Fourier transform

$$\mathcal{F}(y) = \Lambda(H)\mathcal{F}(x).$$

This property also holds for the differentiator system D and the time shifter system T_τ (Exercise 4.1.5). From (4.1.1) and (4.1.2) the spectrum of T_τ and the kth differentiator D^k satisfy

$$\Lambda(T_\tau) = e^{-j2\pi f\tau}, \qquad \Lambda(D^k) = (j2\pi f)^k$$

from which we obtain the **time shift property**,

$$\mathcal{F}\big(T_\tau(x)\big) = \Lambda(T_\tau)\mathcal{F}(x) = e^{-j2\pi f\tau}\mathcal{F}(x),$$

and the **differentiation property**,

$$\mathcal{F}\big(D^k(x)\big) = \Lambda(D^k)\mathcal{F}(x) = (j2\pi f)^k\mathcal{F}(x),$$

of the Fourier transform. These results motivate assigning the following Fourier transforms to the delta "function" and its derivatives

$$\mathcal{F}(\delta) = 1, \qquad \mathcal{L}(\delta^k) = (j2\pi f)^k. \tag{5.0.4}$$

These conventions are common in the literature [Oppenheim et al., 1996].

Similarly to the Laplace transform (4.0.5), the Fourier transform obeys a **frequency shift rule** that relates the transform of a signal $x(t)$ to that of the signal $e^{2\pi j\gamma f t}x(t)$ where $\gamma \in \mathbb{R}$. From (4.0.5), the frequency shift rule asserts that

$$\mathcal{F}\big(e^{2\pi j\gamma t}x(t), f\big) = \mathcal{F}(x, f-\gamma). \tag{5.0.5}$$

Since $\cos(2\pi\gamma t) = \frac{1}{2}e^{2\pi j\gamma t} + \frac{1}{2}e^{-2\pi j\gamma t}$ we also have

$$\mathcal{F}\big(\cos(2\pi\gamma t)x(t), f\big) = \frac{1}{2}\mathcal{F}(x, f-\gamma) + \frac{1}{2}\mathcal{F}(x, f+\gamma). \tag{5.0.6}$$

This is sometimes called the **modulation property** of the Fourier transform [Papoulis, 1977, page 61]. This property is of particular importance in communications engineering [Proakis, 2007].

Like the Laplace transform (4.0.7), the Fourier transform obeys a **time scaling property**. If x is an absolutely integrable signal then the time scaled signal $x(\alpha t)$ with $\alpha \neq 0$ has Fourier transform

$$\mathcal{F}\big(x(\alpha t), f\big) = \frac{1}{|\alpha|}\mathcal{F}\big(x, f/\alpha\big). \tag{5.0.7}$$

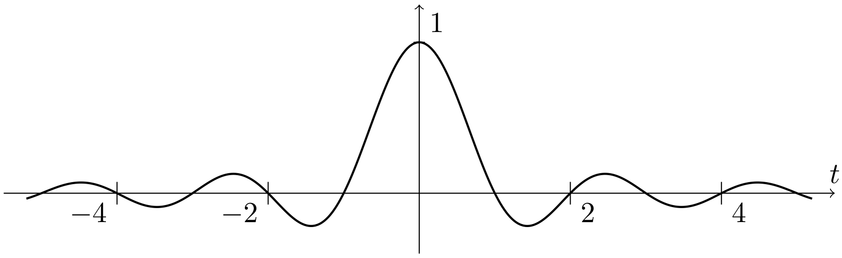

Figure 5.1: The **sinc function** $\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$.

5.1 The inverse transform and the Plancherel theorem

Given a signal x we will often denote its Fourier transform by $\hat{x} = \mathcal{F}(x)$. Observe that $\hat{x}$, like x, is a function that maps a real number to a complex number. Thus, the Fourier transform $\hat{x}$ is a **signal** with independent variable representing **frequency**. It is usual to call x the **time domain** representation of the signal and $\hat{x}$ the **frequency domain** representation. If $\hat{x}$ is absolutely integrable, then x can be recovered using the **inverse Fourier transform**

$$x(t) = \mathcal{F}^{-1}(\hat{x}) = \int_{-\infty}^{\infty} \hat{x}(f) e^{j2\pi f t} df. \tag{5.1.1}$$

For example, let $\hat{x} = \mathcal{F}(x) = \Pi$ be the rectangular pulse. By working analogous to that from (5.0.2),

$$x(t) = \int_{-\infty}^{\infty} \Pi(f) e^{j2\pi f t} df = \mathrm{sinc}(-t) = \mathrm{sinc}(t).$$

We are lead to the conclusion that the Fourier transform of sinc is the rectangular pulse Π.

The rectangular pulse Π is finite and absolutely integrable. The sinc function is not absolutely integrable (Exercise 5.3). Because of this the integral equation that we have used to define the Fourier transform (5.0.1) cannot be directly applied to the sinc function. Although sinc is not absolutely integrable, it is square integrable (Exercise 5.3). It happens that all square integrable signals can be assigned a Fourier transform by interpreting the integral in (5.0.1) as what is called its **Cauchy principal value**. That is, for x a square integrable signal, we assign the Fourier transform

$$\hat{x} = \mathcal{F}(x) = \lim_{T \to \infty} \int_{-T}^{T} x(t) e^{-j2\pi f t} dt.$$

This Fourier transform $\hat{x}$ is itself a square integrable signal and the orignal time domain signal x can be recovered almost everywhere by taking the

Cauchy principal value of the integral in (5.1.1), that is,

$$x = \mathcal{F}^{-1}(\hat{x}) = \lim_{T\to\infty} \int_{-T}^{T} \hat{x}(f) e^{j2\pi f t} df \qquad \text{a.e..}$$

Infact, the energy of x and its Fourier transform $\hat{x}$ are the same, that is,

$$\|x\|_2^2 = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |\hat{x}(f)|^2 \, dt = \|\hat{x}\|_2^2. \tag{5.1.2}$$

These results are known as the **Plancherel theorem** [Rudin, 1986, Th. 9.13]. The equality of energies in (5.1.2) is often called **Parseval's identity**. For our purposes it will suffice to remember only that the Fourier transform of the sinc function is the rectangular pulse Π.

A result more general than (5.1.2) holds. If x and y are square integrable signals with Fourier transforms $\hat{x}$ and $\hat{y}$, then

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \int_{-\infty}^{\infty} \hat{x}(f) \hat{y}^*(f) dt \tag{5.1.3}$$

where the superscript * denotes the complex cojugate. One obtains (5.1.2) by putting $y = x$ in (5.1.3). This more general result often also goes by the name of Parseval's identity.

Let x be a signal with Fourier transform

$$\mathcal{F}(x, f) = \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \tau} d\tau.$$

Evaluating the Fourier transform at $-t$ we find that

$$\mathcal{F}(x, -t) = \int_{-\infty}^{\infty} x(\tau) e^{j2\pi t \tau} d\tau = \mathcal{F}^{-1}(x, t). \tag{5.1.4}$$

This is the called the **duality** property of the Fourier transform. In words, if $\hat{x}$ is the Fourier transform of x, then x is the Fourier transform of $\hat{x}$ reflected in time.

5.2 Analogue filters

For many engineering purposes it is desirable to construct systems that will **pass** (have little affect on) a complex exponential signal $e^{j2\pi f t}$ for certain frequencies f, but will **reject** (significantly attenuate) these signals for other frequencies. Such systems are called **frequency dependent filters**. Those frequencies that the filter intends to pass unaffected are said to be in the **pass band** and those frequencies that the filter intends to reject are said to be in the **stop band**.

An **ideal lowpass filter** with **cuttoff frequency** c is the system L_c with spectrum

$$\Lambda(L_c) = \begin{cases} 1 & |f| < c \\ 0 & \text{otherwise} \end{cases} = \Pi\left(\frac{f}{2c}\right).$$

Applying the inverse Fourier transform to $\Pi\left(\frac{f}{2c}\right)$ gives

$$\int_{-\infty}^{\infty} \Pi\left(\tfrac{f}{2c}\right) e^{j2\pi tf} df = \int_{-c}^{c} e^{j2\pi tf} df = \frac{\sin(2c\pi t)}{\pi t} = 2c \operatorname{sinc}(2ct).$$

We conclude that the ideal lowpass filter L_c is a regular linear time invariant system with impulse response $2c \operatorname{sinc}(2ct)$.

An **ideal highpass filter** with cuttoff frequency c is given by the linear combination $T_0 - L_c$ where T_0 is the identity system. The spectrum is

$$\Lambda(T_0 - L_c) = \Lambda(T_0) - \Lambda(L_c) = 1 - \Pi\left(\frac{f}{2c}\right) = \begin{cases} 0 & |f| < c \\ 1 & \text{otherwise.} \end{cases}$$

This ideal highpass filter is not regular because the system T_0 is not regular. The system does not have an impulse response. Nevertheless, it is common to represent one by $\delta(t) - 2c \operatorname{sinc}(2ct)$ using the delta function as described in Section 3.1.

An **ideal bandpass filter** with upper cuttoff frequency u and lower cuttoff frequency ℓ is given by the linear combination $L_u - L_\ell$. The spectrum is

$$\Lambda(L_u - L_\ell) = \Pi\left(\frac{f}{2u}\right) - \Pi\left(\frac{f}{2\ell}\right) = \begin{cases} 1 & -u < f \leq -\ell \\ 1 & \ell \leq f < u \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the ideal bandpass filter has impulse response $2u \operatorname{sinc}(2ut) - 2\ell \operatorname{sinc}(2\ell t)$. The spectrum and impulse response of the ideal lowpass, highpass, and bandpass filters are plotted in Figure 5.2.

Ideal filters are not realisable in practice. One reason for this is that they are not causal because the sinc function is unbounded in time. We now describe a popular practical low-pass filter discovered by Butterworth [1930]. A **normalised low pass Butterworth filter** of order m, denoted by B_m, has transfer function

$$\lambda(B_m) = \frac{1}{\prod_{i=1}^{m}(\frac{s}{2\pi} - \beta_i)} = \frac{(2\pi)^m}{\prod_{i=1}^{m}(s - 2\pi\beta_i)},$$

where $\beta_1, \dots, \beta_m$ are the roots of the polynomial $s^{2m} + (-1)^m$ that lie strictly in the left half of the complex plane (have negative real part). It is convenient to precisely define these roots as

$$\beta_k = \begin{cases} \exp\left(j\frac{\pi}{2}(1 + \frac{2k-1}{m})\right), & k = 1, \dots, m \\ \exp\left(j\frac{\pi}{2}(1 - \frac{2k-1}{m})\right), & k = m+1, \dots, 2m \end{cases}$$

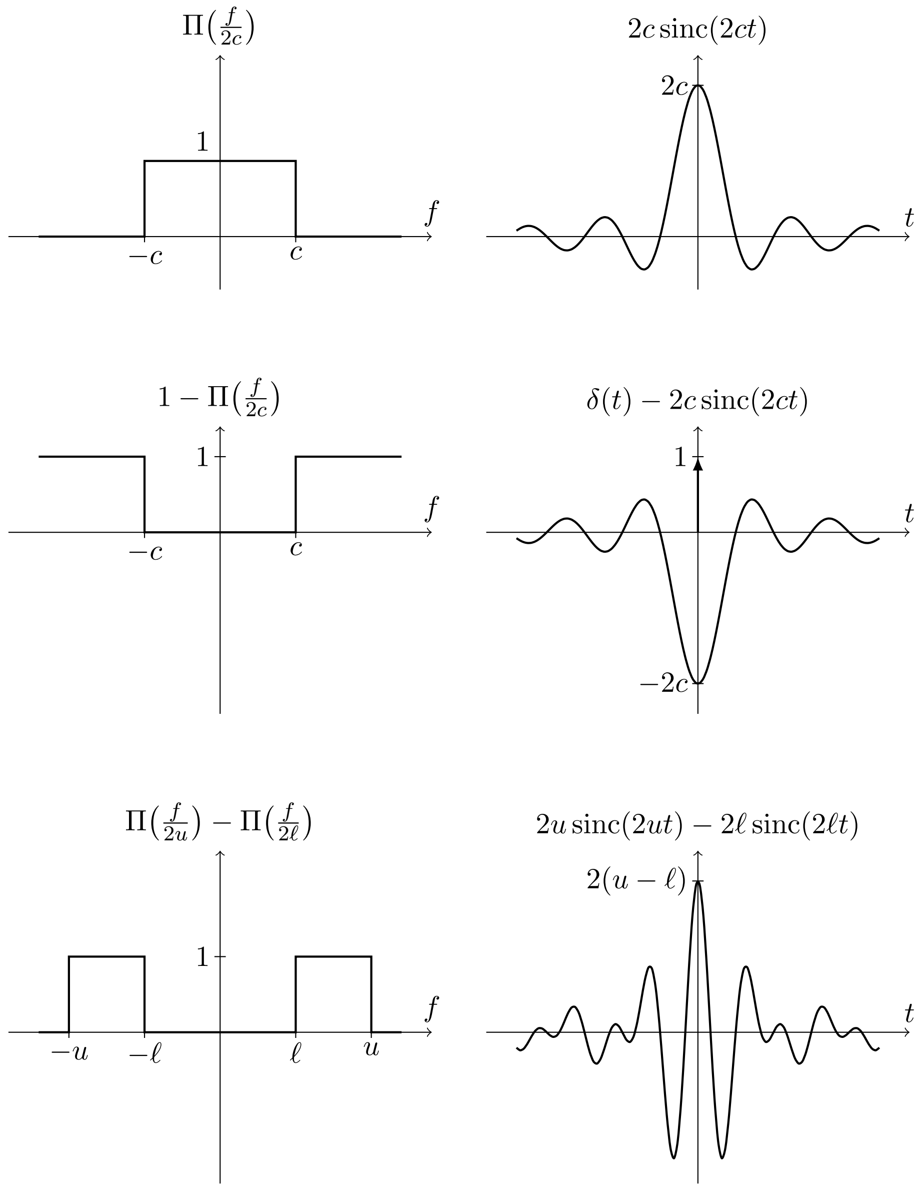

Figure 5.2: Spectrum and impulse response of the ideal lowpass filter L_c (top), the ideal highpass filter $T_0 - L_c$ (middle), and the ideal bandpass filter $L_u - L_\ell$ (bottom). The ideal highpass filter is not regular and does not have an impulse response. We plot the 'pretend' impulse response using the delta function described in Section 3.1.

or equivalently

$$\beta_k = \begin{cases} j\cos\left(\frac{\pi(2k-1)}{2m}\right) - \sin\left(\frac{\pi(2k-1)}{2m}\right), & k = 1, \dots, m \\ j\cos\left(\frac{\pi(2k-1)}{2m}\right) + \sin\left(\frac{\pi(2k-1)}{2m}\right), & k = m+1, \dots, 2m. \end{cases}$$

The roots are plotted in Figure 5.3. Observe that the roots $\beta_{m+1}, \dots, \beta_{2m}$ are given by negating the real parts of $\beta_1, \dots, \beta_m$, that is, $\beta_{m+i} = j(\beta_i/j)^*$.

The spectrum of B_m is

$$\Lambda(B_m) = \frac{1}{\prod_{i=1}^{m}(jf - \beta_i)}.$$

The squared magnitude of the polynomial on the denominator is

$$\begin{aligned} \left|\prod_{i=1}^{m}(jf - \beta_i)\right|^2 &= \left(\prod_{i=1}^{m}(jf - \beta_i)\right)\left(\prod_{i=1}^{m}(jf - \beta_i)\right)^* \\ &= \prod_{i=1}^{m}(jf - \beta_i)(jf - \beta_i)^* \\ &= \prod_{i=1}^{m}(jf - \beta_i)j^*(f - (\beta_i/j)^*) \end{aligned}$$

and because $j^*/j = -1$ we have

$$\begin{aligned} \left|\prod_{i=1}^{m}(jf - \beta_i)\right|^2 &= (-1)^m \prod_{i=1}^{m}(jf - \beta_i)(jf - j(\beta_i/j)^*) \\ &= (-1)^m \prod_{i=1}^{m}(jf - \beta_i)(jf - \beta_{m+i}) \\ &= (-1)^m \prod_{i=1}^{2m}(jf - \beta_i). \end{aligned}$$

Because $\beta_1, \dots, \beta_{2m}$ are the roots of the polynomial $s^{2m} + (-1)^m$ we have

$$\left|\prod_{i=1}^{m}(jf - \beta_i)\right|^2 = (-1)^m\big((jf)^{2m} + (-1)^m\big) = f^{2m} + 1.$$

It follows that the magnitude spectrum of B_m is

$$|\Lambda(B_m)| = \sqrt{\frac{1}{f^{2m} + 1}}.$$

The magnitude and phase spectrum of the filters B_1, B_2, B_3, and B_4 are plotted in Figure 5.4.

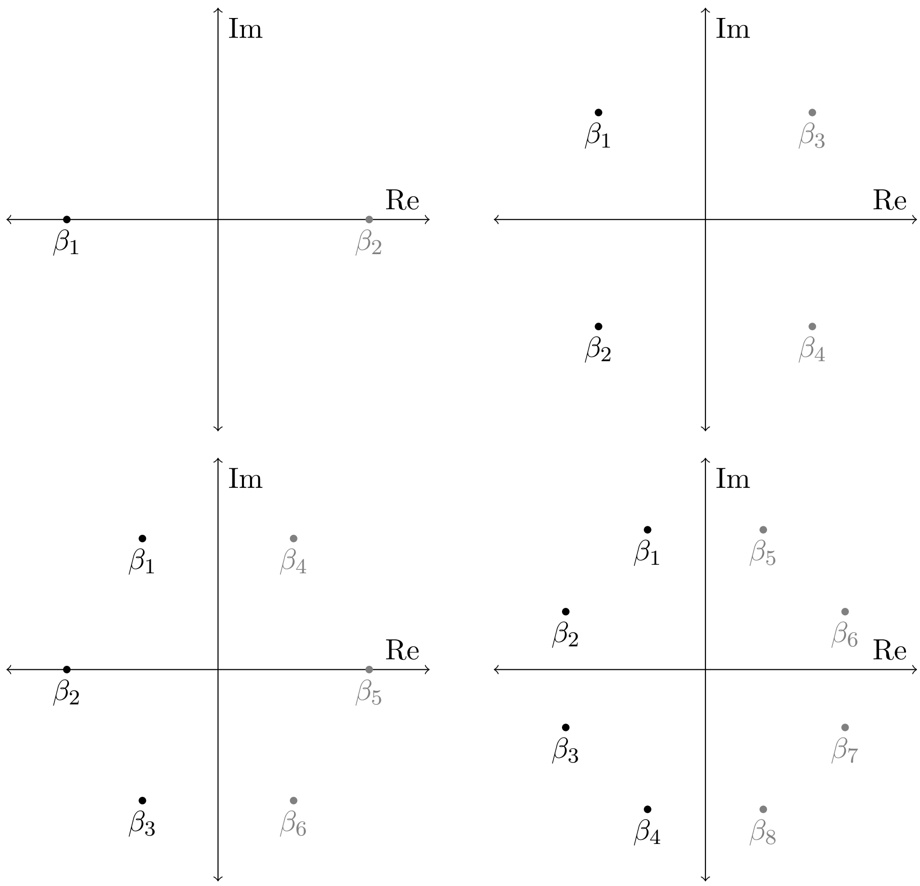

Figure 5.3: Roots of the polynomial $s^{2m} + (-1)^m$ for $m = 1$ (top left), $m = 2$ (top right), $m = 3$ (bottom left), and $m = 4$ (bottom right). All the roots lie on the complex unit circle and have magnitude one. The poles of the normalised Butterworth filter B_m are those roots from the left half of the complex plane (unshaded).

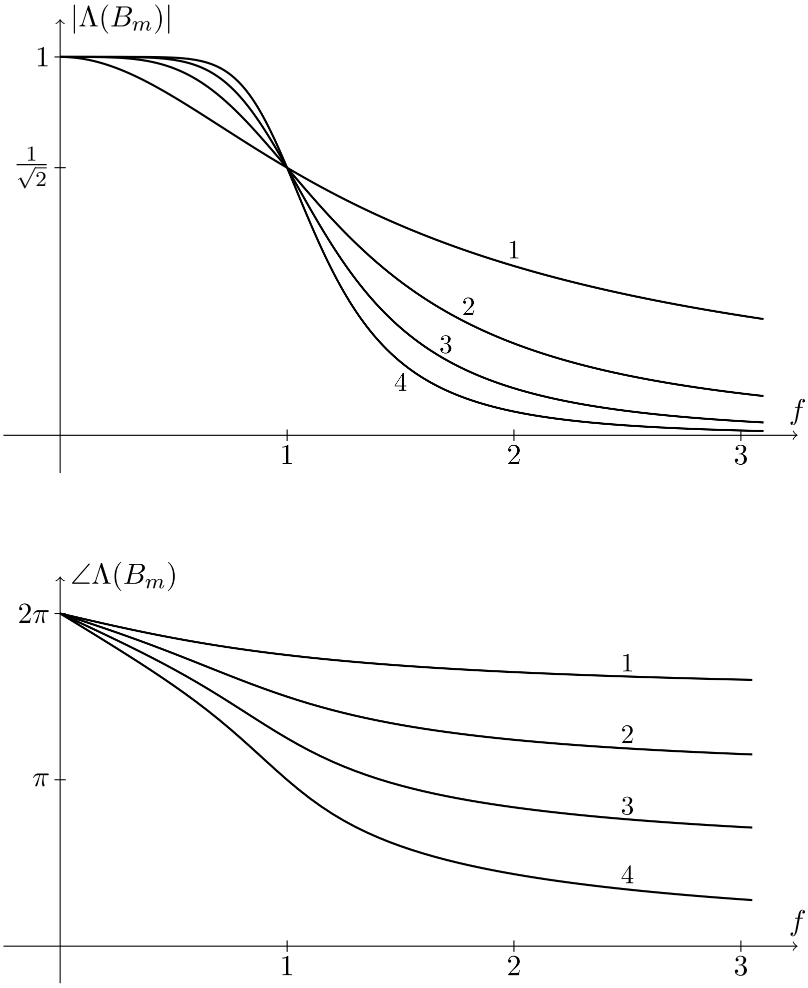

Figure 5.4: Magnitude spectrum (top) and phase spectrum (bottom) of normalised Butterworth filters B_1, B_2, B_3 and B_4.

The **cuttoff frequency** of the lowpass filter B_m is defined as the positive real number c such that $|\Lambda(B_m, f)|^2 < \frac{1}{2}$ for all $f > c$. The normalised Butterworth filters have cuttoff frequency $c = 1$Hz. A lowpass Butterworth filter of order m and cuttoff frequency c, denoted B_m^c, has transfer function

$$\lambda(B_m^c, s) = \lambda(B_m, \tfrac{s}{c}) = \frac{1}{\prod_{i=1}^{m}(\frac{s}{2\pi c} - \beta_i)}.$$

The magnitude spectrum satisfies

$$|\Lambda(B_m^c, f)|^2 = |\Lambda(B_m, \tfrac{f}{c})|^2 = \frac{1}{\left(\frac{f}{c}\right)^{2m} + 1} = \frac{c^{2m}}{f^{2m} + c^{2m}}. \tag{5.2.1}$$

A first order Butterworth filter B_1^c has spectrum

$$\Lambda(B_1^c) = \frac{1}{j\frac{f}{c} + 1} = \frac{c}{jf + c}.$$

Putting $\frac{1}{c} = 2\pi RC$ we find that this is the same as the spectrum of the RC electrical circuit (Figure 2.1) or the active RC circuit after negation (3.5.2). Thus, the RC electrical circuit is a first order Butterworth filter with cuttoff frequency $c = \frac{1}{2\pi RC}$. In Test 4 we constructed the active RC circuit with $R \approx 27\text{k}\Omega$ and $C \approx 10\text{nF}$ and measured its magnitude spectrum. The cuttoff frequency was $c = \frac{5\times 10^4}{27\pi} \approx 589$Hz.

A second order electrical Butterworth filter can be constructed using the Sallen-Key circuit described in Section 2.2 and Figure 2.10. The input voltage x and output voltage y of the Sallen-Key satisfy the differential equation (2.2.9)

$$x = y + C_2(R_1 + R_2)D(y) + R_1R_2C_1C_2D^2(y).$$

The transfer function is

$$\frac{\mathcal{L}(y)}{\mathcal{L}(x)} = \frac{1}{1 + C_2(R_1 + R_2)s + R_1R_2C_1C_2s^2}.$$

The second order Butterworth filter B_2^c has transfer function

$$\Lambda(B_2^c) = \frac{1}{(\frac{1}{2\pi c}s - \beta_1)(\frac{1}{2\pi c}s - \beta_2)},$$

where $\beta_1 = \beta_2^* = e^{j3\pi/4}$. Expanding the quadratic on the denominator gives

$$\Lambda(B_2^c) = \frac{1}{1 + \frac{1}{\sqrt{2}\pi c}s + \frac{1}{4\pi^2c^2}s^2}.$$

Choosing the resistors and capacitors of the Sallen-Key to satisfy

$$C_2(R_1 + R_2) = \frac{1}{\sqrt{2}\pi c}, \qquad R_1R_2C_1C_2 = \frac{1}{4\pi^2c^2}$$

leads to a second order Butterworth filter. A convenient solution is to put $C_1 = 2C_2$ and $R_1 = R_2$. This gives a second order Butterworth filter with cuttoff

$$c = \frac{1}{\sqrt{2}\pi C_2(R_1 + R_2)} = \frac{1}{2\sqrt{2}\pi C_2 R_2}.$$

In Test 6 we construct a second order Butterworth filter using a Sallen-Key and measure its spectrum.

Butterworth filters of orders larger than $m = 2$ can be constructed by concatenating Sallen-Key circuits and RC circuits. If m is even then $m/2$ Sallen-Key circuits are required. Each Sallen-Key is used to construct a conjugate pair of poles, that is, the kth Sallen-Key would have poles $2\pi c\beta_k$ and $2\pi c\beta_k^* = 2\pi c\beta_{m-k+1}$. If m is odd then $(m-1)/2$ Sallen-Key circuits and a single RC circuit (or active RC circuit) can be used. The RC circuit is designed to have the real valued pole $\beta_{(m+1)/2} = 2\pi c$.

Test 6 (**Butterworth filter**)

We construct a second order Butterworth filter using the Sallen-Key circuit from Figure 2.10 with capacitors $C_2 \approx 100\text{nF}$, $C_1 \approx 2C_2 \approx 200\text{nF}$ and resistors $R_1 \approx R_2 \approx 330\Omega$. The cuttoff frequency is

$$c = \frac{1}{2\sqrt{2}\pi C_2 R_2} \approx 3410\text{Hz}.$$

Sinusoids of the form

$$\sin(2\pi f_k t), \qquad f_k = \left\lceil 110 \times 2^{k/2} \right\rfloor, \quad k = 1, 2, \dots, 13$$

are input to the filter using a computer soundcard and the magnitude and phase spectrum are measured using the procedure described in Test 4. Figure 5.5 shows the measurements (dots) plotted alongside the hypothesised magnitude spectrum

$$|\Lambda(B_2^c)| = \sqrt{\frac{1}{(f/c)^4 + 1}}$$

and the hypothesised phase spectrum $\angle\Lambda(B_2^c)$.

5.3 Real and complex valued sequences

Let x be a signal with Fourier transform $\hat{x} = \mathcal{F}(x)$. The signal x is said to be **bandlimited** if there exists a positive real number b such that

$$\hat{x}(f) = \mathcal{F}(x, f) = 0 \qquad \text{for all } |f| > b.$$

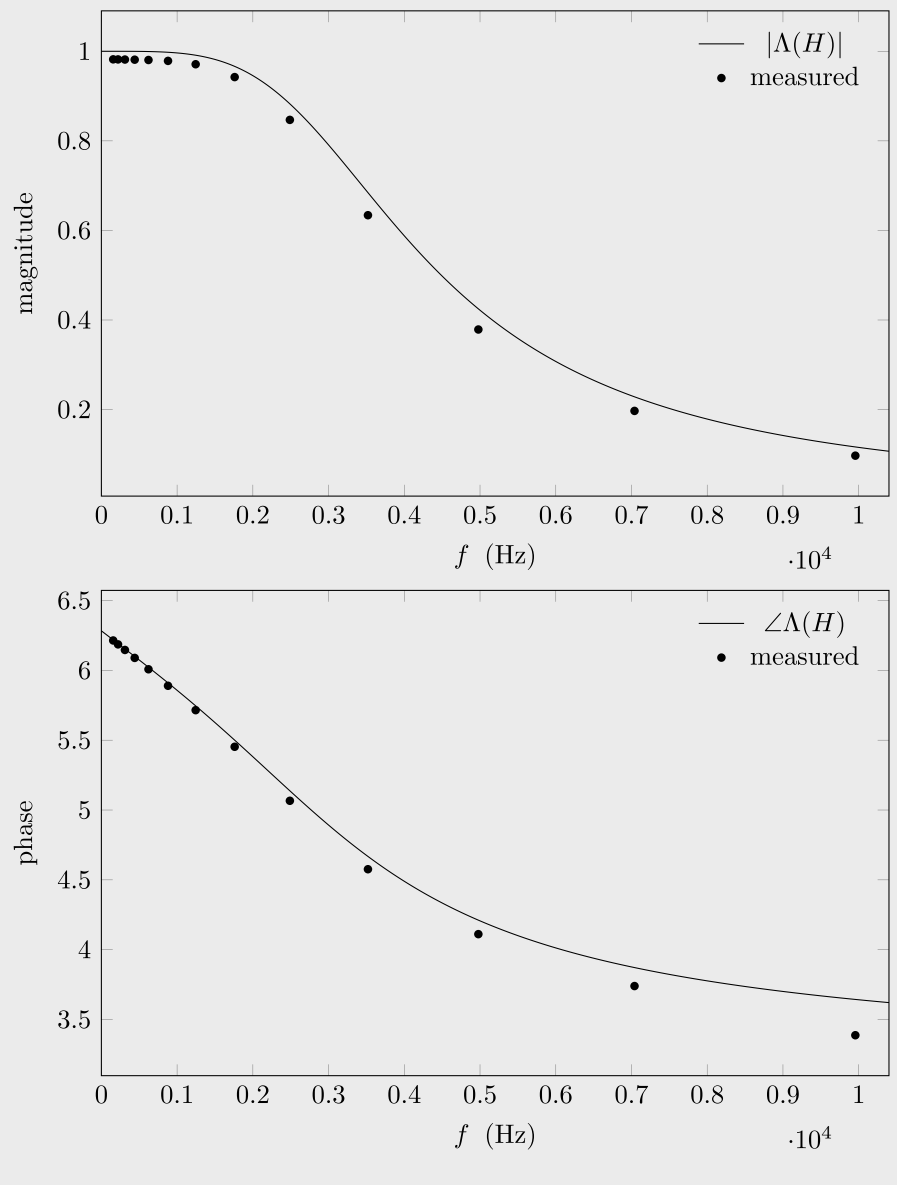

Figure 5.5: Hypothesised magnitude spectrum $|\Lambda(B_m^c)|$ (top) and phase spectrum $\angle\Lambda(B_m^c)$ (bottom) of the second order Butterworth filter and the measured magnitude and phase spectrum of the filter implemented with a Sallen-Key active electrical circuit (dots).

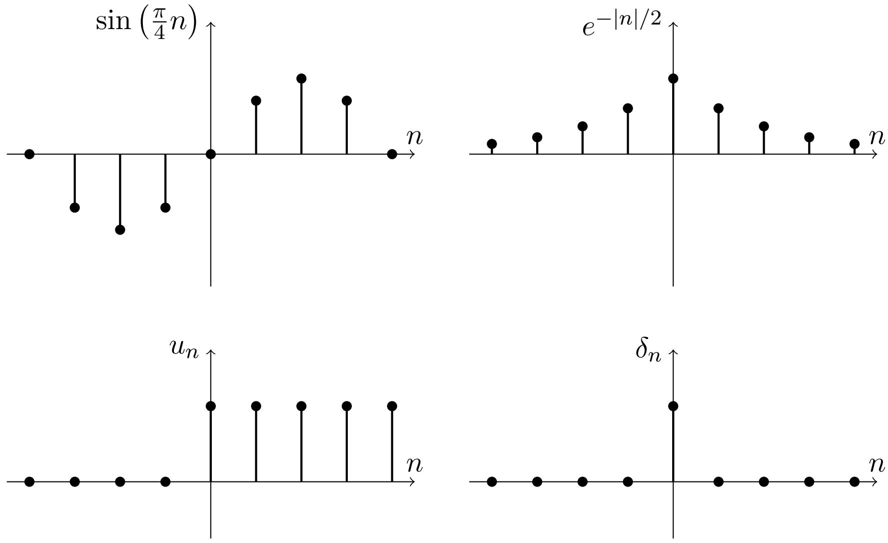

Figure 5.6: Real valued sequences. The bottom plots show that step sequence u and the delta sequence δ.

The value b is called the **bandwidth** of the signal x. For example, the sinc function is bandlimited with bandwidth $\frac{1}{2}$ because its Fourier transform $\mathcal{F}(\text{sinc}, f) = \Pi(f) = 0$ for all $|f| > \frac{1}{2}$. Bandlimited signals have a number of properties that make them suitable for representation and manipulation by a computer. They are of particular importance for this reason. Before we can study bandlimited signals we first require some properties of real and complex valued **sequences**.

A **sequence** is a function with domain given by the integers $\mathbb{Z}$. The value of the sequence corresponding with the integer n can be denoted by $x(n)$ but it is conventional to write x_n. We are primarily interested in sequences that take real or complex values, that is, $x_n \in \mathbb{R}$ or $x_n \in \mathbb{C}$. For example,

$$\sin(\tfrac{\pi}{4}n), \qquad n^3, \qquad e^{-|n|/2}$$

each denote a real valued sequence. In what follows the term **sequence** will always mean a real or complex valued sequence unless otherwise stated. Real and complex valued sequences are commonly called **discrete time signals** and the nth element in the sequence is denoted by $x[n]$ using squared brackets [Oppenheim et al., 1996]. Here, we use the subscript notation x_n. This notation is also common [Vetterli et al., 2014; Rudin, 1986]. Sequences are plotted using vertical lines with dotted ends as in Figure 5.6 and have a number of properties analogous to the properties of signals (Section 1.1).

A sequence x is bounded if there exists a real number M such that

$$|x_n| < M \qquad \text{for all } n \in \mathbb{Z}.$$

Both $\sin(\frac{\pi}{4}n)$ and $e^{-|n|/2}$ are examples of bounded sequences, but n^3 is not bounded because its magnitude grows indefinitely as n moves away from the origin. A sequence x is periodic if there exists a nonnegative integer T such that

$$x_n = x_{n+kT} \qquad \text{for all integers } k \text{ and } n.$$

The smallest such T is called the period. The sequence $\sin(\frac{\pi}{4}n)$ is periodic with period $T = 8$. Neither n^3 or $e^{-n^2/4}$ are periodic. A sequence x is even (or symmetric) if $x_n = x_{-n}$ for all $n \in \mathbb{Z}$ and odd (or antisymmetric) if $x_n = -x_{-n}$ for all $n \in \mathbb{Z}$. Both $\sin(\frac{\pi}{4}n)$ and n^3 are odd and $e^{-|n|/2}$ is even.

A sequence x is **right sided** if there exists a $T \in \mathbb{R}$ such that $x_n = 0$ for all $t < T$. Correspondingly x is **left sided** if $x_n = 0$ for all $T > t$. For example, the **step sequence** u with nth element

$$u_n = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{5.3.1}$$

is right sided (Figure 1.2). The reflected sequence u_{-n} is left sided. A sequence is said to be **finite** if it is both left and right sided. For example the sequence δ with nth element

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise,} \end{cases} \tag{5.3.2}$$

called the **delta sequence**, is finite. The delta sequence is analogous to the delta "function" introduced in Section 3.1. The delta "function" is not actually function, but only a notational device. Contrastingly, the delta sequence is a well defined sequence.

A sequence x is **absolutely summable** if

$$\|x\|_1 = \sum_{n \in \mathbb{Z}} |x_n| < \infty,$$

that is, if the sum of absolute values of the elements in the sequence converges to a finite number. The real number $\|x\|_1$ is commonly called the ℓ^1-norm of x. The sequences $\sin(\frac{\pi}{4}n)$ and n^3 are not absolutely summable, but $e^{-|n|/2}$ is because

$$\sum_{n \in \mathbb{Z}} |e^{-|n|/2}| = \sum_{n \in \mathbb{Z}} e^{-|n|/2} = 1 + \frac{2}{\sqrt{e} - 1}. \qquad \text{(Exercise 5.8)}$$

It is common to denote the set of absolutely summable sequences by ℓ^1. So, $e^{-|n|/2} \in \ell^1$ and $\sin(\frac{\pi}{4}n) \notin \ell^1$.

A sequence x is **square summable** if

$$\|x\|_2^2 = \sum_{n \in \mathbb{Z}} |x_n|^2 < \infty,$$

that is, if the sum of squared magnitudes of the elements converges to a finite number. The real number $\|x\|_2$ is commonly called the ℓ^2-norm and its square $\|x\|_2^2$ the **energy** of x. The sequences $\sin(\frac{\pi}{4}n)$ and n^3 are not square summable, but $e^{-|n|/2}$ is because

$$\sum_{n \in \mathbb{Z}} |e^{-|n|/2}|^2 = \sum_{n \in \mathbb{Z}} e^{-|n|} = 1 + \frac{2}{e-1}. \qquad \text{(Exercise 5.8)}$$

It is common to denote the set of square summable sequences by ℓ^2. So, $e^{-|n|/2} \in \ell^2$ and $\sin(\frac{\pi}{4}n) \notin \ell^2$. If a sequence is absolutely summable then it is also square summable (Exercise 5.9). The corresponding property is not true of signals, that is, absolutely integrable signals are not necessarily square integrable (Exercise 1.4).

5.4 Bandlimited signals

Let b be a positive real number and let x be a signal with Fourier transform $\hat{x} = \mathcal{F}(x)$. The signal x is said to be **bandlimited** with **bandwidth** b if

$$\hat{x}(f) = \mathcal{F}(x, f) = 0 \qquad \text{for all } |f| > b.$$

For example, the sinc function $\operatorname{sinc}(t)$ that has Fourier transform $\Pi(f)$ is bandlimited with bandwidth $b \geq \frac{1}{2}$. Another example is the signal with Fourier transform given by a **raised cosine**

$$\hat{x}(f) = \Pi(f)\big(1 + \cos(2\pi f)\big) = \begin{cases} 1 + \cos(2\pi f) & |f| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

that is bandlimited with bandwidth $b \geq \frac{1}{2}$. The time domain signal is found by applying the inverse Fourier transform

$$x(t) = \operatorname{sinc}(t) + \tfrac{1}{2}\operatorname{sinc}(t+1) + \tfrac{1}{2}\operatorname{sinc}(t-1). \qquad \text{(Exercise 5.6)}$$

Another example is the signal with Fourier transform given by the **triangle pulse**

$$\triangle(f) = \begin{cases} f+1 & -1 < f < 0 \\ 1-f & 0 \leq f < 1 \\ 0 & \text{otherwise} \end{cases}$$

that is bandlimited with bandwidth $b \geq 1$. The corresponding time domain signal is given by the square of the sinc function $\operatorname{sinc}^2(t)$ (Exercises 5.2). These bandlimited signals and their Fourier transforms are plotted in Figure 5.7.

It happens that bandlimited signals are not finite. We can reasonably suppose that all signals ever encountered in practice are finite and so no signals encountered in practice are truely bandlimited. However, many practically occuring signals are approximately bandlimited, that is, their Fourier transform is small for frequencies larger than some positive number b. For example, in Test 7 the Fourier transform of an audio signal taken from a lecture recording is plotted (Figure 5.8). This signal appears approximately bandlimited with bandwidth a little larger than 8 kHz.

A surprising result is that every square integrable bandlimited signal x with bandwidth b can be written as a sum of time scaled and time shifted sinc functions, that is, in the form

$$x(t) = \sum_{n\in\mathbb{Z}} c_n \operatorname{sinc}(Ft - n) \tag{5.4.1}$$

where c is a square integrable complex valued sequence and $F = 2b$. This is a consequence of the **Riesz-Fischer theorem** [Rudin, 1986, page 91]. Evaluating the signal x at integer multiples of $P = \frac{1}{F}$ we find that

$$x(\ell P) = \sum_{n\in\mathbb{Z}} c_n \operatorname{sinc}(\ell - n) = c_\ell$$

because $\operatorname{sinc}(\ell - n)$ is equal to 1 when $\ell = n$ and 0 otherwise. So, the elements of the sequence c correspond with samples of the signal x taken at integer multiples of $P = \frac{1}{F} = \frac{1}{2b}$, that is, $c_n = x(nP)$. The positive real number P is called the **sampling period** and its reciprical F the **sampling rate**. It follows that every square integrable bandlimited signal x with bandwidth b can be reconstructed from samples taken at rate $F = 2b$, that is,

$$x(t) = \sum_{n\in\mathbb{Z}} x(nP) \operatorname{sinc}(Ft - n).$$

This result known as the **Nyquist sampling theorem**. This theorem motivated use of this recontruction method in Tests 1, 2, 3, and 5.

5.5 The discrete time Fourier transform

Let x be a square integrable bandlimited signal with bandwidth b and let c be the sequence containing samples of x at sampling rate $F = \frac{1}{P} = 2b$, that is, $c_n = x(nP)$. From (5.4.1), the Fourier transform of x is

$$\begin{aligned}\hat{x}(f) = \mathcal{F}(x, f) &= \sum_{n\in\mathbb{Z}} c_n \mathcal{F}\big(\operatorname{sinc}(Ft - n)\big) \\ &= P\Pi(fP) \sum_{n\in\mathbb{Z}} c_n e^{-j2\pi Pnf} \\ &= P\Pi(fP)\mathcal{D}(c, Pf) \end{aligned} \tag{5.5.1}$$

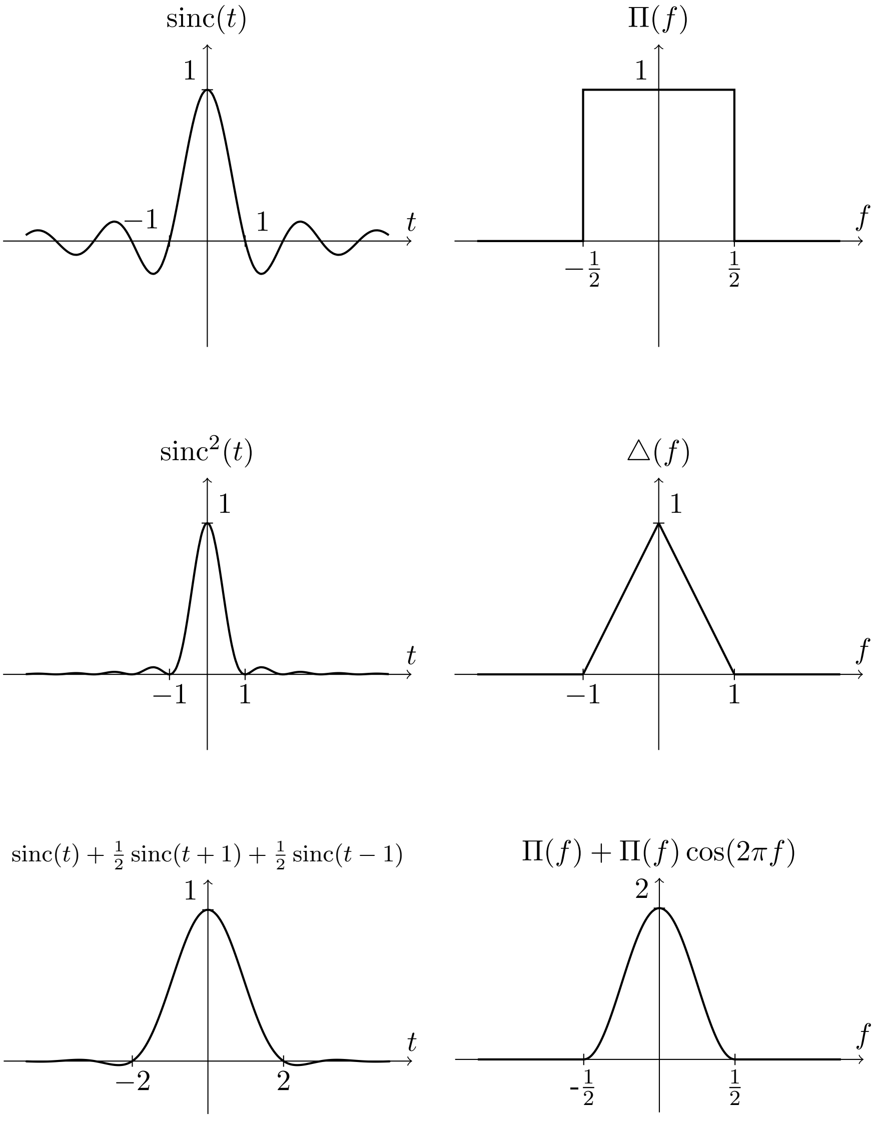

Figure 5.7: Bandlimited signals $\mathrm{sinc}(t)$, $\mathrm{sinc}^2(t)$, and $\mathrm{sinc}(t)+\frac{1}{2}\,\mathrm{sinc}(t+1)+\frac{1}{2}\,\mathrm{sinc}(t-1)$ with bandwidth $\frac{1}{2}$, 1, and $\frac{1}{2}$ respectively.

where

$$\mathcal{D}(c, f) = \sum_{n \in \mathbb{Z}} c_n e^{-j2\pi n f}$$

is called the **discrete time Fourier transform** of the sequence c. We write $\hat{c} = \mathcal{D}(c)$ for the discrete time Fourier transform of c. The discrete time Fourier transform $\hat{c} = \mathcal{D}(c)$ is a periodic signal with period 1. The above equations relate the Fourier transform of the bandlimited signal x to the discrete time Fourier transform of its sequence of samples c. In Test 7 we compute the Fourier transform of a 20 s segment of audio from a lecture recording. In Test 8 we pass the audio signal through the Butterworth filter constructed in Test 6 and plot the Fourier transform of the response.

The sequence of samples $c_n = x(nP)$ can be recovered by evaluating the inverse Fourier transform

$$\begin{aligned} c_n = x(nP) &= \mathcal{F}^{-1}(\hat{x}, nP) \\ &= \int_{-\infty}^{\infty} P\Pi(Pf)\mathcal{D}(c, Pf) e^{j2\pi f n P} df \\ &= P \int_{-F/2}^{F/2} \hat{c}(Pf) e^{j2\pi f n P} df \\ &= \int_{-1/2}^{1/2} \hat{c}(\gamma) e^{j2\pi \gamma n} d\gamma \qquad \text{(change variable } \gamma = fP\text{)}. \end{aligned}$$

We obtain the following relationship between the square integrable sequence c and its periodic discrete time Fourier transform $\hat{c} = \mathcal{D}(c)$,

$$c_n = \int_{-1/2}^{1/2} \hat{c}(f) e^{j2\pi f n} df.$$

The right hand side of this expression is called the **inverse discrete time Fourier transform**. The element c_{-n} is also called the nth **Fourier coefficient** of the periodic function $\hat{c}$.

Test 7 (**The Fourier transform of a lecture recording**)

In this test we consider a 20 s segment of audio taken from the lecture video `ch1sec3.mp4`. This 34.8 MB file contains both compressed video (H.264 codec) and audio (mp3 codec) of duration 23 min and 36 s. The audio is mono and sampled at rate $F = 22\,050$ Hz. The `avconv` program from the `libav` library is used to extract a 20 s segment of audio starting at time 85 s and ending at time 105 s. The segment is decompressed to wav format. The command used is:

```
avconv -i ch1sec3.mp4 -ss 85 -t 20 audio.wav
```

The resulting file `audio.wav` is 882 kB in size and contains $N = 441216$ samples that we denote by $c_0, c_1, \dots, c_{N-1}$. Each sample takes a value in the interval $[-1, 1]$. We put $c_n = 0$ when $n < 0$ or $n \geq N$. The reconstructed audio signal is given by

$$x(t) = \sum_{n \in \mathbb{Z}} c_n \operatorname{sinc}(Ft - n) = \sum_{n=0}^{N-1} c_n \operatorname{sinc}(Ft - n).$$

From (5.5.1) the Fourier transform of this signal is $\hat{x}(f) = P\Pi(Pf)\hat{c}(Pf)$ where

$$\hat{c}(f) = \mathcal{D}(c, f) = \sum_{n \in \mathbb{Z}} c_n e^{-j2\pi n f} = \sum_{n=0}^{N-1} c_n e^{-j2\pi n f} \tag{5.5.2}$$

is the discrete time Fourier transform of the sequence of samples. Figure 5.8 shows a plot of the magnitude of the Fourier transform for frequencies in the interval -12 kHz to 12 kHz. The plot is constructed by evaluting $|\hat{c}(f)|$ at all $K = 1201$ frequencies

$$f_k = -12000 + 20k \qquad k = 0, \dots, K - 1,$$

that is, from -12 kHz to 12 kHz in steps of 20 Hz. It takes approximately 137 s to compute the Fourier transform at all of these frequencies. Evaluating the Fourier transform at a particular frequency requires calculating and accumlating each of the N terms in the sum (5.5.2). We hypothesise it to take approximately

$$\frac{137\,\mathrm{s}}{NK} \approx 260\,\mathrm{ns}$$

to compute each term. The computer used is an Intel Core 2 running at 2.4 GHz and the software is written in the `Scala` programming language.

The audio recording contains human voice that primarily resides at lower frequencies below 5 kHz. Audible in the recording is a faint high pitched hum. The cause of this is unknown. It might be a feature of the (probably low quality) webcam microphone used to record the audio. This hum is represented in Figure 5.8 by the spikes occurring at approximately ± 8 kHz and also by the region between 4900 Hz and 5900 Hz where the magnitude of the Fourier transform is elevated. Figure 5.9 is a plot of the Fourier transform for frequencies from 7998 Hz to 8002 Hz in steps of 5 mHz. This gives a high resolution view of the spike that occurs near 8 kHz. The magnitude of the Fourier transform is precisely zero for frequencies $|f| > F/2 = 11\,025$ Hz due to $\Pi(Pf)$ occurring in the definition of $\hat{x}$. However, in Figure 5.8 it is apparent that the Fourier transform is small if $|f|$ is a little larger than 8 kHz. This audio signal appears approximately bandlimited with bandwidth a little larger 8 kHz.

5.6 The fast Fourier transform

In Test 7 the Fourier transform of a 20 s audio signal consisting of $N = 441216$ consecutive samples was evaluated. This scenario where only a finite number, say N, of consecutive samples of a signal is available is common in practice. Let c be a sequence with elements $c_0, c_1, \dots, c_{N-1}$ equal to the N samples. A convenient assumption is that the remaining samples are equal to zero, that is, $c_n = 0$ when $n < 0$ or $n \geq N$. This assumption was made in Test 7.

With this assumption the discrete time Fourier transform of the sequence c is given by the finite sum

$$\hat{c}(f) = \mathcal{D}(c, f) = \sum_{n \in \mathbb{Z}} c_n e^{-j2\pi n f} = \sum_{n=0}^{N-1} c_n e^{-j2\pi n f}.$$

The values of $\hat{c}(f)$ for f a multiple of $\frac{1}{N}$ have a number of convenient properties. Denote these values by

$$\mathcal{D}_N(c, k) = \mathcal{D}\left(c, \frac{k}{N}\right) = \sum_{n=0}^{N-1} c_n e^{-j2\pi nk/N} \qquad k \in \mathbb{Z}. \tag{5.6.1}$$

This is called the **discrete Fourier transform** (as opposed to the discrete *time* Fourier transform). The discrete Fourier transform $\mathcal{D}_N(c)$ is a sequence (a function of $k \in \mathbb{Z}$) with elements given by the discrete time Fourier transform $\mathcal{D}(c)$ evaluated at multiples of $\frac{1}{N}$. We write either $\mathcal{D}_N(c, k)$ or $\mathcal{D}_N(c)(k)$ to denote the value of $\mathcal{D}_N(c)$ at $k \in \mathbb{Z}$. The positive integer N is called the **length** of the transform. In practical applications N often corresponds with the number of samples of a signal that have been obtained.

The discrete Fourier transform is a periodic sequence with period N as a result of the discrete time Fourier transform $\hat{c} = \mathcal{D}(c)$ having period 1, that is,

$$\mathcal{D}_N(c, k) = \hat{c}\left(\frac{k}{N}\right) = \hat{c}\left(\frac{k + mN}{N}\right) = \mathcal{D}_N(c, k + mN) \quad \text{for all } k, m \in \mathbb{Z}.$$

Because of this it is sufficient to know only $\mathcal{D}_N(c, k)$ for $k = 0, \dots, N-1$ in order to know the entire sequence $\mathcal{D}_N(c)$. Given $\mathcal{D}_N(c)$ the original samples $c_0, \dots, c_{N-1}$ can be recovered by

$$c_n = \mathcal{D}_N^{-1}(\mathcal{D}_N(c), n) = \frac{1}{N} \sum_{k=0}^{N-1} \mathcal{D}_N(c, k) e^{j2\pi nk/N} \qquad n = 0, \dots, N-1.$$

This is called the **inverse discrete Fourier transform** (Excersise 5.11). Taking complex conjugates on both sides gives

$$c_n^* = \frac{1}{N} \sum_{k=0}^{N-1} \mathcal{D}_N^*(c, k) e^{-j2\pi nk/N} = \frac{1}{N} \mathcal{D}_N(\mathcal{D}_N^*(c), n) \qquad n = 0, \dots, N-1.$$

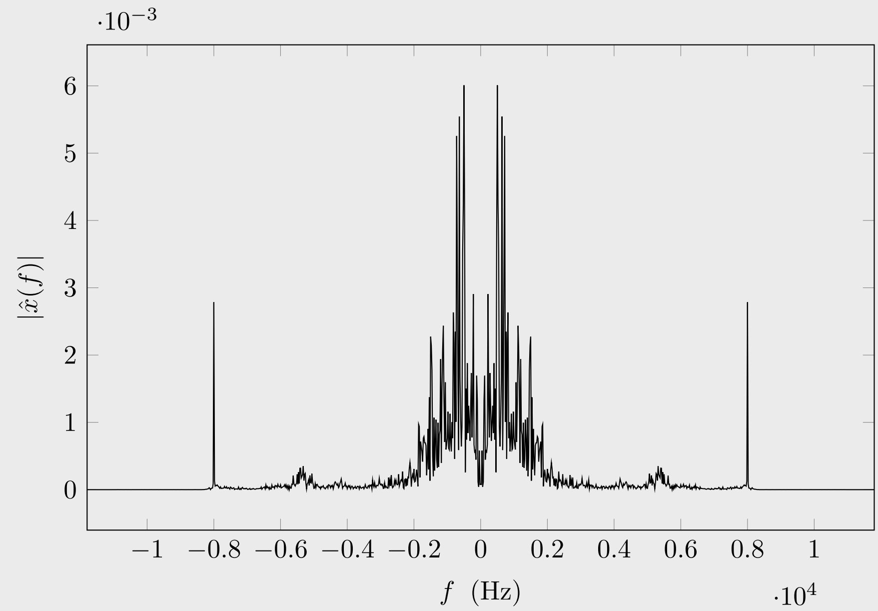

Figure 5.8: Magnitude of the Fourier transform of 20 s of audio from a lecture recording. The human voice signal is primarily contained in the low frequency region below 5 kHz. The spikes occurring at approximately ±8 kHz and the region between 4900 Hz and 5900 Hz where the magnitude is elevated are audible in the recording as a high pitched hum.

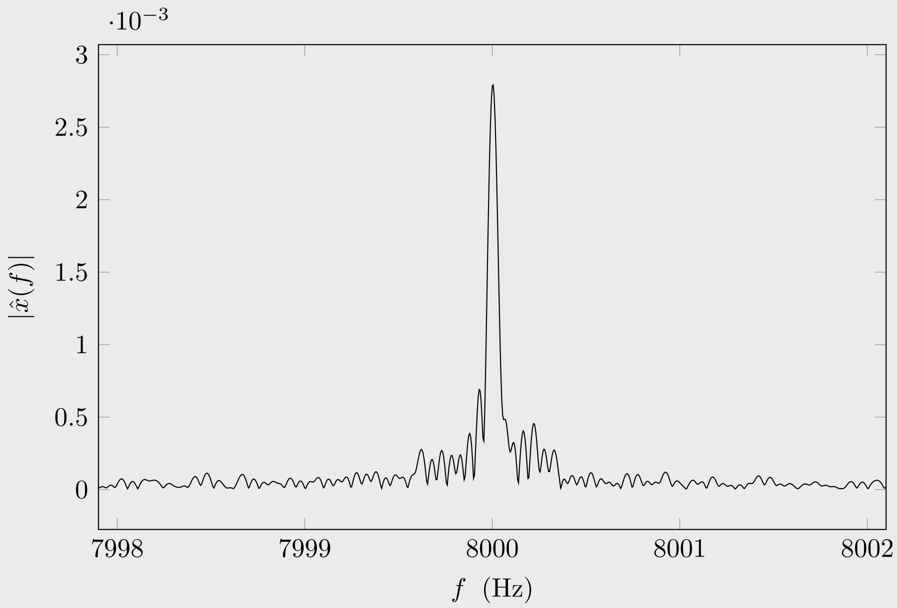

Figure 5.9: A plot of the magnitude of the Fourier transform zoomed in on the spike at 8 kHz.

Test 8 (**Butterworth filtered lecture recording**)
We consider again the 20 s audio signal from Test 7. In this test we pass this signal through the second order Butterworth filter from Test 6 with cuttoff frequency approximately 3041 Hz. The output of the Butterworth filter is fed back to the soundcard input and recorded at 22 050 Hz. The recorded samples are written to the file `filtered.wav`. Listening to `filtered.wav` confirms that the high pitched hum is weaker than it is in the original audio signal contained in the file `audio.wav` from Test 7. The Fourier transform of the Butterworth filtered signal is plotted in Figure 5.10. This figure is constructed by the same procedure as used for Figure 5.8 from Test 7. Observe that the spikes occurring at approximately ±8 kHz are less prominent than in Figure 5.8.

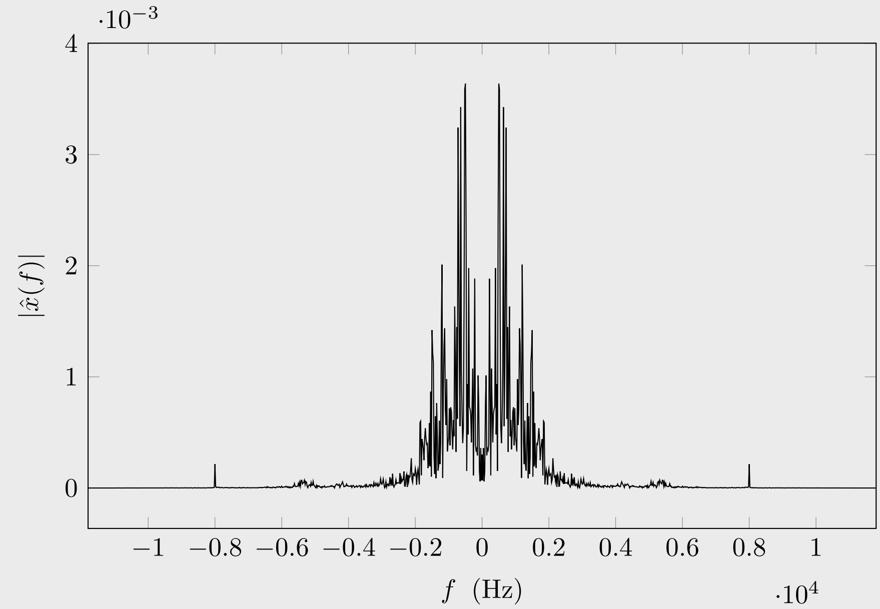

Figure 5.10: Magnitude of the Fourier transform of 20 s of audio from Test 7 after being passed through the second order Butterworth filter from Test 6 with cutoff frequency approximately 3041 Hz. The magnitude of the Fourier transform at higher frequencies is attenuated when compared with the Fourier transform of the original audio signal (Figure 5.8). In particular, the spikes occurring at approximately ±8 kHz are less prominent than in Figure 5.8. The high pitched hum that is audible in the original audio signal is significantly weaker in the Butterworth filtered audio signal.

A practical consequence of this is that the inverse discrete Fourier transform can be evaluted by applying the complex conjugate, taking the discrete Fourier transform, applying the complex conjugate again, and finally dividing by N. That is, if d is a sequence, then $\mathcal{D}_N^{-1}(d) = \frac{1}{N}\mathcal{D}_N^*(d^*)$.

Suppose that we wish to evaluate the discrete Fourier transform $\mathcal{D}_N(c)$ of the 20 s audio signal comprising of $N = 441216$ samples from Test 7. In Test 7 we hypothesised that approximately 260 ns are required to compute each term in the sum (5.5.2). We require to compute the sum for each $k = 0, \dots, N-1$ and so we might expect that

$$N^2 \times 260\,\mathrm{ns} \approx 50\,614\,\mathrm{s} \approx 14 \text{ hours} \tag{5.6.2}$$

will be required to compute $\mathcal{D}_N(c)$ for this 20 s audio signal! A primary cause of this lengthy computation time is the quadratic term N^2 that occurs in the expression above. The amount of time required grows proportionally with the square of the length of the transform. Suppose that instead of 20 s of audio we have 1 hour and $N = 60 \times 60 \times 22050 = 79380000$ samples. The amount of time required in this case is approximated by $N^2 \times 260\,\mathrm{ns} \approx 52$ years!

Computing the discrete Fourier transform by direct application of the formula (5.6.1) is too slow when N is large. Fortunately, much faster algorithms exist. The algorithms are appropriately called **fast Fourier transforms**. The specific algorithm used depends on N. The simplest case is when $N = 2^m$ is a power of 2. In this case an algorithm attributed to Cooley and Tukey [1965] can be used. When $N = 2^m$ is divisible by 2 the sum in (5.6.1) can be split into two parts corresponding with n being even or odd,

$$\mathcal{D}_N(c,k) = \sum_{n=0}^{N/2-1} c_{2n} e^{-j2\pi(2n)k/N} + \sum_{n=0}^{N/2-1} c_{2n+1} e^{-j2\pi(2n+1)k/N}. \tag{5.6.3}$$

Put $M = N/2$ and let p be the sequence with elements $p_n = c_{2n}$, that is, the elements of p are the even indexed elements of c. Now the first term in (5.6.3) can be written in the form

$$\sum_{n=0}^{N/2-1} c_{2n} e^{-j2\pi(2n)k/N} = \sum_{n=0}^{M-1} p_n e^{-j2\pi nk/M} = \mathcal{D}_M(p,k),$$

that is, this term is the discrete Fourier transform of length $M = N/2$ of the sequence p. Let q be the sequence with elements $q_n = c_{2n+1}$, that is, q contains the odd indexed elements of c. The second term in (5.6.3) can be written in the form

$$\begin{aligned}\sum_{n=0}^{N/2-1} c_{2n+1} e^{-j2\pi(2n+1)k/N} &= e^{-j2\pi k/N} \sum_{n=0}^{M-1} q_n e^{-j2\pi nk/M} \\ &= e^{-j2\pi k/N} \mathcal{D}_M(q,k),\end{aligned}$$

that is, this term is the discrete Fourier transform of length $M = N/2$ of the sequence q multiplied by the term $e^{-j2\pi k/N}$. Combining these results we have

$$\mathcal{D}_N(c,k) = \mathcal{D}_{N/2}(p,k) + e^{-j2\pi k/N}\mathcal{D}_{N/2}(q,k).$$

We see that the discrete Fourier transform $\mathcal{D}_N(c)$ can be evaluated by computing two smaller discrete Fourier transforms $\mathcal{D}_{N/2}(p)$ and $\mathcal{D}_{N/2}(q)$ of length $N/2$. Both of these smaller transforms are sequences that are periodic with period $N/2$ and so it is sufficient to know their values only for $k = 0, \dots, \frac{N}{2} - 1$. These $N/2$ length transforms can intern be computed by two transforms of length $N/4$ and so on until transforms of length 1 are obtained. In this case $\mathcal{D}_1(c,k) = \sum_{n=0}^{0} c_n e^{-j2\pi nk} = c_0$ for all $k \in \mathbb{Z}$.

The computational cost of this procedure can be analysed as follows. Suppose that C_N is the number of complex arithmetic operations (complex additions and multiplications) required to compute the discrete Fourier transform $\mathcal{D}_N(c)$ of length $N = 2^m$. The computation requires calculation of two transforms of length $N/2$ followed by N complex multiplications and N additions. The multiplications arise from the multiplication of $\mathcal{D}_{N/2}(q,k)$ by $e^{-j2\pi k/N}$ and the additions arise from summing the result of this product with $\mathcal{D}_{N/2}(p,k)$. The number of operations satisfies

$$C_N = 2C_{N/2} + 2N \qquad N \geq 2.$$

Because $\mathcal{D}_1(c,k) = c_0$ we have $C_1 = 0$, that is, computing a discrete Fourier transform of length 1 requires no complex operations at all. Putting $a_m = C_{2^m}$ we have

$$a_0 = C_1 = 0 \qquad a_m = 2a_{m-1} + 2^{m+1} \quad m \geq 1. \tag{5.6.4}$$

This type of recursive equation is called a **difference equation**. Exercise 6.8 shows that

$$C_N = a_m = 2^{m+1}m = 2N \log_2 N.$$

Observe that the number of operations (and hence the amount of time required) grows proportionally to $N \log_2 N$ rather than N^2. Suppose that each complex operation requires no more than 260 ns. For the 20 s audio signal consisting of $N = 441216$ samples the amount of time required will be less than

$$2N \log_2 N \times 260\,\text{ns} \approx 4.3\,\text{s}. \tag{5.6.5}$$

This is more reasonable than 14 hours! If instead we have 1 hour of audio and $N = 79380000$ samples the amount of time required is hypothesised to be less than $1084\,\text{s} \approx 18\,\text{min}$. This is very reasonable when compared with the 52 years hypothesised to be required by direct application of formula (5.6.1). In practice the computation time will vary based on the computer used and the specific algorithm implementation. Nevertheless, these numbers indicate that a fast Fourier transform of large length can be computed within a reasonable amount of time. This is not possible by direct application of formula (5.6.1). Test 9 compares the practical running time of various discrete Fourier transform implementations.

In the above computation of run times we have neglected that the fast Fourier transform we have described required the length N to be a power of two. Other algorithms exist for the case when N is not a power of two [Rader, 1968; Bluestein, 1968; Frigo and Johnson, 2005]. These algorithms deliver similarly dramatic computational savings. Even so, the restriction of the length to a power of 2 is often not a significant drawback in practical applications. Consider again the example from Test 7 with $N = 441216$ samples. Denote by

$$L = 2^{\lceil \log_2 N \rceil} = 2^{19} = 524288$$

the smallest power of 2 greater than N. We can use the fast Fourier transform algorithm described to compute the discrete Fourier transform $D_L(c)$ of length L. This transform is a sequence with period L and elements

$$\mathcal{D}_L(c, k) = \hat{c}\left(\frac{k}{L}\right) = \sum_{n=0}^{L-1} c_n e^{-j2\pi nk/L} = \sum_{n=0}^{N-1} c_n e^{-j2\pi nk/L} \qquad k \in \mathbb{Z}.$$

The second sum follows from our assumption that $c_n = 0$ for $n \geq N$. The elements of $\mathcal{D}_L(c)$ are the values of the discrete time Fourier transform $\hat{c}$ at multiples of $\frac{1}{L}$ rather than $\frac{1}{N}$. This fact is often of no significant consequence and can even be of benefit for some applications [Quinn and Hannan, 2001; Quinn et al., 2008]. The original samples $c_0, \dots, c_{N-1}$ can still be recovered by application of the inverse transform of length L, that is,

$$c_n = \mathcal{D}_L^{-1}(\mathcal{D}_L(c), n) \qquad n = 0, \dots, N-1.$$

This procedure of increasing the length of the transform is often called **zero padding** on account of the fact that the samples $c_N, c_{N+1}, \dots, c_{L-1}$ are assumed to be zero. Test 10 presents a practical example of zero padding for the purpose of filtering the 20 s audio recording from Test 7.

Test 9 (**Benchmarking the fast Fourier transform**)

In this test we compare the computational complexity of practical implementations of the discrete Fourier transform. Three differenent implementations are compared: a direct implementation by formula (5.6.1), an implementation of the fast Fourier transform of Cooley and Tukey [1965] when the length $N = 2^m$ is a power of 2 as described in Section 5.6, and an implementation from an optimised fast Fourier transform library called

`JTransforms`. The `JTransforms` library contains implementations of fast Fourier transforms of all lengths, not just powers of 2.

Figure 5.10 shows the run-time in seconds versus transform length. For the `JTransforms` library and formula (5.6.1) the length of the transforms is given by the sequence $N_k = \lceil 2^{6+k/2} \rfloor$ for $k = 0, 1, 2, \dots$. For our implementation of the Cooley and Tukey [1965] algorithm the length must be a power of two and is given by $N_k = 2^{6+k/2}$ for $k = 0, 2, 4, \dots$. The dashed lines indicate the approximate running times given by (5.6.2) and by (5.6.5). These approximations appear reasonably accurate on the log scale used in Figure 5.10. The fast Fourier transform algorithms are considerably faster than formula (5.6.1) as expected. For example, when the length is $N = 2^{21} = 2097152$ the `JTransforms` library required approximately 0.58 s whereas formula (5.6.1) is hypothesised by (5.6.2) to require approximately $N^2 \times 260\,\text{ns} \approx 13$ days.

The optimised algorithms from the `JTransforms` library are considerably faster than our implementation of the Cooley and Tukey [1965] algorithm. Observe the jagged nature of the run-time with the `JTransforms` library. The algorithms used by the library for length N_k and odd k appear slower than when k is even so that the length is a power of 2. The computer used is an Intel Core 2 running at 2.4 GHz and the software is written in the `Scala` programming language.

Test 10 (**Filtering a lecture recording by fast Fourier transform**)

We again consider the 20 s segment of audio consisting of $N = 441216$ samples from Test 7. As in Test 7 we let c be the sequence with elements $c_0, \dots, c_{N-1}$ equal to the audio samples and put $c_n = 0$ for $n < 0$ or $n \geq N$. The reconstructed audio signal is given by

$$x(t) = \sum_{n \in \mathbb{Z}} c_n \operatorname{sinc}(Ft - n) = \sum_{n=0}^{N-1} c_n \operatorname{sinc}(Ft - n)$$

where $P = \frac{1}{F}$ is the sample period and $F = 22\,050\,\text{Hz}$ is the sample rate. The Fourier transform of x is $\hat{x} = \mathcal{F}(x) = P\Pi(Pf)\hat{c}(Pf)$ where $\hat{c} = \mathcal{D}(c)$ is the discrete time Fourier transform of the sequence of samples c. Audible in the recording is a faint high pitched hum. This hum appears in the Fourier transform as spikes occurring at $\pm 8\,\text{kHz}$ and also as the region between 4900 Hz and 5900 Hz where the magnitude of the Fourier transform is elevated (Figure 5.8).

In this test we use a fast Fourier transform to remove this hum from the audio while minimally affecting the human voice. To do this we compute an

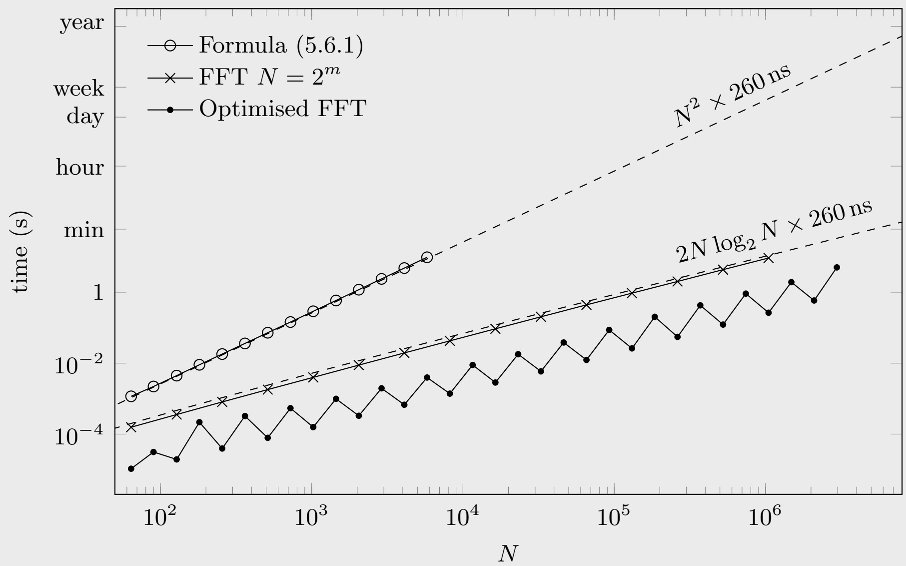

Figure 5.10: Comparison between run-times of the discrete Fourier transform computed directly by formula (5.6.1), by an implementation of the fast Fourier transform (FFT) of Cooley and Tukey [1965] described in Section 5.6, and by the optimised `JTransforms` fast Fourier transform library.

approximation of the bandlimited signal y with Fourier transform

$$\hat{y}(f) = \mathcal{F}(y, f) = \begin{cases} 0 & |f| > 7200 \\ 0 & |f - 5400| < 500 \\ 0 & |f + 5400| < 500 \\ \hat{x}(f) & \text{otherwise.} \end{cases}$$

That is, y is the signal with Fourier transform equal to $\hat{x}$ except for those frequencies between 4900 Hz and 5900 Hz and above 7200 Hz where the Fourier transform is zero. Because y is bandlimited with bandwidth less than F,

$$y(t) = \sum_{n \in \mathbb{Z}} b_n \operatorname{sinc}(Ft - n)$$

where b is the sequence with elements $b_n = y(nP)$ given by samples of y at sample period P. Now $\hat{y} = P\Pi(Pf)\hat{b}(Pf)$ where $\hat{b} = \mathcal{D}(b)$ is the discrete time Fourier transform of b. For f inside the interval $[-\frac{1}{2}, \frac{1}{2})$, the discrete time Fourier transforms $\hat{c}$ and $\hat{b}$ are related by

$$\hat{b}(f) = \begin{cases} 0 & |f| > 7200P \\ 0 & |f - 5400P| < 500P \\ 0 & |f + 5400P| < 500P \\ \hat{c}(f) & \text{otherwise.} \end{cases}$$

For $f \notin [-\frac{1}{2}, \frac{1}{2})$ a similar relationship can be obtained by appealing to the periodicity of $\hat{b}$ and $\hat{c}$. This is easiest to express by introducing the notation $\langle a \rangle = a - \lceil a \rfloor$ called the **centered fractional part** of $a \in \mathbb{R}$. Now

$$\hat{b}(f) = \begin{cases} 0 & |\langle f \rangle| > 7200P \\ 0 & |\langle f - 5400P \rangle| < 500P \\ 0 & |\langle f + 5400P \rangle| < 500P \\ \hat{c}(f) & \text{otherwise} \end{cases}$$

for all $f \in \mathbb{R}$.

Let $L = 2^{19} = 524288$ be the smallest power of 2 less than or equal to N. Using the fast Fourier transform we compute the discrete Fourier transform $\mathcal{D}_L(c)$ of length L of the sequence c. This yields values of $\hat{c}$ at multiples of $\frac{1}{L}$, that is,

$$\mathcal{D}_L(c, k) = \hat{c}\left(\frac{k}{L}\right) \qquad k \in \mathbb{Z}.$$

Let d be the sequence with elements

$$d_k = \mathcal{D}(b, k/L) = \begin{cases} 0 & \left|\left\langle \frac{k}{L} \right\rangle\right| > 7200P \\ 0 & \left|\left\langle \frac{k}{L} - 5400P \right\rangle\right| < 500P \\ 0 & \left|\left\langle \frac{k}{L} + 5400P \right\rangle\right| < 500P \\ \mathcal{D}_L(c, k) = \hat{c}(k/L) & \text{otherwise.} \end{cases}$$

We do not necessarily have $d = \mathcal{D}_L(b)$ because b_n is not necessarily equal to zero for $n < 0$ and $n \geq L$. Nevertheless, we will suppose that $d \approx \mathcal{D}_L(b)$. In this case, application of the inverse discrete Fourier transform yeilds the periodic sequence $\tilde{b} = \mathcal{D}_L^{-1}(d)$ and we expect the first L elements of $\tilde{b}$ to be an approximation of the first L elements of b, that is,

$$\tilde{b}_n \approx b_n = y(nP) \qquad \text{for } n = 0, \dots, L-1.$$

An approximation of the signal y is now given by

$$y(t) \approx \tilde{y}(t) = \sum_{n=0}^{N-1} \tilde{b}_n \operatorname{sinc}(Ft - \ell).$$

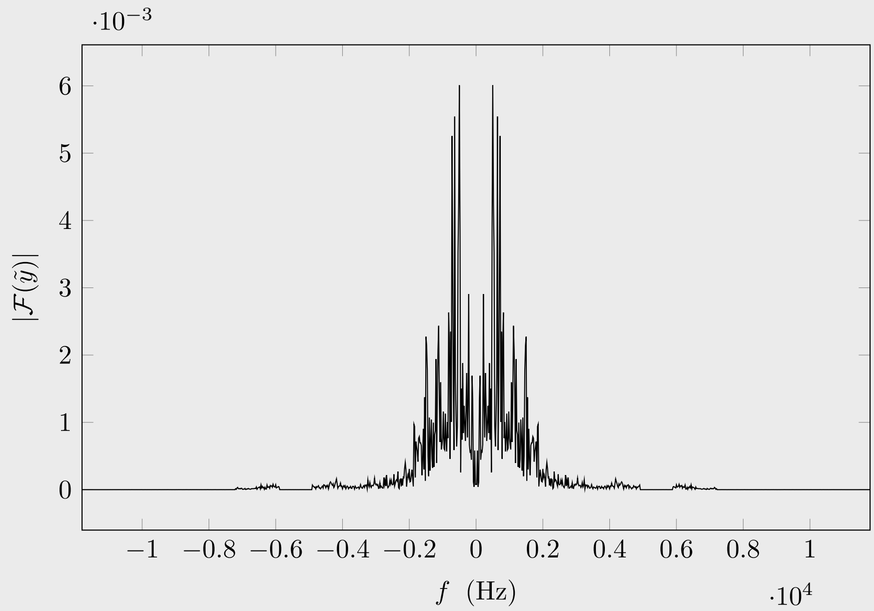

Figure 5.11: Plot of the magnitude of the Fourier transform $\mathcal{F}(\tilde{y})$. The plot looks similar to that of the magnitude of the Fourier transform of the original audio signal (Figure 5.8) except that the spikes at ± 8 kHz and the elevated region between 4900 Hz and 5900 Hz no longer exist.

Figure 5.11 plots the magnitude of Fourier transform $\mathcal{F}(\tilde{y})$. Observe that $|\mathcal{F}(\tilde{y})|$ looks similar to the magnitude of the Fourier transform of the original audio signal x plotted in Figure 5.8 except that the spikes at ± 8 kHz and the elevated region between 4900 Hz and 5900 Hz no longer exist. The samples $\tilde{b}_0, \dots, \tilde{b}_{N-1}$ are written to the audio file `nohum.wav`. Listening to the audio confirms that the human voice signal remains, but the high pitched hum is no longer audible.

5.7 Exercises

5.1. Plot the signal $e^{-\alpha|t|}$ where $\alpha > 0$ and find its Fourier transform.

5.2. Plot the signal

$$\triangle(t) = \begin{cases} t+1 & -1 < t < 0 \\ 1-t & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

and find its Fourier transform.

5.3. Show that the sinc function is square integrable, but not absolutely integrable.

5.4. Find and plot the impulse response of the normalised lowpass Butterworth filters B_1, B_2 and B_3.

5.5. Plot the signal

$$t\Pi(t) = \begin{cases} t & -\frac{1}{2} < t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

and find its Fourier transform.

5.6. Let x be the signal with Fourier transform $\hat{x}(f) = \Pi(f)\big(\cos(2\pi f)+1\big)$. Plot the Fourier transform $\hat{x}$ and find and plot x.

5.7. State whether the following signals are bandlimited and, if so, find the bandwidth.

(a) $\operatorname{sinc}(4t)$,
(b) $\Pi(t/4)$,
(c) $\cos(2\pi t)\operatorname{sinc}(t)$,
(d) $e^{-|t|}$.

5.8. Show that

$$\sum_{n\in\mathbb{Z}} e^{\alpha|n|} = 1 + \frac{2}{e^{-\alpha} - 1}$$

if $\alpha < 0$ (Hint: solve Exercise 3.5 first).

5.9. Show that if a sequence absolutely summable then it is also square summable.

5.10. Show that $\sum_{k=0}^{N-1} e^{j2\pi nk/N}$ is equal to N if n is a multiple of N and zero if n is any integer not a multiple of N. (Hint: use the result from Exersise 3.5)

5.11. Let $d = \mathcal{D}_N(c)$ be the discrete Fourier transform of the sequence c. Show that

$$c_n = \frac{1}{N}\sum_{k=0}^{N-1} d_k e^{j2\pi nk/N} \qquad n = 0, \dots, N-1.$$

(Hint: use the result from Exersize 5.10)

5.12. Plot the sequence $\cos(n)$ and determine whether it is bounded or periodic.

5.13. Find the discrete time Fourier transform of the sequence $\alpha^n u_n$ where $|\alpha| < 1$ and u_n is the step sequence. Plot the sequence and the magnitude of the discrete time Fourier transform when $\alpha = \frac{4}{5}, \frac{1}{2}, \frac{1}{10}$.

Chapter 6

Discrete time systems

We have so far studied linear time invariant systems and inparticular those systems described by linear differential equations with constant coefficients. Such systems are useful for modelling electrical circuits, mechanical machines, electro-mechanical devices, and many other real world devices. One particular linear time invariant system has so far been absent. This is the time shifter T_τ with non zero time shift $\tau \neq 0$.

We now consider systems constructed from linear combinations of time shifters of the form T_{Pn} where $n \in \mathbb{Z}$ and P is a positive real number called the **sample period** or simply **period**. That is, we consider systems of the form

$$H(x) = \sum_{n \in \mathbb{Z}} h_n T_{Pn}(x)$$

where h is a real or complex valued sequence. Such systems are called **discrete time systems**. Discrete time systems are not regular because the time shifter is not regular. However, we will find that the sequence h plays a role analogous to that of the impulse response of a regular system. For this reason h is called the **discrete impulse response** of H.

6.1 The discrete time impulse response

The discrete impulse response h immediately yields some properties of the corresponding discrete time system H. For example, if $h_n = 0$ for all $n < 0$, then H is causal because

$$H(x,t) = \sum_{n \in \mathbb{Z}} h_n T_{Pn}(x,t) = \sum_{n=0}^{\infty} h_n x(t - Pn)$$

only depends on values of the input signal x at times less than or equal to t. A discrete time system is stable if and only if its discrete impulse response is absolutely summable (Exercise 6.2). This is analagous to the property of regular systems that are stable if and only if their impulse response is absolutely integrable (Exercise 3.3).

Let F and G be discrete time systems with equal sample period P and discrete impulse responses f and g. Let

$$H = aF + bG \qquad a, b \in \mathbb{C}$$

be the system formed by a linear combination of F and G. The response of H to input signal x is

$$\begin{aligned} H(x) &= a \sum_{n\in\mathbb{Z}} f_n T_{Pn}(x) + b \sum_{n\in\mathbb{Z}} g_n T_{Pn}(x) \\ &= \sum_{n\in\mathbb{Z}} (af_n + bg_n) T_{Pn}(x), \end{aligned}$$

and so H is a discrete time system with discrete impulse response given by the linear combination of sequences $af + ag$.

Now suppose that

$$H(x) = F\big(G(x)\big)$$

is formed by the composition of discrete time systems F and G. The response of H to x is

$$\begin{aligned} H(x) &= F\big(\sum_{n\in\mathbb{Z}} g_n T_{Pn}(x)\big) \\ &= \sum_{m\in\mathbb{Z}} f_m T_{Pm}\big(\sum_{n\in\mathbb{Z}} g_n T_{Pn}(x)\big) \\ &= \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} f_m g_n T_{Pm}\big(T_{Pn}(x)\big) \\ &= \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} f_m g_n T_{P(m+n)}(x). \end{aligned}$$

By putting $k = m + n$ we have

$$\begin{aligned} H(x) &= \sum_{m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} f_m g_{k-m} T_{Pk}(x) \\ &= \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} f_m g_{k-m} T_{Pk}(x) \qquad \text{(interchange summation)} \qquad (6.1.1) \\ &= \sum_{k\in\mathbb{Z}} h_k T_{Pk}(x) \end{aligned}$$

where h is the sequence with elements given by

$$h_n = \sum_{m\in\mathbb{Z}} f_m g_{n-m} = (f * g)_n.$$

This is called the **discrete convolution** of sequences f and g. The special notation $f * g$ is again used to denote the discrete convolution so that $h = f * g$. The system H constructed by composition of the discrete time systems F and G is a discrete time system. The discrete impulse response of H is the discrete convolution of the discrete impulse responses of F and G. In what follows we will often use the term **convolution** rather than the lengthier term **discrete convolution** whenever there is no chance for confusion.

Consider the convolution $u * u$ of the step sequence (5.3.1) with itself. The nth element of the convolution is

$$(u * u)_n = \sum_{m \in \mathbb{Z}} u_m u_{n-m} = \sum_{m=0}^{n} 1 = \begin{cases} n+1 & n \geq 0 \\ 0 & n < 0. \end{cases}$$

Not all sequences can be convolved. Denote by $\mathbf{1}$ the sequence with all elements equal one. The convolution $u * \mathbf{1}$ is not possible because

$$(u * \mathbf{1})_n = \sum_{m \in \mathbb{Z}} u_m \mathbf{1}_{n-m} = \sum_{m=0}^{\infty} 1 = \infty$$

is not finite for any n. The convolution $u * \mathbf{1}$ is said not to exist. When considering convolution of sequences it is important to make sufficient assumptions for the convolution to exist. For example, the the convoltuion $f * g$ always exists if both f and g are absolutely summable sequences (Exercise 6.4). Similarly, the interchange of summation in (6.1.1) only holds under appropriate assumptions about the sequences f and g. For example, the interchange is valid when f and g are absolutely summable.

Discrete convolution has many properties analogous to that of the convolution of signals described in Section 3.2. For example, discrete convolution is commutative, that is, $f * g = g * f$. Discrete convolution is associative, that is, if f, g, and h are sequences then

$$(f * g) * h = f * (g * h). \qquad \text{(Exercise 6.1)}$$

Discrete convolution distributes with addition and commutes with scalar multiplication, that is,

$$a(f * h) + b(g * h) = (af + bg) * h$$

where a and b are real or complex constants.

6.2 The z-transform

Let c be a sequence. The function

$$\mathcal{Z}(c) = \sum_{n \in \mathbb{Z}} c_n z^{-n}$$

is called the **z-transform** of c. The z-transform $\mathcal{Z}(c)$ is a function of the complex plane. To indicate the value of $\mathcal{Z}(c)$ at $z \in \mathbb{C}$ we write either $\mathcal{Z}(c, z)$ or $\mathcal{Z}(c)(z)$. The z-transform is not necessarily defined for all complex numbers z. Let R_c be the set of nonnegative real numbers such that the sequence $c_n r^{-n}$ is absolutely summable if and only if $r \in R_c$, that is,

$$\sum_{n\in\mathbb{Z}} |c_n|\, r^{-n} < \infty \qquad \text{if and only if } r \in R_c.$$

In this case, $\mathcal{Z}(c, z)$ is finite for all z with magnitude satisfying $|z| \in R_c$ because

$$|\mathcal{Z}(c, z)| \le \sum_{n\in\mathbb{Z}} \left|c_n z^{-n}\right| \le \sum_{n\in\mathbb{Z}} |c_n|\, r^{-n} < \infty.$$

The subset of the complex plane with magnitude from R_c is called the **region of convergence** of the sequence c. The region of convergence of a sequence is analogous to the region of convergence of a signal when considering its Laplace transform. Recall that the region of convergence of a signal was either a half plane, a vertical strip, the entire complex plane, or the empty set (Section 4). We will find that the region of convergence of a sequence is either a circular disk, an annular region in the complex plane, the complex plane with a disc at the origin removed, the entire complex plane, or the empty set.

The step sequence u has z-transform

$$\mathcal{Z}(u) = \sum_{n\in\mathbb{Z}} u_n z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{z}{z-1} \qquad |z| > 1. \qquad \text{(Exercise 6.5)}$$

This sum converges only if the magnitude of z is greater than one, that is, only if $|z| > 1$. The **region of convergence** of the step sequence is the set of complex numbers with magnitude greater than one. Graphically this region of convergence is the complex plane with a disc of radius one centered at the origin removed (Figure 6.2).

Consider the sequence with nth element $(\frac{1}{2})^n u_n$. The z-transform

$$\mathcal{Z}\big((\tfrac{1}{2})^n u_n\big) = \sum_{n\in\mathbb{Z}} (\tfrac{1}{2})^n u_n z^{-n} = \sum_{n=0}^{\infty} (2z)^{-n} = \frac{2z}{2z-1} \qquad |z| > \tfrac{1}{2}$$

converges only if $|z| > \frac{1}{2}$. The region of convergence is the complex plane with a disc of radius $\frac{1}{2}$ removed. Now consider the sequence with elements

$$(\tfrac{3}{2})^n u_{-n} = \begin{cases} (\frac{3}{2})^n & n \le 0 \\ 0 & n > 0. \end{cases}$$

The z-transform

$$\mathcal{Z}\big((\tfrac{3}{2})^n u_{-n}\big) = \sum_{n\in\mathbb{Z}} (\tfrac{3}{2})^n u_{-n} z^{-n} = \sum_{n=0}^{\infty} (\tfrac{2}{3}z)^n = \frac{3}{3-2z} \qquad |z| < \tfrac{3}{2}$$

converges only if $|z| < \frac{3}{2}$. The region of convergence is an open disc of radius $\frac{3}{2}$ centered at the origin of the complex plane. The sequence with nth element $(\frac{1}{2})^n u_n + (\frac{3}{2})^n u_{-n}$ has z-transform

$$\mathcal{Z}\big((\tfrac{1}{2})^n u_n + (\tfrac{3}{2})^n u_{-n}\big) = \frac{2z}{2z-1} + \frac{3}{3-2z} \qquad \tfrac{1}{2} < |z| < \tfrac{3}{2}$$

that converges only if $\frac{1}{2} < |z| < \frac{3}{2}$. The region of convergence is an annulus in the complex plane with inner radius $\frac{1}{2}$ and outer radius $\frac{3}{2}$.

The delta sequence δ with elements

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

has z-transform

$$\mathcal{Z}(\delta) = \sum_{n \in \mathbb{Z}} \delta_n z^{-n} = 1.$$

The region of convergence is the entire complex plane. Finally, consider the sequence $\mathbf{1}$ that has every element equal to 1. In this case $\mathcal{Z}(\mathbf{1}) = \sum_{n\in\mathbb{Z}} z^{-n}$ does not converge for any $z \in \mathbb{C}$ and the region of convergence is the empty set. The sequence $\mathbf{1}$ is said not to have a z-transform.

Given the z-transform $\mathcal{Z}(c)$ the sequence c can be recovered by the inverse z-transform

$$c_n = \frac{1}{2\pi j} \oint_C \mathcal{Z}(c, z) z^{n-1} dz$$

where C is a counterclockwise closed path encircling the origin and within the region of convergence of c. Similarly to the inverse Laplace transform (Section 4), direct calculation of the inverse z-transform requires a form of integration called **contour integration** that we will not consider here [Stewart and Tall, 2004]. For our purposes, and for many engineering purposes, it suffices to remember only the following z-transform pair

$$\mathcal{Z}\big([n]_k u_n\big) = \frac{k!z}{(z-1)^{k+1}} \qquad |z| > 1 \qquad \text{(Exercise 6.6)}$$

where

$$[n]_k = n(n-1)\dots(n-k+1)$$

is called the **falling factorial** [Graham et al., 1994, p. 48]. In the case that $k = 0$ the falling factorial is defined as $[n]_0 = 1$ for all $n \in \mathbb{Z}$.

Let $a \in \mathbb{C}$. If c is a sequence with z-transform $\mathcal{Z}(c)$ and region of convergence R_c then the sequence with nth element $a^n c_n$ has z-transform

$$\mathcal{Z}(a^n c_n) = \sum_{n\in\mathbb{Z}} a^n c_n z^{-n} = \sum_{n\in\mathbb{Z}} c_n (z/a)^{-n} = \mathcal{Z}(c, z/a) \qquad \frac{z}{a} \in R_c.$$

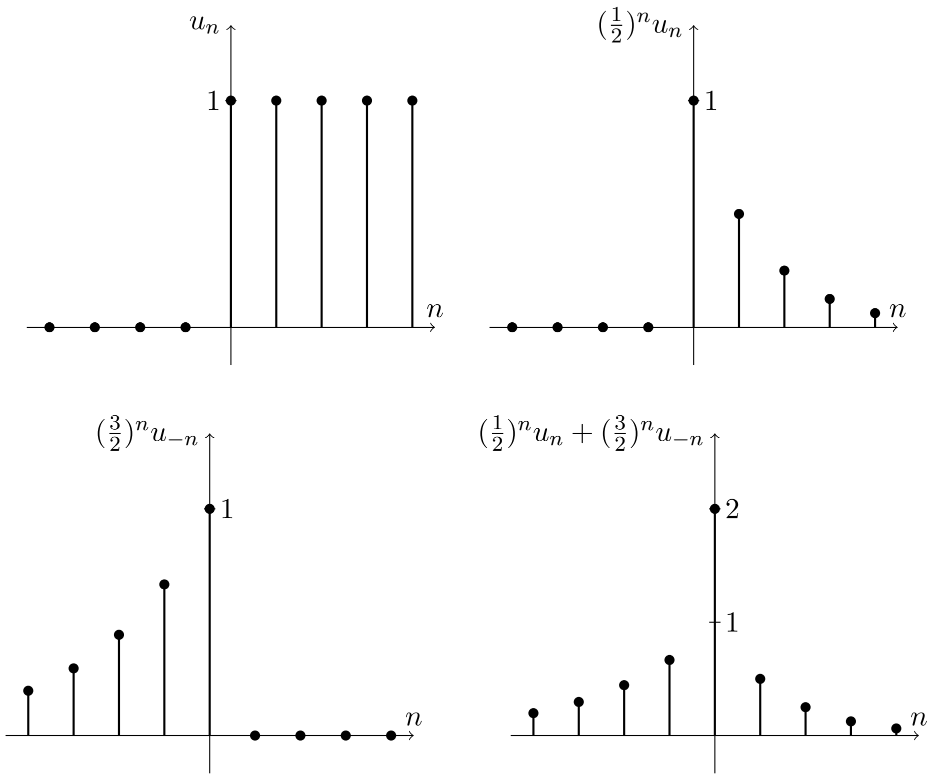

Figure 6.1: Real valued sequences. The top left plot show the step sequence u.

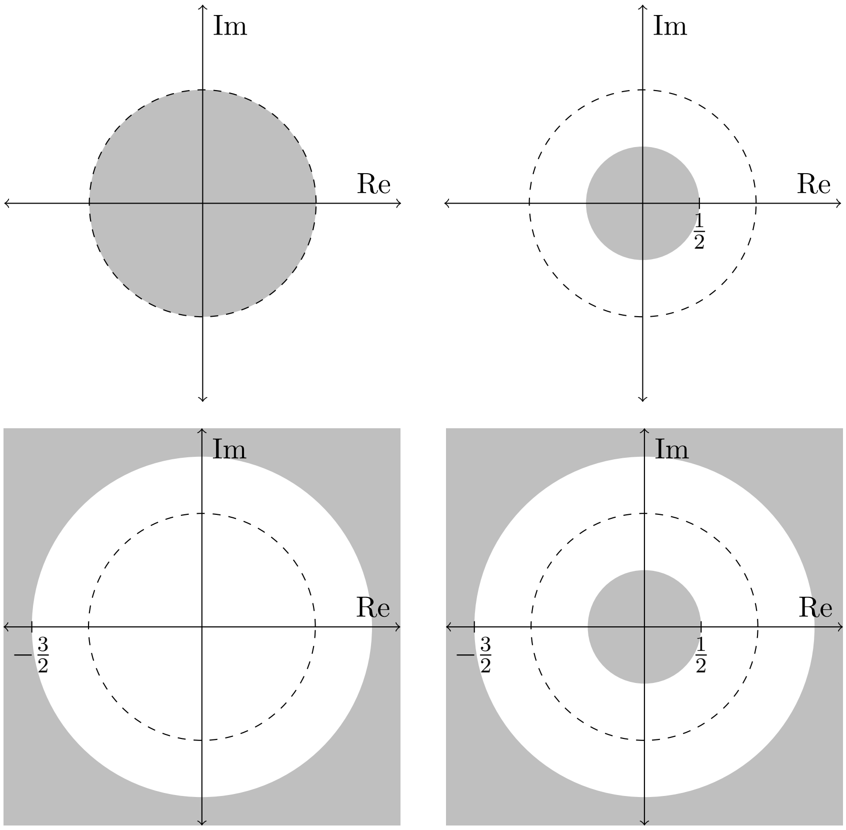

Figure 6.2: Regions of convergence (unshaded) for the step sequence u (top left), the sequence $(\frac{1}{2})^n u_n$ (top right), the sequence $(\frac{3}{2})^n u_{-n}$ (bottom left), and the sequence $(\frac{1}{2})^n u_n + (\frac{3}{2})^n u_{-n}$ (bottom right). The unit circle is indicated by the dashed circle. The region of convergence takes the form of the complex plane with a disc at the origin removed (top), a disc at the origin (bottom left), an annulus (bottom right), the entire complex plane, or the empty set.

This is called the **scaling** property of the z-transform. Using this property the z-transform of the sequence $a^n[n]_k u_n$ is

$$\mathcal{Z}\big(a^n[n]_k u_n\big) = \frac{k!a^k z}{(z-a)^{k+1}} \qquad |z| > |a|\,. \tag{6.2.1}$$

This is the only z-transform pair we require here. We will have particular use of the case when k is 0 or 1. In the case that $k = 0$ we obtain the z-transform pair

$$\mathcal{Z}\big(a^n u_n\big) = \frac{z}{z-a} \qquad |z| > |a|$$

and in the case that $k = 1$ we obtain

$$\mathcal{Z}\big(a^n n u_n\big) = \frac{az}{(z-a)^2} \qquad |z| > |a|\,.$$

Let g be a sequence with region of convergence R_g. The z-transform of the shifted sequence with nth element $g_{n-\ell}$ where $\ell \in \mathbb{Z}$ is related to that of g by

$$\begin{aligned}\mathcal{Z}(g_{n-\ell}) &= \sum_{n\in\mathbb{Z}} g_{n-\ell} z^{-n} \\ &= \sum_{n\in\mathbb{Z}} g_n z^{-(n+\ell)} \\ &= z^{-\ell} \sum_{n\in\mathbb{Z}} g_n z^{-n} = z^{-\ell}\mathcal{Z}(g) \qquad z \in R_g.\end{aligned} \tag{6.2.2}$$

This is called the **time shift** property of the z-transform.

The z-transform of a sequence is related to the transfer function of a discrete time system. Let H be a discrete time system with discrete impulse response h and period P. Because the transfer function of the time shifter T_{Pn} is $\lambda(T_{Pn}) = e^{-sPn}$ (4.1.1) the transfer function of H is

$$\lambda(H,s) = \sum_{n\in\mathbb{Z}} h_n \lambda(T_{Pn}) = \sum_{n\in\mathbb{Z}} h_n e^{-sPn}.$$

Putting $z = e^{Ps}$ we have

$$\mathcal{Z}(h,z) = \lambda(H, \tfrac{1}{P}\log z) = \sum_{n\in\mathbb{Z}} h_n z^{-n} \qquad z \in R_h$$

where R_h is the region of convergence of h. We see that the transfer function of a discrete time system with period P is related to the z-transform of its discrete impulse response by the equation above or equivalently by

$$\lambda(H,s) = \mathcal{Z}(h, e^{Ps}) \qquad e^{sP} \in R_h. \tag{6.2.3}$$

This relationship is analogous to the relationship between the transfer function of a regular linear time invariant system and the Laplace transform of its impulse response (Section 4.1). The set of complex numbers s such that $e^{sP} \in R_h$ is called the region of convergence of the discrete time system H. In this way, both sequences and discrete time systems have a region of convergence.

Let F and G be discrete time systems with periods P and discrete impulse responses f and g having regions of convergence R_f and R_g. Let $H(x) = F(G(x))$ be the discrete time system formed by the composition F and G. As shown in Section 6.1 the discrete impulse response of H is the discrete convolution $f * g$. Recall from (4.1.3) that the transfer function of a composition of linear time invariant systems is given by the product of the transfer functions, that is,

$$\lambda(H) = \lambda(G)\lambda(F) \qquad e^{sP} \in R_f \cap R_g.$$

Because

$$\mathscr{Z}(f, e^{sP}) = \lambda\big(F, s\big), \quad \mathscr{Z}(g, e^{sP}) = \lambda\big(G, s\big), \quad \mathscr{Z}(f * g, e^{sP}) = \lambda(H, s)$$

when $e^{sP} \in R_f \cap R_g$ we have

$$\mathscr{Z}(f * g) = \mathscr{Z}(f)\mathscr{Z}(g) \qquad z \in R_f \cap R_g.$$

That is, the z-transform of a convolution of sequences is the multiplication of the z-transforms of those sequences. The region of convergence of the convolution is the intersection of the regions of convergence. This is called the **convolution property** of the z-transform.

Let H be discrete time system with discrete impulse response h having region of convergence containing the complex unit circle. The spectrum of H is

$$\Lambda(H, f) = \lambda(H, j2\pi f) = \sum_{n \in \mathbb{Z}} h_n e^{-2\pi j f P n}.$$

The spectrum is periodic with period equal to the reciprocal of the sample period $F = \frac{1}{P}$ called the **sample rate**. The spectrum is related to the discrete time Fourier transform of h by

$$\mathcal{D}(h, f) = \Lambda(H, \tfrac{f}{P}) = \sum_{n \in \mathbb{Z}} h_n e^{-2\pi j f n}.$$

We have the following relationships between the transfer function, the spectrum, the discrete time Fourier transform, and the z-transform, of the discrete time system H and its discrete impulse response h,

$$\lambda(H, j2\pi f) = \Lambda(H, f) = \mathcal{D}(h, f) = \mathscr{Z}(h, e^{2\pi j P f}).$$

6.3 Difference equations

We have previously shown that interesting systems are found by consideration of a linear differential equation with constant coefficients. We have used these systems to model electrical and mechanical devices (Chapter 2). We will find that interesting discrete time systems are found by consideration of a linear **difference equation** with constant coefficients. That is, an equation relating two sequences c and d of the form

$$\sum_{\ell=0}^{m} a_\ell c_{n-\ell} = \sum_{\ell=0}^{k} b_\ell d_{n-\ell} \qquad n \in \mathbb{Z} \tag{6.3.1}$$

where $a_0, \dots, a_m$ and $b_0, \dots, b_k$ are real or complex constants.

In order to study this equation it is useful to study the equation

$$\sum_{\ell=0}^{m} a_\ell T_{P\ell}(x) = \sum_{\ell=0}^{k} b_\ell T_{P\ell}(y) \tag{6.3.2}$$

that relates two signals x and y. Zemanian [1965, Sec. 9.5] calls (6.3.2) the **continuous variable case** of a linear difference equation with constant coefficients. If x and y are signals satisfying this equation then the samples of x and y at multiples of P satisfy (6.3.1). That is, if we define sequences c and d by $c_n = x(nP)$ and $d_n = y(nP)$ then c and d satisfy (6.3.1) whenever x and y satisfy (6.3.2).

Suppose that H is a linear time invariant system with the property that the response $y = H(x)$ to input signal x is such that x and y satisfy (6.3.2). The transfer function of H is found to be

$$\lambda(H, s) = \frac{\sum_{\ell=0}^{m} a_\ell e^{-sP\ell}}{\sum_{\ell=0}^{k} b_\ell e^{-sP\ell}} = z^{k-m} \frac{\sum_{\ell=0}^{m} a_\ell z^{m-\ell}}{\sum_{\ell=0}^{k} b_\ell z^{k-\ell}} \qquad \text{(Exercise 6.3)}$$

where $z = e^{sP}$. Suppose that h is a sequence with z-transform

$$\mathcal{Z}(h, z) = \lambda(H, s) = z^{k-m} \frac{a_0 z^m + a_1 z^{m-1} + \cdots + a_m}{b_0 z^k + b_1 z^{k-1} + \cdots + b_k}.$$

It follows from (6.2.3) that H is a discrete time system with discrete impulse response h. By applying the inverse z-transform we can find an explicit expression for h. This procedure is similar to how the impulse response of a system described by a differential equation was found by application of the inverse Laplace transform in Section 4.5. In the case that $m > k$ the term z^{k-m} can be incorporated into the denominator obtaining

$$\mathcal{Z}(h) = \frac{a_0 z^m + a_1 z^{m-1} + \cdots + a_m}{b_0 z^m + b_1 z^{k-1} + \cdots + b_k z^{m-k}}$$

and in the case that $m < k$ the term z^{m-k} can be incorporated into the numerator obtaining

$$\mathcal{Z}(h) = \frac{a_0 z^k + a_1 z^{k-1} + \cdots + a_m z^{k-m}}{b_0 z^k + b_1 z^{k-1} + \cdots + b_k}.$$

In either case the order of the polynomials on the numerator and denominator are the same, that is, the order is $w = \max(m, k)$.

By factorising the polynomials on the numerator and denominator we obtain

$$\mathcal{Z}(h) = \frac{a_0}{b_0} \frac{(z-\alpha_0)(z-\alpha_1)\cdots(z-\alpha_w)}{(z-\beta_0)(z-\beta_1)\cdots(z-\beta_w)}$$

where $\alpha_0, \ldots, \alpha_w$ are the roots of the numerator polynomial and $\beta_0, \ldots, \beta_w$ are the roots of the denominator polynomial. If the numerator and denominator polynomials share one or more roots, then these roots cancel leaving the simpler expression

$$\mathcal{Z}(h) = \frac{a_0}{b_0} \frac{(z-\alpha_d)(z-\alpha_{d+1})\cdots(z-\alpha_w)}{(z-\beta_d)(z-\beta_{d+1})\cdots(z-\beta_w)}, \tag{6.3.3}$$

where d is the number of shared roots, these shared roots being

$$\alpha_0 = \beta_0, \quad \alpha_1 = \beta_1, \quad \ldots, \quad \alpha_{d-1} = \beta_{d-1}.$$

The roots from the numerator $\alpha_d, \ldots, \alpha_w$ are called the **zeros** and the roots from the denominator $\beta_d, \ldots, \beta_w$ are called the **poles**. For a discrete time system, the number of poles and zeros are equal. A pole-zero plot is constructed by marking the complex plane with a cross at the location of each pole and a circle at the location of each zero (Figure 6.3).

The z-transform pair (6.2.1) has the term z on its numerator and so it is convenient to write

$$\mathcal{Z}(h) = \frac{a_0}{b_0} z \frac{(z-\alpha_d)(z-\alpha_{d+1})\cdots(z-\alpha_w)}{z(z-\beta_d)(z-\beta_{d+1})\cdots(z-\beta_w)}.$$

Applying partial fraction to polynomial quotient yields

$$\mathcal{Z}(h) = \frac{a_0}{b_0} z \sum_{\ell \in K} \frac{A_\ell}{(z-\beta_\ell)^{r_\ell}}$$

where r_ℓ are positive integers, A_ℓ are complex constants, and K is a subset of the indices from $\{d, d+1, \ldots, w\}$. We need to consider those terms where $\beta_\ell = 0$ separately. Let K_1 be the subset of indices from K such that $\beta_\ell = 0$ when $\ell \in K_1$ and let K_2 be the subset such that $\beta_\ell \neq 0$ when $\ell \in K_2$. Now

$$\mathcal{Z}(h) = \frac{a_0}{b_0} \sum_{\ell \in K_1} \frac{A_\ell}{z^{r_\ell - 1}} + \sum_{\ell \in K} B_\ell \frac{\beta_\ell^{r_\ell - 1}(r_\ell - 1)! z}{(z-\beta_\ell)^{r_\ell}}.$$

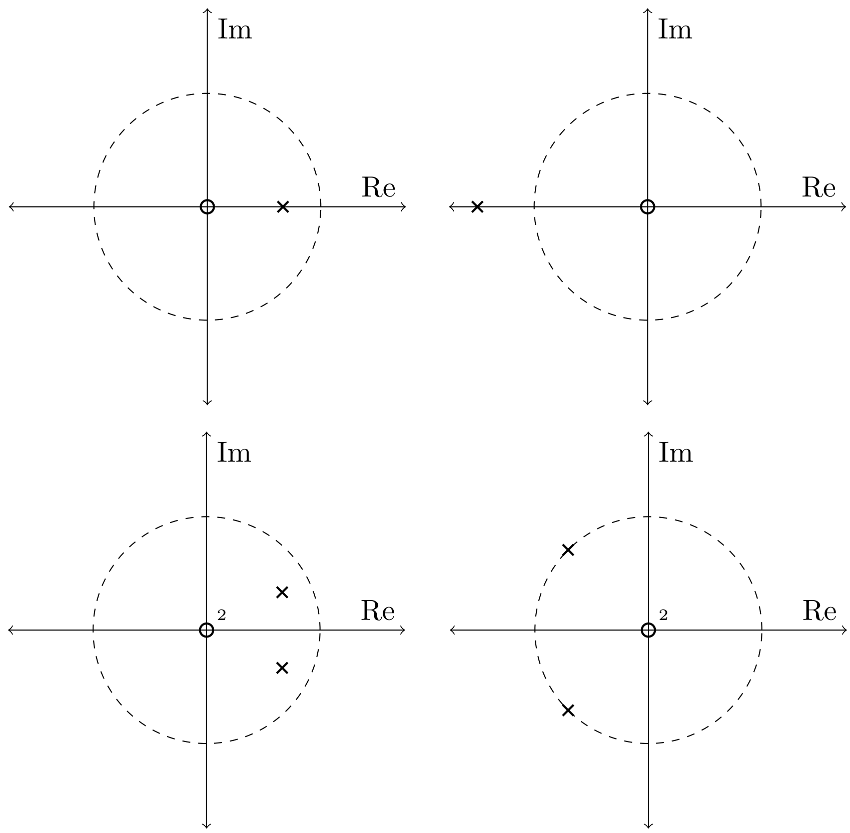

Figure 6.3: Pole zero plots for discrete time systems corresponding with first order difference equations $c_n = d_n - \frac{2}{3}d_{n-1}$ (top left) and $c_n = d_n + \frac{3}{2}d_{n-1}$ (top right) and second order difference equations $z^2 - \frac{4}{3}x - \frac{4}{9}$ (bottom left) and $z^2 + \sqrt{2}z + 1$ (bottom right). The plots of the left correspond with stable systems because all poles are contained inside the complex unit circle (dashed). Plots on the right correspond with unstable systems because there exist poles on or outside the complex unit circle. The small 2's above the zero on the lower plots indicate the existence of two zeros at the origin.

where
$$B_\ell = \frac{a_0 A_\ell}{b_0 \beta_\ell^{r_\ell - 1}(r_\ell - 1)!}.$$
Those terms of the form $A_\ell z^{1-r_\ell}$ correspond with sequences $A_\ell \delta_{n+r_\ell-1}$ where δ is the delta sequence. From (6.2.1) with $k = r_\ell - 1$ those terms of the form
$$\frac{\beta_\ell^{r_\ell-1}(r_\ell - 1)! z}{(z - \beta_\ell)^{r_\ell}}$$
are found to correspond with sequences $B_\ell \beta_\ell^n [n]_{r_\ell - 1} u_n$ where u is the step sequence. Other sequences with the same z-transform are disregarded because they are not right sided and so do not correspond with a causal discrete time system. Combing the above results we find that the discrete impulse response h of the discrete time system H takes the form
$$h_n = \frac{a_0}{b_0} \sum_{\ell \in K_1} A_\ell \delta_{n+r_\ell-1} + \sum_{\ell \in K_2} B_\ell \beta_\ell^n [n]_{r_\ell-1} u_n.$$
The discrete impulse response is absolutely summable only if the poles satisfy $|\beta_\ell| < 1$ for all $\ell = d, \dots, w$ as a result of the terms β_ℓ^n that occur when $\beta_\ell \neq 0$. The system H is stable if and only if h is absolutely summable (Exercise 6.2) and so a discrete time system is stable if and only if no poles lie outside or on the complex unit circle.

We now consider some specific examples of difference equations and their corresponding discrete time systems. Consider the difference equation
$$c_n = d_n + a d_{n-1} \qquad n \in \mathbb{Z} \tag{6.3.4}$$
where $a \in \mathbb{C}$. This is called a **first order difference equation**. Suppose that H is a discrete time system such that the response $y = H(x)$ to input x satisfies
$$x = y - a T_P(y).$$
The transfer function of H is
$$\lambda(H, s) = \frac{1}{1 - a e^{-sP}} = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}$$
where $z = e^{sP}$. The system has a single zero at $z = 0$ and a single pole at $z = a$. The system will be stable if and only if this pole lies strictly inside the complex unit circle, that is, if and only if $|a| < 1$. The discrete impulse response is found to be $h_n = a^n u_n$ by putting $k = 0$ in (6.2.1). Other sequences with this z-transform are discarded because they do not correspond with a causal system. When $|a| < 1$ the region of convergence contains the unit circle and the system has spectrum
$$\Lambda(H, f) = \lambda(H, j2\pi f) = \mathcal{Z}(h, e^{2\pi j P f}) = \frac{e^{2\pi j P f}}{e^{2\pi j P f} - a}.$$

The magnitude and phase spectrum are plotted in Figure 6.4 in the case that $a = \frac{1}{2}$ and $\frac{1}{10}$.

Now consider the difference equation

$$c_n = d_n - a d_{n-1} - b d_{n-2} \qquad n \in \mathbb{Z}.$$

where $a, b \in \mathbb{C}$. This is called a **second order difference equation**. Suppose that H is a discrete time system with reponse $y = H(x)$ satisfying the equation $x = y - aT_P(y) - bT_{2P}(y)$. The transfer function is

$$\lambda(H) = \frac{1}{1 - ae^{-sP} - be^{-2sP}} = \frac{z^2}{z^2 - az - b} = \mathcal{Z}(h)$$

where h is the discrete impulse response of H. The system has two zeros at $z = 0$ and two poles given by the roots of the polynomial $z^2 - az - b$. The z-transform can be inverted to obtain h (Exercise 6.7). The system H is stable if and only if both poles lie strictly inside the complex unit circle (Figure 6.3). In this case, H has spectrum

$$\Lambda(H, f) = \mathcal{Z}(h, e^{2\pi jPf}) = \frac{e^{2\pi jPf}}{e^{4\pi jPf} - ae^{2\pi jPf} - b}.$$

6.4 Exercises

6.1. Show that discrete convolution is associative.

6.2. Show that a discrete time system is stable if and only if its discrete impulse response is absolutely summable.

6.3. Suppose that H is a linear time invariant system such that the response $y = H(x)$ to input x satisfies (6.3.2). Find the transfer function of H.

6.4. Let f and g be absolutely summable sequences. Show that the discrete convolution $f * g$ is also absolutely summable.

6.5. Show that the z-transform of the sequence $a^n u_n$ is $z/(z-a)$ with region of convergence $|z| > |a|$.

6.6. Show that the z-transform of the sequence $[n]_k u_n$ where $[n]_k = n(n-1)\dots(n-k+1)$ is a falling factorial is

$$\mathcal{Z}\big([n]_k u_n\big) = \frac{k!z}{(z-1)^{k+1}} \qquad |z| > 1.$$

6.7. Find the discrete impulse response of the discrete time system corresponding with the second order difference equation $c_n = d_n - ad_{n-1} - bd_{n-2}$.

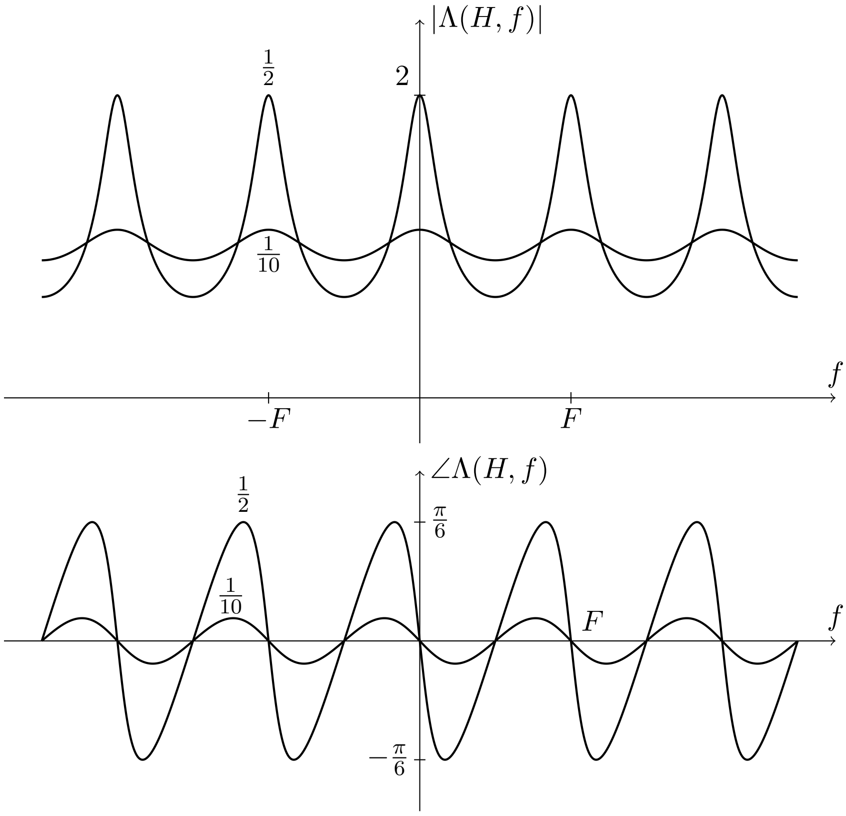

Figure 6.4: Magnitude and phase spectrum of the discrete time H with discrete implulse response $h_n = a^n u_n$ for $a = \frac{1}{2}$ and $\frac{1}{10}$ and period $P = \frac{1}{F}$. The spectrum is periodic with period $F = \frac{1}{P}$. This system corresponds with the first order difference equation $c_n = d_n - 2d_{n-1}$.

6.8. Let d_n be a sequence satisfying $d_n = 2d_{n-1} + 2^{n+1}$ and suppose that $d_0 = 0$. Show that $d_n = 2^{n+1}n$ for $n = 1, 2, \ldots$.

6.9. The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ satisfies the recursive equation $d_0 = 0, d_1 = 1$, and $d_n = d_{n-1} + d_{n-2}$ for $n \geq 2$. Find a closed form expression for the nth Fibonacci number.

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