Signals and Systems

Robby McKilliam

August 27, 2014

Contents

Chapter 1

Signals and systems

A **signal** is a function mapping an input variable to some output variable. For example

$$\sin(\pi t), \qquad \frac{1}{2}t^3, \qquad e^{-t^2}$$

all represent **signals** with real input variable $t \in \mathbb{R}$ and real output variable. These signals are plotted in Figure 1.1. If x is a signal and t an input variable we write $x(t)$ for the output variable corresponding with t. Signals can be multidimensional. This page is an example of a 2-dimensional signal, the independent variables are the horizontal and vertical position on the page, and the signal maps this position to a colour, in this case either black or white. A moving image such as seen on your television or computer monitor is an example of a 3-dimensional signal, the three independent variables being vertical and horizontal screen position and time. The signal maps each position and time to a colour on the screen. In these notes we focus exclusively on 1-dimensional signals such as those in Figure 1.1 and we will only consider signals where the output variable is real or complex valued. Many of the results presented here can be extended to deal with multidimensional signals.

1.1 Properties of signals

A signal x is **bounded** if there exists a real number M such that

$$|x(t)| \leq M \qquad \text{for all } t \in \mathbb{R}$$

where $|\cdot|$ denotes the (complex) magnitude. Both $\sin(\pi t)$ and e^{-t^2} are examples of bounded signals because $|\sin(\pi t)| \leq 1$ and $|e^{-t^2}| \leq 1$ for all $t \in \mathbb{R}$. However, $\frac{1}{2}t^3$ is not bounded because its magnitude grows indefinitely as t moves away from the origin.

A signal x is **periodic** if there exists a positive real number T such that

$$x(t) = x(t + kT) \qquad \text{for all } k \in \mathbb{Z} \text{ and } t \in \mathbb{R}.$$

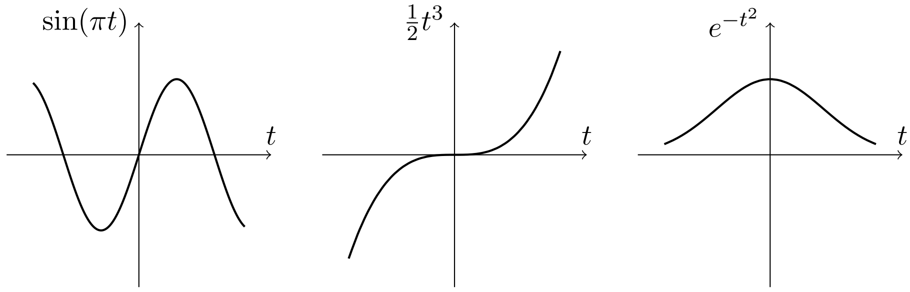

Figure 1.1: 1-dimensional signals

If there exists a smallest such positive T it is called the **fundamental period** or simply the **period**. For example, the signal $\sin(\pi t)$ is periodic with period $T = 2$. Neither $\frac{1}{2}t^3$ or e^{-t^2} are periodic.

A signal x is **right sided** if there exists a $T \in \mathbb{R}$ such that $x(t) = 0$ for all $t < T$. Correspondingly x is **left sided** if $x(t) = 0$ for all $T > t$. For example, the **step function**

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{1.1.1}$$

is right-sided. Its reflection in time $u(-t)$ is left sided (Figure 1.2). A signal x is called **finite in time** if it is both left and right sided, that is, if there exits a $T \in \mathbb{R}$ such that $x(t) = x(-t) = 0$ for all $t > T$. A signal is called **unbounded in time** if it is neither left nor right sided. For example, the signals $\sin(\pi t)$ and e^{-t^2} are unbounded in time, but the **rectangular pulse**

$$\Pi(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{1.1.2}$$

is finite in time.

A signal x is **even** (or **symmetric**) if

$$x(t) = x(-t) \qquad \text{for all } t \in \mathbb{R}$$

and **odd** (or **antisymmetric**) if

$$x(t) = -x(-t) \qquad \text{for all } t \in \mathbb{R}.$$

For example, $\sin(\pi t)$ and $\frac{1}{2}t^3$ are odd and e^{-t^2} is even.

A signal x is **locally integrable** if

$$\int_a^b |x(t)|\, dt < \infty$$

for all finite constants a and b, where by $< \infty$ we mean that the integral evaluates to a finite number. An example of a signal that is not locally

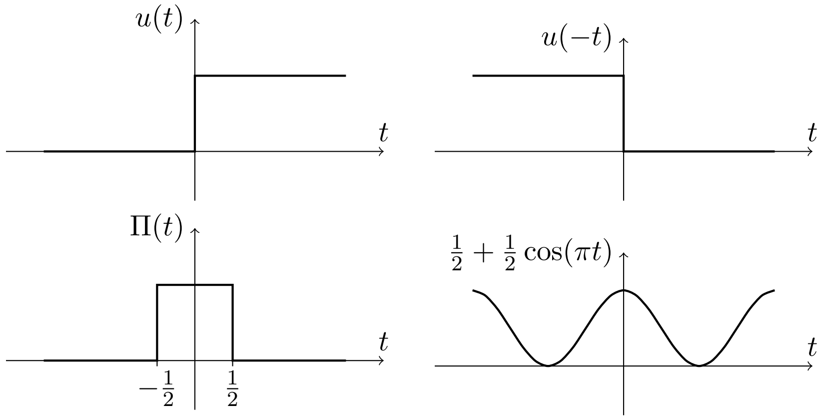

Figure 1.2: The right sided step function $u(t)$, its left sided reflection $u(-t)$, the finite in time rectangular pulse $\Pi(t)$ and the unbounded in time signal $\frac{1}{2}+\frac{1}{2}\cos(x)$.

integrable is $x(t) = \frac{1}{t}$ (Exercise 1.2). A signal x is **absolutely integrable** if

$$\|x\|_1 = \int_{-\infty}^{\infty} |x(t)|\, dt < \infty. \tag{1.1.3}$$

Here we introduce the notation $\|x\|_1$ called the **L^1-norm** of x. For example $\sin(\pi t)$ and $\frac{1}{2}t^3$ are not absolutely integrable, but e^{-t^2} is because [Nicholas and Yates, 1950]

$$\int_{-\infty}^{\infty} |e^{-t^2}| dt = \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}. \tag{1.1.4}$$

It is common to denote the set of absolutely integrable signals by L^1 or $L^1(\mathbb{R})$. So, $e^{-t^2} \in L^1$ and $\frac{1}{2}t^3 \notin L^1$. A signal x is called is **square integrable** if

$$\|x\|_2^2 = \int_{-\infty}^{\infty} |x(t)|^2\, dt < \infty.$$

The real number $\|x\|_2$ is called the **L^2-norm** of x. Square integrable signals are also called **energy signals**, and the squared L^2-norm $\|x\|_2^2$ is called the **energy** of x. For example $\sin(\pi t)$ and $\frac{1}{2}t^3$ are not energy signals, but e^{-t^2} is (Exercise 1.5). The set of square integrable signals is often denoted by L^2 or $L^2(\mathbb{R})$.

We write $x = y$ to indicate that two signals x and y are **equal pointwise**, that is, $x(t) = y(t)$ for all $t \in \mathbb{R}$. This definition of equality is often stronger than we desire. For example, the step function u and the signal

$$z(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}$$

are not equal pointwise because they are not equal at $t = 0$, that is, $u(0) = 1$ and $z(0) = 0$. It is useful to identify signals that differ only at isolated points and for this we use a weaker definition of equality. We say that two signals x and y are equal **almost everywhere** if

$$\int_a^b |x(t) - y(t)| \, dt = 0$$

for all finite constants a and b. So, in the previous example, while $u \neq z$ pointwise we do have $u = z$ almost everywhere. Typically the term almost everywhere is abbreviated to a.e. and one writes

$$x = y \ \text{ a.e.} \qquad \text{or} \qquad x(t) = y(t) \ \text{ a.e.}$$

to indicate that the signals x and y are equal almost everywhere.

1.2 Systems (functions of signals)

A **system** is a function that maps a signal to another signal. For example

$$x(t) + 3x(t-1), \qquad \int_0^1 x(t-\tau)d\tau, \qquad \frac{1}{x(t)}, \qquad \frac{d}{dt}x(t)$$

represent systems, each mapping the signal x to another signal. Consider the electric circuit in Figure 1.3 called a **voltage divider**. If the voltage at time t is $x(t)$ then, by Ohm's law, the current at time t satisfies

$$i(t) = \frac{1}{R_1 + R_2} x(t),$$

and the voltage over the resistor R_2 is

$$y(t) = R_2 i(t) = \frac{R_2}{R_1 + R_2} x(t). \tag{1.2.1}$$

The circuit can be considered as a system mapping the signal x representing the voltage to the signal $i = \frac{1}{R_1+R_2}x$ representing the current, or a system mapping x to the signal $y = \frac{R_2}{R_1+R_2}x$ representing the voltage over resistor R_2.

We denote systems with capital letters such as H and G. A system H is a function that maps a signal x to another signal denoted $H(x)$. We call x the **input signal** and $H(x)$ the **output signal** or the **response** of system H to signal x. The value of the signal $H(x)$ at t is denoted by $H(x, t)$ or $H(x, t)$ and we do not distinguish between these notations. It is sometimes useful to depict systems with a block diagram. Figure 1.4 is a simple block diagram showing the input and output signals of a system H.

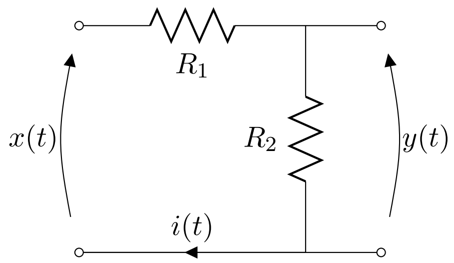

Figure 1.3: A **voltage divider** circuit.

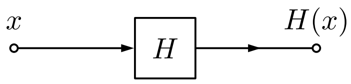

Figure 1.4: System block diagram with input signal x and output signal $H(x)$.

The electric circuit in Figure 1.3 corresponds with the system

$$H(x) = \frac{R_2}{R_1 + R_2} x = y.$$

This system multiplies the input signal x by $\frac{R_2}{R_1+R_2}$. This brings us to our first practical test.

Test 1 (**Voltage divider**) In this test we construct the voltage divider from Figure 1.3 on a breadboard with resistors $R_1 \approx 100\Omega$ and $R_2 \approx 470\Omega$ with values accurate to within 5%. Using a computer soundcard (an approximation of) the voltage signal

$$x(t) = \sin(2\pi f_1 t) \qquad \text{with} \qquad f_1 = 100$$

is passed through the circuit. The approximation is generated by sampling $x(t)$ at rate $F = \frac{1}{P} = 44100\text{Hz}$ to generate samples

$$x(nP) \qquad n = 0, \dots, 2F$$

corresponding to approximately 2 seconds of signal. These samples are passed to the soundcard which starts playback. The voltage over resistor R_2 is recorded (also using the soundcard) that returns a list of samples $y_1, \dots, y_L$ taken at rate F. The voltage over R_2 can be (approximately) reconstructed from these samples as

$$\tilde{y}(t) = \sum_{\ell=1}^{L} y_\ell \operatorname{sinc}(Ft - \ell) \tag{1.2.2}$$

where

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \tag{1.2.3}$$

is the called the **sinc function** and is plotted in Figure **??**. We will justify this reconstruction in Section **??**. Simultaneously the (stereo) soundcard is used to record the input voltage x producing samples $x_1, \dots, x_L$ taken at rate F. An approximation of the input signal is

$$\tilde{x}(t) = \sum_{\ell=1}^{L} x_\ell \operatorname{sinc}(Ft - \ell). \tag{1.2.4}$$

In view of (1.2.1) we would expect the approximate relationship

$$\tilde{y} \approx \frac{R_2}{R_1 + R_2}\tilde{x} = \frac{47}{57}\tilde{x}.$$

A plot of $\tilde{y}$, $\tilde{x}$ and $\frac{47}{57}\tilde{x}$ over a 20ms period from 1s to 1.02s is given in Figure 1.5. The hypothesised output signal $\frac{47}{57}\tilde{x}$ does not match the observed output signal $\tilde{y}$. A primary reason is that the circuitry inside the soundcard itself cannot be ignored. When deriving the equation for the voltage divider we implicitly assumed that current flows through the output of the soundcard without resistance (a short circuit), and that no current flows through the input device of the soundcard (an open circuit). These assumptions are not realistic. Modelling the circuitry in the sound card wont be attempted here. In Section 2.2 we will construct circuits that contain external sources of power (active circuits). These are less sensitive to the circuitry inside the soundcard.

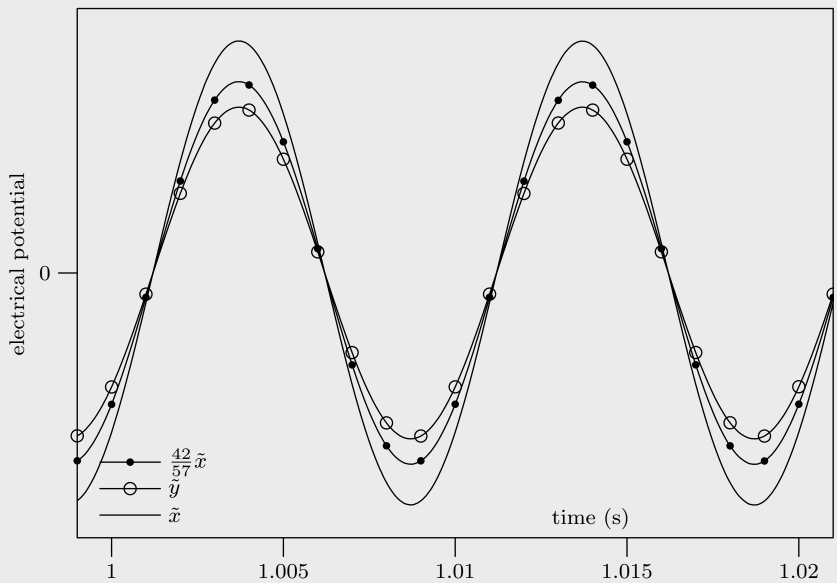

Figure 1.5: Plot of reconstructed input signal $\tilde{x}$ (solid line), output signal $\tilde{y}$ (solid line with circle) and hypothesised output signal $\frac{47}{57}\tilde{x}$ (solid line with dot) for the voltage divider circuit in Figure 1.3. The hypothesised signal does not match $\tilde{y}$. One reason is that the model does not take account of the circuitry inside the soundcard.

Not all signals can be input to all systems. For example, the system

$$H(x,t) = \frac{1}{x(t)}$$

is not defined at those t where $x(t) = 0$ because we cannot divide by zero. Another example is the system

$$I_\infty(x,t) = \int_{-\infty}^{t} x(\tau)d\tau, \tag{1.2.5}$$

called an **integrator**. The signal $x(t) = 1$ cannot be input to the integrator because the integral $\int_{-\infty}^{t} dt$ is not finite for any t.

When specifying a system it is necessary to also specify a set of signals that can be input. This is called a **domain** for the system. We are free to choose the domain at our convenience. For example, a domain for the system $H(x,t) = \frac{1}{x(t)}$ is a the set of signals $x(t)$ which are not zero for any t. An example of a domain for the integrator I_∞ is the set L^1 of absolutely integrable signals because, if x is absolutely integrable, then

$$|I_\infty(x,t)| \leq \left| \int_{-\infty}^{t} x(\tau)d\tau \right| \leq \int_{-\infty}^{t} |x(\tau)|\, d\tau < \int_{-\infty}^{\infty} |x(\tau)|\, d\tau = \|x\|_1 < \infty$$

and so, $I_\infty(x,t)$ is finite for all t. In this text, the domain used for a given system will usually be obvious from the context in which the system is defined. For this reason we will not usually state the domain explicitly. We will only do so if there is chance for confusion.

1.3 Some important systems

The system

$$T_\tau(x,t) = x(t - \tau)$$

is called a **time-shifter**. This system shifts the input signal along the t axis ('time' axis) by τ. When τ is positive T_τ delays the input signal by τ. The time-shifter will appear so regularly in this course that we use the special notation T_τ to represent it. Figure 1.6 depicts the action of time-shifters

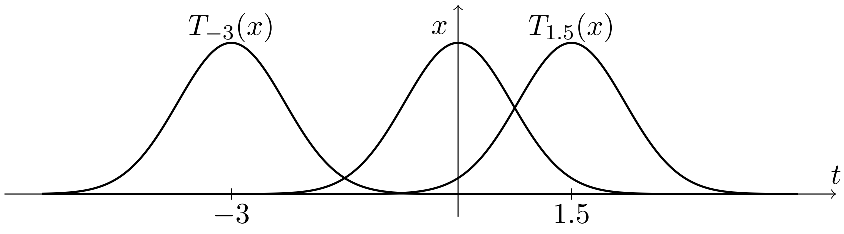

Figure 1.6: Time-shifter system $T_{1.5}(x,t) = x(t-1.5)$ and $T_{-3}(x,t) = x(t+3)$ acting on the signal $x(t) = e^{-t^2}$.

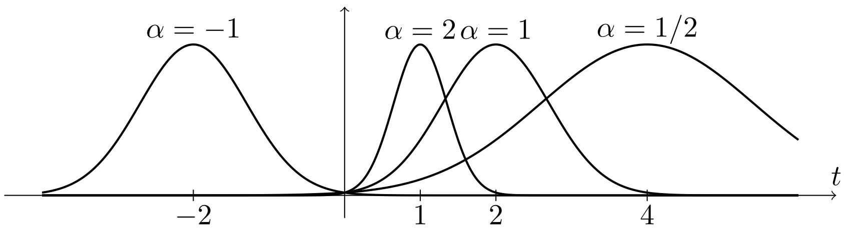

Figure 1.7: Time-scaler system $H(x,t) = x(\alpha t)$ for $\alpha = -1, \frac{1}{2}, 1$ and 2 acting on the signal $x(t) = e^{-(t-2)^2}$.

$T_{1.5}$ and T_{-3} on the signal $x(t) = e^{-t^2}$. When $\tau = 0$ the time-shifter is the **identity system**

$$T_0(x) = x$$

that maps the signal x to itself.

Another important system is the **time-scaler** that has the form

$$H(x,t) = x(\alpha t), \qquad \alpha \in \mathbb{R}.$$

Figure 1.7 depicts the action of a time-scaler with a number of values for α. When $\alpha = -1$ the time-scaler reflects the input signal in the time axis. When $\alpha = 1$ the time-scaler is the identity system T_0.

Another system we regularly encounter is the **differentiator**

$$D(x,t) = \frac{d}{dt}x(t),$$

that returns the derivative of the input signal. We also define a kth differentiator

$$D^k(x,t) = \frac{d^k}{dt^k}x(t)$$

that returns the kth derivative of the input signal.

A related system is the **integrator**

$$I_a(x,t) = \int_{-a}^{t} x(\tau)d\tau.$$

The parameter a describes the lower bound of the integral. In this course it will often be that $a = \infty$. For example, the response of the integrator I_∞ to the signal $tu(t)$ where u is the step function (1.1.1) is

$$\int_{-\infty}^{t} \tau u(\tau) d\tau = \begin{cases} \int_0^t \tau d\tau = \frac{t^2}{2} & t > 0 \\ 0 & t \leq 0. \end{cases}$$

Observe that the integrator I_∞ cannot be applied to the signal $x(t) = t$ because $\int_{-\infty}^{t} \tau d\tau$ is not finite for any t. A domain for I_∞ would not contain the signal $x(t) = t$.

1.4 Properties of systems

In this section we define a number of important properties that systems can possess. In what follows H will be a system and the phrase "for all signals" will mean for all signals inside some domain for H.

A system H is called **memoryless** if the output signal $H(x)$ at time t depends only on the input signal x at time t. For example $\frac{1}{x(t)}$ and the identity system T_0 are memoryless, but

$$x(t) + 3x(t-1) \qquad \text{and} \qquad \int_0^1 x(t-\tau) d\tau$$

are not. A time-shifter T_τ with $\tau \neq 0$ is not memoryless.

A system H is **causal** if the output signal $H(x)$ at time t depends on the input signal only at times less than or equal to t. Memoryless systems such as $\frac{1}{x(t)}$ and T_0 are also causal. Time-shifters T_τ are causal when $\tau \geq 0$, but are not causal when $\tau < 0$. The systems

$$x(t) + 3x(t-1) \qquad \text{and} \qquad \int_0^1 x(t-\tau) d\tau$$

are causal, but the systems

$$x(t) + 3x(t+1) \qquad \text{and} \qquad \int_0^1 x(t+\tau) d\tau$$

are not causal.

A system H is called **bounded-input-bounded-output (BIBO) stable** or just **stable** if the output signal $H(x)$ is bounded whenever the input signal x is bounded. That is, H is stable if for every positive real number M there exists a positive real number K such that for all signals x satisfying

$$|x(t)| < M \qquad \text{for all } t \in \mathbb{R},$$

it also holds that

$$|H(x,t)| < K \qquad \text{for all } t \in \mathbb{R}.$$

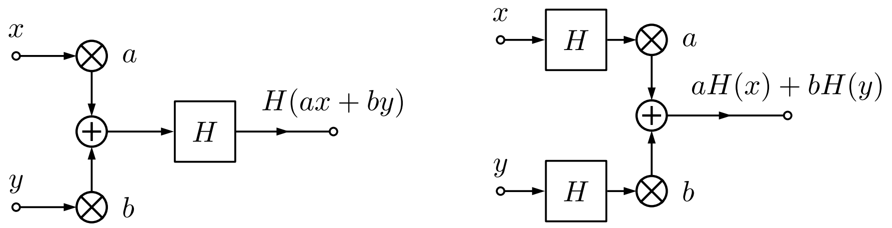

Figure 1.8: If H is a linear system the outputs of these two diagrams are the same signal, i.e. $H(ax+by) = aH(x) + bH(y)$.

For example, the system $x(t) + 3x(t-1)$ is stable with $K = 4M$ since if $|x(t)| < M$ then

$$|x(t) + 3x(t-1)| \leq |x(t)| + 3\,|x(t-1)| < 4M = K.$$

The integrator I_a for any $a \in \mathbb{R}$ and differentiator D are not stable (Exercises 1.6 and 1.7).

A system H is **linear** if

$$H(ax+by) = aH(x) + bH(y)$$

for all signals x and y and all complex numbers a and b. That is, a linear system has the property: If the input consists of a weighted sum of signals, then the output consists of the same weighted sum of the responses of the system to those signals. Figure 1.8 indicates the linearity property using a block diagram. For example, the differentiator is linear because

$$\begin{aligned} D(ax+by,t) &= \frac{d}{dt}\big(ax(t) + by(t)\big) \\ &= a\frac{d}{dt}x(t) + b\frac{d}{dt}y(t) \\ &= aD(x,t) + bD(y,t) \end{aligned}$$

whenever both x and y are differentiable. However, the system $H(x,t) = \frac{1}{x(t)}$ is not linear because

$$H(ax+by,t) = \frac{1}{ax(t)+by(t)} \neq \frac{a}{x(t)} + \frac{b}{y(t)} = aH(x,t) + bH(y,t)$$

in general.

The property of linearity trivially generalises to more than two signals. For example, if $x_1, \dots, x_k$ are signals and $a_1, \dots, a_k$ are complex numbers for some finite k, then

$$H(a_1x_1 + \cdots + a_kx_k) = a_1H(x_1) + \cdots + a_kH(x_k).$$

A system H is **time invariant** if

$$H\big(T_\tau(x), t\big) = H(x, t-\tau)$$

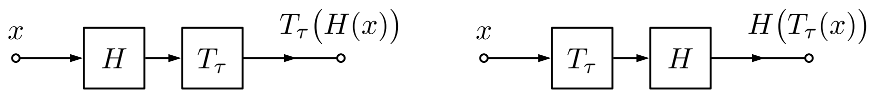

Figure 1.9: If H is a time-invariant system the outputs of these two diagrams are the same signal, i.e. $H\big(T_\tau(x)\big) = T_\tau\big(H(x)\big)$.

for all signals x and all time-shifts $\tau \in \mathbb{R}$. That is, a system is time-invariant if time shifting the input signal results in the same time-shift of the output signal. Equivalently, H is time-invariant if it commutes with the time-shifter T_τ, that is, if

$$H\big(T_\tau(x)\big) = T_\tau\big(H(x)\big)$$

for all $\tau \in \mathbb{R}$ and all signals x. Figure 1.9 represents the property of time-invariance with a block diagram.

1.5 Exercises

1.1. State whether the step function $u(t)$ is bounded, periodic, absolutely integrable, an energy signal.

1.2. Show that the signal t^2 is locally integrable, but that the signal $\frac{1}{t^2}$ is not.

1.3. Plot the signal

$$x(t) = \begin{cases} \frac{1}{t+1} & t > 0 \\ \frac{1}{t-1} & t \leq 0. \end{cases}$$

State whether it is: bounded, locally integrable, absolutely integrable, square integrable.

1.4. Plot the signal

$$x(t) = \begin{cases} \frac{1}{\sqrt{t}} & 0 < t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that x is absolutely integrable, but not square integrable.

1.5. Compute the energy of the signal $e^{-\alpha^2 t^2}$ (Hint: use equation (1.1.4) on page 3 and a change of variables).

1.6. Show that the integrator I_a for any $a \in \mathbb{R}$ is not stable.

1.7. Show that the differentiator system D is not stable.

1.8. Show that the time-shifter is linear and time-invariant and that the time-scaler is linear, but not time invariant

1.9. Show that the integrator I_c with c finite is linear, but not time-invariant.

1.10. Show that the integrator I_∞ is linear and time invariant.

1.11. State whether the system $H(x) = x+1$ is linear, time-invariant, stable.

1.12. State whether the system $H(x) = 0$ is linear, time-invariant, stable.

1.13. State whether the system $H(x) = 1$ is linear, time-invariant, stable.

1.14. Let x be a signal with period T that is not equal to zero almost everywhere. Show that x is not absolutely integrable.

Chapter 2

Systems modelled by differential equations

Systems of particular interest in this text are those where the input signal x and output signal y are related by a linear differential equation with constant coefficients, that is, an equation of the form

$$\sum_{\ell=0}^{m} a_\ell \frac{d^\ell}{dt^\ell} x(t) = \sum_{\ell=0}^{k} b_\ell \frac{d^\ell}{dt^\ell} y(t),$$

where $a_0, \dots, a_m$ and $b_0, \dots, b_k$ are real or complex numbers. In what follows we use the differentiator system D rather than the notation $\frac{d}{dt}$ to represent differentiation. To represent the ℓth derivative we write D^ℓ instead of $\frac{d^\ell}{dt^\ell}$. Using this notation the differential equation above is

$$\sum_{\ell=0}^{m} a_\ell D^\ell(x) = \sum_{\ell=0}^{k} b_\ell D^\ell(y). \tag{2.0.1}$$

Equations of this form can be used to model a large number of electrical, mechanical and other real world devices. For example, consider the resistor and capacitor (RC) circuit in Figure 2.1. Let the signal v_R represent the voltage over the resistor and i the current through both resistor and capacitor. The voltage signals satisfy

$$x = y + v_R,$$

and the current satisfies both

$$v_R = Ri \qquad \text{and} \qquad i = CD(y).$$

Combining these equations,

$$x = y + RCD(y) \tag{2.0.2}$$

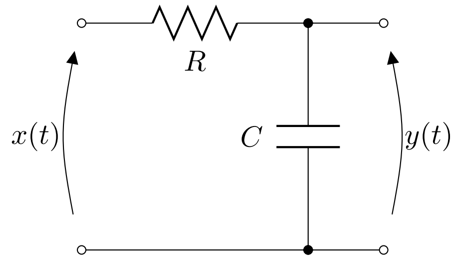

Figure 2.1: An electrical circuit with resistor and capacitor in series, otherwise known as an **RC circuit**.

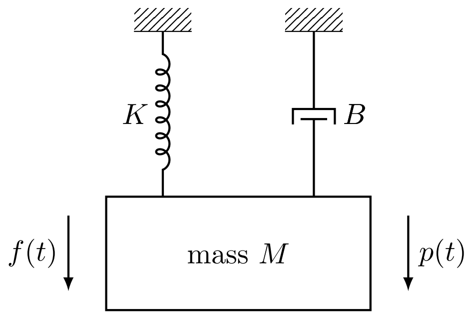

Figure 2.2: A mechanical mass-spring-damper system

that is in the form of (2.0.1).

As another example, consider the mass-spring-damper in Figure 2.2. A force represented by the signal f is externally applied to the mass, and the position of the mass is represented by the signal p. The spring exerts force $-Kp$ that is proportional to the position of the mass, and the damper exerts force $-BD(p)$ that is proportional to the velocity of the mass. The cumulative force exerted on the mass is

$$f_m = f - Kp - BD(p)$$

and by Newton's law the acceleration of the mass $D^2(p)$ satisfies

$$MD^2(p) = f_m = f - Kp - BD(p).$$

We obtain the differential equation

$$f = Kp + BD(p) + MD^2(p) \tag{2.0.3}$$

that is in the form of (2.0.1) if we put $x = f$ and $y = p$. Given p we can readily solve for the corresponding force f. As a concrete example, let the spring constant, damping constant and mass be $K = B = M = 1$. If the position satisfies $p(t) = e^{-t^2}$, then the corresponding force satisfies

$$f(t) = e^{-t^2}(4t^2 - 2t - 1).$$

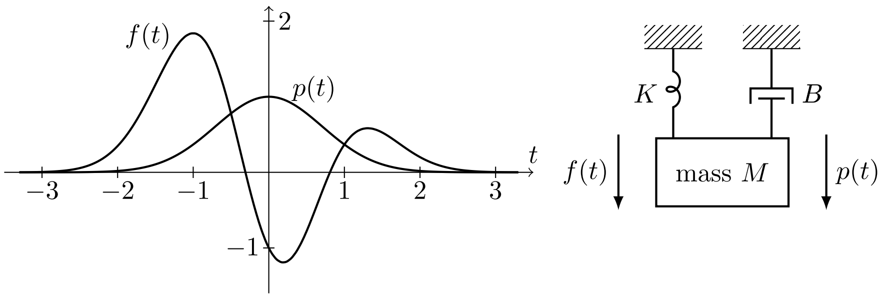

Figure 2.3: A solution to the mass-spring-damper system with $K = B = M = 1$. The position is $p(t) = e^{-t^2}$ with corresponding force $f(t) = e^{-t^2}(4t^2 - 2t - 1)$.

Figure 2.3 depicts these signals.

What happens if a particular force signal f is applied to the mass? For example, say we apply the force

$$f(t) = \Pi(t - \tfrac{1}{2}) = \begin{cases} 1 & 0 < t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the corresponding position signal p? We are not yet ready to answer this question, but will be later (Exercise 4.12).

In both the mechanical mass-spring-damper system in Figure 2.2 and the electrical RC circuit in Figure 2.1 we obtain a differential equation relating the input signal x with the output signal y. The equations do not specify the output signal y explicitly in terms of the input signal x, that is, they do not explicitly define a system H such $y = H(x)$. As they are, the differential equations do not provide as much information about the behaviour of the system as we would like. For example, is the system stable? We will be able to obtain much more information about these systems when the **Laplace transform** is introduced in Chapter 4. The remainder of this chapter details the construction of differential equations that model various mechanical, electrical, and electro-mechanical systems. We will use the systems constructed here as examples throughout the course.

2.1 Passive electrical circuits

Passive electrical circuits require no sources of power other than the input signal itself. For example, the voltage divider in Figure 1.3 and the RC circuit in Figure 2.1 are passive circuits. Another common passive electrical circuit is the resistor, capacitor and inductor (RLC) circuit depicted in Figure 2.4. In this circuit we let the output signal y be the voltage over the resistor. Let v_C represent the voltage over the capacitor and v_L the voltage over the inductor and let i be the current. We have

$$y = Ri, \qquad i = CD(v_C), \qquad v_L = LD(i),$$

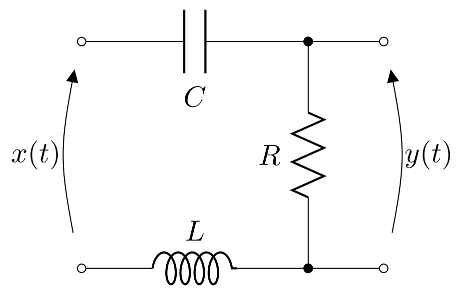

Figure 2.4: An electrical circuit with resistor, capacitor and inductor in series, otherwise known as an **RLC circuit**.

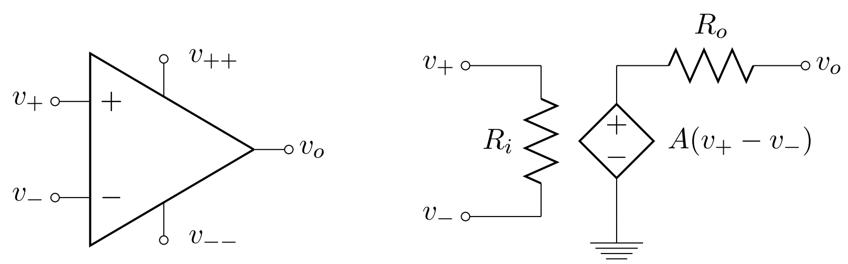

Figure 2.5: Left: triangular component diagram of an **operational amplifier**. The v_{++} and v_{--} connectors indicate where an external voltage source can be connected to the amplifier. These connectors will usually be omitted. Right: model for an operational amplifier including input resistance R_i, output resistance R_o, and open loop gain A. The diamond shaped component is a dependent voltage source. This model is usually only useful when the operational amplifier is in a negative feedback circuit.

leading to the following relationships between y, v_C and v_L,

$$y = RCD(v_C), \qquad Rv_L = LD(y).$$

Kirchhoff's voltage law gives $x = y + v_C + v_L$ and by differentiating both sides

$$D(x) = D(y) + D(v_C) + D(v_L).$$

Substituting the equations relating y, v_C and v_L leads to

$$RCD(x) = y + RCD(y) + LCD^2(y). \tag{2.1.1}$$

We can similarly find equations relating the input voltage with v_C and v_L.

2.2 Active electrical circuits

Unlike passive electrical circuits, an **active electrical circuit** requires a source of power external to the input signal. Active circuits can be modelled and constructed using **operational amplifiers** as depicted in Figure 2.5. The left hand side of Figure 2.5 shows a triangular circuit diagram for an operational amplifier, and the right hand side of Figure 2.5 shows a circuit that can be used to model the behaviour of the amplifier. The v_{++} and v_{--} connectors indicate where an external voltage source can be connected to the amplifier. These connectors will usually be omitted. The diamond shaped component is a dependent voltage source with voltage $A(v_+ - v_-)$ that depends on the difference between the voltage at the **non-inverting input** v_+ and the voltage at the **inverting input** v_-. The dimensionless constant A is called the **open loop gain**. Most operational amplifiers have large open loop gain A, large **input resistance** R_i and small **output resistance** R_o. As we will see, it can be convenient to consider the behaviour as $A \to \infty$, $R_i \to \infty$ and $R_0 \to 0$, resulting in an **ideal operational amplifier**.

As an example, an operational amplifier configured as a **multiplier** is depicted in Figure 2.6. This circuit is an example of an operational amplifier configured with **negative feedback**, meaning that the output of the amplifier is connected (in this case by a resistor) to the inverting input v_-. The horizontal wire at the bottom of the plot is considered to be ground (zero volts) and is connected to the negative terminal of the dependent voltage source of the operational amplifier depicted in Figure 2.5. An equivalent circuit for the multiplier using the model in Figure 2.5 is shown in Figure 2.7. Solving this circuit (Exercise 2.1) yields the following relationship between the input voltage signal x and the output voltage signal y,

$$y = \frac{R_i(R_o - AR_2)}{R_i(R_2 + R_o) + R_1(R_2 + R_i + AR_i + R_o)} x. \tag{2.2.1}$$

For an ideal operational amplifier we let $A \to \infty$, $R_i \to \infty$ and $R_o \to 0$. In this case terms involving the product AR_i dominate and we are left with the simpler equation

$$y = -\frac{R_2}{R_1} x. \tag{2.2.2}$$

Thus, assuming an ideal operational amplifier, the circuit acts as a multiplier with constant $-\frac{R_2}{R_1}$.

The equation relating x and y is much simpler for the ideal operational amplifier. Fortunately this equation can be obtained directly using the following two rules:

1. the voltage at the inverting and non-inverting inputs are equal,
2. no current flows through the inverting and non-inverting inputs.

These rules are only useful for analysing circuits with negative feedback. Let us now rederive (2.2.2) using these rules. Because the non-inverting input is connected to ground, the voltage at the inverting input is zero. So, the voltage over resistor R_2 is $y = R_2 i$. Because no current flows through the inverting input the current through R_1 is also i and $x = -R_1 i$. Combining these results, the input voltage x and the output voltage y are related by

$$y = -\frac{R_2}{R_1}x.$$

In Test 2 the inverting amplifier circuit is constructed and the relationship above is tested using a computer soundcard.

We now consider another circuit consisting of an operational amplifier, two resistors and two capacitors depicted in Figure 2.8. Assuming an ideal operational amplifier, the voltage at the inverting terminal is zero because the non-inverting terminal is connected to ground. Thus, the voltage over capacitor C_2 and resistor R_2 is equal to y and, by Kirchoff's current law,

$$i = \frac{y}{R_2} + C_2 D(y).$$

Similarly, since no current flows through the inverting terminal,

$$i = -\frac{x}{R_1} - C_1 D(x).$$

Combining these equations yields

$$-\frac{x}{R_1} - C_1 D(x) = \frac{y}{R_2} + C_2 D(y). \tag{2.2.3}$$

Observe the similarity between this equation and that for the passive RC circuit (2.0.2) when $R_1 = R_2$ and $C_1 = 0$ (an open circuit). In this case

$$x = -y - R_1 C_2 D(y). \tag{2.2.4}$$

We call this the **active RC circuit**. This circuit is tested in Test 3.

Consider the circuit in Figure 2.9. Assuming an ideal operational amplifier, the input voltage x satisfies

$$-i = \frac{x}{R_1} + C_1 D(x).$$

The voltage over the capacitor C_2 is $y - R_2 i$ and so the current satisfies

$$i = C_2 D(y - R_2 i).$$

Combining these equations gives

$$-\frac{x}{R_1} - C_1 D(x) = C_2 D(y) + \frac{R_2 C_2}{R_1} D(x) + R_2 C_2 C_1 D^2(x),$$

and after rearranging,

$$D(y) = -\frac{1}{R_1 C_2}x - \left(\frac{R_2}{R_1} + \frac{C_1}{C_2}\right) D(x) - R_2 C_1 D^2(x).$$

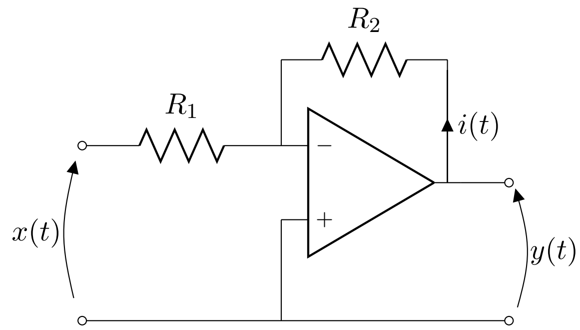

Figure 2.6: Inverting amplifier

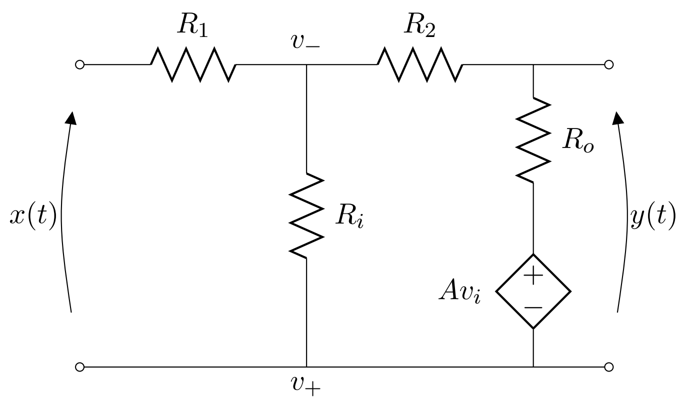

Figure 2.7: An equivalent circuit for the inverting amplifier from Figure 2.6 using the model for an operational amplifier in Figure 2.5. The symbol $v_i = v_+ - v_-$ is the voltage over resistor R_i.

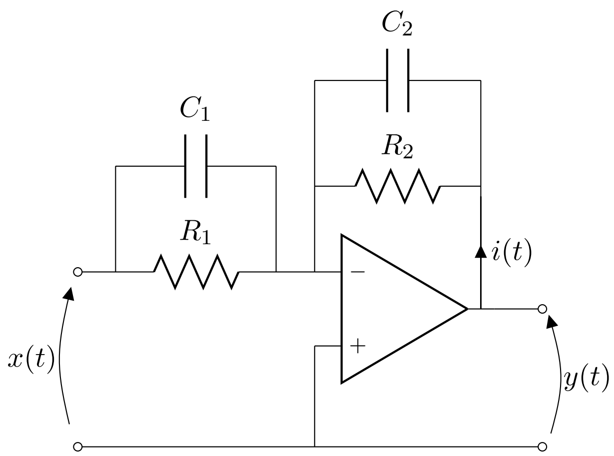

Figure 2.8: Operational amplifier configured with two capacitors and two resistors.

Test 2 (**Inverting amplifier**) In this test we construct the inverting amplifier circuit from Figure 2.6 with $R_2 \approx 22\text{k}\Omega$ and $R_1 \approx 12\text{k}\Omega$ that are accurate to within 5% of these values. The operational amplifier used is the Texas Instruments LM358P. Using a computer soundcard (an approximation of) the voltage signal

$$x(t) = \tfrac{1}{3}\sin(2\pi f_1 t) + \tfrac{1}{3}\sin(2\pi f_2 t)$$

with $f_1 = 100$ and $f_2 = 233$ is passed through the circuit. As in previous tests, the soundcard is used to sample the input signal x and the output signal y. Approximate reconstructions of the input signal $\tilde{x}$ and output signal $\tilde{y}$ are given according to (1.2.4) and (1.2.2). According to (2.1.1) we expect the approximate relationship

$$\tilde{y} \approx -\frac{R_2}{R_1}\tilde{x} = -\frac{11}{6}\tilde{x}.$$

Each of $\tilde{y}$, $\tilde{x}$ and $-\frac{11}{6}\tilde{x}$ are plotted in Figure 2.9.

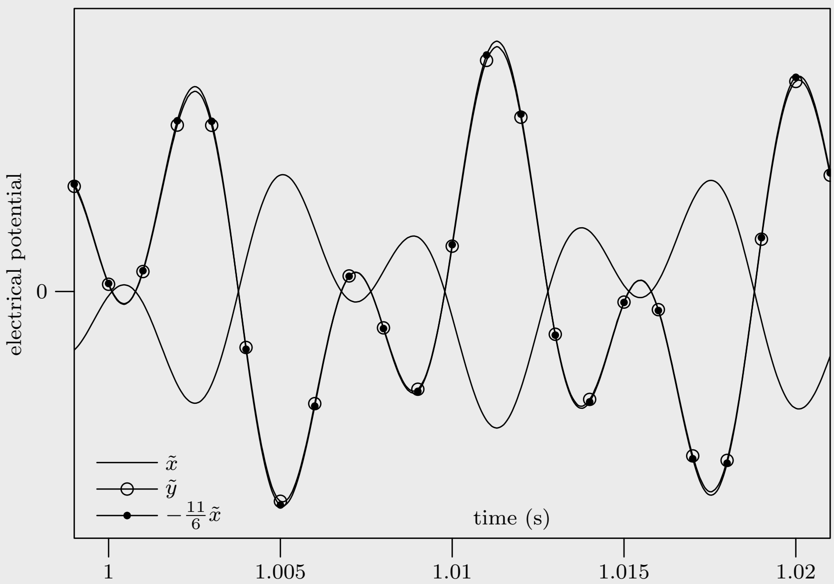

Figure 2.9: Plot of reconstructed input signal $\tilde{x}$ (solid line), output signal $\tilde{y}$ (solid line with circle) and hypothesised output signal $-\frac{11}{6}\tilde{x}$ (solid line with dot).

Test 3 (**Active RC circuit**) In this test we construct the circuit from Figure 2.8 with $R_1 \approx R_2 \approx 27\mathrm{k}\Omega$ and $C_2 \approx 10\mathrm{nF}$ accurate to within 5% of these values and $C_1 = 0$ (an open circuit). The operational amplifier used is a Texas Instruments LM358P. Using a computer soundcard (an approximation of) the voltage signal

$$x(t) = \tfrac{1}{3}\sin(2\pi f_1 t) + \tfrac{1}{3}\sin(2\pi f_2 t)$$

with $f_1 = 500$ and $f_2 = 1333$ is passed through the circuit. As in previous tests, the soundcard is used to sample the input signal x and the output signal y and approximate reconstructions $\tilde{x}$ and $\tilde{y}$ are given according to (1.2.4) and (1.2.2). According to (2.2.4) we expect the approximate relationship

$$\tilde{x} \approx -\frac{R_1}{R_2}\tilde{y} - R_1 C D(\tilde{y}) = -\tilde{y} - \frac{27}{10^5}D(\tilde{y}).$$

The derivative of the sinc function is

$$D(\mathrm{sinc}, t) = \frac{1}{\pi t^2}\big(\pi t\cos(\pi t) - \sin(\pi t)\big), \tag{2.2.5}$$

and so,

$$D(\tilde{y}, t) = \frac{d}{dt}\left(\sum_{\ell=1}^{L} y_\ell \,\mathrm{sinc}(Ft - \ell)\right) = F\sum_{\ell=1}^{L} y_\ell D(\mathrm{sinc}, Ft - \ell). \tag{2.2.6}$$

Each of $\tilde{y}$, $\tilde{x}$ and $-\tilde{y} - \frac{27}{10^5}D(\tilde{y})$ are plotted in Figure 2.9.

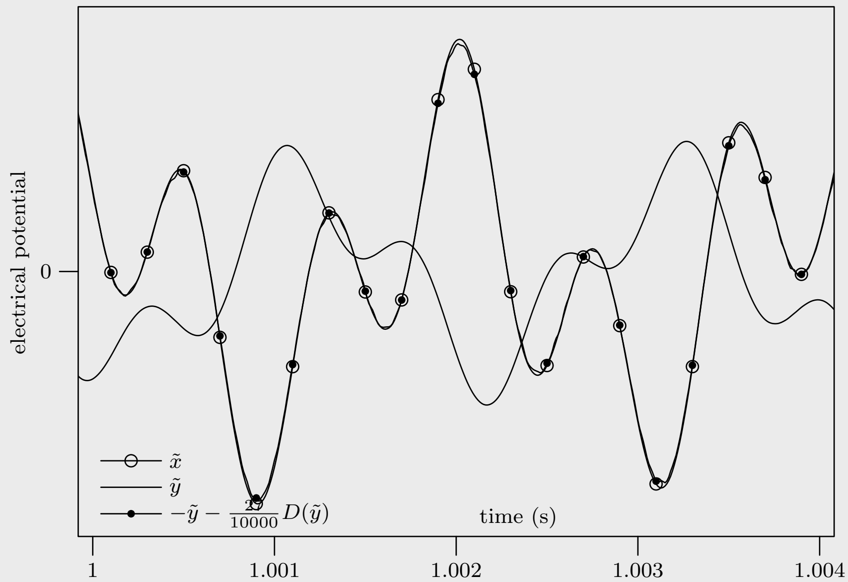

Figure 2.9: Plot of reconstructed input signal $\tilde{x}$ (solid line with circle), output signal $\tilde{y}$ (solid line), and hypothesised input signal $-\tilde{y} - \frac{27}{10^5}D(\tilde{y})$ (solid line with dot).

Put

$$K_i = \frac{1}{R_1C_2}, \qquad K_p = \frac{R_2}{R_1} + \frac{C_1}{C_2}, \qquad K_d = R_2C_1$$

and now

$$D(y) = -K_i x - K_p D(x) - K_d D^2(x). \tag{2.2.7}$$

This equation models what is called a **proportional-integral-derivative controller** or **PID controller**. The coefficients K_i, K_p and K_d are called the **integral gain**, **proportional gain**, and **derivative gain**.

The final active circuit we consider is called a **Sallen-Key** [Sallen and Key, 1955] and is depicted in Figure 2.10. Observe that the output of the amplifier is connected directly to the inverting input and is also connected to the noninverting input by a capacitor and resistor. This circuit has both negative *and* positive feedback. It is not immediately apparent that we can use the simplifying assumptions for an ideal operational amplifier with negative feedback. However, we will do so, and will find that it works in this case.

Let v_{R1}, v_{R2}, v_{C1}, and v_{C2} be the voltages over the components R_1, R_2, C_1, and C_2. Kirchoff's voltage law leads to the equations

$$x = v_{R1} + v_{R2} + v_{C2}, \qquad y = v_{C1} + v_{R2} + v_{C2}.$$

The voltage at the inverting and noninverting terminals is y and so the voltage over the capacitor C_2 is y, that is, $y = v_{C2}$. Using this, the equations above simplify to

$$x = v_{R1} + v_{R2} + y, \qquad v_{C1} = -v_{R2}.$$

The current i_2 through capacitor C_2 satisfies $i_2 = C_2D(v_{C2}) = C_2D(y)$. Because no current flows into the inverting terminal of the amplifier the current through R_2 is also i_2 and so $v_{R2} = R_2i_2 = R_2C_2D(y)$. Substituting this into the equations above gives

$$x = v_{R1} + R_2C_2D(y) + y, \qquad v_{C1} = -R_2C_2D(y). \tag{2.2.8}$$

Kirchoff's current law asserts that $i + i_1 = i_2$. The current i through capacitor C_1 satisfies $i = C_1D(v_{C1}) = -R_2C_1C_2D^2(y)$ and the current through resistor R_1 satisfies

$$v_{R1} = R_1i_1 = R_1(i_2 - i) = R_1C_2D(y) + R_1R_2C_1C_2D^2(y).$$

Substituting this into the equation on the left of (2.2.8) gives

$$x = y + C_2(R_1 + R_2)D(y) + R_1R_2C_1C_2D^2(y). \tag{2.2.9}$$

The Sallen-Key will be useful when we consider the design of analogue electrical filters in Section **??**.

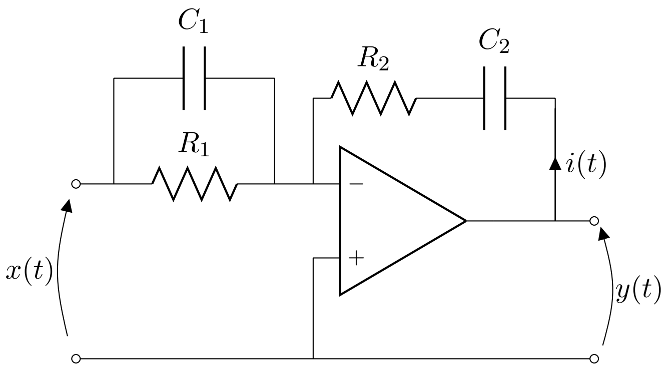

Figure 2.9: Operational amplifier implementing a **proportional-integral-derivative controller**.

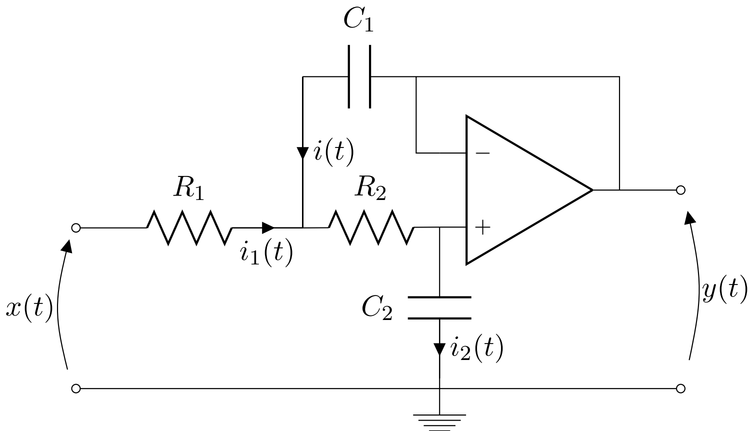

Figure 2.10: Operational amplifier implementing a **Sallen-Key**.

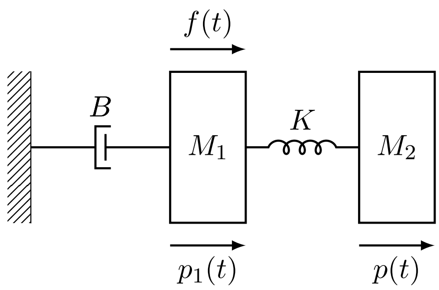

Figure 2.11: Two masses, a spring and a damper

2.3 Masses, springs, and dampers

A mechanical mass-spring-damper system was described in Section 2 and Figure 2.2. We now consider another mechanical system involving a different configuration of masses, a spring and a damper depicted in Figure 2.11. A mass M_1 is connected to a wall by a damper with constant B, and to another mass M_2 by a spring with constant K. A force represented by the signal f is applied to the first mass. We will derive a differential equation relating f with the position p of the second mass. Assume that the spring applies no force (is in equilibrium) when the masses are distance d apart. The forces due to the spring satisfy

$$f_{s1} = -f_{s2} = K(p - p_1 - d)$$

where f_{s1} and f_{s2} are signals representing the force due to the spring on mass M_1 and M_2 respectively. It is convenient to define the signal $g(t) = p_1(t) + d$ so that forces due to spring satisfy the simpler equation

$$f_{s1} = -f_{s2} = K(p - g).$$

The only force applied to M_2 is by the spring and so, by Newton's law, the acceleration of M_2 satisfies

$$M_2 D^2(p) = f_{s2}.$$

Substituting this into the previous equation gives a differential equation relating g and p,

$$Kg = Kp + M_2 D^2(p). \tag{2.3.1}$$

The force applied by the damper on mass M_1 is given by the signal

$$f_d = -BD(p_1) = -BD(g)$$

where the replacement of p_1 by g is justified because differentiation will remove the constant d. The cumulative force on M_1 is given by the signal

$$\begin{aligned} f_1 &= f + f_d + f_{s1} \\ &= f - Kg + Kp - BD(g), \end{aligned} \tag{2.3.2}$$

and by Newton's law the acceleration of M_1 satisfies

$$M_1 D^2(p_1) = M_1 D^2(g) = f_1.$$

Substituting this into (2.3.2) and using (2.3.1) we obtain a fourth order differential equation relating p and f,

$$f = BD(p) + (M_1 + M_2)D^2(p) + \frac{BM_2}{K} D^3(p) + \frac{M_1 M_2}{K} D^4(p). \tag{2.3.3}$$

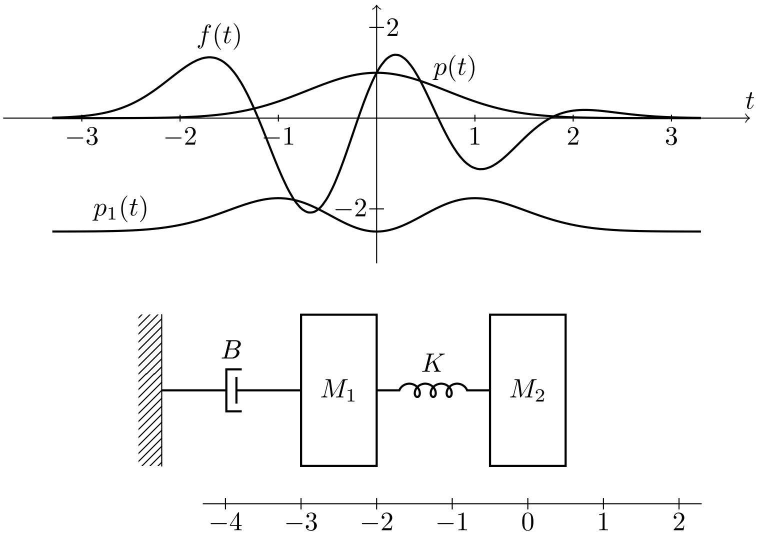

Figure 2.12: Solution of the system describing two masses with a spring and damper where $B = K = 1$ and $M_1 = M_2 = \frac{1}{2}$ and the position of the second mass is $p(t) = e^{-t^2}$.

Given the position of the second mass p we can readily solve for the corresponding force f and position of the first mass p. For example, if the constants $B = K = 1$ and $M_1 = M_2 = \frac{1}{2}$ and $d = \frac{5}{2}$, and if the position of the second mass satisfies

$$p(t) = e^{-t^2}$$

then, by application of (2.3.3) and (2.3.1),

$$f(t) = e^{-t^2}(1 + 4t - 8t^2 - 4t^3 + 4t^4), \qquad \text{and} \qquad p_1(t) = 2e^{-t^2}t^2 - \tfrac{5}{2}.$$

This solution is plotted in Figure 2.12.

2.4 Direct current motors

Direct current (DC) motors convert electrical energy, in the form of a voltage, into rotary kinetic energy [Nise, 2007, page 76]. We derive a differential equation relating the input voltage v to the angular position of the motor θ. Figure 2.13 depicts the components of a DC motor.

The voltages over the resistor and inductor satisfy

$$v_R = Ri, \qquad v_L = LD(i),$$

and the motion of the motor induces a voltage called the **back electromotive force** (EMF),

$$v_b = K_b D(\theta)$$

that we model as being proportional to the angular velocity of the motor. The input voltage now satisfies

$$v = v_R + v_L + v_b = Ri + LD(i) + K_b D(\theta).$$

The torque τ applied by the motor is modelled as being proportional to the current i,

$$\tau = K_\tau i.$$

A load with inertia J is attached to the motor. Two forces are assumed to act on the load, the torque τ applied by the current, and a torque $\tau_d = -BD(\theta)$ modelling a damper that acts proportionally against the angular velocity of the motor. By Newton's law, the angular acceleration of the load satisfies

$$JD^2(\theta) = \tau + \tau_d = K_\tau i - BD(\theta).$$

Combining these equations we obtain the 3rd order differential equation

$$v = \left(\frac{RB}{K_\tau} + K_b\right) D(\theta) + \frac{RJ + LB}{K_\tau} D^2(\theta) + \frac{LJ}{K_\tau} D^3(\theta)$$

relating voltage and motor position. In many DC motors the inductance L is small and can be ignored, leaving the simpler second order equation

$$v = \left(\frac{RB}{K_\tau} + K_b\right) D(\theta) + \frac{RJ}{K_\tau} D^2(\theta). \tag{2.4.1}$$

Given the position signal θ we can find the corresponding voltage signal v. For example, put the constants $K_b = K_\tau = B = R = J = 1$ and assume that

$$\theta(t) = 2\pi(1 + \operatorname{erf}(t))$$

where $\operatorname{erf}(t) = \frac{2}{\pi}\int_{-\infty}^{t} e^{-\tau^2} d\tau$ is the **error function**. The corresponding angular velocity $D(\theta)$ and voltage v satisfy

$$D(\theta, t) = 4\sqrt{\pi}e^{-t^2}, \qquad v(t) = 8\sqrt{\pi}e^{-t^2}(1 - t).$$

These signals are depicted in Figure 2.14. This voltage signal is sufficient to make the motor perform two revolutions and then come to rest.

2.5 Exercises

2.1. Analyse the inverting amplifier circuit in Figure 2.7 to obtain the relationship between input voltage x and output voltage y given by (2.2.1). You may wish to use a symbolic programming language (for example Sage, Mathematica, or Maple).

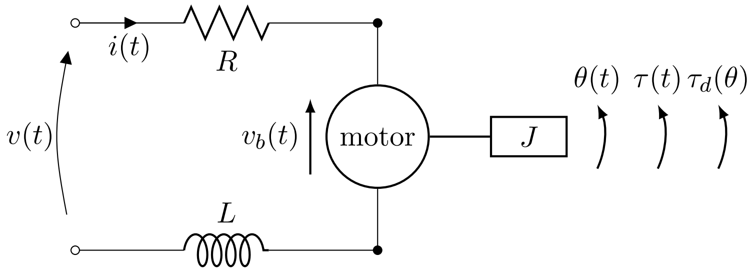

Figure 2.13: Diagram for a rotary direct current (DC) motor

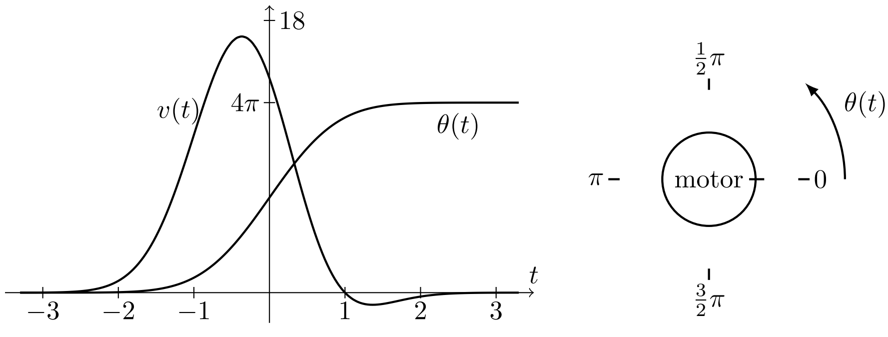

Figure 2.14: Voltage and corresponding angle for a DC motor with constants $K_b = K_\tau = B = R = J = 1$.

Chapter 3

Linear time invariant systems

In the previous section we derived differential equations that model mechanical, electrical, and electro-mechanical systems. The equations themselves often do not provide as much information about these system as we require. For example, we were able to find a signal p representing the position of the mass-spring-damper in Figure 2.2 given a particular force signal f is applied to the mass. However, it is not immediately obvious how to find the force signal f given a particular position signal p. We will be able to solve this problem and, more generally, to describe properties of systems modelled by linear differential equations with constant coefficient, if we make the added assumptions that the systems are **linear** and **time invariant**. We study linear time invariant systems in this chapter. Throughout this chapter H will denote a linear time invariant system.

3.1 Convolution, regular systems and the delta "function"

A large number of linear time invariant systems can be represented by a signal called the **impulse response**. The impulse response of a system H is a signal h such that

$$H(x,t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau,$$

that is, the response of H to input signal x can be represented as an integral equation involving x and the impulse response h. The integral is called a **convolution** and appears so often that a special notation is used for it. We write $h * x$ to indicate the signal that results from convolution of signals h and x, that is, $h * x$ is the signal satisfying

$$h * x = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau.$$

Those systems that have an impulse response we call **regular systems**[1]. Observe that regular systems are linear because

$$\begin{aligned}H(ax+by) &= h*(ax+by) \\ &= \int_{-\infty}^{\infty} h(\tau)\big(ax(t-\tau)+by(t-\tau)\big)d\tau \\ &= a\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau + b\int_{-\infty}^{\infty} h(\tau)y(t-\tau)d\tau \\ &= a(h*x)+b(h*y) \\ &= aH(x)+bH(y).\end{aligned} \tag{3.1.1}$$

The above equations show that convolution commutes with scalar multiplication and distributes with addition, that is,

$$h*(ax+by) = a(h*x)+b(h*y).$$

Regular systems are also time invariant because

$$\begin{aligned}T_\kappa\big(H(x)\big) &= T_\kappa(h*x) \\ &= \int_{-\infty}^{\infty} h(\tau)x(t-\kappa-\tau)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)T_\kappa(x,t-\tau)d\tau \\ &= h*T_\kappa(x) \\ &= H\big(T_\kappa(x)\big).\end{aligned}$$

We can define the impulse response of a regular system H in the following way. First define the signal

$$p_\gamma(t) = \begin{cases}\gamma, & 0<t\leq\frac{1}{\gamma} \\ 0, & \text{otherwise,}\end{cases}$$

that is, a rectangular shaped pulse of height γ and width $\frac{1}{\gamma}$. The signal p_γ is plotted in Figure 3.1 for $\gamma = \frac{1}{2}, 1, 2, 5$. As γ increases the pulse gets thinner and higher so as to keep the area under p_γ equal to one. Consider the response of the regular system H to the signal p_γ,

$$\begin{aligned}H(p_\gamma) &= h*p_\gamma \\ &= \int_{-\infty}^{\infty} h(\tau)p_\gamma(t-\tau)d\tau \\ &= \gamma\int_{t-1/\gamma}^{t} h(\tau)d\tau.\end{aligned}$$

[1]The name **regular system** is motivated by the term **regular distribution** [Zemanian, 1965]

Taking limits as $\gamma \to \infty$,

$$\lim_{\gamma\to\infty} H(p_\gamma) = \lim_{\gamma\to\infty} \gamma \int_{t-1/\gamma}^{t} h(\tau)d\tau = h(t) \;\text{ a.e.}$$

Thus, we define the impulse response of a regular system H as the limit

$$h = \lim_{\gamma\to\infty} H(p_\gamma). \tag{3.1.2}$$

The limit exists when H is regular. If this limit does not exist, the system is not regular and does not have an impulse response.

As an example, consider the integrator system

$$I_\infty(x) = \int_{-\infty}^{t} x(\tau)d\tau \tag{3.1.3}$$

described in Section 1.3. This systems response to p_γ is

$$I_\infty(p_\gamma, t) = \int_{-\infty}^{t} p_\gamma(\tau)d\tau = \begin{cases} 0, & t \leq 0 \\ \gamma t, & 0 < t \leq \frac{1}{\gamma} \\ 1, & t > \frac{1}{\gamma} \end{cases}$$

The response is plotted in Figure 3.1. Taking the limit as $\gamma \to \infty$ we find that the impulse response of the integrator is the step function

$$u(t) = \lim_{\gamma\to\infty} H(p_\gamma) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0. \end{cases} \qquad \text{a.e.} \tag{3.1.4}$$

Some important systems do not have an impulse response. For example, the identity system T_0 does not because

$$\lim_{\gamma\to\infty} T_0(p_\gamma) = \lim_{\gamma\to\infty} p_\gamma$$

does not exist. Similarly, all the time shifters T_τ do not have impulse responses. However, it can be notationally useful to pretend that T_0 *does* have an impulse response and we denote it by the symbol δ called the **delta function**. The idea is to assign δ the property

$$\int_{-\infty}^{\infty} x(t)\delta(t)dt = x(0)$$

so that convolution of x and δ satisfies

$$\delta * x = \int_{-\infty}^{\infty} \delta(\tau)x(t-\tau)d\tau = x(t) = T_0(x).$$

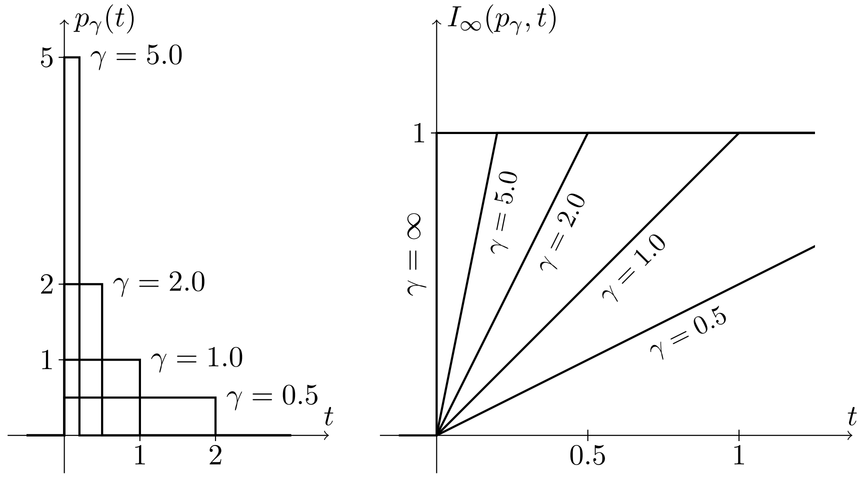

Figure 3.1: The rectangular shaped pulse p_γ for $\gamma = 0.5, 1, 2, 5$ and the response of the integrator (3.1.3) to p_γ for $\gamma = 0.5, 1, 2, 5, \infty$.

We now treat δ as if it were a signal. So $\delta(t-\tau)$ will represent the impulse response of the time shifter T_τ because

$$\begin{aligned} T_\tau(x) &= \delta(t-\tau) * x \\ &= \int_{-\infty}^{\infty} \delta(\kappa-\tau)x(t-\kappa)d\kappa \\ &= \int_{-\infty}^{\infty} \delta(k)x(t-\tau-k)dk \qquad \text{(change variable } k = \kappa - \tau\text{)} \\ &= x(t-\tau). \end{aligned}$$

For $a \in \mathbb{R}$ it is common to plot $a\delta(t-\tau)$ using an arrow of height a at $t = \tau$ as indicated in Figure 3.2. It is important to realise that δ is not actually a signal. It is not a function. However, it can be convenient to treat δ as if it were a function. The manipulations in the last set of equations, such as the change of variables, are not formally justified, but they do lead to the desired result $T_\tau(x) = x(t-\tau)$ in this case. In general, there is no guarantee that mechanical mathematical manipulations involving δ will lead to sensible results.

The only other non regular systems that we have use of are differentiators D^k, and it is convenient to define a similar notation for pretending that these systems have an impulse response. In this case we use the symbol δ^k and assign it the property

$$\int_{-\infty}^{\infty} x(t)\delta^k(t)dt = D^k(x, 0),$$

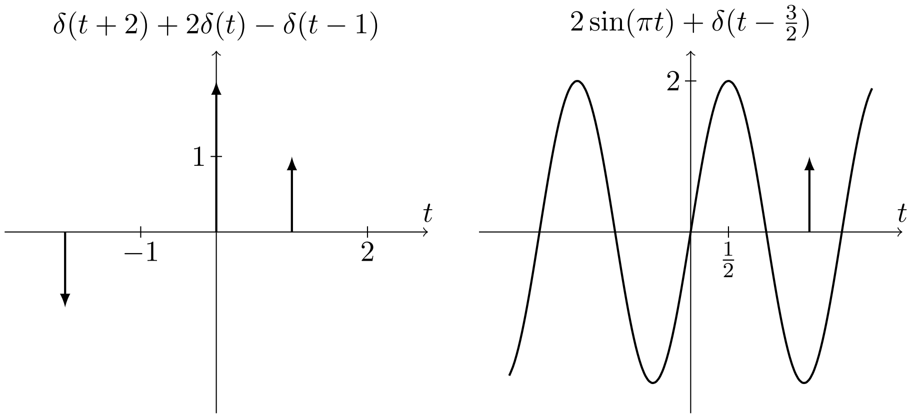

Figure 3.2: Plot of the "signal" $\delta(t+2)+2\delta(t)-\delta(t-1)$ (left) and the "signal" $2\sin(\pi t)+\delta(t-\frac{3}{2})$ (right).

so that convolution of x and δ is

$$\delta^k * x = \int_{-\infty}^{\infty} \delta^k(\tau)x(t-\tau)d\tau = D^k(x,t).$$

As with the delta function the symbol δ^k must be treated with care. This notation can be useful, but purely formal manipulations with δ^k may not lead to sensible results in general.

The impulse response h immediately yields some properties of the corresponding system H. For example, if $h(t)=0$ for all $t<0$, then H is causal because

$$H(x) = h * x = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$$

only depends on values of the input signal x at times less than t, i.e., only times $t-\tau$ with $\tau>0$. The system H is stable if and only if h is absolutely integrable (Exercise 3.3).

Another important signal is the **step response** of a system that is defined as the response of the system to the step function $u(t)$. For example, the step response of the time shifter T_τ is the time shifted step function $T_\tau(u,t)=u(t-\tau)$. The step response of the integrator I_∞ is

$$I_\infty(u) = \int_{-\infty}^{t} u(\tau)d\tau = \begin{cases} \int_0^t d\tau = t & t>0 \\ 0 & t\leq 0. \end{cases}$$

This signal is often called the **ramp function**. Not all systems have a step response. For example, the system with impulse response $u(-t)$ does not because the convolution of the step $u(t)$ and its reflection $u(-t)$ does not exist. If a system H has both an impulse response h and a step response

$H(u)$, then these two signals are related. To see this, observe that the step response is

$$H(u) = h * u = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau = I_\infty(h). \qquad (3.1.5)$$

Thus, the step response can be obtained by applying the integrator I_∞ to the impulse response.

3.2 Properties of convolution

The convolution $x * y$ of two signals x and y does not always exist. For example, if $x(t) = u(t)$ and $y(t) = 1$, then

$$x * y = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau = \int_{-\infty}^{\infty} u(\tau)d\tau = \int_{0}^{\infty} d\tau$$

is not finite for any t. We cannot convolve the step function u and the signal that is equal to 1 for all time. On the other hand, if $x(t) = y(t) = u(t)$, then

$$x * y = \int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau = \begin{cases} \int_0^t d\tau = \tau & t > 0 \\ 0 & t \leq 0, \end{cases}$$

if finite for all t.

We have already shown in (3.1.1) that convolution commutes with scalar multiplication and is distributive with addition, that is, for signals x, y, w and complex numbers a, b,

$$a(x * w) + b(y * w) = (ax + by) * w.$$

Convolution is commutative, that is, $x * y = y * x$ whenever these convolutions exist. To see this, write

$$\begin{aligned} x * y &= \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} x(t-\kappa)y(\kappa)d\kappa \qquad \text{(change variable } \kappa = t - \tau) \\ &= y * x. \end{aligned}$$

Convolution is also associative, that is, for signals x, y, z,

$$(x * y) * z = x * (y * z). \qquad \text{(see Exercise 3.2)}$$

By combining the associative and commutative properties we find that the order in which the convolutions in $x * y * z$ are performed does not mater, that is

$$x * y * z = y * z * x = z * x * y = y * x * z = x * z * y = z * y * x$$

provided that all the convolutions involved exist. More generally, the order in which any sequence of convolutions is performed does not change the final result.

3.3 Linear combining and composition

Let H_1 and H_2 be linear time invariant systems and let H be the system

$$H(x) = cH_1(x) + dH_2(x), \qquad c, d \in \mathbb{C}$$

formed by a linear combination of H_1 and H_2. The system H is linear because for signals x, y and complex numbers a, b,

$$\begin{aligned}
H(ax + by) &= cH_1(ax + by) + dH_2(ax + by) \\
&= acH_1(x) + bcH_1(y) + adH_2(x) + bdH_2(y) \qquad \text{(linearity } H_1, H_2\text{)} \\
&= a\big(cH_1(x) + dH_2(x)\big) + b\big(cH_1(y) + dH_2(y)\big) \\
&= aH(x) + bH(y).
\end{aligned}$$

The system is also time invariant because

$$\begin{aligned}
H\big(T_\tau(x)\big) &= cH_1\big(T_\tau(x)\big) + dH_2\big(T_\tau(x)\big) \\
&= cT_\tau\big(H_1(x)\big) + dT_\tau\big(H_2(x)\big) \qquad \text{(time-invariance } H_1, H_2\text{)} \\
&= T_\tau\big(cH_1(x) + dH_2(x)\big) \qquad \text{(linearity } T_\tau\text{)} \\
&= T_\tau\big(H(x)\big).
\end{aligned}$$

So, we can construct linear time invariant systems by **linearly combining** (adding and multiplying by constants) other linear time invariant systems. If H_1 and H_2 are regular systems this linear combining property can be expressed using their impulse responses h_1 and h_2. We have

$$\begin{aligned}
H(x) &= aH_1(x) + bH_2(x) \\
&= ah_1 * x + bh_2 * x \\
&= (ah_1 + bh_2) * x \qquad \text{(distributivity of convolution)} \\
&= h * x,
\end{aligned}$$

and so, H is a regular system with impulse response $h = ah_1 + bh_2$.

Another way to construct linear time invariant systems is by **composition**. Let H_1 and H_2 be linear time invariant systems and let

$$H(x) = H_2\big(H_1(x)\big),$$

that is, H first applies the system H_1 and then applies the system H_2. The composition $H_2\big(H_1(x)\big)$ only applies to those signals x in the domain of H_1 and such that the signal $H_1(x)$ is in the domain of H_2. The system H is linear because, for signals x, y and complex numbers a, b,

$$\begin{aligned}
H(ax + by) &= H_2\big(H_1(ax + by)\big) \\
&= H_2\big(aH_1(x) + bH_1(y)\big) \qquad \text{(linearity } H_1\text{)} \\
&= aH_2\big(H_1(x)) + bH_2\big(H_1(y)\big)\big) \qquad \text{(linearity } H_2\text{)} \\
&= aH(x) + bH(y).
\end{aligned}$$

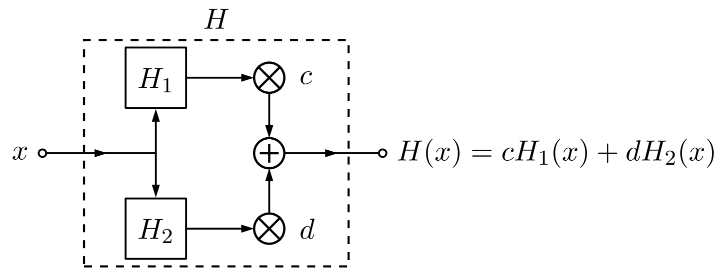

Figure 3.3: Block diagram depicting the linear combining property of linear time invariant systems. The system $cH_1(x) + dH_2(x)$ can be expressed as a single linear time invariant system $H(x)$.

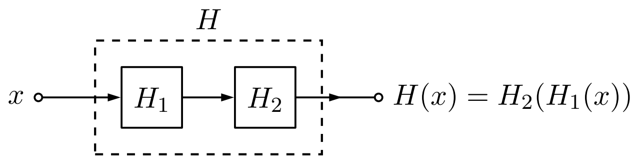

Figure 3.4: Block diagram depicting the composition property of linear time invariant systems. The system $H_2(H_1(x))$ can be expressed as a single linear time invariant system $H(x)$.

The system is also time invariant because

$$\begin{aligned} H\big(T_\tau(x)\big) &= H_2\big(H_1(T_\tau(x)\big) \\ &= H_2\big(T_\tau\big(H_1(x)\big)\big) && \text{(time-invariance } H_1\text{)} \\ &= T_\tau\big(H_2\big(H_1(x)\big)\big) && \text{(time-invariance } H_2\text{)} \\ &= T_\tau\big(H(x)\big). \end{aligned}$$

If H_1 and H_2 are regular systems the composition property can be expressed using their impulse responses h_1 and h_2. It follows that

$$\begin{aligned} H(x) &= H_2(H_1(x)) \\ &= h_2 * (h_1 * x) \\ &= (h_2 * h_1) * x && \text{(associativity of convolution)} \\ &= h * x, \end{aligned}$$

and so, H is a regular system with impulse response $h = h_2 * h_1$.

A wide variety of linear time invariant systems can now be constructed by linearly combining and composing simpler systems.

3.4 Eigenfunctions and the transfer function

Let $s = \sigma + j\omega \in \mathbb{C}$. Complex exponential signals of the form

$$e^{st} = e^{\sigma t} e^{j\omega t} = e^{\sigma t}\big(\cos(\omega t) + j\sin(\omega t)\big)$$

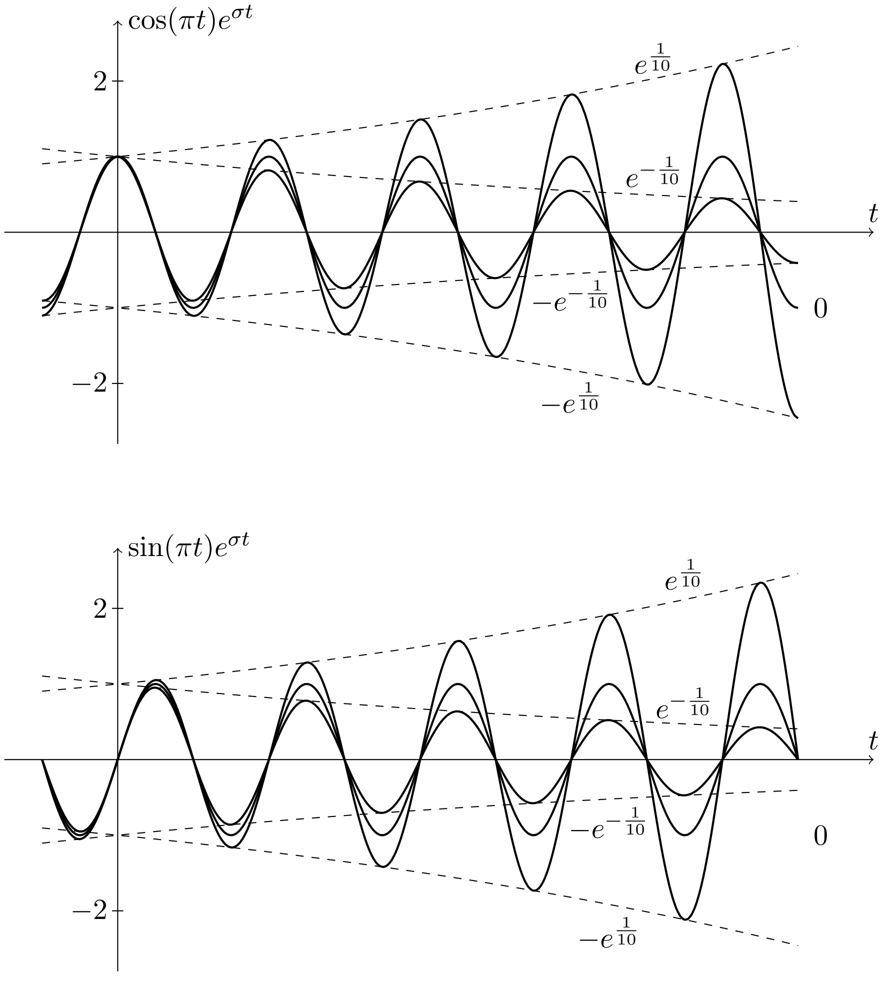

Figure 3.5: The function $\cos(\pi t)e^{\sigma t}$ (top) and $\sin(\pi t)e^{\sigma t}$ (bottom) for $\sigma = -\frac{1}{10}, 0, \frac{1}{10}$.

play an important role in the study of linear time invariant systems. The real and imaginary parts of the signal $e^{(\sigma+j\pi)t}$ with $\sigma = -\frac{1}{10}, 0, \frac{1}{10}$ are plotted in Figure 3.5. The signal is oscillatory when $\omega \neq 0$. The signal converges to zero as $t \to \infty$ when $\sigma < 0$ and diverges as $t \to \infty$ when $\sigma > 0$.

Let H be a linear time invariant system and let $y = H(e^{st})$ be the response of H to the exponential signal e^{st}. Consider the response of H to the time-shifted signal $e^{s(t+\tau)}$ for $\tau \in \mathbb{R}$. By time-invariance

$$H(e^{s(t+\tau)}, t) = H(e^{st}, t+\tau) = y(t+\tau) \qquad \text{for all } t, \tau \in \mathbb{R},$$

and by linearity

$$H(e^{s(t+\tau)}, t) = e^{s\tau} H(e^{st}, t) = e^{s\tau} y(t) \qquad \text{for all } t, \tau \in \mathbb{R}.$$

Combining these equations we obtain

$$y(t+\tau) = e^{s\tau} y(t) \qquad \text{for all } t, \tau \in \mathbb{R}.$$

This equation is satisfied by signals of the form $y(t) = \lambda e^{st}$ where λ is a complex number. That is, the response of a linear time invariant system H to an exponential signal e^{st} is the same signal e^{st} multiplied by some constant complex number λ. Due to this property exponential signals are called **eigenfunctions** of linear time invariant systems. The constant λ does not depend on t, but it does usually depend on the complex number s and the system H. To highlight this dependence on H and s we write $\lambda(H)(s)$ or $\lambda(H, s)$. Considered as a function of s, $\lambda(H, s)$ is called the **transfer function** of the system H. Thus, the transfer function satisfies

$$H(e^{st}) = \lambda(H, s) e^{s\tau}. \tag{3.4.1}$$

We can use these eigenfunctions to better understand the properties of systems modelled by differential equations, such as those in Section 2. As an example, consider the active electrical circuit from Figure 2.8. In the case that the resistors $R_1 = R_2$, and the capacitor $C_1 = 0$ (an open circuit) the differential equation relating the input voltage x and output voltage y is

$$x = -y - R_1 C_2 D(y).$$

We called this the **active RC** circuit. To simplify notation put $R = R_1$ and $C = C_2$ so that $x = -y - RCD(y)$. Observe what occurs when $y = ce^{st}$ is a complex exponential signal with $c \in \mathbb{C}$. We have

$$x = -ce^{st} - cRCse^{st} = -(1 + RCs)ce^{st} = -(1 + RCs)y,$$

and so, x is also a complex exponential signal. We immediately obtain the relationship

$$y = -\frac{1}{1 + RCs} x,$$

that holds whenever y (or equivalently x) is of the form ce^{st} with $c \in \mathbb{C}$. Let H be a system that maps the input voltage x to the output voltage y, i.e., H is a system that describes the active RC circuit. Putting $x = e^{st}$ in the equation above, we find that

$$y = H(x) = H(e^{st}) = -\frac{1}{1 + RCs}e^{st},$$

and so, the transfer function of the system H describing the active RC circuit is

$$\lambda(H, s) = -\frac{1}{1 + RCs}. \tag{3.4.2}$$

3.5 The spectrum

It is often of interest to focus on the transfer function when s is purely imaginary, that is, when $s = j\omega$. In this case the complex exponential signal takes the form

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t).$$

This signal is oscillatory when $\omega \neq 0$ and does not decay or explode as $|t| \to \infty$. The function

$$\Lambda(H, f) = \lambda\big(H, j2\pi f\big)$$

is called the **spectrum** of the system H. It follows from (3.4.1) that the response of the system to the complex exponential signal $e^{j2\pi ft}$ satisfies

$$H(e^{j2\pi ft}) = \lambda(H, j2\pi f)e^{j2\pi ft} = \Lambda(H, f)e^{j2\pi ft}, \qquad f \in \mathbb{R}.$$

It is of interest to consider the **magnitude spectrum** $|\Lambda(H)|$ and the **phase spectrum** $\angle\Lambda(H)$ separately. The notation $\angle$ denotes the **argument** (or **phase**) of a complex number. We have,

$$\Lambda(H, f) = |\Lambda(H, f)|\, e^{j\angle\Lambda(H,f)}$$

and correspondingly

$$H(e^{j2\pi ft}) = |\Lambda(H, f)|\, e^{j(2\pi ft + \angle\Lambda(H,f))}.$$

By taking real and imaginary parts we obtain the pair of real valued solutions

$$H\big(\cos(2\pi ft)\big) = |\Lambda(H, f)| \cos\big(2\pi ft + \angle\Lambda(H, f)\big),$$

$$H\big(\sin(2\pi ft)\big) = |\Lambda(H, f)| \sin\big(2\pi ft + \angle\Lambda(H, f)\big). \tag{3.5.1}$$

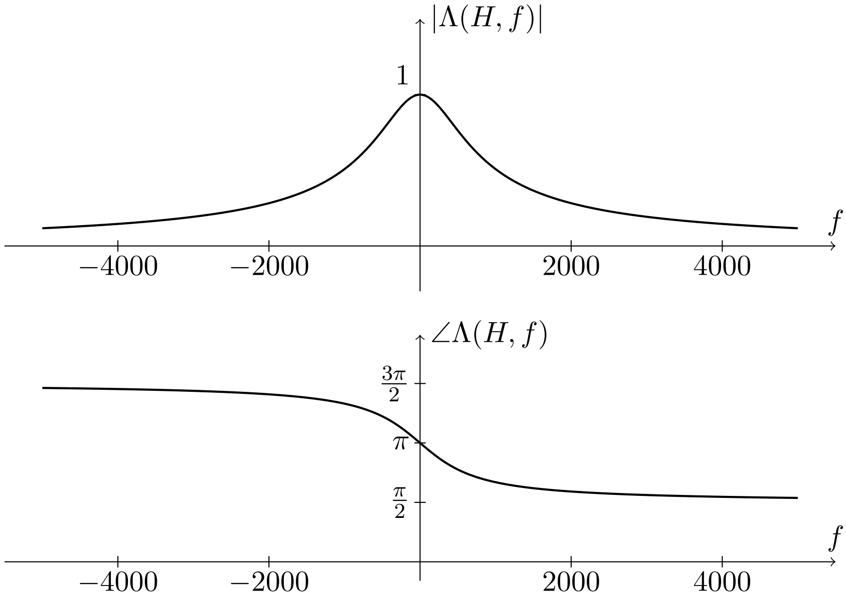

Figure 3.6: Magnitude spectrum (top) and phase spectrum (bottom) of the active RC circuit with $R = 27 \times 10^3$ and $C = 10 \times 10^{-9}$.

Consider again the active RC circuit with H the system mapping the input voltage x to the output voltage y. According to (3.4.2) the spectrum of H is

$$\Lambda(H, f) = -\frac{1}{1 + 2\pi RCfj}. \tag{3.5.2}$$

The magnitude and phase spectrum is

$$|\Lambda(H, f)| = \big(1 + 4\pi^2 R^2 C^2 f^2\big)^{-\frac{1}{2}}, \qquad \angle\Lambda(H, f) = \pi - \operatorname{atan}(2\pi RCf\big).$$

The magnitude and phase spectrum are plotted in Figure 3.6 when $R = 27 \times 10^3$ and $C = 10 \times 10^{-9}$. Observe from the plot of the magnitude spectrum that a low frequency sinusoidal signal, say 100Hz or less, input to the active RC circuit results in a sinusoidal output signal with the same frequency and approximately the same amplitude. However, a high frequency sinusoidal signal, say greater than 1000Hz, input to the circuit results in a sinusoidal output signal with the same frequency, but smaller amplitude. For this reason RC circuits are called **low pass filters**.

Test 4 (**Spectrum of the active RC circuit**) We test the hypothesis that the active RC circuit satisfies (3.5.1). To do this sinusoidal signals at varying frequencies of the form

$$x_k(t) = \sin(2\pi f_k t), \qquad f_k = \left\lceil 110 \times 2^{k/2} \right\rfloor, \quad k = 0, 1, \dots, 12$$

are input to the active RC circuit constructed as in Test 3 with $R = R_1 = 27\text{k}\Omega$ and $C = C_2 = 10\text{nF}$. The notation $\lceil \cdot \rfloor$ denotes rounding to the nearest integer with half integers rounded up. In view of (3.5.1) the expected output signals are of the form

$$y_k(t) = |\Lambda(H, f_k)| \sin\big(2\pi f_k t + \angle\Lambda(H, f_k)\big), \qquad k = 0, 1, \dots, 12.$$

This equality can also be shown directly using the differential equation for the active RC circuit.

Using the soundcard each signal x_k is played for a period of approximately 1 second and approximately $F = 44100$ samples are obtained. On the soundcard hardware used for this test samples near the beginning and end of playback are distorted. This appears to be an unavoidable feature of the soundcard. To alleviate this we discard the first 10^4 samples and use only the $L = 8820$ samples that follow (corresponding to 200ms of signal). After this process we have samples $x_{k,1}, \dots, x_{k,L}$ and $y_{k,1}, \dots, y_{k,L}$ of the input and output signals corresponding with the kth signal x_k. The samples are expected to take the form

$$x_{k,\ell} \approx x_k(P\ell - \tau) = \rho \sin(2\pi f_k P\ell - \theta)$$

and

$$y_{k,\ell} \approx y_k(\ell P - \tau) = |\Lambda(H, f_k)|\, \rho \sin\big(2\pi f_k P\ell - \theta + \angle\Lambda(H, f_k)\big)$$

where $P = \frac{1}{F}$ is the sample period, the positive real number ρ corresponds with the gain on the input and output of the soundcard, and $\theta = 2\pi f_k \tau$ corresponds with delays caused by discarding the first 10^4 samples and also unavoidable delays that occur when starting soundcard playback and recording.

We will not measure the gain ρ nor the delay θ, but will be able to test the properties of the circuit without knowledge of these. To simplify notation put $\gamma = 2\pi f_k P$. From the samples of the input signal $x_{k,1}, \dots, x_{k,L}$ compute the complex number

$$\begin{aligned} A &= \frac{2j}{L} \sum_{\ell=1}^{L} x_{k,\ell} e^{-j\gamma\ell} \\ &\approx \frac{2j}{L} \sum_{\ell=1}^{L} \rho \sin(\gamma\ell - \theta) e^{-j\gamma\ell} \\ &= \alpha + \alpha^* C \end{aligned}$$

where $\alpha = \rho e^{-j\theta}$ and α^* denotes the complex conjugate of α and

$$C = e^{-\gamma(L+1)} \frac{\sin(\gamma L)}{L \sin(\gamma)} \qquad \text{(Excersize 3.6)}.$$

Similarly, from the samples of the output signal $y_{k,1}, \dots, y_{k,L}$ we compute the complex number

$$B = \frac{2j}{L} \sum_{\ell=1}^{L} y_{k,\ell} e^{-j\gamma\ell} \approx \beta + \beta^* C$$

where $\beta = \rho e^{-j\theta} \Lambda(H, f_k) = \alpha \Lambda(H, f_k)$. Now compute the quotient

$$Q_k = \frac{B - B^* C}{A - A^* C} \approx \frac{\beta(1 + |C|^2)}{\alpha(1 + |C|^2)} = \frac{\beta}{\alpha} = \Lambda(H, f_k).$$

Thus, we expect the quotient Q_k to be close to the spectrum of the active RC circuit evaluated at frequency f_k. We will test this hypothesis by observing the magnitude and phase of Q_k individually, that is, we will test the expected relationships

$$|Q_k| \approx |\Lambda(H, f_k)| = \sqrt{\frac{1}{1 + 4\pi^2 R^2 C^2 f_k^2}}$$

and

$$\angle Q_k \approx \angle \Lambda(H, f_k) = \pi - \operatorname{atan}(2\pi R C f_k)$$

for each $k = 0, \dots, 12$.

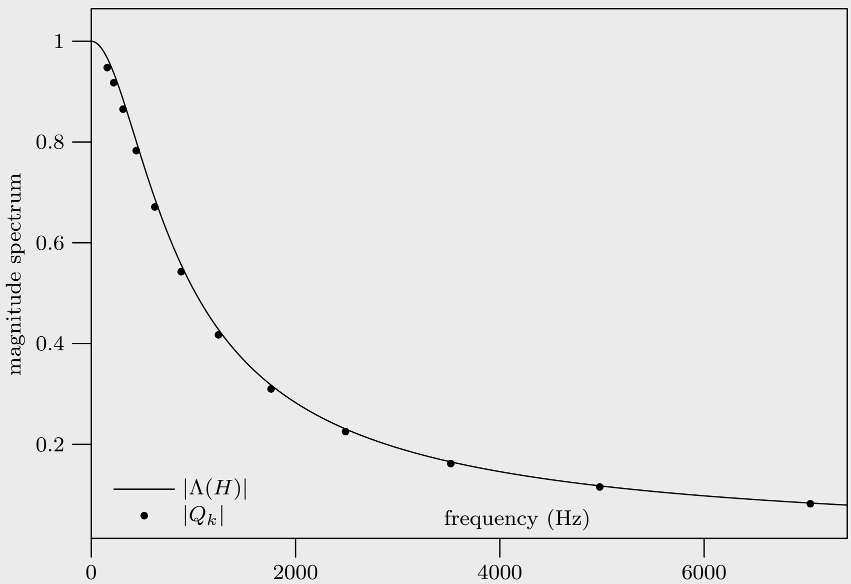

Figure 3.7: Plot of the hypothesised magnitude spectrum $|\Lambda(H, f)|$ (solid line) and the measured magnitude spectrum $|Q_k|$ for $k = 0, \dots, 12$ (dots).

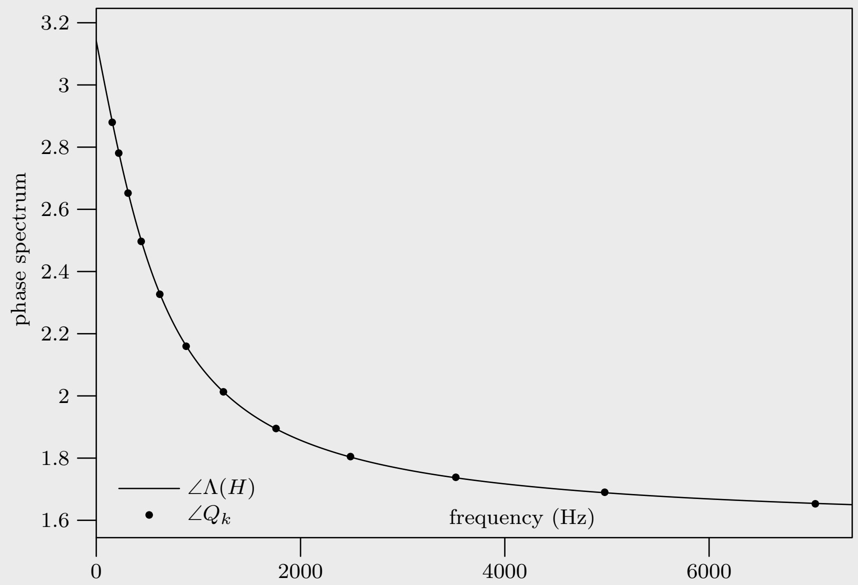

Figure 3.8: Plot of the hypothesised phase spectrum $\angle\Lambda(H, f)$ (solid line) and the measured phase spectrum $\angle Q_k$ for $k = 0, \dots, 12$ (dots).

3.6 Exercises

3.1. Show that convolution distributes with addition and commutes with scalar multiplication, that is, show that $a(x*w)+b(y*w) = (ax+by)*w$.

3.2. Show that convolution is associative. That is, if x, y, z are signals then $x * (y * z) = (x * y) * z$.

3.3. Show that a regular system is stable if and only if its impulse response is absolutely integrable.

3.4. Show that the system $H(x) = \int_{-1}^{1} \sin(\pi\tau)x(t+\tau)d\tau$ is linear time invariant and regular. Find and sketch the impulse response.

3.5. Show that $\sum_{\ell=1}^{L} e^{\beta\ell} = \frac{e^{\beta(L+1)}-e^{\beta}}{e^{\beta}-1}$ (Hint: sum a geometric progression).

3.6. Show that

$$\frac{2j}{L}\sum_{\ell=1}^{L} \sin(\gamma\ell - \theta)e^{-j\gamma\ell} = \alpha + \alpha^* C$$

where $\alpha = e^{-j\theta}$ and $C = e^{-j\gamma(L+1)}\frac{\sin(\gamma L)}{L\sin(\gamma)}$. (Hint: solve Exercise 3.5 first and then use the formula $2j\sin(x) = e^{jx} - e^{-jx}$).

Chapter 4

The Laplace transform

Let $x\colon \mathbb{R} \to \mathbb{C}$ be a complex valued function of the real line (a signal). The function

$$\mathcal{L}(x) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{4.0.1}$$

is called the **Laplace transform** of x. The Laplace transform is a function of the complex parameter s, and if we need to indicate this we write $\mathcal{L}(x)(s)$ or $\mathcal{L}(x, s)$. The Laplace transform $\mathcal{L}(x)$ is not necessarily defined for all values of $s \in \mathbb{C}$. Let R be the set of real numbers such that $x(t)e^{-\sigma t}$ is absolutely integrable if and only if $\sigma \in R$, that is

$$\int_{-\infty}^{\infty} |x(t)|\, e^{-\sigma t}dt < \infty \qquad \text{if and only if } \sigma \in R.$$

In this case, $\mathcal{L}(x, s)$ is finite for all s with real part satisfying $\operatorname{Re}(s) \in R$ because

$$|\mathcal{L}(x, s)| = \left| \int_{-\infty}^{\infty} x(t)e^{-st}dt \right| \leq \int_{-\infty}^{\infty} |x(t)|\, e^{-\operatorname{Re}(s)t}dt < \infty.$$

The subset of the complex plane with real part from R is called the **region of convergence** (ROC) of the signal x.

For example, the Laplace transform of the right sided signal $e^{\alpha t}u(t)$ is

$$\begin{aligned} \mathcal{L}(e^{\alpha t}u(t)) &= \int_{-\infty}^{\infty} e^{\alpha t}e^{-st}u(t)dt \\ &= \int_{0}^{\infty} e^{(\alpha - s)t}dt \\ &= \lim_{t\to\infty} \frac{e^{(\alpha-s)t}}{\alpha - s} - \frac{1}{\alpha - s}. \end{aligned}$$

The limit converges for all s with $\operatorname{Re}(\alpha - s) < 0$. Thus, the Laplace transform of $e^{\alpha t}u(t)$ is

$$\mathcal{L}(e^{\alpha t}u(t)) = \frac{1}{s-\alpha} \qquad \operatorname{Re}(s) > \operatorname{Re}(\alpha)$$

The region of convergence of $e^{\alpha t}u(t)$ is the subset of the complex plane with real part greater than $\mathrm{Re}(\alpha)$. Figure 4.1 shows the region of convergence when $\mathrm{Re}(\alpha) = -2$. Now consider the left sided signal $e^{\beta t}u(-t)$ with Laplace transform

$$\mathcal{L}(e^{\beta t}u(-t)) = \lim_{t \to -\infty} \frac{e^{(\beta - s)t}}{\beta - s} + \frac{1}{\beta - s}.$$

The limit converges when $\mathrm{Re}(\beta - s) > 0$, and so,

$$\mathcal{L}(e^{\beta t}u(-t)) = \frac{1}{\beta - s} \qquad \mathrm{Re}(s) < \mathrm{Re}(\beta).$$

The region of convergence of $e^{\beta t}u(-t)$ is those $s \in \mathbb{C}$ such that $\mathrm{Re}(s) < \mathrm{Re}(\beta)$. The signal $ae^{\alpha t}u(t) + be^{\beta t}u(-t)$ has Laplace transform

$$\begin{aligned}
\mathcal{L}\big(ae^{\alpha t}u(t) + be^{\beta t}u(-t)\big) &= \int_{-\infty}^{\infty} \big(ae^{\alpha t}u(t) + be^{\beta t}u(-t)\big)e^{-st}dt \\
&= a\int_{-\infty}^{\infty} e^{\alpha t}u(t)e^{-st}dt + b\int_{-\infty}^{\infty} e^{\beta t}u(-t)e^{-st}dt \\
&= a\mathcal{L}(e^{\alpha t}u(t)) + b\mathcal{L}(e^{\beta t}u(-t))
\end{aligned}$$

that is finite only when $\mathrm{Re}(\alpha) < \mathrm{Re}(s) < \mathrm{Re}(\beta)$. The corresponding ROC is shown in Figure 4.1 when $\mathrm{Re}(\alpha) = -2$ and $\mathrm{Re}(\beta) = 3$. In the previous equation we have discovered that the Laplace transform is **linear**, that is, for signals x and y and constants a and b,

$$\mathcal{L}(ax + by) = a\mathcal{L}(x) + b\mathcal{L}(y). \tag{4.0.2}$$

In words: the Laplace transform of a linear combination of signals is the same linear combination of the Laplace transforms of those signals.

In the previous example the Laplace transform is guaranteed to be finite for any s if $\mathrm{Re}(\alpha) \geq \mathrm{Re}(\beta)$, and the region of convergence is correspondingly the empty set. Other signals also have this property. For example, the signal $x(t) = 1$ because

$$\mathcal{L}(1) = \int_{\infty}^{\infty} e^{-st}dt = \lim_{t \to -\infty} \frac{e^{-st}}{s} - \lim_{t \to \infty} \frac{e^{-st}}{s}$$

and the limit as $t \to -\infty$ converges only when $\mathrm{Re}(s) < 0$ while the limit as $t \to \infty$ converges only when $\mathrm{Re}(s) > 0$.

As a final example, consider the rectangular pulse

$$\Pi(t) = \begin{cases} 1 & -\frac{1}{2} < t \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Its Laplace transform is

$$\mathcal{L}(\Pi) = \int_{-\infty}^{\infty} \Pi(t)e^{-st}dt = \int_{-1/2}^{1/2} e^{-st}dt = \frac{e^{s/2} - e^{-s/2}}{s} \tag{4.0.3}$$

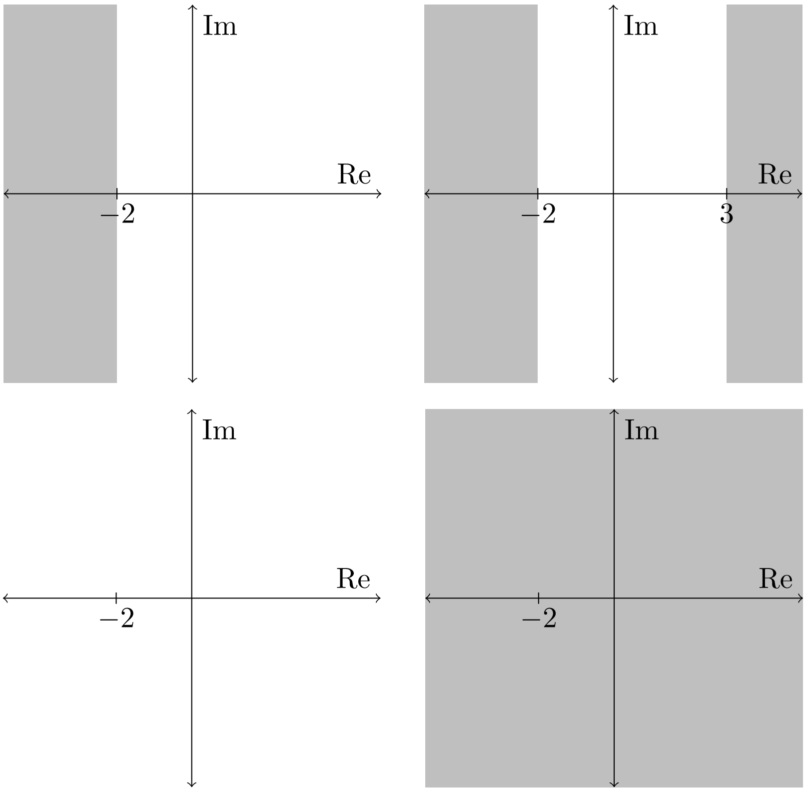

Figure 4.1: Regions of convergence (unshaded) for the signal $e^{-2t}u(t)$ (top left), the signal $e^{-2t}u(t) + e^{3t}u(-t)$ (top right), the rectangular pulse Π (bottom left), and the constant signal $x(t) = 1$ (bottom right).

and is finite for all $s \in \mathbb{C}$. The region of convergence of the rectangular pulse Π is the entire complex plane. The examples just given exhibit all the possible types of regions of convergence. The region of convergence is either the entire complex plane, a left or right half plane, a vertical strip, or the empty set.

Given the Laplace transform $\mathcal{L}(x)$ the signal x can be recovered by the **inverse Laplace transform**

$$x(t) = \mathcal{L}^{-1}(x) = \frac{1}{2\pi j} \lim_{\omega \to \infty} \int_{\sigma - j\omega}^{\sigma - j\omega} \mathcal{L}(x, s) e^{st} ds,$$

where σ is a real number that is inside the region of convergence of x. Solving the integral above typically requires a special type of integration called **contour integration** that we will not consider here [Stewart and Tall, 2004]. For our purposes, and for many engineering purposes, it suffices to remember only the following Laplace transform pair

$$\mathcal{L}\big(t^n u(t)\big) = \frac{n!}{s^{n+1}} \qquad \text{Re}(s) > 0, \tag{4.0.4}$$

where $n \geq 0$ is an integer (Exercise 4.2). Let $x(t)$ be a signal with region of convergence R. The Laplace transforms of the signal $x(t)$ and the signal $e^{\alpha t}x(t)$ are related. To see this write

$$\begin{aligned}\mathcal{L}\big(e^{\alpha t}x(t), s\big) &= \int_{-\infty}^{\infty} e^{\alpha t}x(t)e^{-st}dt \\ &= \int_{-\infty}^{\infty} x(t)e^{-(s-\alpha)t}dt \\ &= \mathcal{L}(x, s-\alpha) \qquad \operatorname{Re}(s-\alpha) \in R. \end{aligned} \tag{4.0.5}$$

This is called the **frequency shift rule**. Combining the frequency shift rule with (4.0.4) we obtain the transform pair

$$\mathcal{L}\big(t^n e^{\alpha t}u(t)\big) = \mathcal{L}\big(t^n u(t), s-\alpha\big) = \frac{n!}{(s-\alpha)^{n+1}} \qquad \operatorname{Re}(s) > \operatorname{Re}(\alpha), \tag{4.0.6}$$

where $n \geq 0$ is an integer. This is the only Laplace transform pair we require here.

A useful relationship exists between the Laplace transform of a signal x and its time scaled version $x(\alpha t)$ where $\alpha \neq 0$. If x is a signal with region of convergence R then the time scaled signal $x(\alpha t)$ with $\alpha \neq 0$ has Laplace transform

$$\mathcal{L}\big(x(\alpha t), s\big) = \frac{1}{|\alpha|}\mathcal{L}(x, s/\alpha), \qquad \operatorname{Re}(s/\alpha) \in R. \tag{4.0.7}$$

This is called the **time scaling property** (Excercise 4.10).

4.1 The transfer function and the Laplace transform

Our purpose for introducing the Laplace transform is to study the response of a linear time invariant system H to exponential signals of the form e^{st}. Recall from Section 3.4 that exponential signals are **eigenfunctions** of linear time invariant systems. That is, if $s \in \mathbb{C}$ such that the complex exponential signal e^{st} is in the domain of H, then response of H to e^{st} is λe^{st} where $\lambda \in \mathbb{C}$ is a constant that does not depend on t, but may depend on s and the system H. To highlight this dependence on H and s we write $\lambda(H, s)$ or $\lambda(H)(s)$ and do not distinguish between these notations. Considered as a function of s, $\lambda(H)$ is called the **transfer function** of the system H. For a given system H, we would like to understand how $\lambda(H, s)$ behaves as s changes. In what follows we regularly drop the argument "(s)" and simply write $\lambda(H)$ as the transfer function of H.

Assume that H is a regular system with impulse response h. In this case,

$$\begin{aligned} H(e^{st}) = e^{st}\lambda(H,s) &= h * e^{st} \\ &= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau \\ &= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \\ &= e^{st}\mathcal{L}(h,s), \end{aligned}$$

and so, $\lambda(H) = \mathcal{L}(h)$. That is, the transfer function of a regular system is precisely the Laplace transform of its impulse response. The region of convergence of the impulse response describes a set of complex exponential signals e^{st} in the domain of the system and we refer to this as the region of convergence of the *system*. In this way, both signals and systems have regions of convergence.

The transfer functions of the time-shifter and differentiator can be obtained by inspection. For the time-shifter

$$T_\tau(e^{st}) = e^{s(t-\tau)} = e^{-s\tau}e^{st} \qquad \text{and so} \qquad \lambda(T_\tau) = e^{-s\tau}. \tag{4.1.1}$$

The region of convergence is the whole complex plane $s \in \mathbb{C}$. For the special case of the identity system T_0 we obtain $\lambda(T_0, s) = 1$. For the differentiator

$$D(e^{st}) = \frac{d}{dt}e^{st} = se^{st} \qquad \text{and so} \qquad \lambda(D) = s.$$

The region of convergence is the whole complex plane $s \in \mathbb{C}$. More generally, for the kth differentiator

$$D^k(e^{st}) = \frac{d^k}{dt^k}e^{st} = s^k e^{st} \qquad \text{and so} \qquad \lambda(D^k) = s^k. \tag{4.1.2}$$

The region of convergence is again the whole complex plane. These results motivate assigning the following Laplace transforms to the delta "function" and its derivatives

$$\mathcal{L}(\delta) = 1, \qquad \mathcal{L}(\delta^k) = s^k.$$

These conventions are common in the literature [Oppenheim et al., 1996].

4.1.1 The transfer function of a linear combination of systems

Let $H = aH_1 + bH_2$ be a linear combination of systems H_1 and H_2. Let $R_1 \subseteq \mathbb{C}$ and $R_2 \subseteq \mathbb{C}$ be the regions of convergence of H_1 and H_2. We have,

$$\begin{aligned} H(e^{st}) &= aH_1(e^{st}) + bH_2(e^{st}) \\ &= a\lambda(H_1)e^{st} + b\lambda(H_2)e^{st} && s \in R_1 \cap R_2, \\ &= \big(a\lambda(H_1) + b\lambda(H_2)\big)e^{st} && s \in R_1 \cap R_2, \\ &= \lambda(H)e^{st} && s \in R_1 \cap R_2, \end{aligned}$$

and so,

$$\lambda(H) = a\lambda(H_1) + b\lambda(H_2) \qquad s \in R_1 \cap R_2.$$

That is, the transfer function of a linear combination of systems is the same linear combination of the transfer functions. The region of convergence of the linear combination is the intersection of the regions of convergence of the systems being combined.

4.1.2 The transfer function of a composition of systems

Let H be the system constructed by composing two systems H_1 and H_2 with regions of convergence R_1 and R_2, that is, $H(x) = H_1\big(H_2(x)\big)$. The response of H to the signal e^{st} is

$$\begin{aligned}
H(e^{st}) &= H_1\big(H_2(e^{st})\big) & \\
&= H_1\big(\lambda(H_2)e^{st}\big) & s \in R_2 \\
&= \lambda(H_2)H_1(e^{st}) & s \in R_2 \\
&= \lambda(H_2)\lambda(H_1)e^{st} & s \in R_1 \cap R_2 \\
&= \lambda(H)e^{st} & s \in R_1 \cap R_2,
\end{aligned}$$

and so,

$$\lambda(H) = \lambda(H_1)\lambda(H_2) \qquad s \in R_1 \cap R_2. \tag{4.1.3}$$

That is, the transfer function of a composition of linear time invariant systems is the multiplication of the transfer functions of those systems. The region of convergence of the composition is the intersection of the regions of convergence of the systems being composed.

4.1.3 The convolution theorem

We showed in Section 3.3 that if H_1 and H_2 are regular systems with impulse responses h_1 and h_2, then the impulse of the system $H(x) = H_1\big(H_2(x)\big)$ is given by the convolution $h = h_1 * h_2$. Because,

$$\lambda(H) = \mathcal{L}(h) \qquad \lambda(H_1) = \mathcal{L}(h_1) \qquad \lambda(H_1) = \mathcal{L}(h_1),$$

and using (4.1.3), we obtain,

$$\mathcal{L}(h_1 * h_2) = \mathcal{L}(h) = \lambda(H) = \lambda(H_1)\lambda(H_2) = \mathcal{L}(h_1)\mathcal{L}(h_2), \qquad s \in R_1 \cap R_2.$$

Putting $x = h_1$, $y = h_2$, $R_x = R_1$, and $R_y = R_2$ we obtain the **convolution theorem**,

$$\mathcal{L}(x * y) = \mathcal{L}(x)\mathcal{L}(y), \qquad s \in R_x \cap R_y. \tag{4.1.4}$$

In words: the Laplace transform of a convolution of signals is the multiplication of their Laplace transforms.

4.1.4 The Laplace transform of an output signal

Let H be a regular system with impulse response h and let $y = H(x) = h * x$ be the response of H to input signal x. Using the convolution theorem, the Laplace transform of the output signal y is

$$\mathcal{L}(y) = \mathcal{L}(h)\mathcal{L}(x) = \lambda(H)\mathcal{L}(x), \qquad s \in R \cap R_x, \tag{4.1.5}$$

where R is the region of convergence of the system H and R_x is the region of convergence of the input signal x. Thus, the Laplace transform of the output signal $y = H(x)$ is the transfer function of the system H multiplied by the Laplace transform of the input signal x. This result also holds when H is a time-shifter or a differentiator (Exercise 4.1.5).

4.2 Solving differential equations

Assume we have a system modelled by a differential equation of the form

$$\sum_{\ell=0}^{m} a_\ell D^\ell(x) = \sum_{\ell=0}^{k} b_\ell D^\ell(y), \tag{4.2.1}$$

where x and y are signals. Taking Laplace transforms of both sides of this equation,

$$\begin{aligned}
\mathcal{L}\left(\sum_{\ell=0}^{m} a_\ell D^\ell(x)\right) &= \mathcal{L}\left(\sum_{\ell=0}^{k} b_\ell D^\ell(y)\right) \\
\sum_{\ell=0}^{m} a_\ell \mathcal{L}\big(D^\ell(x)\big) &= \sum_{\ell=0}^{k} b_\ell \mathcal{L}\big(D^\ell(y)\big) && \text{(linearity (4.0.2))} \\
\sum_{\ell=0}^{m} a_\ell \lambda(D^\ell)\mathcal{L}(x) &= \sum_{\ell=0}^{k} b_\ell \lambda(D^\ell)\mathcal{L}(y) && \text{(using (4.1.5))} \\
\sum_{\ell=0}^{m} a_\ell s^\ell \mathcal{L}(x) &= \sum_{\ell=0}^{k} b_\ell s^\ell \mathcal{L}(y). && \text{(since } \lambda(D^\ell) = s^\ell \text{ by (4.1.2))}
\end{aligned}$$

We have obtained an equation relating the Laplace transforms of x and y,

$$\mathcal{L}(x)(a_0 + a_1 s + \dots a_m s^m) = \mathcal{L}(y)(b_0 + b_1 s + \dots b_k s^k).$$

Rearranging this equation we obtain

$$\mathcal{L}(y) = \frac{a_0 + a_1 s + \dots a_m s^m}{b_0 + b_1 s + \dots b_k s^k}\mathcal{L}(x).$$

Let H be a linear time invariant system such that $y = H(x)$ whenever x and y satisfy the differential equation (4.2.1). According to (4.1.5) the transfer function of H is

$$\lambda(H) = \frac{\mathcal{L}(y)}{\mathcal{L}(x)} = \frac{a_0 + a_1 s + \dots a_m s^m}{b_0 + b_1 s + \dots b_k s^k}.$$

Properties of H can be obtained by inspecting this transfer function. For example, the impulse response of H (if it exists) can be obtained by applying the inverse Laplace transform.

We now apply these results to the differential equations that model the RC electrical circuit from Figure 2.1 and the mass spring damper from Figure 2.2. The RC circuit is an example of what is called a **first order system** and the mass spring damper is an example of what is called a **second order system**.

4.3 First order systems

Recall the passive electrical RC circuit from Figure 2.1. The differential equation modelling this circuit is (2.0.1),

$$x = y + RCD(y),$$

where x is the input voltage signal, y is the voltage over the capacitor, and R and C are the resistance and capacitance. The RC circuit is an example of a **first order system**. Let H be a system mapping the input voltage signal x to the output voltage signal y. We will discover the impulse response of H. Taking the Laplace transform on both sides of the differential equation gives

$$\mathcal{L}(x) = (1 + RCs)\mathcal{L}(y),$$

and it follows that the transfer function of H is

$$\lambda(H) = \frac{\mathcal{L}(y)}{\mathcal{L}(x)} = \frac{1}{1 + RCs} = \frac{r}{r + s},$$

where $r = \frac{1}{RC}$. The value $\frac{1}{r} = RC$ is called the **time constant**. The impulse response of H is given by the inverse of this Laplace transform. There are two signals with Laplace transform $\frac{r}{r+s}$: the right sided signal $re^{-rt}u(t)$ with region of convergence $\operatorname{Re}(s) > -r$, and the left sided signal $-re^{-rt}u(-t)$ with region of convergence $\operatorname{Re}(s) < -r$. The RC circuit (and in fact all physically realisable systems) are expected to be causal. For this reason, the left sided signal $-re^{-rt}u(-t)$ cannot be the impulse response of H. The impulse response is the right sided signal

$$h(t) = re^{-rt}u(t).$$

Given an input voltage signal x we can now find the corresponding output signal $y = H(x)$ by convolving x with the impulse response h. That is,

$$y = H(x) = h * x = \int_{-\infty}^{\infty} re^{-r\tau} u(\tau) x(t-\tau) d\tau = r \int_{0}^{\infty} e^{-r\tau} x(t-\tau) d\tau.$$

If $r \geq 0$ the impulse response is absolutely integrable, that is,

$$\begin{aligned} \|h\|_1 &= \int_{-\infty}^{\infty} \left| re^{-rt} u(t) \right| dt \\ &= r \int_{0}^{\infty} e^{-rt} dt \\ &= 1 - \lim_{t \to \infty} e^{-rt} = 1, \end{aligned}$$

and the system is stable (Exercise 3.3). However, if $r < 0$ the impulse response is not absolutely integrable, and the system is not stable. Figure 4.3 shows the impulse response when $r = -\frac{1}{5}, -\frac{1}{3}, -\frac{1}{2}, 1, 2$. In a passive electrical RC circuit the resistance R and capacitance C are always positive and $r = \frac{1}{RC}$ is positive. For this reason, passive electrical RC circuits are always stable.

From (3.1.5), the step response $H(u)$ is given by applying the integrator I_∞ to the impulse response, that is,

$$H(u) = I_\infty(h) = \int_{-\infty}^{t} re^{-r\tau} u(\tau) d\tau = \begin{cases} r \int_0^t e^{-r\tau} d\tau & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

or more simply

$$H(u) = \left(1 - e^{-rt}\right) u(t). \tag{4.3.1}$$

This step response in plotted in Figure 4.3.

Test 5 (**The active RC circuit again**) In this test we again use the active RC circuit from Test 3 with resistors $R = R_1 = R_2 = 27\mathrm{k}\Omega$ and capacitors $C = C_2 = 10\mathrm{nF}$. In Test 3 we applied the differential equation (2.2.4) to the reconstructed output signal $\tilde{y}$ and asserted that the resulting signal was close to the reconstructed input signal $\tilde{x}$. In this test we instead convolve the input signal $\tilde{x}$ with the impulse response

$$h = -\tfrac{1}{RC} e^{-t/RC} = -re^{-rt}, \qquad r = \tfrac{1}{RC} = \frac{10^5}{27}$$

and assert that the resulting signal is close to the output signal $\tilde{y}$. That is, we test the expected relationship

$$\tilde{y} \approx h * \tilde{x} = -\int_{-\infty}^{\infty} h(\tau) u(\tau) \tilde{x}(t-\tau) d\tau = \int_{0}^{\infty} h(\tau) \tilde{x}(t-\tau) d\tau.$$

From (1.2.4),

$$\begin{aligned}\tilde{y}(t) &\approx \int_0^\infty h(\tau) \sum_{\ell=1}^{L} x_\ell \operatorname{sinc}(Ft - F\tau - \ell) d\tau \\ &= \sum_{\ell=1}^{L} x_\ell \int_0^\infty h(\tau) \operatorname{sinc}(Ft - F\tau - \ell) d\tau \\ &= \sum_{\ell=1}^{L} x_\ell f(Ft - \ell),\end{aligned}$$

where the function

$$f(t) = \int_0^\infty h(\tau) \operatorname{sinc}(t - F\tau) d\tau = -r \int_0^\infty e^{-r\tau} \operatorname{sinc}(t - F\tau) d\tau.$$

An approximation of $f(t)$ is made using the trapezoidal sum

$$f(t) \approx \frac{K}{2N} \left(g(0) + g(K) + 2 \sum_{n=1}^{N-1} g(\Delta n) \right),$$

where $g(\tau) = h(\tau) \operatorname{sinc}(t - F\tau)$, and

$$K = -RC \log\left(10^{-3}\right), \qquad N = \lceil 10FK \rceil, \qquad \Delta = K/N.$$

Figure 4.2 plots the input signal $\tilde{x}$, output signal $\tilde{y}$, and hypothesised output signal $h * \tilde{x}$ over a 4ms window.

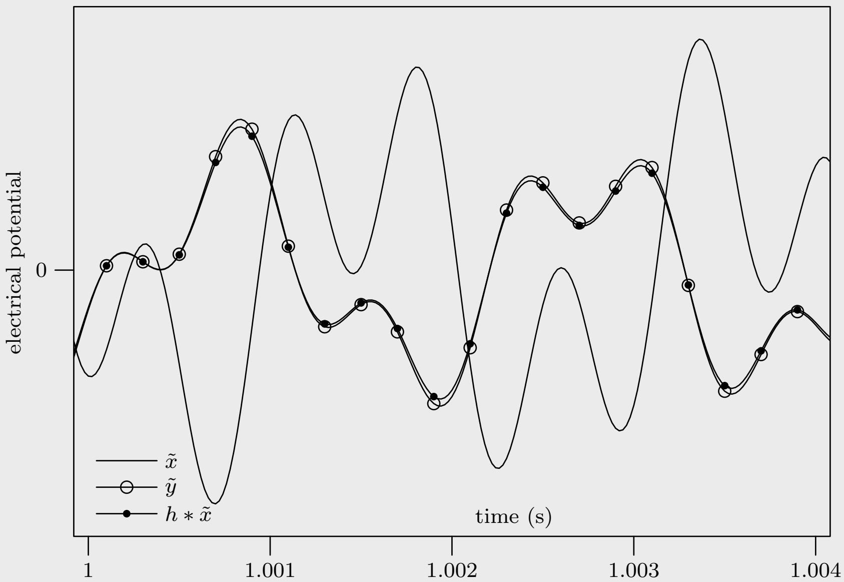

Figure 4.2: Plot of reconstructed input signal $\tilde{x}$ (solid line), output signal $\tilde{y}$ (solid line with circle), and hypothesised output signal $h * \tilde{x}$ (solid line with dot).

4.4 Second order systems

Consider the mass spring damper system from Figure 2.2 that is described by the equation

$$f = Kp + BD(p) + MD^2(p) \tag{4.4.1}$$

where f is the force applied to the mass M and p is the position of the mass and K and B are the spring and damping coefficients. The mass spring damper is an example of a **second order system**. Another example of a second order system is the Sallen-Key active electrical circuit depicted in Figure 2.10. In Section 2 we were able to find the force f corresponding with a given position signal p. Let H be a system mapping f to p, that is, such that $p = H(f)$. We will find the impulse response of H. Taking Laplace transforms on both sides of the differential equation gives

$$\mathcal{L}(f) = (K + Bs + Ms^2)\mathcal{L}(p).$$

Rearranging gives the transfer function of H,

$$\lambda(H) = \frac{\mathcal{L}(p)}{\mathcal{L}(f)} = \frac{1}{K + Bs + Ms^2}.$$

We can invert this Laplace transform to obtain the impulse response. There are three cases to consider depending on whether the quadratic $K+Bs+Ms^2$ has two distinct real roots, is irreducible (does not have real roots), or has two identical real roots.

Case 1: (Distinct real roots) In this case, the roots are

$$\beta - \alpha, \qquad -\beta - \alpha,$$

where

$$\alpha = \frac{B}{2M}, \qquad \beta = \frac{\sqrt{B^2 - 4KM}}{2M}$$

and $B^2 - 4KM > 0$. By a partial fraction expansion (Exercise 4.7),

$$\begin{aligned} \lambda(H) &= \frac{1}{M(s - \beta + \alpha)(s + \beta + \alpha)} \\ &= \frac{1}{2\beta M}\left(\frac{1}{s - \beta + \alpha} - \frac{1}{s + \beta + \alpha}\right). \end{aligned}$$

From (4.0.6) we obtain the transform pairs

$$\mathcal{L}(e^{(\beta-\alpha)t}u(t)) = \frac{1}{s - \beta + \alpha}, \qquad \mathcal{L}(e^{-(\beta+\alpha)t}u(t)) = \frac{1}{s + \beta + \alpha}.$$

As in Section 4.3, other signals with these Laplace transforms are discarded because they do not lead to an impulse response that is zero for $t < 0$. That

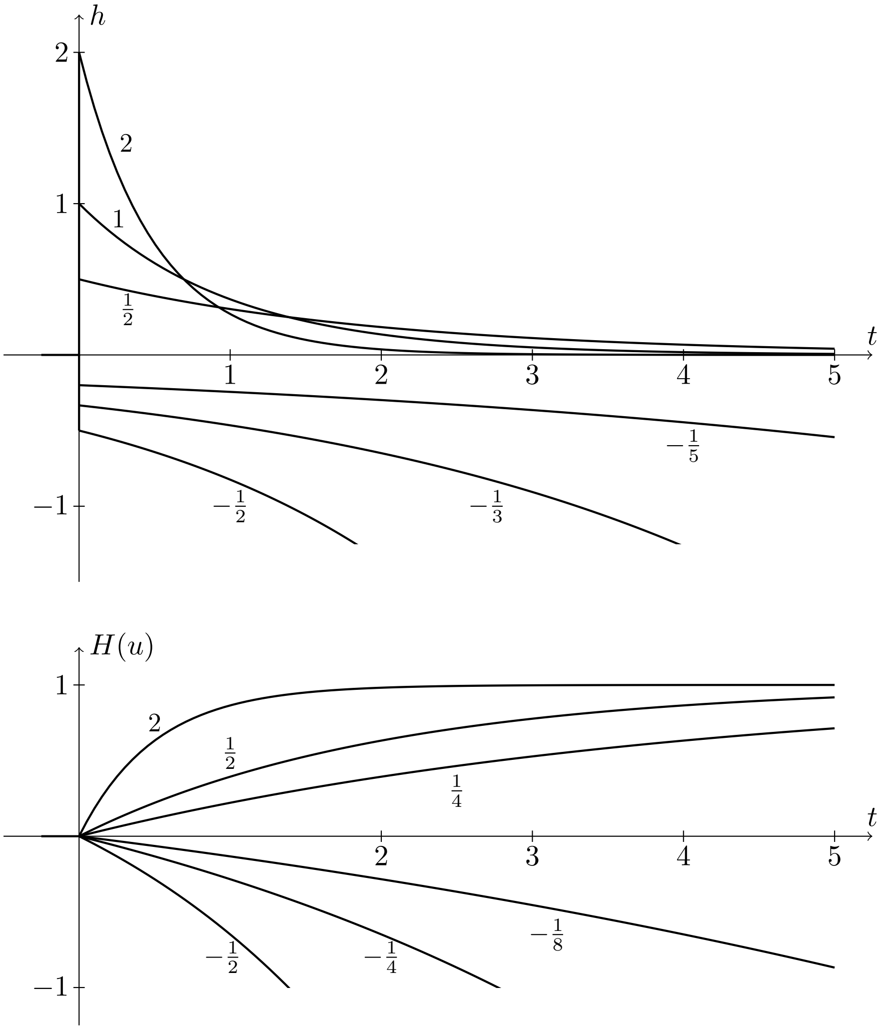

Figure 4.3: Top: impulse response of a first order system with $r = -\frac{1}{2}$, $-\frac{1}{3}$, $-\frac{1}{5}$, $\frac{1}{2}$, 1, 2. Bottom: step response of a first order system with $r = -\frac{1}{2}$, $-\frac{1}{4}$, $-\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$, 2.

is, they do not lead to a causal system H. The impulse response of H is thus

$$h(t) = \frac{1}{2\beta M} u(t) e^{-\alpha t}\big(e^{\beta t} - e^{-\beta t}\big).$$

This is a sum of the impulse responses of two first order systems.

Case 2: (Distinct imaginary roots) The solution is as in the previous case, but now $4KM - B^2 > 0$ and β is imaginary. Put $\theta = \beta/j$ so that

$$e^{\beta t} - e^{-\beta t} = e^{j\theta t} - e^{-j\theta t} = 2j\sin(\theta t).$$

The impulse response of H is

$$h(t) = \frac{1}{\theta M} u(t) e^{-\alpha t}\sin(\theta t).$$

Case 3: (Identical roots) In this case, the two roots are equal to $-\alpha$ and

$$\lambda(H) = \frac{1}{M(s+\alpha)^2}.$$

From (4.0.6) we obtain the transform pair

$$\mathcal{L}\big(te^{-\alpha t}u(t)\big) = \frac{1}{(s+\alpha)^2}$$

and this is the only signal with this Laplace transform that leads to a causal impulse response. The impulse response of H is thus

$$h(t) = \frac{1}{M} te^{-\alpha t}u(t).$$

A second order system is called **overdamped** when there are two distinct real roots, **underdamped** when their are two distinct imaginary roots, and **critically damped** when the roots are identical. The different types of impulse responses for are plotted in Figure 4.4.

With no damping (i.e. damping coefficient $B = 0$) the roots are of the form $\pm\beta$ and have no real part. In this case, the impulse response is

$$h(t) = \frac{1}{\theta M} u(t)\sin(\theta t),$$

where $\theta = \beta/j = \sqrt{KM}$ is called the **natural frequency** of the second order system. This impulse response oscillates for all $t > 0$ without decay or explosion. Two identical roots occur when the damping coefficient $B = \sqrt{4KM}$ and this is sometimes called the **critical damping coefficient**.

The impulse response of a second order system is absolutely integrable when $\alpha = \frac{B}{2M} > 0$, but not when $\alpha \leq 0$. Thus, the system is stable when

$\alpha > 0$ and not stable when $\alpha \leq 0$. For the mass spring damper both the mass M and damping coefficient B are positive and so mass spring dampers are always stable.

From (3.1.5) the step response $H(u)$ is given by applying the integrator I_∞ to the impulse response. There are three cases to consider depending on whether the system is overdamped, underdamped, or critically damped. When the system is overdamped the step response is

$$\begin{aligned} H(u) = I_\infty(h) &= \frac{1}{2\beta M}\int_{-\infty}^{t} e^{-\alpha\tau}\big(e^{\beta\tau} - e^{-\beta\tau}\big)u(\tau)d\tau \\ &= \frac{1}{2\beta M}\int_{0}^{t} e^{-\alpha\tau}\big(e^{\beta\tau} - e^{-\beta\tau}\big)d\tau \\ &= \frac{1}{2\beta M}u(t)\left(\frac{e^{(\beta-\alpha)t}-1}{\beta-\alpha} + \frac{e^{-(\beta+\alpha)t}-1}{\beta+\alpha}\right). \end{aligned}$$

When the system is underdamped the step response is

$$\begin{aligned} H(u) = I_\infty(h) &= \frac{1}{\theta M}\int_{0}^{t} e^{-\alpha\tau}\sin(\theta\tau)dt \\ &= u(t)\left(\frac{\theta - e^{-t\alpha}\big(\theta\cos(t\theta) + \alpha\sin(t\theta)\big)}{M\theta(\alpha^2+\theta^2)}\right). \end{aligned}$$

When the system is critically damped the step response is

$$\begin{aligned} H(u) = I_\infty(h) &= \frac{1}{\theta M}\int_{0}^{t} \frac{1}{M}te^{-\alpha t}dt \\ &= \frac{1}{M\alpha^2}u(t)\big(1 - e^{-t\alpha s}(1+t\alpha)\big). \end{aligned}$$

These step responses are plotted in Figure 4.5.

4.5 Poles, zeros, and stability

As discussed in Section 4.2 the transfer function of a system described by a linear differential equation with constant coefficients is of the form

$$\lambda(H) = \frac{\mathcal{L}(y)}{\mathcal{L}(x)} = \frac{a_0 + a_1 s + \dots a_m s^m}{b_0 + b_1 s + \dots b_k s^k}.$$

Factorising the polynomials on the numerator and denominator we obtain

$$\lambda(H) = C\frac{(s-\alpha_0)(s-\alpha_1)\cdots(s-\alpha_m)}{(s-\beta_0)(s-\beta_1)\cdots(s-\beta_k)},$$

where $\alpha_0, \dots, \alpha_m$ are the roots of the numerator polynomial $a_0 + a_1 s + \dots + a_m s^m$, and $\beta_0, \dots, \beta_k$ are the roots of the denominator polynomial

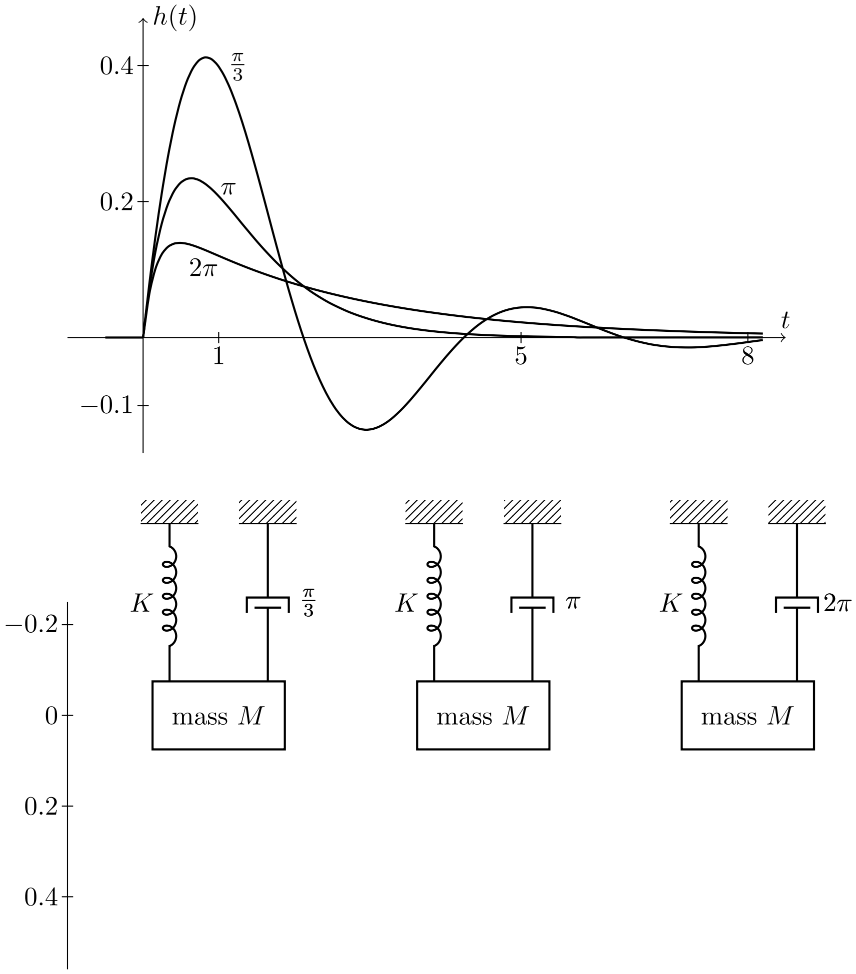

Figure 4.4: Impulse response of the mass spring damper with $M = 1$, $K = \frac{\pi^2}{4}$ and damping constant $B = \frac{\pi}{3}$ (underdamped), $B = \sqrt{4KM} = \pi$ (critically damped), and $B = 2\pi$ (overdamped).

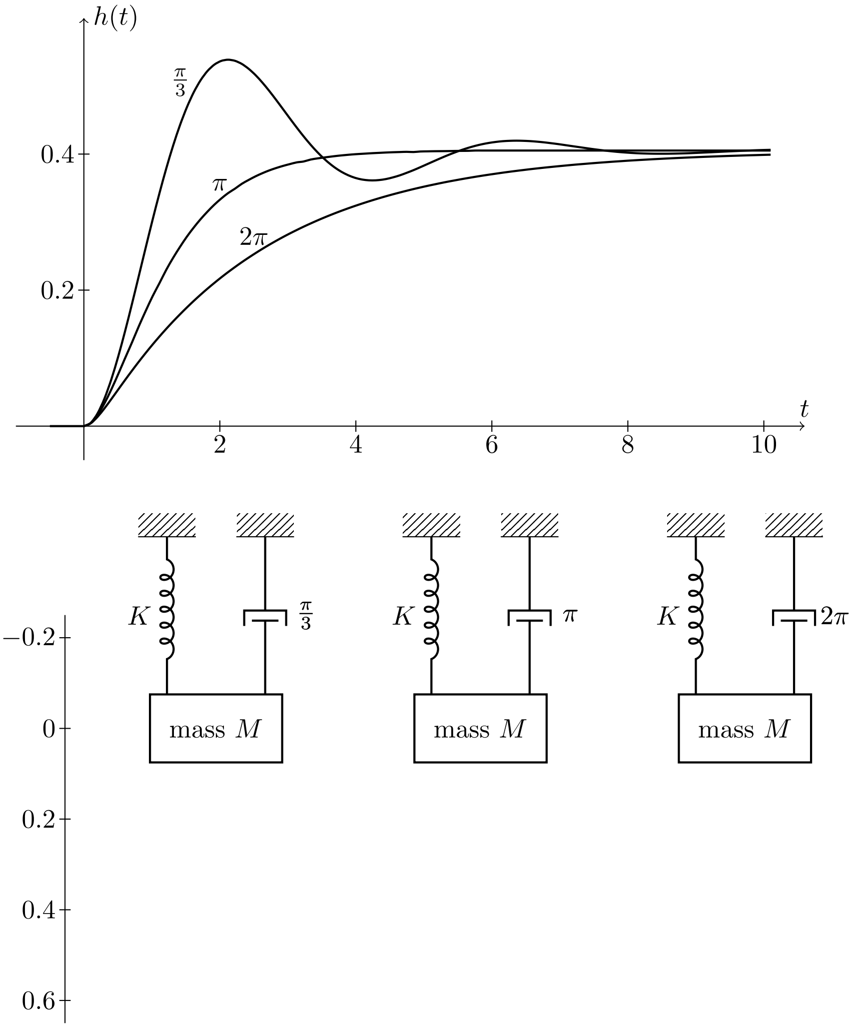

Figure 4.5: Step response of the mass spring damper with $M = 1$, $K = \frac{\pi^2}{4}$ and damping constant $B = \frac{\pi}{3}$ (underdamped), $B = \sqrt{4KM} = \pi$ (critically damped), and $B = 2\pi$ (overdamped).

$b_0+b_1s+\cdots+b_ks^k$, and $C = \frac{a_m}{b_m}$. That such a factorisation is always possible is called the **fundamental theorem of algebra** [Fine and Rosenberger, 1997]. If the numerator and denominator polynomials share one or more roots, then these roots cancel leaving the simpler expression

$$\lambda(H) = C\frac{(s-\alpha_d)(s-\alpha_1)\cdots(s-\alpha_m)}{(s-\beta_d)(s-\beta_1)\cdots(s-\beta_k)}, \tag{4.5.1}$$

where d is the number of shared roots, these shared roots being

$$\alpha_0 = \beta_0, \;\; \alpha_1 = \beta_1, \;\; \dots, \;\; \alpha_{d-1} = \beta_{d-1}.$$

The roots from the numerator $\alpha_d, \dots, \alpha_m$ are called the **zeros** and the roots from the denominator $\beta_d, \dots, \beta_m$ are called the **poles**. A **pole-zero plot** is constructed by marking the complex plane with a cross at the location of each pole and a circle at the location of each zero. Pole-zero plots for the first order system from Section 4.3, the second order system from Section 4.4, and the system describing the PID controller (2.2.7) are shown in Figure 4.6.

It is always possible to apply partial fractions and write (4.5.1) in the form

$$\lambda(H) = p(s) + \sum_{\ell\in K} \frac{A_\ell}{(s-\beta_\ell)^{r_\ell}},$$

where r_ℓ are positive integers, A_ℓ are complex constants, K is a subset of the indices from $\{d, d+1, \dots, k\}$, and $p(s)$ is a polynomial of degree $m-k$. If $k > m$ then $p(s) = 0$. The integer r_ℓ is called the **multiplicity** of the pole β_ℓ. We now restrict attention to the common case when the coefficients of the numerator polynomial $a_0, \dots, a_m$ and the coefficients of the denominator polynomial $b_0, \dots, b_k$ are real. In this case, the coefficients of the polynomial $p(s)$ are real, and the constant A_ℓ is real whenever the corresponding pole β_ℓ is real. If the pole β_ℓ has nonzero imaginary part there always exists another pole β_i such that $\beta_\ell = \beta_i^*$, where β_i^* is the **complex conjugate** of β_i. These poles have the same multiplicity, that is, $r_\ell = r_i$, and also the constants $A_\ell = A_i^*$. Stated another way: the complex poles occur in **conjugate pairs**.

We see that the transfer function contains the summation of two parts: the polynomial $p(s)$, and a sum of terms of the form $\frac{A}{(s-\beta)^r}$. Let $p(s) = \gamma_0 + \gamma_1 s + \cdots + \gamma_{k-m}s^{m-k}$. This polynomial is the transfer function of the nonregular system

$$H_1 = \gamma_0 T_0 + \gamma_1 D + \gamma_2 D^2 + \cdots + \gamma_{m-k}D^{m-k}.$$

This system is a linear combination of the identity system T_0 and differentiators of order at most $m-k$. From (4.0.6),

$$\mathcal{L}\left(\frac{A}{r!}t^{r-1}e^{\beta t}u(t)\right) = \frac{A}{(s-\beta)^r} \qquad \mathrm{Re}(s) > \mathrm{Re}(\beta)$$

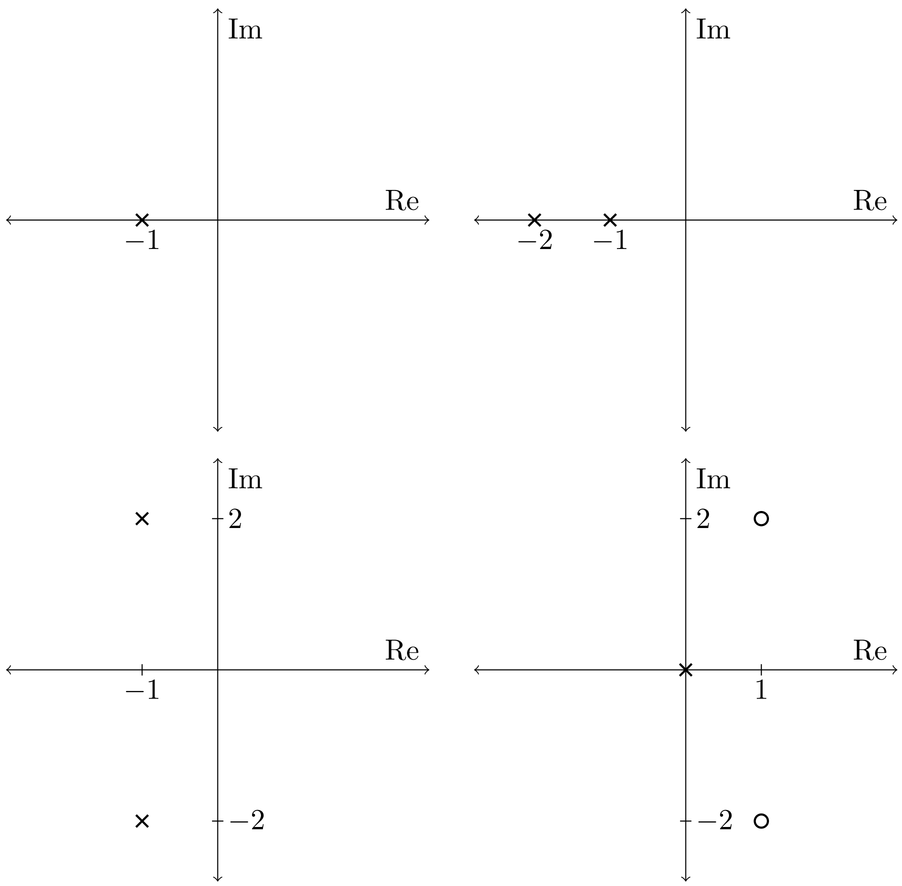

Figure 4.6: Top left: pole zero plot for the first order system $x = y + D(y)$. There is a single pole at -1. Top right: pole zero plot for the overdamped second order system $x = 2y + 3D(y) + D^2(y)$ that has two real poles at -1 and -2. Bottom left: pole zero plot for the underdamped second order system $x = 5y + 2D(y) + D^2(y)$ that has two imaginary poles at $-1 + 2j$ and $-1 - 2j$. The poles form a conjugate pair. Bottom right: pole zero plot for the equation $D(y) = 5x - 2D(x) + D^2(x)$ that models a PID controller (2.2.7). The system has a single pole at the origin and two zeros at $1 + 2j$ and $1 - 2j$.

and so the terms of the form $\frac{A}{(s-\beta)^r}$ correspond with the transfer function of a regular system with impulse response $\frac{A}{r!}t^{r-1}e^{\beta t}u(t)$. Other signals with Laplace transform $\frac{A}{(s-\beta)^r}$ are discarded because they do not correspond with the impulse response of a causal system. Thus, $\sum_{\ell \in K} \frac{A_\ell}{(s-\beta_\ell)^{r_\ell}}$ is the transfer function of the regular system H_2 with impulse response

$$h_2(t) = u(t) \sum_{\ell \in K} \frac{A_\ell}{r_\ell !} t^{r_\ell - 1} e^{\beta_\ell t}.$$

Let $K_r = \{\ell \in K \,;\, \operatorname{Im} \beta_\ell = 0\}$ be the indices from K corresponding with the real poles, and let $K_i = \{\ell \in K \,;\, \operatorname{Im} \beta_\ell > 0\}$ be the indices corresponding with those poles with positive imaginary part. Because the imaginary poles occur in conjugate pairs the impulse response h_2 can be written as

$$h_2(t) = u(t) \sum_{\ell \in K_r} \frac{A_\ell}{r_\ell !} t^{r_\ell - 1} e^{\beta_\ell t} + u(t) \sum_{\ell \in K_i} \frac{t^{r_\ell - 1}}{r_\ell !} \big(A_\ell e^{\beta_\ell t} + A_\ell^* e^{\beta_\ell^* t} \big).$$

The terms

$$\begin{aligned} A_\ell e^{\beta_\ell t} + A_\ell^* e^{\beta_\ell^* t} &= |A_\ell| \, e^{\operatorname{Re} \beta_\ell t} \big(e^{\operatorname{Im} \beta_\ell t + \angle A_\ell} + e^{-\operatorname{Im} \beta_\ell t - \angle A_\ell} \big) \\ &= 2 \, |A_\ell| \, e^{\operatorname{Re} \beta_\ell t} \cos \big(\operatorname{Im} \beta_\ell t + \angle A_\ell \big), \end{aligned}$$

and so, the impulse response is

$$h_2(t) = u(t) \sum_{\ell \in K_r} \frac{A_\ell}{r_\ell !} t^{r_\ell - 1} e^{\beta_\ell t} + u(t) \sum_{\ell \in K_i} \frac{2\,|A_\ell|}{r_\ell !} t^{r_\ell - 1} e^{\operatorname{Re} \beta_\ell t} \cos \big(\operatorname{Im} \beta_\ell t + \angle A_\ell \big).$$

This expression can be simplified by putting

$$B_\ell = \begin{cases} \frac{A_\ell}{r_\ell !} & \operatorname{Im} \beta_\ell = 0 \\ 2\frac{A_\ell}{r_\ell !} & \operatorname{Im} \beta_\ell > 0 \end{cases}$$

so that

$$h_2(t) = u(t) \sum_{\ell \in K_r \cup K_i} B_\ell t^{r_\ell - 1} e^{\operatorname{Re} \beta_\ell t} \cos \big(\operatorname{Im} \beta_\ell t + \angle B_\ell \big). \tag{4.5.2}$$

Observe that the impulse response is a real valued signal (as expected).

The system H mapping x to y is the sum of the regular system H_2 and nonregular system H_1, that is,

$$y = H(x) = H_1(x) + H_2(x).$$

Observe that H is regular only if the system $H_1 = 0$, that is, only if H_1 maps all input signals to the signal $x(t) = 0$ for all $t \in \mathbb{R}$. This occurs only when the polynomial $p(s) = 0$, that is, only when the number of poles exceeds the number of zeros. The system H will be stable if both H_1 and H_2 are stable. Because the differentiator D^ℓ is not stable (Exercise 1.7) the system H_1 is stable if only if the order of the polynomial $p(s)$ is zero, that is, if $p(s) = \gamma_0$ is a constant (potentially $\gamma_0 = 0$). In this case $H_1(x) = \gamma_0 T_0(x)$ is the identity system multiplied by a constant. The polynomial $p(s)$ is a constant only when the order of the denominator polynomial is greater than or equal to the order of the numerator polynomial, that is, when the number of poles is greater than or equal to the number of zeros. The regular system H_2 is stable if and only if its impulse response h_2 is absolutely integrable. This occurs only when the terms $e^{\mathrm{Re}\,\beta_\ell t}$ inside the sum (4.5.2) are decreasing as $t \to \infty$, that is, only if the real part of the poles $\mathrm{Re}\,\beta_\ell$ are negative. Thus, the system H_2 is stable if and only if the real part of the poles are strictly negative.

The stability of the system H can be immediately determined from its pole-zero plot. The system is stable if and only if:

1. the number of poles is greater than or equal to the number of zeros (there are at least as many crosses on the pole-zero plot as circles),
2. all of the poles (crosses) lie strictly in the left half plane.

The pole-zero plots in Figure 4.6 all represent stable systems with the exception of the plot on the bottom right (a PID controller). This system has two zeros and only one pole. The single pole is contained on the imaginary axis. It is not strictly in the left half plane.

4.5.1 Two masses, a spring, and a damper

Consider the system involving two masses, a spring, and a damper in Figure 2.11. From (2.3.3), the equation relating the force applied to the first mass f and the position of the second mass p is

$$f = BD(p) + (M_1 + M_2)D^2(p) + \frac{BM_2}{K}D^3(p) + \frac{M_1M_2}{K}D^4(p),$$

where B is the damping coefficient, K is the spring constant, and M_1 and M_2 are the masses. Taking Laplace transforms

$$\mathcal{L}(f) = s\left(B + (M_1 + M_2)s + \frac{BM_2}{K}s^2 + \frac{M_1M_2}{K}s^3\right)\mathcal{L}(p),$$

from which, we obtain the transfer function of a system H that maps f to p,

$$\lambda(H) = \frac{\mathcal{L}(p)}{\mathcal{L}(f)} = \frac{1}{s\left(B + (M_1 + M_2)s + \frac{BM_2}{K}s^2 + \frac{M_1M_2}{K}s^3\right)}.$$

The system has no zeros and 4 poles. One of these poles always exists at the origin. The system is not stable because this pole is not strictly in the left half of the complex plane.

Consider the specific case when $B = K = M_1 = M_2 = 1$. Factorising the denominator polynomial gives

$$\lambda(H) = \frac{1}{s(s-\beta_1)(s-\beta_2)(s-\beta_2^*)},$$

where

$$\beta_1 = \frac{1}{3}\left(\gamma - \frac{5}{\gamma} - 1\right) \approx -0.56984,$$

$$\beta_2 = \frac{1}{6}\left(\frac{5(1+j\sqrt{3})}{\gamma} - (1-j\sqrt{3})\gamma - \frac{1}{2}\right) \approx -0.21508 + 1.30714j,$$

and $\gamma = \left(\frac{3\sqrt{69}-11}{2}\right)^{1/3}$. Applying partial fractions (Exercise 4.8) gives

$$\lambda(H) = \frac{1}{s(s-\beta_1)(s-\beta_2)(s-\beta_2^*)} = \frac{A_0}{s} + \frac{A_1}{s-\beta_1} + \frac{A_2}{s-\beta_2} + \frac{A_2^*}{s-\beta_2^*},$$

where

$$A_0 = -\frac{1}{\beta_1|\beta_2|^2} = 1, \qquad A_1 = \frac{1}{\beta_1|\beta_1-\beta_2|^2} \approx -0.956611,$$

$$A_2 = \frac{1}{\beta_2(\beta_2-\beta_1)(\beta_2-\beta_2^*)} \approx -0.0216944 + 0.212084j.$$

From (4.5.2), the impulse response of H is

$$h(t) = u(t)\big(A_0 + A_1 e^{\beta_1 t} + 2\,|A_2|\, e^{\mathrm{Re}\,\beta_2 t}\cos(\mathrm{Im}\,\beta_2 t + \angle A_2)\big).$$

This impulse response is plotted in Figure 4.7. Observe that h is not absolutely integrable and the system is not stable. The impulse response $h(t)$ does not converge to zero as $t \to \infty$ and correspondingly the mass M_2 does not come come to rest at position zero in Figure 4.7. In the figure it is assumed that the spring is at equilibrium when the two masses are $d = 1$ apart. From (2.3.1), the position of mass M_1 is given by the signal $p_1 = g - d$ where $g = h + M_2 D^2(h)$.

4.5.2 Direct current motors

Recall the direct current (DC) motor from Figure 2.13 described by the differential equation from (2.4.1),

$$v = \left(\frac{RB}{K_\tau} + K_b\right) D(\theta) + \frac{RJ}{K_\tau} D^2(\theta),$$

where v is the input voltage signal and θ is a signal representing the angle of the motor. The constants R, B, K_τ, K_b, and J are related to components of

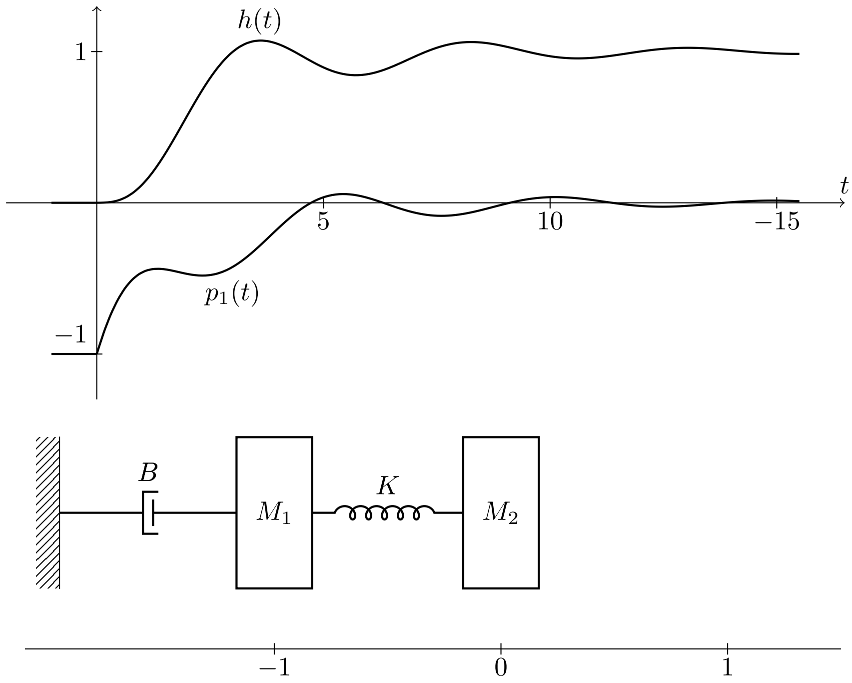

Figure 4.7: Impulse response of the system with two masses, a spring, and a damper, where $B = K = M_1 = M_2 = 1$.

the motor as described in Section 2.4. To simplify the differential equation put $a = \frac{RB}{K_\tau} + K_b$ and $b = \frac{RJ}{K_\tau}$ and the equation becomes

$$v = aD(\theta) + bD^2(\theta).$$

Taking Laplace transforms on both sides of this equation gives the transfer function of a system H that maps input voltage v to motor angle θ,

$$\lambda(H) = \frac{1}{s(a + bs)}.$$

This system has no zeros and two poles. One pole is at $-\frac{a}{b}$ and the other is at the origin. The system is not stable because the pole at the origin is not strictly in the left half of the complex plane.

Applying partial fractions we find that

$$\lambda(H) = \frac{1}{as} - \frac{1}{a(s - \beta)}, \tag{4.5.3}$$

where $\beta = -\frac{a}{b}$. Using (4.0.6), the impulse response of H is

$$h(t) = \frac{1}{a}u(t)\big(1 - e^{\beta t}\big). \tag{4.5.4}$$

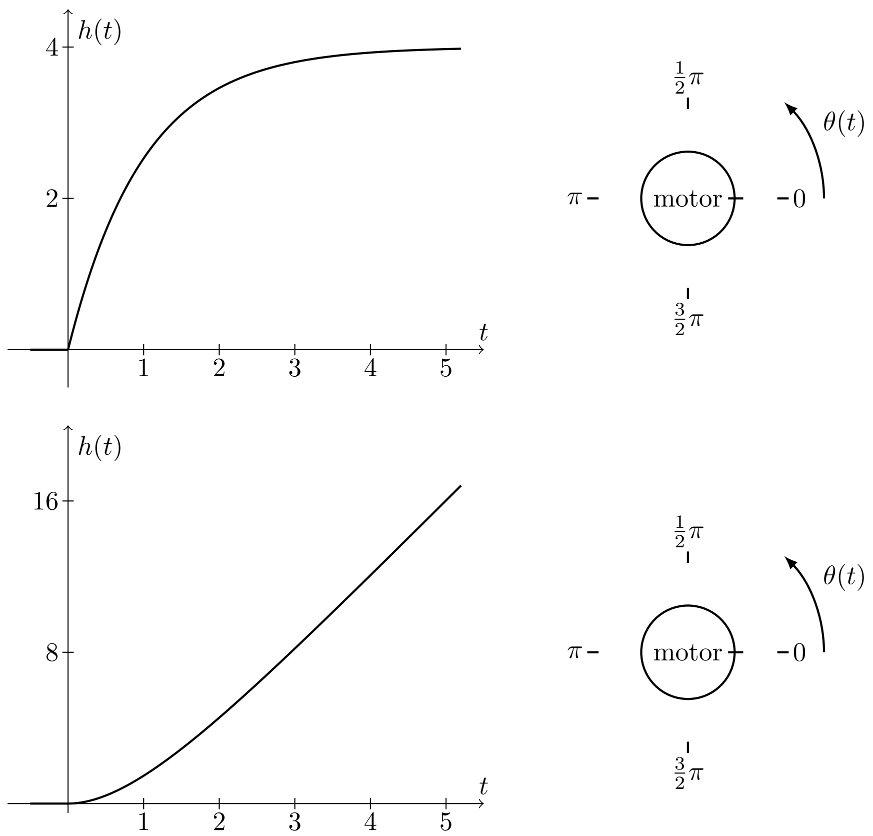

Figure 4.8: Impulse response (top) and step response (bottom) of a DC motor with constants $K_b = \frac{1}{4}$, $K_\tau = 8$ and $B = R = J = 1$.

Other signals with Laplace transform (4.5.3) are discarded because they do not lead to a causal system. The step response $H(u)$ is obtained by applying the integrator system I_∞ to the impulse response, that is

$$H(u) = I_\infty(h) = \frac{1}{a\beta} u(t)\big(\beta t + e^{\beta t} - 1\big).$$

The impulse response and step response are plotted in Figure 4.8 when $K_b = \frac{1}{8}$, $K_\tau = 8$ and $B = R = 1$ and $J = 2$ so that $a = \frac{1}{4}$, $b = \frac{1}{4}$ and $\beta = -1$.

4.6 Exercises

4.1. Sketch the signal

$$x(t) = e^{-2t}u(t) + e^{t}u(-t)$$

where $u(t)$ is the step function. Find the Laplace transform of $x(t)$ and the corresponding region of convergence (ROC). Sketch the region of convergence on the complex plane.

4.2. Find the Laplace transform of the signal $t^n u(t)$ where $n \geq 0$ is an integer.

4.3. Let $n \geq 0$ be an integer. Show that the Laplace transform of the signal $t^n u(-t)$ is the same as the Laplace transform of the signal $t^n u(t)$, but with a different region of convergence.

4.4. Show that equation (4.1.5) on page 51 holds when the system H is the differentiator D^k or the time shifter T_τ.

4.5. What is the transfer function of the integrator system I_∞ and what is its region of convergence?

4.6. By partial fractions, or otherwise, assert that

$$\frac{as}{s+b} = a - \frac{ab}{s+b}$$

4.7. By partial fractions, or otherwise, assert that

$$\frac{s+c}{(s+a)(s+b)} = \frac{a-c}{(a-b)(s+a)} + \frac{c-b}{(a-b)(s+b)}$$

4.8. By partial fractions, or otherwise, assert that

$$\frac{1}{s(s-a)(s-b)(s-b^*)} = \frac{A_0}{s} + \frac{A_1}{s-a} + \frac{A_2}{s-b} + \frac{A_2^*}{s-b^*}$$

where $a, b \in \mathbb{C}$ and $\mathrm{Re}(b) \neq 0$ and

$$A_0 = -\frac{1}{a|b|^2}, \qquad A_1 = \frac{1}{a|a-b|^2}, \qquad A_2 = \frac{1}{b(b-a)(b-b^*)}.$$

You might wish to check your solution using a symbolic programming language (for example Sage, Mathematica, or Maple).

4.9. Let

$$\mathcal{L}(y) = \frac{2s+1}{s^2+s-2}$$

be the Laplace transform of a signal y. By partial fractions, or otherwise, find all possible signals y and their regions of convergence.

4.10. Let x be a signal with region of convergence R. Show that the time scaled signal $x(\alpha t)$ with $\alpha \neq 0$ satisfies equation (4.0.7) on page 48.

4.11. Consider the active electrical circuit from Figure 2.8 described be the differential equation from (2.2.3). Derive the transfer function of this system. Find an explicit system H that maps the input voltage x to the output voltage y. State whether this system is stable and/or regular.

4.12. Given the mass spring damper system described by (4.4.1), find the position signal p given that the force signal

$$f(t) = \Pi\big(t - \tfrac{1}{2}\big) = \begin{cases} 1 & 0 < t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

is the rectangular function time shifted by $\frac{1}{2}$. Consider three cases:

(a) $M = 1$, $K = \frac{\pi^2}{4}$ and $B = \frac{\pi}{3}$,

(b) $M = 1$, $K = \frac{\pi^2}{4}$ and $B = \pi$,

(c) $M = 1$, $K = \frac{\pi^2}{4}$ and $B = 2\pi$,

Plot the solution in each case, and comment on whether the system is underdamped, overdamped, or critically damped.

Bibliography

Fine, B. and Rosenberger, G. [1997]. *The Fundamental Theorem of Algebra.* Undergraduate Texts in Mathematics. Spring-Verlag, Berlin.

Nicholas, C. B. and Yates, R. C. [1950]. The Probability Integral. *Amer. Math. Monthly*, 57, 412–413.

Nise, N. S. [2007]. *Control systems engineering.* Wiley, 5th edition.

Oppenheim, A. V., Willsky, A. S. and Nawab, S. H. [1996]. *Signals and Systems.* Prentice Hall, 2nd edition.

Sallen, R. and Key, E. [1955]. A practical method of designing RC active filters. *Circuit Theory, IRE Transactions on*, 2(1), 74–85.

Stewart, I. and Tall, D. O. [2004]. *Complex Analysis.* Cambridge University Press.

Zemanian, A. H. [1965]. *Distribution theory and transform analysis.* Dover books on mathematics. Dover.