

# Testable linear shift-invariant systems

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# Chapter 1

## Signals and systems

It is assumed that the reader is familiar with the concept of a function! That is, a map from the elements in a set  $X$  to the elements in another set  $Y$ . Consider sets

$$X = \left\{ \begin{array}{l} \text{Mario} \\ \text{Link} \\ \text{Ness} \end{array} \right\} \quad Y = \left\{ \begin{array}{l} \text{Freeman} \\ \text{Ryu} \\ \text{Sephiroth} \\ \text{Conker} \\ \text{Ness} \end{array} \right\}.$$

An example of a function from  $X$  to  $Y$  is

$$f(x) = \begin{cases} \text{Conker} & x = \text{Mario} \\ \text{Sephiroth} & x = \text{Link} \\ \text{Sephiroth} & x = \text{Ness}. \end{cases}$$

The function  $f$  maps Mario to Conker, Link to Sephiroth, and Ness to Sephiroth. The value of  $f$  for input  $x$  is denoted by  $f(x)$  and so, for example,  $f(\text{Mario}) = \text{Conker}$  and  $f(\text{Link}) = \text{Sephiroth}$ . The set  $X$  is called the **domain** of the function  $f$  and the set of values that the functions takes, that is, the set  $\{f(x), x \in X\}$ , is called the **range**. In the above example the range is the set  $\{\text{Conker}, \text{Sephiroth}\}$ . Observe that the range is a subset of  $Y$ . The set of all functions mapping  $X$  to  $Y$  is denoted by  $X \rightarrow Y$  and so  $f \in X \rightarrow Y$  in the example above.

A **signal** is a function that maps a real number to a complex number, that is, a function from the set  $\mathbb{R} \rightarrow \mathbb{C}$ . For example

$$\sin(\pi t), \quad \frac{1}{2}t^3, \quad e^{-t^2}$$

all represent signals with  $t \in \mathbb{R}$ . The real part of these signals is plotted in Figure 1.1. Many physical phenomena, such as sound, light, weather,

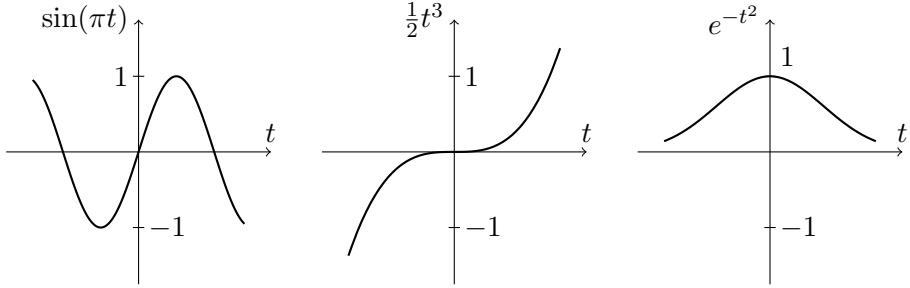


Figure 1.1: Plots of three signals.

and motion, can be modelled using signals. In this text we primarily focus on examples from electrical and mechanical engineering where signals are used to model changes in quantities such as voltage, current, position, angle, force, and torque, over time. In these examples, the independent variable  $t$  represents “time”. However, there is no fundamental reason for this and the techniques developed here can be applied equally well when  $t$  represents a quantity other than time. An example where this occurs is image processing.

## 1.1 Properties of signals

A signal  $x$  is **bounded** if there exists a real number  $M$  such that

$$|x(t)| < M \quad \text{for all } t \in \mathbb{R}$$

where  $|\cdot|$  denotes the (complex) magnitude. Both  $\sin(\pi t)$  and  $e^{-t^2}$  are examples of bounded signals because  $|\sin(\pi t)| \leq 1$  and  $|e^{-t^2}| \leq 1$  for all  $t \in \mathbb{R}$ . However,  $\frac{1}{2}t^3$  is not bounded because its magnitude grows indefinitely as  $t$  moves away from the origin.

A signal  $x$  is **periodic** if there exists a positive real number  $T$  such that

$$x(t) = x(t + kT) \quad \text{for all } k \in \mathbb{Z} \text{ and } t \in \mathbb{R}.$$

The smallest such positive  $T$  it is called the **fundamental period** or simply the **period**. For example, the signal  $\sin(\pi t)$  is periodic with period  $T = 2$ . Neither  $\frac{1}{2}t^3$  or  $e^{-t^2}$  are periodic.

A signal  $x$  is **right sided** if there exists  $T \in \mathbb{R}$  such that  $x(t) = 0$  for all  $t < T$ . Correspondingly  $x$  is **left sided** if  $x(t) = 0$  for all  $T > t$ . For example, the **step function**

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \quad (1.1.1)$$

is right-sided. Its horizontal reflection  $u(-t)$  is left sided (Figure 1.2). A signal  $x$  is said to be **finite** or to have **finite support** if it is both left and

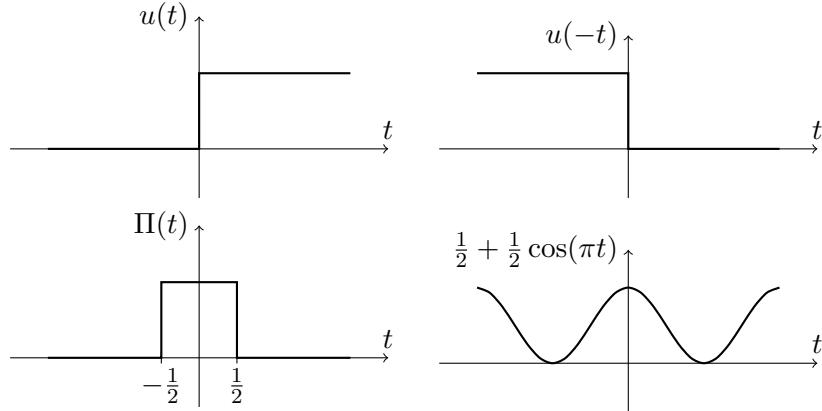


Figure 1.2: The right sided step function  $u(t)$ , its left sided reflection  $u(-t)$ , the finite rectangular pulse  $\Pi(t)$  and the signal  $\frac{1}{2} + \frac{1}{2} \cos(\pi t)$  that is not finite.

right sided, that is, if there exists  $T \in \mathbb{R}$  such that  $x(t) = x(-t) = 0$  for all  $t > T$ . The signals  $\sin(\pi t)$  and  $e^{-t^2}$  do not have finite support, but the **rectangular pulse**

$$\Pi(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad (1.1.2)$$

does.

A signal  $x$  is **even** (or **symmetric**) if

$$x(t) = x(-t) \quad \text{for all } t \in \mathbb{R}$$

and **odd** (or **antisymmetric**) if

$$x(t) = -x(-t) \quad \text{for all } t \in \mathbb{R}.$$

For example,  $\sin(\pi t)$  and  $\frac{1}{2}t^3$  are odd and  $e^{-t^2}$  is even. A signal  $x$  is **conjugate symmetric** if

$$x(t) = x(-t)^* \quad \text{for all } t \in \mathbb{R}$$

and **conjugate antisymmetric** if

$$x(t) = -x(-t)^* \quad \text{for all } t \in \mathbb{R},$$

where  $*$  denotes the complex conjugate of a complex number. Equivalently,  $x$  is conjugate symmetric if its real part  $\operatorname{Re} x$  is an even signal and its imaginary part  $\operatorname{Im} x$  is an odd signal, and  $x$  is conjugate antisymmetric if its real part is odd and its imaginary part is even. For example, the signal  $e^{-t^2} + j \sin(\pi t)$  where  $j = \sqrt{-1}$  is conjugate symmetric and the signal  $\frac{1}{2}t^3 + je^{-t^2}$  is conjugate antisymmetric.

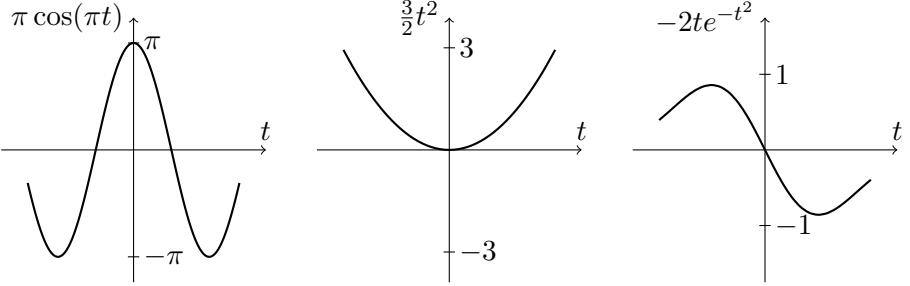


Figure 1.3: Derivatives of the signals  $\sin(\pi t)$ ,  $\frac{1}{2}t^3$ ,  $e^{-t^2}$  from Figure 1.1.

A signal  $x$  is **continuous** at  $t \in \mathbb{R}$  if

$$\lim_{h \rightarrow 0} x(t + h) = \lim_{h \rightarrow 0} x(t - h)$$

and  $x$  is said to be **continuous** if it is continuous at all  $t \in \mathbb{R}$ . The signals  $\sin(\pi t)$ ,  $\frac{1}{2}t^3$ , and  $e^{-t^2}$  are continuous, but the step function  $u$  is not continuous at zero because

$$\lim_{h \rightarrow 0} u(h) = 1 \neq 0 = \lim_{h \rightarrow 0} u(-h).$$

The set of continuous signals is typically denoted by  $C^0(\mathbb{R})$  or just  $C^0$ . A signal  $x$  is **continuously differentiable** or just **differentiable** if

$$\lim_{h \rightarrow 0} \frac{x(t + h) - x(t)}{h} = \lim_{h \rightarrow 0} \frac{x(t) - x(t - h)}{h} \quad \text{for all } t \in \mathbb{R}.$$

Considered as a function of  $t$  this limit is called the **derivative** of  $x$  at  $t$  and is typically denoted by  $\frac{d}{dt}x(t)$ . For example, the signals  $\sin(\pi t)$ ,  $\frac{1}{2}t^3$ ,  $e^{-t^2}$ , and  $t^2$  are differentiable with derivatives

$$\pi \cos(\pi t), \quad \frac{3}{2}t^2, \quad -2te^{-t^2}, \quad 2t,$$

but the step function  $u$  and the rectangular pulse  $\Pi$  are not differentiable (Exercise 1.7). The set of differentiable signals is denoted by  $C^1$  or  $C^1(\mathbb{R})$ . A signal is  **$k$ -times differentiable** if its  $k-1$  th derivative is differentiable. The set of  $k$ -times differentiable signals is typically denoted by  $C^k$  or  $C^k(\mathbb{R})$ .

A signal  $x$  is **locally integrable** if

$$\int_a^b |x(t)| dt < \infty$$

for all finite constants  $a$  and  $b$ , where  $< \infty$  means that the integral evaluates finite complex number. The signals  $\sin(\pi t)$ ,  $\frac{1}{2}t^3$ , and  $e^{-t^2}$  are all locally integrable. An example of a signal that is not locally integrable is  $x(t) = \frac{1}{t}$

(Exercise 1.3). The set of locally integrable signals is typically denoted by  $L_{\text{loc}}$  or  $L_{\text{loc}}(\mathbb{R})$ .

A signal  $x$  is **absolutely integrable** or **Lebesgue integrable** if

$$\|x\|_1 = \int_{-\infty}^{\infty} |x(t)| dt < \infty. \quad (1.1.3)$$

Here we introduce the notation  $\|x\|_1$  called the  **$L^1$ -norm** of  $x$ . For example  $\sin(\pi t)$  and  $\frac{1}{2}t^3$  are not absolutely integrable, but  $e^{-t^2}$  is because [Nicholas and Yates, 1950]

$$\int_{-\infty}^{\infty} |e^{-t^2}| dt = \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}. \quad (1.1.4)$$

It is common to denote the set of absolutely integrable signals by  $L^1$  or  $L^1(\mathbb{R})$ . So,  $e^{-t^2} \in L^1$  and  $\frac{1}{2}t^3 \notin L^1$ . A signal  $x$  is **square integrable** if

$$\|x\|_2^2 = \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty.$$

The real number  $\|x\|_2$  is called the  **$L^2$ -norm** of  $x$ . Square integrable signals are also called **energy signals** and the squared  $L^2$ -norm  $\|x\|_2^2$  is called the **energy** of  $x$ . For example,  $\sin(\pi t)$  and  $\frac{1}{2}t^3$  are not energy signals, but  $e^{-t^2}$  is. It has energy  $\|e^{-t^2}\|_2^2 = \sqrt{\pi/2}$  (Exercise 1.6). The set of square integrable signals is denoted by  $L^2$  or  $L^2(\mathbb{R})$ .

We write  $x = y$  to indicate that two signals  $x$  and  $y$  are **equal pointwise**, that is,  $x(t) = y(t)$  for all  $t \in \mathbb{R}$ . This definition of equality is often stronger than we desire. For example, the step function  $u$  and the signal

$$z(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}$$

are not equal pointwise because they are not equal at  $t = 0$  since  $u(0) = 1$  and  $z(0) = 0$ . It is useful to identify signals that differ only at isolated points and for this we use a weaker definition of equality. We say that two signals  $x$  and  $y$  are equal **almost everywhere** if

$$\int_a^b |x(t) - y(t)| dt = 0$$

for all finite constants  $a$  and  $b$ . So, in the previous example, while  $u \neq z$  pointwise we do have  $u = z$  almost everywhere. Typically the term almost everywhere is abbreviated to a.e. and one writes

$$x = y \text{ a.e.} \quad \text{or} \quad x(t) = y(t) \text{ a.e.}$$

to indicate that the signals  $x$  and  $y$  are equal almost everywhere.

## 1.2 Spaces of signals

We will regularly be interested in subsets of the set of all signals  $\mathbb{R} \rightarrow \mathbb{C}$ . Two important families of subsets are the **linear spaces** and the **shift-invariant spaces**.

Let  $x$  and  $y$  be signals. We denote by  $x+y$  the signal that takes the value  $x(t) + y(t)$  for each  $t \in \mathbb{R}$ , that is, the signal that results from adding  $x$  and  $y$ . For  $a \in \mathbb{C}$  we denote by  $ax$  the signal that takes the value  $ax(t)$  for each  $t \in \mathbb{R}$ , that is, the signal that results from multiplying  $x$  by  $a$  (Figure 1.4). For signals  $x$  and  $y$  and complex numbers  $a$  and  $b$  the signal

$$ax + by$$

is called a **linear combination** of  $x$  and  $y$ .

Let  $X \subseteq \mathbb{R} \rightarrow \mathbb{C}$  be a set of signals. The set  $X$  is a **linear space** (or **vector space**) if for all signals  $x$  and  $y$  from  $X$  and all complex numbers  $a$  and  $b$  the linear combination  $ax + by$  is also in  $X$ . The set of all signals  $\mathbb{R} \rightarrow \mathbb{C}$  is a linear space. Another example is the set of differentiable signals, because, if  $x$  and  $y$  are differentiable, then the linear combination  $ax + by$  is differentiable. The derivative is  $aDx + bDy$ . The set of even signals is another example of a linear space because if  $x$  and  $y$  are even then

$$ax(t) + by(t) = ax(-t) + by(-t)$$

and so the linear combination  $ax + by$  is even. The set of locally integrable signals  $L_{\text{loc}}$ , the set of absolutely integrable signals  $L^1$ , and the set of square integrable signals  $L^2$  are linear spaces (Exercise 1.9). The set of periodic signals is not a linear space (Exercise 1.10).

For a real number  $\tau$  the signal  $x(t - \tau)$  is called a **time-shift** or **shift** or sometimes **translation** of the signal  $x(t)$ . Figure 1.5 depicts the shift  $x(t - \tau)$  for different values of  $\tau$  in the case that  $x(t) = e^{-t^2}$ . A set of signals  $X \subseteq \mathbb{R} \rightarrow \mathbb{C}$  is a **shift-invariant space** if for all  $x \in X$  and all  $\tau \in \mathbb{R}$  the shift  $x(t - \tau)$  is also in  $X$ . Examples of shift-invariant spaces are the set of differentiable signals, the set of periodic signals (Exercise 1.10), and  $L_{\text{loc}}$ ,  $L^1$ , and  $L^2$  (1.9). The set of even signals and the set of odd signals are not shift-invariant spaces.

Linear spaces and shift-invariant spaces of signals will act as domains for the next type of function that we consider called **systems**.

## 1.3 Systems (functions of signals)

A **system** is a function that maps a signal to another signal. For example,

$$x(t) + 3x(t - 1), \quad \int_0^1 x(t - \tau) d\tau, \quad \frac{1}{x(t)}, \quad \frac{d}{dt} x(t)$$

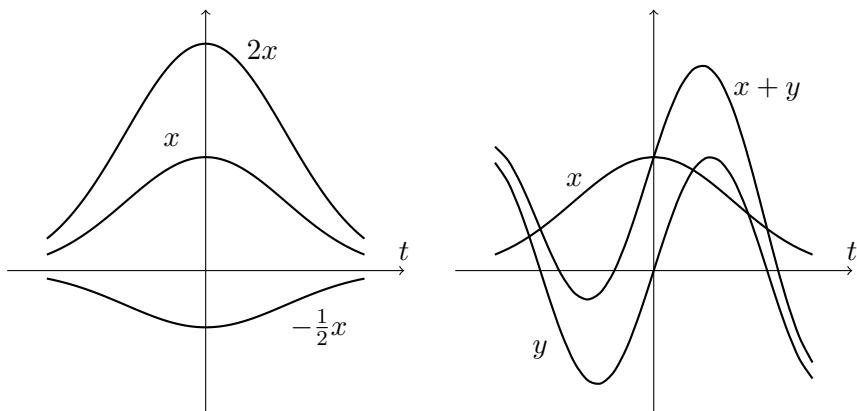


Figure 1.4: The signal  $x(t) = e^{-t^2}$  and the signals  $2x$  and  $-\frac{1}{2}x$  (left). The signals  $x(t) = e^{-t^2}$  and  $y(t) = \sin(\pi t)$  and the signal  $x + y$ .

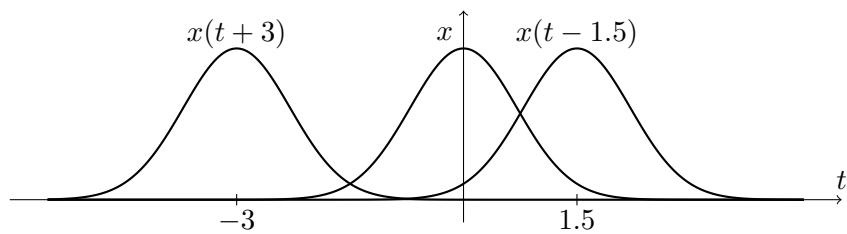


Figure 1.5: The signal  $x(t) = e^{-t^2}$  and shift  $x(t - \tau) = e^{-(t-\tau)^2}$  for  $\tau = 1.5$  and  $-3$ .

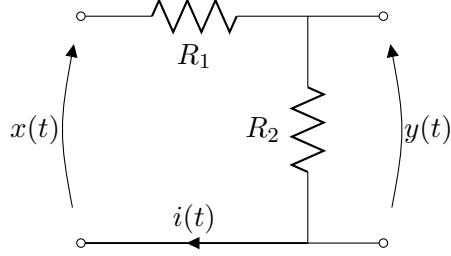
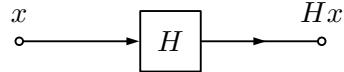
Figure 1.6: A **voltage divider** circuit.

Figure 1.7: System block diagram with input signal \$x\$ and output signal \$H(x)\$.

represent systems, each mapping the signal \$x\$ to another signal. Consider the electric circuit in Figure 1.6 called a **voltage divider**. If the voltage at time \$t\$ is \$x(t)\$ then, by Ohm's law, the current at time \$t\$ satisfies

$$i(t) = \frac{1}{R_1 + R_2} x(t),$$

and the voltage over the resistor \$R\_2\$ is

$$y(t) = R_2 i(t) = \frac{R_2}{R_1 + R_2} x(t). \quad (1.3.1)$$

The circuit can be considered as a system mapping the signal \$x\$ representing the voltage to the signal \$i = \frac{1}{R\_1 + R\_2} x\$ representing the current, or a system mapping \$x\$ to the signal \$y = \frac{R\_2}{R\_1 + R\_2} x\$ representing the voltage over resistor \$R\_2\$.

Let \$X \subseteq \mathbb{R} \rightarrow \mathbb{C}\$ and \$Y \subseteq \mathbb{R} \rightarrow \mathbb{C}\$ be sets of signals. We denote systems with capital letters such as \$H\$ and \$G\$. A system is a function \$H \in X \rightarrow Y\$ that maps each signal from the domain \$X\$ to a signal from \$Y\$. Given **input signal** \$x \in X\$ the **output signal** of the system is denoted by \$H(x)\$. The output signal is often called the **response** of system \$H\$ to signal \$x\$. We will often drop the brackets and write simply \$Hx\$ for the response of \$H\$ to \$x\$<sup>1</sup>. The value of the output signal \$Hx\$ at \$t \in \mathbb{R}\$ is denoted by \$Hx(t)\$ or \$H(x)(t)\$ or \$H(x, t)\$ and we do not distinguish between these notations. It is sometimes useful to depict systems with a block diagram as in Figure 1.7. The electric circuit in Figure 1.6 corresponds with the system

$$Hx = \frac{R_2}{R_1 + R_2} x = y.$$

---

<sup>1</sup>In the literature it is customary to drop the brackets only when \$H\$ is a **linear** system (Section 1.5). In this text we occasionally drop the brackets even when \$H\$ is not linear. Since we deal primarily with linear systems this faux pas will occur rarely.

This system multiplies the input signal  $x$  by  $\frac{R_2}{R_1+R_2}$ . This brings us to our first practical test.

**Test 1 (Voltage divider)** In this test we construct the voltage divider from Figure 1.6 on a breadboard with resistors  $R_1 \approx 100\Omega$  and  $R_2 \approx 470\Omega$  with values accurate to within 5%. Using a computer soundcard (an approximation of) the voltage signal

$$x(t) = \sin(2\pi f_1 t) \quad \text{with} \quad f_1 = 100$$

is passed through the circuit. The approximation is generated by sampling  $x(t)$  at rate  $F = \frac{1}{P} = 44100\text{Hz}$  to generate samples

$$x(nP) \quad n = 0, \dots, 2F$$

corresponding to approximately 2 seconds of signal. These samples are passed to the soundcard which starts playback. The voltage over resistor  $R_2$  is recorded (also using the soundcard) that returns a list of samples  $y_1, \dots, y_L$  taken at rate  $F$ . The voltage over  $R_2$  can be (approximately) reconstructed from these samples as

$$\tilde{y}(t) = \sum_{\ell=1}^L y_\ell \operatorname{sinc}(Ft - \ell) \quad (1.3.2)$$

where

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \quad (1.3.3)$$

is the called the **sinc function** and is plotted in Figure 5.1. We will justify this reconstruction in Section 5.4. Simultaneously the (stereo) soundcard is used to record the input voltage  $x$  producing samples  $x_1, \dots, x_L$  taken at rate  $F$ . An approximation of the input signal is

$$\tilde{x}(t) = \sum_{\ell=1}^L x_\ell \operatorname{sinc}(Ft - \ell). \quad (1.3.4)$$

In view of (1.3.1) we would expect the approximate relationship

$$\tilde{y} \approx \frac{R_2}{R_1 + R_2} \tilde{x} = \frac{47}{57} \tilde{x}.$$

A plot of  $\tilde{y}$ ,  $\tilde{x}$  and  $\frac{47}{57}\tilde{x}$  over a 20ms period from 1s to 1.02s is given in Figure 1.8. The hypothesised output signal  $\frac{47}{57}\tilde{x}$  does not match the observed output signal  $\tilde{y}$ . A primary reason is that the circuitry inside the soundcard itself cannot be ignored. When deriving the equation for the voltage divider

we implicitly assumed that current flows through the output of the soundcard without resistance (a short circuit), and that no current flows through the input device of the soundcard (an open circuit). These assumptions are not realistic. Modelling the circuitry in the sound card wont be attempted here. In Section 2.2 we will construct circuits that contain external sources of power (active circuits). These are less sensitive to the circuitry inside the soundcard.

When specifying a system we are free to choose the domain  $X$  at our convenience. In cases such as the voltage divider it is reasonable to choose the domain  $X = \mathbb{R} \rightarrow \mathbb{C}$ , that is, the domain can contain *all* signals. However, this is not always convenient or possible. For example, the system

$$Hx(t) = \frac{1}{x(t)}$$

is not defined at those  $t$  where  $x(t) = 0$  because we cannot divide by zero. To avoid this we might choose the domain as the set of signals  $x(t)$  that are not zero for any  $t \in \mathbb{R}$ .

Another example is the system  $I_\infty$  defined by

$$I_\infty x(t) = \int_{-\infty}^t x(\tau) d\tau, \quad (1.3.5)$$

called an **integrator**. The signal  $x(t) = 1$  cannot be input to the integrator because the integral  $\int_{-\infty}^t dt$  is not finite for any  $t$ . However, the integrator  $I_\infty$  can operate on absolutely integrable signals because, if  $x$  is absolutely integrable, then

$$I_\infty x(t) = \int_{-\infty}^t x(\tau) d\tau \leq \int_{-\infty}^t |x(\tau)| d\tau < \int_{-\infty}^\infty |x(\tau)| d\tau = \|x\|_1 < \infty$$

for all  $t \in \mathbb{R}$ . We might then choose a domain for  $I_\infty$  as the set of absolutely integrable signals  $L^1$ . The integrator can also be applied to signals that are right sided and locally integrable because, for any right sided signal  $x$  there exists  $T \in \mathbb{R}$  such that  $x(t) = 0$  for all  $t < T$  and so,

$$I_\infty x(t) = \int_{-\infty}^t x(\tau) d\tau = \int_T^t x(\tau) d\tau < \infty$$

for all  $t \in \mathbb{R}$  if  $x$  is locally integrable. So another possible domain for  $I_\infty$  is the set of right sided locally integrable signals. A final possible domain is the subset of locally integrable signals for which  $\int_{-\infty}^0 |x(t)| dt$  is finite. This last example will be the domain we usually choose for the integrator  $I_\infty$ .

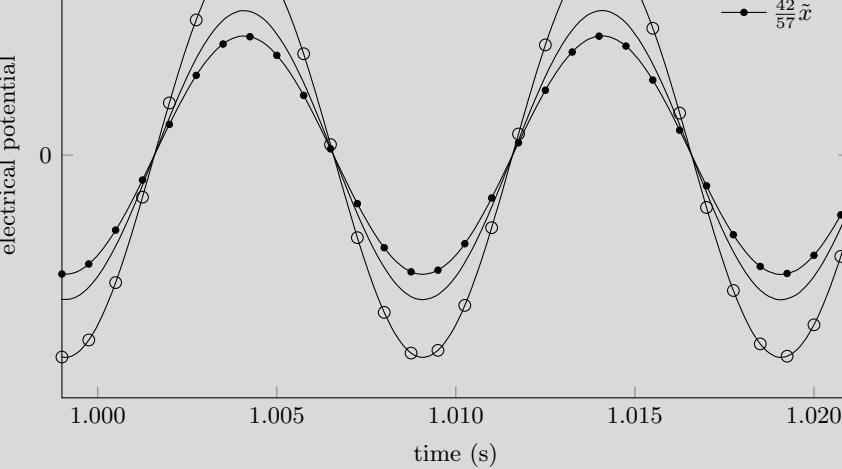


Figure 1.8: Plot of reconstructed input signal  $\tilde{x}$  (solid line), output signal  $\tilde{y}$  (solid line with circle) and hypothesised output signal  $\frac{42}{57}\tilde{x}$  (solid line with dot) for the voltage divider circuit in Figure 1.6. The hypothesised signal does not match  $\tilde{y}$ . One reason is that the model does not take account of the circuitry inside the soundcard.

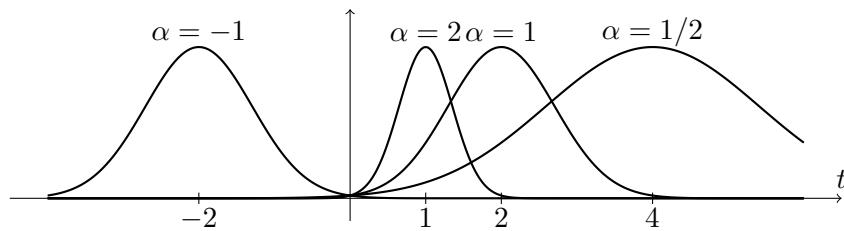


Figure 1.9: Time-scaler system  $Hx(t) = x(\alpha t)$  for  $\alpha = -1, \frac{1}{2}, 1$  and  $2$  acting on the signal  $x(t) = e^{-(t-2)^2}$ .

## 1.4 Some important systems

The system

$$T_\tau x(t) = x(t - \tau)$$

is called a **time-shifter** or simply **shifter**. This system shifts the input signal along the  $t$  axis (“time” axis) by  $\tau$ . When  $\tau$  is positive  $T_\tau$  delays the input signal by  $\tau$ . The shifter will appear so regularly that we use the special notation  $T_\tau$  to represent it. Figure 1.5 depicts the action of shifters  $T_{1.5}$  and  $T_{-3}$  on the signal  $x(t) = e^{-t^2}$ . When  $\tau = 0$  the shifter is the **identity system**  $T_0x = x$  that maps a signal to itself. Another important system is the **time-scaler** that has the form

$$Hx(t) = x(\alpha t), \quad \alpha \in \mathbb{R}.$$

Figure 1.9 depicts the action of time-scalers with different values for  $\alpha$ . When  $\alpha = -1$  the time-scaler reflects the input signal in the  $t$  axis. When  $\alpha = 1$  the time-scaler is the identity system  $T_0$ . Both the shifter and time-scaler are well defined for all signals and so it is reasonable to choose their domains to be the entire set of signals  $\mathbb{R} \rightarrow \mathbb{C}$ . We always assume this is the case unless otherwise stated.

Another regularly encountered system is the **differentiator**

$$Dx(t) = \frac{d}{dt}x(t)$$

that returns the derivative of the input signal. We also define a  $k$ th differentiator

$$D^k x(t) = \frac{d^k}{dt^k}x(t)$$

that returns the  $k$ th derivative of the input signal. The differentiator is only defined for differentiable signals. A domain for  $D$  is the set of differentiable signals  $C^1$  and a domain for  $D^k$  is the set of  $k$ -times differentiable signals  $C^k$ . Unless otherwise stated we will always assume the domain of  $D^k$  to be  $C^k$ .

Another important system is the **integrator**

$$I_a x(t) = \int_{-a}^t x(\tau) d\tau.$$

The parameter  $a$  describes the lower bound of the integral. In this course it will often be that  $a = \infty$ . For example, the response of the integrator  $I_\infty$  to the signal  $tu(t)$  where  $u$  is the step function (1.1.1) is

$$\int_{-\infty}^t \tau u(\tau) d\tau = \begin{cases} \int_0^t \tau d\tau = \frac{t^2}{2} & t > 0 \\ 0 & t \leq 0. \end{cases}$$

Observe that the integrator  $I_\infty$  cannot be applied to the signal  $x(t) = t$  because  $\int_{-\infty}^t \tau d\tau$  is not finite for any  $t$ . A domain for  $I_\infty$  cannot contain the signal  $x(t) = t$ . Unless otherwise stated we will assume the domain of  $I_\infty$  to be the subset of locally integrable signals  $L_{\text{loc}}$  for which  $\int_{-\infty}^0 |x(t)| dt < \infty$  (Exercise 1.15). For finite  $a < \infty$  we will assume, unless otherwise stated, that the domain of  $I_a$  is the set of locally integrable signals  $L_{\text{loc}}$ .

## 1.5 Properties of systems

In this section we define a number of important properties that systems can possess. In what follows  $X \subseteq \mathbb{R} \rightarrow \mathbb{C}$  and  $Y \subseteq \mathbb{R} \rightarrow \mathbb{C}$  are sets of signals. A system  $H \in X \rightarrow Y$  with domain  $X$  is called **memoryless** if, for all input signals  $x \in X$ , the output signal  $Hx$  at time  $t$  depends only on  $x$  at time  $t$ . For example  $\frac{1}{x(t)}$  with domain the set of signals that do not take the value zero and the identity system  $T_0$  with domain the set of all signals  $\mathbb{R} \rightarrow \mathbb{C}$  are memoryless, but

$$x(t) + 3x(t-1) \quad \text{and} \quad \int_0^1 x(t-\tau) d\tau \quad x \in L_{\text{loc}}$$

are not. A shifter  $T_\tau$  with  $\tau \neq 0$  is not memoryless.

A system  $H \in X \rightarrow Y$  is **causal** if, for all input signals  $x \in X$ , the output signal  $Hx$  at time  $t$  depends on  $x$  at times less than or equal to  $t$ . Memoryless systems such as  $\frac{1}{x(t)}$  and  $T_0$  are also causal. The shifter  $T_\tau$  is causal when  $\tau \geq 0$ , but is not causal when  $\tau < 0$ . The systems

$$x(t) + 3x(t-1) \quad \text{and} \quad \int_0^1 x(t-\tau) d\tau \quad x \in L_{\text{loc}}$$

are causal, but the systems

$$x(t) + 3x(t+1) \quad \text{and} \quad \int_0^1 x(t+\tau) d\tau \quad x \in L_{\text{loc}}$$

are not causal.

A system  $H \in X \rightarrow Y$  is called **bounded-input-bounded-output (BIBO) stable** or just **stable** if the output signal  $Hx$  is bounded whenever the input signal  $x$  is bounded. That is,  $H$  is stable if for every positive real number  $M$  there exists a positive real number  $K$  such that for all input signals  $x \in X$  bounded below  $M$ , that is,

$$|x(t)| < M \quad \text{for all } t \in \mathbb{R},$$

it holds that the output signal  $Hx$  is bounded below  $K$ , that is,

$$|Hx(t)| < K \quad \text{for all } t \in \mathbb{R}.$$

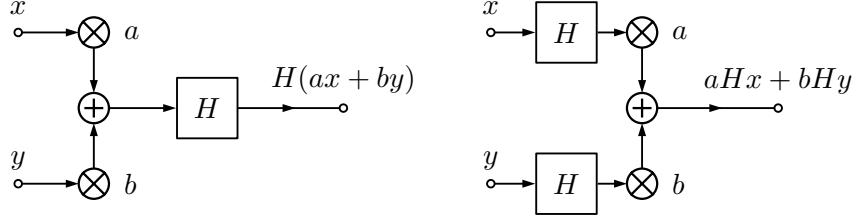


Figure 1.10: If  $H$  is a linear system the outputs of these two diagrams are the same signal, i.e.  $H(ax + by) = aHx + bHy$ .

For example, the system  $x(t) + 3x(t - 1)$  with domain  $\mathbb{R} \rightarrow \mathbb{C}$  is stable with  $K = 4M$  since if  $|x(t)| < M$ , then

$$|x(t) + 3x(t - 1)| \leq |x(t)| + 3|x(t - 1)| < 4M = K.$$

The integrator  $I_a$  for  $a \in \mathbb{R}$  having domain  $L_{\text{loc}}$  and the differentiator  $D$  with domain  $C^1$  are not stable (Exercises 1.16 and 1.17).

Let  $H \in X \rightarrow Y$  be a system with both the domain  $X$  and  $Y$  being linear spaces of signals. The system  $H$  is **linear** if

$$H(ax + by) = aHx + bHy$$

for all signals  $x, y \in X$  and all complex numbers  $a$  and  $b$ . That is, a linear system has the property: if the input consists of a weighted sum of signals, then the output consists of the same weighted sum of the responses of the system to those signals. Figure 1.10 indicates the linearity property using a block diagram. For example, the differentiator is linear because

$$D(ax + by)(t) = \frac{d}{dt}(ax(t) + by(t)) = a\frac{d}{dt}x(t) + b\frac{d}{dt}y(t) = aDx(t) + bDy(t)$$

whenever both  $x$  and  $y$  are differentiable, that is, when  $x, y \in C^1$ . However, the system  $Hx(t) = \frac{1}{x(t)}$  is not linear because

$$H(ax + by)(t) = \frac{1}{ax(t) + by(t)} \neq \frac{a}{x(t)} + \frac{b}{y(t)} = aHx(t) + bHy(t)$$

in general.

Let  $H \in X \rightarrow Y$  be a system with both the domain  $X$  and  $Y$  being shift-invariant spaces. The system  $H$  is **shift-invariant** (or **time-invariant**) if

$$HT_\tau x(t) = Hx(t - \tau)$$

for all signals  $x \in X$  and all shifts  $\tau \in \mathbb{R}$ . That is, a system is shift-invariant if shifting the input signal results in the same shift of the output signal. Equivalently,  $H$  is shift-invariant if it commutes with the shifter  $T_\tau$ , that is, if

$$HT_\tau x = T_\tau Hx$$

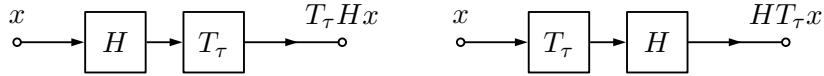


Figure 1.11: If  $H$  is a shift-invariant system the outputs of these two diagrams are the same signal, i.e.  $HT_\tau x = T_\tau Hx$ .

for all  $\tau \in \mathbb{R}$  and all signals  $x \in X$ . Figure 1.11 represents the property of shift-invariance with a block diagram. For example, the differentiator is shift-invariant because

$$DT_\tau x(t) = \frac{d}{dt}x(t - \tau) = T_\tau Dx(t)$$

by the chain rule for differentiation. The integrator  $I_\infty$  is shift-invariant (Exercise 1.20) but the integrator  $I_a$  for finite  $a < \infty$  is not (Exercise 1.19).

We will be primarily interested in systems that are both linear and shift-invariant. Such systems are said to be **linear shift-invariant** or **linear time-invariant** systems. The phrase “Let  $H \in X \rightarrow Y$  be a linear shift-invariant system” will occur regularly and it will always imply that both sets  $X$  and  $Y$  are linear and shift-invariant spaces of signals.

## Exercises

- 1.1. How many distinct functions from the set  $X = \{\text{Mario, Link}\}$  to the set  $Y = \{\text{Freeman, Ryu, Sephiroth}\}$  exist? Write down each function, that is, write down all functions from the set  $X \rightarrow Y$ .
- 1.2. State whether the step function  $u(t)$  is bounded, periodic, absolutely integrable, an energy signal.
- 1.3. Show that the signal  $t^2$  is locally integrable, but that the signal  $\frac{1}{t^2}$  is not.
- 1.4. Plot the signal

$$x(t) = \begin{cases} \frac{1}{t+1} & t > 0 \\ \frac{1}{t-1} & t \leq 0. \end{cases}$$

State whether it is: bounded, locally integrable, absolutely integrable, square integrable.

- 1.5. Plot the signal

$$x(t) = \begin{cases} \frac{1}{\sqrt{t}} & 0 < t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that  $x$  is absolutely integrable, but not square integrable.

- 1.6. Compute the energy of the signal  $e^{-\alpha^2 t^2}$  (Hint: use equation (1.1.4) on page 5 and a change of variables).
- 1.7. Show that the signal  $t^2$  is differentiable, but the step function  $u$  and rectangular pulse  $\Pi$  are not.
- 1.8. Plot the signal  $\sin(t) + \sin(\pi t)$ . Show that this signal is not periodic.
- 1.9. Show that the set of locally integrable signals  $L_{loc}$ , the set of absolutely integrable signals  $L^1$ , and the set of square integrable signals  $L^2$  are linear shift-invariant spaces.
- 1.10. Show that the set of periodic signals is a shift-invariant space, but not a linear space.
- 1.11. Show that the set of bounded signals is a linear shift-invariant space.
- 1.12. Let  $K > 0$  be a fixed real number. Show that the set of signals bounded below  $K$  is a shift invariant space, but not a linear space.
- 1.13. Show that the set of even signals and the set of odd signals are not shift invariant spaces.
- 1.14. Show that the integrator  $I_c$  with finite  $c \in \mathbb{R}$  is not stable.
- 1.15. Show that if the signal  $x$  is locally integrable and  $\int_{-\infty}^0 |x(t)| dt < \infty$  then  $I_\infty x(t) = \int_{-\infty}^t x(t) dt < \infty$  for all  $t \in \mathbb{R}$ .
- 1.16. Show that the integrator  $I_\infty$  is not stable.
- 1.17. Show that the differentiator system  $D$  is not stable.
- 1.18. Show that the shifter  $T_\tau$  is linear and shift-invariant and that the time-scaler is linear, but not time invariant.
- 1.19. Show that the integrator  $I_c$  with finite  $c \in \mathbb{R}$  is linear, but not shift-invariant.
- 1.20. Show that the integrator  $I_\infty$  is linear and shift-invariant.
- 1.21. State whether the system  $Hx = x + 1$  is linear, shift-invariant, stable.
- 1.22. State whether the system  $Hx = 0$  is linear, shift-invariant, stable.
- 1.23. State whether the system  $Hx = 1$  is linear, shift-invariant, stable.
- 1.24. Let  $x$  be a signal with period  $T$  that is not equal to zero almost everywhere. Show that  $x$  is neither absolutely integrable nor square integrable.

## Chapter 2

# Systems modelled by differential equations

Systems of particular interest are those where the input signal  $x$  and output signal  $y$  are related by a linear differential equation with constant coefficients, that is, an equation of the form

$$\sum_{\ell=0}^m a_\ell \frac{d^\ell}{dt^\ell} x(t) = \sum_{\ell=0}^k b_\ell \frac{d^\ell}{dt^\ell} y(t),$$

where  $a_0, \dots, a_m$  and  $b_0, \dots, b_k$  are real or complex numbers. In what follows we use the differentiator system  $D$  rather than the notation  $\frac{d}{dt}$  to represent differentiation. To represent the  $\ell$ th derivative we write  $D^\ell$  instead of  $\frac{d^\ell}{dt^\ell}$ . Using this notation the differential equation above is

$$\sum_{\ell=0}^m a_\ell D^\ell x = \sum_{\ell=0}^k b_\ell D^\ell y. \quad (2.0.1)$$

Equations of this form can be used to model a large number of electrical, mechanical and other real world devices. For example, consider the resistor and capacitor (RC) circuit in Figure 2.1. Let the signal  $v_R$  represent the voltage over the resistor and  $i$  the current through both resistor and capacitor. The voltage signals satisfy

$$x = y + v_R,$$

and the current satisfies both

$$v_R = Ri \quad \text{and} \quad i = CDy.$$

Combining these equations,

$$x = y + RCDy \quad (2.0.2)$$

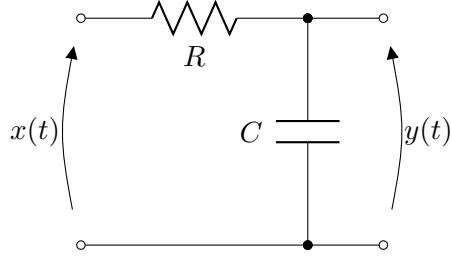


Figure 2.1: An electrical circuit with resistor and capacitor in series, otherwise known as an **RC circuit**.

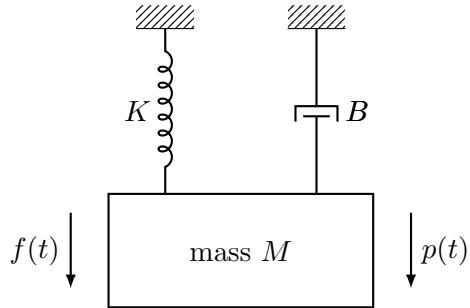


Figure 2.2: A mechanical mass-spring-damper system

that is in the form of (2.0.1).

As another example, consider the mass-spring-damper in Figure 2.2. A force represented by the signal  $f$  is externally applied to the mass, and the position of the mass is represented by the signal  $p$ . The spring exerts force  $-Kp$  that is proportional to the position of the mass, and the damper exerts force  $-BDp$  that is proportional to the velocity of the mass. The cumulative force exerted on the mass is

$$f_m = f - Kp - BDp$$

and by Newton's law the acceleration of the mass  $D^2p$  satisfies

$$MD^2p = f_m = f - Kp - BDp.$$

We obtain the differential equation

$$f = Kp + BDp + MD^2p \quad (2.0.3)$$

that is in the form of (2.0.1) if we put  $x = f$  and  $y = p$ . Given  $p$  we can readily solve for the corresponding force  $f$ . As a concrete example, let the spring constant, damping constant and mass be  $K = B = M = 1$ . If the position satisfies  $p(t) = e^{-t^2}$ , then the corresponding force satisfies

$$f(t) = e^{-t^2}(4t^2 - 2t - 1).$$

Figure 2.3: A solution to the mass-spring-damper system with  $K = B = M = 1$ . The position is  $p(t) = e^{-t^2}$  with corresponding force  $f(t) = e^{-t^2}(4t^2 - 2t - 1)$ .

Figure 2.3 depicts these signals.

What happens if a particular force signal  $f$  is applied to the mass? For example, say we apply the force

$$f(t) = \Pi(t - \frac{1}{2}) = \begin{cases} 1 & 0 < t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the corresponding position signal  $p$ ? We are not yet ready to answer this question, but will be later (Exercise 4.13).

In both the mechanical mass-spring-damper system in Figure 2.2 and the electrical RC circuit in Figure 2.1 we obtain a differential equation relating the input signal  $x$  with the output signal  $y$ . The equations do not specify the output signal  $y$  explicitly in terms of the input signal  $x$ , that is, they do not explicitly define a system  $H$  such  $y = Hx$ . As they are, the differential equations do not provide as much information about the behaviour of the system as we would like. For example, is the system stable? Much more information about these systems will be obtained when the **Laplace transform** is introduced in Chapter 4. The remainder of this chapter details the construction of differential equations that model various mechanical, electrical, and electro-mechanical systems. The systems constructed will be used as examples throughout the text.

## 2.1 Passive electrical circuits

**Passive electrical circuits** require no sources of power other than the input signal itself. For example, the voltage divider in Figure 1.6 and the RC circuit in Figure 2.1 are passive circuits. Another common passive electrical circuit is the resistor, capacitor and inductor (RLC) circuit depicted in Figure 2.4. In this circuit we let the output signal  $y$  be the voltage over the resistor. Let  $v_C$  represent the voltage over the capacitor and  $v_L$  the voltage over the inductor and let  $i$  be the current. We have

$$y = Ri, \quad i = CDv_C, \quad v_L = LDi,$$

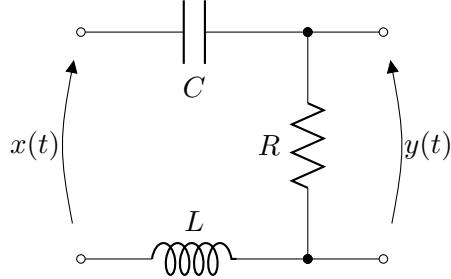


Figure 2.4: An electrical circuit with resistor, capacitor and inductor in series, otherwise known as an **RLC circuit**.

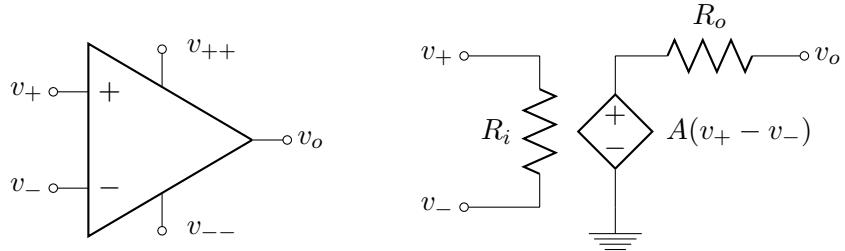


Figure 2.5: Left: triangular component diagram of an **operational amplifier**. The  $v_{++}$  and  $v_{--}$  connectors indicate where an external voltage source can be connected to the amplifier. These connectors will usually be omitted. Right: model for an operational amplifier including input resistance  $R_i$ , output resistance  $R_o$ , and open loop gain  $A$ . The diamond shaped component is a dependent voltage source. This model is usually only useful when the operational amplifier is in a negative feedback circuit.

leading to the following relationships between  $y$ ,  $v_C$  and  $v_L$ ,

$$y = RCDv_C, \quad Rv_L = LDy.$$

Kirchhoff's voltage law gives  $x = y + v_C + v_L$  and by differentiating both sides

$$Dx = Dy + Dv_C + Dv_L.$$

Substituting the equations relating  $y$ ,  $v_C$  and  $v_L$  leads to

$$RCDx = y + RCDy + LCD^2y. \quad (2.1.1)$$

We can similarly find equations relating the input voltage with  $v_C$  and  $v_L$ .

## 2.2 Active electrical circuits

Unlike passive electrical circuits, an **active electrical circuit** requires a source of power external to the input signal. Active circuits can be modelled

and constructed using **operational amplifiers** as depicted in Figure 2.5. The left hand side of Figure 2.5 shows a triangular circuit diagram for an operational amplifier, and the right hand side of Figure 2.5 shows a circuit that can be used to model the behaviour of the amplifier. The  $v_{++}$  and  $v_{--}$  connectors indicate where an external voltage source can be connected to the amplifier. These connectors will usually be omitted. The diamond shaped component is a dependent voltage source with voltage  $A(v_+ - v_-)$  that depends on the difference between the voltage at the **non-inverting input**  $v_+$  and the voltage at the **inverting input**  $v_-$ . The dimensionless constant  $A$  is called the **open loop gain**. Most operational amplifiers have large open loop gain  $A$ , large **input resistance**  $R_i$  and small **output resistance**  $R_o$ . As we will see, it can be convenient to consider the behaviour as  $A \rightarrow \infty$ ,  $R_i \rightarrow \infty$  and  $R_o \rightarrow 0$ , resulting in an **ideal operational amplifier**.

As an example, an operational amplifier configured as a **multiplier** is depicted in Figure 2.6. This circuit is an example of an operational amplifier configured with **negative feedback**, meaning that the output of the amplifier is connected (in this case by a resistor) to the inverting input  $v_-$ . The horizontal wire at the bottom of the plot is considered to be ground (zero volts) and is connected to the negative terminal of the dependent voltage source of the operational amplifier depicted in Figure 2.5. An equivalent circuit for the multiplier using the model in Figure 2.5 is shown in Figure 2.7. Solving this circuit (Exercise 2.1) yields the following relationship between the input voltage signal  $x$  and the output voltage signal  $y$ ,

$$y = \frac{R_i(R_o - AR_2)}{R_i(R_2 + R_o) + R_1(R_2 + R_i + AR_i + R_o)} x. \quad (2.2.1)$$

For an ideal operational amplifier we let  $A \rightarrow \infty$ ,  $R_i \rightarrow \infty$  and  $R_o \rightarrow 0$ . In this case terms involving the product  $AR_i$  dominate and we are left with the simpler equation

$$y = -\frac{R_2}{R_1} x. \quad (2.2.2)$$

Thus, assuming an ideal operational amplifier, the circuit acts as a multiplier with constant  $-\frac{R_2}{R_1}$ .

The equation relating  $x$  and  $y$  is much simpler for the ideal operational amplifier. Fortunately this equation can be obtained directly using the following two rules:

1. the voltage at the inverting and non-inverting inputs are equal,
2. no current flows through the inverting and non-inverting inputs.

These rules are only useful for analysing circuits with negative feedback. Let us now rederive (2.2.2) using these rules. Because the non-inverting input is connected to ground, the voltage at the inverting input is zero. So, the voltage over resistor  $R_2$  is  $y = R_2 i$ . Because no current flows through the

inverting input the current through  $R_1$  is also  $i$  and  $x = -R_1i$ . Combining these results, the input voltage  $x$  and the output voltage  $y$  are related by

$$y = -\frac{R_2}{R_1}x.$$

In Test 2 the inverting amplifier circuit is constructed and the relationship above is tested using a computer soundcard.

We now consider another circuit consisting of an operational amplifier, two resistors and two capacitors depicted in Figure 2.8. Assuming an ideal operational amplifier, the voltage at the inverting terminal is zero because the non-inverting terminal is connected to ground. Thus, the voltage over capacitor  $C_2$  and resistor  $R_2$  is equal to  $y$  and, by Kirchoff's current law,

$$i = \frac{y}{R_2} + C_2Dy.$$

Similarly, since no current flows through the inverting terminal,

$$i = -\frac{x}{R_1} - C_1Dx.$$

Combining these equations yields

$$-\frac{x}{R_1} - C_1Dx = \frac{y}{R_2} + C_2Dy. \quad (2.2.3)$$

Observe the similarity between this equation and that for the passive RC circuit (2.0.2) when  $R_1 = R_2$  and  $C_1 = 0$  (an open circuit). In this case

$$x = -y - R_1C_2Dy. \quad (2.2.4)$$

We call this the **active RC circuit**. This circuit is tested in Test 3.

Consider the circuit in Figure 2.9. Assuming an ideal operational amplifier, the input voltage  $x$  satisfies

$$-i = \frac{x}{R_1} + C_1Dx.$$

The voltage over the capacitor  $C_2$  is  $y - R_2i$  and so the current satisfies

$$i = C_2D(y - R_2i).$$

Combining these equations gives

$$-\frac{x}{R_1} - C_1Dx = C_2Dy + \frac{R_2C_2}{R_1}Dx + R_2C_2C_1D^2x,$$

and after rearranging,

$$Dy = -\frac{1}{R_1C_2}x - \left(\frac{R_2}{R_1} + \frac{C_1}{C_2}\right)Dx - R_2C_1D^2x.$$

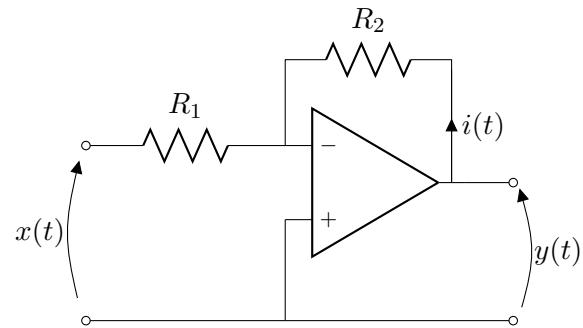


Figure 2.6: Inverting amplifier

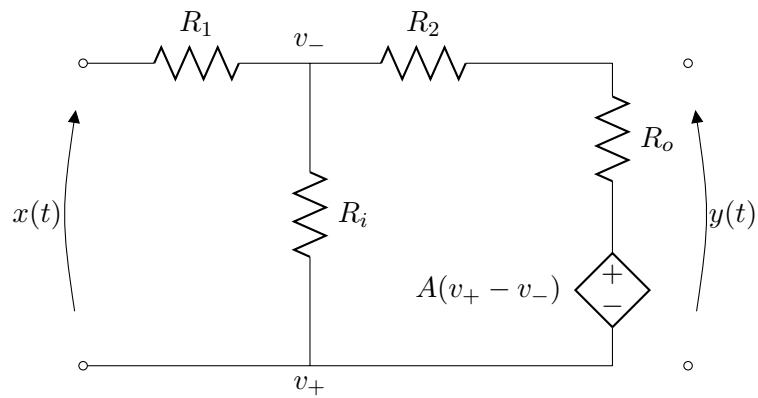


Figure 2.7: An equivalent circuit for the inverting amplifier from Figure 2.6 using the model for an operational amplifier in Figure 2.5.

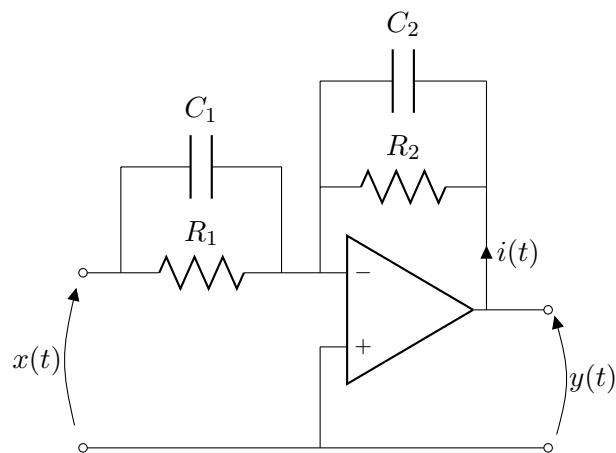


Figure 2.8: Operational amplifier configured with two capacitors and two resistors.

**Test 2 (Inverting amplifier)** In this test we construct the inverting amplifier circuit from Figure 2.6 with  $R_2 \approx 22\text{k}\Omega$  and  $R_1 \approx 12\text{k}\Omega$  that are accurate to within 5% of these values. The operational amplifier used is the Texas Instruments LM358P. Using a computer soundcard (an approximation of) the voltage signal

$$x(t) = \frac{1}{3} \sin(2\pi f_1 t) + \frac{1}{3} \sin(2\pi f_2 t)$$

with  $f_1 = 100$  and  $f_2 = 233$  is passed through the circuit. As in previous tests, the soundcard is used to sample the input signal  $x$  and the output signal  $y$ . Approximate reconstructions of the input signal  $\tilde{x}$  and output signal  $\tilde{y}$  are given according to (1.3.4) and (1.3.2). According to (2.1.1) we expect the approximate relationship

$$\tilde{y} \approx -\frac{R_2}{R_1} \tilde{x} = -\frac{11}{6} \tilde{x}.$$

Each of  $\tilde{y}$ ,  $\tilde{x}$  and  $-\frac{11}{6} \tilde{x}$  are plotted in Figure 2.9.

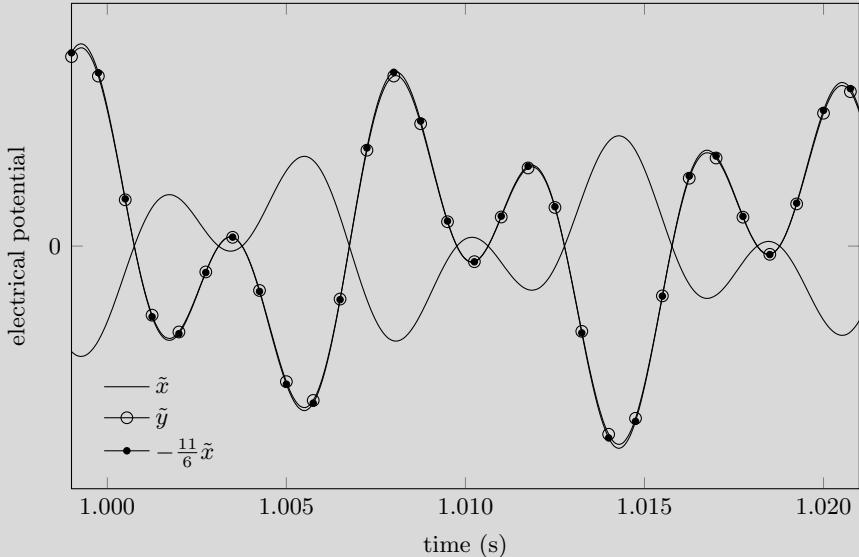


Figure 2.9: Plot of reconstructed input signal  $\tilde{x}$  (solid line), output signal  $\tilde{y}$  (solid line with circle) and hypothesised output signal  $-\frac{11}{6} \tilde{x}$  (solid line with dot).

**Test 3 (Active RC circuit)** In this test we construct the circuit from Figure 2.8 with  $R_1 \approx R_2 \approx 27\text{k}\Omega$  and  $C_2 \approx 10\text{nF}$  accurate to within 5% of these values and  $C_1 = 0$  (an open circuit). The operational amplifier used is a Texas Instruments LM358P. Using a computer soundcard (an approximation of) the voltage signal

$$x(t) = \frac{1}{3} \sin(2\pi f_1 t) + \frac{1}{3} \sin(2\pi f_2 t)$$

with  $f_1 = 500$  and  $f_2 = 1333$  is passed through the circuit. As in previous tests, the soundcard is used to sample the input signal  $x$  and the output signal  $y$  and approximate reconstructions  $\tilde{x}$  and  $\tilde{y}$  are given according to (1.3.4) and (1.3.2). According to (2.2.4) we expect the approximate relationship

$$\tilde{x} \approx -\frac{R_1}{R_2} \tilde{y} - R_1 C D(\tilde{y}) = -\tilde{y} - \frac{27}{10^5} D(\tilde{y}).$$

The derivative of the sinc function is

$$D \operatorname{sinc}(t) = \frac{d}{dt} \operatorname{sinc}(t) = \frac{1}{\pi t^2} (\pi t \cos(\pi t) - \sin(\pi t)), \quad (2.2.5)$$

and so,

$$D\tilde{y}(t) = \frac{d}{dt} \left( \sum_{\ell=1}^L y_\ell \operatorname{sinc}(Ft - \ell) \right) = F \sum_{\ell=1}^L y_\ell D \operatorname{sinc}(Ft - \ell). \quad (2.2.6)$$

Each of  $\tilde{y}$ ,  $\tilde{x}$  and  $-\tilde{y} - \frac{27}{10^5} D\tilde{y}$  are plotted in Figure 2.9.

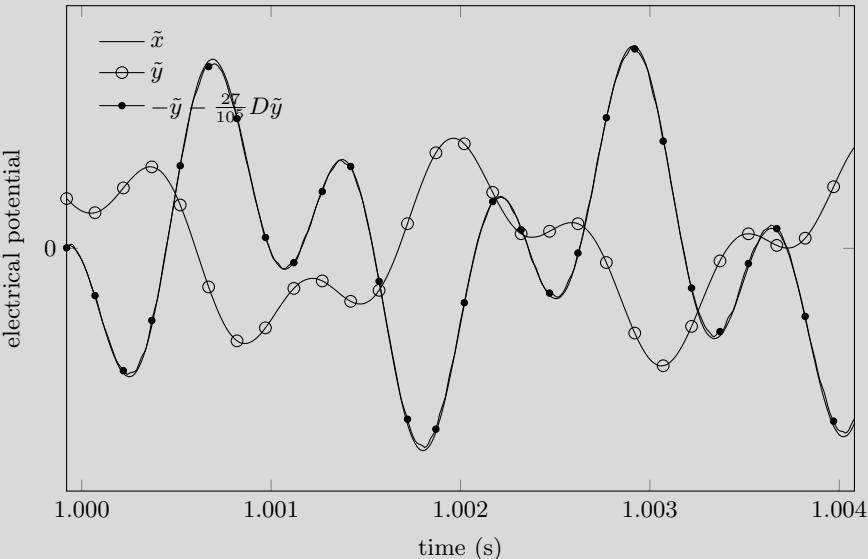


Figure 2.9: Plot of reconstructed input signal  $\tilde{x}$  (solid line with circle), output signal  $\tilde{y}$  (solid line), and hypothesised input signal  $-\tilde{y} - \frac{27}{10^5} D\tilde{y}$  (solid line with dot).

Put

$$K_i = \frac{1}{R_1 C_2}, \quad K_p = \frac{R_2}{R_1} + \frac{C_1}{C_2}, \quad K_d = R_2 C_1$$

and now

$$Dy = -K_i x - K_p Dx - K_d D^2 x. \quad (2.2.7)$$

This equation models what is called a **proportional-integral-derivative controller** or **PID controller**. The coefficients  $K_i$ ,  $K_p$  and  $K_d$  are called the **integral gain**, **proportional gain**, and **derivative gain**.

The final active circuit we consider is called a **Sallen-Key** [Sallen and Key, 1955] and is depicted in Figure 2.10. Observe that the output of the amplifier is connected directly to the inverting input and is also connected to the noninverting input by a capacitor and resistor. This circuit has both negative *and* positive feedback. It is not immediately apparent that we can use the simplifying assumptions for an ideal operational amplifier with negative feedback. However, we will do so and will find that it works in this case.

Let  $v_{R1}$ ,  $v_{R2}$ ,  $v_{C1}$ , and  $v_{C2}$  be the voltages over the components  $R_1$ ,  $R_2$ ,  $C_1$ , and  $C_2$ . Kirchoff's voltage law leads to the equations

$$x = v_{R1} + v_{R2} + v_{C2}, \quad y = v_{C1} + v_{R2} + v_{C2}.$$

The voltage at the inverting and noninverting terminals is  $y$  and so the voltage over the capacitor  $C_2$  is  $y$ , that is,  $y = v_{C2}$ . Using this, the equations above simplify to

$$x = v_{R1} + v_{R2} + y, \quad v_{C1} = -v_{R2}.$$

The current  $i_2$  through capacitor  $C_2$  satisfies  $i_2 = C_2 D v_{C2} = C_2 D y$ . Because no current flows into the inverting terminal of the amplifier the current through  $R_2$  is also  $i_2$  and so  $v_{R2} = R_2 i_2 = R_2 C_2 D y$ . Substituting this into the equations above gives

$$x = v_{R1} + R_2 C_2 D y + y, \quad v_{C1} = -R_2 C_2 D y. \quad (2.2.8)$$

Kirchoff's current law asserts that  $i + i_1 = i_2$ . The current  $i$  through capacitor  $C_1$  satisfies  $i = C_1 D v_{C1} = -R_2 C_1 C_2 D^2 y$  and the current through resistor  $R_1$  satisfies

$$v_{R1} = R_1 i_1 = R_1 (i_2 - i) = R_1 C_2 D y + R_1 R_2 C_1 C_2 D^2 y.$$

Substituting this into the equation on the left of (2.2.8) gives

$$x = y + C_2 (R_1 + R_2) D y + R_1 R_2 C_1 C_2 D^2 y. \quad (2.2.9)$$

The Sallen-Key will be useful when we consider the design of analogue electrical filters in Section 5.2.

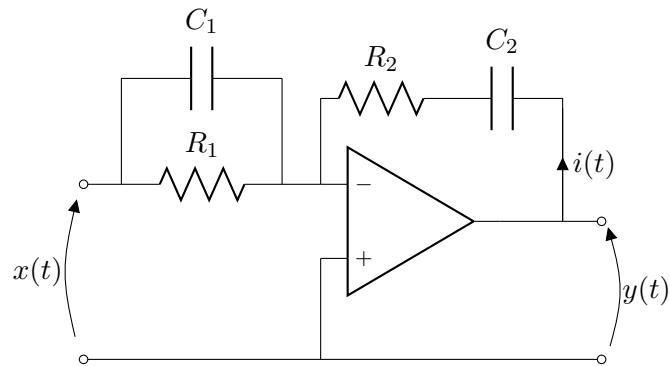


Figure 2.9: Operational amplifier implementing a **proportional-integral-derivative controller**.

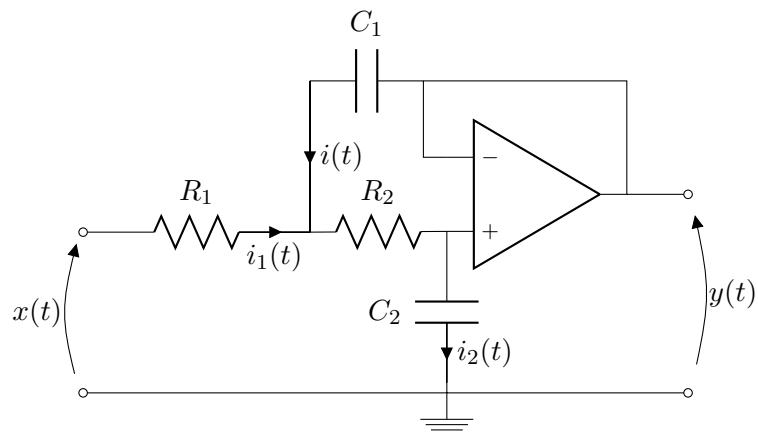


Figure 2.10: Operational amplifier implementing a **Sallen-Key**.

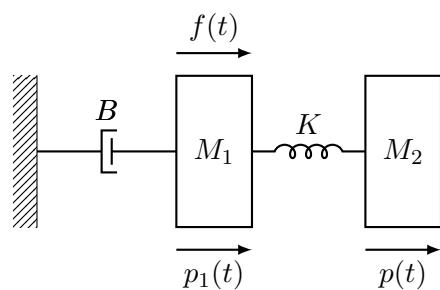


Figure 2.11: Two masses, a spring and a damper

### 2.3 Masses, springs, and dampers

A mechanical mass-spring-damper system was described in Section 2 and Figure 2.2. We now consider another mechanical system depicted in Figure 2.11 involving two masses, a spring and a damper. A mass  $M_1$  is connected to a wall by a damper with constant  $B$ , and to another mass  $M_2$  by a spring with constant  $K$ . A force represented by the signal  $f$  is applied to the first mass. We will derive a differential equation relating  $f$  with the position  $p$  of the second mass. Assume that the spring applies no force (is in equilibrium) when the masses are distance  $d$  apart. The forces due to the spring satisfy

$$f_{s1} = -f_{s2} = K(p - p_1 - d)$$

where  $f_{s1}$  and  $f_{s2}$  are signals representing the force due to the spring on mass  $M_1$  and  $M_2$  respectively. It is convenient to define the signal  $g = p_1 + d$  so that forces due to spring satisfy the simpler equation

$$f_{s1} = -f_{s2} = K(p - g).$$

The only force applied to  $M_2$  is by the spring and so, by Newton's law, the acceleration of  $M_2$  satisfies

$$M_2 D^2 p = f_{s2} = Kg - Kp. \quad (2.3.1)$$

The force applied by the damper on mass  $M_1$  is given by the signal

$$f_d = -BDp_1 = -BDg$$

where the replacement of  $p_1$  by  $g$  is justified because differentiation will remove the constant  $d$ . The cumulative force on  $M_1$  is given by the signal

$$f_1 = f + f_d + f_{s1} = f + f_d - f_{s2} = f - M_2 D^2 p - BDg$$

and by Newton's law the acceleration of  $M_1$  satisfies

$$M_1 D^2 p_1 = M_1 D^2 g = f_1 = f - M_2 D^2 p - BDg.$$

Combining this equation with (2.3.1) we obtain a fourth order differential equation relating the position  $p$  and force  $f$ ,

$$f = BDp + (M_1 + M_2)D^2 p + \frac{BM_2}{K} D^3 p + \frac{M_1 M_2}{K} D^4 p. \quad (2.3.2)$$

Given the position of the second mass  $p$  we can readily solve for the corresponding force  $f$  and position of the first mass  $p$ . For example, if the constants  $B = K = 1$  and  $M_1 = M_2 = \frac{1}{2}$  and  $d = \frac{5}{2}$ , and if the position of the second mass satisfies

$$p(t) = e^{-t^2}$$

then, by application of (2.3.2) and (2.3.1),

$$f(t) = e^{-t^2}(1 + 4t - 8t^2 - 4t^3 + 4t^4), \quad \text{and} \quad p_1(t) = 2e^{-t^2}t^2 - \frac{5}{2}.$$

This solution is plotted in Figure 2.12.

Figure 2.12: Solution of the system describing two masses with a spring and damper where  $B = K = 1$  and  $M_1 = M_2 = \frac{1}{2}$  and the position of the second mass is  $p(t) = e^{-t^2}$ .

## 2.4 Direct current motors

Direct current (DC) motors convert electrical energy, in the form of a voltage, into rotary kinetic energy [Nise, 2007, page 76]. We derive a differential equation relating the input voltage  $v$  to the angular position of the motor  $\theta$ . Figure 2.13 depicts the components of a DC motor.

The voltages over the resistor and inductor satisfy

$$v_R = Ri, \quad v_L = LDi,$$

and the motion of the motor induces a voltage called the **back electromotive force** (EMF),

$$v_b = K_b D\theta$$

that we model as being proportional to the angular velocity of the motor. The input voltage now satisfies

$$v = v_R + v_L + v_b = Ri + LDi + K_b D\theta.$$

The torque  $\tau$  applied by the motor is modelled as being proportional to the current  $i$ ,

$$\tau = K_\tau i.$$

A load with inertia  $J$  is attached to the motor. Two forces are assumed to act on the load, the torque  $\tau$  applied by the current, and a torque  $\tau_d = -BD\theta$

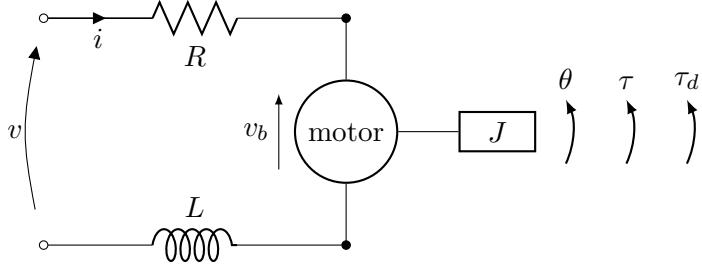


Figure 2.13: Diagram for a rotary direct current (DC) motor

modelling a damper that acts proportionally against the angular velocity of the motor. By Newton's law, the angular acceleration of the load satisfies

$$JD^2\theta = \tau + \tau_d = K_\tau i - BD\theta.$$

Combining these equations we obtain the 3rd order differential equation

$$v = \left( \frac{RB}{K_\tau} + K_b \right) D\theta + \frac{RJ + LB}{K_\tau} D^2\theta + \frac{LJ}{K_\tau} D^3\theta$$

relating voltage and motor position. In many DC motors the inductance  $L$  is small and can be ignored, leaving the simpler second order equation

$$v = \left( \frac{RB}{K_\tau} + K_b \right) D\theta + \frac{RJ}{K_\tau} D^2\theta. \quad (2.4.1)$$

Given the position signal  $\theta$  we can find the corresponding voltage signal  $v$ . For example, put the constants  $K_b = K_\tau = B = R = J = 1$  and assume that

$$\theta(t) = 2\pi(1 + \text{erf}(t))$$

where  $\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-\tau^2} d\tau$  is the **error function**. The corresponding angular velocity  $D\theta$  and voltage  $v$  satisfy

$$D\theta(t) = 4\sqrt{\pi}e^{-t^2}, \quad v(t) = 8\sqrt{\pi}e^{-t^2}(1 - t).$$

These signals are depicted in Figure 2.14. This voltage signal is sufficient to make the motor perform two revolutions and then come to rest.

## Exercises

- 2.1. Analyse the inverting amplifier circuit in Figure 2.7 to obtain the relationship between input voltage  $x$  and output voltage  $y$  given by (2.2.1). You may wish to use a symbolic programming language (for example Maxima, Sage, Mathematica, or Maple).

Figure 2.14: Voltage and corresponding angle for a DC motor with constants  $K_b = K_\tau = B = R = J = 1$ .

- 2.2. Figure 2.15 depicts a mechanical system involving two masses, two springs, and a damper connected between two walls. Suppose that the spring  $K_2$  is at rest when the mass  $M_2$  is at position  $p(t) = 0$ . A force, represented by the signal  $f$ , is applied to mass  $M_1$ . Derive a differential equation relating the force  $f$  and the position  $p$  of mass  $M_2$ . Determine the force  $f$  in the case that the position  $p(t) = e^{-t^2}$  and  $M_1 = M_2 = \frac{1}{2}$  and  $K_1 = K_2 = B = 1$ .
- 2.3. Consider the electromechanical system in Figure 2.16. A direct current motor is connected to a potentiometer in such a way that the voltage at the output of the potentiometer is equal to the angle of the motor  $\theta$ . This voltage is fed back via a unity gain amplifier to the input terminal of the motor. An input voltage  $v$  is applied to the other terminal on the motor. Find the differential equation relating  $v$  and  $\theta$ . What is the input voltage  $v$  if the motor angle satisfies  $\theta(t) = \frac{\pi}{2}(1 + \text{erf}(t))$ ? Plot  $\theta$  and  $v$  in this case when the motor coefficients satisfy  $L = 0$ ,  $R = \frac{3}{4}$ , and  $K_b = K_\tau = B = J = 1$ .

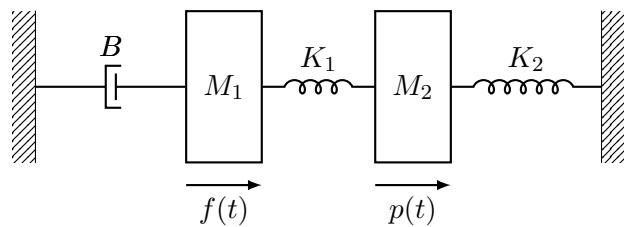


Figure 2.15: Two masses, a spring, and a damper connected between two walls for Exercise 2.2.

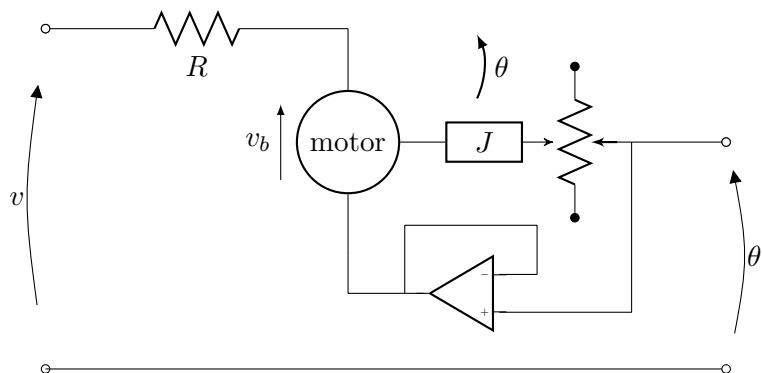


Figure 2.16: Diagram for a rotary direct current (DC) with potentiometer feedback for Exercise 2.3.

## Chapter 3

# Linear shift-invariant systems

In the previous section we derived differential equations that model mechanical, electrical, and electro-mechanical systems. The equations themselves often do not provide sufficient information. For example, we were able to find a signal  $p$  representing the position of the mass-spring-damper in Figure 2.2 given a particular force signal  $f$  is applied to the mass. However, it is not immediately obvious how to find the force signal  $f$  given a particular position signal  $p$ . We will be able to solve this problem and, more generally, to describe properties of systems modelled by linear differential equations with constant coefficient, if we make the added assumptions that the systems are **linear** and **shift-invariant**. We study linear shift-invariant systems in this chapter.

### 3.1 Convolution, regular systems and the delta “function”

A large number of linear shift-invariant systems can be represented by a signal called the **impulse response**. The impulse response of a system  $H$  is a locally integrable signal  $h$  such that

$$Hx(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau,$$

that is, the response of  $H$  to input signal  $x$  can be represented as an integral equation involving  $x$  and the impulse response  $h$ . The integral is called a **convolution** and appears so often that a special notation is used for it. We write  $h * x$  to indicate the signal that results from convolution of signals  $h$  and  $x$ , that is,  $h * x$  is the signal

$$h * x = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau$$

where the right hand side is to be interpreted as a signal, a function of  $t \in \mathbb{R}$ . We write  $(h * x)(t)$  to indicate the value of  $h * x$  at  $t \in \mathbb{R}$ . Those systems that have an impulse response we call **regular systems**<sup>1</sup>.

Some care must be taken when selecting a domain for a regular system. To see what can go wrong it is worth first considering an example. Suppose that  $H$  has discrete impulse response given by the step function  $u$  (1.1.1). The signal  $x(t) = 1$  that takes the value 1 for all  $t \in \mathbb{R}$  will not be in the domain of  $H$  since, in this case,

$$Hx(t) = \int_{-\infty}^{\infty} u(\tau)x(t - \tau)d\tau = \int_0^{\infty} d\tau$$

is not finite for any  $t \in \mathbb{R}$ . Given a signal  $h$ , denote by  $\text{dom } h$  the set of signals  $x$  such that the integral

$$\int_{-\infty}^{\infty} |h(\tau)x(t - \tau)| d\tau < \infty \quad \text{for all } t \in \mathbb{R}.$$

If  $H$  has impulse response  $h$  then, for all signal  $x \in \text{dom } h$  we have

$$|Hx(t)| = \left| \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau \right| \leq \int_{-\infty}^{\infty} |h(\tau)x(t - \tau)| d\tau < \infty$$

for all  $t \in \mathbb{R}$  and so  $Hx(t)$  is finite for all  $t \in \mathbb{R}$ . We take  $\text{dom } h$  as the domain of a regular system  $H$  with impulse response  $h$  unless otherwise stated. It can be shown that  $\text{dom } h$  is a linear shift-invariance space (Exercise 3.4).

Regular systems are linear because, for all  $x, y \in \text{dom } h$  and all  $a, b \in \mathbb{C}$ ,

$$\begin{aligned} H(ax + by) &= h * (ax + by) \\ &= \int_{-\infty}^{\infty} h(\tau)(ax(t - \tau) + by(t - \tau))d\tau \\ &= a \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau + b \int_{-\infty}^{\infty} h(\tau)y(t - \tau)d\tau \\ &= a(h * x) + b(h * y) \\ &= aHx + bHy. \end{aligned} \tag{3.1.1}$$

The above equation shows that convolution commutes with scalar multiplication and distributes with addition, that is,

$$h * (ax + by) = a(h * x) + b(h * y).$$

---

<sup>1</sup>The name **regular system** is motivated by the term **regular distribution** [Zemanian, 1965]

Regular systems are also shift-invariant because for all  $x \in \text{dom } h$

$$\begin{aligned} T_\kappa Hx &= T_\kappa(h * x) \\ &= \int_{-\infty}^{\infty} h(\tau)x(t - \kappa - \tau)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)T_\kappa x(t - \tau)d\tau \\ &= h * (T_\kappa x) \\ &= HT_\kappa x. \end{aligned}$$

The impulse response of a regular system  $H$  can be found in the following way. First define the signal

$$p_\gamma(t) = \begin{cases} \gamma, & 0 < t \leq \frac{1}{\gamma} \\ 0, & \text{otherwise,} \end{cases}$$

that is, a rectangular shaped pulse of height  $\gamma$  and width  $\frac{1}{\gamma}$ . The signal  $p_\gamma$  is plotted in Figure 3.1 for  $\gamma = \frac{1}{2}, 1, 2, 5$ . As  $\gamma$  increases the pulse gets thinner and higher so as to keep the area under  $p_\gamma$  equal to one. Consider the response of the regular system  $H$  to the signal  $p_\gamma$ ,

$$Hp_\gamma = h * p_\gamma = \int_{-\infty}^{\infty} h(\tau)p_\gamma(t - \tau)d\tau = \gamma \int_{t-1/\gamma}^t h(\tau)d\tau.$$

Taking  $\gamma \rightarrow \infty$  we find that

$$\lim_{\gamma \rightarrow \infty} Hp_\gamma = \lim_{\gamma \rightarrow \infty} \gamma \int_{t-1/\gamma}^t h(\tau)d\tau = h \text{ a.e.}$$

As an example, consider the integrator system  $I_\infty$  described in Section 1.4. The response of  $I_\infty$  to  $p_\gamma$  is

$$I_\infty p_\gamma(t) = \int_{-\infty}^t p_\gamma(\tau)d\tau = \begin{cases} 0, & t \leq 0 \\ \gamma t, & 0 < t \leq \frac{1}{\gamma} \\ 1, & t > \frac{1}{\gamma}. \end{cases}$$

The response is plotted in Figure 3.1. Taking the limit as  $\gamma \rightarrow \infty$  we find that the impulse response of the integrator is the step function

$$u(t) = \lim_{\gamma \rightarrow \infty} I_\infty p_\gamma(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0. \end{cases} \quad \text{a.e.} \quad (3.1.2)$$

Some important systems do not have an impulse response and are not regular. For example, the identity system  $T_0$  is not regular. Similarly, the

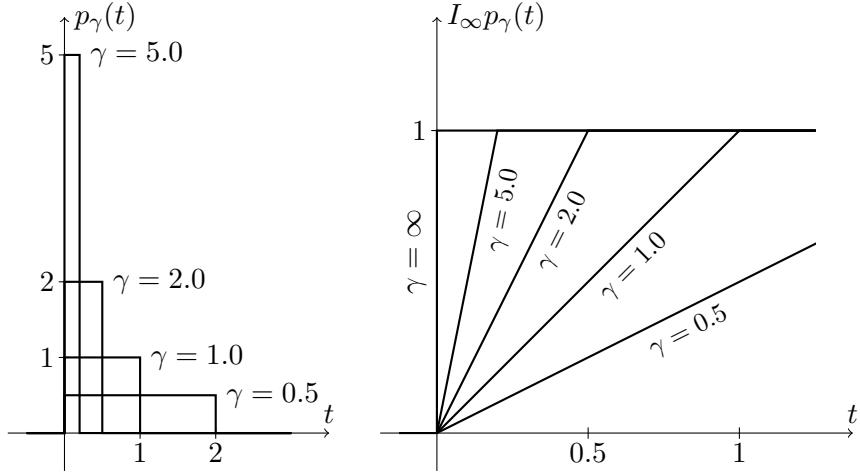


Figure 3.1: The rectangular shaped pulse  $p_\gamma$  for  $\gamma = 0.5, 1, 2, 5$  and the response of the integrator  $I_\infty$  to  $p_\gamma$  for  $\gamma = 0.5, 1, 2, 5, \infty$ .

shifter  $T_\tau$  and differentiators  $D^k$  are not regular. However, it is common to pretend that  $T_0$  *does* have an impulse response and this is typically denoted by the symbol  $\delta$  called the **delta function**. The idea is to assign  $\delta$  the property

$$\int_{-\infty}^{\infty} x(t)\delta(t)dt = x(0)$$

so that convolution of  $x$  and  $\delta$  satisfies

$$\delta * x = \int_{-\infty}^{\infty} \delta(\tau)x(t - \tau)d\tau = x(t) = T_0x.$$

We now treat  $\delta$  as if it were a signal. So  $\delta(t - \tau)$  will represent the impulse response of the shifter  $T_\tau$  because

$$\begin{aligned} T_\tau x &= \delta(t - \tau) * x \\ &= \int_{-\infty}^{\infty} \delta(\kappa - \tau)x(t - \kappa)d\kappa \\ &= \int_{-\infty}^{\infty} \delta(k)x(t - \tau - k)dk \quad (\text{change variable } k = \kappa - \tau) \\ &= x(t - \tau). \end{aligned}$$

For  $a \in \mathbb{R}$  it is common to plot  $a\delta(t - \tau)$  using an arrow of height  $a$  at  $t = \tau$  as indicated in Figure 3.2. It is important to realise that  $\delta$  is not actually a signal. It is not a function in  $\mathbb{R} \rightarrow \mathbb{C}$ . However, it can be convenient to treat  $\delta$  as if it were a signal. The manipulations in the last set of equations, such as the change of variables, are not formally justified, but they do lead to the desired result  $T_\tau x = x(t - \tau)$  in this case. In general, there is no guarantee

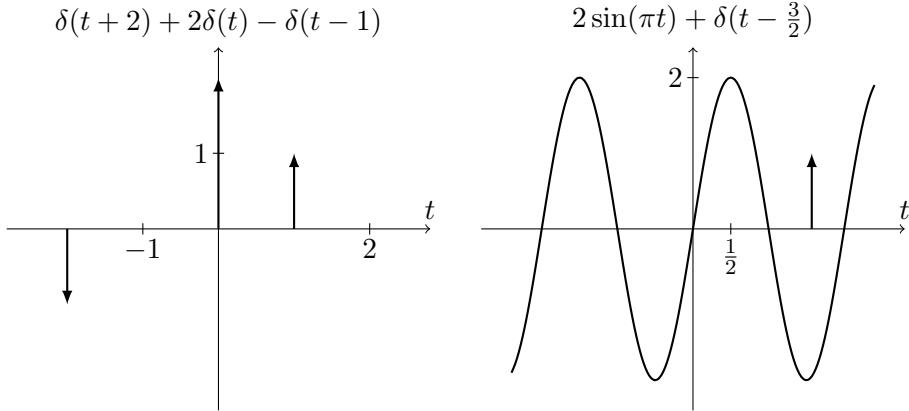


Figure 3.2: Plot of the “signal”  $\delta(t + 2) + 2\delta(t) - \delta(t - 1)$  (left) and the “signal”  $2 \sin(\pi t) + \delta(t - \frac{3}{2})$  (right).

that mechanical mathematical manipulations involving  $\delta$  will lead to sensible results.

The only other non regular systems that we have use of are differentiators  $D^k$  and it is common to define a similar notation for pretending that these systems have an impulse response. In this case, the symbol  $\delta^k$  is assigned the property

$$\int_{-\infty}^{\infty} x(t)\delta^k(t)dt = D^k x(0),$$

so that convolution of  $x$  and  $\delta^k$  is

$$\delta^k * x = \int_{-\infty}^{\infty} \delta^k(\tau)x(t - \tau)d\tau = D^k x(t).$$

As with the delta function the symbol  $\delta^k$  must be treated with care. This notation can be useful, but purely formal manipulations with  $\delta^k$  may not always lead to sensible results.

The impulse response  $h$  immediately yields some properties of the corresponding system  $H$ . For example, if  $h(t) = 0$  for all  $t < 0$ , then  $H$  is causal because

$$Hx(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau = \int_0^{\infty} h(\tau)x(t - \tau)d\tau$$

only depends on values of the input signal  $x$  at  $t - \tau$  with  $\tau > 0$ . The system  $H$  is stable if and only if  $h$  is absolutely integrable (Exercise 3.6).

Another related important signal is the **step response** defined as the response of the system to the step function  $u$ . For example, the step response of the shifter  $T_\tau$  is the shifted step function  $T_\tau u(t) = u(t - \tau)$ . The step

response of the integrator  $I_\infty$  is

$$I_\infty u(t) = \int_{-\infty}^t u(\tau) d\tau = \begin{cases} \int_0^t d\tau = t & t > 0 \\ 0 & t \leq 0. \end{cases}$$

This signal is often called the **ramp function**. A system has a step response only if  $u$  is inside its domain. For example, the regular system with impulse response  $u(-t)$  does not have a step response because  $u \notin \text{dom } u(-t)$ . Convolution of the step function  $u$  and its reflection  $u(-t)$  is not possible. If a system  $H$  has both an impulse response  $h$  and a step response  $Hu$ , then these two signals are related. To see this, observe that the step response is

$$Hu = h * u = \int_{-\infty}^{\infty} h(\tau)u(t - \tau) d\tau = \int_{-\infty}^t h(\tau) d\tau = I_\infty h. \quad (3.1.3)$$

Thus, the step response can be obtained by applying the integrator  $I_\infty$  to the impulse response in the case that both of these signals exist.

### 3.2 Properties of convolution

The convolution  $x * y$  of two signals  $x$  and  $y$  does not always exist. For example, if  $x(t) = u(t)$  and  $y(t) = 1$ , then

$$x * y = \int_{-\infty}^{\infty} x(\tau)y(t - \tau) d\tau = \int_{-\infty}^{\infty} u(\tau)d\tau = \int_0^{\infty} d\tau$$

is not finite for any  $t$ . We cannot convolve the step function  $u$  and the signal that takes the constant value 1. On the other hand, if  $x(t) = y(t) = u(t)$ , then

$$x * y = \int_{-\infty}^{\infty} u(\tau)u(t - \tau) d\tau = \begin{cases} \int_0^t d\tau = \tau & t > 0 \\ 0 & t \leq 0 \end{cases}$$

is finite for all  $t$ . If  $x \in \text{dom } y$  or equivalently  $y \in \text{dom } x$  then the convolution  $x * y$  exists because, in this case,

$$|(x * y)(t)| = \left| \int_{-\infty}^{\infty} x(\tau)y(t - \tau) d\tau \right| \leq \int_{-\infty}^{\infty} |x(\tau)y(t - \tau)| d\tau < \infty$$

for all  $t \in \mathbb{R}$ .

We have already shown in (3.1.1) that convolution commutes with scalar multiplication and distributes with addition, that is, for signals  $x, y, w$  and complex numbers  $a, b$ ,

$$a(x * w) + b(y * w) = (ax + by) * w.$$

The property holds provided that the convolutions  $x * w$  and  $y * w$  exist. This is the case if, for example,  $w$  is in both  $\text{dom } x$  and  $\text{dom } y$ , that is,

$w \in \text{dom } x \cap \text{dom } y$ . Convolution is commutative, that is,  $x * y = y * x$  whenever these convolutions exist. To see this, write

$$\begin{aligned} x * y &= \int_{-\infty}^{\infty} x(\tau)y(t - \tau)d\tau \\ &= \int_{-\infty}^{\infty} x(t - \kappa)y(\kappa)d\kappa \quad (\text{change variable } \kappa = t - \tau) \\ &= y * x. \end{aligned}$$

Convolution is associative under appropriate assumptions, that is, for signals  $x, y, z$ , we have  $x * (y * z) = (x * y) * z$ . To describe conditions under which associativity holds let us first define the set  $\text{dom}(x, y)$  containing all those signals  $z$  such that the double integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x(\tau)y(\kappa)z(t - \kappa - \tau)| d\kappa d\tau < \infty \quad \text{for all } t \in \mathbb{R}.$$

We will often drop the brackets and write simply  $\text{dom } xy$ . If  $z \in \text{dom } xy$  then

$$\begin{aligned} x * (y * z) &= \int_{-\infty}^{\infty} x(\tau)(y * z)(t - \tau)d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} y(\kappa)z(t - \kappa - \tau)d\kappa d\tau. \end{aligned}$$

Because  $z \in \text{dom } xy$  Fubini's theorem [Rudin, 1986, Theorem 8.8] justifies swapping the order of integration leading to

$$x * (y * z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)y(\kappa)z(t - \kappa - \tau)d\tau d\kappa$$

and by the change of variable  $\nu = \kappa + \tau$  we find that

$$\begin{aligned} x * (y * z) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)y(\nu - \tau)z(t - \nu)d\tau d\nu \\ &= \int_{-\infty}^{\infty} (x * y)(\nu)z(t - \nu)d\nu \\ &= (x * y) * z. \end{aligned}$$

By combining the associative and commutative properties we find that the order in which the convolutions in  $x * y * z$  are performed does not matter, that is

$$x * y * z = y * z * x = z * x * y = y * x * z = x * z * y = z * y * x$$

provided that all the convolutions involved exist. For example, this holds if  $z \in \text{dom } xy$  or equivalently  $x \in \text{dom } yz$  or  $y \in \text{dom } xz$ . More generally, the order in which any sequence of convolutions is performed does not change the final result.

### 3.3 Linear combination and composition

Let  $F$  and  $G$  be linear shift-invariant systems with domain  $X_F \subseteq \mathbb{R} \rightarrow \mathbb{C}$  and  $X_G \subseteq \mathbb{R} \rightarrow \mathbb{C}$  respectively. For complex numbers  $c$  and  $d$ , let  $H$  be the system satisfying

$$Hx = cFx + dGx \quad x \in X_F \cap X_G.$$

The system  $H$  is said to be a **linear combination** of  $F$  and  $G$  and its domain is taken to be  $X_G \cap X_F$  unless otherwise stated. The system  $H$  is linear because for all signals  $x, y \in X_G \cap X_F$  and  $a, b \in \mathbb{C}$ ,

$$\begin{aligned} H(ax + by) &= cF(ax + by) + dG(ax + by) \\ &= acFx + bcFy + adGx + bdGy \quad (\text{linearity } F, G) \\ &= a(cFx + dGx) + b(cFy + dGy) \\ &= aHx + bHy. \end{aligned}$$

The system  $H$  is also shift-invariant because, for  $x \in X_G \cap X_F$ ,

$$\begin{aligned} HT_\tau x &= cFT_\tau x + dGT_\tau x \\ &= cT_\tau Fx + dT_\tau Gx \quad (\text{shift-invariance } F, G) \\ &= T_\tau(cFx + dGx) \quad (\text{linearity } T_\tau) \\ &= T_\tau Hx. \end{aligned}$$

So, linear shift-invariant systems can be constructed as linear combinations of other linear shift-invariant systems. If  $F$  and  $G$  are regular systems with impulse responses  $f$  and  $g$  then, for  $x \in \text{dom } f \cap \text{dom } g$ ,

$$\begin{aligned} Hx &= aFx + bGx \\ &= af * x + bg * x \\ &= (af + bg) * x \quad (\text{distributivity of convolution}) \\ &= h * x, \end{aligned}$$

and so,  $H$  is a regular system with impulse response  $h = af + bg$ . Its domain is taken to be  $\text{dom } f \cap \text{dom } g$  unless otherwise stated.

Another way to construct linear shift-invariant systems is by **composition**. Let  $X, Y, Z$  be linear shift-invariant spaces of signals. Let  $F \in X \rightarrow Y$  and  $G \in Y \rightarrow Z$  be linear shift-invariant systems and let  $H \in X \rightarrow Z$  be the system satisfying

$$Hx = GFx,$$

that is,  $H$  first applies  $F$  and then applies  $G$ . The system  $H$  is said to be the **composition** of  $F$  and  $G$ . Observe that the range of  $F$  is contained within the domain  $Y$  of  $G$ . This is necessary for the composition  $GF$  to make sense.

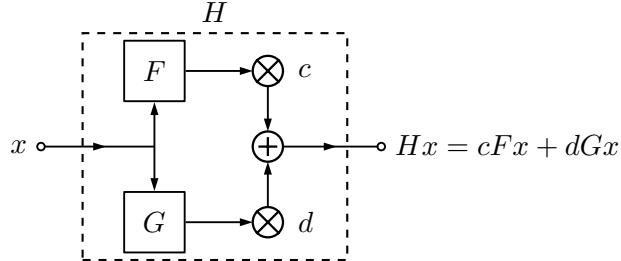


Figure 3.3: Block diagram depicting the linear combination of linear shift-invariant systems. The system  $cFx + dGx$  can be expressed as a single linear shift-invariant system  $Hx$ .

The system  $H$  is linear because, for signals  $x, y \in X$  and complex numbers  $a, b$ ,

$$\begin{aligned} H(ax + by) &= GF(ax + by) \\ &= G(aFx + bFy) && (\text{linearity } F) \\ &= aGFx + bGFy && (\text{linearity } G) \\ &= aHx + bHy. \end{aligned}$$

The system is also shift-invariant because, for  $x \in X$ ,

$$\begin{aligned} HT_\tau x &= GFT_\tau x \\ &= GT_\tau Fx && (\text{shift-invariance } F) \\ &= T_\tau GFx && (\text{shift-invariance } G) \\ &= T_\tau Hx. \end{aligned}$$

If  $F$  and  $G$  are regular systems the composition property can be expressed using their impulse responses  $f$  and  $g$ . For  $x \in \text{dom } f g$ , the associative property of convolution asserts that  $g * (f * x) = (g * f) * x$  (Section 3.2). It follows that, for  $x \in \text{dom } f g$ ,

$$Hx = GFx = g * (f * x) = (g * f) * x = h * x$$

and so  $H$  is a regular system with impulse response  $h = g * f$ . We can take its domain to be  $\text{dom } f g$  unless otherwise stated.

A wide variety of linear shift-invariant systems can be constructed by linear combination and composition of simpler systems.

### 3.4 Eigenfunctions and the transfer function

Let  $s = \sigma + j\omega \in \mathbb{C}$  where  $j = \sqrt{-1}$ . A signal of the form

$$e^{st} = e^{\sigma t} e^{j\omega t} = e^{\sigma t} (\cos(\omega t) + j \sin(\omega t))$$

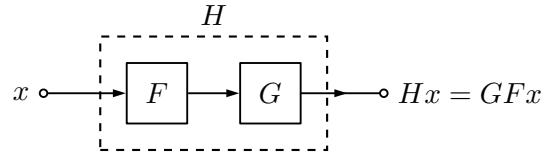


Figure 3.4: Block diagram depicting composition of linear shift-invariant systems. The system  $GF$  can be expressed as a single linear shift-invariant system  $H$ .

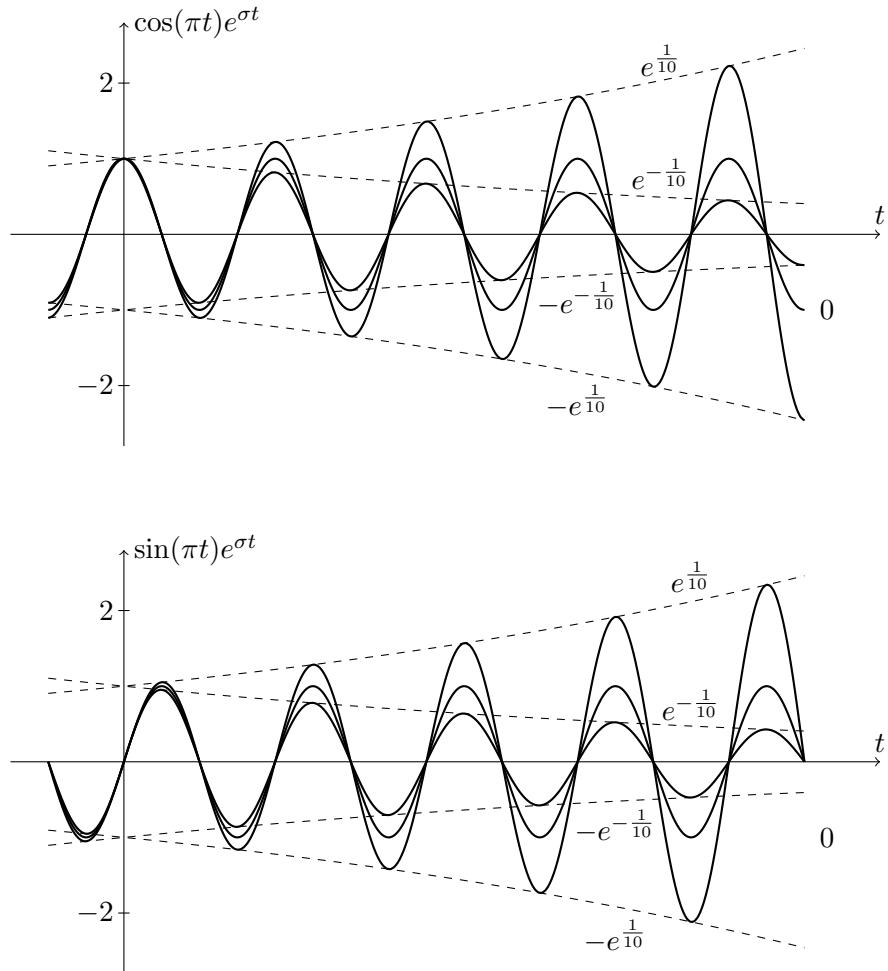


Figure 3.5: The function  $\cos(\pi t)e^{\sigma t}$  (top) and  $\sin(\pi t)e^{\sigma t}$  (bottom) for  $\sigma = -\frac{1}{10}, 0, \frac{1}{10}$ .

is called a **complex exponential signal**. Complex exponential signals play an important role in the study of linear shift-invariant systems. The real and imaginary parts of  $e^{(\sigma+j\pi)t}$  are plotted in Figure 3.5 for  $\sigma = -\frac{1}{10}, 0, \frac{1}{10}$ . The signal is oscillatory when  $\omega \neq 0$ . The signal converges to zero as  $t \rightarrow \infty$  when  $\sigma < 0$  and diverges as  $t \rightarrow \infty$  when  $\sigma > 0$ .

Let  $H \in X \rightarrow Y$  be a linear shift-invariant system. Suppose that  $y = He^{st}$  is the response of  $H$  to the complex exponential signal  $e^{st} \in X$ . Consider the response of  $H$  to the shifted signal  $T_\tau e^{st} = e^{s(t-\tau)}$  for  $\tau \in \mathbb{R}$ . By shift-invariance

$$HT_\tau e^{st} = T_\tau He^{st} = y(t - \tau)$$

and by linearity

$$HT_\tau e^{st} = He^{s(t-\tau)} = e^{-s\tau}He^{st} = e^{-s\tau}y(t).$$

Combining these equations we obtain

$$y(t - \tau) = e^{-s\tau}y(t) \quad \text{for all } t, \tau \in \mathbb{R}.$$

This equation is satisfied by signals of the form  $y(t) = \lambda e^{st}$  where  $\lambda$  is a complex number. That is, the response of a linear shift-invariant system  $H$  to a complex exponential signal  $e^{st}$  is the same signal  $e^{st}$  multiplied by some constant complex number  $\lambda$ . Due to this property complex exponential signals are called **eigenfunctions** of linear shift-invariant systems. The constant  $\lambda$  does not depend on  $t$ , but it does usually depend on the complex number  $s$  and the system  $H$ . To highlight this dependence on  $H$  and  $s$  we write  $\lambda H(s)$  or  $\lambda(H)(s)$  or  $\lambda(H, s)$  and do not distinguish between these notations. Considered as a function of  $s$ , the expression  $\lambda H$  is called the **transfer function** of the system  $H$ . Observe that the transfer function  $\lambda H$  maps a complex number to a complex number.

Denote by  $\text{cep } X$  the set of complex numbers  $s$  such that  $e^{st} \in X$ ,

$$\text{cep } X = \{s \in \mathbb{C} ; e^{st} \in X\}.$$

We take  $\text{cep } X$  as the domain of the transfer function  $\lambda H$ , that is  $\lambda H \in \text{cep } X \rightarrow \mathbb{C}$ . The transfer function satisfies

$$He^{st} = \lambda H(s)e^{st} \quad s \in \text{cep } X. \quad (3.4.1)$$

This is an important equation. Stated in words: the response of a linear shift-invariant system  $H \in X \rightarrow Y$  to a complex exponential signal  $e^{st} \in X$  is the transfer function  $\lambda H$  evaluated at  $s$  multiplied by the same complex exponential signal  $e^{st}$ .

We can use these eigenfunctions to better understand the properties of systems modelled by differential equations, such as those in Section 2. Consider the active electrical circuit from Figure 2.8. In the case that the

resistors  $R_1 = R_2$ , and the capacitor  $C_1 = 0$  (an open circuit) the differential equation relating the input voltage  $x$  and output voltage  $y$  is

$$x = -y - R_1 C_2 D y.$$

We called this the **active RC** circuit. To simplify notation put  $R = R_1$  and  $C = C_2$  so that  $x = -y - RCDy$ . Suppose that  $H$  is a linear shift-invariant system that maps the input voltage  $x$  to the output voltage  $y$ , that is,  $H$  is a system that describes the active RC circuit. If the input voltage is a complex exponential signal  $x = e^{st}$ , then the output voltage is the same complex exponential signal multiplied by the transfer function, that is,  $y = Hx = \lambda H(s)e^{st}$ . Substituting this into the differential equation for the active RC circuit we obtain

$$e^{st} = -\lambda e^{st} - RCD(\lambda e^{st}) = -\lambda e^{st}(1 - RCs)$$

where, to simplify notation, we have written simply  $\lambda$  for  $\lambda H(s)$  above. Solving for  $\lambda$  leads to the transfer function of the system  $H$  describing the active RC circuit

$$\lambda H(s) = -\frac{1}{1 + RCs}. \quad (3.4.2)$$

Now, if  $e^{st}$  is input to the circuit, we expect the output to be

$$He^{st} = \lambda H(s)e^{st} = -\frac{e^{st}}{1 + RCs}.$$

This satisfies the differential equation  $x = -y - RCDy$  for the active RC circuit.

In Chapter 2 we modelled electrical, mechanical, and electromechanical devices by differential equations of the form

$$\sum_{\ell=0}^m a_\ell D^\ell x = \sum_{\ell=0}^k b_\ell D^\ell y. \quad (3.4.3)$$

Suppose that  $H$  is a linear shift-invariant system such that  $y = Hx$  if  $x$  and  $y$  satisfy a differential equation of this form. The response of  $H$  to the complex exponential signal  $e^{st}$  satisfies  $He^{st} = \lambda H(s)e^{st}$ . Substituting  $x(t) = e^{st}$  and  $y = \lambda H(s)e^{st}$  into the differential equation gives

$$\sum_{\ell=0}^m a_\ell s^\ell e^{st} = \sum_{\ell=0}^k b_\ell s^\ell \lambda H(s)e^{st} = \lambda H(s) \sum_{\ell=0}^k b_\ell s^\ell e^{st}.$$

Rearranging we find that the transfer function  $\lambda H$  satisfies

$$\lambda H(s) = \frac{\sum_{\ell=0}^m a_\ell s^\ell e^{st}}{\sum_{\ell=0}^k b_\ell s^\ell e^{st}} = \frac{a_0 + a_1 s + \dots + a_m s^m}{b_0 + b_1 s + \dots + b_k s^k}. \quad (3.4.4)$$

The transfer function takes the form of a polynomial divided by a polynomial. The transfer function associated with a linear differential equation with constant coefficients (3.4.3) always takes this form. We will study these transfer functions in greater detail when we introduce the Laplace transform in Chapter 4.

### 3.5 The spectrum

It is of interest to focus on the transfer function when  $s$  is purely imaginary, that is, when  $s = j\omega$  with  $\omega \in \mathbb{R}$ . In this case the complex exponential signal takes the form

$$e^{j\omega t} = e^{j2\pi f} = \cos(2\pi ft) + j \sin(2\pi ft)$$

where  $\omega = 2\pi f$ . This signal is oscillatory when  $f \neq 0$  and does not decay or explode as  $|t| \rightarrow \infty$ . Let  $H \in X \rightarrow Y$  be a linear shift-invariant system with domain  $X$  containing the signal  $e^{j2\pi ft}$ , that is,  $j2\pi ft \in \text{cep } X$ . We denote by  $\Lambda H$  the function satisfying

$$\Lambda H(f) = \lambda H(j2\pi f) \quad j2\pi f \in \text{cep } X$$

called the **spectrum** of  $H$ . It will typically be the case that  $\text{cep } X$  contains the entire imaginary axis and so the domain of  $\Lambda H$  is the set of real numbers  $\mathbb{R}$ . In this case the spectrum is a signal, that is,  $\lambda H \in \mathbb{R} \rightarrow \mathbb{C}$ . The independent variable  $f$  typically represents frequency in Hertz.

It follows from (3.4.1) that the response of  $H$  to  $e^{j2\pi ft} \in X$  satisfies

$$He^{j2\pi ft} = \lambda H(j2\pi f)e^{j2\pi ft} = \Lambda H(f)e^{j2\pi ft}$$

It is of interest to consider the **magnitude spectrum**  $|\Lambda H(f)|$  and the **phase spectrum**  $\angle \Lambda H(f)$  separately. The notation  $\angle$  denotes the **argument** (or **phase**) of a complex number. We have,

$$\Lambda H(f) = |\Lambda H(f)| e^{j\angle \Lambda H(f)}$$

and correspondingly

$$He^{j2\pi ft} = |\Lambda H(f)| e^{j(2\pi ft + \angle \Lambda H(f))}.$$

Taking real and imaginary parts we obtain the pair of real valued solutions

$$\begin{aligned} H \cos(2\pi ft) &= |\Lambda H(f)| \cos(2\pi ft + \angle \Lambda H(f)), \\ H \sin(2\pi ft) &= |\Lambda H(f)| \sin(2\pi ft + \angle \Lambda H(f)). \end{aligned} \tag{3.5.1}$$

Consider again the active RC circuit with  $H$  the system mapping input voltage  $x$  to output voltage  $y$ . According to (3.4.2) the spectrum of  $H$  is

$$\Lambda H(f) = -\frac{1}{1 + 2\pi RCfj}. \tag{3.5.2}$$

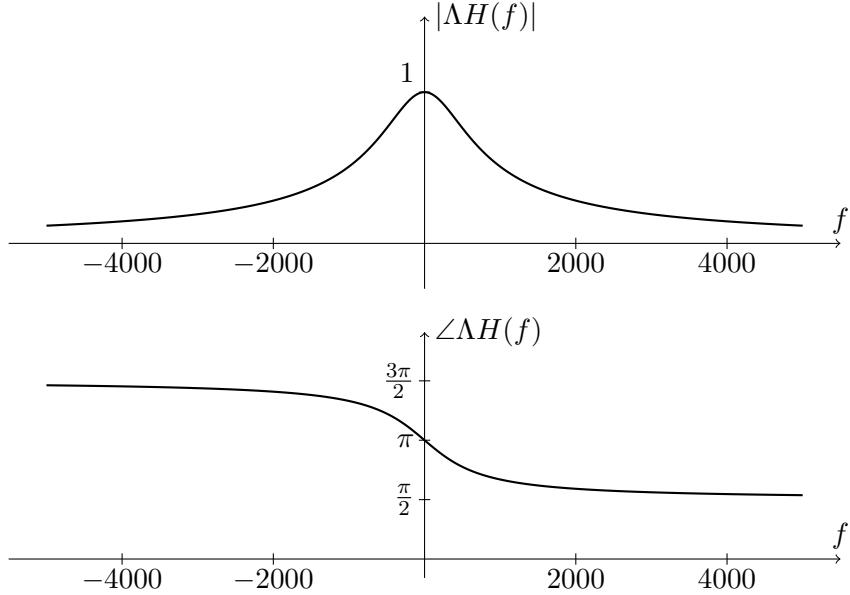


Figure 3.6: Magnitude spectrum (top) and phase spectrum (bottom) of the active RC circuit with  $R = 27 \times 10^3$  and  $C = 10 \times 10^{-9}$ .

The magnitude and phase spectrum is

$$|\Lambda H(f)| = (1 + 4\pi^2 R^2 C^2 f^2)^{-\frac{1}{2}}, \quad \angle \Lambda H(f) = \pi - \text{atan}(2\pi R C f).$$

These are plotted in Figure 3.6 when  $R = 27 \times 10^3$  and  $C = 10 \times 10^{-9}$ . Observe from the plot of the magnitude spectrum that a low frequency sinusoidal signal, say 100Hz or less, input to the active RC circuit results in a sinusoidal output signal with the same frequency and approximately the same amplitude. However, a high frequency sinusoidal signal, say greater than 1000Hz, input to the circuit results in a sinusoidal output signal with the same frequency, but smaller amplitude. For this reason RC circuits are called **low pass filters**.

**Test 4 (Spectrum of the active RC circuit)** We test the hypothesis that the active RC circuit satisfies (3.5.1). To do this sinusoidal signals at varying frequencies of the form

$$x_k(t) = \sin(2\pi f_k t), \quad f_k = \lceil 110 \times 2^{k/2} \rceil, \quad k = 0, 1, \dots, 12$$

are input to the active RC circuit constructed as in Test 3 with  $R = R_1 = 27\text{k}\Omega$  and  $C = C_2 = 10\text{nF}$ . The notation  $\lceil \cdot \rceil$  denotes rounding to the nearest integer with half integers rounded up. In view of (3.5.1) the expected output

signals are of the form

$$y_k(t) = |\Lambda H(f_k)| \sin(2\pi f_k t + \angle \Lambda H(f_k)), \quad k = 0, 1, \dots, 12.$$

This equality can also be shown directly using the differential equation for the active RC circuit.

Using the soundcard each signal  $x_k$  is played for a period of approximately 1 second and approximately  $F = 44100$  samples are obtained. On the soundcard hardware used for this test samples near the beginning and end of playback are distorted. This appears to be an unavoidable feature of the soundcard. To alleviate this we discard the first  $10^4$  samples and use only the  $L = 8820$  samples that follow (corresponding to 200ms of signal). After this process we have samples  $x_{k,1}, \dots, x_{k,L}$  and  $y_{k,1}, \dots, y_{k,L}$  of the input and output signals corresponding with the  $k$ th signal  $x_k$ . The samples are expected to take the form

$$x_{k,\ell} \approx x_k(P\ell - \tau) = \rho \sin(2\pi f_k P\ell - \theta)$$

and

$$y_{k,\ell} \approx y_k(\ell P - \tau) = |\Lambda H(f_k)| \rho \sin(2\pi f_k P\ell - \theta + \angle \Lambda H(f_k))$$

where  $P = \frac{1}{F}$  is the sample period, the positive real number  $\rho$  corresponds with the gain on the input and output of the soundcard, and  $\theta = 2\pi f_k \tau$  corresponds with delays caused by discarding the first  $10^4$  samples and also unavoidable delays that occur when starting soundcard playback and recording.

We will not measure the gain  $\rho$  nor the delay  $\theta$ , but will be able to test the properties of the circuit without knowledge of these. To simplify notation put  $\gamma = 2\pi f_k P$ . From the samples of the input signal  $x_{k,1}, \dots, x_{k,L}$  compute the complex number

$$A = \frac{2j}{L} \sum_{\ell=1}^L x_{k,\ell} e^{-j\gamma\ell} \approx \frac{2j}{L} \sum_{\ell=1}^L \rho \sin(\gamma\ell - \theta) e^{-j\gamma\ell} = \alpha + \alpha^* C$$

where  $\alpha = \rho e^{-j\theta}$  and  $\alpha^*$  denotes the complex conjugate of  $\alpha$  and

$$C = e^{-\gamma(L+1)} \frac{\sin(\gamma L)}{L \sin(\gamma)} \quad (\text{Exercise 3.9}).$$

Similarly, from the samples of the output signal  $y_{k,1}, \dots, y_{k,L}$  we compute the complex number

$$B = \frac{2j}{L} \sum_{\ell=1}^L y_{k,\ell} e^{-j\gamma\ell} \approx \beta + \beta^* C$$

where  $\beta = \rho e^{-j\theta} \Lambda H(f_k) = \alpha \Lambda H(f_k)$ . Now compute the quotient

$$Q_k = \frac{B - B^*C}{A - A^*C} \approx \frac{\beta(1 + |C|^2)}{\alpha(1 + |C|^2)} = \frac{\beta}{\alpha} = \Lambda H(f_k).$$

Thus, we expect the quotient  $Q_k$  to be close to the spectrum of the active RC circuit evaluated at frequency  $f_k$ . We will test this hypothesis by observing the magnitude and phase of  $Q_k$  individually, that is, we will test the expected relationships

$$|Q_k| \approx |\Lambda H(f_k)| = \sqrt{\frac{1}{1 + 4\pi^2 R^2 C^2 f_k^2}}$$

and

$$\angle Q_k \approx \angle \Lambda H(f_k) = \pi - \text{atan}(2\pi R C f_k)$$

for each  $k = 0, \dots, 12$ . Figure 3.7 plots the hypothesised magnitude and phase spectrum alongside the measurements  $Q_k$  for  $k = 0, \dots, 12$ .

## Exercises

- 3.1. Show that the convolution of two absolutely integrable signals is absolutely integrable.
- 3.2. State whether each of the following systems are: causal, linear, shift-invariant, or stable. Plot the impulse and step response of the systems whenever they exist. In each case, assume the domain to be the set of locally integrable signals.
  - (a)  $Hx(t) = 3x(t-1) - 2x(t+1)$
  - (b)  $Hx(t) = \sin(2\pi x(t))$
  - (c)  $Hx(t) = t^2 x(t)$
  - (d)  $Hx(t) = \int_{-1/2}^{1/2} \cos(\pi\tau)x(t+\tau)d\tau$
- 3.3. Show that the system  $Hx(t) = \int_{-1}^1 \sin(\pi\tau)x(t+\tau)d\tau$  is linear shift-invariant and regular. Find and sketch the impulse response and the step response.
- 3.4. Let  $h$  be a locally integrable signal. Show that the set  $\text{dom } h$  defined in Section 3.1 on page 33 is a linear shift-invariant space.
- 3.5. Show that  $\text{dom } u$  where  $u$  is the step function is the subset of locally integrable signals such that  $\int_{-\infty}^0 |x(t)| dt < \infty$ .

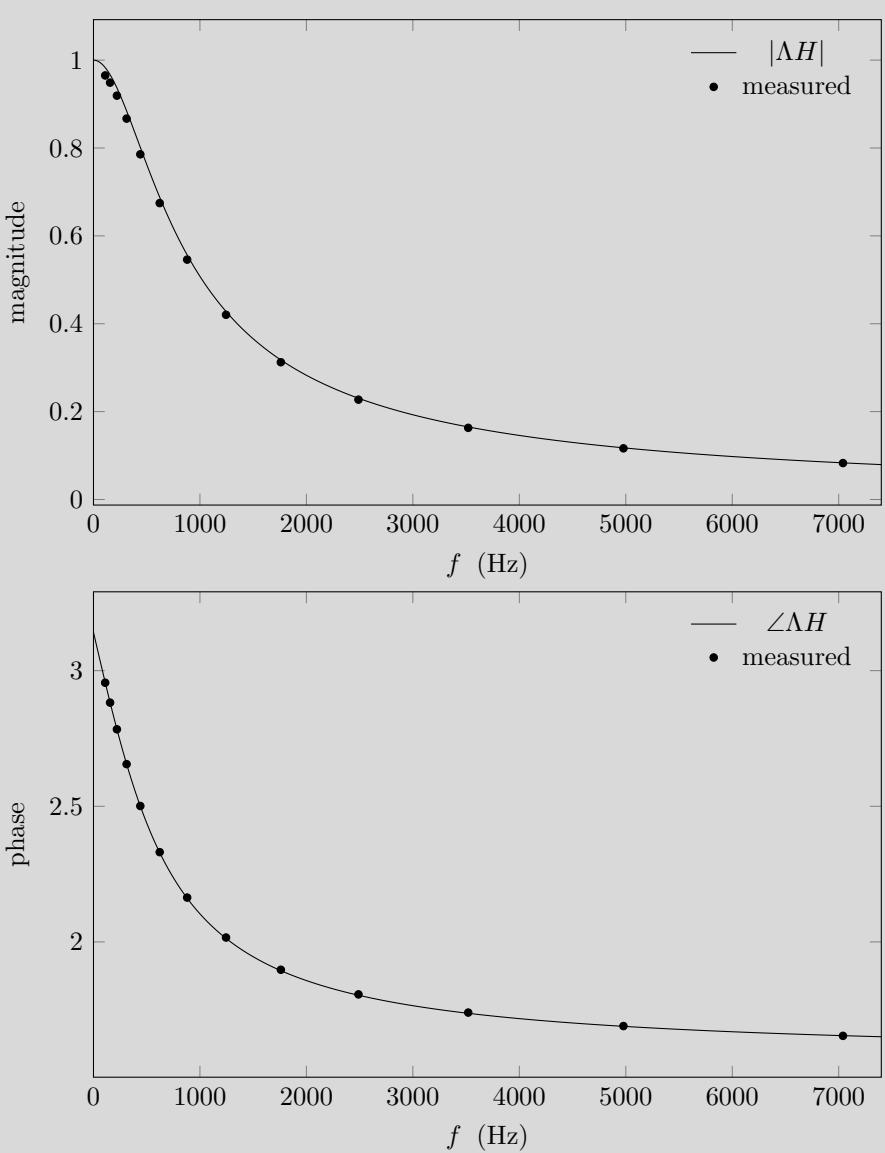


Figure 3.7: Hypothesised magnitude spectrum  $|\Lambda H(f)|$  (top) and phase spectrum  $\angle \Lambda H(f)$  (bottom) and the measured magnitude and phase spectrum  $|Q_k|$  and  $\angle Q_k$  for  $k = 0, \dots, 12$  (dots).

- 3.6. Show that a regular system is stable if and only if its impulse response is absolutely integrable.
- 3.7. Define signals  $x(t) = u(t)$ ,  $y = u(-t)$ , and  $z(t) = \Pi(t) - \Pi(t-1)$  where  $u$  is the step function and  $\Pi$  is the rectangular pulse. Plot  $x$ ,  $y$ , and  $z$  and show that the associative property of convolution does not hold for these signals. That is, show that  $x * (y * z) \neq (x * y) * z$ .
- 3.8. Show that  $\sum_{\ell=1}^L e^{\beta\ell} = \frac{e^{\beta(L+1)} - e^\beta}{e^\beta - 1}$  (Hint: sum a geometric progression).

- 3.9. Show that

$$\frac{2j}{L} \sum_{\ell=1}^L \sin(\gamma\ell - \theta) e^{-j\gamma\ell} = \alpha + \alpha^* C$$

where  $\alpha = e^{-j\theta}$  and  $C = e^{-j\gamma(L+1)} \frac{\sin(\gamma L)}{L \sin(\gamma)}$ . (Hint: solve Exercise 3.8 first and then use the formula  $2j \sin(x) = e^{jx} - e^{-jx}$ ).

## Chapter 4

# The Laplace transform

In the previous chapter we studied the properties of linear shift-invariant systems. We considered regular systems for which the response to input signal  $x$  is given by  $h * x$ , that is, by the convolution of  $x$  with the impulse response  $h$  of the system. We discovered that complex exponential signals of the form  $e^{st}$  were eigenfunctions of linear shift-invariant systems, that is, if  $H$  is the linear-shift-invariant system with domain  $X$ , then the response of  $H$  to input signal  $e^{st} \in X$  satisfies

$$He^{st} = \lambda H(s) e^{st}$$

where  $\lambda H(s)$  is a complex number that depends on  $s \in \mathbb{C}$ , but not on  $t \in \mathbb{R}$ . Considered as a function of  $s$ , the expression  $\lambda H$  is called the transfer function of the system  $H$ . The domain of  $\lambda H$  is the set of complex numbers  $s$  such that  $e^{st} \in X$  and this set is denoted by  $\text{cep } X$ .

Now consider the special case where  $H$  is a regular system with impulse response  $h$  and domain  $\text{dom } h$ . The response of  $H$  to a complex exponential signal  $e^{st} \in \text{dom } h$  is given by convolution of  $h$  with  $e^{st}$ , that is,

$$He^{st} = h * e^{st} = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau.$$

It follows that the transfer function of the regular system  $H$  satisfies

$$\lambda H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau \quad s \in \text{cep dom } h.$$

This is also known as the **Laplace transform** of the signal  $h$ . The set  $\text{cep dom } h$  is called the **region of convergence** of the Laplace transform of  $h$ . We will denote the region of convergence by the simpler notation  $\text{roc } h = \text{cep dom } h$  in what follows.

Let  $x$  be a signal. The Laplace transform of  $x$  is denoted by  $\mathcal{L}x$ . It is the complex valued function of  $\text{roc } x$  satisfying

$$\mathcal{L}x(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \quad s \in \text{roc } x. \quad (4.0.1)$$

The domain of  $\mathcal{L}x$  is the region of convergence  $\text{roc } x = \text{cep dom } x$ .

### 4.1 Regions of convergence

We now study the Laplace transform and the region of convergence in more detail. The region of convergence  $\text{roc } x = \text{cep dom } x$  is precisely the set of complex numbers  $s$  such that  $x(t)e^{-st}$  is absolutely integrable (Exercise 4.18), that is,

$$\int_{-\infty}^{\infty} |x(t)e^{-st}| dt < \infty \quad \text{if and only if } s \in \text{roc } x.$$

For example, consider the right sided signal  $e^{\alpha t}u(t)$ . We have

$$\int_{-\infty}^{\infty} |e^{\alpha t}u(t)e^{-st}| dt = \int_0^{\infty} e^{\text{Re}(\alpha-s)t} dt = \lim_{t \rightarrow \infty} \frac{e^{\text{Re}(\alpha-s)t}}{\text{Re}(\alpha-s)} - \frac{1}{\text{Re}(\alpha-s)}.$$

The limit above converges if and only if  $\text{Re}(\alpha - s) < 0$  and so the region of convergence of the Laplace transform of  $e^{\alpha t}u(t)$  is

$$\text{roc } e^{\alpha t}u(t) = \{s \in \mathbb{C} ; \text{Re } s > \text{Re } \alpha\}.$$

Figure 4.1 shows the region of convergence when  $\text{Re } \alpha = -2$ . Applying (4.0.1) we find that

$$\mathcal{L}(e^{\alpha t}u(t)) = \int_{-\infty}^{\infty} e^{\alpha t}e^{-st}u(t)dt = \lim_{t \rightarrow \infty} \frac{e^{(\alpha-s)t}}{\alpha-s} - \frac{1}{\alpha-s}.$$

The limit converges to zero when  $\text{Re}(\alpha - s) < 0$ , that is, when  $s \in \text{roc } e^{\alpha t}u(t)$ , and so the Laplace transform is

$$\mathcal{L}(e^{\alpha t}u(t)) = \frac{1}{s-\alpha} \quad \text{Re } s > \text{Re } \alpha.$$

Now consider the left sided signal  $e^{\beta t}u(-t)$ . The region of convergence is

$$\text{roc } e^{\beta t}u(-t) = \{s \in \mathbb{C} ; \text{Re } s < \text{Re } \beta\}$$

and the Laplace transform is

$$\mathcal{L}(e^{\beta t}u(-t)) = \lim_{t \rightarrow -\infty} \frac{e^{(\beta-s)t}}{\beta-s} + \frac{1}{\beta-s} = \frac{1}{\beta-s} \quad \text{Re } s < \text{Re } \beta.$$

The signal  $ae^{\alpha t}u(t) + be^{\beta t}u(-t)$  has Laplace transform

$$\begin{aligned} \mathcal{L}(ae^{\alpha t}u(t) + be^{\beta t}u(-t)) &= \int_{-\infty}^{\infty} (ae^{\alpha t}u(t) + be^{\beta t}u(-t))e^{-st}dt \\ &= a\mathcal{L}(e^{\alpha t}u(t)) + b\mathcal{L}(e^{\beta t}u(-t)) \end{aligned}$$

with region of convergence

$$\text{roc}(ae^{\alpha t}u(t) + be^{\beta t}u(-t)) = \{s \in \mathbb{C} ; \operatorname{Re} \alpha < \operatorname{Re} s < \operatorname{Re} \beta\}.$$

This region is shown in Figure 4.1 when  $\operatorname{Re} \alpha = -2$  and  $\operatorname{Re} \beta = 3$ . In the previous equation we have discovered that the Laplace transform is **linear**, that is, for signals  $x$  and  $y$  and non zero complex numbers  $a$  and  $b$ , the Laplace transform of the linear combination  $ax + by$  is

$$\mathcal{L}(ax + by) = a\mathcal{L}x + b\mathcal{L}y \quad s \in \text{roc}(ax + by) = \text{roc } x \cap \text{roc } y. \quad (4.1.1)$$

The region of convergence is the intersection of the regions of convergence of  $\mathcal{L}x$  and  $\mathcal{L}y$ .

In the previous example the region of convergence is the empty set  $\emptyset$  if  $\operatorname{Re} \alpha \geq \operatorname{Re} \beta$ . The Laplace transform is said not to exist in this case. Other signals have this property. For example, the signal  $x(t) = 1$  has no Laplace transform because

$$\int_{-\infty}^{\infty} |e^{-st}| dt = \lim_{t \rightarrow -\infty} \frac{e^{-\operatorname{Re} st}}{s} - \lim_{t \rightarrow \infty} \frac{e^{-\operatorname{Re} st}}{s}$$

and the limit as  $t \rightarrow -\infty$  converges only when  $\operatorname{Re} s < 0$  while the limit as  $t \rightarrow \infty$  converges only when  $\operatorname{Re} s > 0$ .

As a final example, consider the rectangular pulse

$$\Pi(t) = \begin{cases} 1 & -\frac{1}{2} < t \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

The region of convergence is the entire complex plane,  $\text{roc } \Pi = \mathbb{C}$ , because

$$\int_{-\infty}^{\infty} |\Pi(t)e^{-st}| dt = \int_{-1/2}^{1/2} e^{-\operatorname{Re} st} dt = \frac{e^{\operatorname{Re} s/2} - e^{-\operatorname{Re} s/2}}{s} < \infty$$

for all  $s \in \mathbb{C}$ . At  $s = 0$  the above expression is given the value 1 by appealing to continuity (Exercise 4.15). The Laplace transform of  $\Pi$  is

$$\mathcal{L}\Pi = \int_{-\infty}^{\infty} \Pi(t)e^{-st} dt = \int_{-1/2}^{1/2} e^{-st} dt = \frac{e^{s/2} - e^{-s/2}}{s} \quad s \in \mathbb{C}. \quad (4.1.2)$$

The examples just given exhibit all the possible types of regions of convergence. The region of convergence is either the entire complex plane, a left or right half plane, a vertical strip, or the empty set.

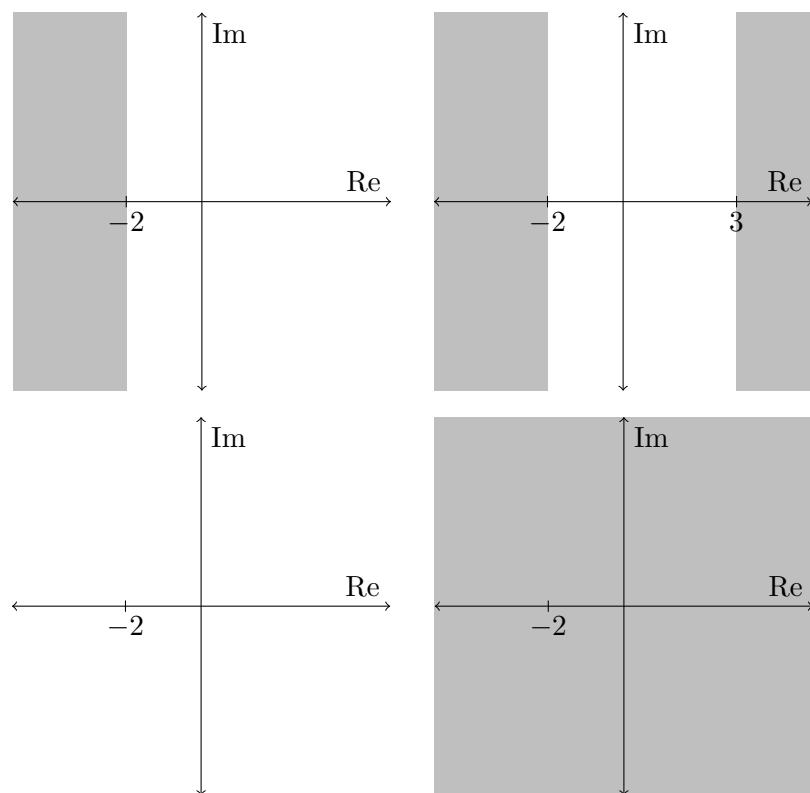


Figure 4.1: Regions of convergence (unshaded) of the Laplace transforms of the signal  $e^{-2t}u(t)$  (top left), the signal  $e^{-2t}u(t) + e^{3t}u(-t)$  (top right), the rectangular pulse  $\Pi$  (bottom left), and the constant signal  $x(t) = 1$  (bottom right).

## 4.2 The inverse Laplace transform

Given the Laplace transform  $\mathcal{L}x \in \text{roc } x \rightarrow \mathbb{C}$  the signal  $x$  can be recovered by the **inverse Laplace transform**

$$x(t) = \frac{1}{2\pi j} \lim_{\omega \rightarrow \infty} \int_{\sigma-j\omega}^{\sigma+j\omega} \mathcal{L}x(s)e^{st} ds,$$

where  $\sigma$  is any real number inside the region of convergence  $\text{roc } x$ . Solving the integral above typically requires a special type of integration called **contour integration** that we will not consider here [Stewart and Tall, 2004]. For our purposes, and for many engineering purposes, it suffices to remember only the following Laplace transform pair

$$\mathcal{L}(t^n u(t)) = \frac{n!}{s^{n+1}} \quad \text{Re } s > 0, \quad (4.2.1)$$

where  $n \geq 0$  is an integer (Exercise 4.2). The Laplace transforms of the signal  $x(t)$  and the signal  $e^{\alpha t}x(t)$  are related,

$$\begin{aligned} \mathcal{L}(e^{\alpha t}x(t))(s) &= \int_{-\infty}^{\infty} e^{\alpha t}x(t)e^{-st} dt \\ &= \int_{-\infty}^{\infty} x(t)e^{-(s-\alpha)t} dt \\ &= \mathcal{L}x(s-\alpha) \quad s - \alpha \in \text{roc } x. \end{aligned} \quad (4.2.2)$$

The region of convergence of  $\mathcal{L}(e^{\alpha t}x(t))$  is all those complex numbers  $s$  such that  $s - \alpha \in \text{roc } x$ . This is called the **frequency-shift rule**. Combining the frequency-shift rule with (4.2.1) we obtain the transform pair

$$\mathcal{L}(t^n e^{\alpha t} u(t)) = \frac{n!}{(s-\alpha)^{n+1}} \quad \text{Re } s > \text{Re } \alpha, \quad (4.2.3)$$

where  $n \geq 0$  is an integer. This is the only Laplace transform pair we require here.

A useful relationship exists between the Laplace transform of a signal  $x(t)$  and its time-scaled version  $x(\alpha t)$  where  $\alpha \neq 0$ ,

$$\mathcal{L}(x(\alpha t))(s) = \frac{1}{|\alpha|} \mathcal{L}x(s/\alpha), \quad \text{Re}(s/\alpha) \in \text{roc } x. \quad (4.2.4)$$

The region of convergence of  $\mathcal{L}(x(\alpha t))$  is those complex numbers  $s$  such that  $s/\alpha \in \text{roc } x$ . This is called the **time-scaling property** of the Laplace transform (Excercise 4.11).

### 4.3 The transfer function and the Laplace transform

We have already discovered that the transfer function  $\lambda H$  of a regular system  $H$  with impulse response  $h$  is equal to the Laplace transform  $\mathcal{L}h$  of the impulse response, that is,

$$\lambda H(s) = \mathcal{L}h(s) \quad s \in \text{roc } h.$$

The transfer functions of the shifter  $T_\tau$  and differentiator  $D$  can be obtained by inspection. For the shifter

$$T_\tau e^{st} = e^{s(t-\tau)} = e^{-s\tau} e^{st} \quad \text{and so} \quad \lambda T_\tau(s) = e^{-s\tau}. \quad (4.3.1)$$

For the special case of the identity system  $T_0$  we obtain  $\lambda T_0 = 1$ . For the differentiator

$$De^{st} = \frac{d}{dt} e^{st} = se^{st} \quad \text{and so} \quad \lambda D(s) = s.$$

More generally, for the  $k$ th differentiator

$$D^k e^{st} = \frac{d^k}{dt^k} e^{st} = s^k e^{st} \quad \text{and so} \quad \lambda D^k(s) = s^k. \quad (4.3.2)$$

The domain of  $\lambda T_\tau$  and of  $\lambda D^k$  is the entire complex plane  $\mathbb{C}$ . These results motivate assigning the following Laplace transforms to the delta “function”  $\delta$ , its shift  $T_\tau \delta = \delta(t - \tau)$ , and the  $\delta^k$  symbol,

$$\mathcal{L}\delta = 1, \quad \mathcal{L}(\delta(t - \tau)) = e^{-s\tau}, \quad \mathcal{L}\delta^k = s^k.$$

These conventions are common in the literature [Oppenheim et al., 1996].

We now study the transfer function of a system formed by composition. Let  $F \in X \rightarrow Y$  and  $G \in Y \rightarrow Z$  be linear shift-invariant systems with domains  $X$  and  $Y$  and suppose that the complex exponential signal  $e^{st} \in X$  if  $e^{st} \in Y$ , that is,  $\text{cep } X \subset \text{cep } Y$ . This will be the only case of interest to us. Let  $H \in X \rightarrow Z$  be the system formed by the composition  $H = GF$ . The response of  $H$  to the signal  $e^{st} \in X$  satisfies

$$He^{st} = GFe^{st} = G(\lambda F(s)e^{st}) = \lambda F(s)Ge^{st}$$

and since  $e^{st}$  is also in  $Y$ ,

$$He^{st} = \lambda F(s)\lambda G(s)e^{st} = \lambda H(s)e^{st}.$$

It follows that

$$\lambda H(s) = \lambda F(s)\lambda G(s) \quad s \in \text{cep } X = \text{cep } X \cap \text{cep } Y. \quad (4.3.3)$$

That is, the transfer function of a composition of linear shift-invariant systems is the multiplication of the transfer functions of those systems.

Now suppose that  $F$  and  $G$  are regular systems with impulse responses  $f$  and  $g$ . We showed in Section 3.3 that the composition  $H = GF$  is a regular system with impulse response given by the convolution  $h = g * f$  and we take its domain to be  $\text{dom } f \cap \text{dom } g$ . It can be shown that  $e^{st} \in \text{dom } f * g$  if and only if  $s \in \text{roc } f \cap \text{roc } g$ , that is,  $\text{cep dom } f * g = \text{roc } f \cap \text{roc } g$  (Exercise 4.19). Thus, for  $s \in \text{roc } f \cap \text{roc } g$  we have

$$\lambda H(s) = \mathcal{L}h(s) \quad \lambda F(s) = \mathcal{L}f(s) \quad \lambda G(s) = \mathcal{L}g(s),$$

and using (4.3.3) and dropping the “(s)”’s for notational clarity, we obtain,

$$\mathcal{L}(f * g) = \mathcal{L}h = \lambda H = \lambda F \lambda G = \mathcal{L}f \mathcal{L}g$$

for  $s \in \text{roc } f \cap \text{roc } g$ . Putting  $x = f$  and  $y = g$ , we obtain the **convolution theorem**,

$$\mathcal{L}(x * y) = \mathcal{L}x \mathcal{L}y \tag{4.3.4}$$

with region of convergence  $\text{roc } x \cap \text{roc } y$ . In words: the Laplace transform of a convolution of signals is the multiplication of their Laplace transforms. The region of convergence of  $\mathcal{L}(x * y)$  is the intersection of the regions of convergence of  $\mathcal{L}x$  and  $\mathcal{L}y$ , that is,  $\text{roc } x \cap \text{roc } y$ .

Let  $H$  be a regular system with impulse response  $h$  and let  $y = Hx$  be the response of the system  $H$  to input signal  $x \in \text{dom } h$ . We have  $y = h * x$  and the convolution theorem asserts that

$$\mathcal{L}y = \mathcal{L}h \mathcal{L}x = \lambda H \mathcal{L}x \tag{4.3.5}$$

with region of convergence  $\text{roc } h \cap \text{roc } x$ . Thus, the Laplace transform of the response  $y = Hx$  is the transfer function of the regular system  $H$  multiplied by the Laplace transform of the input signal  $x$ . This result also holds for the shifter, that is,

$$\mathcal{L}T_\tau x = \lambda T_\tau \mathcal{L}x = e^{-s\tau} \mathcal{L}x$$

with region of convergence  $\text{roc } x$ . This is called the **time-shift property** of the Laplace transform (Exercise 4.4). The result also holds for the differentiator, that is,

$$\mathcal{L}Dx = \lambda D \mathcal{L}x = s \mathcal{L}x$$

with region of convergence  $\text{roc } x$  under the added assumptions that  $x$  is differentiable, i.e.  $x \in C^1$ , and that

$$\lim_{t \rightarrow \infty} x(t)e^{-st} = \lim_{t \rightarrow -\infty} x(t)e^{-st} = 0 \quad \text{when } s \in \text{roc } x.$$

This is called the **differentiation property** of the Laplace transform (Exercise 4.5). Observe that the added assumptions are true if, for example,  $x(t)$  is finite.

In Chapter 2 we modelled electrical, mechanical, and electromechanical devices by linear differential equations with constant coefficients of the form

$$\sum_{\ell=0}^m a_\ell D^\ell x = \sum_{\ell=0}^k b_\ell D^\ell y.$$

Suppose that  $H$  is a linear shift-invariant system such that  $y = Hx$  if  $x$  and  $y$  satisfy a differential equation of this form. In Section 3.4 we found that the transfer function of  $H$  is a ratio of polynomials, that is,

$$\lambda H(s) = \frac{\sum_{\ell=0}^m a_\ell s^\ell e^{st}}{\sum_{\ell=0}^k b_\ell s^\ell e^{st}} = \frac{a_0 + a_1 s + \dots + a_m s^m}{b_0 + b_1 s + \dots + b_k s^k}.$$

Properties of  $H$  can be obtained by inspecting this transfer function. For example, the impulse response of  $H$  (if it exists) can be obtained by applying the inverse Laplace transform.

We now apply these results to the differential equations that model the RC electrical circuit from Figure 2.1 and the mass spring damper from Figure 2.2. The RC circuit is an example of what is called a **first order system** and the mass spring damper is an example of what is called a **second order system**.

#### 4.4 First order systems

Recall the passive electrical RC circuit from Figure 2.1. The differential equation modelling this circuit is (2.0.1),

$$x = y + RCDy$$

where  $x$  is the input voltage signal,  $y$  is the voltage over the capacitor, and  $R$  and  $C$  are the resistance and capacitance. The RC circuit is an example of a **first order system** so called because the highest order derivative that occurs is of order one, that is,  $D^k$  with  $k = 1$ . Let  $H$  be a system mapping the input voltage signal  $x$  to the output voltage signal  $y$ . From (3.4.4) the transfer function  $\lambda H$  is found to satisfy

$$\lambda H(s) = \frac{1}{1 + RCS} = \frac{r}{r + s}$$

where  $r = \frac{1}{RC}$ . The value  $\frac{1}{r} = RC$  is called the **time constant**. The impulse response of  $H$  is given by the inverse of this Laplace transform. There are two signals with Laplace transform  $\frac{r}{r+s}$ : the right sided signal  $re^{-rt}u(t)$  with region of convergence  $\text{Re } s > -r$ , and the left sided signal  $-re^{-rt}u(-t)$  with region of convergence  $\text{Re } s < -r$ . The RC circuit (and in fact all physically realisable systems) are expected to be causal. For this

reason, the left sided signal  $-re^{-rt}u(-t)$  cannot be the impulse response of  $H$ . The impulse response is the right sided signal

$$h(t) = re^{-rt}u(t).$$

Given an input voltage signal  $x$  we can now find the corresponding output signal  $y = Hx$  by convolving  $x$  with the impulse response  $h$ . That is,

$$y = Hx = h * x = \int_{-\infty}^{\infty} re^{-r\tau}u(\tau)x(t - \tau)d\tau = r \int_0^{\infty} e^{-r\tau}x(t - \tau)d\tau.$$

If  $r \geq 0$  the impulse response is absolutely integrable, that is,

$$\begin{aligned}\|h\|_1 &= \int_{-\infty}^{\infty} |re^{-rt}u(t)| dt \\ &= r \int_0^{\infty} e^{-rt}dt \\ &= 1 - \lim_{t \rightarrow \infty} e^{-rt} = 1,\end{aligned}$$

and the system is stable (Exercise 3.6). However, if  $r < 0$  the impulse response is not absolutely integrable and the system is not stable. Figure 4.3 shows the impulse response when  $r = -\frac{1}{5}, -\frac{1}{3}, -\frac{1}{2}, 1, 2$ . In a passive electrical RC circuit the resistance  $R$  and capacitance  $C$  are always positive and  $r = \frac{1}{RC}$  is positive. For this reason, passive electrical RC circuits are always stable.

From (3.1.3), the step response  $Hu$  is given by applying the integrator  $I_{\infty}$  to the impulse response, that is,

$$Hu = I_{\infty}h = \int_{-\infty}^t re^{-r\tau}u(\tau)d\tau = \begin{cases} r \int_0^t e^{-r\tau}d\tau & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

or more simply

$$Hu = (1 - e^{-rt})u(t). \quad (4.4.1)$$

This step response is plotted in Figure 4.3.

**Test 5 (The impulse response of the active RC circuit)** In this test we again use the active RC circuit from Test 3 with resistors  $R = R_1 = R_2 = 27\text{k}\Omega$  and capacitor  $C = C_2 = 10\text{nF}$ . In Test 3 we applied the differential equation (2.2.4) to the reconstructed output signal  $\tilde{y}$  and asserted that the resulting signal was close to the reconstructed input signal  $\tilde{x}$ . In this test we instead convolve the input signal  $\tilde{x}$  with the impulse response

$$h(t) = -\frac{1}{RC}e^{-t/RC}u(t) = -re^{-rt}u(t), \quad r = \frac{1}{RC} = \frac{10^5}{27}$$

and assert that the resulting signal is close to the output signal  $\tilde{y}$ . That is, we test the expected relationship

$$\tilde{y} \approx h * \tilde{x} = \int_{-\infty}^{\infty} h(\tau) \tilde{x}(t - \tau) d\tau.$$

From (1.3.4),

$$\begin{aligned}\tilde{y}(t) &\approx \int_{-\infty}^{\infty} h(\tau) \sum_{\ell=1}^L x_{\ell} \operatorname{sinc}(Ft - F\tau - \ell) d\tau \\ &= \sum_{\ell=1}^L x_{\ell} \int_{-\infty}^{\infty} h(\tau) \operatorname{sinc}(Ft - F\tau - \ell) d\tau \\ &= \sum_{\ell=1}^L x_{\ell} g(Ft - \ell)\end{aligned}$$

where the function

$$g(t) = \int_{-\infty}^{\infty} h(\tau) \operatorname{sinc}(t - F\tau) d\tau = -r \int_0^{\infty} e^{-r\tau} \operatorname{sinc}(t - F\tau) d\tau.$$

An approximation of  $g(t)$  is made by the trapezoidal sum

$$g(t) \approx \frac{K}{2N} \left( p(0) + p(K) + 2 \sum_{n=1}^{N-1} p(\Delta n) \right),$$

where  $p(\tau) = h(\tau) \operatorname{sinc}(t - F\tau)$  and

$$K = -RC \log(10^{-3}), \quad N = \lceil 10FK \rceil, \quad \Delta = K/N.$$

Figure 4.2 plots the input signal  $\tilde{x}$ , output signal  $\tilde{y}$ , and hypothesised output signal  $h * \tilde{x}$  over a 4ms window.

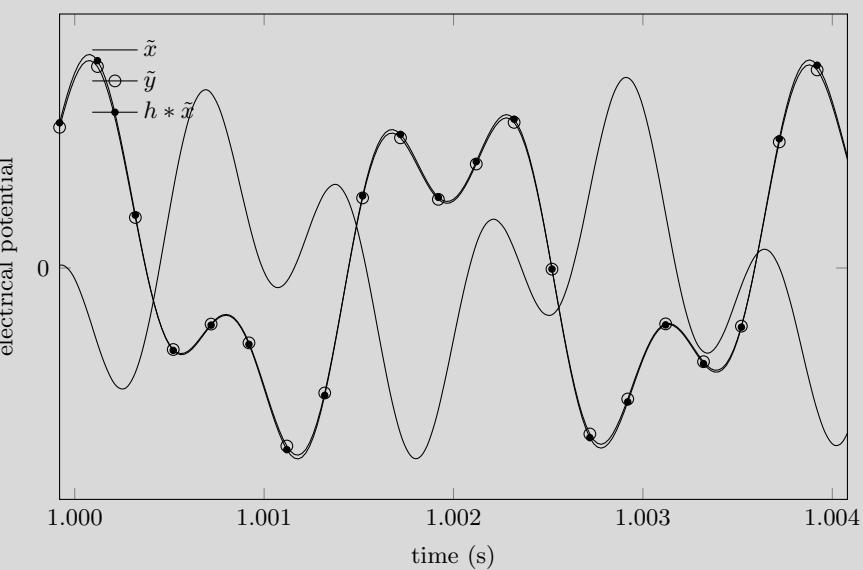


Figure 4.2: Plot of reconstructed input signal  $\tilde{x}$  (solid line), output signal  $\tilde{y}$  (solid line with circle), and hypothesised output signal  $h * \tilde{x}$  (solid line with dot).

## 4.5 Second order systems

Consider the mass spring damper system from Figure 2.2 that is described by the equation

$$f = Kp + BDp + MD^2p \quad (4.5.1)$$

where  $f$  is the force applied to the mass  $M$  and  $p$  is the position of the mass and  $K$  and  $B$  are the spring and damping coefficients. The mass spring damper is an example of a **second order system** because it contains differentiators  $D^2$  of order at most two. Another example of a second order system is the Sallen-Key active electrical circuit depicted in Figure 2.10. In Section 2 we were able to find the force  $f$  corresponding with a given position signal  $p$ . Suppose that  $H$  is a linear shift-invariant system mapping  $f$  to  $p$ , that is, such that  $p = Hf$ . We will find the impulse response of  $H$ . From 3.4.4 the transfer function is found to satisfy

$$\lambda H(s) = \frac{1}{K + Bs + Ms^2}.$$

We can invert this Laplace transform to obtain the impulse response. There are three cases to consider depending on whether the quadratic  $K + Bs + Ms^2$  has two distinct real roots, is irreducible (does not have real roots), or has two identical real roots.

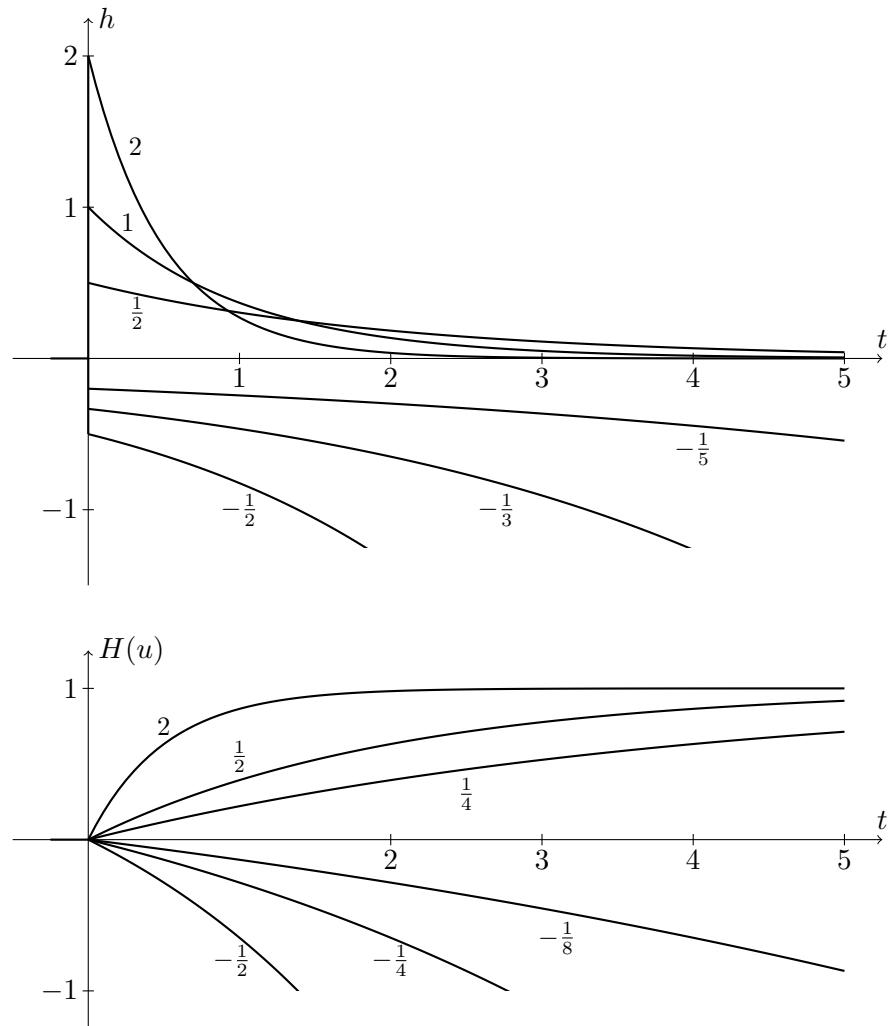


Figure 4.3: Top: impulse response of a first order system with  $r = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{5}, \frac{1}{2}, 1, 2$ . Bottom: step response of a first order system with  $r = -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 2$ .

**Case 1: (Distinct real roots)** In this case, the roots are

$$\beta - \alpha, \quad -\beta - \alpha,$$

where

$$\alpha = \frac{B}{2M}, \quad \beta = \frac{\sqrt{B^2 - 4KM}}{2M}$$

and  $B^2 - 4KM > 0$ . By a partial fraction expansion (Exercise 4.8),

$$\begin{aligned} \lambda H(s) &= \frac{1}{M(s - \beta + \alpha)(s + \beta + \alpha)} \\ &= \frac{1}{2\beta M} \left( \frac{1}{s - \beta + \alpha} - \frac{1}{s + \beta + \alpha} \right). \end{aligned}$$

From (4.2.3) we obtain the transform pairs

$$\mathcal{L}(e^{(\beta-\alpha)t}u(t)) = \frac{1}{s - \beta + \alpha}, \quad \mathcal{L}(e^{-(\beta+\alpha)t}u(t)) = \frac{1}{s + \beta + \alpha}.$$

As in Section 4.4, other signals with these Laplace transforms are discarded because they do not lead to an impulse response that is zero for  $t < 0$ . That is, they do not lead to a causal system  $H$ . The impulse response of  $H$  is thus

$$h(t) = \frac{1}{2\beta M} u(t) e^{-\alpha t} (e^{\beta t} - e^{-\beta t}).$$

This is a sum of the impulse responses of two first order systems.

**Case 2: (Distinct imaginary roots)** The solution is as in the previous case, but now  $4KM - B^2 > 0$  and  $\beta$  is imaginary. Put  $\theta = \beta/j$  so that

$$e^{\beta t} - e^{-\beta t} = e^{j\theta t} - e^{-j\theta t} = 2j \sin(\theta t).$$

The impulse response of  $H$  is

$$h(t) = \frac{1}{\theta M} u(t) e^{-\alpha t} \sin(\theta t).$$

**Case 3: (Identical roots)** In this case, the two roots are equal to  $-\alpha$  and

$$\lambda H(s) = \frac{1}{M(s + \alpha)^2}.$$

From (4.2.3) we obtain the transform pair

$$\mathcal{L}(te^{-\alpha t}u(t)) = \frac{1}{(s + \alpha)^2}$$

and this is the only signal with this Laplace transform that leads to a causal impulse response. The impulse response of  $H$  is thus

$$h(t) = \frac{1}{M}te^{-\alpha t}u(t).$$

A second order system is called **overdamped** when there are two distinct real roots, **underdamped** when their are two distinct imaginary roots, and **critically damped** when the roots are identical. The different types of impulse responses for are plotted in Figure 4.4.

With no damping (i.e. damping coefficient  $B = 0$ ) the roots are of the form  $\pm\beta$  and have no real part. In this case, the impulse response is

$$h(t) = \frac{1}{\theta M}u(t)\sin(\theta t),$$

where  $\theta = \beta/j = \sqrt{KM}$  is called the **natural frequency** of the second order system. This impulse response oscillates for all  $t > 0$  without decay or explosion. Two identical roots occur when the damping coefficient  $B = \sqrt{4KM}$  and this is sometimes called the **critical damping coefficient**.

The impulse response of a second order system is absolutely integrable when  $\alpha = \frac{B}{2M} > 0$ , but not when  $\alpha \leq 0$ . Thus, the system is stable when  $\alpha > 0$  and not stable when  $\alpha \leq 0$ . For the mass spring damper both the mass  $M$  and damping coefficient  $B$  are positive and so mass spring dampers are always stable.

From (3.1.3) the step response  $H(u)$  is given by applying the integrator  $I_\infty$  to the impulse response. There are three cases to consider depending on whether the system is overdamped, underdamped, or critically damped. When the system is overdamped the step response is

$$\begin{aligned} Hu = I_\infty h &= \frac{1}{2\beta M} \int_{-\infty}^t e^{-\alpha\tau} (e^{\beta\tau} - e^{-\beta\tau}) u(\tau) d\tau \\ &= \frac{1}{2\beta M} \int_0^t e^{-\alpha\tau} (e^{\beta\tau} - e^{-\beta\tau}) d\tau \\ &= \frac{1}{2\beta M} u(t) \left( \frac{e^{(\beta-\alpha)t} - 1}{\beta - \alpha} + \frac{e^{-(\beta+\alpha)t} - 1}{\beta + \alpha} \right). \end{aligned}$$

When the system is underdamped the step response is

$$\begin{aligned} Hu = I_\infty h &= \frac{1}{\theta M} \int_0^t e^{-\alpha\tau} \sin(\theta\tau) dt \\ &= u(t) \left( \frac{\theta - e^{-t\alpha} (\theta \cos(t\theta) + \alpha \sin(t\theta))}{M\theta(\alpha^2 + \theta^2)} \right). \end{aligned}$$

When the system is critically damped the step response is

$$\begin{aligned} Hu = I_\infty h &= \frac{1}{\theta M} \int_0^t \frac{1}{M} te^{-\alpha t} dt \\ &= \frac{1}{M\alpha^2} u(t)(1 - e^{-t\alpha s}(1 + t\alpha)). \end{aligned}$$

These step responses are plotted in Figure 4.5.

## 4.6 Poles, zeros, and stability

The transfer function of a system described by a linear differential equation with constant coefficients is of the form of (3.4.4), that is,

$$\lambda H(s) = \frac{a_0 + a_1 s + \dots + a_m s^m}{b_0 + b_1 s + \dots + b_k s^k}.$$

Factorising the polynomials on the numerator and denominator we obtain

$$\lambda H(s) = C \frac{(s - \alpha_0)(s - \alpha_1) \cdots (s - \alpha_m)}{(s - \beta_0)(s - \beta_1) \cdots (s - \beta_k)},$$

where  $\alpha_0, \dots, \alpha_m$  are the roots of the numerator polynomial  $a_0 + a_1 s + \dots + a_m s^m$ , and  $\beta_0, \dots, \beta_k$  are the roots of the denominator polynomial  $b_0 + b_1 s + \dots + b_k s^k$ , and  $C = \frac{a_m}{b_m}$ . That such a factorisation is always possible is called the **fundamental theorem of algebra** [Fine and Rosenberger, 1997]. If the numerator and denominator polynomials share one or more roots, then these roots cancel leaving the simpler expression

$$\lambda H(s) = C \frac{(s - \alpha_d)(s - \alpha_{d+1}) \cdots (s - \alpha_m)}{(s - \beta_d)(s - \beta_{d+1}) \cdots (s - \beta_k)}, \quad (4.6.1)$$

where  $d$  is the number of shared roots, these shared roots being

$$\alpha_0 = \beta_0, \quad \alpha_1 = \beta_1, \quad \dots, \quad \alpha_{d-1} = \beta_{d-1}.$$

The roots from the numerator  $\alpha_d, \dots, \alpha_m$  are called the **zeros** and the roots from the denominator  $\beta_d, \dots, \beta_m$  are called the **poles**. A **pole-zero plot** is constructed by marking the complex plane with a cross at the location of each pole and a circle at the location of each zero. Pole-zero plots for the first order system from Section 4.4, the second order system from Section 4.5, and the system describing the PID controller (2.2.7) are shown in Figure 4.6.

It is always possible to apply partial fractions and write (4.6.1) in the form

$$\lambda H(s) = p(s) + \sum_{\ell \in K} \frac{A_\ell}{(s - \beta_\ell)^{r_\ell}},$$

Figure 4.4: Impulse response of the mass spring damper with  $M = 1$ ,  $K = \frac{\pi^2}{4}$  and damping constant  $B = \frac{\pi}{3}$  (underdamped),  $B = \sqrt{4KM} = \pi$  (critically damped), and  $B = 2\pi$  (overdamped).

Figure 4.5: Step response of the mass spring damper with  $M = 1$ ,  $K = \frac{\pi^2}{4}$  and damping constant  $B = \frac{\pi}{3}$  (underdamped),  $B = \sqrt{4KM} = \pi$  (critically damped), and  $B = 2\pi$  (overdamped).

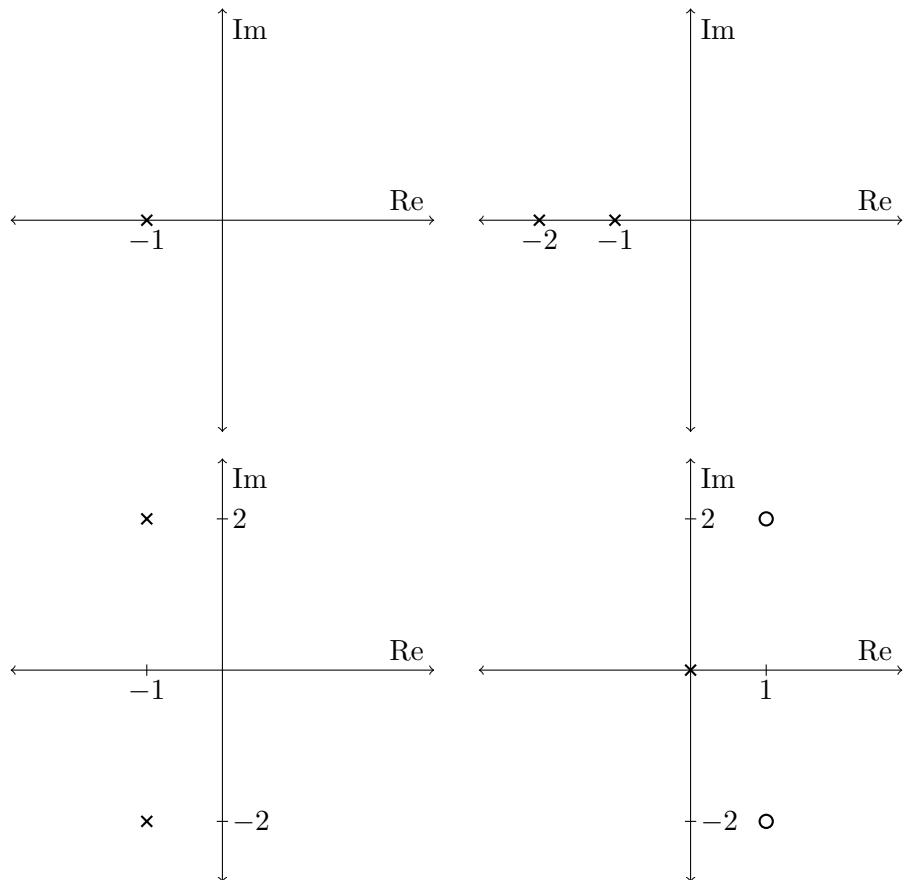


Figure 4.6: Top left: pole zero plot for the first order system  $x = y + Dy$ . There is a single pole at  $-1$ . Top right: pole zero plot for the overdamped second order system  $x = 2y + 3Dy + D^2y$  that has two real poles at  $-1$  and  $-2$ . Bottom left: pole zero plot for the underdamped second order system  $x = 5y + 2Dy + D^2y$  that has two imaginary poles at  $-1 + 2j$  and  $-1 - 2j$ . The poles form a conjugate pair. Bottom right: pole zero plot for the equation  $Dy = 5x - 2Dx + D^2x$  that models a PID controller (2.2.7). The system has a single pole at the origin and two zeros at  $1 + 2j$  and  $1 - 2j$ .

where  $r_\ell$  are positive integers,  $A_\ell$  are complex constants,  $K$  is a subset of the indices from  $\{d, d+1, \dots, k\}$ , and  $p(s)$  is a polynomial of degree  $m-k$ . If  $k > m$  then  $p(s) = 0$ . The integer  $r_\ell$  is called the **multiplicity** of the pole  $\beta_\ell$ . We see that the transfer function contains the summation of two parts: the polynomial  $p(s)$ , and a sum of terms of the form  $\frac{A}{(s-\beta)^r}$ . Let  $p(s) = \gamma_0 + \gamma_1 s + \dots + \gamma_{m-k} s^{m-k}$ . This polynomial is the transfer function of the nonregular system

$$F = \gamma_0 T_0 + \gamma_1 D + \gamma_2 D^2 + \dots + \gamma_{m-k} D^{m-k}.$$

This system is a linear combination of the identity system  $T_0$  and differentiators of order at most  $m-k$ . From (4.2.3),

$$\mathcal{L}\left(\frac{A}{r!} t^{r-1} e^{\beta t} u(t)\right) = \frac{A}{(s-\beta)^r} \quad \operatorname{Re} s > \operatorname{Re} \beta$$

and so the terms of the form  $\frac{A}{(s-\beta)^r}$  correspond with the transfer function of a regular system with impulse response  $\frac{A}{r!} t^{r-1} e^{\beta t} u(t)$ . Other signals with Laplace transform  $\frac{A}{(s-\beta)^r}$  are discarded because they do not correspond with the impulse response of a causal system. Thus,  $\sum_{\ell \in K} \frac{A_\ell}{(s-\beta_\ell)^{r_\ell}}$  is the transfer function of the regular system  $G$  with impulse response

$$g(t) = u(t) \sum_{\ell \in K} \frac{A_\ell}{r_\ell!} t^{r_\ell-1} e^{\beta_\ell t}. \quad (4.6.2)$$

The system  $H$  mapping  $x$  to  $y$  is the sum of the regular system  $G$  and nonregular system  $F$ , that is,

$$y = Hx = Fx + Gx.$$

Observe that  $H$  is regular only if the system  $F = 0$ , that is, only if  $F$  maps all input signals to the signal  $x(t) = 0$  for all  $t \in \mathbb{R}$ . This occurs only when the polynomial  $p(s) = 0$ , that is, only when the number of poles exceeds the number of zeros. The system  $H$  will be stable if both  $F$  and  $G$  are stable. Because the differentiator  $D^\ell$  is not stable (Exercise 1.17) the system  $F$  is stable if and only if the order of the polynomial  $p(s)$  is zero, that is, if  $p(s) = \gamma_0$  is a constant (potentially  $\gamma_0 = 0$ ). In this case  $Fx = \gamma_0 T_0 x$  is the identity system multiplied by a constant. The polynomial  $p(s)$  is a constant only when the order of the denominator polynomial is greater than or equal to the order of the numerator polynomial, that is, when the number of poles is greater than or equal to the number of zeros. The regular system  $G$  is stable if and only if its impulse response  $g$  is absolutely integrable. This occurs only when the terms  $e^{\beta_\ell t}$  inside the sum (4.6.2) are decreasing as  $t \rightarrow \infty$ , that is, only if the real part of the poles  $\operatorname{Re} \beta_\ell$  are negative. Thus, the system  $G$  is stable if and only if the real part of the poles are strictly negative.

The stability of the system  $H$  can be immediately determined from its pole-zero plot. The system is stable if and only if:

1. the number of poles is greater than or equal to the number of zeros (there are at least as many crosses on the pole-zero plot as circles),
2. No poles (crosses) lie on the imaginary axis or in the right half of the complex plane.

The pole-zero plots in Figure 4.6 all represent stable systems with the exception of the plot on the bottom right (a PID controller). This system has two zeros and only one pole. The single pole is contained on the imaginary axis.

#### 4.6.1 Two masses, a spring, and a damper

Consider the system involving two masses, a spring, and a damper in Figure 2.11. From (2.3.2), the equation relating the force applied to the first mass  $f$  and the position of the second mass  $p$  is

$$f = BDp + (M_1 + M_2)D^2p + \frac{BM_2}{K}D^3p + \frac{M_1M_2}{K}D^4p,$$

where  $B$  is the damping coefficient,  $K$  is the spring constant, and  $M_1$  and  $M_2$  are the masses. The transfer function of a system  $H$  that maps  $f$  to  $p$  is

$$\lambda H(s) = \frac{1}{s(B + (M_1 + M_2)s + \frac{BM_2}{K}s^2 + \frac{M_1M_2}{K}s^3)}.$$

The system has no zeros and 4 poles. One of these poles always exists at the origin. The system is not stable because this pole is not strictly in the left half of the complex plane.

Consider the specific case when  $B = K = M_1 = M_2 = 1$ . Factorising the denominator polynomial gives

$$\lambda H(s) = \frac{1}{s(s - \beta_1)(s - \beta_2)(s - \beta_2^*)},$$

where

$$\beta_1 = \frac{1}{3} \left( \gamma - \frac{5}{\gamma} - 1 \right) \approx -0.56984,$$

$$\beta_2 = \frac{1}{6} \left( \frac{5(1 + j\sqrt{3})}{\gamma} - (1 - j\sqrt{3})\gamma - \frac{1}{2} \right) \approx -0.21508 + 1.30714j,$$

and  $\gamma = (\frac{3\sqrt{69}-11}{2})^{1/3}$ . Applying partial fractions (Exercise 4.9) gives

$$\lambda(H) = \frac{1}{s(s - \beta_1)(s - \beta_2)(s - \beta_2^*)} = \frac{A_0}{s} + \frac{A_1}{s - \beta_1} + \frac{A_2}{s - \beta_2} + \frac{A_2^*}{s - \beta_2^*},$$

Figure 4.7: Impulse response of the system with two masses, a spring, and a damper, where  $B = K = M_1 = M_2 = 1$ .

where

$$A_0 = -\frac{1}{\beta_1|\beta_2|^2} = 1, \quad A_1 = \frac{1}{\beta_1|\beta_1 - \beta_2|^2} \approx -0.956611,$$

$$A_2 = \frac{1}{\beta_2(\beta_2 - \beta_1)(\beta_2 - \beta_2^*)} \approx -0.0216944 + 0.212084j.$$

From (4.6.2), the impulse response of  $H$  is

$$h(t) = u(t)(A_0 + A_1 e^{\beta_1 t} + 2|A_2| e^{\operatorname{Re} \beta_2 t} \cos(\operatorname{Im} \beta_2 t + \angle A_2)).$$

This impulse response is plotted in Figure 4.7. Observe that  $h$  is not absolutely integrable and the system is not stable. The impulse response  $h(t)$  does not converge to zero as  $t \rightarrow \infty$  and correspondingly the mass  $M_2$  does not come to rest at position zero in Figure 4.7. In the figure it is assumed that the spring is at equilibrium when the two masses are  $d = 1$  apart. From (2.3.1), the position of mass  $M_1$  is given by the signal  $p_1 = g - d$  where  $g = h + M_2 D^2(h)$ .

### 4.6.2 Direct current motors

Recall the direct current (DC) motor from Figure 2.13 described by the differential equation from (2.4.1),

$$v = \left( \frac{RB}{K_\tau} + K_b \right) D\theta + \frac{RJ}{K_\tau} D^2\theta,$$

where  $v$  is the input voltage signal and  $\theta$  is a signal representing the angle of the motor. The constants  $R, B, K_\tau, K_b$ , and  $J$  are related to components of the motor as described in Section 2.4. To simplify the differential equation put  $a = \frac{RB}{K_\tau} + K_b$  and  $b = \frac{RJ}{K_\tau}$  and the equation becomes

$$v = aD\theta + bD^2\theta.$$

The transfer function of a system  $H$  that maps input voltage  $v$  to motor angle  $\theta$  is

$$\lambda H(s) = \frac{1}{s(a+bs)}.$$

This system has no zeros and two poles. One pole is at  $-\frac{a}{b}$  and the other is at the origin. The system is not stable because the pole at the origin is not strictly in the left half of the complex plane.

Applying partial fractions we find that

$$\lambda H(s) = \frac{1}{as} - \frac{1}{a(s-\beta)}, \quad (4.6.3)$$

where  $\beta = -\frac{a}{b}$ . Using (4.2.3), the impulse response of  $H$  is

$$h(t) = \frac{1}{a} u(t) (1 - e^{\beta t}). \quad (4.6.4)$$

Other signals with Laplace transform (4.6.3) are discarded because they do not lead to a causal system. The step response  $Hu$  is obtained by applying the integrator system  $I_\infty$  to the impulse response, that is

$$Hu = I_\infty h = \frac{1}{a\beta} u(t) (\beta t + e^{\beta t} - 1).$$

The impulse response and step response are plotted in Figure 4.8 when  $K_b = \frac{1}{8}$ ,  $K_\tau = 8$  and  $B = R = 1$  and  $J = 2$  so that  $a = \frac{1}{4}$ ,  $b = \frac{1}{4}$  and  $\beta = -1$ .

### Exercises

4.1. Sketch the signal

$$x(t) = e^{-2t}u(t) + e^t u(-t)$$

where  $u(t)$  is the step function. Find the Laplace transform of  $x(t)$  and the corresponding region of convergence (ROC). Sketch the region of convergence on the complex plane.

Figure 4.8: Impulse response (top) and step response (bottom) of a DC motor with constants  $K_b = \frac{1}{4}$ ,  $K_\tau = 8$  and  $B = R = J = 1$ .

- 4.2. Find the Laplace transform of the signal  $t^n u(t)$  where  $n \geq 0$  is an integer.
- 4.3. Let  $n \geq 0$  be an integer. Show that the Laplace transform of the signal  $-t^n u(-t)$  is the same as the Laplace transform of the signal  $t^n u(t)$ , but with a different region of convergence.
- 4.4. Show that equation (4.3.5) on page 57 holds when the system  $H$  is the time shifter  $T_\tau$ .
- 4.5. Show that equation (4.3.5) on page 57 holds when the system  $H$  is the differentiator under the added assumption that the limits  $\lim_{t \rightarrow \infty} x(t)e^{-st}$  and  $\lim_{t \rightarrow -\infty} x(t)e^{-st}$  both converge to zero when  $s \in \text{roc}(x)$ .
- 4.6. What is the transfer function of the integrator system  $I_\infty$  and what is its region of convergence?
- 4.7. By partial fractions, or otherwise, assert that

$$\frac{as}{s+b} = a - \frac{ab}{s+b}$$

- 4.8. By partial fractions, or otherwise, assert that

$$\frac{s+c}{(s+a)(s+b)} = \frac{a-c}{(a-b)(s+a)} + \frac{c-b}{(a-b)(s+b)}$$

- 4.9. By partial fractions, or otherwise, assert that

$$\frac{1}{s(s-a)(s-b)(s-b^*)} = \frac{A_0}{s} + \frac{A_1}{s-a} + \frac{A_2}{s-b} + \frac{A_2^*}{s-b^*}$$

where  $a \in \mathbb{R}$  and  $b \in \mathbb{C}$  and  $\text{Im}(b) \neq 0$  and

$$A_0 = -\frac{1}{a|b|^2}, \quad A_1 = \frac{1}{a|a-b|^2}, \quad A_2 = \frac{1}{b(b-a)(b-b^*)}.$$

You might wish to check your solution using a symbolic programming language (for example Sage, Mathematica, or Maple).

- 4.10. Let

$$\mathcal{L}y(s) = \frac{2s+1}{s^2+s-2}$$

be the Laplace transform of a signal  $y$ . By partial fractions, or otherwise, find all possible signals  $y$  and their regions of convergence.

- 4.11. Let  $x$  be a signal with region of convergence  $R$ . Show that the time scaled signal  $x(\alpha t)$  with  $\alpha \neq 0$  satisfies equation (4.2.4) on page 55.

- 4.12. Consider the active electrical circuit from Figure 2.8 described by the differential equation from (2.2.3). Derive the transfer function of this system. Find an explicit system  $H$  that maps the input voltage  $x$  to the output voltage  $y$ . State whether this system is stable and/or regular.
- 4.13. Given the mass spring damper system described by (4.5.1), find the position signal  $p$  given that the force signal

$$f(t) = \Pi\left(t - \frac{1}{2}\right) = \begin{cases} 1 & 0 < t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

is the rectangular function time shifted by  $\frac{1}{2}$ . Consider three cases:

- (a)  $M = 1$ ,  $K = \frac{\pi^2}{4}$  and  $B = \frac{\pi}{3}$ ,
- (b)  $M = 1$ ,  $K = \frac{\pi^2}{4}$  and  $B = \pi$ ,
- (c)  $M = 1$ ,  $K = \frac{\pi^2}{4}$  and  $B = 2\pi$ ,

Plot the solution in each case, and comment on whether the system is underdamped, overdamped, or critically damped.

- 4.14. Plot the signal  $x(t) = \sin(te^t)u(t)$  and find and plot its derivative  $Dx$ . Show that the region of convergence of  $x$  contains those complex numbers  $s$  with  $\operatorname{Re}(s) > 0$  and that the region of convergence of  $Dx$  contains those with  $\operatorname{Re}(s) > 1$ .
- 4.15. Show that the limit as  $|s| \rightarrow 0$  of

$$\frac{e^{s/2} - e^{-s/2}}{s}$$

is equal to 1.

- 4.16. Consider the mechanical system in Figure 2.15 from Exercise 2.2. After solving Exercise 2.2, find the transfer function of a linear shift-invariant  $H$  system mapping  $f$  to  $p$ . Now suppose that  $M_1 = K_1 = K_2 = B = 1$  and  $M_2 = 2$ . Find the poles and zeros of  $H$  and draw a pole zero plot. Determine whether  $H$  is stable and/or regular. Find and plot the impulse response and the step response of  $H$  if they exist.
- 4.17. Consider the electromechanical system in Figure 2.16 from Exercise 2.3. After solving Exercise 2.3, find the transfer function of a linear shift-invariant system that maps the input voltage  $v$  to the motor angle  $\theta$ . Under the assumption that the motor coefficients satisfy  $L = 0$  and  $K_b = K_\tau = B = R = J = 1$  draw a pole zero plot and determine whether this system is stable and/or regular. Find and plot the impulse response and step response if they exist.

- 4.18. Let  $x$  be a signal. Show that the complex exponential signal  $e^{st} \in \text{dom } x$  if and only if the signal  $x(t)e^{-st}$  is absolutely integrable.
- 4.19. Show that the complex exponential signal  $e^{st} \in \text{dom } f \circ g$  if and only if  $s \in \text{roc } f \cap \text{roc } g$ , that is  $\text{cep dom } f \circ g = \text{roc } f \cap \text{roc } g$ .

# Chapter 5

## The Fourier transform

Let  $x$  be an absolutely integrable signal. We denote by  $\mathcal{F}x$  the complex valued function satisfying

$$\mathcal{F}x(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \quad (5.0.1)$$

called the **Fourier transform**. The Fourier transform is a complex valued function of the real number  $f$ , that is,  $\mathcal{F}x \in \mathbb{R} \rightarrow \mathbb{C}$ , or in other words,  $\mathcal{F}x$  is a **signal**. For example, the rectangular pulse  $\Pi(t)$  from (1.1.2) is absolutely integrable and has Fourier transform

$$\begin{aligned} \mathcal{F}\Pi(f) &= \int_{-\infty}^{\infty} \Pi(t)e^{-j2\pi ft}dt \\ &= \int_{-1/2}^{1/2} e^{-j2\pi ft}dt \\ &= \frac{e^{j\pi f} - e^{-j\pi f}}{j2\pi f} = \frac{\sin(\pi f)}{\pi f} = \text{sinc}(f). \end{aligned} \quad (5.0.2)$$

The sinc function is plotted in Figure 5.1.

The Fourier transform is closely related to the Laplace transform because

$$\mathcal{F}x(f) = \mathcal{L}x(j2\pi f)$$

for those signals  $x$  with region of convergence containing the imaginary axis, that is, for absolutely integrable  $x$ . The Fourier transform inherits the properties of the Laplace transform that were described in Section 4.3. For example, if  $H$  is a stable regular system with absolutely integrable impulse response  $h$  having Fourier transform  $\mathcal{F}h$ , then the spectrum of  $H$  satisfies

$$\Lambda H(f) = \lambda H(j2\pi f) = \mathcal{L}h(j2\pi f) = \mathcal{F}h(f),$$

that is, the spectrum of a stable regular system is the Fourier transform of its impulse response. Like the Laplace transform, the Fourier transform

obeys the **convolution theorem** (4.3.4), that is,

$$\mathcal{F}(x * y) = \mathcal{F}x\mathcal{F}y \quad (5.0.3)$$

when the signals  $x$  and  $y$  are absolutely integrable. In words: the Fourier transform of a convolution of signals is the multiplication of the Fourier transforms of those signals. The convolution of two absolutely integrable signals is always absolutely integrable and so there is no need to include the assumption that  $x * y$  is absolutely integrable (Exercises 3.1).

It follows from (4.3.5) that if  $H$  is a regular system with spectrum  $\Lambda H$  and if  $x$  is a signal with Fourier transform  $\mathcal{F}x$ , then the signal  $y = Hx$  has Fourier transform

$$\mathcal{F}y = \Lambda H \mathcal{F}x.$$

This property also holds for the shifter  $T_\tau$  (Exercise 4.4) and it holds for the differentiator  $D$  under the added assumption that  $\lim_{t \rightarrow \infty} x(t) = 0$  and  $\lim_{t \rightarrow -\infty} x(t) = 0$  (Exercise 4.5). From (4.3.1) and (4.3.2) the spectrum of  $T_\tau$  and the  $k$ th differentiator  $D^k$  satisfy

$$\Lambda T_\tau = e^{-j2\pi f\tau}, \quad \Lambda D^k = (j2\pi f)^k$$

from which we obtain the **time shift property**,

$$\mathcal{F}T_\tau x = \Lambda T_\tau \mathcal{F}x = e^{-j2\pi f\tau} \mathcal{F}x,$$

and the **differentiation property**,

$$\mathcal{F}D^k x = \Lambda D^k \mathcal{F}x = (j2\pi f)^k \mathcal{F}x,$$

of the Fourier transform. These results motivate assigning the following Fourier transforms to the delta “function”  $\delta$ , its shift  $T_\tau \delta = \delta(t - \tau)$ , and its derivatives

$$\mathcal{F}\delta = 1, \quad \mathcal{F}(\delta(t - \tau)) = e^{-j2\pi f\tau}, \quad \mathcal{F}\delta^k = (j2\pi f)^k. \quad (5.0.4)$$

These conventions are common in the literature [Oppenheim et al., 1996].

Similarly to the Laplace transform (4.2.2), the Fourier transform obeys a **frequency shift rule** that relates the transform of a signal  $x(t)$  to that of the signal  $e^{2\pi j\gamma ft}x(t)$  where  $\gamma \in \mathbb{R}$ . From (4.2.2), the frequency shift rule asserts that

$$\mathcal{F}(e^{2\pi j\gamma t}x(t))(f) = \mathcal{F}x(f - \gamma). \quad (5.0.5)$$

Since  $\cos(2\pi\gamma t) = \frac{1}{2}e^{2\pi j\gamma t} + \frac{1}{2}e^{-2\pi j\gamma t}$  we also have

$$\mathcal{F}(\cos(2\pi\gamma t)x(t))(f) = \frac{1}{2}\mathcal{F}x(f - \gamma) + \frac{1}{2}\mathcal{F}x(f + \gamma). \quad (5.0.6)$$

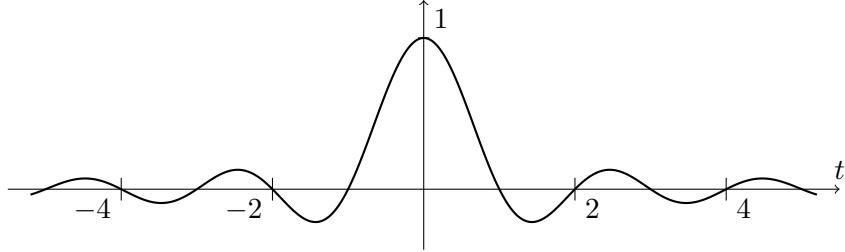


Figure 5.1: The **sinc function**  $\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$ .

and, similarly, since  $\sin(2\pi\gamma t) = \frac{1}{2j}e^{2\pi j\gamma t} - \frac{1}{2j}e^{-2\pi j\gamma t}$  we have

$$\mathcal{F}(\sin(2\pi\gamma t)x(t))(f) = \frac{1}{2j}\mathcal{F}x(f-\gamma) - \frac{1}{2j}\mathcal{F}x(f+\gamma)$$

These results are sometimes called the **modulation property** of the Fourier transform [Papoulis, 1977, page 61]. This property is of particular importance in communications engineering [Proakis, 2007]. Combing the frequency shift rule with the convention  $\mathcal{F}\delta = 1$  motivates assigning the the following Fourier transforms to the complex exponential signal  $e^{2\pi j\gamma t}$  and the cosine and sine signals,

$$\mathcal{F}(e^{2\pi j\gamma t}) = \delta(t - \gamma).$$

$$\begin{aligned}\mathcal{F}(\cos(2\pi\gamma t)) &= \frac{1}{2}\delta(t - \gamma) + \frac{1}{2}\delta(t + \gamma), \\ \mathcal{F}(\sin(2\pi\gamma t)) &= \frac{1}{2j}\delta(t - \gamma) - \frac{1}{2j}\delta(t + \gamma),\end{aligned}$$

These conventions are common in the literature [Oppenheim et al., 1996; Proakis, 2007], but must be treated with caution. There is no guarantee that mechanical mathematical manipulations involving these conventions will lead to sensible results.

Like the Laplace transform (4.2.4), the Fourier transform obeys a **time-scaling property**. If  $x$  is an absolutely integrable signal then the time scaled signal  $x(\alpha t)$  with  $\alpha \neq 0$  has Fourier transform

$$\mathcal{F}(x(\alpha t))(f) = \frac{1}{|\alpha|}\mathcal{F}x(f/\alpha). \quad (5.0.7)$$

## 5.1 The inverse transform and the Plancherel theorem

Given a signal  $x$  we will often denote its Fourier transform by  $\hat{x} = \mathcal{F}x$ . Observe that  $\hat{x}$ , like  $x$ , is a function that maps a real number to a complex number, that is,  $\hat{x}$  is a **signal** with independent variable representing frequency. It is usual to call  $\hat{x}$  the **frequency-domain** representation of the

signal and  $x$  the **time-domain** representation although the signal  $x$  need not be a function of “time”.

If  $\hat{x}$  is absolutely integrable, then  $x$  can be recovered using the **inverse Fourier transform**

$$x(t) = \mathcal{F}^{-1}\hat{x}(t) = \int_{-\infty}^{\infty} \hat{x}(f)e^{j2\pi ft}df. \quad (5.1.1)$$

For example, let  $\hat{x} = \mathcal{F}x = \Pi$  be the rectangular pulse. By working analogous to that from (5.0.2),

$$x(t) = \int_{-\infty}^{\infty} \Pi(f)e^{j2\pi ft}df = \text{sinc}(-t) = \text{sinc}(t).$$

We are lead to the conclusion that the Fourier transform of sinc is the rectangular pulse  $\Pi$ .

The rectangular pulse  $\Pi$  is finite and absolutely integrable. The sinc function is not absolutely integrable (Exercise 5.3). Because of this the integral equation that we have used to define the Fourier transform (5.0.1) cannot be directly applied to the sinc function. Although sinc is not absolutely integrable, it is square integrable (Exercise 5.3). It happens that all square integrable signals can be assigned a Fourier transform by interpreting the integral in (5.0.1) as what is called its **Cauchy principal value**. That is, for  $x$  a square integrable signal, we assign the Fourier transform

$$\hat{x}(f) = \mathcal{F}x(f) = \lim_{T \rightarrow \infty} \int_{-T}^T x(t)e^{-j2\pi ft}dt. \quad (5.1.2)$$

This Fourier transform  $\hat{x}$  is itself a square integrable signal and the original time domain signal  $x$  can be recovered almost everywhere by taking the Cauchy principal value of the integral for the inverse Fourier transform (5.1.1), that is,

$$x(t) = \mathcal{F}^{-1}\hat{x}(t) = \lim_{T \rightarrow \infty} \int_{-T}^T \hat{x}(f)e^{j2\pi ft}df \quad \text{a.e.}$$

Infact, the energy of  $x$  and its Fourier transform  $\hat{x}$  are the same, that is,

$$\|x\|_2^2 = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{x}(f)|^2 df = \|\hat{x}\|_2^2. \quad (5.1.3)$$

These results are known as the **Plancherel theorem** [Rudin, 1986, Th. 9.13]. The equality of energies in (5.1.3) is often called **Parseval's identity**. For our purposes it will suffice to remember only that the Fourier transform of the sinc function is the rectangular pulse  $\Pi$ . If  $x$  and  $y$  are square integrable signals with Fourier transforms  $\hat{x}$  and  $\hat{y}$ , then a consequence of (5.1.3) is

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} \hat{x}(f)\hat{y}^*(f)df \quad (\text{Exercise } ??) \quad (5.1.4)$$

where the superscript \* denotes the complex conjugate. This result often also goes by the name of Parseval's identity.

Let  $x$  be a signal with Fourier transform

$$\mathcal{F}x(f) = \int_{-\infty}^{\infty} x(\tau)e^{-j2\pi f\tau}d\tau.$$

Evaluating the Fourier transform at  $-t$  we find that

$$\mathcal{F}x(-t) = \int_{-\infty}^{\infty} x(\tau)e^{j2\pi t\tau}d\tau = \mathcal{F}^{-1}x(t). \quad (5.1.5)$$

This is the called the **duality** property of the Fourier transform. In words, if  $\hat{x}$  is the Fourier transform of  $x$ , then  $x$  is the Fourier transform of  $\hat{x}$  reflected in time.

## 5.2 Analogue filters

For many engineering purposes it is desirable to construct systems that will **pass** (have little affect on) a complex exponential signal  $e^{j2\pi ft}$  for certain frequencies  $f$ , but will **reject** (significantly attenuate) these signals for other frequencies. Such systems are called **frequency dependent filters**. Those frequencies that the filter intends to pass unaffected are said to be in the **pass band** and those frequencies that the filter intends to reject are said to be in the **stop band**.

An **ideal lowpass filter** with **cutoff frequency**  $c$  is the system  $L_c$  with spectrum

$$\Lambda L_c = \begin{cases} 1 & |f| < c \\ 0 & \text{otherwise} \end{cases} = \Pi\left(\frac{f}{2c}\right).$$

Applying the inverse Fourier transform to  $\Pi\left(\frac{f}{2c}\right)$  gives

$$\int_{-\infty}^{\infty} \Pi\left(\frac{f}{2c}\right) e^{j2\pi tf} df = \int_{-c}^{c} e^{j2\pi tf} df = \frac{\sin(2c\pi t)}{\pi t} = 2c \operatorname{sinc}(2ct).$$

We are lead to the conclude that the ideal lowpass filter  $L_c$  is a regular linear time invariant system with impulse response  $2c \operatorname{sinc}(2ct)$ <sup>1</sup>

---

<sup>1</sup>This conclusion should be taken with scepticism because the signal  $2c \operatorname{sinc}(2ct)$  is not absolutely integrable and so its transfer function with not contain the imaginary axis in its domain. Infact,  $\operatorname{roc} \operatorname{sinc} = \emptyset$ . An exception has been made to allow square integrable signals to have a Fourier transform and a similar exception must be made to allow regular systems with square integrable impulse response to have a spectrum. We will have not use of ideal filters beyond this section and so this caveat with not carry throughout the text. The concept of an ideal filter is nonetheless informative from an intuitive viewpoint.

An **ideal highpass filter** with cutoff frequency  $c$  is given by the linear combination  $T_0 - L_c$  where  $T_0$  is the identity system. The spectrum is

$$\Lambda(T_0 - L_c) = \Lambda T_0 - \Lambda L_c = 1 - \Pi\left(\frac{f}{2c}\right) = \begin{cases} 0 & |f| < c \\ 1 & \text{otherwise.} \end{cases}$$

This ideal highpass filter is not regular because the system  $T_0$  is not regular. The system does not have an impulse response. Nevertheless, it is common to represent one by  $\delta(t) - 2c \operatorname{sinc}(2ct)$  using the delta function as described in Section 3.1.

An **ideal bandpass filter** with upper cutoff frequency  $u$  and lower cutoff frequency  $\ell$  is given by the linear combination  $L_u - L_\ell$ . The spectrum is

$$\Lambda(L_u - L_\ell) = \Pi\left(\frac{f}{2u}\right) - \Pi\left(\frac{f}{2\ell}\right) = \begin{cases} 1 & -u < f \leq -\ell \\ 1 & \ell \leq f < u \\ 0 & \text{otherwise.} \end{cases}$$

We conclude that the ideal bandpass filter has impulse response  $2u \operatorname{sinc}(2ut) - 2\ell \operatorname{sinc}(2\ell t)$ . The spectrum and impulse response of the ideal lowpass, highpass, and bandpass filters are plotted in Figure 5.2.

Ideal filters are not realisable in practice. We now describe a popular practical low-pass filter discovered by Butterworth [1930]. A **normalised low pass Butterworth filter** of order  $m$ , denoted by  $B_m$ , has transfer function

$$\lambda B_m(s) = \frac{1}{\prod_{i=1}^m (\frac{s}{2\pi} - \beta_i)} = \frac{(2\pi)^m}{\prod_{i=1}^m (s - 2\pi\beta_i)},$$

where  $\beta_1, \dots, \beta_m$  are the roots of the polynomial  $s^{2m} + (-1)^m$  that lie strictly in the left half of the complex plane (have negative real part). Specifically, these roots are

$$\beta_k = \begin{cases} \exp(j\frac{\pi}{2}(1 + \frac{2k-1}{m})), & k = 1, \dots, m \\ \exp(j\frac{\pi}{2}(1 - \frac{2k-1}{m})), & k = m+1, \dots, 2m. \end{cases}$$

The roots are plotted in Figure 5.3. The spectrum of  $B_m$  is

$$\Lambda B_m(f) = \frac{1}{\prod_{i=1}^m (jf - \beta_i)}$$

and the magnitude spectrum of  $B_m$  can be shown to satisfy

$$|\Lambda B_m(f)| = \sqrt{\frac{1}{f^{2m} + 1}}. \quad (\text{Exercise 5.4})$$

The magnitude and phase spectrum of the filters  $B_1$ ,  $B_2$ ,  $B_3$ , and  $B_4$  are plotted in Figure 5.4.

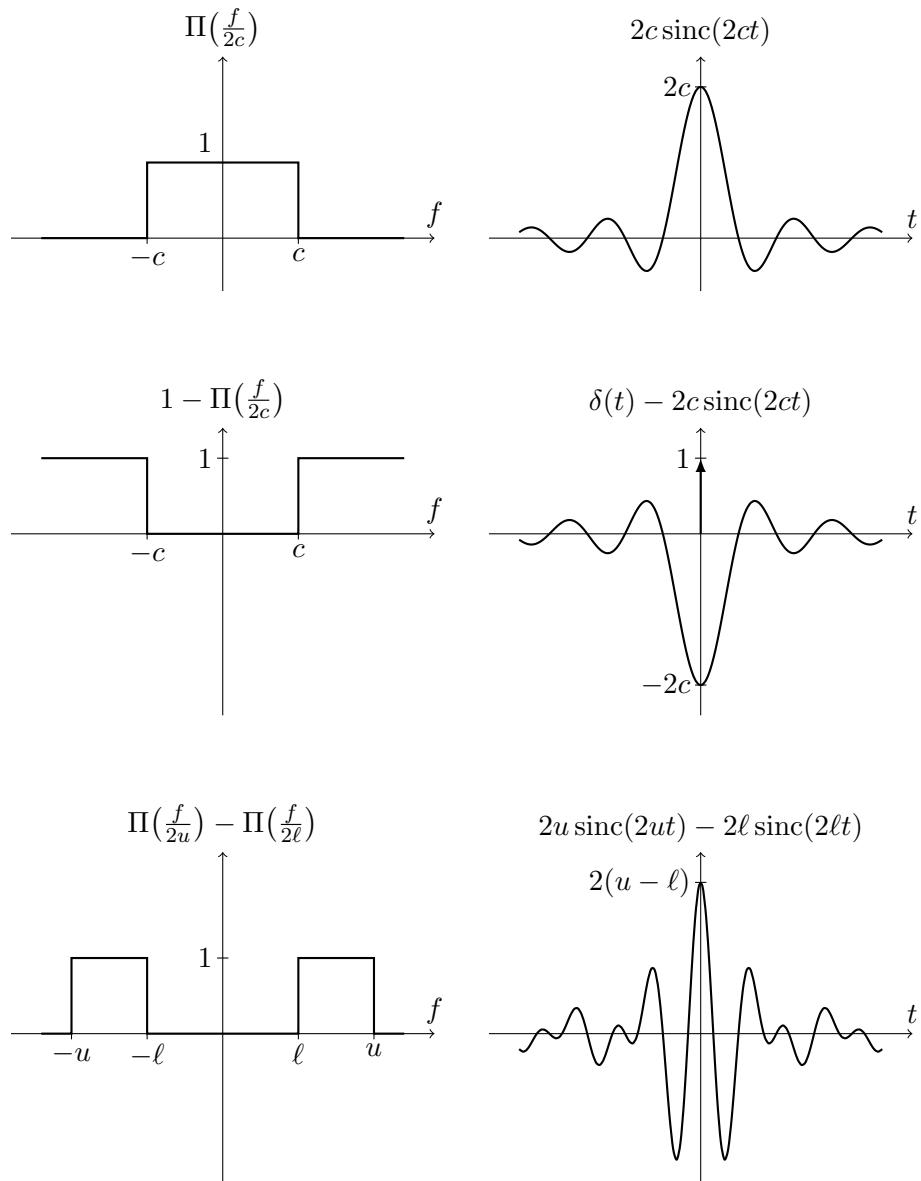


Figure 5.2: Spectrum and impulse response of the ideal lowpass filter  $L_c$  (top), the ideal highpass filter  $T_0 - L_c$  (middle), and the ideal bandpass filter  $L_u - L_\ell$  (bottom). The ideal highpass filter is not regular and does not have an impulse response. We plot the ‘pretend’ impulse response using the delta function described in Section 3.1.

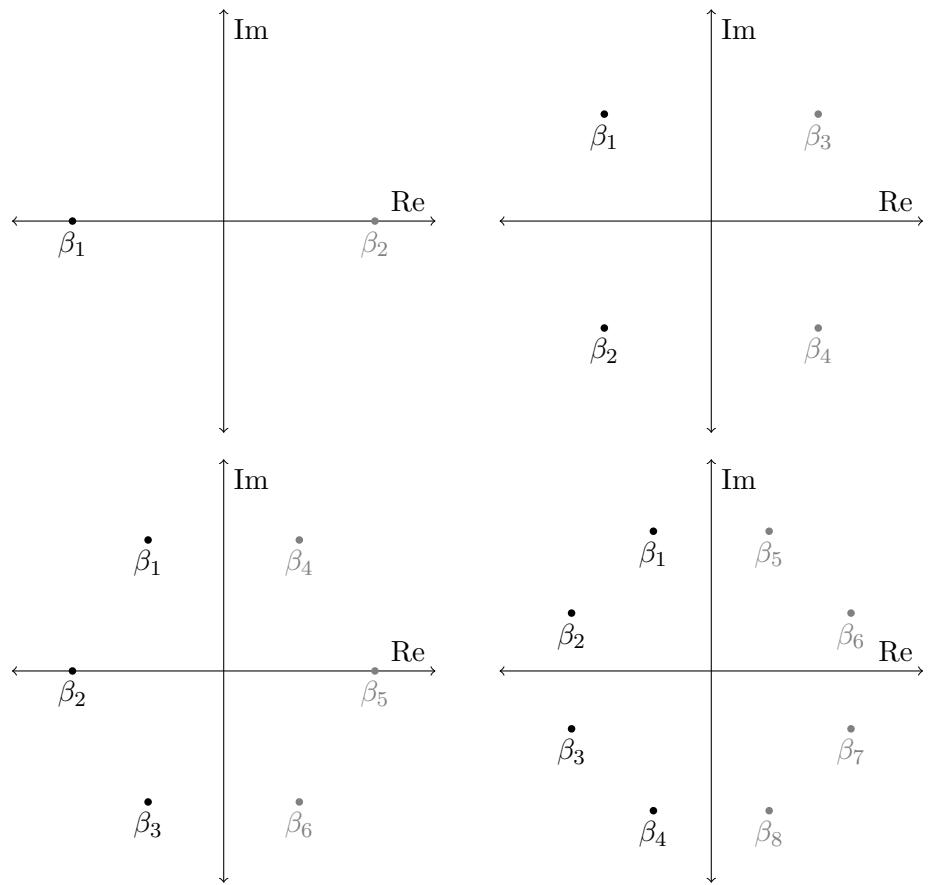


Figure 5.3: Roots of the polynomial  $s^{2m} + (-1)^m$  for  $m = 1$  (top left),  $m = 2$  (top right),  $m = 3$  (bottom left), and  $m = 4$  (bottom right). All the roots lie on the complex unit circle and have magnitude one. The poles of the normalised Butterworth filter  $B_m$  are those roots from the left half of the complex plane (unshaded).

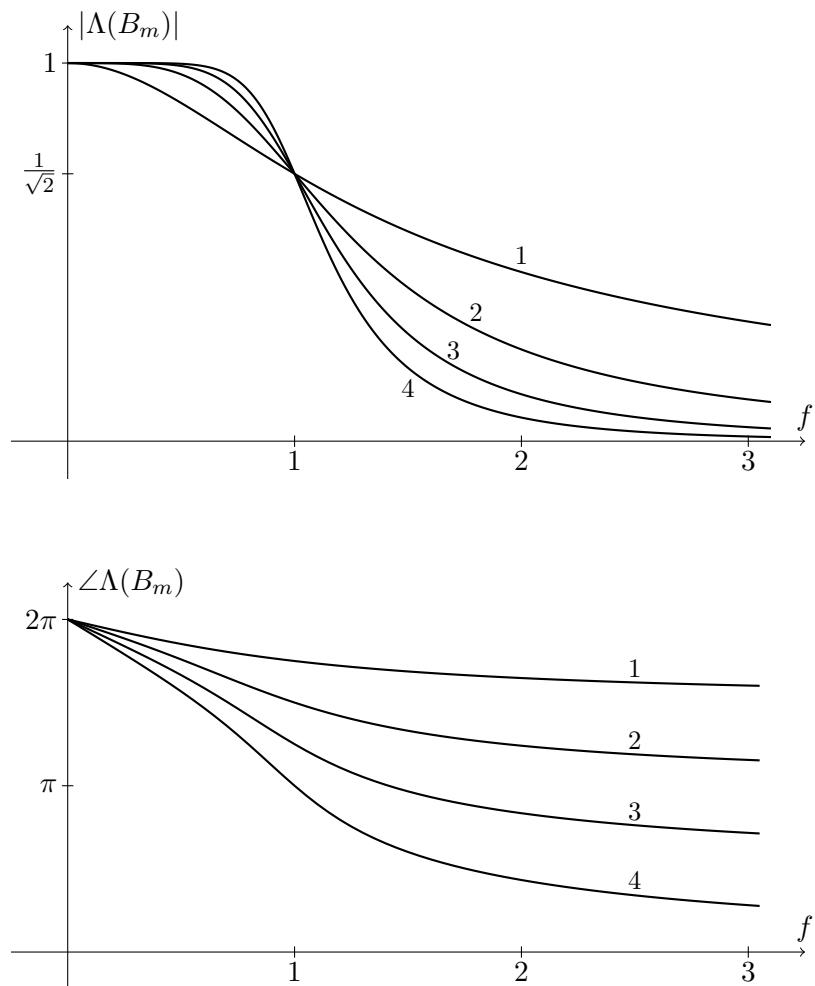


Figure 5.4: Magnitude spectrum (top) and phase spectrum (bottom) of normalised Butterworth filters  $B_1, B_2, B_3$  and  $B_4$ .

The **cutoff frequency** of the lowpass filter  $B_m$  is defined as the positive real number  $c$  such that  $|\Lambda B_m(f)|^2 < \frac{1}{2}$  for all  $f > c$ . The normalised Butterworth filters have cutoff frequency  $c = 1\text{Hz}$ . A lowpass Butterworth filter of order  $m$  and cutoff frequency  $c$ , denoted  $B_m^c$ , has transfer function

$$\Lambda B_m^c(s) = \lambda B_m\left(\frac{s}{c}\right) = \frac{1}{\prod_{i=1}^m (\frac{s}{2\pi c} - \beta_i)}.$$

The magnitude spectrum satisfies

$$|\Lambda B_m^c(f)|^2 = |\Lambda B_m\left(\frac{f}{c}\right)|^2 = \frac{1}{\left(\frac{f}{c}\right)^{2m} + 1} = \frac{c^{2m}}{f^{2m} + c^{2m}}. \quad (5.2.1)$$

A first order Butterworth filter  $B_1^c$  has spectrum

$$\Lambda B_1^c(f) = \frac{1}{j\frac{f}{c} + 1} = \frac{c}{jf + c}.$$

Putting  $\frac{1}{c} = 2\pi RC$  we find that this is the same as the spectrum of the RC electrical circuit (Figure 2.1) or the active RC circuit after negation (3.5.2). Thus, the RC electrical circuit is a first order Butterworth filter with cutoff frequency  $c = \frac{1}{2\pi RC}$ . In Test 4 we constructed the active RC circuit with  $R \approx 27\text{k}\Omega$  and  $C \approx 10\text{nF}$  and measured its magnitude spectrum. The cutoff frequency was  $c = \frac{5 \times 10^4}{27\pi} \approx 589\text{Hz}$ .

A second order electrical Butterworth filter can be constructed using the Sallen-Key circuit described in Section 2.2 and Figure 2.10. The input voltage  $x$  and output voltage  $y$  of the Sallen-Key satisfy the differential equation (2.2.9)

$$x = y + C_2(R_1 + R_2)Dy + R_1R_2C_1C_2D^2y.$$

The transfer function corresponding with this equation is

$$\frac{1}{1 + C_2(R_1 + R_2)s + R_1R_2C_1C_2s^2}.$$

The second order Butterworth filter  $B_2^c$  has transfer function

$$\Lambda B_2^c(s) = \frac{1}{(\frac{1}{2\pi c}s - \beta_1)(\frac{1}{2\pi c}s - \beta_2)},$$

where  $\beta_1 = \beta_2^* = e^{j3\pi/4}$ . Expanding the quadratic on the denominator gives

$$\Lambda B_2^c(s) = \frac{1}{1 + \frac{1}{\sqrt{2}\pi c}s + \frac{1}{4\pi^2 c^2}s^2}.$$

Choosing the resistors and capacitors of the Sallen-Key to satisfy

$$C_2(R_1 + R_2) = \frac{1}{\sqrt{2}\pi c}, \quad R_1R_2C_1C_2 = \frac{1}{4\pi^2 c^2}$$

leads to a second order Butterworth filter. A convenient solution is to put  $C_1 = 2C_2$  and  $R_1 = R_2$ . This gives a second order Butterworth filter with cutoff

$$c = \frac{1}{\sqrt{2}\pi C_2(R_1 + R_2)} = \frac{1}{2\sqrt{2}\pi C_2 R_2}.$$

In Test 6 we construct a second order Butterworth filter using a Sallen-Key and measure its spectrum.

Butterworth filters of orders larger than  $m = 2$  can be constructed by concatenating Sallen-Key circuits and RC circuits. If  $m$  is even then  $m/2$  Sallen-Key circuits are required. Each Sallen-Key is used to construct a conjugate pair of poles, that is, the  $k$ th Sallen-Key would have poles  $2\pi c\beta_k$  and  $2\pi c\beta_k^* = 2\pi c\beta_{m-k+1}$ . If  $m$  is odd then  $(m - 1)/2$  Sallen-Key circuits and a single RC circuit (or active RC circuit) can be used. The RC circuit is designed to have the real valued pole  $\beta_{(m+1)/2} = 2\pi c$ .

#### Test 6 (Butterworth filter)

We construct a second order Butterworth filter using the Sallen-Key circuit from Figure 2.10 with capacitors  $C_2 \approx 100\text{nF}$ ,  $C_1 \approx 2C_2 \approx 200\text{nF}$  and resistors  $R_1 \approx R_2 \approx 330\Omega$ . The cutoff frequency is

$$c = \frac{1}{2\sqrt{2}\pi C_2 R_2} \approx 3410\text{Hz}.$$

Sinusoids of the form

$$\sin(2\pi f_k t), \quad f_k = \left\lceil 110 \times 2^{k/2} \right\rceil, \quad k = 1, 2, \dots, 13$$

are input to the filter using a computer soundcard and the magnitude and phase spectrum are measured using the procedure described in Test 4. Figure 5.5 shows the measurements (dots) plotted alongside the hypothesised magnitude spectrum

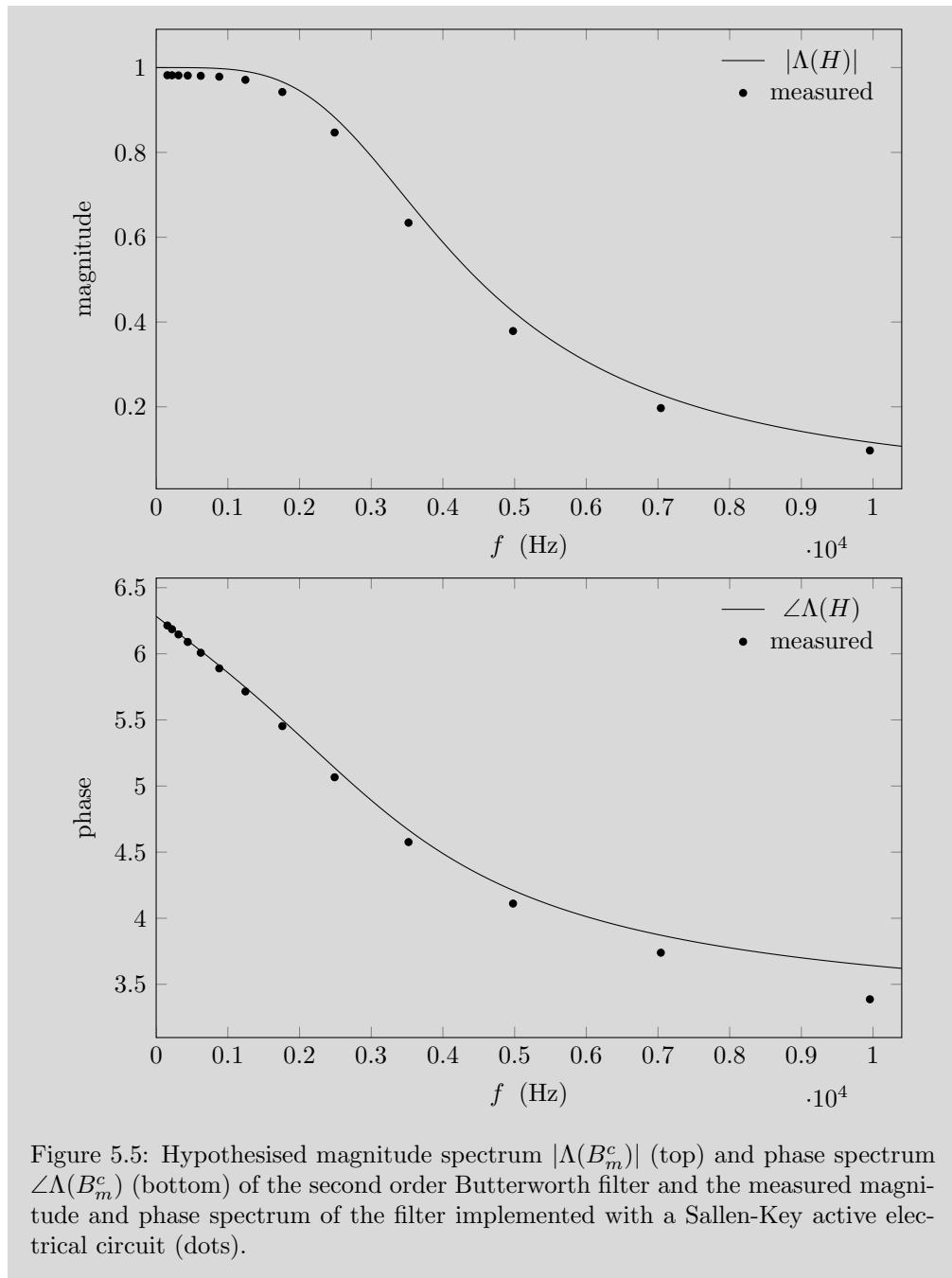
$$|\Lambda B_2^c(f)| = \sqrt{\frac{1}{(f/c)^4 + 1}}$$

and the hypothesised phase spectrum  $\angle \Lambda B_2^c(f)$ .

### 5.3 Real and complex valued sequences

Let  $x$  be a signal with Fourier transform  $\hat{x} = \mathcal{F}x$ . The signal  $x$  is said to be **bandlimited** if there exists a positive real number  $b$  such that

$$\hat{x}(f) = \mathcal{F}x(f) = 0 \quad \text{when } |f| > b.$$



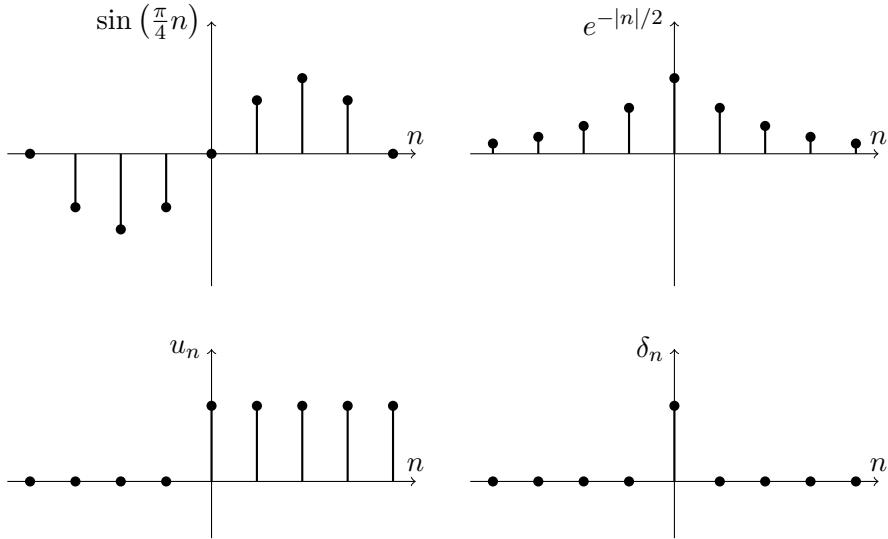


Figure 5.6: Real valued sequences. The bottom plots show that step sequence  $u$  and the delta sequence  $\delta$ .

The value  $b$  is called the **bandwidth** of the signal  $x$ . For example, the sinc function is bandlimited with bandwidth  $\frac{1}{2}$  because its Fourier transform  $\mathcal{F}\text{sinc}(f) = \Pi(f) = 0$  for all  $|f| > \frac{1}{2}$ . Bandlimited signals have a number of properties that make them suitable for representation and manipulation by a computer. They are of particular importance for this reason. Before we can study bandlimited signals we first require some properties of real and complex valued **sequences**.

A **sequence** is a function with domain given by the integers  $\mathbb{Z}$ . The value of a sequence  $x$  at integer  $n$  can be denoted by  $x(n)$  but it is conventional to write  $x_n$ . We are primarily interested in sequences that take complex (or real) values, that is, functions the set  $\mathbb{Z} \rightarrow \mathbb{C}$ . For example,

$$\sin(\frac{\pi}{4}n), \quad n^3, \quad e^{-|n|/2}$$

each denote a real (and so also complex) valued sequence. In what follows the term **sequence** will always mean a complex valued sequence unless otherwise stated. Complex valued sequences are commonly called **discrete time signals** and the  $n$ th element in the sequence is denoted by  $x[n]$  using squared brackets [Oppenheim et al., 1996]. Here, we use the subscript notation  $x_n$ . This notation is also common [Vetterli et al., 2014; Rudin, 1986]. Sequences are plotted using vertical lines with dotted ends as in Figure 5.6 and have a number of properties analogous to the properties of signals (Section 1.1).

A sequence  $x$  is bounded if there exists a real number  $M$  such that

$$|x_n| < M \quad \text{for all } n \in \mathbb{Z}.$$

Both  $\sin(\frac{\pi}{4}n)$  and  $e^{-|n|/2}$  are examples of bounded sequences, but  $n^3$  is not bounded because its magnitude grows indefinitely as  $n$  moves away from the origin. A sequence  $x$  is periodic if there exists a positive integer  $T$  such that

$$x_n = x_{n+kT} \quad \text{for all integers } k \text{ and } n.$$

The smallest such  $T$  is called the period. The sequence  $\sin(\frac{\pi}{4}n)$  is periodic with period  $T = 8$ . Neither  $n^3$  or  $e^{-n^2/4}$  are periodic. A sequence  $x$  is even (or symmetric) if  $x_n = x_{-n}$  for all  $n \in \mathbb{Z}$  and odd (or antisymmetric) if  $x_n = -x_{-n}$  for all  $n \in \mathbb{Z}$ . Both  $\sin(\frac{\pi}{4}n)$  and  $n^3$  are odd and  $e^{-|n|/2}$  is even.

A sequence  $x$  is **right sided** if there exists a  $T \in \mathbb{R}$  such that  $x_n = 0$  for all  $t < T$ . Correspondingly  $x$  is **left sided** if  $x_n = 0$  for all  $T > t$ . For example, the **step sequence**  $u$  with  $n$ th element

$$u_n = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \quad (5.3.1)$$

is right sided (Figure 1.2). The reflected sequence  $u_{-n}$  is left sided. A sequence is said to be of **finite support** or just **finite** if it is both left and right sided. For example the sequence  $\delta$  with  $n$ th element

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise,} \end{cases} \quad (5.3.2)$$

called the **delta sequence**, is finite. The delta sequence is analogous to the delta “function” introduced in Section 3.1. The delta “function” is not actually function, only a notational device. Contrastingly, the delta sequence is a well defined sequence.

A sequence  $x$  is **absolutely summable** if

$$\|x\|_1 = \sum_{n \in \mathbb{Z}} |x_n| < \infty,$$

that is, if the sum of absolute values of the elements in the sequence converges to a finite number. The real number  $\|x\|_1$  is commonly called the  $\ell^1$ -norm of  $x$ . The sequences  $\sin(\frac{\pi}{4}n)$  and  $n^3$  are not absolutely summable, but  $e^{-|n|/2}$  is because

$$\sum_{n \in \mathbb{Z}} |e^{-|n|/2}| = \sum_{n \in \mathbb{Z}} e^{-|n|/2} = 1 + \frac{2}{\sqrt{e} - 1}. \quad (\text{Exercise 5.9})$$

It is common to denote the set of absolutely summable sequences by  $\ell^1$  or  $\ell^1(\mathbb{Z})$ . So,  $e^{-|n|/2} \in \ell^1$  and  $\sin(\frac{\pi}{4}n) \notin \ell^1$ .

A sequence  $x$  is **square summable** if

$$\|x\|_2^2 = \sum_{n \in \mathbb{Z}} |x_n|^2 < \infty,$$

that is, if the sum of squared magnitudes of the elements converges to a finite number. The real number  $\|x\|_2$  is commonly called the  $\ell^2$ -norm and its square  $\|x\|_2^2$  the **energy** of  $x$ . The sequences  $\sin(\frac{\pi}{4}n)$  and  $n^3$  are not square summable, but  $e^{-|n|/2}$  is because

$$\sum_{n \in \mathbb{Z}} |e^{-|n|/2}|^2 = \sum_{n \in \mathbb{Z}} e^{-|n|} = 1 + \frac{2}{e-1}. \quad (\text{Exercise 5.9})$$

It is common to denote the set of square summable sequences by  $\ell^2$ . So,  $e^{-|n|/2} \in \ell^2$  and  $\sin(\frac{\pi}{4}n) \notin \ell^2$ . If a sequence is absolutely summable then it is also square summable (Exercise 5.10). The corresponding property is not true of signals, that is, absolutely integrable signals are not necessarily square integrable (Exercise 1.5).

## 5.4 Bandlimited signals

Let  $b$  be a positive real number and let  $x$  be a signal with Fourier transform  $\hat{x} = \mathcal{F}x$ . The signal  $x$  is said to be **bandlimited** with **bandwidth**  $b$  if

$$\hat{x}(f) = \mathcal{F}x(f) = 0 \quad \text{for all } |f| > b.$$

For example, the sinc function  $\text{sinc}(t)$  that has Fourier transform  $\Pi(f)$  is bandlimited with bandwidth  $b \geq \frac{1}{2}$ . Another example is the signal with Fourier transform given by a **raised cosine**

$$\hat{x}(f) = \Pi(f)(1 + \cos(2\pi f)) = \begin{cases} 1 + \cos(2\pi f) & |f| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

that is bandlimited with bandwidth  $b \geq \frac{1}{2}$ . The time domain signal is found by applying the inverse Fourier transform

$$x(t) = \text{sinc}(t) + \frac{1}{2} \text{sinc}(t+1) + \frac{1}{2} \text{sinc}(t-1). \quad (\text{Exercise 5.7})$$

Another example is the signal with Fourier transform given by the **triangle pulse**

$$\Delta(f) = \begin{cases} f+1 & -1 < f < 0 \\ 1-f & 0 \leq f < 1 \\ 0 & \text{otherwise} \end{cases}$$

that is bandlimited with bandwidth  $b \geq 1$ . The corresponding time domain signal is given by the square of the sinc function  $\text{sinc}^2(t)$  (Exercises 5.2). These bandlimited signals and their Fourier transforms are plotted in Figure 5.7.

It happens that bandlimited signals are never finite. We can reasonably suppose that all signals ever encountered in practice are finite and so no signals encountered in practice are truly bandlimited. However, many practically occurring signals are approximately bandlimited, that is, their Fourier transform is small for frequencies larger than some positive number  $b$ . For example, in Test 7 the Fourier transform of an audio signal taken from a lecture recording is plotted (Figure 5.8). This signal appears approximately bandlimited with bandwidth a little larger than 8 kHz.

A surprising result is that every square integrable bandlimited signal  $x$  with bandwidth  $b$  can be written as a sum of time-scaled and time-shifted sinc functions, that is, in the form

$$x(t) = \sum_{n \in \mathbb{Z}} c_n \operatorname{sinc}(Ft - n) \quad (5.4.1)$$

where  $c$  is a square integrable complex valued sequence and  $F = 2b$ . This is a consequence of a property of the space of square integrable signals  $L_2$  called **completeness** [Rudin, 1986, Theorem 3.11]. This property is sometimes also called the **Riesz-Fischer theorem** [Rudin, 1986, page 91]. Evaluating the signal  $x$  at integer multiples of  $P = \frac{1}{F}$  we find that

$$x(\ell P) = \sum_{n \in \mathbb{Z}} c_n \operatorname{sinc}(\ell - n) = c_\ell$$

because  $\operatorname{sinc}(\ell - n)$  is equal to 1 when  $\ell = n$  and 0 otherwise. So, the elements of the sequence  $c$  correspond with samples of the signal  $x$  taken at integer multiples of  $P = \frac{1}{F} = \frac{1}{2b}$ , that is,  $c_n = x(nP)$ . The positive real number  $P$  is called the **sampling period** and its reciprocal  $F$  the **sampling rate**. It follows that every square integrable bandlimited signal  $x$  with bandwidth  $b$  can be reconstructed from samples taken at rate  $F = 2b$ , that is,

$$x(t) = \sum_{n \in \mathbb{Z}} x(nP) \operatorname{sinc}(Ft - n).$$

This result known as the **Nyquist sampling theorem**. This theorem motivated use of this reconstruction method in Tests 1, 2, 3, and 5.

## 5.5 The discrete time Fourier transform

Let  $x$  be a square integrable bandlimited signal with bandwidth  $b$  and let  $c$  be the square summable sequence containing samples of  $x$  at sampling rate  $F = \frac{1}{P} = 2b$ , that is,  $c_n = x(nP)$ . Suppose temporarily that  $c$  is also

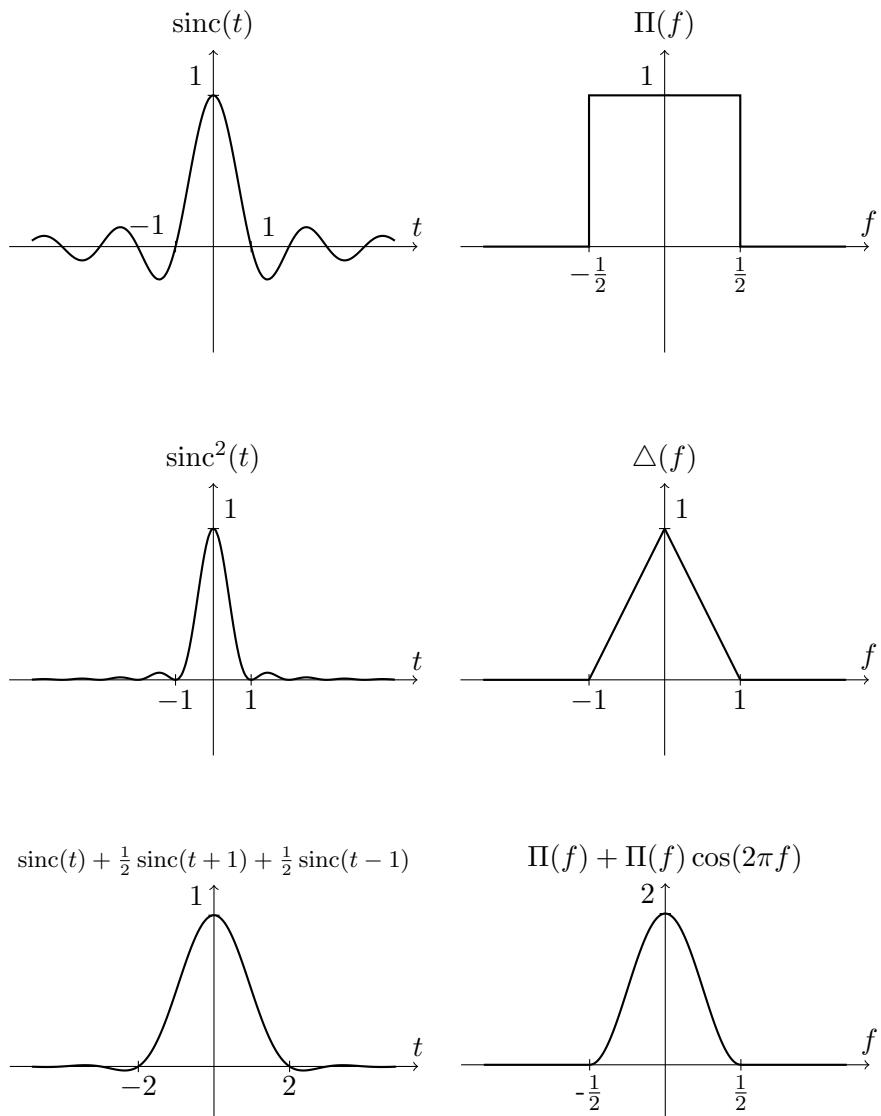


Figure 5.7: Bandlimited signals  $\text{sinc}(t)$ ,  $\text{sinc}^2(t)$ , and  $\text{sinc}(t) + \frac{1}{2}\text{sinc}(t+1) + \frac{1}{2}\text{sinc}(t-1)$  with bandwidth  $\frac{1}{2}$ , 1, and 1 respectively.

absolutely summable. From (5.4.1), the Fourier transform of  $x$  is

$$\hat{x}(f) = \mathcal{F}x(f) = \sum_{n \in \mathbb{Z}} c_n \mathcal{F}(\text{sinc}(Ft - n))(f) \quad (5.5.1)$$

$$\begin{aligned} &= P\Pi(fP) \sum_{n \in \mathbb{Z}} c_n e^{-j2\pi Pnf} \\ &= P\Pi(fP) \mathcal{D}c(Pf) \end{aligned} \quad (5.5.2)$$

where

$$\mathcal{D}c(f) = \sum_{n \in \mathbb{Z}} c_n e^{-j2\pi nf} \quad (5.5.3)$$

is called the **discrete time Fourier transform** of the sequence  $c$ . We write  $\hat{c} = \mathcal{D}c$  for the discrete time Fourier transform of  $c$ . Under our assumption that  $c$  is absolutely summable

$$|\mathcal{D}c(f)| \leq \sum_{n \in \mathbb{Z}} |c_n e^{-j2\pi nf}| = \sum_{n \in \mathbb{Z}} |c_n| = \|c\|_1 < \infty$$

and so discrete Fourier transform  $\mathcal{D}c(f)$  is finite for all  $f$ . A discrete time Fourier transform can also be assigned to sequences that are only square integrable and not necessarily absolutely integrable. In this case one interprets the sum in (5.5.3) as

$$\mathcal{D}c(f) = \lim_{N \rightarrow \infty} \sum_{n=-N}^N c_n e^{-j2\pi nf}.$$

This is analogous to how the Fourier transform of a square integrable signal was assigned using the Cauchy principal value (5.1.2).

The discrete time Fourier transform  $\hat{c} = \mathcal{D}c$  is a periodic function from  $\mathbb{R} \rightarrow \mathbb{C}$ , that is,  $\hat{c}$  is a periodic signal. The fundamental period of  $\hat{c}$  is 1. The above equations relate the Fourier transform of the bandlimited signal  $x$  to the discrete time Fourier transform of its sequence of samples  $c$ . In Test 7 we compute the Fourier transform of a 20 s segment of audio from a lecture recording. In Test 8 we pass the audio signal through the Butterworth filter constructed in Test 6 and plot the Fourier transform of the response.

The sequence of samples  $c_n = x(nP)$  can be recovered by evaluating the inverse Fourier transform

$$\begin{aligned} c_n &= x(nP) = \mathcal{F}^{-1}\hat{x}(nP) \\ &= \int_{-\infty}^{\infty} P\Pi(Pf) \mathcal{D}c(Pf) e^{j2\pi f n P} df \\ &= P \int_{-F/2}^{F/2} \hat{c}(Pf) e^{j2\pi f n P} df \\ &= \int_{-1/2}^{1/2} \hat{c}(\gamma) e^{j2\pi \gamma n} d\gamma \quad (\text{change variable } \gamma = fP). \end{aligned}$$

We obtain the following relationship between the square integrable sequence  $c$  and its periodic discrete time Fourier transform  $\hat{c} = \mathcal{D}c$ ,

$$c_n = \int_{-1/2}^{1/2} \hat{c}(f) e^{j2\pi f n} df.$$

The right hand side of this expression is called the **inverse discrete time Fourier transform**. The element  $c_{-n}$  is also called the *n*th **Fourier coefficient** of the periodic function  $\hat{c}$ .

### Test 7 (The Fourier transform of a lecture recording)

In this test we consider a 20 s segment of audio taken from the lecture video `ch1sec3.mp4`. This 34.8 MB file contains both compressed video (H.264 codec) and audio (mp3 codec) of duration 23 min and 36 s. The audio is mono and sampled at rate  $F = 22\,050$  Hz. The `ffmpeg` program is used to extract a 20 s segment of audio starting at time 85 s and ending at time 105 s. The segment is decompressed to wav format. The command used is:

```
ffmpeg -i ch1sec3.mp4 -ss 85 -t 20 audio.wav
```

The resulting file `audio.wav` is 882 kB in size and contains  $N = 440998$  samples that we denote by  $c_0, c_1, \dots, c_{N-1}$ . Each sample takes a value in the interval  $[-1, 1]$ . We put  $c_n = 0$  when  $n < 0$  or  $n \geq N$ . The reconstructed audio signal is given by

$$x(t) = \sum_{n \in \mathbb{Z}} c_n \operatorname{sinc}(Ft - n) = \sum_{n=0}^{N-1} c_n \operatorname{sinc}(Ft - n).$$

From (5.5.2) the Fourier transform of this signal is  $\hat{x}(f) = P\Pi(Pf)\hat{c}(Pf)$  where

$$\hat{c}(f) = \mathcal{D}c(f) = \sum_{n \in \mathbb{Z}} c_n e^{-j2\pi n f} = \sum_{n=0}^{N-1} c_n e^{-j2\pi n f} \quad (5.5.4)$$

is the discrete time Fourier transform of the sequence of samples. Figure 5.8 shows a plot of the magnitude of the Fourier transform for frequencies in the interval  $-12$  kHz to  $12$  kHz. The plot is constructed by evaluating  $|\hat{c}(f)|$  at all  $K = 1201$  frequencies

$$f_k = -12000 + 20k \quad k = 0, \dots, K-1,$$

that is, from  $-12$  kHz to  $12$  kHz in steps of 20 Hz. It takes approximately 137 s to compute the Fourier transform at all of these frequencies. Evaluating the Fourier transform at a particular frequency requires calculating and

accumulating each of the  $N$  terms in the sum (5.5.4). We hypothesise it to take approximately

$$\frac{137 \text{ s}}{NK} \approx 260 \text{ ns}$$

to compute each term. The computer used is an Intel Core 2 running at 2.4 GHz and the software is written in the **Scala** programming language.

The audio recording contains human voice that primarily resides at lower frequencies below 5 kHz. Audible in the recording is a faint high pitched hum. The cause of this is unknown. It might be a feature of the (probably low quality) webcam microphone used to record the audio. This hum is represented in Figure 5.8 by the spikes occurring at approximately  $\pm 8$  kHz and also by the region between 4900 Hz and 5900 Hz where the magnitude of the Fourier transform is elevated. Figure 5.9 is a plot of the Fourier transform for frequencies from 7998 Hz to 8002 Hz in steps of 5 mHz. This gives a high resolution view of the spike that occurs near 8 kHz. The magnitude of the Fourier transform is precisely zero for frequencies  $|f| > F/2 = 11\,025$  Hz due to  $\Pi(Pf)$  occurring in the definition of  $\hat{x}$ . However, in Figure 5.8 it is apparent that the Fourier transform is small if  $|f|$  is a little larger than 8 kHz. This audio signal appears approximately bandlimited with bandwidth a little larger 8 kHz.

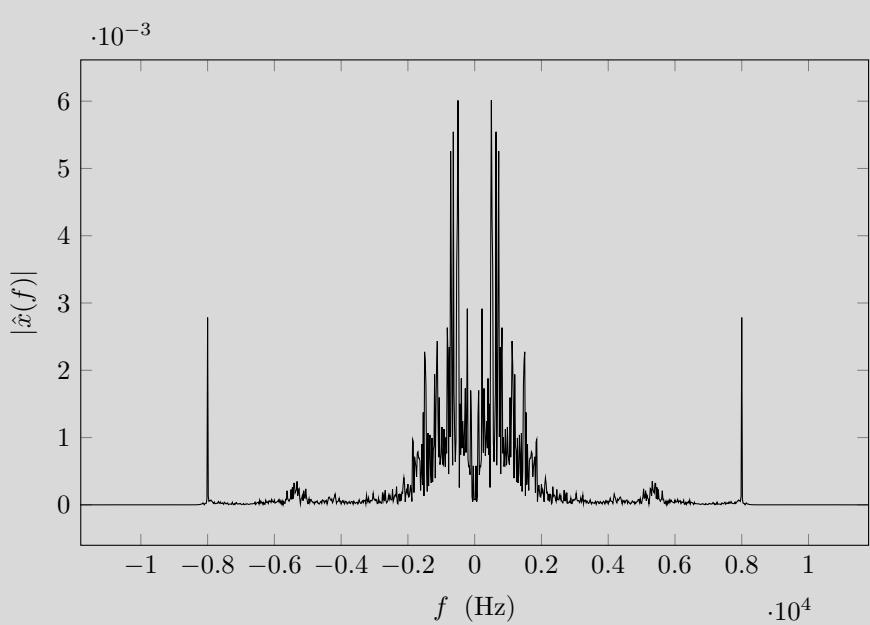


Figure 5.8: Magnitude of the Fourier transform of 20s of audio from a lecture recording. The human voice signal is primarily contained in the low frequency region below 5 kHz. The spikes occurring at approximately  $\pm 8$  kHz and the region between 4900 Hz and 5900 Hz where the magnitude is elevated are audible in the recording as a high pitched hum.

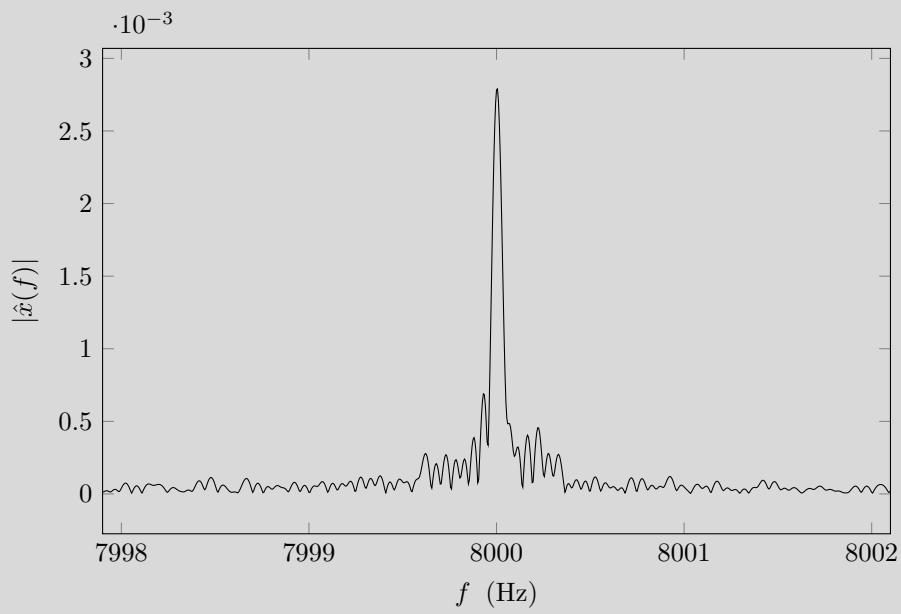


Figure 5.9: A plot of the magnitude of the Fourier transform zoomed in on the spike at 8 kHz.

**Test 8 (Butterworth filtered lecture recording)**

We consider again the 20 s audio signal from Test 7. In this test we pass this signal through the second order Butterworth filter from Test 6 with cutoff frequency approximately 3041 Hz. The output of the Butterworth filter is fed back to the soundcard input and recorded at 22 050 Hz. The recorded samples are written to the file `filtered.wav`. Listening to `filtered.wav` confirms that the high pitched hum is weaker than it is in the original audio signal. The Fourier transform of the Butterworth filtered signal is plotted in Figure 5.10. This figure is constructed by the same procedure as used for Figure 5.8 from Test 7. Observe that the spikes occurring at approximately  $\pm 8$  kHz are less prominent than in Figure 5.8.

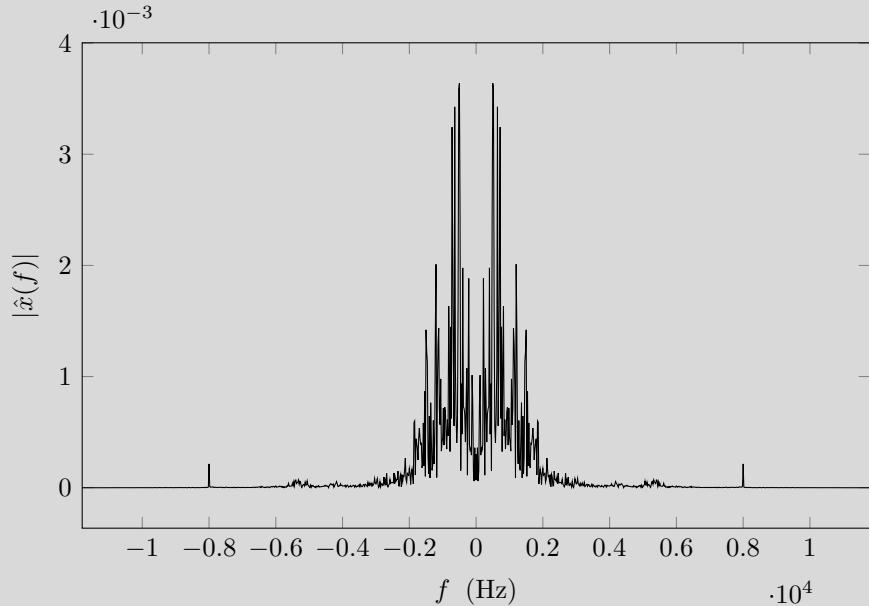


Figure 5.10: Magnitude of the Fourier transform of 20 s of audio from Test 7 after being passed through the second order Butterworth filter from Test 6 with cuttoff frequency approximately 3041 Hz. The magnitude of the Fourier transform at higher frequencies is attenuated when compared with the Fourier transform of the original audio signal (Figure 5.8). In particular, the spikes occurring at approximately  $\pm 8$  kHz are less prominent than in Figure 5.8. The high pitched hum that is audible in the original audio signal is significantly weaker in the Butterworth filtered audio signal.

## 5.6 The fast Fourier transform

In Test 7 the Fourier transform of a 20s audio signal consisting of  $N = 440998$  consecutive samples was evaluated. This scenario where only a finite number, say  $N$ , of consecutive samples of a signal is available is common in practice. Let  $c$  be a sequence with elements  $c_0, c_1, \dots, c_{N-1}$  equal to the  $N$  samples. A convenient assumption is that the remaining samples are equal to zero, that is,  $c_n = 0$  when  $n < 0$  or  $n \geq N$ . This assumption was made in Test 7.

With this assumption the discrete time Fourier transform of the sequence  $c$  is given by the finite sum

$$\hat{c}(f) = \mathcal{D}c(f) = \sum_{n \in \mathbb{Z}} c_n e^{-j2\pi n f} = \sum_{n=0}^{N-1} c_n e^{-j2\pi n f}.$$

The values of  $\hat{c}(f)$  for  $f$  a multiple of  $\frac{1}{N}$  have a number of convenient properties. Denote these values by

$$(\mathcal{D}_N c)_k = \mathcal{D}c\left(\frac{k}{N}\right) = \sum_{n=0}^{N-1} c_n e^{-j2\pi n k / N} \quad k \in \mathbb{Z}. \quad (5.6.1)$$

This is called the **discrete Fourier transform** (as opposed to the discrete *time* Fourier transform). The discrete Fourier transform  $\mathcal{D}_N c$  is a sequence with elements given by the discrete time Fourier transform  $\mathcal{D}c$  evaluated at multiples of  $\frac{1}{N}$ . As usual we include write  $\mathcal{D}_N c_k$  to denote the value of  $\mathcal{D}_N c$  at  $k \in \mathbb{Z}$ . We will regularly include brackets and use  $(\mathcal{D}_N c)_k$  to denote this value. The positive integer  $N$  is called the **length** of the transform. In practical applications  $N$  often corresponds with the number of samples of a signal that have been obtained.

The discrete Fourier transform is a periodic sequence with period  $N$  as a result of the discrete time Fourier transform  $\hat{c} = \mathcal{D}c$  having period 1, that is,

$$(\mathcal{D}_N c)_k = \hat{c}\left(\frac{k}{N}\right) = \hat{c}\left(\frac{k+mN}{N}\right) = (\mathcal{D}_N c)_{k+mN} \quad \text{for all } k, m \in \mathbb{Z}.$$

Because of this it is sufficient to know only  $(\mathcal{D}_N c)_k$  for  $k = 0, \dots, N-1$  in order to know the entire sequence  $\mathcal{D}_N c$ . Given  $d = \mathcal{D}_N c$  the original samples  $c_0, \dots, c_{N-1}$  can be recovered by

$$c_n = (\mathcal{D}_N^{-1} d)_n = \frac{1}{N} \sum_{k=0}^{N-1} d_k e^{j2\pi n k / N} \quad n = 0, \dots, N-1.$$

This is called the **inverse discrete Fourier transform** (Excercise 5.12). Taking complex conjugates on both sides gives

$$c_n^* = \frac{1}{N} \sum_{k=0}^{N-1} d_k^* e^{-j2\pi n k / N} = \frac{1}{N} \mathcal{D}_N(d^*, n) \quad n = 0, \dots, N-1.$$

A practical consequence of this is that the inverse discrete Fourier transform can be evaluated by applying the complex conjugate, taking the discrete Fourier transform, applying the complex conjugate again, and finally dividing by  $N$ . That is, if  $d$  is a sequence, then  $\mathcal{D}_N^{-1}d = \frac{1}{N}\mathcal{D}_N^*d^*$ .

Suppose that we wish to evaluate the discrete Fourier transform  $\mathcal{D}_{NC}$  of the 20 s audio signal comprising of  $N = 440998$  samples from Test 7. In Test 7 we hypothesised that approximately 260 ns are required to compute each term in the sum (5.5.4). We require to compute the sum for each  $k = 0, \dots, N - 1$  and so we might expect that

$$N^2 \times 260 \text{ ns} \approx 50565 \text{ s} \approx 14 \text{ hours} \quad (5.6.2)$$

will be required to compute  $\mathcal{D}_{NC}$  for this 20 s audio signal! A primary cause of this lengthy computation time is the quadratic term  $N^2$  that occurs in the expression above. The amount of time required grows proportionally with the square of the length of the transform. Suppose that instead of 20 s of audio we have 1 hour and  $N = 60 \times 60 \times 22050 = 79380000$  samples. The amount of time required in this case is approximated by  $N^2 \times 260 \text{ ns} \approx 52$  years!

Computing the discrete Fourier transform by direct application of the formula (5.6.1) is too slow when  $N$  is large. Fortunately, much faster algorithms exist. The algorithms are appropriately called **fast Fourier transforms**. The specific algorithm used depends on  $N$ . The simplest case is when  $N = 2^m$  is a power of 2. In this case an algorithm attributed to Cooley and Tukey [1965] can be used. When  $N = 2^m$  is divisible by 2 the sum in (5.6.1) can be split into two parts corresponding with  $n$  being even or odd,

$$(\mathcal{D}_{NC})_k = \sum_{n=0}^{N/2-1} c_{2n} e^{-j2\pi(2n)k/N} + \sum_{n=0}^{N/2-1} c_{2n+1} e^{-j2\pi(2n+1)k/N}. \quad (5.6.3)$$

Put  $M = N/2$  and let  $p$  be the sequence with elements  $p_n = c_{2n}$ , that is, the elements of  $p$  are the even indexed elements of  $c$ . Now the first term in (5.6.3) can be written in the form

$$\sum_{n=0}^{N/2-1} c_{2n} e^{-j2\pi(2n)k/N} = \sum_{n=0}^{M-1} p_n e^{-j2\pi nk/M} = (\mathcal{D}_M p)_k,$$

that is, this term is the discrete Fourier transform of length  $M = N/2$  of the sequence  $p$ . Let  $q$  be the sequence with elements  $q_n = c_{2n+1}$ , that is,  $q$  contains the odd indexed elements of  $c$ . The second term in (5.6.3) can be written in the form

$$\begin{aligned} \sum_{n=0}^{N/2-1} c_{2n+1} e^{-j2\pi(2n+1)k/N} &= e^{-j2\pi k/N} \sum_{n=0}^{M-1} q_n e^{-j2\pi nk/M} \\ &= e^{-j2\pi k/N} (\mathcal{D}_M q)_k, \end{aligned}$$

that is, this term is the discrete Fourier transform of length  $M = N/2$  of the sequence  $q$  multiplied by  $e^{-j2\pi k/N}$ . Combining these results we have

$$(\mathcal{D}_N c)_k = (\mathcal{D}_{N/2} p)_k + e^{-j2\pi k/N} (\mathcal{D}_{N/2} q)_k.$$

We see that the discrete Fourier transform  $\mathcal{D}_N c$  can be evaluated by computing two smaller discrete Fourier transforms  $\mathcal{D}_{N/2} p$  and  $\mathcal{D}_{N/2} q$  of length  $N/2$ . Both of these smaller transforms are sequences that are periodic with period  $N/2$  and so it is sufficient to know their values only for  $k = 0, \dots, \frac{N}{2} - 1$ . These  $N/2$  length transforms can in turn be computed by two transforms of length  $N/4$  and so on until transforms of length 1 are obtained. In this case  $(\mathcal{D}_1 c)_k = \sum_{n=0}^0 c_n e^{-j2\pi n k} = c_0$  for all  $k \in \mathbb{Z}$ .

The computational cost of this procedure can be analysed as follows. Suppose that  $C_N$  is the number of complex arithmetic operations (complex additions and multiplications) required to compute the discrete Fourier transform  $\mathcal{D}_N c$  of length  $N = 2^m$ . The computation requires calculation of two transforms of length  $N/2$  followed by  $N$  complex multiplications and  $N$  additions. The multiplications arise from the multiplication of  $(\mathcal{D}_{N/2} q)_k$  by  $e^{-j2\pi k/N}$  and the additions arise from summing the result of this product with  $(\mathcal{D}_{N/2} p)_k$ . The number of operations satisfies

$$C_N = 2C_{N/2} + 2N \quad N \geq 2.$$

Because  $(\mathcal{D}_1 c)_k = c_0$  we have  $C_1 = 0$ , that is, computing a discrete Fourier transform of length 1 requires no complex operations at all. Putting  $a_m = C_{2^m}$  we have

$$a_0 = C_1 = 0 \quad a_m = 2a_{m-1} + 2^{m+1} \quad m \geq 1. \quad (5.6.4)$$

This type of recursive equation is called a **difference equation** and will be studied further in Section 6.5. Exercise 6.6 shows that

$$C_N = a_m = 2^{m+1}m = 2N \log_2 N.$$

Observe that the number of operations (and hence the amount of time required) grows proportionally to  $N \log_2 N$  rather than  $N^2$ . Suppose that each complex operation requires no more than 260 ns. For the 20 s audio signal consisting of  $N = 440998$  samples the amount of time required will be less than

$$2N \log_2 N \times 260 \text{ ns} \approx 4.3 \text{ s}. \quad (5.6.5)$$

This is more reasonable than 14 hours! If instead we have 1 hour of audio and  $N = 79380000$  samples the amount of time required is hypothesised to be less than 1084 s  $\approx 18$  min. This is very reasonable when compared with the 52 years hypothesised to be required by direct application of formula (5.6.1). In practice the computation time will vary based on the computer used

and the specific algorithm implementation. Nevertheless, these numbers indicate that a fast Fourier transform of large length can be computed within a reasonable amount of time. This is not possible by direct application of formula (5.6.1). Test 9 compares the practical running time of various discrete Fourier transform implementations.

In the above computation of run times we have neglected that the fast Fourier transform we have described required the length  $N$  to be a power of two. Other algorithms exist for the case when  $N$  is not a power of two [Rader, 1968; Bluestein, 1968; Frigo and Johnson, 2005]. These algorithms deliver similarly dramatic computational savings. Even so, the restriction of the length to a power of 2 is often not a significant drawback in practical applications. Consider again the example from Test 7 with  $N = 440998$  samples. Denote by

$$L = 2^{\lceil \log_2 N \rceil} = 2^{19} = 524288$$

the smallest power of 2 greater than  $N$ . We can use the fast Fourier transform algorithm described to compute the discrete Fourier transform  $D_L c$  of length  $L$ . This transform is a sequence with period  $L$  and elements

$$(D_L c)_k = \hat{c}\left(\frac{k}{L}\right) = \sum_{n=0}^{L-1} c_n e^{-j2\pi nk/L} = \sum_{n=0}^{N-1} c_n e^{-j2\pi nk/L} \quad k \in \mathbb{Z}.$$

The second sum follows from our assumption that  $c_n = 0$  for  $n \geq N$ . The elements of  $D_L c$  are the values of the discrete time Fourier transform  $\hat{c}$  at multiples of  $\frac{1}{L}$  rather than  $\frac{1}{N}$ . This fact is often of no significant consequence and can even be of benefit for some applications [Quinn and Hannan, 2001; Quinn et al., 2008]. The original samples  $c_0, \dots, c_{N-1}$  can still be recovered by application of the inverse transform of length  $L$ , that is,

$$c_n = (D_L^{-1} D_L c)_n \quad n = 0, \dots, N-1.$$

This procedure of increasing the length of the transform is often called **zero padding** on account of the fact that the samples  $c_N, c_{N+1}, \dots, c_{L-1}$  are assumed to be zero. Test 10 presents a practical example of zero padding for the purpose of filtering the 20 s audio recording from Test 7.

### Test 9 (Benchmarking the fast Fourier transform)

In this test we compare the computational complexity of practical implementations of the discrete Fourier transform. Three different implementations are compared: a direct implementation by formula (5.6.1), an implementation of the fast Fourier transform of Cooley and Tukey [1965] when the length  $N = 2^m$  is a power of 2 as described in Section 5.6, and an implementation from an optimised fast Fourier transform library called

**JTransforms.** The **JTransforms** library contains implementations of fast Fourier transforms of all lengths, not just powers of 2.

Figure 5.10 shows the run-time in seconds versus transform length. For the **JTransforms** library and formula (5.6.1) the length of the transforms is given by the sequence  $N_k = \lceil 2^{6+k/2} \rceil$  for  $k = 0, 1, 2, \dots$ . For our implementation of the Cooley and Tukey [1965] algorithm the length must be a power of two and is given by  $N_k = 2^{6+k/2}$  for  $k = 0, 2, 4, \dots$ . The dashed lines indicate the approximate running times given by (5.6.2) and by (5.6.5). These approximations appear reasonably accurate on the log scale used in Figure 5.10. The fast Fourier transform algorithms are considerably faster than formula (5.6.1) as expected. For example, when the length is  $N = 2^{21} = 2097152$  the **JTransforms** library required approximately 0.58 s whereas formula (5.6.1) is hypothesised by (5.6.2) to require approximately  $N^2 \times 260 \text{ ns} \approx 13 \text{ days}$ .

The optimised algorithms from the **JTransforms** library are considerably faster than our implementation of the Cooley and Tukey [1965] algorithm. Observe the jagged nature of the run-time with the **JTransforms** library. The algorithms used by the library for length  $N_k$  and odd  $k$  appear slower than when  $k$  is even so that the length is a power of 2. The computer used is an Intel Core 2 running at 2.4 GHz and the software is written in the **Scala** programming language.

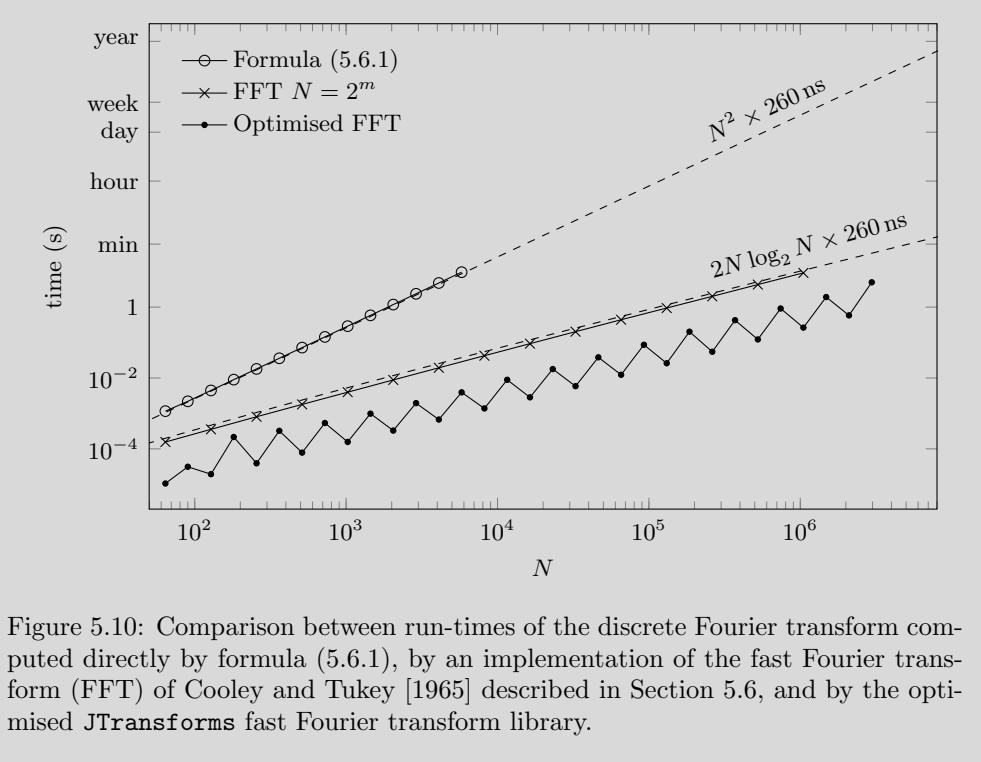
### Test 10 (Filtering a lecture recording by fast Fourier transform)

We again consider the 20 s segment of audio consisting of  $N = 440998$  samples from Test 7. As in Test 7 we let  $c$  be the sequence with elements  $c_0, \dots, c_{N-1}$  equal to the audio samples and put  $c_n = 0$  for  $n < 0$  or  $n \geq N$ . The reconstructed audio signal is given by

$$x(t) = \sum_{n \in \mathbb{Z}} c_n \operatorname{sinc}(Ft - n) = \sum_{n=0}^{N-1} c_n \operatorname{sinc}(Ft - n)$$

where  $P = \frac{1}{F}$  is the sample period and  $F = 22\,050 \text{ Hz}$  is the sample rate. The Fourier transform of  $x$  is  $\hat{x} = \mathcal{F}x = P\Pi(Pf)\hat{c}(Pf)$  where  $\hat{c} = \mathcal{D}c$ . Audible in the recording is a faint high pitched hum. This hum appears in the Fourier transform as spikes occurring at  $\pm 8 \text{ kHz}$  and also as the region between 4900 Hz and 5900 Hz where the magnitude of the Fourier transform is elevated (Figure 5.8).

In this test we use a fast Fourier transform to remove this hum from the audio while minimally affecting the human voice. To do this we compute an



approximation of the bandlimited signal  $y$  with Fourier transform

$$\hat{y}(f) = \mathcal{F}y(f) = \begin{cases} 0 & |f| > 7200 \\ 0 & |f - 5400| < 500 \\ 0 & |f + 5400| < 500 \\ \hat{x}(f) & \text{otherwise.} \end{cases}$$

That is,  $y$  is the signal with Fourier transform equal to  $\hat{x}$  except for those frequencies between 4900 Hz and 5900 Hz and above 7200 Hz where the Fourier transform is zero. Because  $y$  is bandlimited with bandwidth less than  $F$ ,

$$y(t) = \sum_{n \in \mathbb{Z}} b_n \operatorname{sinc}(Ft - n)$$

where  $b$  is the sequence with elements  $b_n = y(nP)$  given by samples of  $y$  at sample period  $P$ . Now  $\hat{y} = P\Pi(Pf)\hat{b}(Pf)$  where  $\hat{b} = \mathcal{D}b$  is the discrete time Fourier transform of  $b$ . For  $f$  inside the interval  $[-\frac{1}{2}, \frac{1}{2})$ , the discrete time

Fourier transforms  $\hat{c}$  and  $\hat{b}$  are related by

$$\hat{b}(f) = \begin{cases} 0 & |f| > 7200P \\ 0 & |f - 5400P| < 500P \\ 0 & |f + 5400P| < 500P \\ \hat{c}(f) & \text{otherwise.} \end{cases}$$

For  $f \notin [-\frac{1}{2}, \frac{1}{2})$  a similar relationship can be obtained by appealing to the periodicity of  $\hat{b}$  and  $\hat{c}$ . This is easiest to express by introducing the notation  $\langle a \rangle = a - \lceil a \rceil$  called the **centered fractional part** of  $a \in \mathbb{R}$ . Now

$$\hat{b}(f) = \begin{cases} 0 & |\langle f \rangle| > 7200P \\ 0 & |\langle f - 5400P \rangle| < 500P \\ 0 & |\langle f + 5400P \rangle| < 500P \\ \hat{c}(f) & \text{otherwise} \end{cases}$$

for all  $f \in \mathbb{R}$ .

Let  $L = 2^{19} = 524288$  be the smallest power of 2 less than or equal to  $N$ . Using the fast Fourier transform we compute the discrete Fourier transform  $\mathcal{D}_L c$  of length  $L$  of the sequence  $c$ . This yields values of  $\hat{c}$  at multiples of  $\frac{1}{L}$ , that is,

$$(\mathcal{D}_L c)_k = \hat{c}\left(\frac{k}{L}\right) \quad k \in \mathbb{Z}.$$

Let  $d$  be the sequence with elements

$$d_k = \mathcal{D}b(k/L) = \begin{cases} 0 & \left|\left\langle \frac{k}{L} \right\rangle\right| > 7200P \\ 0 & \left|\left\langle \frac{k}{L} - 5400P \right\rangle\right| < 500P \\ 0 & \left|\left\langle \frac{k}{L} + 5400P \right\rangle\right| < 500P \\ (\mathcal{D}_L c)_k = \hat{c}(k/L) & \text{otherwise.} \end{cases}$$

We do not necessarily have  $d = \mathcal{D}_L b$  because  $b_n$  is not necessarily equal to zero for  $n < 0$  and  $n \geq L$ . Nevertheless, we will suppose that  $d \approx \mathcal{D}_L b$ . In this case, application of the inverse discrete Fourier transform yeilds the periodic sequence  $\tilde{b} = \mathcal{D}_L^{-1} d$  and we expect the first  $L$  elements of  $\tilde{b}$  to be an approximation of the first  $L$  elements of  $b$ , that is,

$$\tilde{b}_n \approx b_n = y(nP) \quad \text{for } n = 0, \dots, L-1.$$

An approximation of the signal  $y$  is now given by

$$y(t) \approx \tilde{y}(t) = \sum_{n=0}^{N-1} \tilde{b}_n \operatorname{sinc}(Ft - \ell).$$

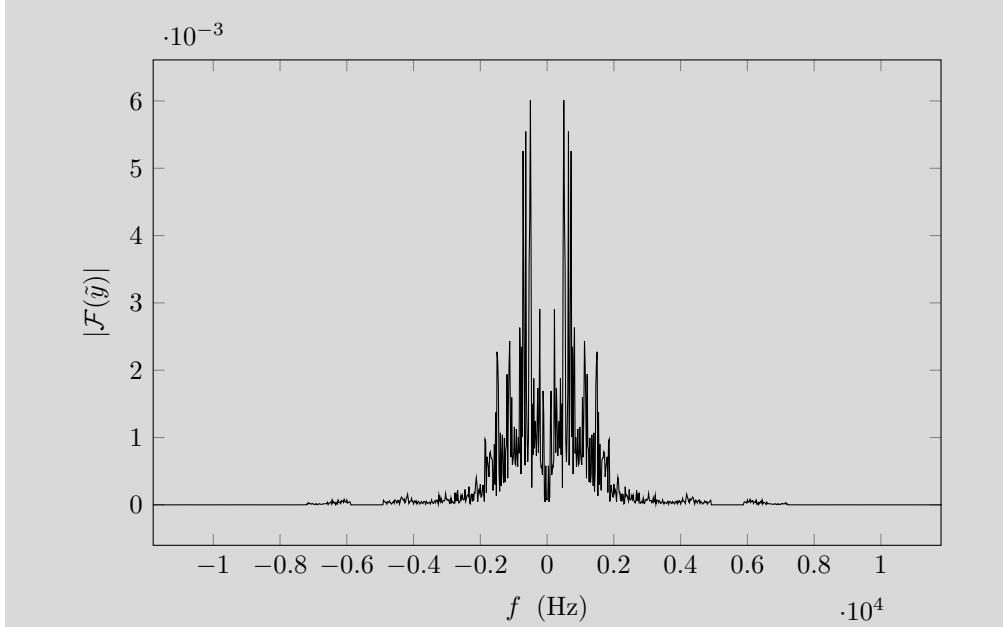


Figure 5.11: Plot of the magnitude of the Fourier transform  $\mathcal{F}\tilde{y}$ . The plot looks similar to that of the magnitude of the Fourier transform of the original audio signal (Figure 5.8) except that the spikes at  $\pm 8$  kHz and the elevated region between 4900 Hz and 5900 Hz no longer exist.

Figure 5.11 plots the magnitude of Fourier transform  $\mathcal{F}\tilde{y}$ . Observe that  $|\mathcal{F}\tilde{y}|$  looks similar to the magnitude of the Fourier transform of the original audio signal  $x$  plotted in Figure 5.8 except that the spikes at  $\pm 8$  kHz and the elevated region between 4900 Hz and 5900 Hz no longer exist. The samples  $\tilde{b}_0, \dots, \tilde{b}_{N-1}$  are written to the audio file `nohum.wav`. Listening to the audio confirms that the human voice signal remains, but the high pitched hum is no longer audible.

## Exercises

5.1. Plot the signal  $e^{-\alpha|t|}$  where  $\alpha > 0$  and find its Fourier transform.

5.2. Plot the signal

$$\Delta(t) = \begin{cases} t + 1 & -1 < t < 0 \\ 1 - t & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

and find its Fourier transform.

- 5.3. Show that the sinc function is square integrable, but not absolutely integrable.
- 5.4. Show that the magnitude spectrum of the normalised Butterworth filter  $B_m$  satisfies

$$|\Lambda B_m(f)| = \sqrt{\frac{1}{f^{2m} + 1}}.$$

- 5.5. Find and plot the impulse response of the normalised lowpass Butterworth filters  $B_1, B_2$  and  $B_3$ .

- 5.6. Plot the signal

$$t\Pi(t) = \begin{cases} t & -\frac{1}{2} < t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

and find its Fourier transform.

- 5.7. Let  $x$  be the signal with Fourier transform  $\hat{x}(f) = \Pi(f)(\cos(2\pi f) + 1)$ . Plot the Fourier transform  $\hat{x}$  and find and plot  $x$ .

- 5.8. State whether the following signals are bandlimited and, if so, find the bandwidth.

- (a)  $\text{sinc}(4t)$ ,
- (b)  $\Pi(t/4)$ ,
- (c)  $\cos(2\pi t) \text{sinc}(t)$ ,
- (d)  $e^{-|t|}$ .

- 5.9. Show that

$$\sum_{n \in \mathbb{Z}} e^{\alpha|n|} = 1 + \frac{2}{e^{-\alpha} - 1}$$

if  $\alpha < 0$  (Hint: solve Exercise 3.8 first).

- 5.10. Show that if a sequence absolutely summable then it is also square summable.

- 5.11. Show that  $\sum_{k=0}^{N-1} e^{j2\pi nk/N}$  is equal to  $N$  if  $n$  is a multiple of  $N$  and zero if  $n$  is any integer not a multiple of  $N$ . (Hint: use the result from Exercise 3.8)

- 5.12. Let  $d = \mathcal{D}_N c$  be the discrete Fourier transform of the sequence  $c$ . Show that

$$c_n = \frac{1}{N} \sum_{k=0}^{N-1} d_k e^{j2\pi nk/N} \quad n = 0, \dots, N-1.$$

(Hint: use the result from Exercise 5.11)

- 5.13. Plot the sequence  $\cos(n)$  and determine whether it is bounded or periodic.
- 5.14. Find the discrete time Fourier transform of the sequence  $\alpha^n u_n$  where  $|\alpha| < 1$  and  $u_n$  is the step sequence. Plot the sequence and the magnitude of the discrete time Fourier transform when  $\alpha = \frac{4}{5}, \frac{1}{2}, \frac{1}{10}$ .
- 5.15. Given (5.1.3) show that (5.1.4) holds.

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