The Measurement of Voting Power: Theory and Practice, Problems and Paradoxes

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Put not yourself into amazement how these things should be: all difficulties are but easy when they are known.

W Shakespeare, *Measure for Measure*, Act IV, Scene ii

The only government which can fully satisfy all the exigencies of the social state is one in which the whole people participate; . . . . But since all can not, in a community exceeding a single small town, participate personally in any but some very minor portions of the public business, it follows that the ideal type of a perfect government must be representative.

J S Mill, *Considerations on Representative Government*

Representative government, therefore, is not an original organic form, but a machinery necessitated by modern civilization and requirements of life to make democratic government possible — a machine more or less perfect in proportion to its success in realizing the democratic idea of a government by the people for the people.

S Sterne, *Representative Government and Personal Representation*

To the two Ilanas,
for all sorts of wonderful reasons

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List of Abbreviations

ABP added blocker postulate

Benelux Belgium, Netherlands, Luxembourg

BPR bipartition rule

BSP blocker's share postulate

Bz Banzhaf

CAP Common Agricultural Policy

CF characteristic function

CFSP Common Foreign and Security Policy

CM Council of Ministers

CMEC Council of Ministers of the European Community

D-P Deegan–Packel

EC European Community

ECSC European Coal and Steel Community

E.D.N.Y. Eastern District, New York

EEC European Economic Community

EMU Economic and Monetary Union

EFTA European Free Trade Association

EU European Union

EURATOM European Atomic Energy Community

GDP gross domestic product

IGC inter-governmental conference

JHA Justice and Home Affairs

JI Johnston index

JS Johnston score

Js Johnston

MMD mean majority deficit

MWC minimal winning coalition

NYS New York State

OPOV one person, one vote

PPV probabilistic proportional voting

PSQRR Penrose's square-root rule

PWP population–weight proportionality

QMV qualified majority voting

SEA Single European Act

S-S Shapley–Shubik

SSQRR second square-root rule

SVG simple voting game

TVR ternary voting rule

UNSC United Nations Security Council

WVG weighted voting game

Preface

We entered the field of voting power some seven years ago. Like most people working in the area of social choice we knew that a priori voting power was commonly measured by two indices — one proposed by Shapley and Shubik [97], the other by Banzhaf [5] — which seemed to be similar to each other. We were also aware that some phenomena associated with these indices were described in the literature as paradoxes. Wanting to learn more about them, we subjected them to closer examination and concluded that these phenomena were not too astonishing. However, in analysing the so-called paradoxes of redistribution and large size (see §§ 7.3 and 7.2, respectively), we began to suspect that the Banzhaf index might display much more extreme paradoxical behaviour. This led to the discovery in [29] of the transfer paradoxes (see § 7.8).

At a deeper level, we remained dissatisfied with the conceptual foundations of the subject. It seemed to us that the underlying notion of voting power — that real magnitude which is supposed to be measured by the indices — was insufficiently analysed and clarified. The technical results we were obtaining seemed to demand such clarification, and at the same time pointed the way to it. Then — particularly in the course of our joint work on [36] with William Zwicker (see § 7.9) — we gradually realized that there were in fact *two* quite distinct notions of voting power, which were commonly conflated with each other. While the Shapley–Shubik index was based on a notion, derived from cooperative game theory, of voting power as expected share in a fixed total prize (see § 6.1), the Banzhaf index (which, as we found out in the meantime, had been proposed much earlier by Lionel Penrose [78]) was based on a notion of voting power as probability of influencing the decision of a voting body — a quantity whose total is not constant (see § 3.1). Yet, the Banzhaf index was inappropriately treated by most writers on voting power as a game-theoretic measure of constant-sum payoff, merely a variant of the Shapley–Shubik index.

As a matter of fact, the distinction between the two notions of voting power had been urged by Coleman [20]; but his objections went unheeded by most researchers, who continued to conflate the two underlying notions. We ourselves had shared the confusion for a while, and in [29] we mistakenly dismissed the Banzhaf index, due to our failure to arrive at a true explanation of its paradoxical behaviour.

When we started to look into a priori voting power, a prevalent view among political scientists was that this subject was somewhat old-fashioned. It may have been all the rage in the 1960s, but by the 1980s it had become an exhausted mine, where it is no longer possible to unearth anything of real value. Meantime, recent research had moved into the field of *de facto* voting power, which depends on the local institutional setting of the decision-making body. This view is still held by those who have not kept up with even more recent work on a priori voting power.

But it became increasingly evident to us that earlier work on a priori voting power had left wide gaps not only in the superstructure of the theory but at its very foundations, which were left in a confused state. It was perhaps not accidental that while this theory was more than half a century old, and the literature on it was fairly large, there was as yet no monograph devoted entirely to it. Meantime, many practitioners (like most scientists who do statistical testing) employed the mathematical formulas mechanically, unconcerned about their justification — much as a cook uses a recipe without worrying about the underlying chemistry.

Moreover, although we would be the last to deny the importance of research into actual, a posteriori, voting power in specific institutions, we believe that a robust theoretical approach to such problems must be based on a sound theory of a priori voting power. Actual voting power is, after all, a result of superposition of specific institutional factors on an a priori ground. Without a clear understanding of the latter, there is little hope of theoretical clarity and empirical coherence in studying the composite edifice as a whole.

The main purpose of this book is therefore to attempt a systematic critical examination and exposition of the foundations and methodological presuppositions of the theory of a priori voting power.

While presenting many technical results, some of them new, we have made no attempt at comprehensive up-to-date coverage of all that is known in this field. And we have resisted the temptation to indulge in mathematical development for its own sake or for purely ornamental purposes. The technical results we do present are those we believe to have foundational significance. Almost all of them are actually used directly or indirectly in the conceptual analysis. Thus, for example, the technical exposition in §§ 3.3 and 3.4 — which contains some new results, including a precise statement and proof of the second square-root rule — is required for analysing issues such as the distinction between equal suffrage and majority rule in a two-tier decision-making system, which is fundamental for representative democracy. Similarly, the novel mathematical framework outlined in Ch. 8, in which abstention is admitted as a distinct option, is in our view an essential extension of the foundation of the theory of voting power. We use it to examine critically two of the stock examples used in the literature as illustrations: the distribution of power in the US legislature and the UN Security Council.

The foundational intent of our project imposes the need for mathematical rigour. The theory of voting power is ineluctably mathematical, and it would be impossible in a book of this sort to relegate all the mathematics to technical appendixes, still less to dumb it down. This would be like Hamlet without the prince, or with the prince in a small walk-on role. Five chapters (2, 3, 6, 7 and 8) are devoted to developing the mathematical foundations of the theory, and subjecting them to critical examination.

Of course, the book is not written primarily for expert mathematicians. We hope that large parts of it will be accessible and useful to students and researchers in the area of social choice, as well as in related areas concerned with aspects of decision-making by vote and distribution of voting power: political science, economics, business administration and constitutional law. We have tried to make the five mathematical chapters as reader-friendly as we could, in two ways. First, the mathematical knowledge that we presuppose is actually quite modest. Apart from general set-theoretic concepts, terminology and notation that are nowadays standard lingua franca in almost all mathematical discourse, we only presuppose a modicum of familiarity with elementary probability theory. In fact, almost all the probability spaces we consider are finite. The sole exception is at the very end of § 6.3; and even there we only use a bit of elementary calculus. Second, the more technical formal definitions and theorems are accompanied by Remarks in which they are explained in plainer language; and they are often illustrated by simple examples.

The reader should note the difference between Remarks and Comments. The former contain, in addition to informal explanations of formal statements, also some supplementary technical, historical and bibliographic details. The latter, on the other hand, contain discussion of methodological and foundational matters. Thus, the 'message' of this book is to be found in the Comments.

A reader who wishes to omit as much of the mathematical development as possible is advised, when reading these five chapters, to concentrate on the Comments, and refer back to the mathematical statements as the need arises. To make this as easy as possible, we have supplied copious cross-references in the text, as well as a Technical Index.

Two chapters are devoted to extensive case studies. These too are not undertaken solely for their own sake, but in order to illustrate how fundamental theoretical issues turn up, and are dealt with for better or worse, in real life. Ch. 4 is devoted to a series of US court cases concerning the application of the principle of equal suffrage ('one person, one vote'). We subject the courts' opinions to critical examination in the light of the theory developed in the preceding chapter. In particular, the frequently cited (and misrepresented) case of weighted voting in Nassau County, NY, is discussed in considerable detail and updated. Ch. 5 outlines the history of weighted voting (aka 'qualified majority voting') in the European Community's Council of Ministers from the Community's foundation in 1958, through its four successive enlargements, up to the time of writing (spring 1998). This weighted voting system is subjected to detailed theoretical analysis.

In view of the main purpose of this book, we could not avoid making outspoken, sometimes even harsh, critical comments on some statements and positions found in the voting-power literature that we believe to be misleading, confused or simply erroneous. In no case are these comments intended personally against the particular authors cited (who, besides, may have changed their views by now). The targets of our criticism are merely a sample: a mistake made by one author was surely shared by many others — in some cases including ourselves. In all fairness we must admit that it has taken us quite long to rid ourselves of some common confusions and misconceptions. And it wasn't easy. Of course, in hindsight 'all difficulties are but easy', as the motto of our book puts it.

Notes (i) For the sake of brevity we often refer to a work listed in the Bibliography by means of its number, without mentioning the author(s). However, in the General Index such references are listed under the authors' names. For example, in fn. 14 on p. 47 we refer to [27] without mentioning the authors. But the Index lists this reference under the names of both authors, Dubey and Shapley.

(ii) The Technical Index is in lieu of a glossary. It lists the locations in the text where a technical term is defined, redefined or explained, but not where it is merely used.

(iii) The US court cases cited are not included in the Bibliography but are listed separately. In quoting from opinions of the US Supreme Court, we follow the style and punctuation of the Lawyers' Edition, but the page references are to the original edition (whose pagination is also indicated in the Lawyers' Edition).

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1. Introduction

1.1 What this Book is About

Roughly speaking, this book is about the measurement of voting power. It will take us some time, space and effort to make this rough statement — and the key concept of *voting power* — more precise; but the general idea can be outlined right away. It concerns any collective body that makes yes-or-no decisions by vote. For the sake of definiteness and brevity, we shall refer here to the decision-making body as a *board* (although it can equally well be a council, a legislature, a committee, a shareholders' meeting or a whole nation participating in a referendum); and we shall refer to a proposed resolution put to the vote as a *bill*. We shall use the term *division* to refer to the collective act whereby each board member casts a vote regarding a given bill.[1] We usually speak of the *voters* or *members* of a board as of individual persons; but as a matter of fact a group of persons may sometimes be regarded as a single member. This is the case, for example, in a legislature in which several political parties are represented: if the representatives of a party can be assumed to vote in unison, with perfect discipline, then they may collectively be regarded as a single voter, wielding a number of mandates.

Every board has a definite rule for passing bills.[2] For any such

[1] We have borrowed this term from English parliamentary usage. The literature on voting power does not always make a careful terminological distinction between the individual and the collective acts, and refers to both as 'voting'.

[2] In reality a voting body may operate several decision rules, applied to different types of resolution; see Rem. 2.1.2(iii) for the way the formal theory deals with this. For the moment, such complexities may be ignored.

rule, we may ask: to what extent is a given member able to control the outcome of a division? This 'extent' is the member's voting power. More precisely, what is to be measured is *a priori* voting power, determined without taking into consideration voters' prior bias regarding the bill voted upon, or the degree of affinity (for example, ideological proximity) between voters.[3]

The question we have just posed can be split into two parts. First, can we quantify the *absolute* voting power of a board member? Is it meaningful to compare the voting powers of members of two different boards, having two different decision rules? Second, what is the *relative share* of a given member in the decision-making power of the entire board? The distinction between absolute and relative voting power is analogous to that between income (what is the size of a given individual's income?) and income distribution (what proportion of the total income of a group accrues to each of its members?). Clearly, a positive solution to the problem of measuring absolute voting power would logically imply a solution to the problem of measuring relative voting power. The converse, however, is not true, as the following discussion demonstrates.

There are cases in which the problem of measuring relative voting power has an obvious solution: if the decision rule is *symmetric*, treating all voters in exactly the same way,[4] then all the voters have equal voting power. In such cases the relative power of each of the n voters is $1/n$. But this tells us nothing about absolute voting power.

1.1.1 Example Consider a board consisting of three members whose decision rule is that of unanimity: a bill is passed iff[5] all three members vote for it. Then the relative voting power of each member is $\frac{1}{3}$.

Now suppose that the board is enlarged by adding one new member. At the same time, the decision rule is changed to that of

[3] For a detailed discussion of this point see Com. 2.2.3.

[4] For a rigorous definition of 'symmetric', see Rem. 2.3.11(ii); but cf. also Def. 2.1.7 and Rem. 2.1.8.

[5] Here and throughout this book, 'iff' is short for 'if and only if'.

absolute majority: a bill is passed iff at least three (*any* three!) of the four members vote for it. Again, for reasons of symmetry, all members have equal relative power: $\frac{1}{4}$.

The *relative* voting power of each member is now smaller than before. But has the *absolute* power of the three old members changed? If so, in what way? The answer is not at all obvious. Moreover, it is not self-evident that the question itself is meaningful.[6]

There is an important class of cases where the determination of relative voting power seems simple enough, but on closer inspection this impression turns out to be deceptive. Here we have in mind *weighted voting* decision rules. Under such a rule, each board member is assigned a non-negative number as *weight*, and a certain positive number is fixed as *quota*. The rule is that a bill is passed iff the total weight of those voting for it is equal to or greater than the quota. It is tempting to jump to the conclusion that the powers of the voters are proportional to their respective weights, so that the relative power of each voter is equal to that voter's relative share of the total weight.[7] But this cannot generally be true.

1.1.2 Example Consider a board with $n+1$ members: a chairperson and n ordinary members. The chairperson has weight 1, while each ordinary member has the same weight $w > 1$. The quota is set at half the total weight: $q = (nw+1)/2$.

First, suppose n is even, say $n = 2m$. Then a bill is passed iff at least $m+1$ members vote for it. This rule is symmetric, because the chairperson's vote has exactly the same effect as an ordinary member's. The voting-power ratio between the chairperson and an ordinary member is $1 : 1$, while their weight ratio is $1 : w$. If w is much greater than 1, the disparity between the two ratios is very large.

Now suppose n is odd, say $n = 2m+1$. In this case a bill is passed iff at least $m+1$ ordinary members vote for it. The

[6]We shall return to this example in Ex. 3.2.6.

[7]This fallacy seems to have the allure of a siren: even seasoned sailors occasionally fall for it. See, for example, below § 4.2, fn. 52.

chairperson's vote makes no difference at all, so her relative voting power is 0. Clearly, all the voting power is shared equally among the n ordinary members, each having relative power $1/n$. This result holds even if w is arbitrarily close to 1.

1.1.3 Example This is a real-life example. The Council of Ministers (CM) of the European Community (EC) operates a weighted voting rule. In 1958, when the EC was first set up (as the European Economic Community), it had six members: France, [the Federal Republic of] Germany, Italy, Belgium, The Netherlands, Luxembourg. The weights in the CM were as follows: France, Germany and Italy had 4 units each; Belgium and The Netherlands had 2 units each; and Luxembourg had 1. The quota was set at 12.

It is evident that each of the three big members had more power than each of the two middle-sized ones, and the latter had more power than Luxembourg. But it would be wrong to assume that the powers were proportional to the respective weights. Luxembourg did *not* have half as much power as Belgium, or a quarter as much as Italy. In fact, Luxembourg had no voting power at all: for a bill would pass iff it were supported at least by all big three, or by two of the big three and both middle-sized members; how Luxembourg voted made absolutely no difference!

Luxembourg apart, what about the other five members, who were evidently not powerless? Would we be justified in inferring that Belgium, with weight 2, had half as much power as France, whose weight was 4? Not necessarily; for, suppose that the weights had been 8 for each of the big three, 6 for each of the middle-sized two, and 1 for Luxembourg, and the quota set at 24. Then power relations in the CM would have not changed at all: for a bill to pass, it would still need the support of all big three, or of two of the big three and both middle-sized members. Whatever the ratio between the voting powers of Belgium and France may have been, it could not possibly be both $1 : 2$ and $3 : 4$.

This example demonstrates that even in a comparatively simple case some effort may be required in order to determine relative voting powers, let alone absolute ones. That the effort is worth making should be clear enough: the measurement of voting power has obvious descriptive and prescriptive uses. From a descriptive viewpoint, citizens of the European Union, for example, ought surely to be interested to know how much voting power is wielded by their representatives in the CM, and in particular how the total power is shared among the various member states. From a prescriptive point of view, knowledge of this kind is vital for a rational and equitable design of decision-making rules. Indeed, as we shall see in Chapter 4, since the late 1960s this aspect of the measurement of voting power assumed legal and political importance in the United States.

As that US story illustrates, what is at issue may often be relative rather than absolute voting power.

Before leaving this section, we should like to say a few words about the relation of our topic to the main body of the theory of voting. Briefly, the measurement of voting power is, so to speak, orthogonal to the concerns of the general theory of voting, which studies the properties of various voting procedures. This is so for two reasons. First, the really interesting problems in the general theory arise in situations where a group of voters (an electorate) must choose between more than two options (candidates). On the other hand, the measurement of voting power is concerned with situations where just two outcomes are possible: a bill is either passed or rejected. Second, the general theory has occupied itself, for the most part, with symmetric decision rules (those which treat all voters in the same way). On the other hand, the study of voting power makes no such restrictions. Moreover, the literature on the subject has concentrated mainly on the problem of relative voting power, which (as we have noted) is trivial in the case of symmetric rules.[8]

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[8]In this connection note that the problem of voting power can also be trivialized by using a probabilistic decision rule. Suppose each voter is assigned a weight, as in weighted voting; and each bill is decided by a single voter chosen at random by weighted lottery, in which the probabilities of being chosen are proportional to the members' weights. Clearly, these probabilities can be regarded as the members' respective voting powers.

1.2 Historical Sketch

In this section we draw a brief bare outline of the history of our subject. We shall list here a few landmarks, which are intended to provide chronological points of reference for the discussion in the sequel.

Many thoughtful observers must have realized that under a weighted voting rule voting power may not be proportional to weight. An early example is Luther Martin, a Maryland delegate to the 1787 Constitutional Convention held in Philadelphia. In a pamphlet published the following year he not only exposes the fallacy of equating power with weight, but makes an attempt — albeit unsystematic and somewhat crude — to measure voting power.[9]

As far as we know, the first scientific work on the measurement of voting power is Lionel Penrose's 1946 short paper [78]. The approach of this trail-blazing paper is entirely probabilistic. In particular, Penrose proposes a probabilistic measure of absolute voting power. Although this measure is defined in the context of a special class of decision rules — 'decisions made by majority vote' — it is clearly much more general, and can be applied to a very broad class of rules. Next, he presents some numerical results concerning the power of a ' "resolute" bloc' of voters 'who always vote together', while the remaining voters are 'an "indifferent" random voting group'.

He then turns to the main theme of the paper: a two-tier voting system, such as 'a federal assembly of nations' — an obvious reference to the newly established United Nations — in which a set of constituencies of different sizes elect one representative each to a decision-making 'assembly of spokesmen'. He argues that an equitable distribution of voting power in the assembly is the *square-root rule*, according to which 'the voting power of each nation in a world assembly should be proportional to the square root of the number of people on each nation's voting list'.

This paper should have been seminal; but the seed fell on stony

[9] For details see Riker [86].

ground. Penrose's ideas were subsequently re-invented by others, and his name is hardly mentioned in the mainstream literature on voting power.[10]

The mainstream's founding paper [97] was published in 1954 by Lloyd S Shapley and Martin Shubik. Their approach is essentially game-theoretic. A year earlier, the first of these authors had published his influential [94], in which he presented the *Shapley value*: a function ϕ that assigns to each cooperative game $\mathbf{v}$ and each player i in the 'universe of players' a numerical value $\phi_i(\mathbf{v})$ — the 'value' for i of playing the game $\mathbf{v}$. In [97], the authors define the *Shapley–Shubik* (briefly, S-S) *index* of voting power as a special case of the Shapley value. This is possible because a decision-making body — or, more precisely, the decision rule of such a body — can often be modelled mathematically as a particularly simple kind of cooperative game.[11] As we shall see in Com. 6.1.5, the S-S index is essentially a *relative* measure: it attributes no real meaning to absolute voting power, and the S-S values of all voters in a given game always add up to 1.[12] In [97], the authors illustrate the use of their index by applying it to several examples, including the real-life cases of the US legislature and the UN Security Council (UNSC). However, they do not specify precisely the class of cooperative games to which their index is applicable; instead, they invoke the notion of *simple game* defined by von Neumann and Morgenstern in [108]. A rigorous redefinition

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[10] The exception that proves the rule is Morriss [70, p. 160], which gives him full credit, but does not belong to the mainstream. In Fielding and Liebeck [37, p. 249], which also does not belong to the mainstream, he is given credit for the square-root rule, but not for his measure of voting power. This attribution of the square-root rule is cited also in Grofman and Scarrow [41, p. 171], which does belong to the mainstream.

[11] See below, Def. 2.1.1 and Rem. 6.2.2(ii). However, this model ignores absences and abstentions by assuming that a voter who does not vote for a bill counts as voting against it. This assumption is violated by many commonly used decision rules — a fact that went largely unmentioned for a very long time (see Com. 2.2.4 and, in greater detail, § 8.1).

[12] In this book we reserve the term *index* of voting power for relative measures, whose values — as in the present case — always add up to 1.

of the class of games in question was published eight years later by Shapley [95]. His class of *simple games* is in fact considerably broader than that of [108]. In [95] the problem of measuring voting power is not addressed. However, the definitions of the notion of simple game and various auxiliary notions presented in this paper (see below, § 2.1) provided the formal mathematical infrastructure for most of the later work on voting-power theory.

The next major landmark was the publication of John F Banzhaf's paper [5] in 1965. The author, a jurist, addresses the issue of voting power from a legal-constitutional point of view: the requirement that in representative assemblies 'equal numbers of citizens have substantially equal representation' ([5, p. 317]). In the US, some state legislatures and county boards had attempted to meet this requirement by using weighted voting and assigning to board members weights proportional to the size of the population each represents. An underlying assumption was, of course, that voting power is proportional to weight. The author shows, using examples of the kind presented by us in § 1.1, that the assumption is fallacious. He then proposes a measure of voting power, based implicitly on a probabilistic approach, which is essentially the same as that proposed 19 years earlier in [78]. However, by the nature of the problem Banzhaf addresses — that of equal representation — he is only interested in determining the *ratio* of the voting power of any board member to that of any other member in the same board. So, although a measure of absolute voting power, virtually identical to that of [78], is implicit in his analysis, what he actually seems to propose is the corresponding relative measure, which became known as the *Banzhaf* (briefly, Bz) *index*.

In a second paper [6] published a year later, Banzhaf presents, essentially, a derivation of the square-root rule — unaware that in this too he is following in the footsteps of Penrose — and explores its consequences for multi-member constituencies.

The legal-constitutional issue addressed in [5] and [6] gained prominence in the US beginning in the late 1960s (see Ch. 4). This highlighted the practical importance of our topic, and no doubt helped to attract theorists to do further research into it.

In a paper [20] published in 1971, James S Coleman subjects the S-S index to a conceptual critique. He shows that the rough, informal characterization of voting power — *the extent to which a given member is able to control the outcome of a division*, as we put it in § 1.1 — can be explicated in more than one way (see below, Com. 2.2.2). He argues that the assumptions underlying the S-S index, inherited from its game-theoretic origins, make it unsuited for application to most real-life situations in which decisions are made by division. He then proposes what he regards as two new measures of voting power — quantifying the (absolute) power of members to *prevent* action and to *initiate* it, respectively — based on an alternative explication of the informal notion of voting power.

In fact, these measures were not entirely new: they were modifications of the Penrose measure, differing from it (and from each other) by a mere scaling factor. The author was apparently unaware not only of [78], as virtually everyone else was, but also of [5]. It was soon pointed out, however, that the index of relative voting power yielded by the Coleman measures is none other than the Bz index (see, for example, [14]). Thus the technical proposals of [20] could be dismissed as having no great novelty value.[13] Unfortunately, the perspicacious conceptual analysis contained in that paper was largely overlooked as well.

The S-S index and the Bz index have, by and large, been accepted as valid measures of a priori voting power. Some authors have a preference for one or another of these two indices; many regard them as equally valid. Although other indices have been proposed — notably by John Deegan and Edward Packel [22] and by R J Johnston [55], both in 1978 — none has achieved anything like general recognition as a valid index.

While the S-S index is relatively easy to handle mathematically, the Bz index is, for technical reasons, quite refractory. These technical reasons disappear if the index is rescaled by multiplying its values by an appropriate factor. In their massive 1979 paper [27], Pradeep

[13] However, the Coleman measures jointly do yield more information than the Penrose–Banzhaf measure, as we shall see in Ex. 3.2.22 and Ex. 3.2.23.

Dubey and Lloyd S Shapley propose such a rescaling as 'being in many respects more natural' than the Bz index itself ([27, p. 102]). This turns out to be essentially the Penrose measure of absolute voting power (in fact, it is precisely that measure multiplied by 2). The authors — who are of course unaware of [78] — point out the probabilistic meaning of this measure and subject it to a searching mathematical examination.

Following [27], other authors have advocated this variant of the Bz index as a measure of absolute voting power (for example, see [100, p. 267]).

From the mid-1970s, a number of investigators have pointed out various 'paradoxes' associated with the measurement of voting power. These are re-examined in [29], where it is shown that all but one of the hitherto discovered paradoxes are only paradoxical in a rather mild and superficial sense: they may be surprising to an uninformed observer but on closer examination turn out to be explicable phenomena inherent in the very notion of voting power, and are therefore displayed by any minimally reasonable method of measuring it. The remaining 'paradox' is a pathology specific to the index proposed in [22], and seems to disqualify this index as a reasonable measure of voting power.

However, two new, closely connected, severe paradoxes are demonstrated in [29], the *bloc* and *donation* paradoxes, which affect all known indices except the S-S (see § 7.8). It is argued that these new paradoxes vindicate the S-S index, at least provisionally, while the remaining indices — the most important of which is the Bz — are, at best, suspect.

This conclusion is modified in [36]. Here yet another severe paradox, the *bicameral* paradox, is shown to afflict the S-S index and all other known indices except the Bz (see § 7.9). On the other hand, the conceptual analysis of [20] is amplified, leading to a distinction between two pre-formal notions of voting power: *I-power* and *P-power*. It is argued that the concept underlying the Bz index is that of I-power, for which the bloc and donation paradoxes may be tolerable; but that as yet there exists no reasonable index of a priori P-power.

2. Groundwork of the Theory

2.1 Simple Voting Games

We begin by defining the most general class of mathematical structures commonly used to model voting decision rules. This, then, is the basic definition of the theory.

2.1.1 Definition A *simple voting game* — briefly, SVG — is a collection $\mathcal{W}$ of subsets of a finite set N, satisfying the following three conditions:

(1) $N \in \mathcal{W}$;

(2) $\emptyset \notin \mathcal{W}$;

(3) *Monotonicity*: whenever $X \subseteq Y \subseteq N$ and $X \in \mathcal{W}$ then also $Y \in \mathcal{W}$.

$\mathcal{W}$ is said to be a *proper* SVG if, in addition, it satisfies the condition

(4) Whenever $X \in \mathcal{W}$ and $Y \in \mathcal{W}$ then $X \cap Y \neq \emptyset$.

Otherwise, $\mathcal{W}$ is said to be *improper*.

We shall refer to N, the largest set in $\mathcal{W}$, as the latter's *assembly*. The members of N are the *voters* of $\mathcal{W}$. A set of voters (that is, a subset of N) is called a *coalition* of $\mathcal{W}$. A coalition S is said to be a *winning* or *losing* coalition, according as $S \in \mathcal{W}$ or $S \notin \mathcal{W}$.

2.1.2 Remarks (i) Apart from some inessential modifications, this definition is the same as that given by Shapley in [95] for what he calls 'simple game'. He attributes the concept to von

Neumann and Morgenstern [108], but as a matter of fact his class of simple games is considerably wider than that admitted by [108]. To prevent confusion with the latter, we use the term 'simple *voting* game' for the broader concept.

In [95], an SVG $\mathcal{W}$ with assembly N is denoted by '$(N, \mathcal{W})$', and that notation is also followed by other authors. However, explicit mention of N is in fact redundant, because N is uniquely determined by $\mathcal{W}$, as the latter's largest member. By the way, N is often referred to as the *grand coalition* of $\mathcal{W}$.

(ii) Conditions (1) and (2) of Def. 2.1.1 jointly imply that the assembly N is nonempty.

(iii) An SVG is designed to be used as a model for a decision-making board[1] in the following way. The set of all board members is represented by the assembly N; the set of 'yes' voters in the division on a given bill is represented by a coalition (subset of N); and the board's decision rule is represented by $\mathcal{W}$ itself: the bill is passed iff the set of board members voting for it is [represented by] a winning coalition. In practice, we often speak loosely, as though a board and the SVG modelling it are the same thing, but in principle the two are quite distinct and must not be conflated: the former is a social structure involving real persons, whereas the latter is an abstract mathematical structure.

Strictly speaking, an SVG models a board *with a given decision rule*; so if a board operates several decision rules, applied to different types of bill, then it requires several SVG models, one for each rule, all having a common assembly.

(iv) At first glance it may seem that an improper SVG is quite useless: if there are two disjoint winning coalitions, S and T, couldn't this lead to two incompatible bills being adopted, one supported by S and the other by T? However, such inconsistencies cannot arise if the improper decision rule governs a restricted class of bills, among which no contradiction is possible. For example, a board may well have a special rule for putting proposed items on its agenda: an

[1] Cf. § 1.1.

item is included iff at least two members (say a proposer and a seconder) are in favour. This rule cannot lead to contradiction, irrespective of the size of the board. A rule of this sort applies to a grant of *certiorari* by the US Supreme Court,[2] which requires approval by four out of the nine justices of this court.

Besides, excluding improper SVGs would be technically inconvenient because they are sometimes required as building blocks for constructing proper SVGs. Nevertheless, since decision rules modelled by improper SVGs are somewhat unusual, we shall confine our examples to proper SVGs as far as possible.

Def. 2.1.1 has the great virtue of conceptual simplicity. But for certain purposes it is convenient to use other, somewhat more complex, representations of SVGs, equivalent to that of Def. 2.1.1. Here we shall define two such representations.

2.1.3 Definition Let $\mathcal{W}$ be an SVG with assembly N. The *characteristic function* — briefly *CF* — of $\mathcal{W}$ is the map $\mathbf{w}$ from the set of all coalitions of $\mathcal{W}$ (the power set of N) to $\{0, 1\}$ such that, for any coalition X,

$$\mathbf{w}X = \begin{cases} 1 & \text{if } X \in \mathcal{W}, \\ 0 & \text{otherwise.} \end{cases}$$

$\mathbf{w}X$ is called the *worth* of X (according to $\mathbf{w}$).

2.1.4 Remarks (i) An SVG $\mathcal{W}$ is faithfully represented by its CF $\mathbf{w}$, in the sense that the former is uniquely determined by the latter:

$$\mathcal{W} = \mathbf{w}^{-1}[\{1\}] = \{X : \mathbf{w}X = 1\}.$$

(ii) An arbitrary map $\mathbf{w}$ from the power set of a finite set N to $\{0, 1\}$ is the CF of some (unique) SVG, iff it satisfies the obvious

[2] This is a writ ordering a case to be transferred from a lower court for consideration by the (full) Supreme Court.

translation of Def. 2.1.1. In particular, the monotonicity condition 2.1.1(3) translates into

$$X \subseteq Y \subseteq N \Rightarrow \mathbf{w}X \leq \mathbf{w}Y.$$

(iii) Historically, the translation went in the reverse direction. The notion of CF (of a general cooperative game) goes back to von Neumann and Morgenstern [108]; Def. 2.1.1 was obtained by translating the conditions satisfied by CFs of the appropriate special kind.

An SVG (or its CF) is a 'one-sided' model of a decision rule: a board's division on a bill is represented by the coalition S of the members voting 'yes'. Of course, this provides all the information needed, since all the remaining members are presumed to be 'no' voters. But for some purposes it is convenient to have a more 'even-handed' representation. We define such a representation here, for the record, but it will not be used for a while; so the reader may skip this and go on directly to Def. 2.1.7.

2.1.5 Definition A *bipartition* of a set N is a map B from N to $\{-1, 1\}$. We denote by 'B^-' and 'B^+' the inverse images of $\{-1\}$ and $\{1\}$ respectively under B:

$$B^- = \{x \in N : Bx = -1\}, \quad B^+ = \{x \in N : Bx = 1\}.$$

If $\mathcal{W}$ is an SVG with assembly N, the *bipartition rule* — briefly, *BPR* — of $\mathcal{W}$ is the map W from ${}^N\{-1, 1\}$ — the set of all bipartitions of N — to $\{-1, 1\}$, such that, for any $B \in {}^N\{-1, 1\}$,

$$\mathsf{W}B = \begin{cases} 1 & \text{if } B^+ \in \mathcal{W}, \\ -1 & \text{otherwise.} \end{cases}$$

We call $\mathsf{W}B$ the *outcome* of the bipartition B (according to W).

2.1.6 Remarks (i) A bipartition B represents a division of a board: the sets B^- and B^+ represent the sets of 'no' and 'yes' voters, respectively; and the outcome $\mathsf{W}B$ is -1 or 1 according as the bill in question is defeated or passed.

(ii) An SVG $\mathcal{W}$ is uniquely determined by its BPR W:

$$\mathcal{W} = \{B^+ : B \in {}^N\{-1, 1\} \text{ and } \mathsf{W}B = 1\}.$$

(iii) Clearly, an arbitrary map W from the set ${}^N\{-1, 1\}$ of all bipartitions of a finite set N to $\{-1, 1\}$ is the BPR of some (unique) SVG, iff it satisfies conditions that are translations of conditions (1)–(3) of Def. 2.1.1. The first two conditions have obvious direct translations:

(1′) $B^+ = N \Rightarrow \mathsf{W}B = 1$;

(2′) $B^- = N \Rightarrow \mathsf{W}B = -1$.

To translate the third condition, we first define a partial ordering among bipartitions. If B_1 and B_2 are two bipartitions of N, we put:

$$B_1 \leq B_2 \Leftrightarrow_{\text{def}} B_1x \leq B_2x \text{ for all } x \in N.$$

Note that

$$B_1 \leq B_2 \Leftrightarrow B_1^+ \subseteq B_2^+.$$

Now we can translate condition (3) of Def. 2.1.1:

(3′) *Monotonicity*: $B_1 \leq B_2 \Rightarrow \mathsf{W}B_1 \leq \mathsf{W}B_2$.

Being a reasonable class of mathematical structures, the class of SVGs admits of a natural notion of isomorphism. Here, for the record, is its obvious definition.

2.1.7 Definition Let $\mathcal{W}$ and $\mathcal{W}'$ be SVGs, with respective assemblies N and N'. An *isomorphism* from $\mathcal{W}$ to $\mathcal{W}'$ is a bijection f from N to N' (that is, a 1-1 map from N onto N'), such that for any $X \subseteq N$,

$$X \in \mathcal{W} \Leftrightarrow f[X] \in \mathcal{W}'.$$

(Here $f[X] = \{fx : x \in X\}$.) If such f exists, we say that $\mathcal{W}$ is *isomorphic* to $\mathcal{W}'$ — briefly: $\mathcal{W} \cong \mathcal{W}'$.

An isomorphism from an SVG $\mathcal{W}$ to itself is called an *automorphism* or *symmetry* of $\mathcal{W}$. If a and b are voters of $\mathcal{W}$ such that $fa = b$ for some automorphism f of $\mathcal{W}$, then a is said to be *symmetric* to b. An SVG is said to be *pairwise symmetric*, if every one of its voters is symmetric to every other.

2.1.8 Remark Isomorphism is evidently an equivalence relation among SVGs. According to a universal principle of mathematical methodology, the theory of SVGs must treat isomorphic SVGs as interchangeable. More precisely: any concept or construct involving an SVG and its voters can only be regarded as theoretically legitimate if it is invariant under all isomorphisms (and in particular under all automorphisms) of SVGs. For example, for any measure ξ of the voting power of a voter in an SVG — whatever this may mean — to be a legitimate concept in the theory of SVGs, it must be invariant under isomorphisms: if f is an isomorphism from $\mathcal{W}$ to $\mathcal{W}'$ and if a is a voter of $\mathcal{W}$ and $fa = a'$, then ξ must ascribe the same power to a' in $\mathcal{W}'$ as it does to a in $\mathcal{W}$. In particular, symmetric voters of $\mathcal{W}$ must be ascribed equal power.

2.2 Comments on the Basic Definition

This section is in the nature of an extended health warning. The simplicity of Def. 2.1.1 conceals a number of pitfalls which must be guarded against. We point out four such pitfalls: the first two have to do with the game-theoretic origin of this basic definition; the remaining two concern the applicability of the SVG model to real decision-making boards.

2.2.1 Comment The first pitfall is a simple terminological one. In normal parlance as well as in the discourse of qualitative political science, 'coalition' denotes an alliance of persons, political parties, or States, which may be temporary but is generally not ephemeral. A typical example is a coalition of parties that forms a government and will then vote in unison, not just in one division of the legislature but on a whole series of issues.[3] Our technical term closest in meaning to this is not 'coalition' but 'bloc' (see Def. 2.3.23). Following Def. 2.1.1, we use the term 'coalition' in its quite different, game-theoretic sense, as denoting a set of players, in our case voters, who may act together in one play of the game, which in the case of an SVG corresponds to a single division of a board. In fact, they need not even do that: *any* set of voters is referred to as a 'coalition'. This sometimes leads to awkwardness of expression, such as when one says that a coalition is 'formed' when one wishes to convey the idea that its members have agreed to act together on a particular occasion.

The reader must be warned that some writers on voting power have not been too careful about the distinction between the two technical terms, 'coalition' and 'bloc'.

2.2.2 Comment The term 'simple voting *game*' is not entirely simple or innocent: the word 'game' carries with it some conceptual game-theoretic baggage that must be scrutinized. In game theory, in order to specify a game completely, one must set down not only its procedural rules but also its payoffs. In particular, a cooperative game is specified by setting down, for each coalition S, a number 'called the *worth* of coalition S, [which] represents the total amount of transferable utility that the members of S could earn without any help from the players outside of S'.[4]

Thus, the terminology of Def. 2.1.1 suggests that when an SVG is specified, it provides us not merely with a decision rule (telling us which coalitions can carry a bill) but also with a system of payoffs. According to this view of voting, the characteristic function $\mathbf{w}$ of an SVG is not merely a formal artefact that distinguishes winning from losing coalitions by attaching to them the respective values 1 and 0 as arbitrary labels; rather, $\mathbf{w}S$ ought to be interpreted as representing the total payoff that the members of the coalition S earn when S is the set of 'yes' voters in a division.

What is this total payoff? Shapley's answer is quite explicit:

[3] For an example of this usage see Laver and Schofield [63].

[4] Myerson [71, p. 422].

'the acquisition of power is the payoff'.[5] The idea that the game-theoretic terminology of Def. 2.1.1 is meant to convey is that upon winning a division the winning coalition captures a fixed purse — the prize of power — which it then proceeds to divide among its members. The formation of the winning coalition as well as the distribution of the spoils among its members are consequent upon a process of bargaining. The motivation of voting behaviour that this view assumes has been called 'office seeking' by political scientists.

The alternative motivation is 'policy seeking'.[6] In his 1971 critique [20, p. 272] of the S-S index, Coleman points out that the latter motivation is the more usual,

> ... for the usual problem is not one in which there is a division of the spoils among the winners, but rather the problem of controlling the action of the collectivity. The action is ordinarily one that carries its own consequences or distribution of utilities, and these cannot be varied at will, i.e. cannot be split up among those who constitute the winning coalition. Instead, the typical question is ... the passage of a bill, a resolution, or a measure committing the collectivity to an action.

Indeed, this seems a realistic account of voting in, say, the UNSC. Incidentally, the UNSC is one of the two examples given by Shapley in [95, p. 59] of a 'body in which the acquisition of power is the payoff'; but it is not clear how passing a resolution in this body amounts to acquisition of power by those voting 'yes'.

As Coleman admits in a note to the passage just quoted ([20, n. 3, p. 299]), there are cases in which the office-seeking assumption is reasonable. The example he cites is quintessentially American: a party convention in which a candidate (presumably for the US presidency) is nominated,

> ... for there are spols [*sic*] to be distributed among those delegations that support the winner, and particularly those

[5] Abstract of his [95, p. 59].

[6] For a discussion of these alternative motivations in a political context, see Laver and Schofield [63, especially Ch. 3].

> delegations that cast the deciding ballots in favor of the winner. But this is an unusual case, in which there is a winning nominee, who does have spoils to distribute.

Another, typically European, example is perhaps a parliamentary vote of confidence in a multi-party government.[7]

One way of thinking about the difference between the office-seeking and policy-seeking motivations of voting is that under the former, *ceteris paribus*, a ‘yes’ voter should want to win with the help of few rather than many partners, because the spoils of office must be shared with them. A policy-seeking voter, on the contrary, only wants the bill to pass (or fail, as the case may be) and, subject to this single aim, does not mind if many others also vote likewise; on the contrary — the more, the merrier. Also, office-seeking motivation cannot operate where voting is secret, because the winners must be clearly identified. However, there are undoubtedly many real-life situations in which voting behaviour is driven by a mixture of the two motives.

In any case, we believe that it would be wrong to approach the problem of measuring voting power with the preconception that voting is always, or usually, motivated by office seeking. In order not to fall out of step with the mainstream literature on the subject, we shall stick to the game-theoretic terminology of Def. 2.1.1, but without necessarily taking on board its conceptual baggage. In particular, we shall not presuppose that an SVG is a fully specified game, in the technical sense explained above, with the payoffs given by its CF.

The difference between the two kinds of voting behaviour will form the basis for our distinction between two pre-formal notions of voting power, discussed in §§ 3.1 and 6.1.

2.2.3 Comment By adopting the sparse conceptual framework of SVGs as the setting for the theory of voting power, we are ruling out by default all considerations based on data not provided by that framework. In real political situations, the formation of a

[7]For a critical discussion and empirical evaluation of the office-seeking assumption in the European context, see Laver and Schofield [63].

coalition may depend not only on whether the coalition is a winning one, but also on its ideological cohesiveness or the mutual affinity between its members. The bare structure of an SVG — which merely classifies coalitions into winners and losers — is inadequate for formalizing situations of this kind. This has been repeatedly pointed out in the literature, beginning from Shapley and Shubik [97, p. 791]. Here is how Roth puts it:

> Analyzing voting rules that are modeled as [SVGs] abstracts from the particular personalities and political interests present in particular voting environments, but this abstraction is what makes the analysis focus on the rules themselves rather than on the other aspects of the political environment. This kind of analysis seems to be just what is needed to analyze the voting rules in a new constitution, for example, long before the specific issues to be voted on arise or the specific factions and personalities that will be involved can be identified.[8]

Another way of putting it is that an SVG is an abstract shell, uninhabited by real agents, with real likes and dislikes, mutual attractions and repulsions. And the 'bills' on which divisions are supposed to take place are — as far as the theory of SVGs is concerned — generic issues without any predetermined relation to the interests of this or that voter.

In particular, in view of Rem. 2.1.8, any measure of voting power defined in this setting can at best measure *a priori* voting power, which ignores factors such as an individual voter's prior bias regarding the issues voted upon, or affinities and disaffinities between voters.

In order to estimate the actual (a posteriori) voting power distribution in a board, one would need to make use of empirical data on actual divisions, and perhaps also to enrich the SVG structure; for example, by assigning each coalition a coefficient of cohesiveness or a priori probability of being formed, or by assigning each pair of voters a coefficient of mutual attraction/repulsion.[9] How-

[8] [90, p. 9].

[9] For such an elaboration of the SVG structure see, for example, [73], [102]

ever, even if we were to possess reliable knowledge of actual voting power, there would still be an important descriptive-analytic role for a measure of a priori voting power. The latter would be a benchmark, showing how much of the actual voting power of a given voter is due to the decision rule itself. The disparity between the two could then 'serve as a measure of, for example, political solidarity, or regional or sociological factionalism, in an assembly'.[10]

The prescriptive application of the a priori theory — for example, in designing a new constitution for a board — requires a great deal of caution. An SVG model can be used to produce a 'level playing field'. However, in a case where there are known long-term systematic real factors favouring some voters and disfavouring others, it may be appropriate to compensate for those factors by a system of 'handicaps'. All the same, the distinction between the impartial basic model and the superimposed handicaps must not be blurred.

2.2.4 Comment All mathematical models involve some idealization and simplification of reality. The question in each case is whether this has gone too far, resulting in unrealistic over-simplification. In our view, the most striking simplification that the SVG model involves is its *binary* character: in each division a voter is assumed to have just two possible courses of action — voting 'yes' or voting 'no'. In the BPR representation of an SVG, this is made more explicit: all those not voting 'yes' are assumed to vote 'no' (see Rem. 2.1.6(i)).

This is a realistic model of some factual decision rules. But many — probably most — real-life decision rules treat abstention as a distinct *tertium quid*, whose effect is not always the same as 'yes' or always the same as 'no'.

In the mainstream literature on voting power the problem of abstention goes virtually unrecognized.[11] Banzhaf [5, p. 332] dis-

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and [19].

[10] Shapley and Shubik [97, p. 791].

[11] Again, the exception that proves the rule is Morriss [70, Ch. 23], which is outside the mainstream.

misses it with a brief (and in our view inadequate) remark encaved in a footnote. This is more than most authors do: they simply ignore the whole issue.

Now, it is true that from the office-seeking perspective on voting, abstention may be regarded as irrational: if by voting 'yes' you can get a share of the spoils as a member of a winning coalition that acquires power, then vote 'yes'; otherwise vote 'no'. You'll never get a prize for sitting on the fence. But authors do not even bother to supply this rationalization. More surprising still: when dealing with real-life decision rules in bodies such as the US legislature and the UNSC, most authors mis-report the rules as though abstention were not a distinct option.

We will say no more on this subject here, as we shall deal with it in detail in Ch. 8.

2.3 Supplementary Definitions

In this section we define various concepts belonging to the general theory of SVGs, which will be used in subsequent sections. The material and the notation introduced here are largely based on [95]. The definitions are collected here together for ease of reference. A reader who gets bored after reading part of the section is advised to skip the rest and return to it later, as the need arises.[12]

2.3.1 Conventions (i) From now on, unless stated otherwise, we shall assume that $\mathcal{W}$ is an SVG with assembly N, characteristic function $\mathbf{w}$ and BPR W; and that the number of voters, $|N|$, is n.

We shall use a similar convention in the presence of subscripts and superscripts. So, for example, if an SVG is denoted by '$\mathcal{W}_1$' then, unless stated otherwise, the reader must take it for granted that 'N_1', '$\mathbf{w}_1$' and 'W_1' denote its assembly, CF and BPR, respectively; and that $|N_1| = n_1$.

[12] A great wealth of material on the general theory of SVGs, particularly on WVGs (Def. 2.3.14) and their various generalizations, can be found in a monograph [105] by Taylor and Zwicker.

(ii) When defining a notion relating to an arbitrary SVG $\mathcal{W}$, we introduce qualifying phrases such as 'in $\mathcal{W}$' or 'of $\mathcal{W}$'. But subsequently we usually drop these qualifying phrases where it is clear from the context which SVG is being referred to.

2.3.2 Definition A winning coalition (that is, a member of $\mathcal{W}$) that does not include any other winning coalition is called a *minimal* winning coalition — briefly, *MWC* — of $\mathcal{W}$.

A *blocking* coalition of $\mathcal{W}$ is a coalition S whose complement is a losing coalition; that is, $S \subseteq N$ and $N - S \notin \mathcal{W}$.

We put

$$\mathcal{W}^* =_{\text{def}} \{X \subseteq N : N - X \notin \mathcal{W}\}.$$

$\mathcal{W}^*$ — the family of all blocking coalitions of $\mathcal{W}$ — is called the *dual* of $\mathcal{W}$.

2.3.3 Remarks (i) $\mathcal{W}$ is uniquely determined by its assembly and its family $\mathcal{W}^{\min}$ of MWCs:

$$\mathcal{W} = \{X \subseteq N : M \subseteq X \text{ for some } M \in \mathcal{W}^{\min}\}.$$

This provides a useful — and sometimes the most economical — way of specifying an SVG.

(ii) In [95], the term 'blocking' is defined in a narrower sense, whereby S is said to be a blocking coalition if not only $N - S$ but also S itself is a losing coalition. That sense agrees with common political parlance, in which the term is used to refer to a coalition that is able to stop a bill being passed but cannot force one through. However, subsequent usage in the voting-power literature has shifted to the broader sense of 'blocking', which we adopt here.[13] This has the technical advantage of making the concepts *winning* and *blocking* mutually dual.

(iii) It is easy to see that $\mathcal{W}^*$ is an SVG with the same assembly as $\mathcal{W}$ and that $(\mathcal{W}^*)^* = \mathcal{W}$.

[13] See, for example, [27, p. 102] and [100, p. 262].

The CFs of $\mathcal{W}^*$ and $\mathcal{W}$ (see Def. 2.1.3) are connected by the identity

$$\mathbf{w}^*X = 1 - \mathbf{w}(N - X) \text{ for all } X \subseteq N;$$

and the respective BPRs (see Def. 2.1.5) are connected by the identity

$$\mathsf{W}^*B = -\mathsf{W}(-B) \text{ for all } B \in {}^N\{-1, 1\}.$$

Here $-B$ is the bipartition obtained from B in the obvious way: $(-B)x = -(Bx)$ for all $x \in N$.

The operation of duality is related to the Boolean-logical operation of negation: applying $\mathcal{W}^*$ as a decision rule to a bill is the opposite of applying $\mathcal{W}$ to the negation of the bill, because a bill is passed under $\mathcal{W}^*$ iff its negation fails under $\mathcal{W}$.

2.3.4 Definition Let a be a voter of $\mathcal{W}$. If S is a coalition such that $S \notin \mathcal{W}$ but $S \cup \{a\} \in \mathcal{W}$, then a is said to be $\mathcal{W}$-*critical outside* S.

If, on the contrary, $S \in \mathcal{W}$ but $S - \{a\} \notin \mathcal{W}$, then a is said to be $\mathcal{W}$-*critical in* S.

If a is not critical in any $S \in \mathcal{W}$, then a is called a *dummy* in $\mathcal{W}$.

If $\{a\}$ is a blocking coalition, then a is called a *blocker* (or a *vetoer*) in $\mathcal{W}$.

If $\{a\}$ is the sole MWC of $\mathcal{W}$, then a is called a *dictator* in $\mathcal{W}$.

2.3.5 Remarks (i) A dummy is a voter who is never able to affect the outcome of a division, and therefore does not play an essential role in the SVG. Now, it is easy to see that a dummy can be characterized as a voter who does not belong to any MWC. For this reason, if two SVGs have the same family of MWCs, they ought to be treated in theory as not differing essentially from each other. (Cf. Rem. 2.3.3(i).)

(ii) A blocker can be characterized as a voter who belongs to every MWC, and hence to every winning coalition. A dictator, if there is one, can be characterized as the sole voter who is not a dummy.

2.3.6 Definition Let a be a voter of $\mathcal{W}$ and let B be a bipartition of N. If $Ba = \mathsf{W}B$, we say that a *agrees* with the outcome of B in $\mathcal{W}$. Moreover, we say that a agrees with the outcome of B *negatively* or *positively*, according as the common value of Ba and $\mathsf{W}B$ is -1 or 1.

If a is $\mathcal{W}$-critical outside B^+ or in B^+, we say that a is $\mathcal{W}$-*critical for* B. Moreover, we say that a is *negatively* or *positively* critical for B, according as a is critical outside B^+ or in B^+.

2.3.7 Remark Intuitively, agreement of a with the outcome of B means that the decision goes a's way: in the case of negative agreement a votes 'no' and the bill fails, and in the case of positive agreement a votes 'yes' and the bill is passed. And a's being (negatively or positively) critical for B means that a not only agrees with the outcome of B (negatively or positively, respectively); but also that if a's vote were reversed, the outcome would likewise be reversed.

2.3.8 Definition For any bipartition $B \in {}^N\{-1, 1\}$ we put

$$\mathsf{S}B =_{\text{def}} \sum_{x \in N} Bx, \quad \mathsf{M}B =_{\text{def}} |\mathsf{S}B|.$$

We call $\mathsf{S}B$ and $\mathsf{M}B$ respectively the *sum* and *margin* of B.

2.3.9 Remarks (i) $\mathsf{S}B$ is the number of 'yes' voters minus the number of 'no' voters in B.

The 'yes' voters or 'no' voters are in the majority according as $\mathsf{S}B$ is positive or negative; and the two camps are evenly matched if $\mathsf{S}B = 0$.

The margin $\mathsf{M}B$ — the absolute value of $\mathsf{S}B$ — is the excess or 'edge' of the majority over the minority.

(ii) In common political parlance, the term 'majority' is used ambiguously to denote the larger of the two camps, or its size, or the margin (as in: 'the bill was defeated by a majority of 17').

2.3.10 Definition Let n be a positive integer. We put

$$I_n =_{\text{def}} \{1, 2, \ldots, n\}.$$

For any positive integer k such that $k \leq n$, we define $\mathcal{M}_{n,k}$ as the SVG whose winning coalitions are just those subsets of I_n that have at least k members:

$$\mathcal{M}_{n,k} =_{\text{def}} \{X \subseteq I_n : |X| \geq k\}.$$

Further, we put

$$\mathcal{M}_n =_{\text{def}} \mathcal{M}_{n,[n/2]+1}, \quad \mathcal{B}_n =_{\text{def}} \mathcal{M}_{n,n}.$$

(Here $[r]$ is the *integral part* of r: the greatest integer that does not exceed r.)

An SVG isomorphic to $\mathcal{M}_n$ for some n is said to be a *majority* SVG.

An SVG isomorphic to $\mathcal{B}_n$ for some n is said to be a *unanimity* SVG.

2.3.11 Remarks (i) We use I_n as the *canonical* assembly with n members. It is sometimes technically convenient to assume that a particular SVG has a canonical assembly; no generality is lost thereby, since every SVG is isomorphic to an SVG of this special kind.

(ii) $\mathcal{M}_{n,k}$ has the following property, which may be called [*full*] *symmetry*: any two voters are interchangeable. More precisely, if a and b are any two voters, then the map f that interchanges them and leaves all other voters fixed[14] is an automorphism of the SVG.

Conversely, it is easy to see that any symmetric SVG is isomorphic to $\mathcal{M}_{n,k}$ for some n and k.

A symmetric SVG is clearly pairwise symmetric (see Def. 2.1.7). But, as we shall soon see (Ex. 2.3.18 and Ex. 2.3.19), the converse is false.

(iii) It is easy to verify that $(\mathcal{M}_{n,k})^* = \mathcal{M}_{n,n-k+1}$. In particular, $\mathcal{B}_n{}^* = \mathcal{M}_{n,1}$.

[14]That is: $fa = b$, $fb = a$ and $fx = x$ for any other voter x.

2.3.12 Definition Let m be a positive integer and let $\mathcal{V}$ be an SVG with assembly I_m. For each $i \in I_m$, let $\mathcal{W}_i$ be an arbitrary SVG. We now define an SVG

$$\mathcal{V}[\mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_m],$$

called *the composite of* $\mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_m$ (in this order!) *under* $\mathcal{V}$. We stipulate that the assembly N of $\mathcal{V}[\mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_m]$ is the union of the assemblies N_i of the $\mathcal{W}_i$:

$$N = \bigcup_{i=1}^{m} N_i\,;$$

and that, for any $X \subseteq N$,

$$X \in \mathcal{V}[\mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_m] \;\Leftrightarrow\; \{i \in I_m : X \cap N_i \in \mathcal{W}_i\} \in \mathcal{V}.$$

We refer to $\mathcal{V}$ as the *top* and to $\mathcal{W}_i$ as the *i-th component* of the composite SVG $\mathcal{V}[\mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_m]$.

We define the following two special sorts of composite. First, we put

$$\mathcal{W}_1 \wedge \mathcal{W}_2 \wedge \cdots \wedge \mathcal{W}_m =_{\text{def}} \mathcal{B}_m[\mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_m].$$

This SVG is called the *meet* of the $\mathcal{W}_i$. If the assemblies N_i are pairwise disjoint then it is also denoted by

$$`\mathcal{W}_1 \times \mathcal{W}_2 \times \cdots \times \mathcal{W}_m'$$

and called the *product* of the $\mathcal{W}_i$.

Second, we put

$$\mathcal{W}_1 \vee \mathcal{W}_2 \vee \cdots \vee \mathcal{W}_m =_{\text{def}} {\mathcal{B}_m}^*[\mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_m].$$

This SVG is called the *join* of the $\mathcal{W}_i$. If the assemblies N_i are pairwise disjoint then it is also denoted by

$$`\mathcal{W}_1 + \mathcal{W}_2 + \cdots + \mathcal{W}_m'$$

and called the *sum* of the $\mathcal{W}_i$.

2.3.13 Remarks (i) The technical advantage of requiring the top of a composite to have a canonical assembly is that in the notation '$\mathcal{V}[\mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_m]$' the order in which the components are listed corresponds automatically to the natural order of the integers in I_m. (See also Rem. 2.3.11(i).)

(ii) The composite SVG $\mathcal{V}[\mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_m]$ can be used to model a 'federal' or two-tier voting system. Such a system is made up of a 'bottom tier' of m boards — each with its own decision rule, represented by the $\mathcal{W}_i$ — and a rule, represented by $\mathcal{V}$, for collating the m bottom-tier decisions into a final decision. To decide a bill, first each of the m boards divides on it and arrives at its own decision, according to its own rule. Then these m decisions are fed as m respective votes to the top, $\mathcal{V}$. For example, we can imagine that the m voters of $\mathcal{V}$ are delegates, one from each of the bottom-tier boards, instructed to vote according to the decisions made by their respective boards. In this way the final decision is reached.

Clearly, the bill in question will pass finally iff the set of bottom-tier 'yes' voters is sufficient to pass the bill in enough bottom-tier boards, so that their delegates make up a winning coalition at the top. This explains the definition of $\mathcal{V}[\mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_m]$ above.

(iii) $\mathcal{W}_1 \wedge \mathcal{W}_2 \wedge \cdots \wedge \mathcal{W}_m$ can be defined directly as the collection of all unions $\bigcup_{i=1}^m X_i$ such that $X_i \in \mathcal{W}_i$ for all $i \in I_m$.

Similarly, $\mathcal{W}_1 \vee \mathcal{W}_2 \vee \cdots \vee \mathcal{W}_m$ can be defined directly as the collection of all unions $\bigcup_{i=1}^m X_i$ such that $X_i \subseteq N_i$ for all $i \in I_m$ and $X_i \in \mathcal{W}_i$ for at least one i.

It is not difficult to see that in the special case where all the N_i coincide,

$$\mathcal{W}_1 \wedge \mathcal{W}_2 \wedge \cdots \wedge \mathcal{W}_m = \mathcal{W}_1 \cap \mathcal{W}_2 \cap \cdots \cap \mathcal{W}_m,$$
$$\mathcal{W}_1 \vee \mathcal{W}_2 \vee \cdots \vee \mathcal{W}_m = \mathcal{W}_1 \cup \mathcal{W}_2 \cup \cdots \cup \mathcal{W}_m.$$

(iv) A meet $\mathcal{W}_1 \wedge \mathcal{W}_2$ can be used to model a *bicameral* system, say a legislature in which a new law needs the approval of each of two houses. If a political party is regarded as a single voter, then the two assemblies may well have voters in common. But if voters are taken to be individual persons, and no person can be a member of both houses, then the meet becomes a product, $\mathcal{W}_1 \times \mathcal{W}_2$.

(v) In the voting-power literature, the terms 'product' and 'sum' are sometimes used loosely, instead of 'meet' and 'join' respectively, even when the assemblies of the components are not pairwise disjoint.

(vi) The BPR W of the composite $\mathcal{W} = \mathcal{V}[\mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_m]$ has the following elegant natural characterization.

First observe that for each $i \in I_m$ there is a natural *projection* p_i that maps each bipartition B of N to a bipartition $\mathrm{p}_i B$ of N_i: $\mathrm{p}_i B$ is simply the restriction of B to N_i.

Next, observe that there is a natural *quotient* map t that maps each bipartition B of N to a bipartition $\mathrm{t}B$ of I_m: namely, $(\mathrm{t}B)i = \mathsf{W}_i(\mathrm{p}_i B)$ for each $i \in I_m$.

Now, W is characterized by the identity $\mathsf{W}B = \mathsf{V}(\mathrm{t}B)$ for all bipartitions B of N.

2.3.14 Definition A *weighting system* on a finite set N is an ordered pair $\langle q, w \rangle$, where q is a real number and w is a mapping that assigns to every $x \in N$ a non-negative real value w_x, such that

$$0 < q \leq \sum_{x \in N} w_x.$$

The set N, the number q and the mapping w are called, respectively, the *domain*, the *quota* and *weight function* of the weighting system $\langle q, w \rangle$.

Given such a weighting system, we put, for any $S \subseteq N$,

$$wS =_{\text{def}} \sum_{x \in S} w_x$$

and we call wS the *weight* of S. The weighting system is said to be *normalized* if $wN = 1$.

Further, the SVG

$$\{X \subseteq N : wX \geq q\}$$

is called the *weighted voting game* — briefly, the *WVG* — of $\langle q, w\rangle$.

If the domain of the weighting system $\langle q, w\rangle$ is I_n, then its WVG is denoted by

$$`[q; w_1, w_2, \dots, w_n]'$$

and called a *canonical* WVG.

2.3.15 Remarks (i) The condition $0 < q \leq wN$ imposed on a weighting system implies that its domain N must be nonempty.

(ii) We could have required the quota and weights of a weighting system to be non-negative integers; no generality would have been lost by this restriction, since it would have yielded exactly the same class of WVGs.

(iii) The correspondence between weighting systems and WVGs is evidently many-to-one. In fact, it can be proved that any given WVG is obtainable from infinitely many *normalized* weighting systems.

Here is an outline of a proof. Start from a given normalized canonical WVG, $[q; w_1, w_2, \dots, w_n]$. Without loss of generality, you may assume that there is no coalition S such that $wS = q$ (otherwise, you can reduce q slightly without altering the WVG). Now, for reasons of continuity, any sufficiently small variation of the w_i, subject to the normalization condition $w(I_n) = 1$, will not alter the WVG.

(iv) The advantage of confining the notation '$[q; w_1, w_2, \dots, w_n]$' to canonical WVGs is that the order in which the weights are listed corresponds automatically to the natural order of the integers in I_n (cf. Rem. 2.3.13(i).)

However, since every WVG is isomorphic to a canonical WVG, we can use the square-bracket notation also to represent a non-canonical WVG $\mathcal{W}$. To do this, we indicate (explicitly or implicitly) a particular total ordering of the assembly of $\mathcal{W}$. For example, suppose the assembly of $\mathcal{W}$ is $\{a, b, c\}$. By saying

$$`\mathcal{W} \cong [4; 3, 2, 1] \quad \textit{in alphabetic order}',$$

we mean that $\mathcal{W}$ is the WVG obtained from the weighting system having quota $q = 4$ and weights $w_a = 3$, $w_b = 2$, $w_c = 1$.

2.3.16 Example The CMEC voting rule which we discussed in Ex. 1.1.3 can be modelled by the WVG that has six voters: France, Germany, Italy, Belgium, The Netherlands, Luxembourg; and, with its voters so ordered, is isomorphic to

$$[12; 4, 4, 4, 2, 2, 1].$$

2.3.17 Example Every symmetric SVG (see Rem. 2.3.11(ii)) is obviously a WVG:

$$\mathcal{M}_{n,k} = [k; \underbrace{1, 1, \ldots, 1}_{n \text{ times}}].$$

But the converse is false: for example, $[1; 1, 0]$ and, less trivially, $[4; 3, 2, 1]$ are clearly not symmetric.

Note however that, in view of Rem 2.3.15(iii), a symmetric WVG can also be produced by a weighting system that assigns unequal weights to the voters; for example, $[3; 2, 1] = [2; 1, 1] = \mathcal{B}_2$, and $[5; 4, 3, 2] = [2; 1, 1, 1] = \mathcal{M}_3$.

2.3.18 Example Not every SVG is a WVG. For instance, consider the SVG

$$\mathcal{W} = \mathcal{V} \times \mathcal{U},$$

where $\mathcal{V}$ is the majority SVG with assembly $\{a, b, c\}$ and $\mathcal{U}$ is the majority SVG with assembly $\{d, e, f\}$ (all six voters are assumed distinct). Suppose $\mathcal{W}$ were produced by a weighting system $\langle q, w \rangle$. Since $\{a, b, d, e\} \in \mathcal{W}$ but $\{a, b, c, e\} \notin \mathcal{W}$, we would have $w_c < w_d$. On the other hand, since $\{b, c, e, f\} \in \mathcal{W}$ but $\{b, d, e, f\} \notin \mathcal{W}$, it would follow that $w_d < w_c$.

Note, however, that $\mathcal{W}$ is evidently pairwise symmetric, although it is not a WVG and hence, *a fortiori*, not symmetric.

2.3.19 Example Here is another example of a pairwise symmetric SVG $\mathcal{W}$ that is not a WVG (and hence not symmetric): it has an assembly of five voters, $\{a, b, c, d, e\}$ and five MWCs: $\{a, b, c\}$, $\{b, c, d\}$, $\{c, d, e\}$, $\{d, e, a\}$ and $\{e, a, b\}$. An argument similar to that in Ex. 2.3.18 shows that $\mathcal{W}$ cannot be a WVG: $\{a, b, c\} \in \mathcal{W}$ but $\{e, b, c\} \notin \mathcal{W}$, hence we would have $w_e < w_a$; on the other hand, $\{d, c, e\} \in \mathcal{W}$ but $\{d, c, a\} \notin \mathcal{W}$, hence it would follow that $w_a < w_e$.

In Ex. 2.3.18, $\mathcal{W}$, while not itself a WVG, was a product of two WVGs. But in the present example $\mathcal{W}$ is a *prime* SVG: it is not a product of two SVGs, weighted or otherwise. Indeed, suppose $\mathcal{W} = \mathcal{W}_1 \times \mathcal{W}_2$. Then each MWC of $\mathcal{W}$ would be a union of an MWC of $\mathcal{W}_1$ and an MWC of $\mathcal{W}_2$. But $\mathcal{W}$ has exactly five MWCs. Hence one of the $\mathcal{W}_i$ would have to have exactly one MWC, say S (and the other $\mathcal{W}_i$ would then have five MWCs). Therefore all five MWCs of $\mathcal{W}$ would include the same nonempty set, S — which is evidently false.

2.3.20 Remark In [104] Taylor and Zwicker prove an interesting characterization of WVGs, which we summarize here.

Let $\langle X_1, X_2, \ldots, X_n \rangle$ be an ordered n-tuple of sets (not necessarily distinct). *Migration* is the operation of moving a member from some X_i to some X_j. More precisely, we say that the n-tuple $\langle Y_1, Y_2, \ldots, Y_n \rangle$ is obtained from $\langle X_1, X_2, \ldots, X_n \rangle$ by *migration* if there are i, j and x such that $1 \leq i, j \leq n$ and $x \in X_i - X_j$; and such that $Y_i = X_i - \{x\}$, $Y_j = X_j \cup \{x\}$, and $Y_k = X_k$ for all $k \neq i, j$.

Further, let us say that an SVG $\mathcal{W}$ is *migration robust* if whenever $\langle X_1, X_2, \ldots, X_n \rangle$ is an ordered n-tuple of winning coalitions (with $n \geq 1$) and the n-tuple $\langle Y_1, Y_2, \ldots, Y_n \rangle$ is obtained from $\langle X_1, X_2, \ldots, X_n \rangle$ by a finite sequence of migrations, then Y_i is a winning coalition for at least one i.

In [104] it is proved that *an SVG is a WVG iff it is migration robust.*[15]

2.3.21 Definition We say that $\mathcal{V}$ is a *subgame* of an SVG $\mathcal{W}$ if

$$\mathcal{V} = \{X \in \mathcal{W} : X \subseteq M\},$$

[15] For this and related material see also [105].

for some winning coalition M of $\mathcal{W}$.

2.3.22 Remark Clearly, in this case $\mathcal{V}$ is an SVG and M is its assembly.

We conclude this section with a definition of the operation of bloc formation, whereby a coalition of an SVG forms itself into a single voter.

2.3.23 Definition Let S be a coalition of the SVG $\mathcal{W}$. We introduce a new object, distinct from all the voters of $\mathcal{W}$, which we call the *bloc of* S and denote by '$\&_S$'. We define $N|\&_S$ to be the set obtained from the assembly N of $\mathcal{W}$ by excluding all the members of S and adding $\&_S$ as a new voter:

$$N|\&_S =_{\text{def}} (N - S) \cup \{\&_S\}.$$

Let f be the *natural surjection* (onto map) from N to $N|\&_S$:

$$fx = \begin{cases} x & \text{if } x \in N - S, \\ \&_S & \text{if } x \in S. \end{cases}$$

We define an SVG $\mathcal{W}|\&_S$, with $N|\&_S$ as assembly:

$$\mathcal{W}|\&_S =_{\text{def}} \{Y : Y \subseteq (N|\&_S) \text{ and } f^{-1}[Y] \in \mathcal{W}\}.$$

(Here $f^{-1}[Y] = \{x : fx \in Y\}$.) We say that $\mathcal{W}|\&_S$ is *obtained from* $\mathcal{W}$ *by the formation of the bloc* $\&_S$.

If a and b are two voters of $\mathcal{W}$, we denote the bloc $\&_{\{a,b\}}$ by '$a\&b$' and write '$N|a\&b$' and '$\mathcal{W}|a\&b$' instead of '$N|\&_{\{a,b\}}$' and '$\mathcal{W}|\&_{\{a,b\}}$' respectively.

2.3.24 Remarks (i) In the definition we did not specify precisely what the new voter $\&_S$ is, because this really does not matter: any object will do, provided it is new — that is, distinct from all the voters of $\mathcal{W}$.

(ii) Intuitively speaking, $\mathcal{W}|\&_S$ comes about as a result of the members of S amalgamating — by mutual consent or through hostile takeover — into a single entity that henceforth votes as one.

This creates a new SVG, because now the members of S no longer exist as distinct voters, but are replaced by a single new voter, $\&_S$.

What are the coalitions of the new SVG? Consider any $T \subseteq N$. If T is disjoint from S, then T is also a coalition of $\mathcal{W}|\&_S$; and it is a winning coalition in this new SVG iff it was a winning coalition in the original $\mathcal{W}$. Next, if T contains some, but not all, members of S, then T is ignored in the new SVG, because here the votes of the bloc of S cannot be split. Finally, if $S \subseteq T$, then T gives rise to a new coalition, $(T - S) \cup \{\&_S\}$, whose status (winning or losing) in the new SVG is the same as that of T in the original $\mathcal{W}$.

(iii) In the special case $S = \emptyset$, the new voter $\&_\emptyset$ is a dummy in $\mathcal{W}|\&_\emptyset$. Each coalition T of $\mathcal{W}$ gives rise to two coalitions in the new SVG: T and $T \cup \{\&_\emptyset\}$, both of which have the same status in this SVG as T had in $\mathcal{W}$. Observe that $\mathcal{W}$ is a subgame of $\mathcal{W}|\&_\emptyset$.

(iv) The operation of bloc formation is quite transparent when $\mathcal{W}$ is a WVG. Suppose $\mathcal{W}$ results from the weighting system $\langle q, w\rangle$ on N. Then $\mathcal{W}|\&_S$ is the WVG of the weighting system $\langle q, w'\rangle$ on $N|\&_S$, where $w'_x = w_x$ for all $x \in N - S$, and $w'_{\&_S} = wS$. All that happens is simply that the new voter $\&_S$ inherits the total weight of S.

3. Power as Influence

3.1 I-Power: A Probabilistic Notion

In § 1.1 we said, by way of informal explanation, that the voting power of a member of a board, under a given decision rule, is *the extent to which the member is able to control the outcome of a division of the board.* We should now like to make this rough explanation more precise. In the voting-power literature there are in fact two different interpretations of a voter's 'ability to control the outcome of a division', corresponding to the two motivations of voting behaviour discussed in Com. 2.2.2. However, these two interpretations are seldom clearly distinguished; more often than not they are conflated with each other.[1]

From the policy-seeking viewpoint on voting, the outcome of a division is simply the passage or failure of the bill in question. A bill having been proposed, each member forms a position — for or against it — and votes accordingly.[2] Rational voters presumably form their positions by comparing the expected payoff of the bill's passage with that of its failure (here 'payoff' must of course be understood in a very broad sense). These payoffs vary from member to member and from bill to bill; but — for a given member and a given bill — they are quite independent of the decision rule operated by the board.

[1] A notable exception is [20], where the distinction is made quite explicitly (see passages quoted in Com. 2.2.2). This distinction is amplified in [36]. In [70, pp. 157–166] a broadly similar distinction is attempted, but is undermined by an apparent lack of clarity about the office-seeking viewpoint.

[2] For the time being we exclude a position of indecision or indifference and do not admit abstention. This restriction will be lifted in Ch. 8.

From this policy-seeking viewpoint, a member's voting power is the degree to which that member's vote is able to *influence* the outcome of a division: whether the bill in question will pass or fail. We shall refer to this notion of voting power as *I-power*.

From the rival, office-seeking, viewpoint on voting, passage or failure of the bill is merely the ostensible and proximate outcome of a division; the real and ultimate outcome is the distribution of a fixed purse — the prize of power — among the victors. From this viewpoint, a member's voting power, the extent of his or her control of the ultimate outcome, is to be measured by that member's expected or estimated share in the fixed purse. We shall refer to this notion of voting power as *P-power*.

For the moment we shall put the notion of P-power on one side, to be elaborated in § 6.1. First we shall pursue the idea of I-power, which is historically prior as well as conceptually less problematic. How can this idea be explicated mathematically? An obvious way — arguably the *only* reasonable way — of doing so is in terms of probabilities: the voting power of voter a can be formally defined as the probability of a being in a position to affect the outcome of a division.[3] Indeed, all three authors who, independently of each other, offered a mathematical explication of I-power — Penrose [78], Banzhaf [5] and Coleman [20] — adopted, in effect, a probabilistic approach.[4] Now, the use of probability requires a definite probability space, serving as a probabilistic model. All three authors just mentioned assume one and the same model, which suggests itself as the most natural for the purpose, although none of them actually spells it out.[5] A concise and transparent way of describing this model is in terms of bipartitions (see Def. 2.1.5), as follows.

[3]Since we are concerned here with I-power, 'outcome' must be understood accordingly, as passage or failure of the bill in question.

[4]This is explicit in [78], where the term 'probability' is actually used. In [5] and [20] the probabilistic reasoning is tacit.

[5]This model is spelt out by other authors; see, for example, [27, pp. 102–103].

3.1.1 Definition Let N be a finite set, with $|N| = n$. The *Bernoulli model* $\mathbf{B}_N$ is the probability space consisting of the set ${}^N\{-1, 1\}$ of all bipartitions of N, with each bipartition assigned the same probability: $1/2^n$.

Where there is no risk of confusion, we omit the subscript 'N' in '$\mathbf{B}_N$' and write simply '$\mathbf{B}$'.

3.1.2 Remarks (i) The N for which we shall use the Bernoulli model $\mathbf{B}_N$ will always be the assembly of some SVG $\mathcal{W}$. In this context, $\mathbf{B}_N$ has the following interpretation: divisions of the board are generated by n 'yes'/'no' equiprobable Bernoulli trials. In other words, each member of the board votes 'yes' and 'no' with equal probability of $\frac{1}{2}$; and members act independently of each other.

Note that we have here a conjunction of two assumptions: equiprobability of each member voting either way; and independence between members.

(ii) In several of his papers, Straffin discusses what he calls the *Independence Assumption*:

> Every proposal has a probability p_i of appealing to the i-th member. Each of the p_is is chosen uniformly and independently from the interval $[0, 1]$.[6]

Thus the p_i themselves are taken to be independent random variables, each distributed uniformly in $[0, 1]$. But this boils down to the model $\mathbf{B}$, because choosing a value of p_i from a uniform distribution in $[0, 1]$ and then proceeding to vote 'yes' with probability p_i amounts to voting 'yes' with probability $\frac{1}{2}$. The same would clearly be true for other distributions of the p_i (not necessarily identical for all i), provided the expected values of all these random variables are equal to $\frac{1}{2}$.[7]

3.1.3 Comment Use of the Bernoulli model $\mathbf{B}$ requires some explanation and justification. We must stress that this model is not offered as a realistic representation of actual real-life voting

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[6] [100, p. 299].

[7] This fairly obvious point is made rather circuitously in [64].

behaviour. What it is supposed to represent — in the spirit of Com. 2.2.3 — is a state of *a priori ignorance* regarding the nature of the bills to be voted on, the voters' personalities and interests, and relations of affinity or disaffinity between voters.

In the classical tradition of probability theory, this use of the Bernoulli model might be justified by invoking the so-called *Principle of Insufficient Reason*: in the state of a priori ignorance just described, we have no reason to regard any one bipartition as more likely than any other.[8] This argument can be broken down into two parts, corresponding to the two assumptions of Rem. 3.1.2(i). First, taking any particular voter, a: if we have no information as to a's personality and interests, or the nature of the bill to be voted upon, then we ought to assign equal a priori probabilities to a voting either way. Second, in ignorance of any concurrence or opposition of interests between voters, the most rational a priori assumption to adopt is that of independence.

In a more modern vein, we may justify the Bernoulli model in terms of the notion of *entropy*, borrowed from information theory. The set ${}^N\{-1, 1\}$ of all bipartitions represents all distinguishable elementary events that can occur when a board divides. The probability distribution on this set of 2^n points that yields maximal entropy — and hence embodies greatest 'disorder' and least information — is the one that assigns equal probability to all 2^n points.

3.2 The Banzhaf Measure

3.2.1 Convention From now until the end of the present chapter, whenever we use probabilistic terms without stating explicitly the probability space to which they refer, the space we have in mind is $\mathbf{B}_N$, where N is the assembly of the SVG under consideration. In particular, 'P' will denote the probability of events in

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[8] For a critical discussion of the Principle and its history, see Keynes [56, Chs. IV and VI]. He points out various fallacies that arise from improper use of the Principle (which he renames 'Principle of Indifference') and lays down conditions under which it can be used safely. The case we are concerned with here satisfies these conditions.

$\mathbf{B}_N$; and 'E' will denote the mathematical expectation (expected value) operator in this space.

3.2.2 Definition The *Bz score* is the function η that assigns to any SVG $\mathcal{W}$ and any voter a of $\mathcal{W}$ a value $\eta_a[\mathcal{W}]$ — called the *Bz score of* a in $\mathcal{W}$ — equal to the number of coalitions of $\mathcal{W}$ in which a is critical.

The *Bz index of voting power* is the function β defined by

$$\beta_a[\mathcal{W}] =_{\text{def}} \frac{\eta_a[\mathcal{W}]}{\sum_{x \in N} \eta_x[\mathcal{W}]}.$$

(Here, as usual, N is the assembly of $\mathcal{W}$.) We refer to $\beta_a[\mathcal{W}]$ as the *Bz index of* a *in* $\mathcal{W}$.

The *Bz measure of voting power* is the function β' defined by

$$\beta'_a[\mathcal{W}] =_{\text{def}} \frac{\eta_a[\mathcal{W}]}{2^{n-1}}.$$

(Here, as usual, $n = |N|$ is the number of voters of $\mathcal{W}$.) We refer to $\beta'_a[\mathcal{W}]$ as the *Bz power of* a *in* $\mathcal{W}$.

3.2.3 Remarks (i) Although there is a good case for naming the concepts just defined after Penrose (see § 1.2), the practice of naming them after Banzhaf is by now well established. Note, however, that in [5] neither β nor β' are actually defined. Instead, the ratio $\eta_a[\mathcal{W}] : \eta_b[\mathcal{W}]$ is proposed as expressing the ratio of the power of voter a to that of b.

The measure of voting power proposed in [78] is not β' but $\beta'/2$.

(ii) In the literature, β' is often referred to as 'the Bz index', or 'the *absolute* Bz index' (to distinguish it from β, which is referred to as 'the *relative* Bz index'). However, in this book we reserve the term 'index' for measures whose values for all voters of an SVG always add up to 1. This condition is obviously satisfied by β; but as we shall soon see (Ex. 3.2.6) it is not satisfied by β'.

(iii) In accordance with Conv. 2.3.1(ii), where there is no risk of confusion we abbreviate '$\eta_a[\mathcal{W}]$', '$\beta_a[\mathcal{W}]$' and '$\beta'_a[\mathcal{W}]$' as 'η_a', 'β_a'

and 'β'_a' respectively. Similar abbreviations will be used for other expressions of the same kind.

(iv) Clearly, η is invariant under isomorphism (cf. Rem. 2.1.8): if f is an isomorphism from $\mathcal{W}$ to $\overline{\mathcal{W}}$ and if a is a voter of $\mathcal{W}$ and $fa = \bar{a}$, then $\eta_a[\mathcal{W}] = \eta_{\bar{a}}[\overline{\mathcal{W}}]$. The same applies also to β and β'.

(v) It is easy to see that η_a is also equal to the number of coalitions *outside* which voter a is critical (see Def. 2.3.4). Likewise, it equals the number of bipartitions for which a is positively critical, as well as the number of those for which it is negatively critical (see Def. 2.3.6). Thus a is critical for $2\eta_a$ bipartitions. The definition of β'_a therefore yields the following theorem.

3.2.4 Theorem
$$\begin{aligned}\beta'_a &= \mathrm{P}(a \text{ is critical})\\ &= \mathrm{P}(a \text{ is critical} \mid a \text{ votes 'yes'})\\ &= \mathrm{P}(a \text{ is critical} \mid a \text{ votes 'no'}).\end{aligned}$$ ∎

3.2.5 Comment Here 'a is critical', 'a votes 'yes' ' and 'a votes 'no' ' denote the events

$$\{B : a \text{ is critical for } B\}, \quad \{B : Ba = 1\}, \quad \{B : Ba = -1\}$$

respectively in the space $\mathbf{B}$.

Put less formally, β'_a is the a priori probability that, in a division on a bill, the votes will be so disposed that if a's vote were to be reversed then the fate of the bill would also be reversed. Likewise β'_a is the conditional probability — given that we know how a will vote — that the fate of the bill would be reversed were a to vote otherwise.

It is this interpretation that serves as the main argument for regarding β' as a measure of absolute I-power: voter a's ability to influence the outcome of a division is to be measured by the a priori probability that a will be in a position to tip the balance. Note that from this point of view β' is the primary concept, while β is merely derivative: it is obtained by *normalizing* β' — rescaling it so as to produce an index, for which $\sum_{x \in N} \beta_x = 1$. What β_a measures is not the *amount* of power wielded by a but a's *relative* *share* of total power. It is therefore a mistake to interpret this normalization as though it somehow presupposes that the total amount of power is always a unity. This would be like thinking that the study of relative income distribution presupposes that total income is always a unity. A closely related mistake is to believe that the normalization of the Bz index makes it an index of P-power rather than relative I-power.[9]

3.2.6 Example Consider the two boards of Ex. 1.1.1. The first of these can be modelled by $\mathcal{B}_3$ (see Def. 2.3.10). Here the whole assembly, $\{1, 2, 3\}$, is the sole winning coalition, so we have $\eta_i = 1$ for $i = 1, 2, 3$. Since there are three voters, the Bz power of each is $1/2^2 = \frac{1}{4}$.

The Bz index of each voter is obviously $\frac{1}{3}$. This is as we anticipated in Ex. 1.1.1. Indeed, although we have not yet defined an index of P-power, we can tell in advance that, for reasons of symmetry, any such index ought also to assign a value of $\frac{1}{3}$ to each voter of $\mathcal{B}_3$.

The second, enlarged, board can be modelled by $\mathcal{M}_4$. Here there are five winning coalitions: the whole assembly, $\{1, 2, 3, 4\}$, and four coalitions of size 3, obtained by dropping any one of the four voters. In the assembly no member is critical. But in the coalitions of size 3 every member is critical. Each voter belongs to three such coalitions, so $\eta_i = 3$ for $i = 1, 2, 3, 4$. And since there are now four voters, the Bz power of each is $3/2^3 = \frac{3}{8}$.

The Bz index of each voter is now evidently $\frac{1}{4}$. Again, for reasons of symmetry, any legitimate index of P-power is also going to assign a value of $\frac{1}{4}$ to each voter of $\mathcal{M}_4$.

We see that in the move from $\mathcal{B}_3$ to $\mathcal{M}_4$, each of the three old voters has *gained* Bz power; but at the same time their *relative* shares in power have *declined*. So, where are these voters better

[9]Both mistakes are made by Barry [9, p. 191]. Being in any case hostile to the very notion of P-power, he dismisses the Bz index as 'a mere gimmick'. He says nothing about the Bz *measure*, which he then proceeds to reinvent under the name of 'decisiveness'. More precisely, his 'decisiveness' is the original Penrose form of the measure. Cf. the rejection of the Bz index by Morriss [70, p. 166], who however is clear about its relation to Penrose's measure.

off, in $\mathcal{B}_3$ or in $\mathcal{M}_4$? The answer depends on the motivation of voting, and therefore ultimately on what sort of issue the voting is going to decide. If it is a matter of carving up a fixed prize, then these voters are better off in $\mathcal{B}_3$, where each of them can expect to get $\frac{1}{3}$ of the loot. On the other hand, in policy-seeking voting it is absolute I-power that matters, so they are better off in $\mathcal{M}_4$, where each of them stands a greater chance a priori of affecting the outcome of a division.[10]

From Def. 2.3.2 it is clear that voter a is $\mathcal{W}^*$-critical in a coalition S iff a is $\mathcal{W}$-critical *outside* $N - S$. We therefore have the following result, which can be expressed by saying that the three functions η, β and β' defined in Def. 3.2.2 are *self-dual*.

3.2.7 Theorem *If $\mathcal{W}$ is any SVG and a is any voter in it, then $\eta_a[\mathcal{W}^*] = \eta_a[\mathcal{W}]$, $\beta_a[\mathcal{W}^*] = \beta_a[\mathcal{W}]$ and ${\beta'}_a[\mathcal{W}^*] = {\beta'}_a[\mathcal{W}]$.* ∎

We shall now deduce some interesting probabilistic facts involving β'. To this end, we first introduce some useful notation.

3.2.8 Definition For any SVG $\mathcal{W}$ we put

$$\omega[\mathcal{W}] =_{\text{def}} |\mathcal{W}|;$$

and for any voter a of $\mathcal{W}$:

$$\omega_a[\mathcal{W}] =_{\text{def}} |\{X : a \in X \in \mathcal{W}\}|.$$

3.2.9 Remarks (i) Thus ω is the number of winning coalitions. It is clearly also the number of bipartitions whose outcome is positive (see Def. 2.1.5). Similarly, ω_a is the number of winning coalitions to which a belongs, as well as the number of bipartitions with whose outcome a agrees positively (see Def. 2.3.6).

(ii) It follows from a theorem quoted by Dubey and Shapley and attributed by them to C K Chow, that any two WVGs that have the same set of ω_a values are isomorphic. Moreover, if $\mathcal{V}$ is a WVG

[10] For a more nuanced assessment see Ex. 3.2.22.

and $\mathcal{W}$ is an SVG that has the same ω and ω_a values as $\mathcal{V}$, then $\mathcal{W} \cong \mathcal{V}$. Lists have been compiled in which one can find the WVG with given values of these parameters. For further details see [27, pp. 126–127].

3.2.10 Lemma *Let a be a voter of $\mathcal{W}$. Let $\mathbf{Y}_a$ be $\{B : Ba = 1\}$, the half of the space $\mathbf{B}$ in which a votes 'yes'. Then in $\mathbf{Y}_a$ there are exactly ω_a bipartitions with positive outcome according to W.*

Moreover, in $\mathbf{B} - \mathbf{Y}_a$, the other half of $\mathbf{B}$, there are exactly $\omega_a - \eta_a$ bipartitions with positive outcome according to W.

Proof We pair each $B \in \mathbf{Y}_a$ with the bipartition B' in which a votes 'no' but which is otherwise the same as B; that is, $B'a = -1$ but $B'x = Bx$ for any other voter x. Of course, $B' \notin \mathbf{Y}_a$. If B has a negative outcome (that is, $\mathsf{W}B = -1$) then the same applies to B', because $B' < B$ (see Rem. 2.1.6(ii)).

Now focus on those $B \in \mathbf{Y}_a$ whose outcome is positive; that is, $\mathsf{W}B = 1$. They are exactly the bipartitions with whose outcome a agrees positively, so by Rem. 3.2.9(i) there are ω_a of them.

They are of two kinds. First, there are these for which a is (positively) critical; there are exactly η_a of these. If B is of this kind, then of course B' has a negative outcome. The remaining $\omega_a - \eta_a$ bipartitions $B \in \mathbf{Y}_a$ with positive outcome are those for which a is not critical. If B is of that kind, then clearly B' still has a positive outcome.

Hence in $\mathbf{B} - \mathbf{Y}_a$ there are exactly $\omega_a - \eta_a$ bipartitions with positive outcome. ∎

According to the lemma, in the whole of $\mathbf{B}$ there are altogether $2\omega_a - \eta_a$ bipartitions with positive outcome. But by Rem. 3.2.9(i) this number is ω. This yields the following result, which is often useful; for example, it serves as a helpful check when computing Bz scores.

3.2.11 Corollary (Dubey and Shapley [27, p. 127]) *For any voter a,*

$$\eta_a = 2\omega_a - \omega.$$

Hence η_a has the same parity as ω: both are even or both are odd. In particular, the Bz scores of all voters in an SVG have the same parity. ∎

Given an SVG $\mathcal{W}$, its BPR W (see Def. 2.1.5) is a random variable defined on the space $\mathbf{B}$; it takes the value 1 or -1, according as the outcome is positive or negative (the bill is passed or defeated). Thus $\mathrm{P}(\mathsf{W} = 1)$ is the probability that the outcome is positive. Now, suppose that we know that voter a is going to vote 'yes'. (More formally: the event $\{B : Ba = 1\}$ is going to occur.) Subject to this information, the *conditional* probability of positive outcome is $\mathrm{P}(\mathsf{W} = 1 \mid a \text{ votes 'yes'})$. Clearly, this must be equal to or greater than the *un*conditional $\mathrm{P}(\mathsf{W} = 1)$; but by how much?

3.2.12 Theorem

$$\mathrm{P}(\mathsf{W} = 1 \mid a \text{ votes 'yes'}) - \mathrm{P}(\mathsf{W} = 1) = \frac{\beta'_a}{2}.$$

Proof By Lemma 3.2.10, there are ω_a bipartitions with positive outcome in the half-space $\mathbf{Y}_a$, where a votes 'yes'; and $2\omega_a - \eta_a$ such bipartitions in the whole of $\mathbf{B}$. Hence

$$\mathrm{P}(\mathsf{W} = 1 \mid a \text{ votes 'yes'}) = \frac{\omega_a}{2^{n-1}}$$

and

$$\mathrm{P}(\mathsf{W} = 1) = \frac{2\omega_a - \eta_a}{2^n}.$$

The difference is $\eta_a/2^n = \beta'_a/2$. ∎

3.2.13 Comment We see that $\beta'_a/2$ is the amount by which a's 'yes' vote *increases* the probability of the bill in question passing. By the same token, it is also the amount by which a's 'no' vote *decreases* that probability. This may be the reason why Penrose proposed $\beta'/2$ (rather than β' itself) as a measure of voting power.

The following theorem concerns the mathematical expectation of the random variable W — both unconditional and subject to the condition that a votes 'yes'.

3.2.14 Theorem $\mathrm{E}(\mathsf{W} \mid a \text{ votes 'yes'}) - \mathrm{E}(\mathsf{W}) = \beta'_a$.

Proof Consider the random variable $\mathsf{X} = (\mathsf{W} + 1)/2$. It takes the value 1 or 0 according as the outcome is positive or negative. Hence its expected value $\mathrm{E}(\mathsf{X})$ equals $\mathrm{P}(\mathsf{W} = 1)$, the probability that the outcome is positive. But $\mathsf{W} = 2\mathsf{X} - 1$; so

$$\mathrm{E}(\mathsf{W}) = 2\mathrm{P}(\mathsf{W} = 1) - 1.$$

Similarly,

$$\mathrm{E}(\mathsf{W} \mid a \text{ votes 'yes'}) = 2\mathrm{P}(\mathsf{W} = 1 \mid a \text{ votes 'yes'}) - 1.$$

The claim of our theorem follows at once from Thm. 3.2.12. ∎

3.2.15 Comment Thm. 3.2.14 may be interpreted as follows.[11] Suppose an outsider (not one of the voters) stands to gain one unit of transferable utility if a board were to pass a certain bill, and to lose one unit if the board were to defeat that bill. Let us assume that the outsider knows nothing about the voters' attitudes to the bill and their mutual interactions. Then the payoff of this person is a random variable; in fact, it is W. The *expected* payoff is therefore $\mathrm{E}(\mathsf{W})$. Now suppose the outsider considers buying the vote of a member a of the board.[12] By Thm. 3.2.14, the certainty that a will vote for the bill raises the buyer's expected payoff by β'_a, so the buyer ought to be prepared to pay any amount short of β'_a for a's vote.

The next result concerns the probability that voter a will agree with the outcome of the bipartition. Recall that this means $Ba = \mathsf{W}B$, so that the decision goes a's way.

3.2.16 Theorem (Penrose)

$$\mathrm{P}(a \text{ agrees with outcome}) = \frac{1 + \beta'_a}{2}.$$

[11] Cf. Morriss [70, p. 226].

[12] To avoid any suggestion of impropriety, we may suppose that the board is a shareholders' meeting or a similar body, where sale of votes is legally and morally acceptable.

Proof We use Lemma 3.2.10. In the half-space $\mathbf{Y}_a$, in which a votes 'yes', a agrees with the outcome of a bipartition iff the outcome is positive. By the lemma there are exactly ω_a such bipartitions in $\mathbf{Y}_a$.

In the other half-space, $\mathbf{B} - \mathbf{Y}_a$, a votes 'no', hence agrees with the outcome iff it is negative. According to the lemma, in this half-space there are exactly $\omega_a - \eta_a$ bipartitions with positive outcome, so the number of those with negative outcome is $2^{n-1} - \omega_a + \eta_a$.

Altogether a agrees with the outcomes of $2^{n-1} + \eta_a$ bipartitions. Therefore

$$\mathrm{P}(a \text{ agrees with outcome}) = \frac{2^{n-1} + \eta_a}{2^n} = \frac{1 + \beta'_a}{2},$$

as claimed. ∎

3.2.17 Remarks (i) For any voter a in an SVG $\mathcal{W}$, the number of bipartitions with whose outcome a agrees is denoted by '$\rho_a[\mathcal{W}]$'. The function ρ is named after Douglas Rae, who re-invented it in his 1969 paper [81]. The identity

$$\rho_a = 2^{n-1} + \eta_a,$$

which is the essence of Thm. 3.2.16, is proved in [27, pp. 124–125], whose authors comment that the close connection between ρ and β' 'was not noticed for several years' after 1969. In fact, the connection had been noticed a good many years before, in 1946; it is stated in [78, p. 53] without proof, as something the readers are expected to work out for themselves:

> In general, the power of the individual vote [*sic*] can be measured by the amount by which his chance of being on the winning side exceeds one half. The power, thus defined, is the same as half the likelihood of a situation in which an individual can be decisive . . . .

(In the terminology of [78], 'being on the winning side' means agreeing with the outcome.)[13]

[13] Thm. 3.2.16 has apparently been rediscovered several times. It appears as the equation 'success = luck + decisiveness' in Barry [9, p. 338], who, as we mentioned in fn. 9, fails to realize the connection between his 'decisiveness' and the Bz index, to which he objects vehemently.

(ii) The function $\rho/2^n$ — which by Thm. 3.2.16 equals $(1+\beta')/2$ — may be used as a measure of a board member's presumed contentment with the decision rule. For instance, looking back at Ex. 3.2.6 we observe that in the first board, modelled by $\mathcal{B}_3$, each member can expect to agree with 5 out of 8 decisions; whereas members of the board modelled by $\mathcal{M}_4$ can expect to agree with 11 out of 16 decisions, so (from the policy-seeking perspective) they ought to feel more contented here.[14]

So why not use $\rho/2^n$ as a measure of voting power? The answer is that by Thm. 3.2.16 this probability is always at least $\frac{1}{2}$; it assumes this minimal value in the case of a dummy. To get a more reasonable measure, which assigns 0 to a dummy, we can take $\frac{1}{2}$ off $\rho/2^n$; the result is $\beta'/2$. The trouble with this is that $\beta'/2$ can never exceed $\frac{1}{2}$; it assumes this maximal value for a dictator. The advantage of β' is that it assigns 0 power to a dummy and 1 to a dictator.

Our next result, taken from [29], says that when two voters form a bloc (see Def. 2.3.23) the Bz power of this bloc is the sum of the Bz powers the partners had in the original SVG.

3.2.18 Theorem *For any two distinct voters a and b of $\mathcal{W}$,*

$$\beta'_{a\&b}[\mathcal{W}|a\&b] = \beta'_a[\mathcal{W}] + \beta'_b[\mathcal{W}].$$

Proof Let $M = N - \{a,b\}$. Note that every coalition of $\mathcal{W}$ has one of the four forms X, $X \cup \{a\}$, $X \cup \{b\}$ or $X \cup \{a,b\}$, with $X \subseteq M$. We classify all subsets X of M into six mutually exclusive classes as follows.

$$\begin{aligned}
\mathcal{A} &= \{X \subseteq M : X \in \mathcal{W}\},\\
\mathcal{B} &= \{X \subseteq M : X \cup \{a\} \in \mathcal{W},\ X \cup \{b\} \notin \mathcal{W}\},\\
\mathcal{C} &= \{X \subseteq M : X \cup \{b\} \in \mathcal{W},\ X \cup \{a\} \notin \mathcal{W}\},\\
\mathcal{D} &= \{X \subseteq M : X \cup \{a\} \in \mathcal{W},\ X \cup \{b\} \in \mathcal{W},\ X \notin \mathcal{W}\},\\
\mathcal{E} &= \{X \subseteq M : X \cup \{a,b\} \in \mathcal{W},\ X \cup \{a\} \notin \mathcal{W},\ X \cup \{b\} \notin \mathcal{W}\},\\
\mathcal{F} &= \{X \subseteq M : X \cup \{a,b\} \notin \mathcal{W}\}.
\end{aligned}$$

[14] Note however the caveat in [27, fn. 45] against using utilitarian terminology in this context.

First, let us count $\eta_a[\mathcal{W}]$, the score of a in $\mathcal{W}$. We count the coalitions in which a is critical. Clearly, if $X \in \mathcal{B}$ then X contributes two points to $\eta_a[\mathcal{W}]$ because a is critical both in $X \cup \{a\}$ and in $X \cup \{a, b\}$. If $X \in \mathcal{D}$ then X contributes just one point to $\eta_a[\mathcal{W}]$ because a is critical in $X \cup \{a\}$ but not in $X \cup \{a, b\}$. Also, if $X \in \mathcal{E}$ then X contributes one point to $\eta_a[\mathcal{W}]$ because a is critical in $X \cup \{a, b\}$ but not in $X \cup \{a\}$. But if X is in $\mathcal{A}$, $\mathcal{C}$ or $\mathcal{F}$ then X contributes nothing to $\eta_a[\mathcal{W}]$ because in these cases a is critical neither in $X \cup \{a\}$ nor in $X \cup \{a, b\}$. Therefore

$$\eta_a[\mathcal{W}] = 2|\mathcal{B}| + |\mathcal{D}| + |\mathcal{E}|.$$

To count $\eta_b[\mathcal{W}]$ we proceed similarly, but now the roles of $\mathcal{B}$ and $\mathcal{C}$ are interchanged:

$$\eta_b[\mathcal{W}] = 2|\mathcal{C}| + |\mathcal{D}| + |\mathcal{E}|.$$

Next, let us count $\eta_{a\&b}[\mathcal{W}|a\&b]$. Note that every coalition of $\mathcal{W}|a\&b$ has one of the two forms X or $X \cup \{a\&b\}$, with $X \subseteq M$. It is easy to see that if X is in any one of the classes $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ or $\mathcal{E}$ then X contributes one point to this score because in these four cases $a\&b$ is critical in $X \cup \{a\&b\}$. But if X is in $\mathcal{A}$ or $\mathcal{F}$ then X makes no contribution to the score. Therefore

$$\eta_{a\&b}[\mathcal{W}|a\&b] = |\mathcal{B}| + |\mathcal{C}| + |\mathcal{D}| + |\mathcal{E}| = \frac{1}{2}(\eta_a[\mathcal{W}] + \eta_b[\mathcal{W}]).$$

The claim of our theorem now follows from the definition of β' (Def. 3.2.2), since $\mathcal{W}|a\&b$ has one voter less than $\mathcal{W}$. ∎

3.2.19 Warning Thm. 3.2.18 seems quite unremarkable: when two voters form a bloc, their voting powers are simply pooled. But this is deceptive: the theorem does not generally apply to more than two voters! An attempt to generalize it by induction soon shows what 'goes wrong': when a and b form a bloc, the Bz powers of the other voters do not generally stay as they were; so $\beta'_c[\mathcal{W}|a\&b]$ is in general not equal to $\beta'_c[\mathcal{W}]$.

We conclude this section with a definition and discussion of the Coleman measures.[15]

[15] In the literature they are often referred to as *indices*, but see Rem. 3.2.3(ii).

3.2.20 Definition The *Coleman measures of voting power* are the functions γ and γ^* defined by

$$\begin{aligned}\gamma_a[\mathcal{W}] &=_{\text{def}} \mathrm{P}(a \text{ is critical} \mid \mathsf{W} = 1),\\ \gamma^*_a[\mathcal{W}] &=_{\text{def}} \mathrm{P}(a \text{ is critical} \mid \mathsf{W} = -1).\end{aligned}$$

3.2.21 Remarks (i) In [20] Coleman phrases the definitions of γ and γ^* in terms of relative frequencies rather than conditional probabilities; but the underlying logic is surely probabilistic. Moreover, the Bernoulli model **B** is tacitly assumed.

(ii) Coleman argues that γ_a measures voter a's power to *prevent action.* The idea is that, given that the board makes a positive decision, γ_a is the conditional probability that a would have been able to prevent it by changing sides. Similarly, he regards γ^*_a as measuring voter a's power to *initiate action.*

(iii) From Rems. 3.2.3(v) and 3.2.9(i) it follows at once that

$$\gamma_a = \frac{\eta_a}{\omega}, \quad \gamma^*_a = \frac{\eta_a}{2^n - \omega}.$$

Thus γ and γ^* are rescalings of β'. Moreover, an easy calculation shows that β' is the harmonic mean of the two Coleman measures:

$$\beta'(\gamma + \gamma^*) = 2\gamma\gamma^*.$$

(iv) From Thm. 3.2.7 and the fact that $\mathcal{W}^*$ has as many winning coalitions as $\mathcal{W}$ has losing ones we get

$$\gamma_a[\mathcal{W}^*] = \gamma^*_a[\mathcal{W}].$$

In this sense, γ and γ^* are *mutually dual.*

(v) By (iii), γ_a and γ^*_a uniquely determine ω and η_a and hence (via Cor. 3.2.11) also ω_a. Hence by Rem. 3.2.9(ii) if $\mathcal{V}$ is a WVG and $\mathcal{W}$ is an SVG that has the same γ and γ^* values as $\mathcal{V}$, then $\mathcal{W} \cong \mathcal{V}$.

3.2.22 Example Consider the two SVGs of Ex. 3.2.6. We saw there that in $\mathcal{B}_3$ the Bz score of each voter is 1. The number ω of winning coalitions is also 1. Hence by Rem. 3.2.21(iii) we have $\gamma_i[\mathcal{B}_3] = 1$ and $\gamma^*_i[\mathcal{B}_3] = \frac{1}{7}$ for $i = 1, 2, 3$.

We also found that in $\mathcal{M}_4$ the Bz score of each voter is 3 and the number ω of winning coalitions is 5. Hence by Rem. 3.2.21(iii) we have $\gamma_i[\mathcal{M}_4] = \frac{3}{5}$ and $\gamma^*_i[\mathcal{M}_4] = \frac{3}{11}$ for $i = 1, 2, 3, 4$.

In Ex. 3.2.6 we concluded that policy-seeking voters are better off in $\mathcal{M}_4$, where they have greater I-power (as measured by β') than the voters of $\mathcal{B}_3$. This was based on the assumption that the voters in question are a priori equally likely to vote 'yes' as 'no'. Now we see that an obstructionist voter, who tends to vote 'no' more often than 'yes', may be better off in $\mathcal{B}_3$, where voters' power to prevent action (as measured by γ) is considerably greater than in $\mathcal{M}_4$. On the other hand, a complaisant voter, who tends to vote 'yes' more often than 'no', would be much better off in $\mathcal{M}_4$, where a voter's power to initiate action (as measured by γ^*) is almost twice as great as in $\mathcal{B}_3$.

3.2.23 Example Compare the following two WVGs:

$$\mathcal{U} = [4; 3, 2, 1, 1], \quad \mathcal{V} = [5; 3, 2, 1, 1].$$

A simple calculation yields:[16]

$$\beta'_1[\mathcal{U}] = \tfrac{3}{4}, \quad \beta'_2[\mathcal{U}] = \beta'_3[\mathcal{U}] = \beta'_4[\mathcal{U}] = \tfrac{1}{4};$$
$$\beta'_1[\mathcal{V}] = \tfrac{5}{8}, \quad \beta'_2[\mathcal{V}] = \tfrac{3}{8}, \quad \beta'_3[\mathcal{V}] = \beta'_4[\mathcal{V}] = \tfrac{1}{8}.$$

First let us look at voter 1. Her relative voting power according to the Bz index is the same in both WVGs, namely $\frac{1}{2}$; and her *absolute* Bz power is slightly smaller in $\mathcal{V}$ than in $\mathcal{U}$.

But this does not tell the whole story. Note that in $\mathcal{V}$ (but not in $\mathcal{U}$) voter 1 is a blocker: she can veto any bill there; thus we expect her to have much more 'obstructive' power in $\mathcal{V}$ than in $\mathcal{U}$. Indeed, a simple calculation yields $\gamma_1[\mathcal{U}] = \frac{3}{4}$, whereas $\gamma_1[\mathcal{V}] = 1$. (The value of γ for a blocker is of course always 1.) So in going from

[16] The calculation for a WVG isomorphic to $\mathcal{V}$ is done in detail in Ex. A.1.

$\mathcal{U}$ to $\mathcal{V}$ voter 1 gains quite a lot of power to prevent action. On the other hand, she loses much power to initiate action: $\gamma^*{}_1[\mathcal{U}] = \frac{3}{4}$, as compared to $\gamma^*{}_1[\mathcal{V}] = \frac{5}{11}$. So her small loss of Bz power is in fact a consequence of two mutually contrary changes.

Turning to voter 2, we see that in going from $\mathcal{U}$ to $\mathcal{V}$ he gains Bz power. In fact, he also gains in terms of γ (from $\frac{1}{4}$ in $\mathcal{U}$ to $\frac{3}{5}$ in $\mathcal{V}$) as well as γ^* (from $\frac{1}{4}$ in $\mathcal{U}$ to $\frac{3}{11}$ in $\mathcal{V}$).

The remaining two voters lose Bz power in going from $\mathcal{U}$ to $\mathcal{V}$. They also lose in terms of γ (from $\frac{1}{4}$ in $\mathcal{U}$ to $\frac{1}{5}$ in $\mathcal{V}$) as well as γ^* (from $\frac{1}{4}$ in $\mathcal{U}$ to $\frac{1}{11}$ in $\mathcal{V}$).

3.2.24 Example A somewhat different behaviour is displayed by the following pair of WVGs:

$$\mathcal{U} = [6; 5, 3, 1, 1, 1], \quad \mathcal{V} = [7; 5, 3, 1, 1, 1].$$

Here the values of the Bz measure are:

$$\begin{gathered} \beta'_1[\mathcal{U}] = \tfrac{7}{8}, \quad \beta'_2[\mathcal{U}] = \beta'_3[\mathcal{U}] = \beta'_4[\mathcal{U}] = \beta'_5[\mathcal{U}] = \tfrac{1}{8}; \\ \beta'_1[\mathcal{V}] = \tfrac{3}{4}, \quad \beta'_2[\mathcal{V}] = \tfrac{1}{4}, \quad \beta'_3[\mathcal{V}] = \beta'_4[\mathcal{V}] = \beta'_5[\mathcal{V}] = \tfrac{1}{8}. \end{gathered}$$

In going from $\mathcal{U}$ to $\mathcal{V}$, voter 1 becomes a blocker, but loses Bz power, as in the previous example. However, here she also loses a certain amount *relative* voting power as measured by the Bz index: $\beta_1[\mathcal{U}] = \frac{7}{11}$, whereas $\beta_1[\mathcal{V}] = \frac{6}{11}$.

Also, as in the previous example, in going from $\mathcal{U}$ to $\mathcal{V}$ voter 1 gains some 'obstructive' power: from $\gamma_1[\mathcal{U}] = \frac{7}{8}$ to $\gamma_1[\mathcal{V}] = 1$. But here her loss of power to initiate action is very considerable: from $\gamma^*{}_1[\mathcal{U}] = \frac{7}{8}$ to $\gamma^*{}_1[\mathcal{V}] = \frac{3}{5}$.

This example shows even more clearly than the preceding one that a voter may not benefit by becoming a blocker as a result of an increase of the quota in a WVG.

3.2.25 Comment In view of Rem. 3.2.21(iii), the *relative* measure of voting power derived from both of the Coleman measures is simply the Bz index. However, as Exs. 3.2.22, 3.2.23 and 3.2.24 illustrate, γ and γ^* can give you information that you cannot get by looking at β' alone.[17]

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[17] We shall return to the same theme in Com. 3.3.21.

3.3 Sensitivity[18]

Since β'_x is the probability that voter x is able to tip the balance, the sum of all β'_x can be regarded as a measure of the *sensitivity*[19] of the decision rule: the ease with which it responds to fluctuations in the voters' wishes. More formally:

3.3.1 Definition For any SVG $\mathcal{W}$ we put:

$$\mathrm{H}[\mathcal{W}] =_{\mathrm{def}} \sum_{x \in N} \eta_x[\mathcal{W}],$$

$$\Sigma[\mathcal{W}] =_{\mathrm{def}} \frac{\mathrm{H}[\mathcal{W}]}{2^{n-1}} = \sum_{x \in N} \beta'_x[\mathcal{W}], \qquad \sigma[\mathcal{W}] =_{\mathrm{def}} \frac{\Sigma[\mathcal{W}]}{n}.$$

We refer to $\mathrm{H}[\mathcal{W}]$ and $\sigma[\mathcal{W}]$ respectively as the *total Bz score* and *mean Bz power* in $\mathcal{W}$. We call $\Sigma[\mathcal{W}]$ the *sensitivity* of $\mathcal{W}$.

3.3.2 Comment It is easy to verify that if a is a voter of $\mathcal{W}$ then a's score in $\mathcal{W}|\&_\emptyset$ — the SVG obtained[20] from $\mathcal{W}$ by adding a new dummy voter $\&_\emptyset$ — is double a's score in $\mathcal{W}$. This is because for every coalition T in which a is $\mathcal{W}$-critical, there are two, T and $T \cup \{\&_\emptyset\}$, in which a is $\mathcal{W}|\&_\emptyset$-critical.

Conversely, if d is a dummy in $\mathcal{W}$ then in the subgame[21]

$$\mathcal{V} = \{X \in \mathcal{W} : d \notin X\},$$

obtained from $\mathcal{W}$ by removing d, the Bz score of each voter is half of what it was in $\mathcal{W}$.

Therefore, when a dummy is added or removed from an SVG, the Bz powers of all other voters, as well as the sensitivity of the whole SVG, are unchanged. (In this connection, cf. Rm. 2.3.5(i).) We can express this by saying that β' and Σ *ignore dummies.*

[18] The material in this section is partly drawn from [27].

[19] Other terms that have been used in the literature for this or for closely related concepts are 'volatility', 'degree of suspense', 'democratic participation' and 'responsiveness'.

[20] See Rem. 2.3.24(iii).

[21] See Def. 2.3.21.

On the other hand, σ does not ignore dummies: if a new dummy is added to an SVG with n voters, then the mean Bz power shrinks by a factor of $n/(n+1)$.

Σ and σ are of interest in their own right: they convey important information — the former about an SVG globally, and the latter about the 'average voter' in it. H by itself does not mean very much, but it is useful as a stepping stone in computing Σ and σ. Note that if a new dummy is added to an SVG, the value of H is doubled.

To compute H directly from Def. 3.3.1, you first have to compute η_x for all $x \in N$. Instead, it may be easier to use one of the following two identities.

3.3.3 Theorem *For any SVG $\mathcal{W}$,*

$$\mathrm{H}[\mathcal{W}] \;=\; \sum_{X \in \mathcal{W}} (2|X| - n) \;=\; \sum_{Y \subseteq N \,:\, Y \notin \mathcal{W}} (n - 2|Y|),$$

where N is the assembly of $\mathcal{W}$ and $n = |N|$.

Proof Take the identity in Cor. 3.2.11 and sum both sides over all $a \in N$. The left-hand side adds up to H.

On the right-hand side, the term $2\omega_a$ adds up to $\sum_{a \in N} 2\omega_a$. To this sum each winning coalition contributes 2 units for each of its members; so this sum equals $\sum_{X \in \mathcal{W}} 2|X|$.

The second term, $-\omega$, adds up to $-n\omega$, which is the same as adding up $-n$ as many times as there are winning coalitions. This proves the first identity claimed by our theorem.

To prove the second identity, observe that $\mathrm{H}[\mathcal{W}] = \mathrm{H}[\mathcal{W}^*]$ by Thm. 3.2.7. Hence by the first identity, which we have just proved,

$$\mathrm{H}[\mathcal{W}] \;=\; \sum_{X \in \mathcal{W}^*} (2|X| - n).$$

For any coalition X, let $Y = N - X$. Then $X \in \mathcal{W}^*$ iff $Y \notin \mathcal{W}$; also, $|X| = n - |Y|$. Substituting this in the last identity, we get

$$\mathrm{H}[\mathcal{W}] \;=\; \sum_{Y \subseteq N \,:\, Y \notin \mathcal{W}} (n - 2|Y|),$$

as claimed. ∎

The following result throws additional light on the significance of $\Sigma[\mathcal{W}]$. First we define an important random variable.

3.3.4 Definition For any SVG $\mathcal{W}$ we define the random variable $\mathsf{Z}[\mathcal{W}]$ by stipulating that for any bipartition B of the assembly N of $\mathcal{W}$, the value of $\mathsf{Z}[\mathcal{W}]$ at B is the number of voters who agree with the outcome of B in $\mathcal{W}$ minus the number of voters who do not agree with the outcome of B in $\mathcal{W}$.

We can now prove

3.3.5 Theorem *For any SVG $\mathcal{W}$,*

$$\mathrm{E}(\mathsf{Z}[\mathcal{W}]) = \Sigma[\mathcal{W}].$$

Proof For each voter x and bipartition B, put

$$\mathsf{Z}_x B =_{\mathrm{def}} (Bx)(\mathsf{W}B).$$

Thus $\mathsf{Z}_x B$ equals 1 or -1 according as x agrees or does not agree with the outcome of B. From Thm. 3.2.16 it follows by an easy calculation that $\mathrm{E}(\mathsf{Z}_x) = \beta'_x$. But clearly

$$\mathsf{Z}[\mathcal{W}] = \sum_{x \in N} \mathsf{Z}_x.$$

Hence

$$\mathrm{E}(\mathsf{Z}[\mathcal{W}]) = \sum_{x \in N} \beta'_x = \Sigma,$$

as claimed. ∎

Majority SVGs are particularly important both in theory and in applications; so we would like to calculate their sensitivity. First let us introduce a convenient notation.

3.3.6 Definition For any positive integer n we put:

$$\mathrm{H}_n =_{\mathrm{def}} \mathrm{H}[\mathcal{M}_n], \quad \Sigma_n =_{\mathrm{def}} \Sigma[\mathcal{M}_n], \quad \sigma_n =_{\mathrm{def}} \sigma[\mathcal{M}_n].$$

3.3.7 Remarks (i) Since $\mathcal{M}_n$ is symmetric, σ_n is in fact the Bz power of each of its voters, as well as of each voter in any other majority SVG with exactly n voters.

(ii) In the special case where $\mathcal{W}$ is a majority SVG, the random variable $\mathsf{Z}[\mathcal{W}]$ coincides with the margin[22] M. So in this special case Thm. 3.3.5 yields: $\mathrm{E}(\mathsf{M}) = \Sigma_n$.

This means that in any set N of n voters, the expected value of the margin is Σ_n.

To calculate the quantities we have just defined, let m be the least integer greater than $n/2$ (thus, using the conventional square bracket notation, $m = [n/2] + 1$). A voter i of $\mathcal{M}_n$ is critical in a coalition S iff $i \in S$ and $|S| = m$. The Bz score of i is equal to the number of different ways the remaining $m - 1$ members of S can be chosen, which is $\binom{n-1}{m-1}$. Hence the *total* Bz score in $\mathcal{M}_n$ is $n\binom{n-1}{m-1}$, which equals $m\binom{n}{m}$. We have thus proved

3.3.8 Theorem *For every positive integer n*

$$\mathrm{H}_n = m\binom{n}{m}, \quad \Sigma_n = \frac{m}{2^{n-1}}\binom{n}{m}, \quad \sigma_n = \frac{m}{n2^{n-1}}\binom{n}{m},$$

where m is the least integer greater than $n/2$. ∎

3.3.9 Remark From the closed formula we have just obtained for σ_n the following recursion formulas can be derived by straightforward induction on n:

$$\sigma_1 = 1, \quad \sigma_2 = \tfrac{1}{2}; \quad \sigma_{2n+1} = \sigma_{2n}, \quad \sigma_{2n+2} = \frac{2n+1}{2n+2}\sigma_{2n}.$$

We recall Stirling's well-known approximation formula for $n!$:

$$n! \sim n^n e^{-n}\sqrt{2\pi n},$$

where '$\sim$' means that the ratio of the two sides tends to 1 as n increases. From this it is a routine matter to obtain the following

[22] See Def. 2.3.8.

3.3.10 Approximations

$$\mathrm{H}_n \sim 2^n \sqrt{\frac{n}{2\pi}}, \quad \Sigma_n \sim \sqrt{\frac{2n}{\pi}}, \quad \sigma_n \sim \sqrt{\frac{2}{n\pi}}.$$

The relative error of these approximations is roughly in inverse proportion to n. For $n = 10$, the approximation is well within 3% of the true value (actually, more like 2.6%). For $n = 100$, the relative error is well under 0.3%, and so on.[23]

Among all SVGs with a given number of voters, which are least sensitive?

3.3.11 Theorem *For any SVG $\mathcal{W}$ with exactly n voters,*

$$\mathrm{H}[\mathcal{W}] \geq n;$$

and the minimal value n is achieved iff $n = 2$ or $\mathcal{W}$ is isomorphic to $\mathcal{B}_n$ or to its dual $\mathcal{M}_{n,1}$.

Proof We use induction on n.

For $n = 1$, our claim is true because all SVGs with exactly one voter are isomorphic to $\mathcal{B}_1$, for which $\mathrm{H} = 1$. (By the way, $\mathcal{B}_1$ coincides with $\mathcal{M}_{1,1}$.)

For $n = 2$, observe that if $\mathcal{W}$ has exactly two voters then it must be isomorphic to $\mathcal{B}_2$ or $\mathcal{M}_{2,1}$; or else it has one dummy and one dictator. In each of these three cases it is easy to calculate directly that $\mathrm{H}[\mathcal{W}] = 2$, as claimed.

Now assume that $n > 2$ and that our claim is true for $n - 1$. Let $\mathcal{W}$ be an SVG with exactly n voters. There are two cases to consider.

Case 1: $\mathcal{W}$ has no dummies. In this case $\eta_a \geq 1$ for every $a \in N$, so $\mathrm{H} \geq n$.

Also, if $\mathcal{W}$ is isomorphic to $\mathcal{B}_n$ or to $\mathcal{M}_{n,1}$, it is easy to see that $\eta_a = 1$ for every $a \in N$, so $\mathrm{H} = n$.

[23] A proof of Stirling's formula can be found in books on the calculus or on probability theory; for example, see [28, vol. I, pp. 50–53]. Note also formula (12.20) in [28, vol. I, p. 63], from which our approximations can be derived more directly.

Conversely, suppose $\mathrm{H} = n$, so that $\eta_a = 1$ for every $a \in N$. We must show that $\mathcal{W}$ is isomorphic to $\mathcal{B}_n$ or to $\mathcal{M}_{n,1}$. Observe that each $a \in N$ belongs to some MWC, because a is not a dummy. But a cannot belong to two distinct MWCs, because then a would be critical in both and have Bz score > 1. Thus each voter belongs to *exactly* one MWC.

If $\mathcal{W}$ has just one MWC, then all the voters must belong to it, hence $\mathcal{W} \cong \mathcal{B}_n$. On the other hand, suppose $\mathcal{W}$ has more than one MWC. Then these must all be singletons. For let S and T be distinct MWCs and suppose T had two distinct members, say b and c. Then any $a \in S$ would be critical in both $S \cup \{b\}$ and $S \cup \{c\}$, and have Bz score > 1. Thus all the MWCs of $\mathcal{W}$ are indeed singletons, so that $\mathcal{W} \cong \mathcal{M}_{n,1}$.

Case 2: $\mathcal{W}$ has at least one dummy, d. Let $\mathcal{V}$ be the subgame obtained from $\mathcal{W}$ by removing d. Then by our induction hypothesis

$$\mathrm{H}[\mathcal{V}] \geq n - 1.$$

As we noted in Com. 3.3.2, $\mathrm{H}[\mathcal{W}] = 2\mathrm{H}[\mathcal{V}]$. Hence

$$\mathrm{H}[\mathcal{W}] \geq 2(n-1) = n + (n-2).$$

But $n > 2$, so $\mathrm{H}[\mathcal{W}] > n$. ∎

Our next task is to find, among all SVGs with n voters, those which are most sensitive. We start by proving a very simple but quite powerful combinatorial lemma, which tells us what happens to the Bz scores of voters when one of the winning coalitions of an SVG is eliminated. If $\mathcal{W}$ is an SVG and $T \in \mathcal{W}$, then $\mathcal{W} - \{T\}$ may not be an SVG. However, if T is an MWC of $\mathcal{W}$ but is not its whole assembly N, then it is easy to see that $\mathcal{W} - \{T\}$ satisfies all three conditions of Def. 2.1.1 and is therefore an SVG. In this case, how do the Bz scores of voters in $\mathcal{W} - \{T\}$ compare with their scores in $\mathcal{W}$?

3.3.12 Lemma *Assume that T is an MWC of an SVG $\mathcal{W}$ and that T is not the whole assembly N of $\mathcal{W}$. Let $\mathcal{V} = \mathcal{W} - \{T\}$. Then*

$$\eta_a[\mathcal{V}] = \begin{cases} \eta_a[\mathcal{W}] - 1 & \text{if } a \in T, \\ \eta_a[\mathcal{W}] + 1 & \text{if } a \in N - T. \end{cases}$$

Proof Look again at that useful identity in Cor. 3.2.11, in the case of $\mathcal{V}$ as well as of $\mathcal{W}$.

First note that since $\mathcal{V}$ has one winning coalition less than $\mathcal{W}$, it follows that $\omega[\mathcal{V}] = \omega[\mathcal{W}] - 1$.

Next, take any $a \in T$. Then $\omega_a[\mathcal{V}] = \omega_a[\mathcal{W}] - 1$, because T is no longer a winning coalition in $\mathcal{V}$. Hence by Cor. 3.2.11 we have

$$\eta_a[\mathcal{V}] = \eta_a[\mathcal{W}] - 1,$$

as claimed.

On the other hand, if $a \notin T$ then $\omega_a[\mathcal{V}] = \omega_a[\mathcal{W}]$, so Cor. 3.2.11 yields in this case

$$\eta_a[\mathcal{V}] = \eta_a[\mathcal{W}] + 1,$$

again, as claimed. ∎

Observe that in Lemma 3.3.12 T is a nonempty *maximal* losing coalition of $\mathcal{V}$, that is, it is a losing coalition that is not included in any other losing coalition. In fact, by turning the lemma around and interchanging the names of the two SVGs concerned, we get

3.3.13 Corollary *Assume that T is a nonempty maximal losing coalition of an SVG $\mathcal{W}$. Let $\mathcal{V} = \mathcal{W} \cup \{T\}$. Then*

$$\eta_a[\mathcal{V}] = \begin{cases} \eta_a[\mathcal{W}] + 1 & \text{if } a \in T, \\ \eta_a[\mathcal{W}] - 1 & \text{if } a \in N - T. \end{cases}$$ ∎

We are now ready to determine which SVGs with a given number of voters have maximal sensitivity.

3.3.14 Theorem *Let N be a finite set with $|N| = n \geq 1$. Among all SVGs $\mathcal{W}$ that have N as assembly,* $\mathrm{H}[\mathcal{W}]$ *attains its maximal value iff $\mathcal{W}$ satisfies the following two conditions:*

(1) *Every coalition T such that $|T| < n/2$ loses in $\mathcal{W}$;*

(2) *Every coalition T such that $|T| > n/2$ wins in $\mathcal{W}$.*

Moreover, this maximal value is H_n.

Proof First let us show that conditions (1) and (2) are necessary. Suppose that (1) is violated: $\mathcal{W}$ has a winning coalition with less than $n/2$ members. Then, included in such a coalition, there must be a *minimal* winning coalition T, and the number of members of T is certainly $< n/2$. Now look at the SVG $\mathcal{V}$ of Lemma 3.3.12: according to that lemma, in going from $\mathcal{W}$ to $\mathcal{V}$, each member of T loses one point of Bz score, and each member of $N - T$ gains one point. So, all in all, the total Bz score in $\mathcal{V}$ must be higher than in $\mathcal{W}$, hence $\mathrm{H}[\mathcal{W}]$ cannot be maximal.

Similarly, if $\mathcal{W}$ violates (2) then, using Cor. 3.3.13, we get an SVG $\mathcal{V}$ with the same assembly but with greater total Bz score than $\mathcal{W}$.

So the SVGs with assembly N that have the greatest possible total Bz score must be some, or all, of those SVGs that do satisfy (1) and (2).

If n is odd, there is just one such SVG: the majority SVG, whose total Bz score is H_n — which proves our theorem in this case.

If n is even, the majority SVG is only one of several satisfying (1) and (2), because these conditions say nothing about the status of coalitions T having exactly $n/2$ members. Suppose however that $\mathcal{W}$ satisfies (1) and (2) and has a winning coalition T with exactly $n/2$ members. By (1), T must be a *minimal* winning coalition. In this case the $\mathcal{V}$ of Lemma 3.3.12 has the same total Bz score as $\mathcal{W}$, because gains in Bz score are exactly balanced by losses. And $\mathcal{V}$ of course still satisfies (1) and (2).

Proceeding in this way, we can eliminate all winning coalitions of size $n/2$, without altering the total Bz score. We end up with the majority SVG. Thus all SVGs satisfying (1) and (2) have the same total Bz score, H_n, which must be the maximum. ∎

3.3.15 Remarks (i) Instead of using Lemma 3.3.12, you can deduce Thm. 3.3.14 from Thm. 3.3.3. In a moment we shall give an even simpler proof of Thm. 3.3.14, using a rather different approach (see Rem. 3.3.18(i)). We have presented the longer proof above (borrowed from [27, p. 107]) because it can be readily adapted to yield a more general result that we intend to prove later on (see Thm. 3.4.9).

(ii) Thm. 3.3.14 could also be stated and proved in terms of bipartitions rather than coalitions. Conditions (1) and (2) then assume a very simple form: *For every bipartition B, if the sum $\mathsf{S}B$ is negative then $\mathsf{W}B = -1$; and if $\mathsf{S}B$ is positive then $\mathsf{W}B = 1$.*

3.3.16 Definition For any SVG $\mathcal{W}$ with assembly N, we define the random variable $\mathsf{D}[\mathcal{W}]$, called the *majority deficit* [*of* $\mathcal{W}$], on the Bernoulli space $\mathbf{B}_N$ by stipulating that the value of $\mathsf{D}[\mathcal{W}]$ at any bipartition B of N equals the size of the majority in B minus the number of voters who agree with the outcome of B in $\mathcal{W}$.

Thus, if the voters who agree with the outcome of B in $\mathcal{W}$ *are* the majority, or if the voters are evenly divided, then the value of $\mathsf{D}[\mathcal{W}]$ at B is 0; but if the voters who agree with the outcome of B in $\mathcal{W}$ are the minority then the value of $\mathsf{D}[\mathcal{W}]$ at B equals that of the margin $\mathsf{M}B$.

Further, we put $\Delta[\mathcal{W}] =_{\text{def}} \mathrm{E}(\mathsf{D}[\mathcal{W}])$. We call $\Delta[\mathcal{W}]$ the *mean majority deficit* of $\mathcal{W}$.

3.3.17 Theorem *For any SVG $\mathcal{W}$ with n voters,*

$$\Delta[\mathcal{W}] = \frac{\Sigma_n - \Sigma[\mathcal{W}]}{2}.$$

Proof By Rem. 2.3.9(i), $\mathsf{M}B$ is the margin by which the size of the majority in B exceeds that of the minority. Hence the size of the majority camp itself is given by

$$\frac{n + \mathsf{M}B}{2}.$$

Similarly, the number of voters who agree with the outcome of B is

$$\frac{n + \mathsf{Z}B}{2},$$

where Z is the $\mathsf{Z}[\mathcal{W}]$ of Def. 3.3.4. Thus by Def. 3.3.16 we have:

$$\mathsf{D}B = \frac{n + \mathsf{M}B}{2} - \frac{n + \mathsf{Z}B}{2} = \frac{\mathsf{M}B - \mathsf{Z}B}{2}.$$

We now apply the operator E to both sides. By Def. 3.3.16 we obtain on the left-hand side $\Delta[\mathcal{W}]$; whereas by Thm. 3.3.5 and Rem. 3.3.7(ii) we obtain on the right-hand side

$$\frac{\Sigma_n - \Sigma[\mathcal{W}]}{2},$$

as claimed. ∎

3.3.18 Remarks (i) By Thm. 3.3.17, maximizing $\Sigma[\mathcal{W}]$ is equivalent to minimizing $\Delta[\mathcal{W}]$. From this, Thm. 3.3.14 can be easily derived. Indeed, the majority deficit D is by definition always non-negative; and it is easy to see that D vanishes everywhere in $\mathbf{B}$ — and thus $\Delta[\mathcal{W}]$ is minimized — iff $\mathcal{W}$ satisfies conditions (1) and (2) of Thm. 3.3.14.

(ii) $\Delta[\mathcal{W}]$ may be regarded as a measure of the mean absolute difference between $\mathcal{W}$ and the majority SVG with the same assembly.

By Thms. 3.3.11 and 3.3.14 and Appx. 3.3.10, the ratio between the highest and lowest sensitivity possible for an n-voter SVG $\mathcal{W}$ is approximately exponential in n. Hence a rough measure of the *relative* (or *comparative*) sensitivity of $\mathcal{W}$ is obtained by locating it on the logarithmic scale between the two extreme values. Equivalently, we locate $\mathrm{H}[\mathcal{W}]$ on the logarithmic scale between the two extreme values of H. This leads to the following

3.3.19 Definition We put

$$\mathrm{S}[\mathcal{W}] =_{\mathrm{def}} \frac{\log \mathrm{H}[\mathcal{W}] - \log n}{\log \mathrm{H}_n - \log n},$$

where n is the number of voters of $\mathcal{W}$. We call $\mathrm{S}[\mathcal{W}]$ the *relative sensitivity* of $\mathcal{W}$.

3.3.20 Remarks (i) This quantity is defined in [31], where it is denoted by 'R' and called the *responsiveness index.*

(ii) Clearly, $\mathrm{S}[\mathcal{W}]$ is always between 0 and 1, assuming its minimal value 0 for the least sensitive SVGs and its maximal value 1 for the most sensitive ones.

3.3.21 Comment While sensitivity is a very important global characteristic of an SVG, it leaves out some vital information about the latter's overall nature.

As we saw in Exs. 3.2.22 and 3.2.23, the Bz power of a voter is a synthesis of power to initiate action and power to prevent it. The Bz power does not distinguish between these two aspects of power, which it treats as interchangeable. Technically, this is expressed by Thm. 3.2.7: the Bz score, index and power do not distinguish between an SVG $\mathcal{W}$ and its dual $\mathcal{W}^*$. The same applies also to sensitivity.

Consequently, SVGs that have similar sensitivities can differ radically in their propensities to approve bills. Suppose that a given SVG $\mathcal{W}$ is very 'complaisant', in the sense that the a priori probability p of a positive outcome is very close to 1. Then its dual, $\mathcal{W}^*$, is very resistant: here that probability is $1 - p$, which is close to 0. Yet the two SVGs have the same sensitivity.

For this reason we introduce a *resistance coefficient*, which is another important global characteristic of an SVG, alongside sensitivity.

3.3.22 Definition For any SVG $\mathcal{W}$ we put

$$\mathrm{R}[\mathcal{W}] =_{\mathrm{def}} \frac{2^{n-1} - \omega[\mathcal{W}]}{2^{n-1} - 1}.$$

We call $\mathrm{R}[\mathcal{W}]$ the *resistance coefficient* of $\mathcal{W}$.

3.3.23 Remarks (i) R is closely related to what Coleman [20] called 'the power of a collectivity to act' and denoted by 'A'. His A was equal to the a priori probability of a positive outcome, which is $\omega/2^n$. Our R is obtained from A by a simple linear transformation.

While A can be regarded as measuring the complaisance of an SVG, R is designed to measure the opposite of complaisance.

(ii) It is easy to see that

$$\mathrm{R}[\mathcal{W}] = -\mathrm{R}[\mathcal{W}^*].$$

Moreover, $\mathrm{R}[\mathcal{W}]$ assumes its greatest value, 1, iff $\mathcal{W}$ is a unanimity SVG; hence it assumes its least value, -1, iff $\mathcal{W}$ is the dual of a unanimity SVG.

3.3.24 Examples (i) For the SVGs considered in Ex. 3.2.22 we find that $\mathrm{R}[\mathcal{B}_3] = 1$ whereas $\mathrm{R}[\mathcal{M}_4] = \frac{3}{7}$.

(ii) For the SVGs of Ex. 3.2.23 we get $\mathrm{R}[\mathcal{U}] = 0$ ($\mathcal{U}$ is self-dual!) and $\mathrm{R}[\mathcal{V}] = \frac{3}{7}$.

Note that $\mathcal{M}_4$ and $\mathcal{V}$ have the same coefficient of resistance. However, they differ in sensitivity: $\Sigma[\mathcal{M}_4] = \Sigma_4 = \frac{3}{2}$ whereas $\Sigma[\mathcal{V}] = \frac{5}{4}$.

(iii) Let $\mathcal{W}$ be an SVG whose number n of voters is even. Suppose that $\mathcal{W}$ has an MWC T such that $|T| = n/2$. Let $\mathcal{V} = \mathcal{W} - \{T\}$. Then from Lemma 3.3.12 (or from Thm. 3.3.3) it follows that $\mathcal{V}$ has the same sensitivity as $\mathcal{W}$. But clearly $\mathcal{V}$ has a greater resistance coefficient than $\mathcal{W}$.

3.4 The Two Square-Root Rules[24]

In modern democratic states, citizens' participation in decision making is largely *indirect*: most political decisions are made by a variety of governing bodies composed of representative agents elected by the citizens. In this two-stage process, citizens exercise direct power at the first stage, in electing their representatives; but only indirect power, via these representatives, at the second stage, when actual political decisions are made.

A widely accepted principle, regarded as necessary for ensuring fairness of the decision-making process, is expressed by the catchphrase 'one person, one vote' (abbreviated 'OPOV'). This means that suffrage ought to be not only *universal*,[25] but also *equal*: one citizen's vote ought to be worth as much as another's. Equality of 'worth' may be taken in a narrow sense, as pertaining exclusively to the first stage of the political process, that of electing the representatives. But surely this stage is only a means to an end; so the ideal of fairness is subverted if there are great disparities in

[24]The material in this section is partly drawn from [34].

[25]The intended connotation of universality explains the shift from the older form of the catchphrase — 'one *man*, one vote' — to the present one.

the indirect decision-making powers of citizens. This suggests a broader sense of the principle of equal suffrage: the power wielded by elected representatives in a decision-making body ought to reflect the respective numbers of their electors.

Let us consider systems in which the representative agents are individuals, elected by constituencies whose size is known in advance but which are in general politically heterogeneous. Typically, a constituency is the population of a district or — in the case of federal or international bodies — a whole state.[26]

If all constituencies are of equal size, and each has a single representative on a decision-making council, then the principle of equal suffrage can evidently be satisfied in a straightforward way, by using a pairwise symmetric decision rule in the council, thus ensuring that all representatives have equal a priori power.[27] Then — provided all citizens have equal a priori degree of influence on the way their respective representatives vote in each division of the council — the ultimate (indirect) voting power of all citizens will be equal as well.

In practice, constituencies of *exactly* equal size can rarely be achieved; but the same solution works also, as a good approxima-

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[26]We are thus leaving out of consideration systems of *proportional representation*, which can arguably also serve as a means of achieving equal suffrage in the broad sense. The underlying idea of such a system is that each sector of opinion, each group of like-minded citizens, should be represented by a bloc of decision-makers whose weight is proportional to the group's size. A legislature elected in this way is a microcosm of the electorate; so each of its meetings may be regarded as a scaled-down general assembly of all citizens. When the legislature divides on some issue, the outcome is presumed to be the same as it would have been in a referendum.

A standard way of implementing thorough-going proportional representation is the party-list system: instead of voting for an individual candidate, each citizen votes for a list of candidates standing for some political party. Seats in the legislature are apportioned among the lists in proportion to the respective number of votes cast for them. In such a system the role of representing agent is played by a party bloc rather than by an individual; and each such agent represents a 'constituency' that is politically homogeneous to a great extent. The purpose of the election is to determine the *size* of this constituency.

[27]See Def. 2.1.7 and Rem. 2.1.8.

tion, if the constituencies are not exactly of equal size, provided the differences in size are sufficiently small to be negligible.

But what of the general case, in which the constituencies may vary quite widely in size? In order to provide an (approximate) answer to this problem, we first introduce the following theoretical model of a two-tier voting system.

3.4.1 Model We consider the composite SVG

$$\mathcal{W} = \mathcal{V}[\mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_m].$$

In this connection we use the notation and terminology introduced in Def. 2.3.12. Recall, in particular, that the assembly of the top, $\mathcal{V}$, is $I_m = \{1, 2, \dots, m\}$ and that the assembly N of the composite $\mathcal{W}$ is the union of the assemblies N_i of the components $\mathcal{W}_i$. Note also that in accordance with Conv. 2.3.1(i) we put $n = |N|$ and $n_i = |N_i|$ for each $i \in I_m$.

We further assume that the N_i are pairwise disjoint. We refer to N_i as the *i-th constituency* of the composite $\mathcal{W}$. We refer to any member of N as a *citizen*. Thus a citizen is a voter of $\mathcal{W}$, and a member of a unique constituency. For each $i \in I_m$ we refer to i as the *delegate* of the component $\mathcal{W}_i$. We refer to $\mathcal{V}$ as the *council* of $\mathcal{W}$.

Next, we assume that each of the components $\mathcal{W}_i$ is a majority SVG: $\mathcal{W}_i \cong \mathcal{M}_{n_i}$.

Finally, we assume that the sizes n_i of the constituencies are so large that the errors in the approximations

$$\sigma_{n_i} \sim \sqrt{\frac{2}{n_i \pi}}$$

(see Appx. 3.3.10) are negligibly small.

3.4.2 Remarks (i) Regarding the use of a composite SVG to model a two-tier voting system, see Rem. 2.3.13(ii). The additional assumption made in the present model, that the constituencies are pairwise disjoint, is self-explanatory.

(ii) In view of our assumption that each $\mathcal{W}_i$ is a majority SVG, the working of Model 3.4.1 may be interpreted as follows. Decisions of the composite $\mathcal{W}$ are made by division of the council $\mathcal{V}$. Before a division on a given bill, the views of the citizens in each constituency are ascertained by plebiscite (or by some other reliable means). In the division of the council, delegate i votes 'yes' if a majority of the citizens in N_i are for the bill; otherwise, i votes 'no'.

(iii) If n_i is odd, the a priori probability that delegate i will vote 'yes' on a given bill is of course exactly $\frac{1}{2}$. But by virtue of our final assumption that the n_i are very large, we may also assume that where n_i is even the probability that delegate i will vote 'yes' is as close to $\frac{1}{2}$ as makes no difference, because the probability that the citizens of N_i will be evenly split is negligibly small.

Also, the votes of the delegates are mutually independent, since the vote of delegate i depends only on the votes of citizens in the i-th constituency, and the constituencies are pairwise disjoint.

Now, as we saw in Rem 2.3.13(vi), there is a natural quotient map t that assigns to each bipartition B of N a bipartition $\mathrm{t}B$ of I_m. The reasoning in the preceding two paragraphs proves that the probability of an event $\mathbf{G}$ in $\mathbf{B}_{I_m}$ is as near as makes no difference to the probability of $\mathrm{t}^{-1}[\mathbf{G}]$ in $\mathbf{B}_N$.

Stated less formally: if $\mathbf{G}$ is any event defined in terms of division of the council and votes of the delegates, without direct reference to the votes of citizens, then the probability $\mathrm{P}(\mathbf{G})$ can be computed, with negligible error, as though $\mathcal{V}$ were a stand-alone SVG rather than the top of the composite.

3.4.3 Theorem (Penrose's Square-Root Rule) *In Model 3.4.1, the citizens' indirect Bz powers $\beta'_x[\mathcal{W}]$ are equal for all $x \in N$ (with negligible error) iff the Bz powers $\beta'_i[\mathcal{V}]$ of the delegates i in the council are proportional to the respective $\sqrt{n_i}$.*

Proof Consider a particular citizen x in constituency N_i. We claim that

$$\beta'_x[\mathcal{W}] \sim \beta'_i[\mathcal{V}]\sqrt{\frac{2}{n_i \pi}}. \qquad (*)$$

By Thm. 3.2.4, $\beta'_x[\mathcal{W}]$ is equal to the probability of the event **E** that x is $\mathcal{W}$-critical. Stated more fully, **E** is the event that all other citizens vote in such a way that x's vote will decide the final fate of the bill.

Now, x's vote can have this effect iff the following two things happen: first, x's vote is able to tip the balance within x's own constituency, N_i; and, second, the vote of delegate i in the council is able to tip the balance there. Therefore **E** is the conjunction (intersection) of two events, **F** and **G**, where **F** is the event that x is $\mathcal{W}_i$-critical, and **G** is the event that delegate i's vote can tip the balance in the council.

Moreover, **F** and **G** are independent, because **F** is completely determined by the votes of the citizens of N_i, whereas **G** is determined by the votes in council of all delegates other than i, which in turn are determined by the votes of citizens of all constituencies other than N_i. Therefore,

$$\beta'_x[\mathcal{W}] = \mathrm{P}(\mathbf{F})\mathrm{P}(\mathbf{G}).$$

But $\mathrm{P}(\mathbf{F}) = \beta'_x[\mathcal{W}_i]$ by Thm. 3.2.4. And since $\mathcal{W}_i$ is a majority SVG, it follows from Rem. 3.3.7(i) that $\beta'_x[\mathcal{W}_i] = \sigma_{n_i}$. Hence by Appx. 3.3.10 we have

$$\mathrm{P}(\mathbf{F}) \sim \sqrt{\frac{2}{n_i \pi}}.$$

Also, by Rem. 3.4.2(iii), $\mathrm{P}(\mathbf{G})$ is equal, or as close as makes no difference, to $\beta'_i[\mathcal{V}]$. This proves (∗), from which our theorem follows at once. ∎

3.4.4 Remarks (i) For brevity, we shall refer to Thm. 3.4.3 as 'PSQRR'.

(ii) As far as we know, the first statement of PSQRR (which we have quoted in § 1.2) is in Penrose's paper [78, p. 57]. However, he does not give a rigorous proof of the rule, but a semi-heuristic (and, in our view, inconclusive) argument based on the fact that $\mathrm{E}(\mathsf{M}) = \Sigma_n$ (see Rem. 3.3.7(ii)). On the other hand, Banzhaf's paper [6] contains all the ingredients of a rigorous proof, but not a precise statement of PSQRR itself.

In the literature, PSQRR is often misstated as though it prescribes a weighted voting system in which the *weights*, rather than voting powers, of the delegates ought to be proportional to the square roots of the sizes of their respective constituencies. Penrose himself seems to be guilty of this ([78, p. 55]). For more recent misstatements see, for example, [37, pp. 249, 254], [41, p. 171] and [60, p. 226]. This confusion is particularly unfortunate because, as we shall see below,[28] there is in fact quite a different rule that does indeed require delegates' weights to be proportional to the $\sqrt{n_i}$.

(iii) PSQRR may not provide a unique solution to the problem of equalizing the indirect voting powers of the citizens. On the one hand, there may be *several different* decision rules in the council that conform equally well to PSQRR. For example, in the trivial case where all constituencies are of equal size, *any* pairwise symmetric SVG with assembly I_m can serve as $\mathcal{V}$. In such cases, where PSQRR provides several solutions, supplementary criteria are needed for choosing between them. For example, a proper SVG may normally be regarded as a solution superior to an improper one; a WVG may be preferred (on grounds of simplicity) to an SVG that is not a WVG; and SVGs whose sensitivity lies in a particular range may be more desirable than others.

On the other hand, as there are only finitely many SVGs $\mathcal{V}$ with assembly I_m, it may well happen that none of them yields $\beta'_i[\mathcal{V}]$ that are proportional, or very nearly proportional, to the respective $\sqrt{n_i}$. In this case the best one can do is to select a $\mathcal{V}$ that comes as close to proportionality as possible. This of course raises the question as to how 'closeness' is to be measured.[29]

The problems we have just discussed are *methodological*: they concern *how* PSQRR may be applied to equalize citizens' a priori voting powers. Quite apart from these there is a *normative* question: whether, or in what circumstances, PSQRR ought to be applied. In order to keep the two issues separate, we shall first present an

[28] Thm. 3.4.9 and Cor. 3.4.10.

[29] We discuss this issue further in Com. 4.6.4.

example in which no methodological difficulty arises. We shall then use this example to illustrate the normative issue.

3.4.5 Example Consider the instance of Model 3.4.1 in which $m = 4$, $n_1 = 9 \cdot 10^6$ and $n_i = 10^6$ for $i = 2, 3, 4$.

PSQRR prescribes an SVG $\mathcal{V}$ whose four voters — the delegates $1, 2, 3, 4$ — have Bz powers in the proportion $3 : 1 : 1 : 1$. A solution is provided by the SVG $\mathcal{V}$ whose MWCs are $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$ and $\{2, 3, 4\}$. It is in fact a WVG:

$$\mathcal{V} = [3; 2, 1, 1, 1].$$

As the reader can easily verify, here $\beta'_1[\mathcal{V}] = \frac{3}{4}$ and $\beta'_i[\mathcal{V}] = \frac{1}{4}$ for $i = 2, 3, 4$.

Moreover, this solution is unique: no other SVG comes anywhere close to the required proportion.[30]

From formula $(*)$ in the proof of Thm. 3.4.3 we obtain

$$\beta'_x[\mathcal{W}] \approx \frac{1}{4000}\sqrt{\frac{2}{\pi}}$$

for each citizen $x \in N$, where '$\approx$' denotes approximate equality. Hence for the sensitivity of the composite SVG $\mathcal{W}$ we get

$$\Sigma[\mathcal{W}] \approx 3000\sqrt{\frac{2}{\pi}} \approx 2394$$

to the nearest unit. In view of Thm. 3.3.5, this means that on the average the number of citizens who agree with a decision made by $\mathcal{W}$ (through the council $\mathcal{V}$) exceeds by approximately 2394 the number of those who disagree. In other words, out of the 12m citizens, the average number of those agreeing with the outcome is approximately 6 001 197.

[30]See [100, pp. 310–312], where all SVGs with at most four non-dummy voters are catalogued. Of course, as pointed out in Rem. 2.3.15(iii), the representation of $\mathcal{V}$ by means of a weighting system is by no means unique; here we have used the simplest one.

These figures should be compared with the corresponding ones for the 'direct democracy' majority model: the majority SVG with the 12m citizens as voters. From Appx. 3.3.10 we obtain

$$\Sigma_{12\cdot 10^6} \approx 1000\sqrt{\frac{24}{\pi}} \approx 2764$$

to the nearest unit. By Rem. 3.3.7(ii), this means that the average margin (the number by which the majority camp in a bipartition exceeds the minority camp) is approximately 2764. In other words, the average size of the majority camp is 6 001 382.

We note that in the present example the figures for the composite $\mathcal{W}$ are relatively close to those for the direct democracy model. The two models deliver results that are not far apart. Of course, they are not identical. Under $\mathcal{W}$ it may happen that only a minority of the citizens agree with the outcome. The most extreme such case is that in which 0.5m citizens in each of the three smaller constituencies are opposed to a bill, while all the remaining 10.5m citizens support it; delegates 2, 3 and 4 will then vote against the bill, and it will be defeated. Under direct democracy, the bill would obviously be passed by a margin of 9m — an enormous landslide.

But the probability of large disparities, on anything like this scale, is extremely small. To see this, consider the majority deficit D (Def. 3.3.16). Recall that whenever those citizens who agree with the outcome are the majority of all citizens, the value of D is 0; but when they are the minority, the value of D equals that of the margin M. The greatest possible value of D, corresponding to the most extreme case described in the preceding paragraph, is 9m. But by Thm. 3.3.17 the *expected* value of D, the mean majority deficit Δ, is only $(2764 - 2394)/2 = 185$.

So in the present example PSQRR delivers a unique solution that, in addition to equalizing the indirect Bz powers of all citizens, is on the average quite close in its effects to the direct democracy majority model.

3.4.6 Comment Ought PSQRR to serve as a norm in assessing or designing a particular real-life representative decision-making council? This can be a rather tricky question, the answer to which must surely depend on the extent to which the assumptions underlying Thm. 3.4.3 are realistic in the particular circumstances.

First, there is the assumption that the vote of a delegate in the council accords with the majority opinion in his or her constituency. This may be problematic.

The second assumption is even more problematic. In using the probability space $\mathbf{B}_N$, it is tacitly assumed that all citizens, irrespective of constituency, vote independently of each other; in particular, this implies that in the long run the votes of citizens of the same constituency are not more highly correlated than votes of citizens of different constituencies. This is quite realistic if the division of the citizenry into constituencies is more or less random, say a matter of mere administrative convenience, unconnected in any systematic way with the attitudes of citizens to the issues that are to be decided by the council.

As a counter-example, where this assumption is highly unrealistic, consider a county divided into four constituencies of the same sizes as in Ex. 3.4.5. But suppose that the large constituency (of 9m citizens) is urban, whereas the three smaller constituencies (of 1m citizens each) are rural. Suppose also that most of the issues to be decided by the county's council are of the kind on which opinions and interests tend to be polarized between townspeople and country-folk. If PSQRR is implemented in this situation, the three rural delegates — representing one quarter of the entire citizenry — will be able *systematically* to outvote the urban delegate. It is clear that in cases of this sort implementation of PSQRR is quite unjustified, as it may lead to a virtual dictatorship of a minority.

Note that in these circumstances ordinary majority rule under direct democracy would result in a virtual dictatorship of the 9m majority. This is bad, but not nearly so bad as minority dictatorship — particularly one that is an artefact of the decision rule used by the council.

Let us point out that minority dictatorship may also come about if the Bz powers of the delegates in the council are proportional to the sizes of their respective constituencies rather than to their square roots. As an example of this kind, consider a county divided into one urban constituency of 7m citizens and four rural constituencies, each of 1m citizens. If we take

$$\mathcal{V} = [4; 3, 1, 1, 1, 1],$$

then $\beta'_1[\mathcal{V}] = \frac{7}{8}$ and $\beta'_i[\mathcal{V}] = \frac{1}{8}$ for $i = 2, 3, 4, 5$. Here the four rural delegates, representing $\frac{4}{11}$ of the citizenry, can systematically outvote the urban delegate.

In cases of this sort, where each constituency is homogeneous and there is strong built-in polarization between constituencies, it may be argued that the *weights* assigned to delegates in the council ought to be proportional to the sizes of their constituencies. We shall refer to this arrangement as *population–weight proportionality*, briefly, PWP.

In such cases, an equivalent result would be achieved if the constituencies were subdivided to produce constituencies of equal size, each of which has one delegate in the council, and all delegates are assigned equal weight, creating, in effect, a council which is a microcosm of the citizenry.[31]

Whereas in theory we can envisage cases where PSQRR is clearly applicable as well as those where its application is wholly unwarranted, most real-life cases lie somewhere between these two extremes. Simple clear-cut prescriptions can rarely be offered without reservation.

Let us now return to the study of our theoretical two-tier system, Model 3.4.1. We pose the following

3.4.7 Problem In Model 3.4.1, with given m and n_i, set up the council $\mathcal{V}$ so as to maximize the sensitivity $\Sigma[\mathcal{W}]$ and thus minimize the mean majority deficit $\Delta[\mathcal{W}]$.

Recall that by Def. 3.3.1, $\Sigma[\mathcal{W}]$ equals the sum of the Bz powers of all the citizens in the composite SVG $\mathcal{W}$.

[31] An equivalent result would also be achieved by using proportional representation; see fn. 26, p. 64.

Going back to $(*)$ in the proof of Thm. 3.4.3, and summing over all $x \in N_i$, we get

$$\sum_{x \in N_i} \beta'_x[\mathcal{W}] \sim \kappa \beta'_i[\mathcal{V}]\sqrt{n_i},$$

where κ is a numerical constant. Summing over all $i \in I_m$, we have

$$\Sigma[\mathcal{W}] \sim \kappa \sum_{i=1}^{m} \beta'_i[\mathcal{V}]\sqrt{n_i},$$

and multiplying both sides by 2^{n-1} we obtain the following approximation for the total Bz score in $\mathcal{W}$:

$$\mathrm{H}[\mathcal{W}] \sim \kappa' \sum_{i=1}^{m} \eta_i[\mathcal{V}]\sqrt{n_i}, \tag{†}$$

where κ' is a new constant. Maximizing $\Sigma[\mathcal{W}]$ is obviously tantamount to maximizing $\mathrm{H}[\mathcal{W}]$, and it will be more convenient to deal with the latter quantity. For brevity, we introduce the following

3.4.8 Definition In connection with Model 3.4.1 we put

$$v_i =_{\mathrm{def}} \sqrt{n_i},$$

for all $i \in I_m$. Further, for any coalition X of $\mathcal{V}$ (that is, any $X \subseteq I_m$) we put

$$v(X) =_{\mathrm{def}} \sum_{i \in X} v_i.$$

Also:

$$\bar{v} =_{\mathrm{def}} v(I_m).$$

Finally, we put

$$\Theta[\mathcal{W}] =_{\mathrm{def}} \sum_{i=1}^{m} \eta_i[\mathcal{V}]v_i.$$

In the notation we have just introduced, (†) can be re-written:

$$\mathrm{H}[\mathcal{W}] \sim \kappa' \Theta[\mathcal{W}]. \qquad (\ddagger)$$

We shall now give a precise solution to the problem of maximizing $\Theta[\mathcal{W}]$.

3.4.9 Theorem *In Model 3.4.1, the quantity $\Theta[\mathcal{W}]$ attains its maximal value for given m and n_i, iff the council $\mathcal{V}$ satisfies the following two conditions:*

(1) *Every coalition T such that $v(T) < \bar{v}/2$ loses in $\mathcal{V}$;*

(2) *Every coalition T such that $v(T) > \bar{v}/2$ wins in $\mathcal{V}$.*

In particular, $\Theta[\mathcal{V}]$ attains its maximal value if

$$\mathcal{V} = [\bar{v}/2 + \epsilon; v_1, v_2, \ldots, v_m],$$

where ϵ is a sufficiently small non-negative real.

Proof We proceed as in the proof of Thm. 3.3.14. Suppose T is an MWC of $\mathcal{V}$ other than the whole assembly I_m; let

$$\mathcal{W}' = \mathcal{V}'[\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_m],$$

where $\mathcal{V}' = \mathcal{V} - \{T\}$. Then it follows from Lemma 3.3.12 that

$$\Theta[\mathcal{W}'] = \Theta[\mathcal{W}] - v(T) + v(I_m - T).$$

On the other hand, if T is a nonempty maximal losing coalition of $\mathcal{V}$ and $\mathcal{V}' = \mathcal{V} \cup \{T\}$, then by Cor. 3.3.13

$$\Theta[\mathcal{W}'] = \Theta[\mathcal{W}] + v(T) - v(I_m - T).$$

An argument similar to that used in the proof of Thm. 3.3.14 establishes the present theorem. ∎

In view of (‡), we now have:

3.4.10 Corollary (Morriss) *If the n_i are sufficiently large, the conditions of Thm. 3.4.9 provide a solution to Prob. 3.4.7, with vanishing or negligibly small error.* ∎

3.4.11 Remarks (i) We shall sometimes refer to Cor. 3.4.10 as the *second square-root rule* (SSQRR). We have attributed it to Morriss because in [70, pp. 187–189] he states the most salient part of this result: namely, that $\Sigma[\mathcal{W}]$ is maximized by taking the council to be a WVG in which delegates' weights are proportional to the square roots of the sizes of their respective constituencies. (He takes it for granted that the quota should equal, or be just greater than, half the total weight; and he omits to mention that this solution need not be the only one. In fact, there may be other solutions because Thm. 3.4.9 imposes no condition on coalitions T for which $v(T) = \bar{v}/2$; such a coalition may win or lose — cf. Ex. 3.4.13 below.) He provides no rigorous proof, but some heuristic arguments that we do not find convincing.[32]

He himself attributes his result to Penrose, citing [78]; but this seems to us mistaken, as the issue of *maximizing* total voting power is never broached in [78]. The source of the error appears to be a misstatement of PSQRR in [78], to which we have alluded in Rem. 3.4.4(ii).

(ii) In view of Rem. 3.3.18(ii), a solution to Prob. 3.4.7 may be regarded as a two-tier system which, for the given m and $\mathcal{W}_i$, comes on the average as close as possible in its effects to the direct democracy majority model: the majority SVG with all citizens as direct voters. However, the SSQRR cannot in general be recommended in a normative capacity. The assumptions underlying this theoretical result are the same as those underlying Thm. 3.4.3, which were discussed in Com. 3.4.6. But here the trouble is not caused by a possible lack of realism of these assumptions, but by the fact — pointed out in [70] — that maximizing $\Sigma[\mathcal{W}]$ may lead to extremely *unequal* distribution of indirect voting power, turning the citizens of whole constituencies into dummies. The following example illustrates this.

3.4.12 Example Consider a county made up of a constituency numbering 9m citizens and two constituencies, each numbering 1m.

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[32] See also op. cit., Note 9 on p. 249, and pp. 229–231.

Let us suppose that the assumptions underlying Thm. 3.4.3 and Cor. 3.4.10 are realistic in this instance.

First let us see how PSQRR may be applied here. It prescribes that the Bz powers of delegates 1, 2 and 3 in $\mathcal{V}$ should be in the proportion $3 : 1 : 1$. By trial and error it is easy — and by consulting the table in [100, pp. 310–312] it is even easier — to verify that there is just one acceptable solution, namely

$$\mathcal{V} = [3; 2, 1, 1],$$

for which $\beta'_1[\mathcal{V}] = \frac{3}{4}$ and $\beta'_i[\mathcal{V}] = \frac{1}{4}$ for $i = 2, 3$. (The same Bz powers are also obtained for the dual of this SVG, $\mathcal{V}^* = [2; 2, 1, 1]$, but this is an improper SVG, which is normally unacceptable.)

With this $\mathcal{V}$ we obtain $\Sigma[\mathcal{W}] \approx 2194$ to the nearest unit. This should be compared with the sensitivity of the 'direct democracy' majority model: the majority SVG with the 11m citizens as voters. Using Appx. 3.3.10 we obtain $\Sigma_{11\cdot 10^6} \approx 2646$ to the nearest unit. Hence

$$\Delta[\mathcal{W}] \approx 226.$$

Now let us see what happens if we maximize $\Sigma[\mathcal{W}]$ and minimize $\Delta[\mathcal{W}]$. Cor. 3.4.10 yields a unique solution for $\mathcal{V}$, namely

$$\mathcal{V} = [3; 3, 1, 1].$$

Here delegate 1 is a dictator, and the other two delegates, and consequently also their constituents, are dummies! This is surely unacceptable, precisely because it is purely an artefact of the choice of $\mathcal{V}$: subject to our present assumptions, the citizens of the two smaller constituencies would not be dummies under PSQRR or under direct democracy. It is no consolation that with this maximizing $\mathcal{V}$ we obtain a higher sensitivity, $\Sigma[\mathcal{W}] \approx 2394$ to the nearest unit; and for $\Delta[\mathcal{W}]$ we now have the absolute minimum for the given set-up,

$$\Delta[\mathcal{W}] \approx 126,$$

which is very small indeed.

Of course, the solutions provided by the SSQRR need not always be unacceptable. This is illustrated by the following example.

3.4.13 Example Let us go back to Ex. 3.4.5: a county that has one constituency with 9m citizens, as in Example 3.4.12, but *three* (instead of two) constituencies with 1m. We saw that PSQRR prescribes a unique solution, namely

$$\mathcal{V} = [3; 2, 1, 1, 1],$$

for which $\Sigma[\mathcal{W}] \approx 2394$ and $\Delta[\mathcal{W}] \approx 185$ to the nearest unit.

Happily, this $\mathcal{V}$ also satisfies conditions (1) and (2) of Thm. 3.4.9. Indeed, from Def. 3.4.8 we have here $v_1 = 3000$ and $v_i = 1000$ for $i = 2, 3, 4$; and, as pointed out in Rem. 3.4.11, Thm. 3.4.9 imposes no condition on coalitions T for which $v(T) = v/2$, so it allows $\{1\}$ to be a losing coalition and $\{2, 3, 4\}$ to be a winning one. Thus, according to Cor. 3.4.10 this $\mathcal{V}$ maximizes Σ and minimizes Δ.

But Cor. 3.4.10 admits also another proper solution, namely

$$\widetilde{\mathcal{V}} = [4; 3, 1, 1, 1],$$

(as well as its improper dual, $\widetilde{\mathcal{V}}^* = [3; 3, 1, 1, 1]$). Here $\beta'_1[\widetilde{\mathcal{V}}] = \frac{7}{8}$ and $\beta'_i[\widetilde{\mathcal{V}}] = \frac{1}{8}$ for $i = 2, 3, 4$. This solution yields of course the same (maximal and minimal, respectively) values for Σ and Δ but does not equalize the citizens' indirect voting powers (as measured by β'); indeed, it follows from $(*)$ in the proof of Thm. 3.4.3 that the ratio between the power of a member of the large constituency and that of her fellow-citizen in a small constituency is $7 : 3$. So on egalitarian grounds $\mathcal{V}$ is to be preferred.

3.4.14 Comment In cases where a two-tier system of decision-making is preferred or obligatory (as in federal or international bodies), it may sometimes be possible, as in Ex. 3.4.13, to satisfy the OPOV principle and at the same time minimize the MMD; but as Ex. 3.4.12 shows, the prescriptions of majoritarianism and equality of suffrage do not necessarily coincide, and may in fact sharply clash. The MMD cannot be made to vanish (except in some trivial cases, with degenerate constituencies), but it may be possible to reduce it considerably. However, this may result in an unequal — sometimes extremely unequal — distribution of voting power. How much inequality, or how high a value of the MMD, one is prepared to tolerate depends on the relative values one attaches to egalitarianism and majoritarianism.

Be that as it may, in our view the main value of Cor. 3.4.10 is descriptive rather than prescriptive. It provides a benchmark for the sensitivity of a composite SVG $\mathcal{W} = \mathcal{V}[\mathcal{W}_1, \ldots, \mathcal{W}_m]$ where the number m of constituencies and their respective sizes n_i are given. It is surely always of interest to find out how close $\mathcal{W}$ is to majority rule. To this end, the sensitivity $\Sigma[\mathcal{W}]$ of any such $\mathcal{W}$ should be compared with the maximal value achievable with the given m and n_i, specified by Cor. 3.4.10, as well as with the sensitivity Σ_n of the corresponding direct majority rule. As explained in connection with Def. 3.3.19, a logarithmic scale is more appropriate for these comparisons than a linear one.

4. Weighted Voting in the US

4.1 One Person, One Vote

Weighted voting is widely used with the intention of implementing the principle of OPOV in a two-tier decision-making system: the representatives (or delegates) in a council are assigned weights that reflect the respective sizes of their constituencies. Commonly, weights are roughly proportional to constituency sizes. However, the theory developed in § 3.4 shows that, if the indirect voting powers of the citizens are to be equalized, the voting *powers* of the representatives must be addressed; and as we pointed out in § 1.1, the latter voting powers may not be proportional to the representatives' weights. Moreover, according to PSQRR (Thm. 3.4.3) the voting powers of the representatives should be proportional to the *square roots* of the respective constituency sizes.

The question as to whether a representative's voting power ought to be considered — and if so how exactly it should be measured — became a legally contested issue in the US in the early 1960s and some cases were still before the courts in the early 1990s. In this chapter we shall outline the history of some of the more prominent cases and the courts' judgments.[1]

The US Congress is a bicameral system consisting, at present, of 435 members of the House of Representatives and 100 Senators.[2]

[1]Many decision-making bodies throughout the world use weighted voting; however, we know of no country except the US where this practice has been subject to litigation and court rulings. For a long (and rather old) list of international organizations using weighted voting see [8].

[2]In fact the Senate has 101 members because by Article I, Section 3(4),

According to Article I, Sections 2(3) and 3(1), of the Constitution, each state — regardless of the size of its population — has at least one Representative, and two Senators. The present practice is that each of the Representatives is elected in one congressional district within a state; and the total number of Representatives for each state is approximately proportional to the state's population, as counted in a national census. The census also serves to determine whether the boundaries of the various congressional and state legislative districts within a state should be reapportioned in order to ensure that all such districts have approximately equal populations.[3]

US congressional districts and state legislative districts did not always have nearly equal populations: until about 1950 it was not uncommon for districts within a state to vary greatly in this respect.

During the years 1962–64 the US Supreme Court established, in a series of four famous reapportionment cases, the unconstitutionality of maintaining grossly unequal districts.

The Supreme Court expressed its opinion on this issue for the first time in 1962, in the case of *Baker v Carr*.[4] The plaintiff voters challenged a 1901 Tennessee statute apportioning the Tennessee General Assembly. Their contention was that inequalities in the populations of Assembly districts debased the votes of constituents in large districts, thereby violating the Equal Protection clause of the 14th Amendment, Section 1, of the US Constitution: 'No State shall ... deny to any person within its jurisdiction the equal protection of the laws.'[5] A three-judge federal district court dismissed the complaint. On appeal the Supreme Court ruled that

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of the US Constitution, the US vice-president serves also as president of the Senate, with (only) a tie-breaking vote.

[3] A national census is only conducted decennially, as the Constitution prescribes, although presidential, congressional, as well as state elections are held at shorter intervals.

[4] 369 US Reports (1962), pp. 186 ff.

[5] Ibid., pp. 186–187.

the district court had jurisdiction over the subject matter,[6] that the plaintiffs, as qualified voters in the allegedly debased districts, had standing to bring suit,[7] and that the alleged denial of equal protection presented a 'justiciable constitutional cause of action'.[8]

Justice Felix Frankfurter dissented from the Court's opinion, arguing that the federal courts should not undertake to restructure state electoral processes, and warned specifically 'of the mathematical quagmire (apart from divers judicially inappropriate and elusive determinants) into which this Court today catapults the lower courts of the country'.[9]

The immediate effect of the Court's ruling was a torrent of reapportionment cases in the federal courts within the following five months, and the remand of two cases then before the Supreme Court.[10]

The next occasion on which the US Supreme Court expressed an opinion on malapportionment, albeit of a somewhat different kind, was one year later, in 1963, in the case of *Gray v Sanders*.[11] This case involved primary elections in the State of Georgia, through which the two main parties chose their nominees for statewide offices, such as governor and US senator. The parties used in their Georgia primaries a two-tier county-based system. Each party held a convention at which its nominee was chosen by weighted voting: every county had a specified weight, a number of 'unit votes', *all* of which were cast for the candidate who had received the plurality (that is, the largest number) of the votes of the party's members in that county.

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[6] Ibid., p. 204.

[7] Ibid., p. 206.

[8] Ibid., p. 237.

[9] Ibid., p. 268. This warning was to be quoted later in the *Iannucci v Board of Supervisors* 1967 case, and in Justice John M Harlan's separate opinion in the 1971 *Whitcomb v Chavis* case, which we discuss in § 4.4.

[10] For details regarding these cases see [53, fn. 11–12, p. 3] and sources cited therein.

[11] 372 US Reports (1963), pp. 368 ff.

This is similar to the winner-take-all practice in the election of a US president by the federal Electoral College. As the analogy is directly relevant, let us turn for a moment to a discussion of the system used in US presidential elections.

As is well known, the president of the US is not elected directly by the people, but by an Electoral College. The US Constitution (Article II, Section 1(2)) prescribes: 'Each State shall appoint, in such Manner as the Legislature thereof may direct, a Number of Electors, equal to the whole Number of Senators and Representatives to which the State may be entitled in the Congress . . . .' Also, by the 23rd Amendment of the Constitution, Washington DC appoints, 'in such manner as the Congress may direct', three Electors (equal to the minimal number of Electors to which a state is entitled). So at present the 50 states and Washington DC appoint altogether 538 Electors, each of whom has one vote in the Electoral College. (California, the most populous state, has 54 Electors in the College.) According to the 12th Amendment of the Constitution, a presidential candidate who gets an absolute majority of the Electors' votes (currently 270) is elected president; but if no candidate gains such a majority then the president must be elected by the House of Representatives from amongst the (up to three) leading candidates, 'the representation from each state having one vote'.[12]

Although the US Constitution does not prescribe how each state legislature should appoint its Electors, the long-standing general practice is that a state casts *all* its Electoral College votes for the presidential and vice-presidential candidates who received the plurality of the popular vote in that state. This practice of winner-

[12] So far this has happened just once: in the 1824 presidential elections, when none of the four candidates — John Quincy Adams, Andrew Jackson, William H Crawford and Henry Clay — had an absolute majority in the Electoral College. Thereupon the House of Representatives elected John Quincy Adams as the sixth president of the US. Andrew Jackson — understandably aggrieved, as he had gained the plurality of the votes in the Electoral College — accused Clay of a 'corrupt bargain' whereby the latter had granted his support to Adams in exchange for being appointed Secretary of State.

takes-all, called 'the unit rule', if applied without exception,[13] would in effect turn the Electoral College into a weighted voting system with 51 members — the 50 states and Washington DC.[14]

Let us now return to *Gray v Sanders*. The plaintiffs alleged that the votes of citizens in different counties were counted differently because the ratio between a county's population and its number of unit votes (its voting weight at the party convention) varied dramatically.[15] The district court who first heard the case ruled

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[13]There are in fact two minor exceptions: Maine and Nebraska. Since 1969, two of Maine's four Electoral votes are decided according to the plurality in each of the state's two Representative districts, and the other two votes are decided according to the statewide plurality. Similarly, since 1993, three of Nebraska's five Electoral votes are decided according to the plurality in each of its three Representative districts, and the remaining two according to the statewide plurality.

[14]*If* the US Constitution were amended so as to formalize the winner-take-all practice in presidential elections, then in view of Thm. 3.4.3 it could reasonably be argued that, in order to implement the OPOV principle, the voting *power* (as measured by β') of each state in the Electoral College should be made proportional to the square root of the number of popular votes actually cast in that state, so as to equalize the a priori voting powers of participating citizens across states. Alternatively, from a majoritarian point of view it could be argued, on the basis of Cor. 3.4.10, that the mean majority deficit of the system should be minimized by giving each state a *number* of Electors proportional to the square root of the number of popular votes cast in it. But since the US Constitution does not link the election of the president with any form of popular vote, there exists no *constitutional* basis for applying the OPOV principle or the majoritarian principle to presidential elections and using it to challenge the actual manner in which the Electoral College operates. It is therefore not surprising that the serious defects of the Electoral College (analysed, for example, in [7] and [12]) have not been subject to litigation in any US court. It seems that the only way to correct these defects is by amending the Constitution. In 1969 the US House of Representatives approved a constitutional amendment to abolish the Electoral College and link the election of the US president directly to the results of the popular vote. However, after a decade of sporadic debate this proposed amendment was defeated in the Senate on July 9, 1979: it did get majority support (51 to 48), but fell 15 votes short of the two-thirds majority needed to approve a constitutional amendment. No proposal to abolish the Electoral College has since been considered by the US Congress.

[15]In 1960 Fulton, the most populous county, with a population of 556 226, had 6 unit votes — 92 704 persons per unit. Echols, the least populous, with

that this counting system was admissible, as long as the disparity in the ratio of unit votes per population between any two counties was not larger than the current largest disparity in the analogous ratio between any two states in the federal Electoral College.[16]

However, the US Supreme Court disagreed with the district court's opinion. It ruled that the two-tier county-based system used in the Georgia primaries be scrapped, on the grounds that, as the nominations are for statewide offices, the primaries ought also to be held on a direct statewide basis.

> Once the *geographical unit for which a representative is to be chosen* is designated, all who participate in the election are to have an equal vote — whatever their race, whatever their sex, whatever their occupation, whatever their income, and wherever their home may be in that geographical unit.[17]

Justice William O Douglas, delivering the Court's opinion, pointed out that

> [t]he county unit system, even in its amended form ... would allow the candidate winning the popular vote in the county to have the entire unit vote of the county. Hence the weighting of votes would continue, even if unit votes were allocated strictly in proportion to population. Thus if a candidate won 6,000 of 10,000 votes in a particular county, he would get the entire unit vote, the 4,000 other votes for a different candidate being worth nothing and being counted only for the purpose of being discarded.[18]

1876 persons, had 2 units — 938 persons per unit. The disparity between these two ratios is about 99 to 1. In 1962 the system was amended so as to give the bigger counties more unit votes. Fulton's number of units was raised to 40 — 13 908 persons per unit; the disparity between this and the Echols ratio was therefore reduced to about 15 to 1.

[16] At that time, the largest disparity in the Electoral College was between New York (16 782 304 population, 43 Electoral votes) and Alaska (226 167 population, 3 Electoral votes) — a disparity of about 5 to 1.

[17] 372 US Reports (1963), p. 379; emphasis added.

[18] Ibid., p. 381, n. 12.

Indeed, due to such wastage of votes, a two-tier system can distort the outcome by allowing a candidate who wins a majority of the unit votes to defeat another candidate who wins a plurality or even a majority of the statewide popular vote. But what of the analogy with the two-tier system used in presidential elections, which can lead to exactly the same kind of distortion?[19] The Supreme Court rejected the analogy on the grounds that 'this conception of political equality [underlying the Electoral College] belongs to a bygone day', and its validation for the presidency, by being specifically included in the Constitution, provided no measure of state duty under subsequently enacted constitutional amendments.[20] A new conception of political equality had arisen through a cumulative process:

> The conception of political equality from the Declaration of Independence to Lincoln's Gettysburg Address, to the Fifteenth, Seventeenth, and Nineteenth Amendments can only mean one thing — one person, one vote.[21]

In 1964 the US Supreme Court formulated this principle somewhat more explicitly. In the case of *Wesberry v Sanders*,[22] which involved federal congressional districts demarcated by state legislatures, the Court said: 'as nearly as practicable one man's vote in a congressional election is to be worth as much as another's.'[23]

Finally, in the 1964 case *Reynolds v Sims*[24] and five accompanying cases, the US Supreme Court also ruled that a 'state ... [must]

[19] This actually happened in the 1876 and 1888 presidential elections. In 1876 Rutherford B Hayes defeated Samuel J Tilden, who had won a majority of the popular vote. In 1888 Benjamin Harrison defeated Grover Cleveland, who had won a plurality of the popular vote.

[20] Ibid., p. 376, n. 8.

[21] Ibid., p. 381.

[22] 376 US Reports (1964), pp. 1 ff.

[23] Ibid., p. 8.

[24] 377 US Reports (1964), pp. 533 ff.

make an honest and good faith effort to construct districts, in both houses of its legislature, as nearly of equal population as practicable', and that the 'equal-population principle' must not be diluted in any 'significant way'.[25]

It seems quite clear that in its rulings in all the above four cases the Supreme Court intended to equalize the 'worth' of citizens' votes. If this is to mean anything at all, it must mean equalizing their voting power. Of course, at that stage neither the litigants nor the judges had a clear idea as to how voting power (or 'worth') might be generally defined or quantified. They approached the matter intuitively, using their common sense.

Indeed, in three of these cases common sense was a reliable and sufficient guide. In *Baker v Carr*, *Wesberry v Sanders* and *Reynolds v Sims*, the issue was elections of representatives to a state or federal legislative chamber in which the decision rule was given in advance, not subject to dispute; and in all these cases it was symmetric. Thus the voting powers of the representatives were obviously equal.[26] Also, it was not in dispute that the elections were to be held in one-member constituencies. In such circumstances you do not need to have a sophisticated theory of voting power to see that in order to equalize the indirect voting powers of the citizens it is necessary and sufficient to make all constituencies equal in size.

The case of *Gray v Sanders* is rather different. Here there was a two-tier system in which voting powers at the top — those of the counties' delegates at a party's all-Georgia convention — were clearly *unequal*. In order to equalize indirect voting powers at the bottom tier, it was necessary to readjust the voting powers at the

[25] Ibid., pp. 577–578.

[26] This is *not* something that the Supreme Court aimed or needed to achieve. For one thing, the plaintiffs in all these cases were citizen-voters rather than legislators; moreover, as pointed out in [6, fn. 23, pp. 1318 f.], '[t]he rights protected by the Constitution are those of the citizen-voter and not those of the legislator. The legislator's voting power is significant only insofar as it affects his constituents.' The equality of the representatives' voting powers was simply a given and self-evident *fact*.

top. The district court attempted to do so, but its judgment was misguided by naïve common sense: it apparently assumed that delegates' voting powers are proportional to their voting weights; and that the citizens' indirect voting powers would be equalized if each county's delegate had voting power in proportion to its population. But in cases of this sort common sense is unreliable; you really need a sophisticated theory.

The Supreme Court side-stepped the problem by ordering the parties to scrap the two-tier system itself and hold the primaries on a statewide basis using a symmetric decision rule, which would automatically equalize the participants' voting powers. This ruling may well be justifiable; but in our view the justification actually offered for it by the Court, and the implied criticism of the two-tier system used in presidential elections, were partly mistaken in that they confused two separate issues.

It is perfectly reasonable to require that an election for a statewide post should be conducted on a statewide basis, unless there is good cause to do otherwise. The justification for this is that *any* two-tier voting system — whatever decision rule is used at the top tier — may sometimes produce an outcome contrary to that of 'direct democracy' majority rule: an absolute majority of citizens may support outcome a, but outcome b wins in the composite two-tier system. Technically speaking, any two-tier voting system is bound to have a positive mean majority deficit.[27] So, *if majority rule is desired*, two-tier systems must be avoided as far as possible.

But this issue is quite separate from equal suffrage, or any 'conception of political equality', whether belonging 'to a bygone day' or to our own more enlightened times. Majority rule and OPOV are not at all the same thing, as the Supreme Court apparently supposed. Recalling the theory of SVGs and Bz power, we note that an SVG in which all voters have the same Bz power need not be a majority SVG. In particular, *any pairwise symmetric* SVG equalizes voting power (by any reasonable measure).[28] Moreover, as we saw in § 3.4, the Bz power of citizens can well be equalized in

[27] See Def. 3.3.16.

[28] Cf. Rem. 2.1.8.

composite (two-tier) SVGs, even those with constituencies of very disparate sizes. Thus in Ex. 3.4.5 we obtained a two-tier SVG satisfying the principle of OPOV, which can (albeit with very small probability) produce an outcome supported by 1.5 million citizens and opposed by 10.5 million.[29]

Similarly, Justice Douglas is simply mistaken in alleging that 'the collegiate principle ... [involves] *inherent* numerical inequality'.[30] The indirect election of the US president and the winner-take-all practice of the Electoral College cannot be faulted on the basis of the OPOV principle alone; although of course the actual allocation of Electoral votes to states can be so faulted. The system itself can however be attacked on the grounds that it is contrary to the principle of majority rule. As we have pointed out,[31] a reallocation of the Electoral votes could make the Electoral College conform to the OPOV principle. Another reallocation can minimize its mean majority deficit, but cannot possibly eliminate that deficit altogether.[32]

The judgments of the Supreme Court in these four cases left several questions wide open. For one thing, two-tier systems with unequal constituencies could not always be avoided. To make constituencies approximately equal in population numbers,[33] bound-

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[29] A two-tier system with given constituencies can be made as close to direct-democracy majority rule as possible by minimizing its mean majority deficit; but, as we saw in Ex. 3.4.12, this is not at all the same thing as equalizing the citizens' voting powers.

[30] 372 US Reports, p. 378; emphasis added.

[31] See fn. 14, p. 83.

[32] The Founding Fathers opted for indirect election of the president not because their conception of equal suffrage was old fashioned — although they demonstrably did not share our present notion of *universal* suffrage — but simply because they were positively averse to the prospect of a president elected by popular vote. Today the president *is* in effect elected by popular vote, except that the two-tier mechanism presents a potential threat that the majority will may be thwarted.

[33] The term 'number' in this context was originally defined in Article I, Section 2(3), of the US Constitution with respect to direct taxation and congres-

aries have to be redrawn from time to time; but this may meet with several obstacles. First, redistricting must often cut across time-hallowed political or civic boundaries. Second, it may increase the total number of districts — and their representatives — requiring, in turn, additional salaries, which taxpayer-voters are reluctant to approve. Third, it is likely to create opportunities for political gerrymandering. Fourth, natural barriers, such as mountains or rivers, may make it difficult to group certain areas into one constituency, on pain of causing transportation or communication hardships. For these and similar reasons, redistricting is more easily ordered than accomplished.

Due to these difficulties, unequal constituencies were often retained in elections to state and local decision-making bodies; but two methods were used in attempting to compensate for the inequality in population numbers.

First, *weighted voting*: in the legislative body the *single* representative of each district is given a weighted vote, a number of unit votes. A resolution is passed if it gains a certain quota — say a simple majority, or two-thirds — of the total weight. A legally binding answer had yet to be given to the question whether any such system could be deemed to satisfy the principle of OPOV; and if so, what mathematical relationship there should be between weights and population numbers. (As we have seen, in the case of

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sional representation only. At the insistence of the southern states, wishing to boost the number of their Representatives, three-fifths of the number of slaves (referred to as 'other Persons') was to be added to the number of free persons ('excluding Indians not taxed'). In Section 2 of the 14th Amendment, enacted following the Civil War, the 'other Persons' phrase was omitted, but 'numbers' were still defined with respect to congressional representation only. These numbers included not only citizens entitled to vote, but 'the whole number of persons in each State' (except 'Indians not taxed', who were still excluded). In the sequel we shall use the terms 'people', 'population', 'voters', 'residents' 'citizens' and 'inhabitants' interchangeably when referring to state and local elections. For a detailed discussion regarding cases where constituents or registered voters rather than 'the whole number of persons' are used as a representation base see [53, pp. 7–8, fn. 39] and sources cited therein.

Gray v Sanders the district court attempted to lay down a rule, but the US Supreme Court side-stepped the issue.)

A second method of compensating for inequalities in population numbers is to allow *multi-member districts*: larger districts have several representatives, with one vote each.[34] Again, it remained to be decided whether this system is compatible with the principle of OPOV; and if so, a legally binding norm had yet to be set for the mathematical relationship between a district's number of representatives and the size of its population. Here there is a complicating factor: if the several representatives of a given district are likely to vote as a single bloc, then this is formally equivalent to a single representative with a multiple (weighted) vote;[35] otherwise the two cases are quite distinct.

Another question left open was whether the principle of OPOV must apply to elections of bodies of local government — counties, cities, school boards and the like — as well as to state and federal legislatures.

All these legal questions were to be addressed in a series of judgments in the late 1960s and the early 1970s, which we discuss in §§ 4.3 and 4.4. In the next section, however, we would like to focus on a person who played a major role in developing the mathematical and jurisprudential theory of voting power as well as in the legal campaigns surrounding OPOV.

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[34]In [6, fn. 2, p. 1309] it is noted that, as of March 1962, almost 46 per cent of the seats in the lower Houses of the various states were filled from multi-member districts.

A variant of a multi-member district is a *floterial* district. In this arrangement — common in southern states — each of several districts has its own independent representation and, in addition, these districts together elect one or more 'shared' representatives. Cf. fn. 13, p. 83.

For a more detailed discussion of the use of multi-member and floterial districts, see [24, pp. 503–515], and the cases cited and discussed therein.

[35]There are, however, very real practical differences. On the one hand, the salaries of several representatives and their staffs impose a heavier burden upon the taxpayer. On the other hand, a single representative of a large district may be less familiar with his or her constituents than at least some of the representatives of the same district under the multi-member system.

4.2 The Impact of John Banzhaf[36]

John F Banzhaf III (b. 1940) received his B Sc degree in Electrical Engineering from MIT in 1962 and his LL B degree from Columbia University's Law School in 1965. At the time of writing, he is Professor of Law and Legal Activism at George Washington University in Washington DC.

Banzhaf has been involved in many individual and class action suits and public campaigns on behalf of consumers and victims of discrimination. He was, for example, instrumental in founding the National Center for Law and the Deaf, and he also founded (in 1968) the advocacy group Action on Smoking and Health (ASH), which eventually was very influential in getting smoking banned in most public places throughout the US. However, in the literature on voting power he is known primarily for the index which bears his name.[37]

When Banzhaf was in his second year of law school at Columbia University, the *Columbia Law Review* received a manuscript from one of the School's distinguished professors (who was later to become a well-known US Federal District judge). The paper suggested that if representatives in legislatures were assigned voting weights in direct proportion to the numbers of citizens they represented, this might well satisfy the requirements of the US Supreme Court in implementing the principle of OPOV. Banzhaf was asked to review the manuscript. Since his MIT training in engineering had equipped him with substantial mathematical skill, he came to realize that the position advocated in the manuscript was gravely flawed. The editors of the *Columbia Law Review* suggested that he should confront the author with his critique. The professor, none too graciously, dismissed the young man from his office with a remark to the effect: 'If you don't like what I proposed, then why don't you do better?'. Banzhaf took up this challenge in [5].

Early use of weighted voting, aiming to implement OPOV in single-member constituencies of unequal size, had been based on

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[36]This section is partly based on Lucas [67, pp. 49–54].

[37]See § 1.2.

two common-sense premisses, which were in fact fallacies:

4.2.1 Premiss *Representatives' voting powers are proportional to the respective voting weights assigned to them.*

4.2.2 Premiss *Citizens' indirect voting powers are equalized (as required by the OPOV principle) provided the voting powers of representatives in a legislature are proportional to the sizes of their respective constituencies.*

From these two premisses it follows that in order to implement the OPOV principle, the weights assigned to representatives ought to be proportional to the sizes of their constituencies. This is the arrangement we have called 'PWP' (see Com. 3.4.6).[38]

In [5] Banzhaf maintains an attitude of studied non-committal towards Prem. 4.2.2; his target here is Prem. 4.2.1, which he refutes conclusively.[39] This leads naturally to the question how voting power is to be measured. Banzhaf was aware of the S-S index,[40] but rejects it on the grounds that 'it ... attaches an importance to the order in which legislators appear in each minimal voting coalition rather than simply to the number of minimal voting coalitions

[38] As we noted on p. 87, the district court in the case of *Gray v Sanders* seems to have endorsed PWP because it implicitly accepted Prems. 4.2.1 and 4.2.2.

However, as we suggested in Com. 3.4.6, in certain circumstances PWP may be justified on other grounds, independent of these false premisses. We shall return to this topic in Com. 4.6.2.

[39] He *states* both premisses on p. 323. In fn. 22, ibid., Prem. 4.2.2 is split into a conjunction of two propositions. The first is, in effect, that the indirect voting power of a citizen is the product of the citizen's direct voting power and the voting power of the citizen's representative. The second proposition is that a citizen's direct voting power is inversely proportional to the size of the constituency. Of the first proposition (which is true — see the proof of Thm. 3.4.3) he says that its analysis is 'beyond the scope of this paper'. Of the second proposition (which is false) he says that it is 'certainly deserving of further study' — probably a veiled reference to his next paper, [6]; and, no doubt with tongue in cheek, he challenges 'proponents of weighted voting ... to justify it'.

[40] We shall discuss this index in § 6.3.

in which each appears'.[41] He adds that the S-S index, 'based as it is upon mathematical game theory in which each "player" seeks to maximize his "expected winnings," seems to make unnecessary and unreasonable assumptions about the legislative process . . . .'[42] Instead, he proposes what came to be called 'the Bz index'.

He then uses his index to analyze and criticize several imaginary and real weighted voting decision rules, including that used in the Board of Supervisors of Nassau County NY, which we shall discuss in detail in § 4.5. His conclusion, as indicated in the very title of the paper, is that in general weighted voting does not work as a means of implementing OPOV.

When his [5] was still in press, Banzhaf read about a weighted voting case that was before the New York Court of Appeals. He submitted an *amicus curiae* brief, which included the galley proofs of his paper. In a 4 to 3 split decision, the court upheld two lower court rulings that had found two PWP-based weighted voting systems proposed for Washington and Saratoga counties unconstitutional. The court's judgment borrowed heavily from Banzhaf's paper, and included the statement: ' . . . a legislator's voting power, as Mr. Banzhaf points out, is not the number or fraction of votes which he may cast but, rather, his ability, by his vote, to affect the passage or defeat of a measure.'[43] Furthermore, in its 1967 ruling in the case *Iannucci v Board of Supervisors of Washington County*,[44] this same court accepted Banzhaf's definition of voting

[41] See [5, p. 331, fn. 32]. In our view, Banzhaf was right to reject the S-S index for the purpose at hand, because the concept of voting power he was trying to explicate was evidently what we have called 'I-power' (see § 3.1); but his main criticism of the S-S index is based on something of a misunderstanding, as it addresses a particular *representation* of that index (which is convenient for describing it) rather than its essence. We discuss this misunderstanding in Coms. 6.2.8 and 6.3.9.

[42] Loc. cit. Here, we believe, the criticism gets much closer to the bone. This point was to be articulated by Coleman in his critique of the S-S index, quoted in Com. 2.2.2.

[43] Quoted in [67, p. 54].

[44] 282 NYS, 2nd US Circuit Court of Appeals (1967), pp. 502 ff.

power and ruled that a computerized analysis based on his index be submitted in order to validate the apportionment plan.

In 1966 Banzhaf published another paper [6], in which he challenges Prem. 4.2.2. As he had pointed out in [5, fn. 22, p. 323], this premiss rests, in part, on the following:

4.2.3 Premiss *A citizen's direct voting power within his or her constituency is inversely proportional to the size of the constituency.*[45]

Now in [6] he refutes this false premiss by proving the approximation formula for σ_n, which shows that the direct voting power of a citizen in a sufficiently large constituency, modelled as a majority SVG, is inversely proportional to the *square root* of the constituency's size.[46]

From this PSQRR can be deduced quite easily.[47] But though he must have been aware of this implication, he does not spell out PSQRR precisely in [6]. This may be due to the fact that the paper is concerned with multi-member constituencies. *If one assumes that the several representatives of each district vote as a single bloc*, in accordance with the majority opinion in the district, this case is mathematically equivalent to that of single members with multiple votes, and PSQRR is theoretically valid.[48] However, Banzhaf cautiously refrains from making this assumption here;[49]

[45]See above, fn. 39, p. 92.

[46]Cf. Appx. 3.3.10.

[47]See Thm. 3.4.3 and its proof.

[48]This assumption holds for the Electoral College, which is the subject of Banzhaf's [7].

[49]See [6, fn. 33, pp. 1326 f.]. On the other hand, earlier [6, fn. 25, p. 1320] he points out that the vast majority of elections in multi-member districts in the US are in effect winner-take-all contests between slates of the two major parties. In these circumstances it is in our view realistic to assume that all the representatives of a district do tend to vote as a bloc, in accordance with the majority opinion in their district. So he is perhaps over-cautious in not making this assumption.

and without some *precise* alternative assumption it is impossible to deduce a general condition under which the indirect voting powers of the citizens are equalized. He considers tentatively the solution of making the *number* of representatives of a constituency proportional to the square root of the latter's size,[50] but makes no firm recommendation.[51]

On 14 July 1967, Banzhaf presented testimony to the Subcommittee on Constitutional Amendments of the US Senate, in which he analysed the distortions of the Electoral College. This material is incorporated in a third paper [7], published in 1968. Until then it had been widely assumed that the Electoral College works to the advantage of the less populous states, as the numbers of their Electors are disproportionately large compared to their respective populations. Banzhaf proved that the opposite was true: a more populous state is greatly advantaged, because the *voting power* of its bloc of Electors is disproportionately large compared to the *square root* of the size of the state's population.[52]

[50]See [6, fn. 42, p. 1333]. We know of no theoretical justification for this hybrid solution. In cases where the multi-member system can be regarded as equivalent to a single-member system with weighted voting (as suggested in fn. 49 above) and decisions require the assent of a simple majority of representatives, implementation of this hybrid rule will *maximize the the sum* of the indirect Bz powers of the citizens (Cor. 3.4.10) rather than *equalizing* them.

[51]The paper contains a definite theoretical error, which however does not invalidate the main argument. On p. 1317 he claims that his own Bz index 'is substantially in accord with' the S-S index. See also, p. 1318, fn. 21, where the two indices are said to be 'essentially the same'; and Riker and Shapley are approvingly reported as asserting that 'there are no significant qualitative differences between the two measures'. As we shall argue in Com. 7.10.2, although the two indices may be fairly close together in some instances, they are in fact quite different in meaning and behaviour, and may sometimes yield very divergent values.

[52]This paper too contains a couple of theoretical errors, which again do not seriously affect the validity of its main argument. First, as in [6], the author asserts that his own Bz index and the S-S index are 'essentially similar' (see p. 312, fn. 28, where he also reports that Riker, Shapley and Mann had predicted that 'the two techniques yield substantially similar results for large

Although the New York Court of Appeals adopted in the *Iannucci* case Banzhaf's method for measuring voting power, it persisted in the fallacy he refuted in [6]. The US Supreme Court explicitly rejected the argument of [6] four years later in its 1971 decision in the case of *Whitcomb v Chavis*.[53] In the following two sections we cite and discuss the judgments of both courts.

4.3 Local Government

We observed in § 4.1 that at first it was not clear whether the doctrine of OPOV laid down by the US Supreme Court in its 1962–1964 rulings was to apply also at the local level: that of counties, cities, towns and villages. Local governments exercise a large measure of a state's power and, because of the nature of the services they provide, are the organs of government with which the people are most often in direct contact. So there would appear to be a strong case for the view that the apportionment standards applied by the US Supreme Court at the federal and state levels should apply as well at the various levels of local government.

However, some doubt remained because whereas all federal and state legislatures were elected by the people, local officials in some states were appointed by the state. If the latter practice had been *universal*, then it might be argued that such appointments, as long as they were properly made, did not violate the 'equal protection'

numbers of voting units'). Moreover, in Table I, p. 329, he uses the S-S index to measure the voting powers of the state blocs in the Electoral College.

Second, and much more surprising: at one point (pp. 314–315) he slips into the very fallacy which he had exposed a few years earlier — that of assuming voting power to be proportional to voting weight. Here he considers a weighted voting system having one bloc voter with weight 3 (or 5) and an unspecified number of individual voters with weight 1 each; the quota is just over half the total weight. He then infers that the voting power of the bloc voter is 3 (or 5, respectively) times that of an individual voter. This is false; indeed, if there is one voter with weight 5 and four voters with weight 1 each, then these four are obviously dummies.

[53] 403 US Reports (1971), pp. 124 ff.

clause of the 14th Amendment of the Constitution. But in fact many local officials were directly elected by the voters — in which case the 'equal protection' clause presumably must apply. So it remained to be clarified whether the 1962–1964 reapportionment rulings of the Supreme Court should extend to local governments universally.

At the time, the nucleus of a typical county government in the US was a Board of Supervisors, consisting of one supervisor from each town in the county and one or more supervisors from each city. Each supervisor had a single vote on the board, regardless of the size of his or her constituency.[54] The board made most of its decisions by simple majority vote,[55] although some types of resolution might require special majorities (say $\frac{3}{5}$ or $\frac{2}{3}$ of the total membership).

Cases prior to *Baker v Carr* and *Reynolds v Sims* requiring reapportionment at the local level were decided by state courts on state grounds. Federal courts had refused to intervene where complaints regarding improper apportionment involved either municipal or state governments.

The first post-*Reynolds* court case calling for application at county level of federal rules for equal representation was *Brouwer v Bronkema*.[56] This litigation, challenging the constitutionality of the wide variance in per capita representation on the Board of Supervisors of Kent County, Michigan, was commenced on 23 June

[54]For example, in the 1965 board of Saint Lawrence County, NY, the supervisor from the town of Clare (1963 population: 87) and the supervisor from the town of Massena (1963 population: 17 730) had one vote each. For a complete table of county malapportionments in 1965 in the State of New York see [109, pp. 51–53].

[55]Thus Section 153(4) of the NY County Law (quoted in [53, p. 27]) stipulated: 'Whenever ... the board of supervisors is authorized or required to act, and no proportion of the voting strength for such action is otherwise prescribed, such action shall be taken by the affirmative vote of a majority of the total membership of the board.'

[56]Circuit Court of Kent County, MI, No. 1855, 11 September 1964; cited in [109, pp. 23–24] and in [24, p. 545, fn. 9].

1964, only eight days after the Supreme Court's ruling in *Reynolds*. Although city representatives on the board were either appointed by city authorities or served *ex officio*, the state court held that Michigan's system, as applied in Kent County, was unconstitutional. It ruled that '[t]he Fourteenth Amendment applies to the State and to every governmental agency or instrumentality of the State which exercises powers delegated to it by the State'. Similar rulings were rendered shortly thereafter by other state courts.[57]

The next relevant case was *Iannucci v Board of Supervisors*,[58] mentioned in § 4.2, which we shall now recount in greater detail.

In the mid-1960s, Washington County in upstate New York attempted to implement a plan of permanent reapportionment based on weighted voting. The Supreme Court of the county issued an order invalidating the plan as unconstitutional.

On appeal, the Appellate Division, Third Department, held that weighted voting at the county level as presented could not satisfy the OPOV mandate of *Reynolds v Sims*.[59] The court went on to assert that weighted voting, 'although not as unrepresentative as the historical representation by town supervisors, is unconstitutional and the only practical solution to the problem of reapportionment is by the method of district representatives based on population or, perhaps, by county wide elections of supervisors at large'.[60] In explaining its decision, the court opined that while weighted voting cures the previous favouritism in areas of smaller population, 'it by the very system itself creates new inequities in terms of power'.[61] The board was directed to prepare and submit a permanent plan of reapportionment within 10 days. It adopted an 'adjusted weighted voting plan'.

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[57] See [53, fn. 29, 38, pp. 6–7].

[58] *Iannucci v Board of Supervisors of Washington County*, NYS, 2nd District, 3rd Dept. (1967), pp. 458 ff.

[59] See above, text to fn. 25, p. 86.

[60] NYS, 2nd District, 3rd Dept., p. 459.

[61] Loc. cit.

The New York State Court of Appeals thereafter reviewed the Washington County case, along with a reapportionment action from Saratoga County, and found that both boards as then constituted were 'malapportioned' and in violation of the OPOV principle.[62] As mentioned in § 4.2, in its judgment the court quoted Banzhaf [5, p. 318], noting that 'a significant standard for measuring a legislator's voting power ... is not the number or fractions of votes which he may cast but, rather, his ability ... by his vote, to affect the passage or defeat of a measure'.[63]

The court did not hold that weighted voting plans are unconstitutional *per se*, but pointed out their potential flaws, to which it offered its own remedy:

> The principle of one man – one vote is violated, however, when the power of a representative to affect the passage of a legislation by his vote, rather than by influencing his colleagues, does not roughly correspond to the proportion of the population in his constituency. Thus, for example, a particular weighted voting plan would be invalid if 60% of the population were represented by a single legislator who was entitled to cast 60% of the votes. Although his vote would apparently be weighted only in proportion to the population he represented, he would actually possess 100% of the voting power whenever a simple majority was all that was necessary to enact legislation. Similarly, a plan would be invalid if it was 'mathematically impossible' for a particular legislator representing, say, 5% of the population to ever cast a decisive vote. Ideally, in any weighted voting plan, *it should be mathematically possible for every member of the legislative body to cast the decisive vote on legislation in the same ratio which the population of his constituency bears to the total population.* Only then would a member representing 5% of the population have, at least in theory, the same voting power (5%) under a weighted voting plan as he would have in a legislative body which did not use weighted voting — e.g., as a member of a 20-member body with each member

[62] 20 NY, 2d, p. 249; 282 NYS, 2d p. 506.

[63] 20 NY, 2d, p. 251, quoted in [67, p. 54].

> entitled to cast a single vote. *This is what is meant by the one man – one vote principle as applied to weighted voting plans for municipal governments. A legislator's voting power, measured by the mathematical possibility of his casting a decisive vote, must approximate the power he would have in a legislative body which did not employ weighted voting.*[64]

Here the court is saying that a representative's Bz power should be proportional to the number of his constituents. Lured by common sense, the court has fallen right into the fallacy of Prem. 4.2.2, which had been refuted by Banzhaf in [6]. Its prescription for implementing the OPOV principle is a sort of hybrid between PWP and PSQRR; inasmuch as it contradicts the latter, it must in general result in *inequality* of citizens' indirect voting powers.

On the credit side, the court's judgment shows a firm grasp of two fundamental points. First, that the OPOV principle is concerned not with actual but with a priori ('mathematical') voting power. Second, that Prem. 4.2.1 is false.

A major difference between Prem. 4.2.1 and Prem. 4.2.2 is that whereas the former can be refuted using simple examples that are easily comprehesible to any intelligent person,[65] the latter can only be disproved by a rather technical argument, which to a lay person may well seem a veritable 'mathematical quagmire'. It is therefore not too surprising that the court, while accepting the Bz measure of voting power, contradicted PSQRR and endorsed Prem. 4.2.2.

The court found that in order to ascertain the 'mathematical' (that is, a priori) voting power of each member of the county boards of supervisors, it would be necessary to seek computer-aided expert opinion. The boards were not entitled simply to presume that their legislative acts were constitutional: such presumption

> … is derived from the principle that it is improper for a court, in passing upon a constitutional question, to lightly disregard the considered judgment of a legislative body

[64] 20 NY, 2d, p. 252; 282 NYS, 2d, pp. 508 f.; emphases added.

[65] Cf. our treatment of this matter in § 1.1.

> which is also charged with a duty to uphold the Constitution. With respect to weighted voting, however, a 'considered' judgment is impossible without computer analyses and, accordingly, if the boards choose to reapportion themselves by the use of weighted voting, there is no alternative but to require them to come forward with such analyses and demonstrate the validity of their reapportionment plans.[66]

The court ended with a note of caution, borrowing Justice Frankfurter's phrase in *Baker v Carr*: such reapportionment cases had the potential to drag the courts unnecessarily into a 'mathematical quagmire'.[67] Ironically, the court was unaware that two pages earlier it had committed a serious mathematical error in its faulty interpretation of the OPOV principle.

The question of applicability to local governments of the OPOV doctrine was decided by the US Supreme Court in the 1968 case of *Avery v Midland County*, which it considered on appeal from the Texas Supreme Court.[68] The case involved the Commissioners Court of Midland County, TX, which was a five-person governing body: each of the county's four districts elected one commissioner, and the fifth was elected by the county at large. The five had one vote each, although one district contained 95 per cent of the county's population. The US Supreme Court found this Commissioners Court to be in violation of the 'equal protection' clause of the 14th Amendment of the US Constitution. It ruled that '[t]he Equal Protection Clause reaches the exercise of state power however manifested, whether exercised directly *or through subdivisions of the State*',[69] and therefore '[local] units with general governmental powers over an entire geographic area' are included under the OPOV mandate.[70]

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[66]20 NY, 2d, p. 254.

[67]Loc. cit.

[68]309 US Reports (1968), pp. 474 ff.

[69]Ibid., p. 476; emphasis added.

[70]Ibid., pp. 485 f.

4.4 Basic Rulings

Although the *Avery* decision made it crystal-clear that the OPOV doctrine applies also to local governments, the question remained whether the Supreme Court would approve the use of weighted voting, or of multi-member districts, or indeed of any method other than equal redistricting as a remedy for malapportioned districts.

Speaking for the Court in the *Avery* case, Justice Byron R White said:

> We hold today only that the Constitution permits no substantial variation from equal population in drawing districts for units of local government . . . .[71]

But the *Avery* case concerned a local government using single-member districts, with one vote per member; so in this factual context the pronouncement just quoted could be read as applying to such set-ups only. This narrow interpretation seemed to be borne out by another statement in the opinion, appearing just two paragraphs later:

> The Constitution and this Court are not roadblocks in the path of innovation, experiment and development among units of local government . . . . Our decision today is only that the Constitution imposes . . . a requirement that [local government] units . . . not be apportioned among single-member districts of substantially unequal population.[72]

So it appeared that the Supreme Court had yet to rule on the constitutionality of weighted voting.

As pointed out in § 4.2, early weighted voting systems in the US were based on two intuitively appealing but false premisses, 4.2.1 and 4.2.2. We saw that the first of these was demolished by Banzhaf in [5], where he also proposed his method for measuring a priori voting power. And this was accepted by the New York

[71] 309 US Reports, pp. 484 f.

[72] Ibid., pp. 485 f.

Court of Appeals in its 1967 decision in the *Iannucci v Board of Supervisors* case.

We have also seen that in his 1966 paper [6] Banzhaf refuted Prem. 4.2.3, and hence by implication Prem. 4.2.2 as well.

Prem. 4.2.3 and its refutation by Banzhaf were addressed explicitly by the US Supreme Court on 7 June 1971, in its judgment in the case of *Whitcomb v Chavis et al.*[73] The case, an appeal against the decision rendered in 1969 by the US District Court for the Southern District of Indiana, involved Indiana's scheme of single- and multi-member districts for its state legislature.[74] In that lower court, the plaintiffs, Patrick Chavis and others — mostly residents of the Black ghetto in Marion county — demanded that the multi-member district comprising the whole of that county be partitioned into several equal single-member districts, of which the ghetto would be one. The lower court agreed with their claim that the multi-member district of Marion County illegally minimized and cancelled out the voting power of the 'cognizable racial minority group' in the ghetto, as evidenced by the relatively small number of ghetto-resident legislators compared to the ghetto's share of the county's population.[75] But the court went even further: it found that, in general, the State of Indiana's mixed system of single- and multi-member districts created excessive variations in the ratios of population per senator and population per assemblyman; and therefore ordered that the *entire* state be repartitioned into single-member districts. Indiana's Governor Edgar Whitcomb appealed against the district court's decision to the US Supreme Court.[76] The Supreme Court reversed the judgment of the district

[73] 403 US Reports (1971), pp. 143 ff.

[74] 307 Federal Supplement (1968), pp. 1362 ff.

[75] Ibid., p. 1365.

[76] By the time the appeal was considered, the main issue had been overtaken by events: in 1970 Indiana's legislature passed a law partitioning the entire state into single-member districts. Nevertheless, for reasons that need not concern us here, the Supreme Court proceeded with the case.

court and remanded the case to that court for 'further proceedings consistent with [the Supreme Court's] opinion'.[77]

In their brief the plaintiffs invoked Banzhaf's analysis in [6] to show that the old mixed system of single- and multi-member districts discriminated against citizens in the smaller districts: that system was based on the false Prem. 4.2.3, and therefore over-compensated the larger districts.[78] This argument was addressed both in the Court's majority opinion and in the separate opinion of Justice John M Harlan.

The Supreme Court's majority opinion as to the validity of the multi-member election district of Marion County, was delivered (again) by Justice Byron R White. Regarding the plaintiffs' argument refuting Prem. 4.2.3 he said:

> In asserting discrimination against voters outside Marion County, plaintiffs recognize that Fortson, Burns and Kilgarlin[79] proceeded on the assumption that the dilution of voting power suffered by a voter who is placed in a district 10 times the population of another is cured by allocating 10 legislators to the larger district instead of the one assigned to the smaller district. Plaintiffs challenge this assumption at both

[77]403 US Reports, p. 163.

[78]That mixed system was in effect equivalent to a weighted voting system because, as was explicitly recognized, the several representative of a multi-member district tended to vote as a bloc.

[79]These are allusions to earlier Supreme Court cases which dealt with issues concerning multi-member districts. The 1965 case *Fortson v Dorsey* (379 US Reports, pp. 433 ff.) was the first of the post-*Baker* cases challenging multi-member districts. In it the Supreme Court held (as it had in *Reynolds v Sims*) that the 'equal protection' clause does not necessarily rule out multi-member districts in a state's legislative apportionment scheme. The next case to come before the Supreme Court on this issue (in 1966) was *Burns v Richardson* (384 US Reports, pp. 73 ff.) in which the Supreme Court reiterated the standard it had set in *Fortson*: that 'the legislative choice of multi-member districts is subject to constitutional challenge only upon showing that the plan was designed to or would operate to minimize or cancel out the voting strength of racial or political groups' (ibid., p. 74). Finally, in the 1967 case *Kilgarlin v Hill* (386 US Reports, pp. 120 ff.) the Supreme Court's majority opinion held that no such showing had been made.

> the voter and legislator level. They demonstrate mathematically that in theory voting power does not vary inversely with the size of the district and that to increase legislative seats in proportion to increased population gives undue voting power to the voter in the multi-member district since he has more chances to determine election outcomes than does the voter in the single-member district. This consequence obtains wholly aside from the quality or effectiveness of representation later furnished by the successful candidates. The District Court did not quarrel with plaintiffs' mathematics, nor do we. But like the District Court we note that the position remains a theoretical one[80] and, as plaintiffs' witness conceded, knowingly avoids and does "not take into account any political or other factors which might affect the actual voting power of the residents, which might include party affiliation, race, previous voting characteristics or any other factor which goes into the entire political voting situation."[81] The real-life impact of multi-member districts on individual voting power has not been sufficiently demonstrated, at least on this record, to warrant departure from prior cases.[82]

What does White mean by the phrase 'the position remains a *theoretical* one'? Clearly, he means that citizens' voting powers as calculated by the plaintiffs are *a priori*. Indeed, the explanation he offers of the term 'theoretical' — the phrase borrowed from the plaintiffs' witness in the passage just quoted — is exactly in the same vein as the many explanations in the voting-power literature of the sense of the term 'a priori' in this context.[83] White, on behalf of the Supreme Court majority, evidently implies that the aprioristic ('theoretical') character of the plaintiffs' analysis weak-

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[80] Here there is a long footnote (fn. 23 in the original), which states and proves the approximation formula for σ_n as it appears in [6, pp. 1320–1322].

[81] [Footnote 24 in the original:] Plaintiffs' brief in this Court recognized the issue: "The obvious question which the foregoing presentation gives rise to is that of whether the fact that a voter in a large multi-member district has greater mathematical chance to cast a crucial vote has any practical significance." Brief of Appellees (Plaintiffs) [p.] 14.

[82] 403 US Reports, pp. 144–146.

[83] Cf. our own explanation in Com. 2.2.3.

ens their position. Against this, three arguments may be offered.

First, the OPOV principle means equalizing citizens' a priori voting powers, not their actual voting powers. Therefore, when the issue is the proper implementation of the OPOV principle, it is perfectly proper to apply an aprioristic analysis of voting power. As we put it in Com. 2.2.3, this is like creating a level playing field. A level playing field does not give a weak team and its strong opponent equal *actual* chances of winning. It only equalizes their *a priori* chances: ensuring that whichever team wins does so by virtue of its greater skill, not because the physical conditions are rigged a priori in its favour. If you see that the playing field is not level, then — *even if you do not know which teams are going to play on it* — you can foretell a priori ('theoretically') that one of these teams will be disadvantaged: it will have an uphill struggle. This is admittedly a theoretical prediction; but it would be rash to doubt its practical implication. Clearly, the theoretically disadvantaged team may or may not lose in the end; but it surely will find it harder to win than if the field had been level.

True, as we pointed out in Com. 2.2.3, there may be cases where there are known long-term systematic real factors favouring some voters and disfavouring others; and it may arguably be appropriate to compensate for those factors by a system of 'handicaps'. But such remedial compensation must be regarded as a sort of affirmative action, superimposed on the a priori levelling of OPOV, not as part of the OPOV principle *per se*. Besides, a proper aprioristic theoretical analysis is needed in order to ensure that the remedial compensation has the desired effect.[84]

Second, the old common-sense premisses on which systems of weighted voting or multi-member constituencies had been based are every bit as aprioristic ('theoretical') as the plaintiffs' analysis refuting Prem. 4.2.3. Those naïve premisses also ignore 'any political or other factors which might affect the actual voting power of

[84]This is illustrated by the Electoral College. As we observed in § 4.2, the common view — based on Prems. 4.2.1 and 4.2.2 — had been that the allocation of Electors to states is biased in favour of the smaller states; but Banzhaf's theoretical analysis in [7] proved that the opposite was true.

the residents, which might include party affiliation, race, previous voting characteristics or any other factor which goes into the entire political voting situation'. So in the case under consideration, as both alternative approaches are equally aprioristic ('theoretical'), there is nothing to choose between them in *this* respect. But in another respect there is a not insignificant difference: the naïve common-sense approach is fallacious, whereas the mathematical argument refuting Prem. 4.2.3 is sound.

Third, in previous rulings enforcing the OPOV principle — for example in the cases of *Wesberry v Sanders* and *Reynolds v Sims* — the Supreme Court endorsed the most non-controversial two-tier system: that of equal-sized districts, each electing a single representative having a single vote. This is the one exceptional (mathematically trivial) case in which both common sense and mathematical analysis concur: *citizens' voting powers are equal and the OPOV principle is satisfied*. But this apparently obvious proposition, with which few would quarrel, is also purely aprioristic ('theoretical'); it too ignores 'any political or other ... factor which goes into the entire political voting situation'.[85]

In a separate opinion, Justice Harlan reiterated his long-standing opposition to the Court's OPOV philosophy. He had expressed this view in his dissenting opinions in the cases of *Baker v Carr* and *Gray v Sanders*, and was consistent in his opposition to judicial intervention aimed at enforcing the OPOV principle, which he regarded as constitutionally untenable. His polemic in the present case is directed not so much against the Court's decision, with which he largely agrees, but against what he sees as inconsistencies in the Supreme Court's general attitude on these matters, as it had evolved over the preceding decade. He welcomes the Court's dismissal of Banzhaf's theoretical position — and, as we shall see, adds anti-Banzhaf arguments of his own — but points out that this undermines the Court's previous decisions in which it sought to enforce the OPOV principle, and vindicates his own dissent from those decisions.

[85] As we shall see shortly, Justice Harlan was not slow to point this out.

It is of course not for us to say whose interpretation of the US Constitution is correct and, in particular, whether or not the Constitution sanctions judicial action enforcing the OPOV principle. But we would like to comment on the *logic* of Harlan's arguments against OPOV and Banzhaf's theory.

In Harlan's view, a whole line of wrong interventionist decisions of the Supreme Court majority, beginning with *Gray v Sanders*, stemmed from a majoritarian philosophy, which is contrary to the Constitution.

> That line of cases can best be understood, I think, as reflections of deep *personal* commitments by some members of the Court to the principles of pure majoritarian democracy.[86]

It is indeed undeniable that the opinion of the Supreme Court in the *Gray* case, for example, reflected majoritarian commitments:[87] Justice Douglas's arguments (on behalf of the Court) against the county unit system, *even in an amended form*,[88] and his obvious distaste for the winner-take-all practice of the Electoral College are clear manifestations of majoritarianism. But as we explained in § 4.1, majoritarianism and the OPOV principle are not at all the same thing: the latter does not presuppose the former. Douglas and his co-thinkers, who supported both causes, made a logical error in conflating them.

Their error is mirrored, on the opposite side of the philosophical fence, by Harlan: invoking the federal scheme of the Constitution, he argues against majoritarianism, and implies that this is *ipso facto* an argument against the OPOV principle. After the sentence just quoted, he goes on:

> This majoritarian strain and its non-constitutional sources are most clearly revealed in Gray v Sanders, ... where my Brother Douglas, speaking for the Court, said: "The conception of political equality from the Declaration of independence, to Lincoln's Gettysburg Address, to the Fifteenth,

[86]403 US Reports, p. 166; emphasis in original.

[87]Whether these were 'personal' in an improper sense, and contrary to the Constitution, is another matter.

[88]See text to fn. 18, p. 84.

> Seventeenth, and Nineteenth Amendments can only mean one thing — one person, one vote." ... It is a philosophy which ignores or overcomes the fact that the scheme of the Constitution is not one of majoritarian democracy, but of federal republics, with equality of representation a value subordinate to many others, as both the body of the Constitution and the Fourteenth Amendment itself show on their face.[89]

The wording of this passage implies that equality of representation is the same thing as majoritarian democracy, or at least that the former depends on the latter. But we have shown that they are not.[90]

In Harlan's view, the Supreme Court's decision in the present case amounts to an implicit rejection of majoritarianism; and it undermines the entire philosophy adopted by the Court in the line of cases since its decision nine years earlier in the *Baker v Carr* case:

> If majoritarianism is to be rejected as a rule of decision, as the Court implicitly rejects it today, then an alternative principle must be supplied if this earlier line of cases just referred to is still to be regarded as good law. The reapportionment opinions of this Court provide little help. They speak in conclusory terms of "debasement" or "dilution" of the "voting power" or "representation" of citizens without explanation of what these concepts are. The answers are hardly apparent, for as the Court observes today: "As our system has it, one candidate wins, the others lose. Arguably the losing candidates' supporters are without representation since the men they voted for have been defeated; arguably they have been denied equal protection of the laws since they

[89] 403 US Reports, pp. 166 f.

[90] The example of the Electoral College illustrates this most clearly. It may be reasonable to invoke the federal principle in defence of the two-tier collegiate system and its winner-take-all practice (although this practice is not explicitly prescribed by the Constitution). And it is quite true that this system — *whatever the allocation of Electors to states* — must have a positive mean majority deficit and is therefore inherently incompatible with pure majoritarian democracy. But it *is* compatible with the OPOV principle.

> have no legislative vote of their own. ... But we have not yet deemed it a denial of equal protection to deny legislative seats to losing candidates, even in those so-called 'safe' districts where the same party wins year after year."[91]

The observation quoted here by Harlan from the Court's majority opinion is a truism: of course, the OPOV principle cannot and is not intended to give all voters equal satisfaction with the actual outcome. A supporter of OPOV might go on to say that it *is* designed to give all voters equal a priori influence over the outcome. But for Harlan the truism is a means of casting doubt on the OPOV principle itself. He now goes on:

> A coherent and realistic notion of what is meant by "voting power" might have restrained some of the extreme lengths to which this Court has gone in pursuit of the will-o'-the-wisp of "one man, one vote."
>
> An interesting illustration of the light which a not implausible definition of "voting power" can shed on [the] reapportionment doctrine is provided by the theoretical model created by Professor Banzhaf, to which the Court refers ... .[92]

He is keen to show that Banzhaf's theory does not provide a 'coherent and realistic notion of what is meant by "voting power" '. If this theory is discredited, the Court is left without *any* coherent theoretical basis for implementing the OPOV principle. Before propounding his own arguments against Banzhaf's theory, he points out in an incisive footnote that the reason given by the Court majority for ignoring that theory can be turned against the previous interventionist judgments of the Court itself:

> The Court, though stating that it does "not quarrel with plaintiffs' mathematics," nevertheless implies that it may be ignored because the "position remains a theoretical one ... and does not take into account any political or other factors which might affect the actual voting power of residents" ... .

[91] 403 US Reports, pp. 167 f. The Court's observation quoted by Harlan is ibid., p. 153.

[92] Ibid., p. 168.

> Precisely the same criticism applies, with even greater force, to the "one man, one vote" opinions of this Court. The only relevant difference between the elementary arithmetic on which the Court relies and the elementary probability theory on which Professor Banzhaf relies is that calculations in the latter field cannot be done on one's fingers.[93]

The logic of this remark is undeniable: as we have noted, some of the Supreme Court's previous judgments had been based implicitly on assumptions that are as 'theoretical' as Banzhaf's.[94] Harlan now launches into a direct critique of Banzhaf's model:

> This model uses as a measure of voting power the probability that a given voter will cast a tie-breaking ballot in an election. Two further assumptions are made: first, that the voting habits of all members of the electorate are alike; and second, that each voter is equally likely to vote for either candidate before him. On these assumptions, and taking the voting population of Marion County as roughly 300,000, it can be shown that the probability of an individual voter's casting a decisive vote in a given election is approximately .00146. This provides a standard to which "voting power" of residents in other districts may be compared. . . .
>
> However, Professor Banzhaf's model also reveals that minor variations in assumption can lead to major variations in results. For instance, if the temper of the electors are changed by one-half of one percent,[95] each individual's voting power is reduced by a factor of approximately 1,000,000. Or if a few of the 300,000 voters are committed — say 15,000 to candidate A and 10,000 to candidate B [96] — the probability of any individual's casting a tie-breaking vote is reduced by a factor of $120,000,000,000,000,000,000$. Obviously in comparison with the astronomical differences in voting power which can

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[93] Loc. cit., fn. 2.

[94] See text to fn. 85, p. 107.

[95] [Footnote 3 in the original:] More precisely, the result follows if the second of Professor Banzhaf's assumptions is altered so that the probability of each voter selecting candidate A over candidate B is 50.5% rather than 50%.

[96] [Footnote 4 in the original:] The text assumes that each of the remaining 275,000 voters is equally likely to vote for A or for B.

> result from such minor variations in political characteristics, the effects of the ... 28% population variations considered in ... this case are de minimis, and even the extreme deviations from the norm presented in Baker v Carr ... and Avery v Midland County ... pale into insignificance.[97]

And he adds a sarcastic remark, borrowed from Mark Twain, implying that Banzhaf's scientific approach (or perhaps science as such) is suspect:

> "There is something fascinating about science. One gets such wholesale returns of conjecture out of such a trifling investment of fact." Mark Twain, Life on the Mississippi 109 (Harper & Row, 1965).[98]

Harlan's argument against Banzhaf's model is founded on a failure to distinguish between *actual* and *a priori* voting power. As we have pointed out on several occasions,[99] the model is concerned solely with the latter. If it were offered as an empirical model of the *actual* voting behaviour of voters in, say, Marion County, then its key assumptions — that they all vote independently of one another, with a probability of $\frac{1}{2}$ for either party — would have been quite arbitrary and unjustified. But, as we pointed out in Com. 3.1.3, the model is designed to represent a state of *a priori ignorance* regarding the nature of the issues to be voted on, the voters' 'tempers', and relations of affinity or disaffinity between voters. In this context, the assumptions of independence and equiprobability are the *only* ones that are non-arbitrary. They are justified as aprioristic rather than empirical assumptions.

Having discharged his broadside against Banzhaf's model, Harlan turns his guns on their main target:

> It is not surprising therefore that the Court in this case declines to embrace the measure of voting power suggested

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[97] 403 US Reports, pp. 168 f.

[98] Ibid., p. 169, fn. 5.

[99] See Com. 2.2.3, Com. 3.1.3 and our discussion earlier in this section (pp. 105 ff.) of the Court's dismissal of the model.

> by Professor Banzhaf. But it neither suggests an alternative nor considers the consequences of its inability to measure what it purports to be equalizing. ... Instead it becomes enmeshed in the haze of slogans and numerology which for 10 years has obscured its vision in this field, and finally remands the case "for further proceedings consistent with [its] opinion." ...
>
> This case is nothing short of a complete vindication of Mr. Justice Frankfurter's warning [in *Baker v Carr*] nine years ago .... With all respect, it also bears witness to the morass into which the Court has gotten itself by departing from sound constitutional principle in the electoral field. ... I hope the day will come when the Court will frankly recognize the error of its ways in ever having undertaken to restructure state electoral processes.[100]

On the same day (7 June 1971) that the US Supreme Court rendered its decision in the *Whitcomb* case, in which the Bz index was rejected, it also pronounced on another case, *Abate v Mundt*,[101] where it upheld a multi-member district plan in which the number of representatives per district was determined (with some permissible variation) in proportion to population size.

In 1969, in a suit brought by taxpayers in the New York Supreme Court of Rockland County, to compel the County Board of Supervisors to reapportion in accordance with constitutional requirements, the court approved a plan calling for a county legislature of 18 members chosen from five districts corresponding to the county's five constituent towns. The smallest district with a population of 12 114 was assigned one representative in the legislature, and the number of representatives of the other districts was determined by dividing the population of each remaining district by that of the smallest district, with any fractional results rounded to the nearest integer. This plan resulted in variations of population per legislator of between 11 577 in one town to 13 020 in another town, with consequent deviation from the mean of +4.8% in the former town and −7.1% in the latter, a total variation range of 11.9%.

[100] 403 US Reports, pp. 169 f.

[101] 403 US Reports (1971), pp. 182 ff.

The Appellate Division affirmed this plan. The New York Court of Appeals also affirmed it. This court stated that 'the issue is not to be resolved merely in terms of a sterile mathematical exercise' and held that a 12% variation in the number of people per legislator was not itself 'insufficient to render a plan constitutionally defective'.[102]

The US Supreme Court granted *certiorari* in the case and affirmed the New York Court of Appeals decision upholding the plan. While cautioning against apportionment structures that contain a built-in bias, the court found that there was no such bias in the Rockland County plan.

By implication, the US Supreme Court's affirmation of the Rockland County plan constituted an endorsement of the fallacious Prems. 4.2.1 and 4.2.2, on which that plan was clearly based.

The US Supreme Court persisted in dismissing Banzhaf's analysis as too theoretical also in subsequent cases involving weighted voting based on single-member districts, in which plaintiffs invoked that analysis in order to demonstrate violation of the 'equal protection' clause. An instance of this was the 1989 US Supreme Court's majority decision in *Morris et al. v Board of Estimate of City of New York et al.*[103] on appeal from the US Court of Appeals for the NY Second Circuit. In this case the plaintiffs contended, using the Banzhaf model, that the structure of the New York City Board of Estimate afforded vastly unequal representation to the citizens of New York City's five boroughs.[104]

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[102] 25 NY, 2d, pp. 314 f.

[103] 489 US Reports (1989), pp. 688 ff.

[104] Since 1898, The Board of Estimate of the City of New York had been a powerful political institution in New York City, in charge of numerous key allocatory and regulatory functions for the city. (In 1989 it was responsible for zoning, municipal contracts, land-use decisions, and water and sewer rates.) It consisted of three citywide officials — the mayor, comptroller and City Council president — each having two votes on the board; and the five borough presidents, each having one vote on the board, who were elected by the residents of their boroughs. Most of the board's decisions were passed by simple majority of the members' votes (6 out of 11). The distribution of New York City's population among its five boroughs was considerably unequal:

We quote from the Supreme Court's majority opinion, delivered (once again) by Justice Byron R White:

> As described by the Court of Appeals [in 1987], ... the method urged by the city to determine an individual voter's power to affect the outcome of a board vote first calculates the power of each member of the board to affect a board vote, and then calculates voters' power to cast the determining vote in the election of that member. This method, termed the Banzhaf Index, applies as follows .... A citizen's voting power through each representative is calculated by dividing the representative's voting power by the square root of the population represented; ....
>
> ... We note also that we have once before, although in a different context, declined to accept the approach now urged by the city. ... In [the *Whitcomb*] case we observed that the Banzhaf methodology "remains a theoretical one" and is unrealistic in not taking into account "any political or other factors which might affect the actual voting power of the residents, which might include party affiliation, race, previous voting characteristics or any other factors which go into the entire political voting situation." ...
>
> The personal right to vote is a value in itself, and a citizen is ... shortchanged if he may vote for only one representative when citizens in a neighboring district, of equal population, vote for two; or to put it another way, if he may vote for one representative and the voters in another district half the size also elect one representative. Even if a desired outcome is the motivating factor bringing voters to the polls, the Court of Appeals in this case considered the Banzhaf index an unrealistic approach to determine whether citizens have an equal voice in electing their representatives because the

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in 1983 the percentages of the city's population residing in Bronx, Brooklyn, Manhattan, Queens and Staten Island were 16.3, 31.5, 20.1, 26.9 and 5.2, respectively. Thus the presidents of Brooklyn and Staten Island had one vote each on the Board of Estimate although the former represented more than six times as many constituents as the latter. Following the Supreme Court's decision, the board was replaced, after two years of study and a referendum, by an expanded 51-member City Council, each of whose members is elected from equal-size single-member districts.

> approach tends to ignore partisanship, race, voting habits or other characteristics having an impact on election outcomes.
>
> The Court of Appeals also thought that the city's approach was "seriously defective in the way it measures Board members' power to determine the outcome of a Board vote." ... The difficulty was that this method did not reflect the way the board actually works in practice; rather, the method is a theoretical explanation of each board member's power to affect the outcome of board actions. It may be that in terms of assuring fair and effective representation, the equal protection approach reflected in the *Reynolds v Sims* line of cases is itself imperfect, but it does assure that legislators will be elected by and represent citizens in districts of substantially equal size. It does not attempt to inquire whether, in terms of how the legislature actually works in practice, the districts have equal power to affect a legislative outcome. This would be a difficult and ever-changing task, and its challenge is hardly met by a mathematical calculation that itself stops short of examining the actual day-to-day operations of the legislative body. The Court of Appeals in any event thought there was insufficient reason to depart from our prior cases, and we agree.[105]

In this remarkable piece of reasoning, White concedes that the simple system of unweighted voting based on single-member districts of equal size is an 'imperfect' implementation of the OPOV principle. The grounds for this admission is, in effect, the proposition that perfect implementation of OPOV means equalizing citizens' *actual* rather than *a priori* voting powers. In our view this proposition is, to say the least, very difficult to maintain. But even if it is maintained, the alternative system, based on PSQRR — which he rejects — is not *less* perfect in his sense than the simple system, which he continues to uphold. As Harlan put it some 18 years earlier, '[t]he only relevant difference between the elementary arithmetic on which the Court relies and the elementary probability theory on which Professor Banzhaf relies is that calculations in the latter field cannot be done on one's fingers'.[106]

[105] 489 US Reports, pp. 697–699.

[106] Cf. text to fn. 93, p. 111.

Although in our opinion White's reasoning here is unconvincing, the Court's decision might be justified by other arguments. For one thing, the system of equal single-member districts has the advantage of simplicity and transparency. Thus, it could be justified on grounds of acceptability to the general public. There is another, more theoretical justification, which we shall present in Com. 4.6.1.

In view of the US Supreme Court's rejection of the Bz index in both the *Whitcomb* and *Morris* cases, it was only to be expected that sooner or later lower courts would also start rejecting reapportionment plans based on this index. This indeed happened in April 1993, when the US District Court for the District of Eastern New York in *Jackson v Nassau County Board of Supervisors*[107] struck down the weighted voting system which had existed since 1972 in Nassau County, the second-largest county in New York State. This case, which has a relatively long history and some unique features, is often cited — sometime inaccurately — in the literature on weighted-voting. We give an updated and fairly detailed account of it in the next section.

4.5 Nassau County, NY: A Case Study

Nassau County is situated in the western part of Long Island, NY. In 1899 the towns of Hempstead, North Hempstead and Oyster Bay petitioned the state of New York to establish a county government. The towns had previously belonged to Queens County, but in 1898 Queens joined four other counties to form New York City, leaving these towns without the umbrella government that a county provides. The petition was granted in 1900 and Nassau County was formed.

Until 1917 the Board of Supervisors — the county's legislature — had three members, one from each of the three towns, each casting one vote on the board. In 1917, however, a system of weighted voting was introduced, giving each supervisor a number of votes

[107] 818 Federal Supplement (1993), pp. 509 ff.

reflecting the size of the population he represented.

By 1922, Hempstead comprised over half of the county's population, and a second position of supervisor-at-large was created specifically for the purpose of doubling that town's representation. Also, the cities of Glen Cove and Long Beach had joined the county. So during the period 1922–1936 there were six representatives on the board with a total of 14 votes, distributed as follows: the two representatives of Hempstead had 4 votes each; the representative of North Hempstead and that of Oyster Bay had 2 votes each; and the representative of Glen Cove and that of Long Beach had 1 vote each.[108]

In 1935 New York State passed the Fearon Amendment to its constitution, allowing the counties of the state to adopt a home-rule charter that would devolve certain governmental functions — including the right to establish county courts — to the counties. Accordingly, Nassau County adopted a new charter in 1936. Article I, Section 104, of this charter prescribed:

> [Each supervisor may cast] a number of votes equal to the quotient in whole numbers obtained by dividing the number of inhabitants, excluding aliens, as determined by the latest federal or state census, whichever is the later, of the town or city from which they may have been elected by ten thousand . . . provided that no supervisor shall have less than one vote, nor shall the supervisor or supervisors of any town or city be entitled to cast more than fifty percentum of the total vote.[109]

This is a modified form of PWP: according to the prescription of 1 : 10 000, the supervisor(s) representing a town whose inhabitants comprise more than 50 per cant of the total inhabitants (as was the case for Hempstead) should get more than 50 per cent of the votes on the board, but this is overridden by the final proviso.

However, it soon transpired that a literal implementation of this truncating proviso would lead to somewhat paradoxical results. According to Grofman and Scarrow [41, p. 178], on the basis of the

[108] Data cited in Grofman and Scarrow [41, p. 178].

[109] Quoted in Pollio [80, p. 7].

1936 Nassau population, an unmodified 1 : 10 000 formula would have assigned the two supervisors of Hempstead 9 votes each; the supervisor of North Hempstead 6; that of Oyster Bay 3; those of Glen Cove and of Long Beach 1 vote each. This would have given the two Hempstead supervisors more than half the total number of votes, 18 out of 29; so their votes had to be truncated. But by how much? Since all the other supervisors together had 11 votes, it would appear that each Hempstead supervisor must be given 5 votes — *less* than the North Hempstead supervisor, who represented a *smaller* population; surely, the former should have *more* votes than the latter! This incongruity was resolved in December 1937 by adopting a (legally questionable) interpretation proposed by the Nassau County attorney. He suggested that the number of votes of each Hempstead supervisor be reduced from 9 to 7 (thereby reducing the total number of votes from 29 to 25), but that simple-majority resolutions should be carried with 15 votes (instead of 13) — as would have been required if no truncation had taken place. Similarly, he suggested that for $\frac{2}{3}$-majority resolutions the required number of votes should be 20 (instead of 17). Grofman and Scarrow [41, p. 179] report that '[t]his "interpretation" of the 1936 charter was accepted by the board and went into effect in January 1938 although corrected census figures which excluded aliens changed the actual vote allocations slightly. The county's attorney recommended procedure was used to determine votes in the three subsequent reapportionments: 1942, 1962, and 1972.'[110]

[110]Grofman and Scarrow [41, fn. 8, p. 179] note that '[t]hese vote reductions and special majority requirement procedures are not well known; most authors who have discussed weighted voting in Nassau County Board of Supervisors were unaware of them. For example, Thomas [1960], in a book on Nassau County government mistakenly asserts that the 1942 apportionment violates the Charter provision that no town be given voting majority in that "Hempstead has 18 out of the 30 votes ... clearly more than fifty percent of the total".' They also claim in [41, loc. cit.] that the same mistake is made by Banzhaf [5] (who asserts that both the 1942 and 1962 apportionments resulted in three of the six Nassau County Board members having zero voting power and the remaining three having equal power according to the Bz index), as well as by Brams [11] and others. Moreover, it should be noted that all these authors, except [41], say that the number of votes assigned to the North Hemp-

In 1968, five qualified voters, one from each of the County's five municipalities, petitioned Nassau's Supreme Court to declare Article I, Section 104, of Nassau's charter (quoted above) unconstitutional, as it violated the OPOV principle. This case is known as *Franklin v Mandeville*.[111] Citing *Iannucci*, Nassau's Supreme Court held that the concept of OPOV applied to local legislative bodies. Specifically, the court found that the scheme of apportionment mandated by the county charter failed to comply with the 'equal protection' clause of the state and federal constitutions — and was therefore invalid in its entirety — because

> ... each citizen of the Town of Hempstead enjoys a voting status inferior to that of any other voter in Nassau County. These citizens cumulatively constitute substantially over 50 percent of the County's population, yet their representatives have less than 50 percent of the Board's total vote.[112]

The court directed the Board of Supervisors defendants to submit within six months a reapportionment plan that would be consistent with the OPOV principle.

On appeal, in 1969 the NY Appelate Division, Second Department, agreed with the findings of Nassau's Supreme Court that the weighted voting scheme of the county's Board of Supervisors violated the OPOV principle as stated in *Reynolds v Sims*.[113] In affirming the Nassau Supreme Court decision, this court ruled that it was unnecessary to take testimony on whether the plan was also

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stead representative was 7 (out of 30). In a private communication we received on 28 January 1997 from Howard A Scarrow (professor at the Department of Political Science, State University of New York at Stony Brook) he stated that the figure given in [41, p. 178] (6 out of 29) 'was based on data from the Nassau county archives', and added: 'Since other authors were unaware of the county attorney's modification of the votes derived strictly from the census counts they were unable to see that the number 7 for North Hempstead makes no sense. I have no idea where they got [that] number 7.'

[111] 294 NYS, 2d (1968), pp. 141 ff.

[112] Ibid., p. 148.

[113] *Franklin v Mandeville*, 299 NYS (1969), 2d, p. 954.

invalid under the prescription laid down in the *Iannucci* case.[114]

The New York Court of Appeals affirmed in 1970 this decision in principle but modified the order of Nassau's Supreme Court: it directed that, rather than using the 1960 census data, a valid reapportionment plan be adopted by the board within six months after the public announcement of the relevant results of the 1970 federal census.[115] In referring to Article I, section 104 of Nassau's charter, this court observed that 'inequality in some degree is mandated and, indeed, perpetuated by the charter provision … a vital factor which distinguishes the case from *Abate v Mundt* recently decided'.[116]

The Nassau County Board of Supervisors did not adopt a plan within the six-month period mandated by the Court of Appeals, but rather introduced a plan on 14 August 1972, which it adopted on 25 September 1972, as Local Law 13-1972.[117] Thereupon the plaintiff voters brought another action in Nassau's Supreme Court, for an order appointing a non-partisan commission to prepare a plan. They stated that since the Board of Supervisors had failed to act within the specified time, it 'forfeited the right to adopt a plan of its own'.[118] The plaintiffs further contended that the plan adopted by the board on 25 September 1972 was only a 'warmed over version of one previously held unconstitutional by the Court of Appeals' and was itself unconstitutional.

In setting the standard for the allocation of votes, Nassau's Local Law 13-1972 prescribed that

> the "voting power" of a Supervisor shall be measured "by the mathematical possibility of his casting a decisive vote on

[114] Ibid., p. 955. For the prescription referred to, see above, text to fn. 64, p. 100.

[115] *Franklin v Mandeville*, 26 NY, 2d (1970), pp. 69 f.

[116] Ibid., p. 69.

[117] *Franklin v Krause*, 338 NYS, 2d, (1972), p. 563.

[118] Ibid., p. 562.

4.5.1 Table
Weighted voting in Nassau County, 1972 ($q = 71$)

	Pop %	η	100β	Deviation
Hempstead #1	28.1	15	27.8	−0.3
Hempstead #2	28.1	15	27.8	−0.3
Oyster Bay	23.1	11	20.4	−2.7
North Hempstead	16.5	7	13.0	−3.5
Long Beach	2.3	3	5.6	+3.3
Glen Cove	1.8	3	5.6	+3.8

Range of deviations: $3.8 - (-3.5) = 7.3\%$.

Note In all four tables in this Section, the column headed 'Pop %' shows the population of each municipality as a percentage of the county's total population; the column headed 'η' shows the Bz score of the municipality's representative; and the column headed '100β' shows the representative's Bz index in percentage terms. The last column is obtained by subtracting the population column from the Bz index column.

> a particular matter." ... [T]he percentages of voting power "shall approximate" the corresponding percentages of population and it further guarantees that no town or city shall be wholly without voting power. Finally, in establishing its general standards for the system, the new plan requires that in preparing each reapportionment of votes defendant-Board shall employ "an independent computerized mathematical analysis" and any other methods which shall "most nearly analyze" the percentages of voting power and population.[119]

As far as we can see, this accords with the 1967 decision of the NY State Court of Appeals in the *Iannucci* case, which we quoted and discussed in § 4.3. In particular, the scheme for the allocation of votes in Law 13-1972 is the hybrid prescribed in that decision.

Assisted by an independent computerized analysis, the Board arrived at a total of 130 votes to be divided as follows among the

[119]Summary of the law, ibid., p. 563.

4.5.2 Table
Weighted voting in Nassau County, 1972 ($q = 92$)

	Pop %	η	100β	Deviation
Hempstead #1	28.1	12	25.0	−3.1
Hempstead #2	28.1	12	25.0	−3.1
Oyster Bay	23.1	10	20.8	−2.3
North Hempstead	16.5	10	20.8	+4.3
Long Beach	2.3	2	4.2	+1.9
Glen Cove	1.8	2	4.2	+2.4

six supervisors: Hempstead #1 and #2 — 35 votes each; Oyster Bay — 32 votes; North Hempstead — 23 votes; Long Beach — 3 votes; Glen Cove — 2 votes.[120] The quota for simple-majority decisions was set at 71, and for $\frac{2}{3}$-majority decisions at 92.[121]

In terms of relevant percentages and (absolute) deviations, this plan resulted in the figures shown in Table 4.5.1 for simple-majority decisions — assuming that it was indeed justified to view the two Hempstead representatives as two independent voters.[122]

Similarly, for a $\frac{2}{3}$-majority decisions the results were as shown in Table 4.5.2.

According to the Board of Supervisors, the scheme satisfied the standard set in *Iannucci*.

[120]Ibid., p. 564.

[121]The terms 'simple majority' and '2/3-majority' are misleading in this case: with a total of 130 votes, 66 votes are needed for true simple majority and 87 for a 2/3-majority. So in fact the 71 and 92 quotas correspond to two levels of what is called 'special majority'.

[122]If the two Hempstead representatives were committed or presumed to vote in tandem, then they should have been viewed as a single bloc voter, wielding 70 of the 130 votes (and thus a blocker). If this is the case, then for a quota of 71 the actual allocation of weights gave this Hempstead bloc a Bz score of 15, and 78.95% of the total Bz power — far more than intended! — whereas each of the other four supervisors got a Bz score of 1, and 5.26% of the total Bz power. For lack of evidence regarding the method of election of the two Hempstead representatives and their voting behaviour, we are unable to say whether they should have been viewed as two independent representatives or as a bloc.

However, Nassau's Supreme Court disapproved. It noted that the plan, 'by giving Hempstead 70 votes and requiring 71 votes for a majority allows Hempstead about 54% of the votes but requires about 54.6% of the votes to carry'.[123] This would require at least one vote from another municipality on the Board for a measure to carry, even though Hempstead had 56.27% of the population. So, 'under the plan now adopted, the Hempstead Supervisors are still barred from exercising majority vote'.[124] In addition, under this plan, the City of Glen Cove was given 'voting power equal to the City of Long Beach, which has one-third greater population than the City of Glen Cove.'[125]

Let us pause to analyse these comments. We have already noted that in our view the provisions of Nassau's Law 13-1972 are in line with the *Iannucci* decision and in particular with the hybrid scheme prescribed by that decision. The Nassau court's main criticism of this shows clearly that the court assumes PWP to be the correct scheme. Indeed, PWP would give the Hempstead supervisors more than half of the total number of votes, enough to carry on their own any simple majority decision. Compared to PWP, the hybrid scheme indeed under-represents Hempstead. But compared with PSQRR the hybrid scheme *over-represents* Hempstead: PSQRR is relatively less generous to larger constituencies.[126]

The court implies that because Hempstead has over half of the county's population, its supervisors ought to exercise on their own a majority vote on the board. At first sight, this proposition seems plausible, at least from a majoritarian point of view. However, it can only be justified provided the Hempstead municipality is

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[123] 338 NYS, 2d, p. 564.

[124] 338 NYS, 2d, p. 565.

[125] Ibid.

[126] Cf. Banzhaf's demonstration in [7] (discussed in § 4.2) that the allocation of votes in the federal Electoral College is biased in favour of the larger states rather than against them, as had been commonly believed.

By the way, if the two Hempstead supervisors voted as a single bloc, then — as suggested in fn. 122, p. 123 — the actual allocation of weights was all the more strongly biased in favour of Hempstead.

assumed to be a monolith, all whose citizens are normally of one mind with regard to issues decided by the board. Only then can it be presumed that the votes of the Hempstead supervisors reflect the uniform view of the whole of Hempstead, and hence of the majority of Nassau's citizens. On the other hand, on the more likely assumption that the citizens of Hempstead are themselves divided on most issues, the votes of the Hempstead supervisors can, at best, be presumed to reflect the views of a majority — in some cases a slender majority — within Hempstead, not necessarily a majority of the whole county.

If each constituency is assumed to be monolithic, so that all its citizens can be regarded as a single bloc with respect to the issues in question, then (as suggested in Com. 3.4.6) a good case can indeed be made for the PWP scheme.[127] On the other hand, under other, very different conditions (discussed in Com. 3.4.6), implementation of PSQRR is theoretically justified as a way of implementing the OPOV principle. But in any case the *Iannucci* hybrid is, as far as we can see, without theoretical justification.

The Nassau Supreme Court went on to remark that weighted voting was looked upon by most courts, including the Court of Appeals, as an 'interim' or 'stopgap' measure, not constitutionally acceptable as a permanent plan of reapportionment. The court then directed the board to propose a new plan 'based upon a system not utilizing weighted voting' within sixty days.[128]

But upon appeal, the NY Court of Appeals reviewed the plan in 1973 and noted that although 'the smaller communities are superenfranchised to a somewhat greater extent than the larger communities are disenfranchised ... the range of deviation [for simple majority decisions] is only 7.3% and the plan fits comfortably within the intendment of *Iannucci* ... as affected by subsequent case law.'[129] Consequently the Court of Appeals held that there was no constitutional infirmity in Nassau's Local Law 13-1972.

[127] We shall return to this point in Com. 4.6.2.

[128] 338 NYS, 2d, p. 570.

[129] 32 NY, 2d p. 237.

4.5.3 Table
Weighted voting in Nassau County, 1982 ($q = 65$)

	Pop %	η	100β	Deviation
Hempstead #1	27.94	15	28.85	+0.91
Hempstead #2	27.94	13	25.00	−2.94
Oyster Bay	23.14	11	21.15	−1.99
North Hempstead	16.54	9	17.31	+0.77
Long Beach	2.58	3	5.77	+3.19
Glen Cove	1.86	1	1.92	+0.66

Range of deviations: $3.19 - (-2.94) = 6.13\%$.

The US Supreme Court received the case in 1974 under its mandatory appelate jurisdiction, but dismissed it for want of a substantial federal question.[130]

In 1982, five registered voters along with the League of Women Voters of Nassau County brought an action in the Federal Court for the Eastern District of New York, seeking a declaratory judgment that Nassau's Local Law 2-1982, which amended Section 104(5) of the Nassau County Government Law, was unconstitutional because it violated the 'equal protection' clause of the US Constitution.

Nassau's Board of Supervisors had adopted Local Law 2-1982 on 8 March 1982, amending Section 104(5) by providing for a total of 108 votes (to reflect the decrease in the county's population revealed by the 1980 census), with a quota of 65 votes for so-called simple-majority decisions and 72 votes for $\frac{2}{3}$-majority decisions.[131] The 108 votes were distributed among the six supervisors as follows: Hempstead #1 (presiding supervisor) — 30 votes; Hempstead #2 — 28 votes; Oyster Bay — 22 votes; North Hempstead — 15 votes; Long Beach — 7 votes; Glen Cove — 6 votes. The revised percentages of populations, Bz power and deviations are shown in Tables 4.5.3 and 4.5.4.

[130] *Franklin v Krause*, 415 US Reports (1974), pp. 904 ff.

[131] *League of Women Voters of Nassau County v Nassau County Board of Supervisors*, CV 82-1607 E.D.N.Y. May 18, 1983.

4.5.4 Table
Weighted voting in Nassau County, 1982 ($q = 72$)

	Pop %	η	100β	Deviation
Hempstead #1	27.94	13	28.26	+0.31
Hempstead #2	27.94	11	23.91	−4.03
Oyster Bay	23.14	9	19.57	−3.57
North Hempstead	16.54	9	19.57	+3.03
Long Beach	2.58	3	6.52	+3.94
Glen Cove	1.86	1	2.17	+0.31

In discussing the impact of the 1973 ruling of the NY Court of Appeals in *Franklin v Krause*, the NY Eastern District Court now noted that the plaintiffs themselves conceded that the current range of (absolute) deviation for simple majority decisions (6.13%) is smaller than the corresponding 1972 range of deviation (7.3%), which the former court had upheld in *Krause*. However, the plaintiffs argued that in *Franklin v Krause* and *League of Women Voters* the court had inadvertently established a new standard by calculating the range of absolute instead of relative deviations, whereas had the latter range been used — as in *Abate v Mundt* — then the resulting range would have been 232%, thereby far exceeding the 12% (maximal) range established in *Abate v Mundt*.[132]

The judge, however, rejected the plaintiffs' argument. Invoking *Abate v Mundt* he stated:

> Under Local Law 2-1982, population is the controlling though perhaps not the only factor in the apportioning of weighted voting among the municipalities in Nassau County.

[132]The plaintiffs were of course correct. The appropriate measure of deviation is not the difference between the percentages of voting power and population, but rather the ratio of this difference to the latter percentage. Thus although the absolute deviations for Glen Cove and North Hempstead (which, as Table 4.5.1 shows, were the two extremes) were +3.8% and −3.5%, respectively, the relative deviations were +211.11% and −21.21%, respectively — yielding a relative deviation range of 232.33%. However, the plaintiffs proposed no alternative weighting scheme for the Nassau Board which would yield a smaller range of relative deviation. We shall return to this issue in Com. 4.6.4.

> Moreover, the relative voting strength of the municipalities parallels their relative populations. The Constitution does not require more. The County acted properly in permitting some variance from ideal population to voting power ratios in order to preserve existing subdivision lines and the existing legislative structure.[133]

On appeal in 1984, the Second Circuit Court also dismissed the case.[134]

However, this was not the end of the story. As we saw in § 4.4, the US Supreme Court in its 1989 decision in *Morris v Board of Estimate* reiterated its position of 18 years earlier in *Whitcomb v Chavis*, rejecting the Bz index as 'unrealistic'. In view of the *Morris* decision, the New York Civil Liberties Union persuaded in October 1991 eight registered voters from Hempstead and Glen Cove to challenge again the weighted voting system of Nassau County. This case, known as *Jackson v Nassau County Board of Supervisors*, was decided by Judge Arthur D Spatt of the US District Court, Eastern District, NY, on 11 April 1993.[135]

The plaintiffs asked the court to declare Nassau's weighted voting scheme unconstitutional for four reasons. First, the plaintiffs alleged that Nassau County's weighted voting system, using the Bz index, allocated weight to each municipality not in direct proportion to its population, thus violating on its face the OPOV principle mandated by the 'equal protection' clause of the 14th Amendment of the US Constitution. Second, the plaintiffs argued that the US Supreme Court had thoroughly repudiated the Banzhaf methodology in *Morris*; and that the defendants' contention that *Morris* did not signal a new doctrinal development sufficient to diminish the force of *Franklin v Krause* is simply erroneous. Third, that even if the Banzhafian methodology of weighted voting is not unconstitutional, the manner in which it was applied by Nassau County

[133] CV 82-1607 E.D.N.Y., 18 May 1983, pp. 9 f.

[134] *League of Women Voters &c.*, 737 F. 2d, pp. 155 ff.

[135] 818 Federal Supplement (1993), pp. 509 ff.

is flawed.[136] Fourth, the plaintiffs contended that the use of a weighted voting scheme which permits the creation of large districts as well as the election of representatives on an at-large basis (as in Hempstead) has the effect of diluting the electoral opportunities of the minority communities in Nassau County.

The court did not address the fourth claim. However, it did address the first three claims at length, surveying all relevant previous cases decided by various US courts, and agreed with the arguments made in an affidavit written on July 8, 1992 by Steven J Brams,[137] Professor of Politics at New York University, in support of the plaintiffs' arguments.[138]

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[136]The court's decision (818 Federal Supplement, pp. 534 f.) mentions that Banzhaf had sent the court a letter on 14 November 1991, in which he observed that neither the plaintiffs nor the defendants in this case had completely understood his index. He added that his index had not been properly applied by Nassau County and yet, ironically, even if it had been, 'the distribution of voting power among the six legislators under the current system is constitutionally flawed.' Having reviewed the pleadings, Banzhaf concluded that the plaintiffs had reached the correct conclusion, but for the wrong reasons.

[137]Mistakenly referred to in the court's decision as 'Abrams'.

[138]Steven Brams has kindly let us have a copy of this document, in which he made three main claims. First, that the quotas needed to pass resolutions (65 for simple-majority decisions and 72 for 2/3-majority) resulted in disproportionate representation for Hempstead when compared to the smaller towns. Second, that for simple-majority decisions Long Beach has an enormous overallocation of voting power in relation to Glen Cove, and that there is similarly a serious misallocation of voting power between Oyster Bay and North Hempstead because, for 2/3-majority decisions, they have equal voting power (but very unequal populations). Third, that the manner in which Nassau County measured deviations of the supervisors' voting power from the percentage of population they represented was inappropriate (for the reason explained above, see fn. 132).

The last point is undoubtedly correct; but it is not at all clear whether there exist other WVGs (for the two decision rules) yielding a smaller range of relative deviations. Anyway, Brams did not propose any such alternatives, but merely suggested that the plaintiffs should be given an opportunity to examine the computerized analysis made by L Papayanopolous (who was responsible for recommending the WVGs used by the Board). Moreover, commenting on the equal power of North Hempstead and Oyster Bay under the 2/3-majority WVG, Brams said (p. 6 of his affidavit): '[N]o model, in my opinion, can justify

The court agreed in effect with Brams that if the Bz index were to be used for measuring voting power, and if Nassau's supervisors were to have voting power proportional, as closely as possible, to the size of the population they represented, then the WVGs used by Nassau were improper. However, the court's more fundamental concern was whether the US Supreme Court's opinions regarding the Bz index, as expressed in *Whitcomb* and, especially, in *Morris*, compelled it to disqualify Nassau's use of the Bz index regardless of the particular WVGs it employed. The court decided that this was indeed the case:

> What is clear from [*Morris*], and what the defendants seek to minimize, is that the Supreme Court firmly rejected weighted voting, not only because of the mathematical quagmire such a system engenders, but just as importantly because the methodology fails to take into account other critical factors related to the actual daily operations of a governing body. There is no question that the Supreme Court took the opportunity to express not only preference, but a directive that legislators be elected by and represent citizens in districts of substantially equal size.
>
> ... In this Court's view, there is only an artificial and non-substantive distinction between the basis for rejecting weighted voting in the *Morris* case and for retaining weighted voting as a method of operation for the Board of Supervisors in this case.
>
> ... Having reviewed not only the case itself but also the briefs submitted to the Supreme Court by both parties, this Court finds the reasoning in *Morris* to be determinative of the issue of weighted voting.[139]

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allocating the same voting power to the Supervisors of North Hempstead and Oyster Bay. The fact that this allocation occurs at the higher decision rule, at which more important items are voted on the Board, makes it all the more egregious.' This seems to us somewhat doubtful. It is only to be expected, under any reasonable measure of voting power, that as the quota needed to pass decisions is set higher, thus approaching a unanimity SVG, an increasing number of members having different weights will have equal power. Brams's claim would have been much more persuasive had he been able to prove that some alternative WVG for 2/3-majority decisions is superior, by his criteria, to the one used by the Nassau Board.

[139] 818 Federal Supplement, p. 532.

Accordingly, the court ruled that 'the plaintiffs are entitled to partial summary judgment as a matter of law on their first and second causes of action' and that the court would confer with the parties on 26 April 1993 in order to determine 'the specific relief to be afforded to the plaintiffs', and to discuss 'the status of the case' and 'future course of litigation' as well as the fourth complaint made by the plaintiffs.[140]

According to a press report,[141] on 26 April 1993 Judge Spatt instructed Nassau County officials to present before him on 10 June 1993 plans to replace the county's Board of Supervisors, on which he would hold a hearing on 23 June 1993. The submissions were to include information on the demographic makeup of the county and suggestion on who should sit on a charter review commission and what timetable should be adopted for studying the proposed changes and submitting them to a referendum.

This ruling of Judge Spatt ended the weighted voting system which had existed in Nassau County, in various forms, for 75 years — longer than in any other county in the US. Like New York City's Board of Estimate, the Nassau Board of Supervisors was eventually replaced by an expanded county council, each of whose members is currently elected from 19 equal-size single-member districts.[142]

Since weighted voting systems based on the Bz index were instituted during the 1970s not only in Nassau County but also in many other counties in New York State,[143] it might have been expected

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[140] Ibid., p. 536.

[141] *New York Times*, 27 April 1993, p. B6.

[142] Private communication from Howard A Scarrow, cited above in fn. 110, p. 120.

[143] By June 1977, 24 of the 57 counties in New York State (excluding the five boroughs of New York City) had adopted simple or computerized weighted voting systems for their boards of supervisors, based on the Bz index. For the list of these counties (which included Nassau) see [41, p. 175–176]. A detailed outline of the weighted voting system that was used in one of these counties (Tompkins) in 1982 appears in [67, p. 55]. All these counties but one use single-member unequal districts and apply the *Iannucci* hybrid, adjusting the representatives' weights so that their Bz power is approximately proportional

that as a result of the above rulings by the US Supreme Court and by Judge Spatt, all those other counties would also abolish their weighted voting systems. However, this has not happened; except for the Board of Estimate of New York City and the Board of Supervisors of Nassau County, all other 23 counties in New York State (and perhaps also elsewhere) that had instituted weighted voting systems based on the Bz index have gone on using them. Moreover, four of these New York counties (Delaware, Jefferson, Oswego and Sullivan) have had their weighted voting schemes approved by a New York State court.[144]

4.6 Concluding Comments

To conclude this chapter, we bring together, reiterate and amplify some comments we have made earlier on. We also add some new comments on related matters. In all but one of our comments we take it for granted that the decision-making systems referred to can reasonably be modelled by SVGs. The one exception is Com. 4.6.3, which is a digression concerning a different kind of two-tier system, not discussed elsewhere in this book.

4.6.1 Comment The simplest sort of two-tier decision-making scheme — single-member constituencies of equal size, with a symmetric decision rule[145] at the top tier — has much to recommend it. Pragmatically speaking, it is straightforward to operate and does not require complex calculations. Also, as we remarked in § 4.4 in

to their districts' population as determined by the national census conducted at the beginning of every decade. The remaining county (Cortland) uses the same system but reapportions its districts every 10 years so that their population sizes do not deviate from one another by more than 5%. (Private communication from John Maceli, professor at the Department of Mathematics and Computer Science, Ithaca College, Ithaca, NY, 31 January, 1996.)

[144]Private communication from Howard A Scarrow, cited above in fn. 110, p. 120. See also [67, p. 55] where it is reported that '[a]djusted weighted voting has now [1992] been in common use in many county governments in New York State for two decades'.

[145]As we saw in Rem. 2.3.11(ii), this simply means a WVG in which all voters — in this case the delegates — have the same weight.

connection with the *Morris* case, the very simplicity of this scheme makes it transparent and acceptable to the general public.

Theoretically speaking, this simple scheme is a special case of both PSQRR and PWP; therefore it shares any theoretical justification possessed by either of these systems. This is a considerable advantage, because the circumstances which justify prescriptive use of PSQRR are in general quite different from those in which use of PWP can be justified.

Certainly, where the choice is between the scheme of equal single-member districts and the *Iannucci* hybrid with unequal districts, the former wins hands down, for the simple reason that the latter lacks any coherent theoretical justification. The hybrid system is a half-baked product of incomplete grasp of the Penrose–Banzhaf theory of voting power: it accepts the measure of voting power offered by this theory, but fails to appreciate its implications for two-tier systems.

So those court rulings that rejected the *Iannucci* hybrid and led to the eventual establishment of equal single-member districts, most notably in Nassau County, NY, were justifiable, although some of the arguments used by the judges in support of those rulings were incorrect.

However, as noted in § 4.1, instituting and maintaining a system of single-member equal-sized districts has its disadvantages, such as the need for periodical re-apportionment of the districts in order to maintain their equal size. This, in turn, may increase the number of districts as well as the risk of gerrymandering.

Besides, there are some decision-making bodies — particularly international organizations such as the CMEC or the International Monetary Fund — where unequal-sized (national) units must be represented.[146] If an organization of this kind wishes to allocate to national representatives on its board a priori I-powers that are approximately proportional to some agreed-upon magnitude (for example, the wealth of their respective countries, or the square root of their population) then the only realistic option is to employ an appropriate SVG based on the Bz index.

[146] The weighted voting system in the CMEC is discussed in Chapter 5.

4.6.2 Comment Consider a two-tier system with a given set of constituencies, not necessarily of equal size. According to the theory developed in § 3.4, PSQRR provides the solution to the problem of equalizing the citizens' indirect a priori voting powers. However, as we pointed out in Com. 3.4.6, normative application of PSQRR as a means of implementing the OPOV principle in real-life situations, is fully justified only under certain conditions, the most important of which can realistically be taken to hold if the division of the citizenry into constituencies (each of which is internally heterogeneous) is more or less random, unconnected in any systematic way with the attitudes of citizens to the issues that are to be decided by the council.

Let us now assume that the opposite extreme is the case: each constituency is a homogeneous monolith, whose citizens can all be taken to act as a single voting bloc. As we observed in Com. 3.4.6, application of PSQRR in these circumstances may lead to unacceptable results.

Moreover, it is arguable that where citizens are grouped into well-defined monolithic blocs, which are relatively stable in the long term, the very notion of OPOV becomes inoperable. This is because the OPOV principle is a principle of *individual* equality: it aims to equalize the voting powers of the *individual* citizen-voters. However, as we pointed out in Rem. 2.3.24(ii), when a bloc $\&_S$ is formed, this creates a new SVG, in which the original members of S no longer exist as distinct voters, but are replaced by a single new voter, $\&_S$. Thus, in the situation we are now considering (that of stable monolithic blocs) the individual citizens do not feature as voters; the blocs are the 'voters'. Moreover, as we shall see in § 7.2, voting power (under *any* reasonable interpretation of this concept that is applicable to SVGs) is not additive: if a bloc $\&_S$ consists of k formerly separate individual voters, it simply does not make sense to say that each of these 'possesses' a k-th of the bloc's voting power. So in *these* circumstances Justice Harlan's phrase, dismissing OPOV as a will-o'-the-wisp, becomes quite apt.[147]

[147] This is not to say that Harlan was necessarily justified in so deprecating the OPOV principle *in the context in which he uttered this phrase.* He would indeed

So, if the OPOV principle is inoperable in the presence of stable monolithic blocs, must the ideal of equal suffrage be given up altogether? Not necessarily. It is arguable that while the OPOV principle seeks to achieve equality of suffrage at the individual level, that of the individual citizen-voter, the principle of *proportional representation* seeks to achieve equality, or at least equity, of representation at the collective or group level, that of the voting bloc. Under proportional representation, the upper tier (council) is a scaled-down replica of the citizenry at large, with each bloc of like-minded citizens represented in proportion to its size.

If this line of reasoning is accepted, then it provides a justification for the PWP system — not as a means of implementing the OPOV principle, but rather as a sort of proportional representation in circumstances where constituencies are stable monolithic blocs, so that the OPOV principle is inoperable.[148]

4.6.3 Comment For real-life two-tier systems of the kind considered so far, the upshot of the analysis in Coms. 3.4.6 and 4.6.2 is that under one set of conditions, application of PSQRR is justified as a way of implementing the OPOV principle, but not proportional representation; and under the polar opposite of these conditions PWP may be justified as a means of achieving a sort of proportional representation, but the OPOV principle is inoperable. This polarity holds even if the constituencies are of equal size, although in this case the PWP and PSQRR *formulas* happen to coincide.

In this connection it may be of interest to describe briefly a different kind of two-tier decision-making system, which has been proposed in the literature (but as far as we know never used in practice) and which in a broad spectrum of cases is capable of implementing simultaneously both proportional representation and the OPOV principle.

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be justified only if we make the assumption (which seems to us quite doubtful) that the cases before the Supreme Court were closer to the monolithic districts pattern than to the situation in which PSQRR is reasonably applicable.

[148] Cf. Com. 3.4.6.

The system, proposed and examined in considerable detail by the jurist Akhil R Amar in [2], may be called *probabilistic proportional voting*, briefly PPV. It differs from the systems considered so far in one crucial respect: within each constituency, decisions are made by lottery. When electing a delegate, each citizen of a constituency casts a ballot for one candidate in the usual way. Then a ballot is drawn at random, and the candidate marked on it wins the election. The probability of a given candidate being elected as the constituency's delegate is proportional to the number of votes cast for that candidate.[149] (Alternatively, when a bill comes up for decision, a citizen is chosen at random in each constituency to serve as the constituency's delegate, and is allowed to vote in the council as he or she sees fit.)

The decision rule at the top tier (council) is a normal deterministic one, so that the council can be modelled as an SVG. But the composite two-tier system as a whole clearly cannot be so modelled.

Now, without any need for a sophisticated theory of voting power it is patently clear that the direct voting power of a citizen within the i-th constituency can be taken to be $1/n_i$, where n_i is the constituency's size.[150] Hence, if the Bz power of the i-th delegate in the council is β'_i, the indirect I-power of a citizen of the i-th constituency can be taken to be β'_i/n_i. Thus, citizens' indirect voting powers are equalized iff the Bz power of the i-th delegate is directly proportional to the size n_i of the i-th constituency.[151]

This implementation of the OPOV principle works whether the constituencies are random divisions that are internally heterogeneous, or monolithic blocs, or anything in between these two extremes; because under PPV the vote of each citizen always counts individually in the same way (that is, with the same probability) irrespective of the political make-up of the constituency or of the

[149] In the social-choice literature, this is sometimes called the *random dictator* election procedure.

[150] Cf. p 5, fn 8.

[151] This is the same formula as in the *Iannucci* hybrid. However, in the present very different context this formula has a firm theoretical basis.

system as a whole.

On the other hand, if the number of constituencies is sufficiently large, and if the council operates a weighted decision rule with weights proportional to constituency sizes, then PPV yields, with very high probability, representation which is as near proportionality as makes no difference. This is so whether the political groups of like-minded citizens coincide with the constituencies or cut across them.[152]

Thus PPV with equal-sized constituencies implements simultaneously both the OPOV principle and proportional representation.

4.6.4 Comment We shall now address, in the light of the US experience, some technical problems that arise in trying to find a voting rule whose distribution of Bz index values is 'as close as possible' to a prescribed distribution.

The general form of this problem is as follows. We are given an m-tuple of positive numbers p_i (where $i = 1, \dots, m$) such that $\sum_{i=1}^{m} p_i = 1$. We must find a *solution* SVG $\mathcal{V}$ with assembly $I_m = \{1, \dots, m\}$, such that the m-tuple of Bz index values $\beta_i[\mathcal{V}]$ is as close as possible to the m-tuple p_i.

For example, if the aim is to implement PSQRR in a county divided into m given constituencies, then the solution $\mathcal{V}$ will model the decision rule to be used by the council; and

$$p_i = \frac{\sqrt{n_i}}{\sum_{j=1}^{m} \sqrt{n_j}},$$

where n_i is the size of the i-th constituency.

On the other hand, if the aim is to implement the *Iannucci* hybrid then

$$p_i = \frac{n_i}{\sum_{j=1}^{m} n_j}.$$

[152] For this and other admirable properties of PPV, particularly from a US political and constitutional standpoint, see [2]. However, despite its many virtues, PPV does not stand much chance of being adopted for decision making in situations where delegates (members of the upper tier) are professional politicians. This is because under PPV even a successful delegate, enjoying wide support among his or her constituents, cannot be sure of being re-elected next time round.

Now, as mentioned in Rem. 3.4.4(iii), in general it may be impossible to find a solution whose β_i are exactly equal to the given p_i; so an approximate solution must be used. In order to choose between several (otherwise equally acceptable) approximate solutions, we need some reasonable measure of the distance between the two m-tuples. The smaller the distance, the better the solution, other things being equal.[153] This issue came up in several of the court cases we discussed earlier in this chapter: for example, in *Abate v Mundt*, *Morris v Board of Estimate of City of New York*, *Franklin v Krause*, *League of Women Voters v Nassau County Board of Supervisors* and *Jackson v Nassau County Board of Supervisors*.[154]

The raw deviations $\beta_i - p_i$ do not provide a reasonable measure of distance. For example, if for some i we have $p_i = 0.50$ then a deviation of ± 0.01 is relatively insignificant. But if $p_i = 0.02$ then a deviation of ± 0.01 will give the i-th delegate half, or half as much again, the required relative voting power — which is much less acceptable.

Therefore we must use the *relative* deviations,

$$d_i = \frac{\beta_i - p_i}{p_i} = \frac{\beta_i}{p_i} - 1.$$

Note that unless a solution is exact, some of its d_i are positive and some negative.

Various functions of the d_i may be used as a measure of distance. For example, in some of the above-mentioned court cases it was suggested, quite reasonably, that the *range* of the d_i,

$$\max_i d_i - \min_i d_i,$$

be used as measure of the distance. (Note that $\max_i d_i$ and $\min_i d_i$ are respectively the most extreme positive and negative relative deviations.) Another measure that could reasonably be used is the

[153] The qualification 'other things being equal' is essential, because, as mentioned in Rem. 3.4.4(iii), there may be other criteria for preferring one kind of solution to another.

[154] See, in particular, fn. 132, p. 127 and text to that footnote.

mean square relative deviation,

$$\frac{\sum_{i=1}^{m} d_i{}^2}{n}.$$

However, in our view the most appropriate measure is the absolute magnitude of the most extreme relative deviation:

$$\max_i |d_i|.$$

The reason for preferring this as measure of distance is that the extreme (positive or negative) value among the d_i is most invidious. So, for example, a single constituency that happens to be landed with an extreme negative value of d_i is not compensated by the fact that several other constituencies get small positive values.

4.6.5 Comment The most thorny technical problem is searching for a suitable solution. For small values of m, there are tables that list all possible (isomorphism types of) SVGs with m voters.[155] But as m increases, the number of different (non-isomorphic) SVGs increases at an enormous rate, at least exponentially if not faster. It is out of the question to list or to search systematically for all of them, even using the fastest computer.

In practice, certain kinds of SVGs are ruled out as solutions. Thus, an improper SVG is normally unacceptable. Also, for most practical purposes non-weighted SVGs are undesirable for various reasons, primarily because their operation is not so transparent.[156] Certainly if the American experience and that of the CMEC is anything to go by, the solution must normally be a WVG whose quota is about $\frac{1}{2}$ or $\frac{2}{3}$ of the total weight. Even so, systematic

[155] See Rem. 3.2.9(ii).

[156] This may mean that an exact unweighted solution may have to be discarded in favour of an approximate weighted one. For example, let $m = 5$ and $p_1 = 7/19$, $p_i = 3/19$ for $i = 2, 3, 4, 5$. An exact solution is the SVG whose MWCs are $\{1, 2, 3\}$ and $\{1, 4, 5\}$, which is easily seen to be unweighted by an argument similar to that used in Ex. 2.3.18. By trial and error it is not difficult to see that no WVG yields exactly the required β_i. The nearest thing is $[7; 3, 2, 2, 2, 2]$, which yields $\beta_1 = 5/13$ and $\beta_i = 2/13$ for $i = 2, 3, 4, 5$.

listing or search for all such potential solutions may be out of the question, except for fairly small m. The method actually used is that of successive approximation by trial and error. This is how it is described by Lucas in [67, p. 44]:

> First, make a "good" guess at what weights $w_A, w_B, \dots$ and quota q (or else minimal winning coalitions) you feel might work. One could, for example, begin with the given population $p_A, p_B, \dots$ (or their approximates) as weights and try various quota $q > w/2$ [where w is the total weight]. Next, one calculates the resulting Banzhaf power indices ... for the choice of weight and q. Compare these indices with the respective populations. If they fail to be a good approximation, then alter the initial guess for the weights or q accordingly. That is, if a voter's power index is too small, then increase his or her weight, and if one's power index is too large, then decrease this voter's weight. Use these new weights and quota as one's second guess, and again calculate the resulting power index for this second trial.
>
> Note that one must alter the weights (that gave the unsatisfactory approximation) by a sufficient amount so as to alter the set of minimal winning coalitions. This is necessary because the Banzhaf index depends upon the minimal winning coalitions, and many different weights may give rise to the same winning coalitions.
>
> Finally, one continues in this manner of repeatedly guessing weights and q, calculating the resulting power, and comparing power with populations until one arrives at a reasonably satisfactory approximation of the ratios for power to the corresponding population. ... Courts usually approve of adjusted weighted voting systems when the ratio of the power indices to populations is within a few percent, say 2% or 3%, (and sometime up to 5%.)

Although this sort of procedure may usually work in practice, it does not rest on a sufficiently firm theoretical foundation: there is no theoretical guarantee that it will home in on the best solution. This is because the Bz index suffers from the *fattening paradox*: there are cases where by *increasing* the weight of one voter (while leaving the other voters' weights and the quota unchanged), that voter's Bz index actually *decreases*.[157] So it would be very desirable to find a successive approximation procedure that has a sounder theoretical basis.

The title of Banzhaf's 1965 paper [5] asserts bluntly: *weighted voting doesn't work*. Evidently, what he means is that it doesn't work as a method of implementing the OPOV principle. In our view this assessment is too pessimistic. True, weighted voting is not actually working, at least so far as the US is concerned; but then to the best of our knowledge it has never been applied correctly there. However, we believe that it can be made to work — under suitable circumstances. It is certainly no panacea: the foregoing comments suggest that choosing a system of representation is no easy matter. Every system of representation has some disadvantages, and none is fair and reasonable in all circumstances.

[157] We shall discuss this paradox in Rem. 7.8.7. As far as we know, it has not been noticed or suspected so far.

5. Weighted Voting in the CMEC

5.1 Legislative Process of the EC

The Council of Ministers (CM) is the principal law-making organ of the European Community (EC).[1] At present it is perhaps the best-known international body that uses a system of weighted voting for passing most of its decisions.

This is how the legislative process of the EC operates. The European Parliament, the CM and the Commission — acting jointly — make regulations and issue directives, take decisions, make recommendations and deliver opinions. A *regulation* is binding and directly applicable in all member states. A *directive* is binding as far as the result to be achieved is concerned, but leaves to the national authorities the choice of forms and methods for achieving

[1]The other top organs of the EC are the Commission, the European Parliament, the European Court of Justice, and the Court of Auditors.

The EC is the most important of three technically separate but interconnected 'pillars' that constitute the European Union (EU). The EU was formed by the Maastricht Treaty which was signed on 7 February 1992 and took effect in November 1993. The task of the EC is to promote harmonious and balanced development of economic activities among the (currently 15) member states. The other two 'pillars' of the EU are common foreign and security policy (CFSP), and common justice and home affairs policies (JHA).

The EC comprises three formerly separate but interrelated European Communities: the European Coal and Steel Community (ECSC), the European Economic Community (EEC), and the European Atomic Energy Community (EURATOM).

it. A *decision* is binding on whom it is addressed to. *Recommendations* and *opinions* have no binding force.

When the CM acts on a proposal from the Commission, unanimity is required for an amendment to that proposal. As long as the CM has not acted, the Commission may alter its proposal at any time during the procedures leading to the adoption of an EC act.

There are two procedures through which the European Parliament can participate in EC decision making, depending on whether or not the subject matter is one in which the Parliament has 'co-decisionmaking' power — the power of veto. In either case, the Commission submits its proposal to the European Parliament as well as to the CM.[2] The CM, having obtained the opinion of the Parliament, adopts — using a weighted voting system to be described in § 5.2 — a 'common position'. This is then sent to the Parliament, and if approved by the latter, it is finally passed by the CM.

The Parliament can also reject or amend the proposal; for this an absolute majority is needed. From this point on, the two procedures differ.

If the proposal is in an area where the Parliament has no 'co-decisionmaking' power and is rejected by the Parliament, the CM can still adopt it, but only by unanimous vote. If the Parliament attaches amendments to the proposal, the Commission may modify it within a month, taking the amendments into account, and sends it back to the CM. The CM can then either adopt the proposal (by its weighted voting system) or amend it on a unanimous vote. The Commission also forwards to the CM any parliamentary amendments it has not accepted. The CM may adopt these, but again only by a unanimous vote.

For proposals that fall in areas that do allow 'co-decisionmaking' the rules are different. If the Parliament rejects the proposal, the

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[2]The Commission has the sole prerogative of formally initiating legislation in EC policy areas. However, in practice much of the policy is suggested by national governments of the member states or by the CM.

CM may convene the conciliation committee — a joint Parliament–CM body — to explain its position. After this, the Parliament must either propose amendments or confirm its rejection. In the latter case, the bill is dead.

If the Parliament proposes amendments, the amended text is then sent to the CM and to the Commission, which deliver their opinions. If the CM approves (by its weighted voting system) all the amendments of the Parliament, the common position is adopted. If the Commission objects to the amendments, a unanimous vote is required by the CM for adoption. If the CM rejects the Parliament's amendments, the president of the CM and the president of the Parliament must convene the conciliation committee to seek an agreement. If the committee cannot produce an agreed text, the proposed bill is dead. If the committee does produce an agreed text, but either the Parliament (by a majority vote), or the CM (by its weighted voting system) still rejects it, then again the proposed bill is dead.

In 1974 it was agreed that the heads of state or of government[3] of the EC member states, together with their foreign ministers, would hold summit conferences at regular intervals. In time these meetings were formalized as the 'European Council'.[4]

In 1986 they attained official status in EC law by virtue of Article 2 of the Single European Act (SEA), which confirms the right of the president of the Commission and one other Commissioner to attend. Meetings are held at least twice a year at the capital of the member state holding the CM presidency (which rotates every six months) and are chaired by the host head of state (or government). The discussions of the European Council may cover matters concerning all three 'pillars' of the EU. However, when it discusses EC matters it acts in the capacity of the CM.

[3]Up to the time of writing this has meant: the president of France and the prime ministers of all the remaining member states.

[4]Not to be confused with the CM or with the Council of Europe; the latter is not part of the EU structure.

In practice, however, it seems that even when dealing with EC matters the European Council takes only general political decisions; these are then translated into legal form at normal meetings of the CM held at ministerial level.

5.2 Evolution of Weighted Voting in the CM[5]

The historical foundations of the EC were laid in what are called 'the three founding treaties': the treaty establishing (in 1951) the European Coal and Steel Community (ECSC),[6] the Treaty of Rome (25 March 1957) establishing the European Economic Community (EEC), and the treaty establishing (in 1958) the European Atomic Energy Community (EURATOM).

The draft of the Schuman Plan was signed in Paris by the foreign ministers of France, Germany,[7] Italy and the three Benelux countries (Belgium, The Netherlands and Luxembourg) on 18 April 1951; and following its ratification by the parliaments of the signatory countries, the High Authority (later to become the Commission) of the ECSC took office on 10 August 1952, with Jean Monnet as chairman. On 8 September 1952 the CM held its organizational meeting in Luxembourg; and two days later, in Strasbourg, the Common Assembly of the ECSC (later to become the European Parliament) met for the first time. The first trainload of coal crossed the Franco-German frontier without the traditional customs formalities in February 1953.

On 20 May 1955 the Benelux governments sent formal proposals to France, Italy and Germany for new integrative moves 'which may best be taken in the economic field'. The Benelux Memoran-

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[5]This section is based partly on [42, pp. 107–110], [43, pp. 14–15], [107, pp. 264–265], Hartley [44, pp. 16–21], and Teasdale [106].

[6]Known as the Schuman Plan, after the French foreign minister, Robert Schuman, who proposed it on behalf of the French government on 9 May 1950.

[7]Here and in the sequel, 'Germany' in the pre-unification period refers to the German Federal Republic, commonly known as 'West Germany'.

dum proposed a conference for the purpose of beginning work on a treaty on the pooling of transport, power, and atomic energy; a treaty on general economic integration; and a treaty defining the European institutions necessary to carry out the entire program. The foreign ministers of the ECSC appointed a committee (chaired by Paul-Henri Spaak, the Belgian foreign minister) to study these proposals. The work of this committee led to the the EEC and EURATOM treaties.[8]

From 1958 the three Communities were served jointly by an Assembly (which has since become the European Parliament), the Council of Ministers (CM), and the European Court of Justice. In July 1964 the CM agreed on the merger of the executive authorities of the three Communities in a single consolidated Commission, to be followed by a merger of the Communities themselves. The Communities were actually merged into a single European Community (EC) by the Single European Act (SEA), signed in February 1986 with effect from 1 July 1987, which also institutionalized — outside the normal EC procedure — the European Political Co-operation.

Let us now outline the evolution of the weighted voting system used by the CM.

The six original signatories of the ECSC treaty were also the original signatories of the EEC treaty: France, Germany, Italy, and the three Benelux countries. Articles 145–149 of the EEC treaty established the responsibilities of the CM and the manner in which it can take decisions, as follows.[9]

> *Article* 145
>
> With view to ensuring achievement of the objectives laid down in the present Treaty, and under the conditions provided for therein, the Council shall

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[8] Cf. Palmer and Perkins [76, pp. 455–458].

[9] Quoted from Friedmann [40, pp. 434–435]. The position is similar under the EURATOM treaty (Art. 118); but under the ECSC treaty (Art. 28) the CM had much less extensive powers.

— be responsible for coordinating the general economic policies of Member States;

— exercise powers of decision.

Article 146

The Council shall be composed of representatives of Member States, each Government appointing to it one of its Members.

Each of the members of the Council shall act as Chairman for a period of six months in rotation, following the alphabetical order of the Member States.

Article 147

Meetings of the Council shall be called by the Chairman on his own initiative, or at the request of one of its members or of the Commission.

Article 148

1. Except where otherwise provided in the present Treaty, decisions of the Council shall be taken by a majority of its members.

2. In the case of Council decisions requiring a prescribed majority, the votes of its members shall be weighted as follows:

Belgium	2
France	4
Germany	4
Italy	4
Luxembourg	1
Netherlands	2

Decisions receiving at least the following number of votes shall be regarded as adopted:

— in the case of decisions which the present Treaty requires to be taken on proposal by the Commission: twelve votes;

> — in other cases: twelve votes representing a favourable vote by at least four members.
>
> 3. Abstentions by members either present in person or represented shall not prevent the adoption of Council decisions requiring unanimity.[10]
>
> *Article* 149
>
> When the present Treaty requires that a Council decision be taken on proposal by the Commission, the Council may amend such proposal by a unanimous vote.
>
> So long as the Council has not taken its decision, the Commission may amend its original proposal, in particular, in cases where the Assembly has been consulted on the proposal in question.

From the wording of Art. 148(1) it might seem as though decisions of the CM could normally be passed by the votes of a simple majority of the members. In fact, however, the specific provisions of the EEC treaty, dealing with almost every matter of importance, require decisions of the CM to be taken either according to the rules specified in Art. 148(2) or unanimously.[11]

The original Art. 148(2), quoted above, laid down *two* different decision rules: a pure weighted rule — in effect, a WVG — in which

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[10]While a member-state's abstention does not block a decision requiring unanimity, its absence does — as France's did during the crisis of 1965–66 described below.

[11]The EEC treaty stipulated that after the completion of a 12-year transitional period (which ended on 31 December 1969) two categories of measure would require unanimity. First, *constitutional measures*: enlargement of the Community (Art. 237), association agreements (Art. 238), institutional adjustments (Art. 165), extension of powers (Art. 235), and revision of the treaty (Art. 236). Second, *derogation from the treaty*: transport provisions (Arts. 76, 84(2)), state aids (Art. 93(2)), conjunctural policy (Art. 103(2)), and the approximation of laws (Art. 100).

Simple majority decisions were required by the EEC treaty in only a few cases, namely, in relation to the movement of workers (Art. 49), vocational training (Art. 128), the CM's Rules of Procedure (Merger Treaty, Art. 5), rules of procedure for the CM's committees (Art. 153), and the collection of information by the CM (Art. 213).

the quota needed to pass a decision was approximately 71% of the total weight (12 out of 17); and a composite rule in which, in addition to this quota, a prescribed number of the member states (four out of six) must assent in order for the decision to pass.[12] In subsequent amendments of Art. 148(2), which followed the accession of new members, the quota and the weights assigned to each of the member states in the CM were adjusted so that the quota was pegged at about 71% of the total weight.[13] Also, the words 'prescribed majority' in the original Art. 148(2) were replaced by 'qualified majority'.

In the remainder of this chapter we shall be concerned solely with the pure weighted rule, commonly known as 'qualified majority voting' (QMV) — a term we shall also use from now on.

During the entire history of the CM up to the time of writing (spring 1998), the weights assigned to member states always varied roughly in line with population size. But, rather than being in strict proportion, the weight distribution was skewed in favour of the smaller and medium-sized countries: their share in the total weight was larger than their share in the total population of the EU.[14]

The Treaty of Rome provided for a transitional period of 12 years (divided into three 4-year stages) by the end of which all articles of the treaty would be fully implemented. It stipulated that QMV in the CM would be implemented at the beginning of the third stage, on 1 January 1966. Until that date, so it was agreed, every member state could veto any proposed CM decision which it considered contrary to its vital interests. Thus, in effect, all decisions of the CM during the first eight years of the EEC were supposed to be unanimous.[15]

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[12]This composite rule also happened to be a WVG: it was isomorphic to [10; 3, 3, 3, 2, 2, 1]. The composite rule was adjusted with each expansion of the EC. At the time of writing, it states that for a resolution to be adopted it must get at least 62 (out of 87) votes cast by 10 (of the 15) member states.

[13]See Table 5.3.2.

[14]See Table 5.3.3.

[15]As observed in Ex. 1.1.3, Luxembourg was to become a dummy in QMV,

However, from the middle of 1965 France boycotted the EEC institutions (the so-called 'empty chair' policy) in protest against various developments, including the imminent onset of QMV in the CM.[16] Thereafter France refused to participate in the work of the CM whilst otherwise observing its EEC obligations. The CM continued to meet in the absence of the French representative and urged France to resume her place. In response, General de Gaulle (who was then conducting his campaign for re-election as president of France) argued that a reform of the institutional structure of the EEC was necessary, because he believed that implementation of QMV in the CM would have unacceptable federalizing effects, subjecting France to the strictures of a 'foreign majority'.

Eventually an informal meeting of the CM held in January 1966 in Luxembourg ended the dispute by an agreement to disagree,

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according to the original Art. 148(2) of the treaty. In so far as Luxembourg was at all aware of this fact, the agreement to postpone the actual implementation of QMV for eight years may have induced it to accept its theoretically intolerable position in the CM, in the hope that within those eight years the situation might change with the accession of new members. As Table 5.3.4 makes clear, Luxembourg indeed ceased being a dummy in QMV in 1973, when Britain, Denmark and Ireland joined the EEC.

[16] In the background there was lingering friction between the CM and the Commission, as well as between France and all the other members, ensuing from President de Gaulle's speech of 14 January 1963: on behalf of France, he vetoed Britain's application for membership in the three European Communities — on the grounds that Britain was politically, psychologically and constitutionally unready for membership — despite the fact that all other member states, to various degrees, favoured Britain's admission. (De Gaulle's invocation of a community of continental European interests and aspirations, as against the essentially different and alien character of the 'Anglo-Saxons', was quite ironic, as no statesman has been more contemptuous than him of moves towards any diminution of national sovereignty within the EEC. Thus the defiant nationalist attitude of Gaullist France rekindled nationalist differences within the EEC.) However, a package of unrelated proposals brought the conflict to a head. These included a deal on farm-price regulations of particular interest to France and the Commission's proposals relating to the change of EEC financing from direct contributions to its own resources, as well as increased budgetary powers of the European Parliament. The CM agreed to resolve these matters by 30 June 1965; and when, by midnight, no agreement had been reached the French President of the CM refused to continue the meeting.

known as the 'Luxembourg Accords'. Its main provisions regarding QMV were the following:

> I. Where, in the case of decisions which may be taken by majority vote on a proposal of the Commission, very important interests of one or more partners are at stake, the Members of the Council will endeavour, within a reasonable time, to reach solutions that can be adopted by all Member States of the Council while respecting their mutual interests and those of the Community, in accordance with Article 2 of the Treaty.
>
> II. With regard to the preceding paragraph, the French delegation considers that where very important interests are at stake the discussion must be continued until unanimous agreement is reached.
>
> III. The six delegations note that there is a divergence of views on what should be done in the event of a failure to reach complete agreement.[17]

There was a fourth paragraph which stated that some specified matters, all concerning agriculture, should be decided by common consent.

Although it was quite clear to the six member states that the Luxembourg Accords had no legal validity,[18] decisions of the CM in most policy fields continued to be unanimous even where no 'very important interests' were asserted by any member state.

However, despite the Luxembourg Accords, the CM applied QMV as of 1966 in a limited number of policy areas — including decisions on the Community's budget and the operation of the

[17] Quoted in Hartley [44, p. 18].

[18] The Accords could not be regarded as amending the EEC treaty since they had not been reached in accordance with the amendment procedure laid down in the treaty, and had not been ratified by the member states in accordance with their constitutional practice. Nor could the Accords be regarded as a decision concerning the interpretation of the treaty, since the CM has no competence in this respect, the exclusive power of interpretation of Community law having been vested in the European Court of Justice. To date the Court has never been asked by any member state to render its opinion on the legality of the Accords — which is not surprising, as it can quite confidently be predicted that the Court would nullify them.

Common Agricultural Policy (CAP). This was politically feasible since France, while otherwise objecting to QMV, wanted the CAP and its associated spending to survive anyone else's veto; therefore use of QMV in these fields suited France's interests.

A summit meeting of the European Council held in Paris on 9–10 December 1974 attempted a further shift in favour of QMV: the heads of government of the member states declared that it was necessary to renounce the practice of insisting on unanimity on all questions; and that, instead, the decision rules in the CM as prescribed in the treaties should be implemented more fully. However, in practice nothing much changed for several years.

The next development occurred in 1982 when the CM adopted an agricultural price increase in the face of an attempted British veto. In what appeared to have been a pre-planned move, the Belgian president of the CM called for a division, disregarding the British representative's protestation that very important interests of Britain were at stake. In the division Britain, Denmark and Greece abstained while the other seven members voted in favour, thus ensuring the necessary majority required by (the revised) Art. 148(2) of the EEC treaty.[19] Britain was outraged; but a month later, after a meeting of the CM called specially to discuss the matter, the British foreign secretary Francis Pym claimed that the right of veto remained intact.[20] And indeed — despite another call by the European Council (meeting in Stuttgart in June 1983) to activate the EEC provisions regarding QMV — Germany

[19] As can be seen from Table 5.3.2, Art. 148(2) was revised twice during the period 1958–1982: in 1973, with the accession of Britain, Denmark and Ireland and again in 1981, with the accession of Greece.

[20] We do not know what was said in the meeting, but a possible explanation for his claim is that the meeting reaffirmed the right of a member state to veto a measure affecting its vital interests. However, in the particular case of the attempted British veto it was known that the price increase was not in itself unacceptable to the British government, who only tried to block it in order to force concessions on Britain's budgetary contributions. Britain was presumably told in that meeting that such tactics did not constitute a bona fide exercise of members' veto rights, and that the majority of member states had therefore been entitled to press the matter to a vote.

decided, somewhat surprisingly, to veto farm price increases when this issue came again before the CM in 1985.

The veto culture prevailing in the CM since 1958 was dealt a decisive body-blow when the European Council convened in Milan in June 1985. In this meeting three (of the then 10) member states (Britain, Denmark and Greece) objected to the proposal to convene an Inter-Governmental Conference (IGC) in order to amend the three Communities' treaties. The Italian president of the Council decided to disregard their objection. In response, the three claimed that the president's decision to disregard their objection raised 'very important national interests', which come under the Luxembourg Accords. The president rejected this claim, arguing that the mere convening of an IGC is a procedural matter that cannot in itself be viewed as possibly affecting 'very important national interests' of any member state. Only if the IGC were to decide to amend the treaties could 'important national interests' be pleaded; but in such an eventuality there would be no need to invoke the (informal) Luxembourg Accords, because the member states' 'important national interests' are *formally* protected by the treaties — which stipulate that their amendment is subject to a unanimous decision of the CM as well as that the amendments must be ratified according to the laws of each of the member states.

The CM president's view prevailed, and the IGC duly convened in Luxembourg and drafted the SEA. According to the SEA (which took effect on 1 July 1987) QMV was henceforth to apply to the (single) market; the free movement, health and safety of workers; capital movements with third countries; the remainder of transport policy (air and sea); and implementing decisions on regional policy, research and development policy, and environment policy.

In some of these areas the SEA inaugurated a 'Cooperation Procedure' between the CM and the European Parliament, whereby the CM must accept a policy that is proposed by the Commission and adopted by an absolute majority in the European Parliament, unless it was rejected unanimously by all CM members. The empowerment of the European Parliament through this kind of 'reverse veto' forced the CM to change the atmosphere of inaction that had persisted in it since the Luxembourg Accords into an atmosphere of enhanced cooperation.[21]

In addition, it could be argued that Art. 18 of the SEA implicitly put an end to the Luxembourg Accords veto because paragraph 4 of this article allowed member states, under certain conditions, to opt out of harmonization measures adopted by the CM by QMV — hence making it impossible for a member state to argue that its vital interests are affected.

The Maastricht Treaty (which took effect in November 1993) extended the use of QMV by the CM to a number of additional policy areas, including consumer protection, trans-European networks, education and vocational training, public health, a wider range of environmental matters, aspects of economic and monetary union (EMU), development policy, and (from 1996) visa policy. Thus QMV has finally become the norm in the CM's decision-making.[22]

While QMV was not used in practice by the CM for a substantial part of the period since its inauguration in 1958, the quota needed to pass decisions under QMV has remained almost constant at about 71% of the member states' votes.[23] Also, the functional relationship between population size and weight has (with some few exceptions) been remarkably stable and uniform.[24] However, with the increasing implementation of QMV and its extension to a growing number of policy areas, and especially in view of the prospect that additional (mainly Eastern European) countries will

[21] The Maastricht Treaty later replaced the Cooperation Procedure between the Parliament and the CM with a 'Co-decision Procedure' which further empowered the Parliament (see § 5.1).

[22] The Treaty of Amsterdam, signed on 2 October 1997, extends the use of QMV, as well as the composite decision rule, to additional areas, in particular to matters relating to common defense, as well as to police and judicial cooperation in criminal matters.

[23] See Table 5.3.2.

[24] This is particularly clear from Table 5.3.8; the main exceptions are Luxembourg on the one hand, and post-unification Germany on the other.

eventually join the EC, there is growing pressure for changes in the structure of the decision rule, so as to give the four most populous countries (Germany, Britain, France and Italy) — who are also the main paymasters of the EC — greater voting power in the CM. Several suggestions as to how this should be done have been put forward, but at the time of writing (spring 1998) nothing definite has emerged.[25]

Nevertheless, one revision of QMV in this direction has already taken place with the 1995 enlargement of the EC. As can be seen from Table 5.3.2, during the period 1986–94 it was possible for member states whose total number of votes (weight) was at least 23 to bloc a QMV decision in the CM. Britain, while supporting the accession of the EFTA countries, nevertheless threatened to veto their accession if QMV were to be adjusted — as was proposed by the majority of member states — so that following the enlargement a total of at least 26 votes would be required to block CM decisions. Eventually a compromise was reached. According to this 'Ioannina Compromise' (named after the Greek city in which the foreign ministers of the EC met informally in March 1994), the quota would be such that a total weight of at least 26 would be needed to block a decision, but in cases where member states controlling at least 23 votes object to a proposed CM decision, the CM would try to reach, within a reasonable time, a satisfactory solution. This compromise was later embodied in a formal

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[25] As reported by Teasdale [106, pp. 110–111], the ideas being studied are twofold. First, *reweighting*: assigning to the most populous member states a considerably greater weight and adjusting the quota accordingly. Second, introducing a system of *double majority*: in effect a meet of two WVGs, one essentially like the present QMV and the other using weights strictly proportional to population and a quota of at least half the total population-weight — which would make it easier for the large member states to block a CM decision regardless of how large the EC grows. Note, however, that perhaps because the authors of these suggestions are unfamiliar with the Bz index, they do not phrase their ideas directly and explicitly in terms of the a priori voting power which in their view ought to be assigned to the various member states as a function of the size of their population and wealth. The actual Bz power of the EC member states in the CM is discussed in § 5.3.

CM decision, and has so far been invoked only once (by Britain in October 1995) resulting in an amended CM decision which passed with Britain's abstention.[26]

5.3 Weight, Population and Voting Power[27]

In Ch. 4 we commented several times on the widespread fallacy, Prem. 4.2.1, which equates a voter's *relative weight* in a WVG with that voter's *relative voting power*. This fallacy is prevalent not only among the general public, but also among reporters, politicians and writers on (qualitative) political science.[28] Here, for example, is an excerpt from an otherwise well-informed paper published in 1996, in which the author points out what he regards as an 'imbalance between member states' in their representation on the CM.

> The scale of current over-representation of small countries in the Council is striking. ... [T]he four largest member states — France, Germany, Italy and the United Kingdom — together possess only 46 percent of the Council's votes, even though they represent 69 percent of the EU's total population and 73 percent of its total GDP. The corresponding figures for the five largest states (that is, including Spain) are 79 percent of population, 81 percent of GDP and 55.2 percent of votes. Post-unification Germany, with just under a quarter of the EU's population (22 percent) and just over a quarter of its GDP (26 percent), enjoys only 11.5 percent of the votes in the Council, approximately a half of what its proportionate weight might justify.
>
> Conversely, the four smallest member states — Denmark, Finland, Ireland and Luxembourg — make up only 4 percent of the EU's population (and 4 percent of its GDP), but

[26]Under QMV, abstention amounts, in substance, to a 'no' vote. However, a member opposing a decision but realizing that it will be passed anyway may prefer to abstain in order not to lose face.

[27]This section is mostly drawn from [31].

[28]Indeed, even experts on voting power are not immune, and have occasionally succumbed to it: cf. p. 95, fn. 52 and Rem. 3.4.4(ii).

5.3.1 Table
Population of EU member states (thousands)

Country	1958–72	1973–80	1981–85	1986–94	1995–
France	44 790	51 920	54 136	55 476	58 150
Germany	54 290	61 970	61 660	61 010	81 640
Italy	49 040	54 788	56 501	56 821	57 290
Belgium	9 050	9 740	9 853	9 876	10 140
Neth'lnds	11 190	13 401	14 213	14 583	15 450
Lux'mbrg	310	353	365	370	400
Britain		55 988	55 387	56 776	58 260
Denmark		5 007	5 121	5 119	5 230
Ireland		3 086	3 431	3 542	3 580
Greece			9 701	9 994	10 460
Spain				38 632	39 210
Portugal				9 897	9 900
Austria					8 050
Sweden					8 830
Finland					5 110
Total	168 670	256 253	270 368	322 096	371 700

Notes In all tables of this chapter, 'Germany' in the pre-unification period denotes West Germany. Figures in the last column are 1995 mid-year estimates; all other data pertain to the beginning of each period.
Sources: Data for 1958 are taken from [52]; population data for subsequent years except 1995 are taken from [113]; data for mid-1995 are taken from [51].

> enjoy no less than 13 percent of total Council votes. The three least prosperous states — Greece, Ireland and Portugal — with 3 percent of the EU's GDP and 7 percent of its population, enjoy 15 percent of Council votes.[29]

Although we are entitled to assume that in any WVG a voter having greater weight shall not have less voting power, according to any reasonable index, than a voter whose weight is smaller,[30] we

[29] Teasdale [106, pp. 108–109].

[30] This is the *monotonicity condition*, which we discuss in § 7.6.

5.3.2 Table
QMV weights and quota

Country	1958	1973	1981	1986	1995
France	4	10	10	10	10
Germany	4	10	10	10	10
Italy	4	10	10	10	10
Belgium	2	5	5	5	5
Neth'lnds	2	5	5	5	5
Lux'mbrg	1	2	2	2	2
Britain		10	10	10	10
Denmark		3	3	3	3
Ireland		3	3	3	3
Greece			5	5	5
Spain				8	8
Portugal				5	5
Sweden					4
Austria					4
Finland					3
Total	17	58	63	76	87
Quota	12	41	45	54	62
Quota %	70.59	70.69	71.43	71.05	71.26

Note The last row gives the quota as percentage of the total weight.

certainly must not expect the two magnitudes to be strictly proportional to each other. In this section we examine the relationship between voting powers and weights of CM members under QMV during the five phases of the EC's growth. Since the allocation of weights was evidently based on population size, we must look at the interrelationship between these three variables: population, weight, and voting power.

Our first table, Table 5.3.1, concerns population: we give the population of each member state in each of the five periods.

Next, in Table 5.3.2 we give the weight (number of votes) assigned to each member in the CM. The quota needed to pass a decision under QMV is listed in the last two rows of this table: in

5.3.3 Table
QMV weight/population index

Country	1958	1973	1981	1986	1995
France	0.886	0.851	0.793	0.764	0.735
Germany	0.731	0.713	0.696	0.695	0.523
Italy	0.809	0.806	0.760	0.746	0.746
Belgium	2.193	2.268	2.178	2.146	2.107
Neth'lnds	1.773	1.648	1.510	1.453	1.383
Lux'mbrg	32.996	25.032	23.515	22.909	21.362
Britain		0.789	0.775	0.746	0.733
Denmark		2.647	2.514	2.484	2.451
Ireland		4.295	3.752	3.589	3.580
Greece			2.212	2.120	2.042
Spain				0.878	0.872
Portugal				2.141	2.158
Austria					2.123
Sweden					1.935
Finland					2.508
ρ	0.9837	0.9848	0.9826	0.9813	0.9593

Notes This table gives, for each member state and each of the five periods, the ratio between the member's share in the total weight (votes) under QMV and that member's share in the total population. Thus the quantity shown in this table is $(wP)/(Wp)$, where w = the given member's weight under QMV, W = the total weight of all CM members, p = the member's population and P = the total population of the EC. The ρ in the bottom row is Pearson's product moment correlation coefficient between weight and population.

absolute terms (penultimate row) and as a percentage of the total weight (bottom row). Note that the quota has been pegged since 1958 at approximately 71% of the total weight.

In Table 5.3.3 we give, for each member state and each of the five periods, the ratio between the member's share in the total weight and that member's share in the total population of the EC. This ratio may be regarded as an *index of relative per caput weight.* Putting it a bit crudely, we can say for example that in 1986, with the accession of Portugal and Spain, the weight assigned to the average German was 69.5% of the all-EC average, whereas the weight assigned to the average Luxembourgeois was just over 23 times the all-EC average. This table shows clearly the 'imbalance' noted in the passage we have quoted from [106]: weight was not assigned in proportion to population, but skewed in favour of the medium-sized and small states. The four most populous countries (Britain, France, Germany and Italy) as well as Spain have weight/population ratios smaller than 1. The most extreme case is post-unification Germany, whose weight has been kept at the pre-unification level. The medium-sized and small countries all have weight/population ratios greater than 1; the smaller the country, the higher the ratio, with Luxembourg as a far-out extreme.

What is the principle underlying this allocation of weights? We shall return to this question later.

Meantime, let us turn to consider voting power. Throughout, we use the Bz measure of a priori voting power. Most experts writing on the distribution of voting power in the CM use the Bz index, as we do.[31] However, they make little or no attempt to justify their choice of index.[32] Perhaps some of those who have used only the Bz index did so for reasons of convenience: it is easier to compute than its most important rival, the S-S index (which we discuss in § 6.3); and they may have assumed (mistakenly, in our view) that in any case there is not a great deal of difference between the two.[33] For our part, our choice of the Bz measure is based on theoretical grounds: I-power is, predominantly if not exclusively, the type of power relevant to decision-making in the CM; and we consider the Bz measure to be the only appropriate measure of

[31]See, for example, [13], [15], [45], [49], [50], [54], [60], [79], [110], [111], [112]. Note however that, unlike all these authors, we do not confine ourselves to the [relative] Bz *index* β but also go into questions of sensitivity and resistance, for which the Bz *measure* β' is needed.

[32]Johnston [55] puts forward some kind of justification, but is so diffident about it that he decides to 'alter' the Bz index. Cf. Com. 6.4.7.

[33]As we saw in § 4.2, Banzhaf himself thought so; cf. p. 95, fn. 52.

5.3.4 Table
Bz score (η) under QMV

Country	1958	1973	1981	1986	1995
France	10	53	100	286	1849
Germany	10	53	100	286	1849
Italy	10	53	100	286	1849
Belgium	6	29	52	148	973
Neth'lnds	6	29	52	148	973
Lux'mbrg	0	5	26	40	375
Britain		53	100	286	1849
Denmark		21	26	102	595
Ireland		21	26	102	595
Greece			52	148	973
Spain				242	1531
Portugal				148	973
Austria					793
Sweden					793
Finland					595
Total	42	317	634	2222	16565

I-power known at present.

Some of the authors cited above[34] use, alongside the Bz index, also the S-S index. Since the latter, as we shall see in Ch. 6, presupposes 'office-seeking' voting behaviour, we regard it as inappropriate for measuring I-power and hence for analyzing the distribution of power in the CM.

The Bz scores (η) and the values of the Bz index (β) for each CM member during each of the five stages of the EC's growth are given in Tables 5.3.4 and 5.3.5, respectively. The values of β (but not those of η) have been reported, for some or all five periods, in [15], [13, pp. 102, 104], [49], [110], and [60]. We calculated the η values from scratch, and used them to re-calculate the β values. Our

[34]For example, [45], [49], [79], [110], [111], [112].

5.3.5 Table
Bz voting-power index (β) under QMV

Country	1958	1973	1981	1986	1995
France	0.238	0.167	0.158	0.129	0.112
Germany	0.238	0.167	0.158	0.129	0.112
Italy	0.238	0.167	0.158	0.129	0.112
Belgium	0.143	0.091	0.082	0.067	0.059
Neth'lnds	0.143	0.091	0.082	0.067	0.059
Lux'mbrg	0.000	*0.016	*0.041	0.018	*0.023
Britain		0.167	0.158	0.129	0.112
Denmark		0.066	0.041	*0.046	0.036
Ireland		0.066	0.041	*0.046	0.036
Greece			0.082	0.067	0.059
Spain				0.109	0.092
Portugal				0.067	0.059
Austria					0.048
Sweden					0.048
Finland					0.036
Total	1.000	1.000	1.000	1.000	1.000

Note An asterisk indicates an occurrence of the so-called *paradox of new members*: a member-state's relative power has *increased* although its relative weight has decreased as a result of the accession of new members. We shall discuss this apparent paradox in § 7.4.

findings with respect to the latter completely agree with those of the sources just cited.

Table 5.3.6 should be compared with Table 5.3.3, which has a similar structure, except that here *weight* is replaced by *a priori voting power*. So the figures in Table 5.3.6 provide us with what may be called an *index of relative per caput voting power*.[35]

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[35]From a naïve viewpoint, this ratio might be regarded as a sort of 'index of empowerment' of the average citizen of each member state, compared to the average citizen of the EC as a whole. But this is quite misleading: as we know from § 3.4, the ratio which better deserves to be used as an 'index of empowerment' of a citizen is that between the voting power of a member-state

5.3.6 Table
QMV power/population index

Country	1958	1973	1981	1986	1995
France	0.897	0.825	0.788	0.747	0.713
Germany	0.740	0.691	0.692	0.680	0.508
Italy	0.819	0.782	0.755	0.730	0.724
Belgium	2.663	2.407	2.251	2.172	2.153
Neth'lnds	2.153	1.749	1.560	1.471	1.413
Lux'mbrg	0.000	11.450	30.377	15.671	21.036
Britain		0.765	0.770	0.730	0.712
Denmark		3.390	2.165	2.888	2.553
Ireland		5.501	3.232	4.174	3.729
Greece			2.286	2.147	2.087
Spain				0.908	0.876
Portugal				2.168	2.205
Austria					2.210
Sweden					2.015
Finland					2.613
ρ	0.8990	0.9578	0.9766	0.9438	0.9534

Notes This table gives, for each member state and each of the five periods, the ratio between the member's Bz index β and that member's share in the total population. Thus the quantity shown in this table is $(\eta P)/(\mathrm{H}p)$, where η = the given member's Bz score under QMV, H = the sum of Bz scores of all CM members, p = the member's population and P = the total population of the EC. The ρ in the bottom row is Pearson's product moment correlation coefficient between Bz power and population size.

The deliberations leading to the allocation of voting weights were held in secret; so we have no first-hand direct evidence of the criteria that the EC politicians and officials used in this allocation. But the circumstantial evidence of our tables makes it very hard to believe that they could have been even dimly aware of the voting-power distribution resulting from their allocation of weights. For one thing, it is highly improbable that in 1958 they *intended*

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and the *square root* of its population. We shall look at this ratio later on.

5.3.7 Table
QMV power/weight index

Country	1958	1973	1981	1986	1995
France	1.012	0.970	0.994	0.978	0.971
Germany	1.012	0.970	0.994	0.978	0.971
Italy	1.012	0.970	0.994	0.978	0.971
Belgium	1.214	1.061	1.033	1.012	1.022
Neth'lnds	1.214	1.061	1.033	1.012	1.022
Lux'mbrg	0.000	0.457	1.292	0.684	0.985
Britain		0.970	0.994	0.978	0.971
Denmark		1.281	0.861	1.163	1.042
Ireland		1.281	0.861	1.163	1.042
Greece			1.033	1.012	1.022
Spain				1.035	1.005
Portugal				1.012	1.022
Austria					1.041
Sweden					1.041
Finland					1.042

Note This table gives, for each member state and each of the five periods, the ratio between the member's Bz index β and that member's share in the total weight under QMV. Thus the quantity shown in this table is $(\eta W)/(\mathrm{H}w)$, where η = the given member's Bz score under QMV, H = the sum of Bz scores of all CM members, w = the given member's weight under QMV and W = the total weight of all CM members.

making Luxembourg a dummy, devoid of any voting power in QMV. If that had been their intention, why bother giving Luxembourg 1 vote? It is equally improbable that in 1981 they knowingly gave Luxembourg exactly the same voting power as Denmark, whose population was more than 14 times as large. If this is what they intended, why did they give Denmark 3 votes and Luxembourg only 2?

It is therefore not unreasonable to conjecture that the politicians and officials who designed and re-designed the QMV rule naïvely assumed that the voting powers of members would be more or less proportional to their respective weights. If so, the allocation of weights is roughly what was *intended* as a distribution of voting power, whereas the *actual* distribution of voting power is an *unintended* and *unforeseen* outcome of that allocation.

If we compare Tables 5.3.3 and 5.3.6 in this light, we can see that the bias in favour of the small and medium-sized members is noticeably greater in the latter than in the former. Thus it seems as though the bias in voting power turned out to be even greater than intended. The bottom rows of these two tables tell a similar story: the correlations in Table 5.3.6 are consistently smaller than the corresponding ones in Table 5.3.3. This implies that the deviations of voting power from proportionality to population size were always somewhat greater than intended.

However, from Table 5.3.7 it can be seen that, on the whole, the outcome (the distribution of voting power) happened to be quite close to the intention (the distribution of weight): most of the values shown in this table are quite close to 1. The one obvious exception was Luxembourg: at first it got no voting power at all; then in the next stage it got something, but probably much less than intended; then too much; then again too little. However the fit between weight and voting power generally improved with time, and as from 1995 even Luxembourg's share of voting power has been in close agreement with its share of the weight. For this reason, the discrepancies between Tables 5.3.3 and 5.3.6 are on the whole (with few obvious exceptions) not very large.

Tables 5.3.8 and 5.3.9 are analogous to Tables 5.3.3 and 5.3.6 respectively, except that population numbers are now replaced by their square roots.

Table 5.3.8 is particularly remarkable: with few exceptions, most of the values shown in it are quite close to 1. This justifies the observation made by Lane and Mæland [60, p. 224], that the EC since 1973 'in a roundabout fashion practiced the principle that the allocation of votes to each state should be proportional to the square root of its population'. In fact, the same also holds for the period 1958–72. It is a matter for conjecture whether this 'principle' was adopted deliberately, or came about inadvertently

5.3.8 Table
QMV weight/population-square-root index

Country	1958	1973	1981	1986	1995
France	0.983	0.989	0.969	0.964	0.972
Germany	0.893	0.905	0.908	0.919	0.821
Italy	0.940	0.963	0.948	0.952	0.980
Belgium	1.094	1.142	1.136	1.142	1.164
Neth'lnds	0.984	0.973	0.945	0.940	0.943
Lux'mbrg	2.995	2.399	2.360	2.360	2.344
Britain		0.952	0.958	0.953	0.971
Denmark		0.955	0.945	0.952	0.973
Ireland		1.217	1.155	1.151	1.176
Greece			1.144	1.135	1.146
Spain				0.924	0.947
Portugal				1.141	1.178
Austria					1.045
Sweden					0.998
Finland					0.984
ρ	0.9934	0.9945	0.9941	0.9939	0.9856

Notes This table gives, for each member state and each of the five periods, the ratio between the member's share in the total weight (votes) under QMV and that member's share in the sum of the square roots of populations. Thus the quantity shown in this table is $(wS)/(Ws)$, where w = the given member's weight under QMV, W = the total weight of all CM members, s = the square root of the member's population and S = the sum of these square roots. The ρ in the bottom row is Pearson's product moment correlation coefficient between weight and the square root of population.

('in a roundabout fashion') as a result of diplomatic horse-trading; but the outcome is consistent with the former possibility.

The most obvious exceptions to this 'principle' are easily explained by political-diplomatic considerations. First, Luxembourg has always been assigned 'too much' weight, probably as a gesture of generosity to this minuscule state, which might otherwise feel

5.3.9 Table
QMV power/population-square-root index

Country	1958	1973	1981	1986	1995
France	0.995	0.959	0.963	0.943	0.944
Germany	0.904	0.878	0.902	0.899	0.797
Italy	0.951	0.934	0.942	0.932	0.951
Belgium	1.333	1.212	1.174	1.156	1.190
Neth'lnds	1.195	1.033	0.977	0.952	0.964
Lux'mbrg	0.000	1.097	3.049	1.615	2.309
Britain		0.924	0.952	0.932	0.943
Denmark		1.224	0.814	1.107	1.013
Ireland		1.559	0.994	1.339	1.224
Greece			1.183	1.149	1.171
Spain				0.956	0.952
Portugal				1.155	1.204
Austria					1.088
Sweden					1.039
Finland					1.025
ρ	0.9692	0.9889	0.9844	0.9953	0.9853

Notes This table gives, for each member state and each of the five periods, the ratio between the member's Bz index (β) under QMV and that member's share in the sum of the square roots of populations. Thus the quantity shown in this table is $(\eta S)/(\mathrm{H}s)$, where η = the given member's Bz score under QMV, H = the sum of Bz scores of all CM members, s = the square root of the member's population and S = the sum of these square roots. The ρ in the bottom row is Pearson's product moment correlation coefficient between Bz power and the square root of population.

overwhelmed by its much larger partners. Ironically, this gesture was at first totally frustrated by an embarrassing glitch: due to insufficient theoretical analysis, Luxembourg was made a dummy in QMV. On the other hand, it seems clear that in 1981–85 the gesture overshot its mark, as no-one could have intended giving Luxembourg as much power as Denmark. Second, post-unification

Germany got too little weight, probably because it would have been invidious to raise Germany's weight above those of Britain, France and Italy. Germany can live with this, at least for the time being, because its real influence (as distinct from a priori voting power) is in any case massive.

Turning to Table 5.3.9, we see that here too the values are on the whole quite close to 1, although slightly less so than in Table 5.3.8. This is again a consequence of the fact, which we have observed in connection with Table 5.3.7, that the distribution of voting power happened to be on the whole rather close to the distribution of weight.

The upshot is that the distribution of Bz power in the CMEC under its QMV has accorded fairly well with Penrose's square-root rule.

The contrast between this state of affairs and the position regarding weighted voting in the US is replete with paradox. As we saw in Ch. 4, in the US the issue was mainly that of decision rules of parochial bodies, such as County Boards of Supervisors. But it was debated in full democratic daylight, argued in public in the highest court of the land before learned judges, who had the benefit of advice from some of the most distinguished experts on voting power. The outcome, alas, was far from satisfactory from the viewpoint of voting-power theory. On the other hand, the CMEC is undoubtedly one of the world's most important decision-making bodies. But its QMV rules were designed and re-designed in the proverbial smoke-filled room, away from public gaze, by a process of political horse-trading between politicians and officials who (as far as we can tell) had no expert advice on the theory of voting power. Yet the outcome in this case accords remarkably well with Penrose's idealistic theoretical prescription for equal representation:[36] 'the voting power of each nation in a world assembly should be proportional to the square root of the number of people on each nation's voting list'!

[36] [78, p. 57].

5.3.10 Table
QMV sensitivity and resistance

Period	n	ω	H	H_n	S	R
1958–72	6	14	42	60	0.845	0.581
1973–80	9	75	317	630	0.838	0.710
1981–85	10	140	634	1 260	0.858	0.728
1986–94	12	402	2 222	5 544	0.851	0.804
1995–	15	2 549	16 565	51 480	0.861	0.845

Notes In this table n is the number of EC members; ω is the number of winning coalitions under QMV (Def. 3.2.8); H is the sum of Bz scores of all members (Def. 3.3.1); H_n is the sum of the Bz scores in the majority SVG with n voters (Def. 3.3.6), which by Thm. 3.3.14 is the maximal value of H for an SVG with n voters; S is the relative sensitivity of the QMV decision rule (Def. 3.3.19); and R is the resistance coefficient of this rule (Def. 3.3.22).

In view of this analysis, the oft-heard claim that the bigger member-states have too little voting power and the smaller ones too much (see above, pp. 156 f) is unjustified except in the case of Germany and Luxembourg.[37]

As we know from § 3.3 the distribution of *relative* voting power is by no means the whole story about an SVG. Let us therefore turn to other aspects of QMV in the CM. Table 5.3.10 gives the values of our relative sensitivity index, S, and resistance coefficient, R, for QMV during the five periods of the EC.[38] The table shows that S has been quite stable, staying in the 84% to 86% range. Roughly speaking, this may be interpreted by saying that the relative sensitivity (or responsiveness) of QMV has always been located at about the 85% mark along the logarithmic scale from the most rigid rule (unanimity) to the most sensitive (simple majority rule).

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[37] *Added in proof*: For a similar but more nuanced conclusion, see Laruelle and Widgrén [61]. Their paper reached us as we were going to press, and we cannot go here into a detailed assessment of their methodology.

[38] For the definitions of these magnitudes see Defs. 3.3.19 and 3.3.22.

From the fact that the voting quota was always kept very close to 71% of the total weight we may perhaps infer that the implicit intention was to keep the responsiveness of the QMV at a fixed level. If so, the intention has been remarkably successful.

Of course, unless the politicians or their advisers actually computed the values of the Bz and sensitivity indices in advance — which we very much doubt — their successes, such as they were, were largely a matter of luck.

Finally, Table 5.3.10 shows that although the relative sensitivity of QMV in the CM has remained more or less steady, the CM as a decision-making body has become progressively more resistant to making new decisions: the coefficient of resistance R has steadily and markedly increased with each enlargement of the EC. Since R measures the relative a priori resistance to passing a resolution, this is a worrying process. If the trend is allowed to continue with future enlargements, then the CM will increasingly please its members by humouring their resistance to change rather than by complying with their wishes for change.

6. Power as a Prize

6.1 P-Power: A Game-Theoretic Notion

The notion of voting power studied in Ch. 3 was that held by Penrose [78], Banzhaf [5] and Coleman [20]; this notion is what we have called 'I-power' (see § 3.1). It assumes a policy-seeking motivation of voting behaviour: the way a voter votes on a given bill is determined by his or her attitude to the bill — an attitude which the voter presumably forms by comparing the expected payoff (to him or her) of the bill's passage with that of its failure. These payoffs are independent of the decision rule and exogenous to it. Thus an SVG $\mathcal{W}$ *by itself* provides no information whatever as to how any voter might vote on an unspecified bill. This state of total a priori ignorance was encapsulated in the Bernoulli model $\mathbf{B}_N$, with which $\mathcal{W}$ must be supplemented. Notice that one and the same $\mathbf{B}_N$ is shared by all SVGs with the same assembly N.

In this chapter we shall examine an alternative notion of voting power, which we have termed *P-power*, first adopted by Shapley and Shubik [97]. This posits an office-seeking motivation of voting behaviour. We begin by sketching the intended meaning of P-power, which we shall then amplify and clarify in a series of comments.

6.1.1 Sketch The basic idea is that a division of a board is a play of a game whose rules are given by the decision rule operated by the board. If the outcome of the division is positive, the winning coalition that voted 'yes' gains collective possession of a fixed amount of transferable utility — the *purse*, or the *prize of power* — which it proceeds to distribute among its members: each of these winning voters receives a non-negative payoff, all adding up to the total purse, while the remaining voters (those who voted 'no') get 0 as payoff. If the outcome of the division is negative, *all* voters get 0. The P-power of a voter is meant to be that voter's expected payoff.

6.1.2 Comment As Coleman [20] pointed out, this sketch and the office-seeking voting behaviour that it posits are by no means a reasonable realistic description of most real-life situations of decision-making by vote.[1] But in defining and attempting to quantify P-power we need not take the sketch as factual. Rather, we can — and in most cases must — regard it as purely hypothetical or even counterfactual. Given an SVG, we may deal with it *as if* it relates to the kind of situation described in Sketch 6.1.1.

For example, we may hypothesize, in a purely 'as-if' mode, that the bill to be decided will be proposed collectively by a coalition S and will decree how the fixed prize is to be divided among all voters. Moreover, any coalition is allowed to propose a bill of this form. According to the usual game-theoretic assumption of self-interested behaviour, the bill will give nothing to non-members of S and divide the whole prize among members of S, allocating to each a share agreed upon through a process of bargaining, prior to the members of S collectively undertaking to table their bill. The non-members of S, being left out, will vote 'no'; so only a *winning* coalition S will bother proposing such a bill.

We stress once more that throughout our theoretical discussion of P-power the assumption of such office-seeking behaviour must be understood in a hypothetical 'as if' sense.

6.1.3 Comment In connection with P-power we assume that before a division takes place a winning coalition is *formed* — that is, its members conclude a pact to vote for the bill in question;[2] and that the pact includes a binding agreement as to how the fixed prize is to be carved up.

[1] See Com. 2.2.2.

[2] Cf. Com. 2.2.1.

In general we cannot assume that the bargaining process is deterministic, resulting in a certain unique outcome. As a simple example, consider $\mathcal{M}_3$ (see Def. 2.3.10). For reasons of symmetry, if this SVG had a certain outcome, it could only be the formation of the grand coalition $\{1, 2, 3\}$, dividing the purse equally among its three members. But it is not really plausible that this should be the only possible outcome: it is too fragile and open to challenge, because any two voters can always do better by getting rid of the third.[3]

Thus there is usually no single winning coalition that is certain to be formed; and, assuming that a particular winning coalition will be formed, the division of the prize is normally again uncertain. The best we can hope for is a representation of the payoff of each voter a as a random variable Y_a with a well-defined probability distribution (depending of course on the given SVG $\mathcal{W}$). We could then obtain the P-power of voter a as $\mathrm{E}\mathsf{Y}_a$, the mathematical expectation of Y_a.

We may hope to determine the distributions of the Y_a in the following way. First, for each winning coalition S, we may be able to assign a definite probability, $\mathrm{P}(S \text{ is formed})$, to the event that S will be formed. Next, for each winning coalition S and each voter $a \in S$, we may be able to assign a *conditional* probability distribution to the payoff Y_a, subject to the condition that S will be formed. From these conditional distributions and the probabilities $\mathrm{P}(S \text{ is formed})$ it would then be easy to compute the a priori (unconditional) distribution of each Y_a.

A *bargaining model* would provide us with such probabilistic data, hence with a solution to the problem of measuring P-power. We shall return to this point a bit later. Meantime we warn our readers not to raise their hopes too high.

6.1.4 Comment In the present context the Bernoulli space $\mathbf{B}_N$ can no longer be used to capture the a priori state of information

[3] We could have put this argument in more technical game-theoretic form; but we refrain from doing so in order to avoid giving the impression that the problem is caused by some technical shortcoming of game theory rather than being inherent.

regarding voting propensities. A voter's propensity to vote 'yes' must depend a priori on the decision rule; and the same applies to the degree of coordination between two or more voters.

For example, in an SVG whose MWCs are $\{a, b, c\}$ and $\{a, d\}$, the blocker a will presumably always vote 'yes'; and the twins b and c will presumably be more likely to vote in tandem than on oposite sides.

The aprioricity of a measure of P-power consists in this: the voting behaviour which it presupposes depends *solely* on the SVG and not on any information extraneous to it.

6.1.5 Comment By a suitable choice of units, the fixed prize to be appropriated by a successful winning coalition can always be taken as 1 unit. Therefore in measuring P-power the primary concept is the *relative* measure, the index. An absolute measure can be derived from such an index by multiplying by an appropriate constant numerical factor; but this is of little interest.[4]

6.1.6 Comment At first blush it may perhaps seem that the distinction between I-power and P-power is too fussy, or purely philosophical. Why should the Bz index not serve as an index of P-power as well? We saw in Com. 3.2.5 that β'_a is the a priori probability that voter a will be critical, in a position to tip the balance in a division. Is it not reasonable to assume that voters' expected payoffs as shares in the purse are proportional to their respective a priori influences over the outcome of a division?

A similar argument can be phrased in terms of buying and selling votes. In Com. 3.2.15 we considered an outsider who stands to gain one unit of transferable utility if a board were to pass a certain bill, and to lose one unit if the board were to defeat that bill. We assumed that the outsider wishes to buy voter a's vote but has no prior information as to the voters' intentions. We saw that such a buyer ought to be prepared to pay any sum short of β'_a. On the other hand, from a's point of view the expected payoff of participation in the SVG is equal to a's index of P-power; so a

[4] For contrast with I-power in this respect, cf. Com. 3.2.5.

ought to be ready to accept any price greater than that quantity. Is it not reasonable to assume that [the supremum of] the buyer's price is proportional to [the infimum of] the seller's price? But if so, then β is an index of P-power!

But these arguments are easily rebutted. First, as we have observed a moment ago in Com. 6.1.4, the probability space $\mathbf{B}_N$, which we used in connection with I-power, is not necessarily applicable in reasoning about P-power; so the arguments in the last two paragraphs do not compare like with like. Second, although it may be reasonable to expect that I-power and P-power are positively correlated, it is naïve to assume that reward (expected payoff) must be strictly proportional to influence over the outcome, or that the buyer's demand prices must be proportional to the sellers' supply prices.

Of course, this rebuttal does not clinch the case; it does not exclude the possibility that β may yet turn out, as it were by a piece of good luck, to be a valid index of P-power. However, such a fortunate coincidence is ruled out by the mathematical behaviour of β. As we shall see in Com. 7.10.1, β has some properties which for a P-power index would be so extremely paradoxical as to be quite unacceptable.

6.1.7 Comment As we noted in Com. 2.2.2, in cooperative game theory the CF $\mathbf{w}$ of a cooperative game is interpreted as assigning to each coalition S a 'worth' $\mathbf{w}S$ equal to the total amount of transferable utility that the members of S can collectively win, irrespective of the actions of non-members of S. In the case of an SVG, $\mathbf{w}S$ is 1 or 0, according as S is a winning or losing coalition.

Now suppose S and T are two distinct winning coalitions, both of which simultaneously make pacts to capture the prize. Which of them will get it? If S and T have a member in common, then such simultaneous pacts can be ruled out because we may reasonably decree that no voter may subscribe simultaneously to two conflicting binding agreements. But what if S and T are disjoint, as may happen in an improper SVG? In this case no double-dealing is involved; but the problem of allocating the prize becomes indeterminate.

So from the perspective of P-power improper SVGs are quite puzzling anomalies. As Shapley [95, p. 60] puts it:

> Games of this kind ... play a role in the theory somewhat analogous to that of "imaginary" numbers in algebra and analysis. Although they may not have a direct interpretation, they are useful in the factorization of proper games ... and in other ways.

Recall that in Rem. 2.1.2(iv) we gave examples of reasonable decision rules that must be modelled by improper SVGs; but those examples make little sense from the viewpoint of P-power.

As we noted in Com. 6.1.3, a bargaining model would provide us with all the probabilistic data required for computing an index of P-power. Of course, if the index is to owe whatever plausibility and intuitive justification it may have to the bargaining model from which it is derived, then that model itself had better be persuasive: it must predict convincingly the probability distributions of the outcomes of the bargaining process described in Coms. 6.1.2 and 6.1.3, assuming all the voters are rational.[5]

The trouble is that where there are more than two voters no such model is available.[6] In the sequel we shall meet several models that are *mathematically* well-defined; but none of them is really persuasive in the sense just outlined. Nor is there any known way to obtain such a normative model.

This predicament is by no means confined to the theory of simple games, but afflicts *a fortiori* the general theory of cooperative games. In this more general context, Shapley [94] proposed what came to be known as the *Shapley value*, which is in effect an ingenious way of getting round the obstacle posed by the absence of a credible bargaining model. The S-S index of voting power is a spin-off, a special case of the Shapley value, which we discuss in the next section.

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[5]Here 'rational' is understood in its game-theoretic sense: pursuing self-interest in the best possible way, given the available information.

[6]For one or two voters the problem is trivial: either there is a dictator, who always gets the entire purse and hence has all the P-power; or there are two symmetric voters, whose expected shares must be equal.

6.2 The Shapley Value

In this section we make a brief excursion into cooperative game theory, from which we shall borrow a well-known result, Thm. 6.2.14, as well as its variant, Thm. 6.2.15. For their proofs (which are fairly easy) we refer the reader to the game-theoretic literature cited in the sequel.

We shall look at *cooperative games, with transferable utility, in characteristic function form*, to which we will refer simply as 'games', for the sake of brevity. We start by defining *game* and some associated notions.

6.2.1 Definition (i) By a *game* on a nonempty finite set N we shall mean a real-valued function $\mathbf{w}$ whose domain is the power set (that is, the set of all subsets) of N such that $\mathbf{w}\emptyset = 0$.

We refer to any member of N as a *player* [of $\mathbf{w}$], and to any subset of N as a *coalition* [of $\mathbf{w}$]. N itself is called the *grand coalition* [of $\mathbf{w}$]. If S is a coalition, the real number $\mathbf{w}S$ is called the *worth of* S [in $\mathbf{w}$].

(ii) A player d is said to be a *dummy* [in $\mathbf{w}$] if

$$\mathbf{w}(X \cup \{d\}) = \mathbf{w}X,$$

for every coalition X.

(iii) The game $\mathbf{w}$ is said to be *superadditive* if

$$\mathbf{w}(X \cup Y) \geq \mathbf{w}X + \mathbf{w}Y,$$

for any disjoint coalitions X and Y.

The game $\mathbf{w}$ is said to be *monotone* if

$$X \subseteq Y \subseteq N \Rightarrow \mathbf{w}X \leq \mathbf{w}Y.$$

(iv) The *dual* of $\mathbf{w}$ is the game $\mathbf{w}^*$, with the same grand coalition N as $\mathbf{w}$, such that, for every $X \subseteq N$,

$$\mathbf{w}^*X = \mathbf{w}N - \mathbf{w}(N - X).$$

(v) Let $\mathbf{w}$ and $\mathbf{w}'$ be games, on respective sets N and N'. An *isomorphism* from $\mathbf{w}$ to $\mathbf{w}'$ is a bijection f from N to N' (that is, a 1-1 map from N onto N'), such that for any $X \subseteq N$,

$$\mathbf{w}'(f[X]) = \mathbf{w}X.$$

(Here $f[X] = \{fx : x \in X\}$.) If such f exists, we say that $\mathbf{w}$ is *isomorphic* to $\mathbf{w}'$ — briefly: $\mathbf{w} \cong \mathbf{w}'$.

An isomorphism from a game $\mathbf{w}$ to itself is called an *automorphism* or *symmetry* of $\mathbf{w}$.

(vi) Let $\mathbf{u}$ and $\mathbf{v}$ be games on the same set N. Their *sum* $\mathbf{u}+\mathbf{v}$ is the game $\mathbf{w}$ on N such that, for every $X \subseteq N$,

$$\mathbf{w}X = \mathbf{v}X + \mathbf{u}X.$$

6.2.2 Remarks (i) As usual, we omit qualifications such as 'in $\mathbf{w}$' (enclosed in square brackets in Def. 6.2.1) when they are self-evident from the context. As before, we shall always assume that $|N| = n$.

(ii) Obviously, a CF $\mathbf{w}$ of an SVG $\mathcal{W}$ (see Def. 2.1.3) is a special case of a game (in the sense just defined); in fact, it is just a monotone game that assumes exactly two values: 0 and 1. Moreover, such $\mathbf{w}$ is superadditive iff $\mathcal{W}$ is proper (in the sense of Def. 2.1.1).

(iii) In cooperative game theory, games are used to model interactive set-ups in which each participant ('player') has a set of available strategies. Before a play of the game, each player chooses a strategy; in the course of the play the players apply their respective chosen strategies; and at the end of the play each player receives a payoff, which is an amount (positive, negative or 0) of transferable utility (often thought of as money or money-equivalent). The payoff of each player depends on the respective strategies chosen by all players. Before a play, any set of players can bargain and form a coalition by making a binding agreement to coordinate their respective choices of strategies and to re-distribute their eventual payoffs. As we noted in Com. 2.2.2, the worth $\mathbf{w}S$ of a coalition

S 'represents the ... least payoff that S can guarantee itself no matter what the other players (that are not in S) do.'[7]

Clearly, in order to admit of such an interpretation, $\mathbf{w}$ must be superadditive. Thus, '[n]on-superadditive games ... pose problems in interpretation, but they are nevertheless useful in the mathematical theory.'[8]

Next, we define the general concept of *value assignment* and two of its special instances, the Shapley and Bz value assignments. The former of these value assignments is the star of this section. The latter will play a walk-on role: it is not, properly speaking, a game-theoretic concept, except in a formal sense.

6.2.3 Definition A *value assignment* is any function ξ that assigns to each game $\mathbf{w}$ and each player a of $\mathbf{w}$ a real number $\xi_a(\mathbf{w})$ called the *value for* a *of* $\mathbf{w}$ [according to ξ].

If N is the grand coalition of $\mathbf{w}$, we regard $\xi(\mathbf{w})$ as a vector whose components are $\xi_a(\mathbf{w})$ for $a \in N$ (that is, a vector in the n-dimensional vector space of all maps from N to the reals). Thus, for example, if $\mathbf{u}$ and $\mathbf{v}$ are games on N we take $\xi(\mathbf{u}) + \xi(\mathbf{v})$ to be the vector whose components are $\xi_a(\mathbf{u}) + \xi_a(\mathbf{v})$ for all $a \in N$.

The *Shapley value assignment* ϕ and the *Bz value assignment* β' are defined as follows:

$$\phi_a(\mathbf{w}) =_{\text{def}} \frac{1}{n!} \sum_{X \subseteq N} \big(|X| - 1\big)!\big(n - |X|\big)!\big(\mathbf{w}X - \mathbf{w}(X - \{a\})\big),$$

$$\beta'_a(\mathbf{w}) =_{\text{def}} \frac{1}{2^{n-1}} \sum_{X \subseteq N} \big(\mathbf{w}X - \mathbf{w}(X - \{a\})\big),$$

for any game $\mathbf{w}$ and any player a of $\mathbf{w}$. (Here, as usual, N is the grand coalition of $\mathbf{w}$ and $n = |N|$.)

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[7] Dubey [26, p. 131].

[8] Dubey and Shapley [27, p. 101]. However, as we pointed out in Com. 6.1.7, improper SVGs, which are problematic from the game-theoretic viewpoint, are quite reasonable from the viewpoint of I-power.

6.2.4 Remarks (i) Following widespread usage, we shall refer to ϕ and β' as the 'Shapley value' and 'Bz value' respectively, omitting the word 'assignment'. This usage is somewhat imprecise, as it fails to distinguish between the assignment itself, as a function, and the value it assigns to a particular game for a particular player; but we shall rely on the context to resolve this ambiguity.

(ii) The Bz value is easily seen to be a natural extension of the Bz measure.[9] Indeed, if $\mathbf{w}$ is the CF of an SVG $\mathcal{W}$, then $\beta'_a(\mathbf{w})$ as defined in Def. 6.2.3 coincides with $\beta'_a[\mathcal{W}]$ as defined in Def. 3.2.2.

(iii) Both the Bz value and the Shapley value are *self-dual* in the sense that $\beta'_a(\mathbf{w}) = \beta'_a(\mathbf{w}^*)$ and $\phi_a(\mathbf{w}) = \phi_a(\mathbf{w}^*)$. To see this, observe that for any coalition X such that $a \in X$, the term with X in the expression for $\beta'_a(\mathbf{w})$ (or $\phi_a(\mathbf{w})$) is equal to the term with $(N - X) \cup \{a\}$ in the expression for $\beta'_a(\mathbf{w}^*)$ (or $\phi_a(\mathbf{w}^*)$, respectively).

(iv) The quantity $\beta'_a(\mathbf{w})$ has an obvious probabilistic interpretation: it is the expected marginal contribution of player a to the worth of a coalition containing a. Let us make this more precise in terms of the Bernoulli space $\mathbf{B}$ (Def. 3.1.1). Let $\mathbf{w}$ be any game on N and let a be a player of $\mathbf{w}$. Define a random variable X_a by putting

$$\mathsf{X}_a B = \mathbf{w}(B^+) - \mathbf{w}(B^+ - \{a\})$$

for any bipartition B of N. (For the definition of B^+ see Def. 2.1.5.) Then $\mathsf{X}_a B$ is the contribution of a to $\mathbf{w}(B^+)$, the worth of the coalition B^+. Of course, if this coalition does not contain a, then the contribution is 0. It is now easy to see that

$$\beta'_a(\mathbf{w}) = \mathrm{E}(\mathsf{X}_a \mid a \in B^+)$$

in the Bernoulli space $\mathbf{B}$.

[9] As far as we know, the first to define this extension in print was Owen in [74] and [75].

The quantity $\phi_a(\mathbf{w})$ can be given an analogous probabilistic interpretation. For this we need to replace $\mathbf{B}$ by another probability space, which we shall now define.

6.2.5 Definition Let N be a finite set. By a *queue of N* we mean any bijection from N to the set $I_n = \{1, 2, \dots, n\}$, where (as usual) $n = |N|$.

If Q is a queue of N and $a \in N$, we refer to the positive integer Qa as the *place of a in Q*. We also put

$$\mathrm{h}_a Q =_{\mathrm{def}} \{x \in N : Qx \le Qa\}.$$

Thus $\mathrm{h}_a Q$ contains just a and those members of N placed ahead of a in Q. We refer to $\mathrm{h}_a Q$ as the *head of a in Q*.

The *queue space* $\mathbf{Q}_N$ is the probability space consisting of the set of all queues of N, with each queue assigned probability $1/n!$.

Where there is no risk of confusion, we omit the subscript 'N' and write simply '$\mathbf{Q}$'.

6.2.6 Theorem *Let $\mathbf{w}$ be a game on N and let a be a player of $\mathbf{w}$. In the queue space $\mathbf{Q}$, define the random variable Y_a to be the contribution of a to the worth of the head of a in a queue; thus*

$$\mathsf{Y}_a Q = \mathbf{w}(\mathrm{h}_a Q) - \mathbf{w}(\mathrm{h}_a Q - \{a\})$$

for any queue Q of N. Then

$$\mathrm{E}(\mathsf{Y}_a) = \phi_a(\mathbf{w}).$$

Proof Let $a \in X \subseteq N$. We count the queues Q for which $\mathrm{h}_a Q = X$. In such a queue, the members of X other than a must be placed ahead of a; these can be ordered in $\big(|X| - 1\big)!$ distinct ways. And the non-members of X must be placed behind a; these can be ordered in $\big(n - |X|\big)!$ distinct ways. So there are exactly $\big(|X| - 1\big)!\big(n - |X|\big)!$ distinct queues of the required kind — out of a total of $n!$ queues.

The claim of the theorem now follows at once from the definition of ϕ (Def. 6.2.3). ∎

The following corollary is obtained directly from the definition of Y_a.

6.2.7 Corollary *Let* $\mathbf{w}$ *and the* Y_a *be as in Thm. 6.2.6. Then*

$$\sum_{a \in N} \mathsf{Y}_a Q = \mathbf{w}N$$

for each queue Q *of* N*.* ∎

6.2.8 Comment In a formal sense, the queue space $\mathbf{Q}$ together with the random variables Y_a of Thm. 6.2.6 can be regarded as a bargaining model for [cooperative] games. We shall refer to it as the *queue bargaining model*. This is how Shapley himself puts it:

> The players constituting ... N agree to play the game [$\mathbf{w}$] in a grand coalition, formed in the following way: (1) Starting with a single member, the coalition adds one player at a time until everyone has been admitted. (2) The order in which the players are to join is determined by chance, with all arrangements equally probable. (3) Each player, on his admission, demands and is promised the amount which his adherence contributes to the [worth] of the coalition (as determined by the function [$\mathbf{w}$]). The grand coalition then plays the game "efficiently" so as to obtain the amount [$\mathbf{w}N$]: exactly enough to meet all the promises.[10]

In the queue Q, the payoff 'promised' to player a is exactly $\mathsf{Y}_a Q$. That all these 'promises' can be met is guaranteed by Cor. 6.2.7. And by Thm. 6.2.6 the expected payoff of a in the queue bargaining model is $\phi_a(\mathbf{w})$.

An oft-heard critique of the Shapley value (or of its special case, the S-S index) as a predictor of a player's expected payoff in a game (or an SVG) asserts that the queue bargaining model is in general quite unrealistic: for one thing, it seems to attach special importance to the order in which the grand coalition is formed; and therefore use of the Shapley value (or the S-S index) is only

[10] [94, § 6]. We have substituted our present notation and terminology [in square brackets] for those used in the original.

legitimate in real-life situations in which the order of coalition formation plays an essential role.[11]

In our view, this sort of critique is off the mark, as it stems from a fundamental misconception about the role of the queue bargaining model. The justification of the Shapley value as a predictor of a player's expected payoff does not in fact depend on this model; nor does Shapley claim that it does, or that the model is realistic. In his [94] the model makes its appearance in a very brief coda, after the main work of the paper has been done. Much of that work would have been almost unnecessary had he had available to him a convincingly realistic theoretical bargaining model: it would have sufficed to present such a model and compute the expected payoffs that it yields.

The point is that no genuinely realistic general theoretical bargaining model for [cooperative] games was — or is — available. To get round this difficulty, Shapley proceeded *axiomatically*: he listed a few postulates, or conditions, which the expected payoffs yielded by a realistic bargaining model — had one existed — presumably ought to obey; and then proved that ϕ is the unique value assignment satisfying these conditions.[12] Any valid critique of the Shapley value must challenge the adequacy of Shapley's postulates, not some particular representation of the Shapley value. We shall return to this issue later on, in Com. 6.2.26 and Com. 6.2.27.

So what purpose does the queue bargaining model serve?[13] It is primarily a mathematical artefact: it may not be genuinely realistic, but it is a mathematically well-defined model that yields the Shapley value. It is not the *only* such model — we shall present another one below — but it is the simplest. As such, it supplies at the same time a useful visualization and a formal aid in reasoning

[11] For a remark in this vein by Banzhaf, see above, § 4.2, text to fn. 41.

[12] See Thm. 6.2.14 below.

[13] In [94, § 6] Shapley points out that '[t]he form of our model, with its chance move, lends support to the view that the [Shapley] value is best regarded as an *a priori* assessment of the situation, based on either ignorance or disregard of the social organization of the players.' Yes; but the aprioristic nature of the Shapley value is in any case evident from the nature of its axiomatic characterization and the absence of any assumption regarding 'social organization'.

about the Shapley value. For example, it provides an immediate guarantee that the $\phi_a(\mathbf{w})$ behave mathematically as expectations (in the probabilistic sense) of the players' respective shares in a fixed total prize, $\mathbf{w}N$.

In order to state Shapley's characterization of ϕ as well as an interesting variant of Shapley's result, due to Young, we shall first need to formulate several conditions (or postulates) that may be imposed on a value assignment.

6.2.9 Definition Let ξ be a value assignment and let $\boldsymbol{\mathcal{G}}$ be a class of games.

(i) We say that ξ is *invariant under isomorphism* — briefly, *iso-invariant* — on $\boldsymbol{\mathcal{G}}$ if, whenever $\mathbf{w} \in \boldsymbol{\mathcal{G}}$ and $\mathbf{w}' \in \boldsymbol{\mathcal{G}}$ and f is an isomorphism from $\mathbf{w}$ to $\mathbf{w}'$, then $\xi_{fa}(\mathbf{w}') = \xi_a(\mathbf{w})$ for every player a of $\mathbf{w}$.

(ii) We say that ξ is *efficient* on $\boldsymbol{\mathcal{G}}$ if, whenever $\mathbf{w} \in \boldsymbol{\mathcal{G}}$, then

$$\sum_{a \in N} \xi_a(\mathbf{w}) = \mathbf{w}N,$$

where N is the grand coalition of $\mathbf{w}$.

(iii) We say that ξ *vanishes for dummies* on $\boldsymbol{\mathcal{G}}$ if, whenever $\mathbf{w} \in \boldsymbol{\mathcal{G}}$, then $\xi_d(\mathbf{w}) = 0$ for every dummy d of $\mathbf{w}$.

(iv) We say that ξ is *additive* on $\boldsymbol{\mathcal{G}}$ if, whenever $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \boldsymbol{\mathcal{G}}$ and $\mathbf{w} = \mathbf{u} + \mathbf{v}$, then $\xi(\mathbf{w}) = \xi(\mathbf{u}) + \xi(\mathbf{v})$. (Here the last '+' denotes vector addition, according to Def. 6.2.3.)

(v) We say that ξ is *marginal* on $\boldsymbol{\mathcal{G}}$ if, whenever $\mathbf{w} \in \boldsymbol{\mathcal{G}}$ and $\mathbf{w}' \in \boldsymbol{\mathcal{G}}$ and both games have the same grand coalition, and a is a player such that

$$\mathbf{w}X - \mathbf{w}(X - \{a\}) = \mathbf{w}'X - \mathbf{w}'(X - \{a\})$$

for every coalition X, then $\xi_a(\mathbf{w}) = \xi_a(\mathbf{w}')$.

6.2.10 Remarks (i) The iso-invariance, efficiency, vanishing for dummies and additivity conditions are adapted (with some inessential modifications) from Shapley [94]. Although he does not say so explicitly, he is widely taken to imply that a value assignment ξ must satisfy these conditions on the class of all superadditive games, if $\xi_a(\mathbf{w})$ is to serve as an estimate of the expected payoff of player a in the game $\mathbf{w}$ or, equivalently, as the the price that a should be prepared to pay to take part in that game.

The condition of marginality is borrowed from Young [114] (the term 'marginality', referring to the marginal contribution of a player to the worth of a coalition, is used in his [115]). He postulates it not for a payoff predictor, but (in effect) for the closely related concept of *arbitration scheme* — the players of $\mathbf{w}$ forgo bargaining and appoint a fair impartial arbitrator; $\xi_a(\mathbf{w})$ is to be interpreted as the amount of transferable utility allocated to a by such an arbitrator.

We shall discuss the plausibility of these conditions at the end of this section.

(ii) Instead of the two conditions of efficiency and vanishing for dummies, Shapley [94] postulates a single condition. A *carrier* of a game $\mathbf{w}$ is any coalition that contains all the non-dummies (and may also contain some dummies). Shapley's *carrier* postulate requires that

$$\sum_{a \in S} \xi_a(\mathbf{w}) = \mathbf{w}S$$

for each carrier S. It is easy to verify that, for any ξ and $\mathbf{w}$, this single condition is equivalent to the conjunction of efficiency and vanishing for dummies. Following common practice, we have replaced Shapley's single condition by these two, which are somewhat more transparent.

6.2.11 Theorem *The Shapley value ϕ is iso-invariant, efficient, vanishes for dummies, is additive and marginal on the class of all games.*

Proof Efficiency is easily deduced from Thm. 6.2.6 in conjunction with Cor. 6.2.7. The other four conditions follow directly from Def. 6.2.3. ∎

6.2.12 Remark The Bz value β' is iso-invariant, vanishes for dummies, is additive and marginal on the class of all games, as can be verified directly from Def. 6.2.3. However, it is not generally efficient. To show this it suffices — in view of Rem. 6.2.4(ii) — to give an example of an SVG in which the Bz powers of all voters do not add up to 1. In Ex. 3.2.6 we met two such SVGs: the Bz powers of the three voters of $\mathcal{B}_3$ add up to $\frac{3}{4}$; and the Bz powers of the four voters of $\mathcal{M}_4$ add up to $\frac{2}{3}$.

We might try to get an efficient value assignment from β' by normalizing it — multiplying the Bz values for all players by a constant, so they add up to $\mathbf{w}N$ — as we did in the special case of SVGs to obtain the Bz *index* β. This is not always possible: if some coalitions have negative worth and some positive, then the $\beta'_a(\mathbf{w})$ may add up to 0, while $\mathbf{w}N \neq 0$. But even leaving such cases aside, normalization may destroy additivity. As an example of this, let $\mathbf{u}$ be the CF of the SVG whose assembly is $\{1, 2, 3\}$ and whose MWCs are $\{1, 2\}$ and $\{1, 3\}$; and let $\mathbf{v}$ be the CF of the SVG with the same assembly and with $\{2, 3\}$ as sole MWC. As the reader can easily verify, the normalized Bz values of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u} + \mathbf{v}$ for voter 1 are $\frac{3}{5}$, 0 and $\frac{2}{3}$ respectively, showing the failure of additivity.

Marginality can also be destroyed by normalizing the Bz value. As an example of this, let $\mathbf{w}$ be the CF of $\mathcal{M}_3$ and let $\mathbf{w}'$ be the game on $\{1, 2, 3\}$ such that $\mathbf{w}'\{1\} = 0$, $\mathbf{w}'\{2\} = \mathbf{w}'\{3\} = 1$, and $\mathbf{w}'X = 2$ for any coalition containing at least two members. We leave it to the reader to verify that the marginal contribution of player 1 to the worth of every coalition is the same under $\mathbf{w}$ as under $\mathbf{w}'$; yet the normalized Bz values of $\mathbf{w}$ and $\mathbf{w}'$ for this player are $\frac{1}{3}$ and $\frac{1}{2}$ respectively.

As we shall see in a moment, these phenomena are by no means accidental.

6.2.13 Convention In order to avoid tedious repetitions, we shall use the following abbreviated mode of expression. When we say that a value assignment ψ is *characterized on a class $\boldsymbol{\mathcal{G}}$ of games by such-and-such conditions*, we shall mean that, first, ψ satisfies these conditions on $\boldsymbol{\mathcal{G}}$;[14] and, second, that if ξ is any value assignment that satisfies these conditions on $\boldsymbol{\mathcal{G}}$, then $\xi(\mathbf{w}) = \psi(\mathbf{w})$ for every $\mathbf{w} \in \boldsymbol{\mathcal{G}}$.

The following is the main result in Shapley [94]. A slightly different proof, as well as some variants of the result, can be found in Dubey [26].

6.2.14 Theorem (Shapley) *For any nonempty finite set N, ϕ is characterized on the class of all games on N, as well as on the class of superadditive games on N, by the conditions of iso-invariance, efficiency, vanishing for dummies and additivity.* ∎

The following is the main result in Young [114].

6.2.15 Theorem (Young) *For any nonempty finite set N, ϕ is characterized on the class of all games on N by the conditions of iso-invariance, efficiency and marginality.* ∎

These characterizations of ϕ are very powerful. They explain the mathematical robustness and ubiquity of the Shapley value. As an illustration of this, we shall deduce two results about ϕ. For the first of our results we shall first need to introduce yet another probability space.

6.2.16 Definition Let N be a finite set. By a *roll-call* of N we mean an ordered pair $R = \langle Q, B\rangle$, where Q is a queue of N and B is a bipartition of N. In this connection, we denote Q and B respectively by '$\mathrm{q}R$' and '$\mathrm{b}R$'; and we refer to them respectively as the *queue* and *bipartition of* R.

If R is a roll-call of N and $a \in N$, we say that a is *positive* or *negative* in R according as $(\mathrm{b}R)a = 1$ or $(\mathrm{b}R)a = -1$. We also put

$$\mathrm{h}_a^+ R =_{\mathrm{def}} \{x \in \mathrm{h}_a(\mathrm{q}R) : (\mathrm{b}R)x = 1\},$$
$$\mathrm{h}_a^- R =_{\mathrm{def}} \{x \in \mathrm{h}_a(\mathrm{q}R) : (\mathrm{b}R)x = -1\}.$$

[14] In all cases, this first part of a characterization claim will be quite easy to prove, and will in fact have been established beforehand.

Thus h_a^+R contains just those members of the head of a in $\mathrm{q}R$ that are positive in R; and h_a^-R contains just those members of the head of a in $\mathrm{q}R$ that are negative in R. We refer to h_a^+R and h_a^-R respectively as the *positive head* and *negative head of a in R*.

The *roll-call space* $\mathbf{R}_N = \mathbf{Q}_N \times \mathbf{B}_N$ is the probability space consisting of the set of all roll-calls of N, with each roll-call assigned the same probability: $1/(n!2^n)$, where (as usual) $n = |N|$.

Where there is no risk of confusion, we omit the subscript 'N' and write simply '$\mathbf{R}$'.

6.2.17 Remark It is helpful to visualize a roll-call R of N as the membership of N queuing up in random order (this is the first component, $\mathrm{q}R$); and at the same time each member a of N saying at random 'yes' or 'no' (that is, $(\mathrm{b}R)a = \pm 1$) with equal probability of $\frac{1}{2}$.

In the following theorem, whose proof is taken from [30], we consider a game $\mathbf{w}$ and a player a. We define a random variable Y_a in the roll-call space $\mathbf{R}$ such that if a is positive in R then Y_aR is the contribution of a to the worth in $\mathbf{w}$ of the *positive* head of a in R; but if a is negative in R then Y_aR is the contribution of a to the worth in the *dual game* $\mathbf{w}^*$ of the *negative* head of a in R.

6.2.18 Theorem *Let* $\mathbf{w}$ *be a game on* N *and let* a *be a player of* $\mathbf{w}$. *In the roll-call space* $\mathbf{R}$, *let* Y_a *be the random variable such that*

$$\mathsf{Y}_aR = \begin{cases} \mathbf{w}(\mathrm{h}_a^+R) - \mathbf{w}(\mathrm{h}_a^+R - \{a\}) & \text{if } (\mathrm{b}R)a = 1, \\ \mathbf{w}^*(\mathrm{h}_a^-R) - \mathbf{w}^*(\mathrm{h}_a^-R - \{a\}) & \text{if } (\mathrm{b}R)a = -1, \end{cases}$$

for any roll-call R *of* N. *Then*

$$\mathrm{E}(\mathsf{Y}_a) = \phi_a(\mathbf{w}).$$

Proof Define a value assignment ξ by putting

$$\xi_a(\mathbf{w}) =_{\mathrm{def}} \mathrm{E}(\mathsf{Y}_a)$$

for every game $\mathbf{w}$ and every player a of $\mathbf{w}$. We claim that ξ is iso-invariant, efficient and marginal on the class of all games.

Iso-invariance is obvious. To prove efficiency, note that by the definition of Y_a and Def. 6.2.1(iv) we have, for every roll-call R:

$$\sum_{a \in N} \mathsf{Y}_a R = \mathbf{w}P + \mathbf{w}^*(N - P) = \mathbf{w}N,$$

where P is the set of players that are positive in R. The efficiency of ξ follows at once. Finally, to prove marginality, observe that Y_a depends on a only through a's contributions to quantities of the form $\mathbf{w}P$ and $\mathbf{w}^*S$ where $a \in P$ and $a \in S$. But then by Def. 6.2.1(iv) a's contribution to $\mathbf{w}^*S$ is the same as to $\mathbf{w}T$, where $T = (N - S) \cup \{a\}$. This establishes marginality.

It now follows from Thm. 6.2.15 that $\xi = \phi$. ∎

6.2.19 Remarks (i) Instead of Thm. 6.2.15 we could just as easily have used Thm. 6.2.14. In our next proof, that of Thm. 6.2.23, we shall use Thm. 6.2.14, but Thm. 6.2.15 would do just as well.

(ii) For the sake of simplicity, we have assumed that each player is positive with probability $\frac{1}{2}$. But we can generalize $\mathbf{R}$ by choosing any p such that $0 \leq p \leq 1$ and letting each player be positive with probability p.[15] The probability of a roll-call R then equals $p^k(1-p)^{(n-k)}/n!$, where k is the number of positive players in R. We shall denote this space by '$\mathbf{R}_N^{(p)}$'; and when there is no risk of confusion we shall write simply '$\mathbf{R}^{(p)}$'. In this more general space Thm. 6.2.18 still holds — with exactly the same proof. Thm. 6.2.6 is obtained as a special case: for $p = 1$.

(iii) The roll-call space $\mathbf{R}$ (or, more generally, $\mathbf{R}^{(p)}$) together with the random variables Y_a of Thm. 6.2.18 can be regarded, in a formal sense, as a bargaining model. This time the game played is not $\mathbf{w}$ (except when $p = 1$) but a constant-sum combination of $\mathbf{w}$ and $\mathbf{w}^*$, in which the coalition P consisting of the positive players gets collectively $\mathbf{w}P$, and the counter-coalition $N - P$ gets $\mathbf{w}^*(N - P) = \mathbf{w}N - \mathbf{w}P$.

[15] In other words, in $\mathbf{R} = \mathbf{Q} \times \mathbf{B}$ we can replace our special Bernoulli space $\mathbf{B}$, with its equiprobable 'yes' and 'no', by an arbitrary Bernoulli space of n trials.

We leave it to the reader to judge how realistic this model is, compared to the queue bargaining model. We make no claim on this score, but merely stress that both models yield the Shapley value.

In order to state our next result we shall need two definitions. First, we extend the definition of bloc formation from SVGs (Def. 2.3.23) to all games. Then we define the *Bz transformation*, which may be applied to any game $\mathbf{w}$ to yield a new game.

6.2.20 Definition Let $\mathbf{w}$ be a game on N and let S be a coalition of $\mathbf{w}$. Let the object $\&_S$ (the *bloc of* S), the set $N|\&_S$ and the natural surjection f from N to $N|\&_S$ be as in Def. 2.3.23. We define a game $\mathbf{w}|\&_S$ on $N|\&_S$ by putting

$$(\mathbf{w}|\&_S)Y =_{\text{def}} \mathbf{w}(f^{-1}[Y])$$

for every $Y \subseteq N|\&_S$. We say that $\mathbf{w}|\&_S$ is *obtained from* $\mathbf{w}$ *by the formation of the bloc* $\&_S$.

6.2.21 Remarks (i) Intuitively, $\mathbf{w}|\&_S$ can be described as the game resulting from $\mathbf{w}$ if the members of S form a stable alliance that plays as a single player. In $\mathbf{w}|\&_S$ the members of S do not bargain with each other; in fact, they no longer exist as individual players, but are submerged in the bloc player $\&_S$, who may bargain with other players.

(ii) Thm. 3.2.18 can quite easily be generalized to all games:

$$\beta'_{a\&b}(\mathbf{w}|a\&b) = \beta'_a(\mathbf{w}) + \beta'_b(\mathbf{w}),$$

for any game $\mathbf{w}$ and distinct players a and b.

6.2.22 Definition Let $\mathbf{w}$ be a game on N. We define the *Bz transform* of $\mathbf{w}$ to be the game $\mathbf{w}^\beta$ on N such that

$$\mathbf{w}^\beta S =_{\text{def}} \beta'_{\&_S}(\mathbf{w}|\&_S)$$

for any $S \subseteq N$. Thus in $\mathbf{w}^\beta$ the worth of a coalition S equals the Bz value to the bloc player $\&_S$ of the game obtained from $\mathbf{w}$ by the formation of this very bloc.

6.2.23 Theorem *Let* $\mathbf{w}$ *be a game on* N*. Then the Shapley value of* $\mathbf{w}^\beta$ *for any* $a \in N$ *is the same as the Shapley value of* $\mathbf{w}$*:*

$$\phi_a(\mathbf{w}^\beta) = \phi_a(\mathbf{w}).$$

Proof Define a value assignment ξ by putting

$$\xi_a(\mathbf{w}) =_{\text{def}} \phi_a(\mathbf{w}^\beta) \tag{$*$}$$

for any game $\mathbf{w}$ and any player a of $\mathbf{w}$. We claim that ξ is iso-invariant, efficient, vanishes for dummies and is additive on the class of all games.

Iso-invariance is obvious. To establish efficiency, vanishing for dummies and additivity, we unpack Def. 6.2.22 via Defs. 6.2.3 and 6.2.20. We obtain the identity:

$$\mathbf{w}^\beta S = \frac{1}{2^{n-|S|}} \sum_{X\,:\,S\subseteq X\subseteq N} \big(\mathbf{w}X - \mathbf{w}(X - S)\big) \tag{$\dagger$}$$

for all $S \subseteq N$, where N is the grand coalition of $\mathbf{w}$ and $n = |N|$.

From ($\dagger$) it is clear that $\mathbf{w}^\beta N = \mathbf{w}N$. The efficiency of ξ now follows directly from ($*$) and the efficiency of ϕ.

From ($\dagger$) it is likewise easy to see that any dummy of $\mathbf{w}$ is also a dummy in $\mathbf{w}^\beta$. Hence the vanishing of ξ for dummies follows at once, in view of ($*$) and the vanishing of ϕ for dummies.

Finally, from ($\dagger$) it is also evident that

$$\mathbf{w} = \mathbf{u} + \mathbf{v} \Rightarrow \mathbf{w}^\beta = \mathbf{u}^\beta + \mathbf{v}^\beta,$$

hence the additivity of ξ follows from that of ϕ.

By Thm. 6.2.14 we have $\xi = \phi$. ∎

6.2.24 Comment Let $\mathcal{W}$ be an SVG, with $\mathbf{w}$ as CF. In this case $\mathbf{w}^\beta$ can be interpreted as a *vote-selling game*. To this end, recall the vote-buying scenario of Com. 3.2.15. In that scenario we found that the buying price of voter a's vote was $\beta'_a[\mathcal{W}]$, which in our present notation is $\beta'_a(\mathbf{w})$ (see Rem. 6.2.3(ii)).

Now suppose that the members of any coalition S can collaborate and offer their votes for sale *en bloc*. From the buyer's viewpoint, buying this bloc vote is equivalent to buying the vote of the single voter $\&_S$ in the SVG $\mathcal{W}|\&_S$; so the buying price is $\beta'_{\&_S}(\mathbf{w}|\&_S)$, which is $\mathbf{w}^\beta S$ according Def. 6.2.22. This is the payoff that the members of S can obtain by cooperating as sellers. By Thm. 6.2.23, for any voter a the Shapley value of this vote-selling game is the same as that of the original voting game $\mathbf{w}$.[16]

6.2.25 Example Consider the CF $\mathbf{w}$ of the WVG $\mathcal{W}$ with assembly $N = \{a, b, c, d\}$ that is isomorphic, in alphabetic order, to $[5; 3, 2, 1, 1]$. First let us look at the values $\phi_x(\mathbf{w})$.[17] A simple calculation (for the details of which see Ex. A.3) yields:

$$\phi_a = \tfrac{7}{12},\ \phi_b = \tfrac{1}{4},\ \phi_c = \phi_d = \tfrac{1}{12}.$$

Now let us determine the game $\mathbf{w}^\beta$, the Bz transform of $\mathbf{w}$. We must find $\mathbf{w}^\beta S$ for each coalition S of $\mathcal{W}$. Apart from the obvious $\mathbf{w}^\beta \emptyset = 0$, the easiest cases are those in which S is any one of the five winning coalitions of $\mathcal{W}$. When such a coalition forms a bloc $\&_S$, this bloc becomes the dictator of the resulting SVG $\mathcal{W}|\&_S$. As we noted in Rem. 3.2.17(ii), the Bz power of a dictator is always 1. Therefore by Def. 6.2.22 we have $\mathbf{w}^\beta S = 1$ for each winning coalition S of $\mathcal{W}$.

Next consider $\mathbf{w}^\beta S$ where S is one of the four singletons. When $\{x\}$ forms itself into a bloc $\&_{\{x\}}$, the WVG $\mathcal{W}$ is essentially unchanged, except that the voter x is formally replaced by the bloc $\&_{\{x\}}$. Therefore $\mathbf{w}^\beta\{x\}$ is simply $\beta'_x[\mathcal{W}]$. These Bz powers are calculated in Ex. A.1, from which we get

$$\mathbf{w}^\beta\{a\} = \tfrac{5}{8},\ \mathbf{w}^\beta\{b\} = \tfrac{3}{8},\ \mathbf{w}^\beta\{c\} = \mathbf{w}^\beta\{d\} = \tfrac{1}{8}.$$

We have so far accounted for 10 coalitions. The remaining 6 require just a little more effort. First let us look at the case where S is a coalition of size 2. We have already accounted for $\{a, b\}$, which

[16] This observation is made, without proof, by Morriss [70, p. 165]. See also ibid., p. 226, where he claims: 'I am sure such a proof could be obtained, but I have not spent much time looking for it, since I am not primarily addressing mathematicians.'

[17] According to Def. 6.3.1 below, these are the values of the *S-S index* for the voters in $\mathcal{W}$.

is a winning coalition of $\mathcal{W}$; but there are five other coalitions of size 2. For these we can use Thm. 3.2.18, which now yields

$$\mathbf{w}^\beta\{x, y\} = \mathbf{w}^\beta\{x\} + \mathbf{w}^\beta\{y\}$$

for any two distinct voters x and y. Hence

$$\mathbf{w}^\beta\{a, c\} = \mathbf{w}^\beta\{a, d\} = \tfrac{3}{4},$$
$$\mathbf{w}^\beta\{b, c\} = \mathbf{w}^\beta\{b, d\} = \tfrac{1}{2},\ \mathbf{w}^\beta\{c, d\} = \tfrac{1}{4}.$$

This leaves only one coalition unaccounted for: $\{b, c, d\}$. When this coalition forms a bloc, the resulting WVG is isomorphic to $[5; 3, 4]$, in which the two voters are symmetric to each other, each having Bz power $1/2^{2-1} = \frac{1}{2}$. Hence $\mathbf{w}^\beta\{b, c, d\} = \frac{1}{2}$.

We have now accounted for all 16 coalitions of $\mathcal{W}$, and have determined $\mathbf{w}^\beta$ completely.

The game $\mathbf{w}^\beta$ is in fact the same as the $\mathbf{v}$ of Ex. A.2. The Shapley values of this game, calculated there in detail, are identical to those of $\mathbf{w}$. Thm. 6.2.23 explains why $\mathbf{w}$ and $\mathbf{v}$, which do not look at all alike, nevertheless have the same Shapley values.

If we accept the Shapley value as a valid predictor of a player's expected payoff, then we may interpret this result as saying that each voter should be indifferent between participating in the voting in $\mathcal{W}$ (in an office-seeking mode) and playing the vote-selling game $\mathbf{w}^\beta$: greed's mean reward will be the same either way.

We end this section with two comments regarding the vulnerability of the Shapley value to two kinds of critique.

6.2.26 Comment As we pointed out in Com. 6.2.8, the Shapley value — as a predictor of a player's average payoff, or as an arbitration scheme — is not justified in terms of this or that general theoretical bargaining model; for no sufficiently realistic and compelling model of this kind is available. Rather, the justification of the Shapley value rests on its axiomatic characterization. This makes the Shapley value potentially vulnerable to criticism of two different and mutually complementary kinds.

The first kind of criticism is directed at the postulates characterizing the Shapley value. If any of these is found to be unreasonable, or unreasonably strong, then the claim of the Shapley value to uniqueness is threatened.

Criticism of this kind has been voiced quite early, notably by Luce and Raiffa in [68], although they (or, as far as we know, anyone else) were not able to propose anything better than the Shapley value.

Following Luce and Raiffa, let us examine Shapley's postulates one by one.

The iso-invariance postulate is not problematic. What we said on this score in Rem. 2.1.8 in the context of SVGs holds just as well in the present more general context. This postulate merely expresses the requirement of aprioricity: an a priori value assignment should not depend on anything other than the abstract structure of the given game and the abstract position of the given player in the game.

The postulate of vanishing for dummies is likewise unproblematic: it is self-evidently compelling.

The efficiency postulate is not quite so unproblematic. For a non-negative superadditive game $\mathbf{w}$ — and we need not concern ourselves here with any other kind — efficiency of an average payoff predictor amounts to the assumption that in every play of the game some coalition whose worth is as much as that of the grand coalition N will be formed. One may attempt to justify this on the grounds that if a coalition S were formed such that $\mathbf{w}S < \mathbf{w}N$ then the members of S could do at least as well, and some may perhaps do better, if they agreed to extend this coalition. But in our view Luce and Raiffa [68, p. 247] have a point in commenting that 'if value to a player is to be interpreted as an *a priori* expectation, it seems that the equality sign in [the efficiency postulate] is *much too strong* — indeed, it represents an n-fold combination of wishful thinking.'[18]

[18] Emphasis in the original. On the other hand, they point out (ibid., p. 251) that 'there is certainly no objection to [this postulate] as an arbitration condition.'

The additivity postulate is hardest to justify. This is what Luce and Raiffa have to say about it:

> The last condition is not nearly so innocent as the other[s]. For, although [$\mathbf{u}+\mathbf{v}$] is a game composed of [$\mathbf{u}$] and [$\mathbf{v}$], we cannot in general expect it to be played as if it were two separate games. It will have its own structure which will determine a set of equilibrium outcomes which may be very different from those for [$\mathbf{u}$] and [$\mathbf{v}$]. Therefore, one might very well argue that its *a priori* value should not necessarily be the sum of the values of the two component games. This strikes us as a flaw in the concept of value, but we have no alternative to suggest.[19]

In view of Thm. 6.2.15, the additivity postulate may be replaced by Young's marginality postulate. But this is also far from compelling for an average payoff predictor. It stipulates that the interaction among players is not too strong, so that the expected payoff of any player a is affected only by a's contributions to the worth of coalitions; any change in the relative positions of the other players that does not affect those contributions of a cannot affect a's expected payoff. It is not at all clear why this must be so.[20]

6.2.27 Comment While criticism of the characterizing postulates can threaten the *uniqueness* of the Shapley value as an average payoff predictor, there is an even greater potential peril. Suppose someone were to put forward a new postulate that would be intuitively compelling for any such predictor, but which the Shapley value failed to satisfy.[21] That would effectively overthrow

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[19] [68, p. 248]; our notation substituted [in square brackets].

[20] For an arbitration scheme, which is the context in which Young stated the marginality condition, this condition is in our opinion not unreasonable, but not compelling either.

[21] In this connection we note that in [68, pp. 250–252] Luce and Raiffa illustrate a piece of behaviour by the Shapley value that may be regarded as unacceptable for an arbitration scheme. However, as they admit at the outset, the Shapley value was not originally proposed as such a scheme. Perhaps more damaging are examples of the kind discussed by Raiffa in [82, pp. 271–273], where a game has a core consisting of a unique allocation that differs from the

the Shapley value as an acceptable predictor. In this event — which can never be ruled out in advance — we would be faced with the following choice. If we accept the criticism of the first kind, discussed in Com. 6.2.26, and therefore agree to dismiss or dilute Shapley's postulates (particularly the additivity postulate), then we could look for a more acceptable predictor. On the other hand, if we dismiss that criticism and accept all Shapley's postulates as compelling, we would have an *impossibility theorem*, stating that these postulates are inconsistent with the new postulate. This would imply that the very notion of average payoff predictor is incoherent.

6.3 The Shapley–Shubik Index

6.3.1 Definition The *S-S score* is the function κ that assigns to any SVG $\mathcal{W}$ and any voter a of $\mathcal{W}$ a value $\kappa_a[\mathcal{W}]$ — called the *S-S score of* a [in $\mathcal{W}$] — given by

$$\kappa_a[\mathcal{W}] =_{\text{def}} \sum_{X \subseteq N} \big(|X| - 1\big)!\big(n - |X|\big)!\big(\mathbf{w}X - \mathbf{w}(X - \{a\})\big),$$

where $\mathbf{w}$ is the CF of $\mathcal{W}$.

The *S-S index* is the function ϕ defined by

$$\phi_a[\mathcal{W}] =_{\text{def}} \frac{\kappa_a[\mathcal{W}]}{n!}.$$

(Here, as usual, N is the assembly of $\mathcal{W}$ and $n = |N|$.) We refer to $\phi_a[\mathcal{W}]$ as *the S-S index of* a [in $\mathcal{W}$].

6.3.2 Remarks (i) As before, we shall often suppress mention of $\mathcal{W}$ where no ambiguity is likely.

(ii) Evidently, the S-S index defined here is a restriction to the class of SVGs of the Shapley value assignment as defined in Def. 6.2.3.

vector of Shapley values. For those who consider the core as a valid predictor, particularly when it consists of a unique allocation, such examples seriously undermine the Shapley value.

Indeed, if $\mathcal{W}$ is an SVG with $\mathbf{w}$ as CF, then $\phi_a[\mathcal{W}]$ as defined in Def. 6.3.1 coincides with $\phi_a[\mathbf{w}]$ as defined in Def. 6.2.3.

Therefore the results proved in § 6.2 for the Shapley value apply in particular to the S-S index. When invoking those results, we shall now assume that $\mathbf{w}$ is the CF of an SVG $\mathcal{W}$. Recall (Rem. 6.2.2(ii)) that this means that $\mathbf{w}$ is monotone and assumes just two values: 0 for losing coalitions and 1 for winning ones. In particular, $\mathbf{w}\emptyset = 0$ and $\mathbf{w}N = 1$.

(iii) In the definition of κ_a, the summation is over all coalitions X. However, $\mathbf{w}X - \mathbf{w}(X - \{a\})$ equals 1 or 0 according as a is or is not critical in X; so only those coalitions in which a is critical make a non-zero contribution to the S-S score of a. If X is such a coalition, its contribution is $\big(|X| - 1\big)!\big(n - |X|\big)!$. The sum of these contributions divided by $n!$ is ϕ_a. Thus

$$\phi_a = \sum_{X\,:\,a \text{ critical in } X} \frac{\big(|X| - 1\big)!\big(n - |X|\big)!}{n!}.$$

For a detailed illustration of how the S-S index is calculated see Ex. A.3. In the following example the computational details are omitted.

6.3.3 Example We return to the WVGs of Ex. 3.2.24:

$$\mathcal{U} = [6; 5, 3, 1, 1, 1], \quad \mathcal{V} = [7; 5, 3, 1, 1, 1].$$

We leave it to the reader to verify that

$$\phi_1[\mathcal{U}] = \tfrac{3}{5}, \quad \phi_2[\mathcal{U}] = \phi_3[\mathcal{U}] = \phi_4[\mathcal{U}] = \phi_5[\mathcal{U}] = \tfrac{1}{10};$$
$$\phi_1[\mathcal{V}] = \tfrac{13}{20}, \quad \phi_2[\mathcal{V}] = \tfrac{3}{20}, \quad \phi_3[\mathcal{V}] = \phi_4[\mathcal{V}] = \phi_5[\mathcal{V}] = \tfrac{1}{15}.$$

In Ex. 3.2.24 we saw that although in going from $\mathcal{U}$ to $\mathcal{V}$ voter 1 becomes a blocker, she loses Bz power both absolutely and relatively, in terms of the Bz index. Now we see that her S-S index actually increases. Thus the Bz and S-S indices yield conflicting results.[22]

[22] This shows that whatever is measured by the S-S index, it cannot possibly be the same kind of thing as is measured by the Bz index. Cf. Com. 7.9.19(i) below.

6.3.4 Observation (i) Look at the random variable Y_a defined in Thm. 6.2.6. Since $\mathbf{w}$ is now the CF of an SVG $\mathcal{W}$, it is clear that $\mathsf{Y}_a Q$ must be 0 or 1. Moreover, since $\mathbf{w}N = 1$, it follows from Cor. 6.2.7 that for each queue Q of N there is a unique voter a of $\mathcal{W}$ for whom $\mathsf{Y}_a Q = 1$.

(ii) Look at the random variable Y_a defined in Thm. 6.2.18. Since $\mathbf{w}$ and $\mathbf{w}^*$ are now the respective CFs of $\mathcal{W}$ and its dual, $\mathcal{W}^*$, it is clear that $\mathsf{Y}_a R$ must be 0 or 1. Moreover, in the proof of Thm. 6.2.18 it was shown that $\sum_{a \in N} \mathsf{Y}_a R = \mathbf{w}N$ for every roll-call R. But now $\mathbf{w}N = 1$, so there is a unique voter a of $\mathcal{W}$ for whom $\mathsf{Y}_a R = 1$.

6.3.5 Definition Let $\mathcal{W}$ be an SVG with assembly N.

(i) For each queue Q of N, the unique voter a of Obs. 6.3.4(i) will be called *the $\mathcal{W}$-pivot of Q* and denoted by '$\mathrm{piv}(Q; \mathcal{W})$'.

(ii) For each roll-call R of N, the unique voter a of Obs. 6.3.4(ii) will be called *the $\mathcal{W}$-pivot of R* and denoted by '$\mathrm{piv}(R; \mathcal{W})$'.

6.3.6 Remarks (i) As usual, where there is no risk of ambiguity we omit reference to $\mathcal{W}$ and write simply '$\mathrm{piv}Q$' and '$\mathrm{piv}R$'.

(ii) If Q is a queue, $\mathrm{piv}Q$ can be characterized as the voter a with least Qa such that $\mathrm{h}_a Q \in \mathcal{W}$.

More intuitively, $\mathrm{piv}Q$ is the earliest voter a in the queue Q such that a together with those standing ahead of a constitute a winning coalition. Thus, if we imagine all the voters queuing up as in Q to cast a 'yes' vote, then $\mathrm{piv}Q$ is the voter whose 'yes' seals the (positive) outcome: once the pivot's 'yes' vote is cast — and only then — the bill is as good as passed, for it would pass even if all subsequent voters were to change their minds and vote 'no'.

(iii) If R is a roll-call, $\mathrm{piv}R$ can be characterized as the voter a with least $(\mathrm{q}R)a$ such that $\mathrm{h}_a^+ R \in \mathcal{W}$ or $\mathrm{h}_a^- R \in \mathcal{W}^*$.

More intuitively, $\mathrm{piv}R$ is the voter whose 'yes' or 'no' vote seals the outcome one way or the other. Imagine all the voters queuing up as in $\mathrm{q}R$ to cast their votes, which may be 'yes' or 'no', according to the bipartition bR. If the pivot is a 'yes' voter, then once this 'yes' vote is cast — and only then — the bill is as good as passed, for it would pass even if all subsequent voters were to vote 'no'; but if the pivot is a 'no' voter, then once this 'no' vote is cast — and only then — the bill is blocked, for it would fail even if all subsequent voters were to vote 'yes'.

From Thm. 6.2.6 and Def. 6.3.5(i) we obtain at once:

6.3.7 Theorem *Let $\mathcal{W}$ be an SVG with assembly N and let a be a voter of $\mathcal{W}$. Then*

$$\begin{aligned}\phi_a[\mathcal{W}] &= \mathrm{P}(a \text{ is the } \mathcal{W}\text{-pivot})\\ &= \frac{1}{n!}\big|\{Q : Q \text{ is a queue of } N \text{ and } \mathrm{piv}(Q;\mathcal{W}) = a\}\big|,\end{aligned}$$

where P *is the probability distribution in the queue space $\mathbf{Q}_N$.* ∎

As we pointed out in [30], the next theorem — stated without proof by Mann and Shapley in [69, fn. 3, p. 153] — follows at once from Rem. 6.2.19(ii) and Def. 6.3.5(ii).

6.3.8 Theorem *Let $\mathcal{W}$ be an SVG with assembly N; for any roll-call R of N let $k(R)$ be the number of positive voters in R; let $a \in N$; and let p be any real in $[0,1]$. Then*

$$\begin{aligned}\phi_a[\mathcal{W}] &= \mathrm{P}(a \text{ is the } \mathcal{W}\text{-pivot})\\ &= \frac{1}{n!}\sum_{R\,:\,\mathrm{piv}(R;\mathcal{W})=a} p^{k(R)}(1-p)^{n-k(R)},\end{aligned}$$

where P *is the probability distribution in the generalized roll-call space $\mathbf{R}_N^{(p)}$. In particular, for $p = \frac{1}{2}$*

$$\begin{aligned}\phi_a[\mathcal{W}] &= \mathrm{P}(a \text{ is the } \mathcal{W}\text{-pivot})\\ &= \frac{1}{n!2^n}\big|\{R : R \text{ is a roll-call of } N \text{ and } \mathrm{piv}(R;\mathcal{W}) = a\}\big|,\end{aligned}$$

where P *is the probability distribution in the roll-call space $\mathbf{R}_N$.* ∎

In the following two comments we take up the theme of Coms. 6.2.8, 6.2.26 and 6.2.27, and discuss the justification of the S-S index.

6.3.9 Comment Thm. 6.3.7 is very well known. Shapley and Shubik [97, p. 788] actually use it to *define* their index; and they have been followed in this by many others.[23]

The advantage of this approach is that it is the simplest and least technical way to present the S-S index, most suited for readers who have low tolerance for mathematical formalities — precisely the kind of readership Shapley and Shubik were addressing in [97].[24]

But this has had the regrettable effect of creating the very widespread false impression that the S-S index depends for its *justification* on the queue bargaining model, and that this model therefore purports to be a realistic description of how voting coalitions are formed and how they divide their spoils. Shapley and Shubik [97, p. 790] expressly warn against this misconception: they point out 'that the scheme we have been using (arranging the individuals in all possible orders, etc.) is just a convenient conceptual device'; but their caveat went largely unheeded.[25]

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[23] In this connection note that the term 'minimal winning coalition' (ibid., fn. 4) is not used by them in the precise sense that is now more usual. What they imply by this term is that the pivot in any queue Q is the *earliest* voter in Q whose head in Q is a winning coalition. This non-standard use of the term has been replicated by some other authors, giving rise to an unfortunate terminological confusion. See, for example, our quotation from Banzhaf in § 4.2, text to fn. 41.

[24] Compare Shapley's very different presentation in [94], where (as we pointed out in Com. 6.2.8) the queue bargaining model appears as a brief afterthought.

[25] They themselves are not entirely blameless: on p. 788 they try to give the queue model some realistic gloss by suggesting, for example, that 'the randomly chosen order of voting [might better be thought of] as an indication of degrees of support by the different members, with the most enthusiastic members "voting" first, etc.' In [87, 200–205] Riker and Shapley go even further in an attempt — which we find quite unconvincing — to dress up the queue model as a realistic picture of how voting takes place. That paper argues for the S-S index as a measure of *influence* (or, to use our terminology, an index of a priori I-power). This seems to be based on the tacit but unwarranted assumption that expected payoff is proportional to influence. Moreover, their crucial argument (at op. cit., p. 202, item 4) depends on the quite arbitrary assumption that any division of a board can be represented by a queue whose pivot is initially (when the bill is first proposed) undecided whether to vote 'yes' or 'no', but whose support is eventually bought by someone called 'the manager', using various inducements. This rules out divisions in which those prepared to vote for the bill, without

This has given rise to a great deal of misguided criticism of the S-S index as well as to some equally misguided defence of it.[26]

As a matter of fact, in the context of SVGs there are at least two versions of the queue bargaining model. The more simplistic version is the one described by Shapley and Shubik in [97, p. 788]: all voters are 'willing to vote for some bill', they queue up and vote in random order, and the pivot 'is given credit for having passed [the bill]' and receives the entire purse. This clearly cannot be taken seriously as a realistic bargaining model. Assuming, as we must here, that voting is motivated by office seeking, why on earth should the voter at the top of the queue, who will get nothing, be more enthusiastic than the pivot, who will get the whole purse?[27] Also, why should the pivot get all the credit for passing the bill, and the whole purse? Why should we assume that all voters always vote 'yes' or at least are willing to do so? And why should the order in which the grand coalition is formed matter so much?

For these and similar reasons, the simplistic version of the queue bargaining model has been widely criticized as highly artificial.[28]

There is, however, an apparently more sophisticated version of the queue bargaining model.[29] According to this version, each queue Q represents an event in which the coalition that is formed is the head of the pivot of Q: the coalition consisting of the pivot and all those placed ahead of the pivot in Q. In our notation, the coalition formed in Q is $\mathrm{h}_{\mathrm{piv}Q}Q$. This winning coalition will vote for the bill, causing it to pass, and will lay its hands on the whole purse. The remaining voters (those standing behind the pivot),

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any further incentive, already constitute a winning coalition; or in which those determined to vote against it, spurning the manager's bribes, are already a blocking coalition.

[26] As an example of the former, see Barry [9, pp. 187–188]. As an example of the latter, beside [87], see also Laver [62, pp. 904–905].

[27] In a proper SVG, the voter at the top of the queue cannot be the pivot, unless she is a dictator; but even then, if $n > 1$ there are queues at the top of which stands a dummy.

[28] See, for example, Brams [11, pp. 168–169].

[29] This version is available in the special context of SVGs but not in the more general context of games discussed in § 6.2.

being offered nothing, will vote against the bill. The probability that a given coalition S will be formed is equal to the probability of its being the head of the pivot in a queue; in our notation, this is

$$\frac{1}{n!}\big|\{Q : \mathrm{h}_{\mathrm{piv}Q}Q = S\}\big|. \tag{$*$}$$

Given that coalition S is formed, how will the purse be divided? In the simplistic version, the pivot always got the whole purse. But now there is no need to make such a crude assumption. Instead, it is enough to stipulate that, subject to the condition that S is formed, the conditional expected payoff of any member a of S is equal to the conditional probability that a is the pivot. Now, it is easy to see that the pivot is always a critical member of the coalition S that is formed. Moreover, subject to the condition that S is formed, all critical members of S have equal conditional probability of being pivotal. Thus our stipulation is satisfied if, for example, whenever S is formed the purse is divided equally among its critical members. (This makes the order in which S is formed irrelevant, except for the proviso that the last member to join up is critical in S.)

To sum up: in the 'sophisticated' queue bargaining model, the probability that coalition S will be formed is given by ($*$); and, subject to the condition that S is formed, the purse is divided among its critical members in such a way that their conditional expected payoffs are equal.

At first glance this version of the queue bargaining model seems fairly realistic.[30] But this first impression is dispelled on closer examination. For one thing, the probability ($*$) depends on the relevant data — the total number of voters, the size of S and the number of its critical members — in a rather peculiar way. Indeed, let $|S| = k$ and let S have m critical members; let us denote by '$p(k, m, n)$' the probability ($*$) that S is formed according to this model. (Here, as usual, n is the total number of voters.) Then

[30] Indeed, Laver seems to take it quite seriously, and relies on it for his defence of the S-S index (mentioned above, fn. 26, p. 201).

$p(k, m, n)$ is easy to calculate: it is the probability that in a random queue the first k voters are exactly the members of S, the k-th of which is *critical* in S. Hence

$$p(k, m, n) = \frac{k!(n-k)!}{n!} \cdot \frac{m}{k} = \frac{(k-1)!(n-k)!m}{n!}.$$

Now suppose that d is a non-member of S such that $S \cup \{d\}$ has the same number m of critical members as S; for example, d can be a dummy. Then the probability that the coalition $S \cup \{d\}$ will be formed is $p(k+1, m, n)$. This can be markedly different from $p(k, m, n)$. The ratio $p(k+1, m, n)/p(k, m, n)$ equals $k/(n-k)$, which can be arbitrarily near 0 if k is very small compared to n and arbitrarily large if n is large and k is close to n. It seems to us quite unreasonable that the addition of a mere dummy should have such dramatic and erratic effects.

The same effect can be obtained without any dummies. For example, take a WVG having 303 voters: a 'heavy' voter a with weight 302, and 302 'light' voters $b_1, b_2, \ldots, b_{302}$, each of whom has weight 1; the quota is set at 303. Then according to the present model a coalition consisting of a and three of the b_i is 100 times less likely to form than a coalition consisting of a and two of the b_i. But a coalition consisting of a and 300 of the b_i is 100 times *more* likely to form than a coalition consisting of a and 299 of the b_i. This does not seem particularly plausible.

Moreover, the model assigns a positive probability to the formation of any coalition containing a and one or more of the b_i, although if it contains two or more b_i none of them will get any payoff, because a is the only critical member. Why should two or more b_i ever agree to participate in forming such a coalition? On the other hand, if a forms a coalition with just one of the b_i then a's expected payoff will be $\frac{1}{2}$, equal to that of her lightweight partner. This is hardly credible in view of the fact that a is a vetoer and can choose her lightweight partner out of 302 willing suitors, whose only chance of getting any payoff at all is to be the unique one chosen. Surely a can exploit the stiff competition between the b_i and get one of them to form a coalition with her for much less than half the total payoff.

The more you probe this model, the less persuasive its claim to realism becomes.

Thm. 6.3.8 is much less widely known than Thm. 6.3.7. Had it been better known, it might have helped to dispel the false impression that the S-S index depends for its justification on the queue bargaining model (in either version). The latter would have been seen for what it is: one of several 'convenient conceptual devices' that yield the S-S index. As such, it is by no means unique, but happens to be the simplest and the easiest to describe. Of course, the roll-call bargaining models suggested by the spaces $\mathbf{R}^{(p)}$ are not realistic either. True, unlike the simplistic version of the queue bargaining model they do not assume that all voters vote 'yes';[31] but on close examination they do no better than the 'sophisticated' version of the queue bargaining model. We shall not demonstrate this in detail. In a sense, it is unnecessary to do so, because $\mathbf{R}^{(p)}$ has an obvious feature that makes it unsuited to serve as the basis for a realistic model of office-seeking voting behaviour, which underlies the notion of P-power. As we pointed out in Com. 6.1.4, in the context of P-power we must assume that a voter's propensity to vote 'yes' and the degree of coordination between any two given voters depend a priori on the decision rule (that is, on the particular SVG). Yet $\mathbf{R}^{(p)}$ is the same for all SVGs with given assembly, and incorporates the assumption that all voters vote 'yes' with fixed probability p, and act independently of each other.

In fact the roll-call model, like the queue model, is 'just a convenient conceptual device': a mathematical artefact that aids visualization and reasoning, but provides no justification for the S-S index.

6.3.10 Comment While there is no shortage of *mathematically correct* bargaining models that yield the S-S index, we know of none that is sufficiently realistic to provide a compelling justification for the S-S index.

As far as we can see, the sole *theoretical* justification of the S-S index as an index of a priori P-power is the one it inherits as

[31] Except in the degenerate case $p = 1$, in which $\mathbf{R}^{(p)}$ reduces to $\mathbf{Q}$.

a specialization (restriction) of the Shapley value.[32] And as we explained in Coms. 6.2.8 and 6.2.26, the theoretical justification of the Shapley value rests entirely on its axiomatic characterization. Thus, in so far as you accept the Shapley value as as a valid predictor of a player's expected payoff in *any* cooperative game, you must also accept its restriction to SVGs, the S-S index, as a valid measure of a priori P-power. On the other hand, the S-S index inherits the vulnerabilities of the Shapley value, discussed in Coms. 6.2.26 and 6.2.27.

In this respect the S-S index is inextricably game-theoretic, not only because the notion of P-power that underlies it is essentially game-theoretic, but also because Thm. 6.2.14, the characterization on which its theoretical justification ultimately rests, just does not apply to the class of SVGs: it only holds for broader classes of games.[33] Without the background of the *general* theory of cooperative games, it is extremely unlikely that the S-S index would have ever occurred to anyone. This contrast between the S-S and

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[32]This leaves open the possibility of justification via *empirical* corroboration. One attempt at such justification is made by Robert Aumann. In [3, p. 13] he addresses the formation of Israeli multi-party governments since 1977 and says: 'The theory that I am testing is very simple, almost naive. It is that the leader [of the party charged with forming the government] — the one with the initiative — tries to maximize the influence of his party within the government. So one takes each possible government that he can form and one looks at the Shapley value of his party within that government; ... the hypothesis that the leader aims to maximize his Shapley value seems a reasonable hypothesis to test, and it works not badly. ... I have been looking at [the formation of Israeli governments] since 1977, and on the whole, the predictions based on the Shapley value have done quite well.'

The idea seems to be that a multi-party government is regarded as an SVG whose winning coalitions are those sets of parties in the government that jointly command a majority in the Knesset. If so, our calculations do not bear out Aumann's claim. We analyzed all governments formed in Israel during the period 1977–1996 and found that in all these cases the party charged with forming the government could have increased its S-S index within the government by either narrowing or widening the government that it actually formed. Of course, it is possible that analysis of the experience of other countries may yet lend better support to Aumann's hypothesis.

[33]See Dubey's example in [26, p. 136] and our Ex. B.2. Thus Shapley and Shubik's remark at [97, pp. 789–790], which suggests that Shapley's characterization theorem applies to the class of SVGs, is misleading.

Bz indices is crucial: despite the *formal* similarity of their definitions, the latter owes little — historically or conceptually — to game theory. True, the Bz measure can be generalized to yield a value assignment (see Def. 6.2.3), which may then be subjected to some analytic tools and procedures developed by game theorists. But this does not make the Bz measure, or the Bz index which is derived from it, game-theoretic; nor does it lend them any additional conceptual justification.[34]

As far as *uniqueness* is concerned, the S-S index is in principle *more* vulnerable than the Shapley value in general. For, even if we were to accept that the latter is completely vindicated by Thm. 6.2.14, that would only guarantee that any 'reasonable' value assignment *defined for all cooperative games* (*or all superadditive ones*) must coincide with the Shapley value. But it would not rule out the existence of a reasonable index of P-power, *defined for SVGs only*, that differs from the S-S index.[35]

Dubey's characterization of the S-S index (Thm. B.8) is mathematically elegant but provides little *added* justification to the S-S index. This is because the measure-additivity postulate (Def. B.5) has little independent persuasive force. In the wider context of cooperative games, it is deducible from ordinary additivity (see B.7). But viewed in its own terms, in the restricted context of SVGs, the best that can be said for it is that it is not implausible for a measure of voting power. No really compelling intuitive justification has been offered for it, as far as we know.[36]

We next present and discuss yet another representation of the S-S

[34]The mistaken view that the Bz index is game-theoretic is however quite widespread, and appears to be held even by authoritative writers; see, for example Straffin [102, p. 1130]. This view is closely connected with a conflation of P-power and I-power: the former is essentially a game-theoretic concept, while the latter is not.

[35]This possibility seems to have escaped Shapley and Shubik in their misleading claim to which we alluded a moment ago in fn. 33.

[36]Dubey and Shapley [27, p. 106] virtually admit as much. Straffin [100, p. 298], concurring with Roth [88], comments that ' ... if we try to think of how to interpret [measure-additivity] as a statement about power in political situations, I think the best we can say is that it appears "somewhat opaque" '.

index, due to Straffin. Recall that in Def. 3.1.1 we used the set ${}^N\{-1,1\}$ of all bipartitions of N to construct a probability space $\mathbf{B}_N$, by assigning the same probability, $1/2^n$, to each bipartition. We now introduce another probability distribution on the same set, thus obtaining from it a different probability space.

6.3.11 Definition Let N be a finite set, with $|N| = n$. The *clone model* $\mathbf{C}_N$ is the probability space consisting of ${}^N\{-1,1\}$ (the set of all bipartitions of N) with the probability distribution P that assigns to each bipartition B of N the probability[37]

$$\mathrm{P}\{B\} = \frac{|B^+|!|B^-|!}{(n+1)!}.$$

Where there is no risk of confusion, we omit the subscript 'N' in '$\mathbf{C}_N$' and write simply '$\mathbf{C}$'.

6.3.12 Remarks (i) We are not proposing $\mathbf{C}$ as a bargaining model. We are putting it forward simply as a hypothetical scheme of voting behaviour, different from that modelled by $\mathbf{B}$. Later on, in Com. 6.3.16, we shall discuss the heuristic assumptions underlying this model, and the circumstances under which its use may be justified. The reason for the curious name 'clone model' will then become clear.

(ii) At first sight the probability distribution of $\mathbf{C}$ may look puzzling; but it can be described quite simply. Let the 2^n bipartitions be classified into $n+1$ mutually exclusive events, $\mathbf{Y}^0, \mathbf{Y}^1, \ldots, \mathbf{Y}^n$: the event $\mathbf{Y}^k$ consists of just those bipartitions in which exactly k voters vote 'yes'. Then let all these $n+1$ events be equiprobable, so that each has probability $1/(n+1)$. Furthermore, let all the bipartitions within each $\mathbf{Y}^k$ be equiprobable.

Now, the number of bipartitions in $\mathbf{Y}^k$ is equal to the number of ways in which k 'yes' voters can be chosen out of n, which is $\binom{n}{k}$. Hence the probability assigned to each of these bipartitions is

$$\left[(n+1)\binom{n}{k}\right]^{-1} = \frac{k!(n-k)!}{(n+1)!},$$

[37]For the definition of the sets B^+ and B^- see Def. 2.1.5 and Rem. 2.1.6(i).

precisely as specified in Def. 6.3.11.[38]

6.3.13 Theorem (Straffin) *Let $\mathcal{W}$ be an SVG. Then, for any voter a,*

$$\phi_a[\mathcal{W}] = \mathrm{P}(a \text{ is } \mathcal{W}\text{-critical})$$

in the space **C**.

Proof Let us calculate $\mathrm{P}(a \text{ is } \mathcal{W}\text{-critical})$. Take any coalition X in which a is critical. X gives us exactly two bipartitions for which a is critical: the bipartition B with $B^+ = X$, for which a is *positively* critical; and the bipartition $\widetilde{B}$ with $\widetilde{B}^+ = X - \{a\}$, for which a is *negatively* critical.[39] According to Def. 6.3.11,

$$\begin{aligned}\mathrm{P}\{B\} + \mathrm{P}\{\widetilde{B}\} &= \frac{|X|!\big(n-|X|\big)!}{(n+1)!} + \frac{\big(|X|-1\big)!\big(n-|X|+1\big)!}{(n+1)!}\\ &= \frac{\big(|X|-1\big)!\big(n-|X|\big)!}{n!}.\end{aligned}$$

To obtain $\mathrm{P}(a \text{ is } \mathcal{W}\text{-critical})$ we must add up these probabilities for all X in which a is critical:

$$\mathrm{P}(a \text{ is } \mathcal{W}\text{-critical}) = \sum_{X\,:\,a\ \mathcal{W}\text{-critical in } X} \frac{\big(|X|-1\big)!\big(n-|X|\big)!}{n!},$$

which equals $\phi_a[\mathcal{W}]$ by Rem. 6.3.2(iii). ∎

Straffin does not present Thm. 6.3.13 in quite this form. Instead, in several of his papers he considers — alongside his so-called Independence Assumption, discussed in Rem. 3.1.2(ii) — the following alternative model of voting behaviour, which he calls the 'Homogeneity Assumption':[40]

[38] Readers familiar with statistical mechanics may have noticed that while the Bernoulli model **B** is an instance of Maxwell–Boltzmann statistics, the clone model **C** is the corresponding Bose–Einstein statistics, in which the 'particles' — in this case the voters — are treated as mutually indistinguishable.

[39] See Def. 2.3.6.

[40] See, for example, his [100, p. 298] and [102, p. 1136]. Our formulation here is not a verbatim quote but (we hope) a faithful paraphrase.

6.3.14 Model Whenever a division is called, a number p is selected at random, as the value of a random variable P distributed uniformly on the unit interval $[0, 1]$. All voters then vote independently of one another, each of them voting 'yes' with the same probability p.

He then proves $\phi_a[\mathcal{W}] = \mathrm{P}(a \text{ is } \mathcal{W}\text{-critical})$ in *this* model. However, Straffin's result is an immediate consequence of Thm. 6.3.13, by virtue of the following lemma.

6.3.15 Lemma *Model 6.3.14 yields the probability space* **C**.

Proof Take any bipartition B, and let $|B^+| = k$. If all voters vote 'yes' with the same probability p, independently of one another, then *subject to this condition* the probability of obtaining B is $p^k(1-p)^{n-k}$.

Under Model 6.3.14, the *prior* (unconditional) probability of $\{B\}$ is therefore

$$\mathrm{P}\{B\} = \int_0^1 p^k(1-p)^{n-k}\mathrm{d}p.$$

The mathematically erudite reader will surely have recognized the right-hand side as $\mathrm{B}(k+1, n-k+1)$, where B is Euler's beta function. But in the present case, in which k and n are natural numbers, this integral is readily evaluated by routine elementary means. Integrating by parts, we get (for $k > 0$):

$$\int_0^1 p^k(1-p)^{n-k}\mathrm{d}p = \frac{k}{n-k+1}\int_0^1 p^{k-1}(1-p)^{n-k+1}\mathrm{d}p.$$

Hence, using induction on k, it is easy to show

$$\mathrm{P}\{B\} = \frac{k!(n-k)!}{(n+1)!},$$

as claimed. ∎

6.3.16 Comment What is the meaning of Thm. 6.3.13? Comparison of this theorem with Thm. 3.2.4 suggests very strongly that although the original justification of ϕ was as an a priori index of P-power, it is (also) a measure of I-power.

However — quite unlike β' — ϕ cannot be regarded as an *a priori* measure of I-power. The reason for this is revealed by comparing the Bernoulli model **B** with the clone model **C**. Whereas the former encapsulates a state of total a priori ignorance regarding relations of affinity or disaffinity between voters,[41] the latter definitely does not. Note that although Model 6.3.14 seems to say that the voters act independently of one another, this independence is only *conditional*, subject to the value p being given. But **C** in fact implies quite strong *prior dependence* between votes of different voters. To see this formally, let a and a' be two distinct voters, and consider Ba and Ba' as random variables in the space **C**. It is easy to see that each of these random variables has mean 0 and variance 1. Therefore their correlation coefficient $\rho(Ba, Ba')$ is equal to $\mathrm{E}(Ba \cdot Ba')$, the mean of their product. This mean can be calculated directly from Def. 6.3.11, but it is easier to make use of Model 6.3.14. For a given value of p, the product $Ba \cdot Ba'$ equals 1 with probability $p^2 + (1-p)^2$, and -1 with probability $2p(1-p)$. Hence the conditional expectation of $Ba \cdot Ba'$ is readily seen to be $(1-2p)^2$. Therefore

$$\rho(Ba, Ba') = \mathrm{E}(Ba \cdot Ba') = \int_0^1 (1-2p)^2 \mathrm{d}p = \frac{1}{3}.$$

In fact, Model 6.3.14 and Lemma 6.3.15 show that the model **C** is appropriate if we assume that all the voters are identical clones, with the same interests and identical [probabilistic] propensities, formalized by the common random variable P, which in each division produces the same probability p for all of them. Any *difference* between the actual votes of two voters on a given bill is attributable to random 'noise', which intervenes between their identical propensities and the actual respective acts of voting.[42]

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[41] Cf. Com. 3.1.3.

[42] Cf. fn. 38, p. 208. Readers familiar with quantum statistical mechanics will recall that the use of Bose–Einstein statistics for photons, mesons, deutrons and other particles with integral spin is commonly justified by the heuristic

Of course, the a priori information built into **C** is incomplete: the voters are assumed to be clones, but the nature of the bills to be decided, as well as the (shared) attitude of the voters to any future bill, remain shrouded in ignorance.

6.4 Other Proposed Indices

The Bz and S-S indices are by far the most important and most widely used indices of a priori voting power, but there have been several other attempts to define alternative indices. In this section we discuss the Deegan–Packel (D-P) and Johnston (Js) indices, which have received some attention in the literature, and subject their underlying assumptions to critical examination. We shall do so at some length, because in our view the cases of these two attempts hold general lessons concerning the conceptual foundations of the theory of a priori voting power. This analysis will be supplemented in Ch. 7 by examples illustrating the extremely counter-intuitive 'pathological' behaviour of these indices.[43]

Deegan and Packel proposed their index in [22]. Their presentation makes it abundantly clear that the D-P index is intended as a measure of a priori P-power rather than I-power. In [23], which covers roughly the same ground, they announce this in slogan form in the paper's very title: 'To the (minimal winning) victors go the (equally divided) spoils … '. The index is based directly on the following simple bargaining model.[44]

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argument that such particles behave as indistinguishable clones, two or more of which can be in the same micro-state (unlike particles with semi-integer spin, which are subject to Pauli's exclusion principle). In the present discussion, voters are the analogues of particles, and 'yes' and 'no' votes are analogous to micro-states.

[43] In order to avoid tedious repetition, we shall not discuss here another index — that proposed by Holler in [47]. Suffice it to say that this index suffers from conceptual and technical shortcomings similar to those of the two indices examined here, except for violation of the bicameral postulate.

[44] The present formulation is almost a verbatim quote from [22, p. 114].

6.4.1 Model It is assumed that

(1) Only *minimal* winning coalitions will emerge victorious.

(2) Each MWC has an equal probability of forming.

(3) Voters in a victorious MWC divide the 'spoils' equally.

The D-P index is then defined as a voter's expected share in the 'spoils', whose (constant) total value is 1. Let us state this formally.

6.4.2 Definition The *D-P index* is the function D-P that assigns to any SVG $\mathcal{W}$ and any voter a of $\mathcal{W}$ a value

$$\text{D-P}_a[\mathcal{W}] =_{\text{def}} \frac{1}{|\mathcal{W}^{\min}|} \sum_{X\,:\,a \in X \in \mathcal{W}^{\min}} \frac{1}{|X|},$$

where $\mathcal{W}^{\min}$ is the class of all MWCs of $\mathcal{W}$. We refer to $\text{D-P}_a[\mathcal{W}]$ as the *D-P index of a* [in $\mathcal{W}$].

6.4.3 Remarks (i) As usual, we shall often suppress mention of $\mathcal{W}$ where no ambiguity is likely.

(ii) According to Model 6.4.1, voter a receives non-zero payoff iff some MWC X containing a is formed (in the sense of Com. 6.1.3) and is therefore victorious. According to assumptions (1) and (2) of the model, the probability that such X will be formed is $1/|\mathcal{W}^{\min}|$; and according to assumption (3) a will, in this event, receive a payoff of $1/|X|$. Thus D-P_a is indeed a's mean (or expected) payoff. From this it follows at once that $\sum_{x \in N} \text{D-P}_x = 1$.

For a detailed illustration showing how the D-P index is calculated, see Ex. A.4.

6.4.4 Comment Deegan and Packel seek to justify their index as a valid measure of a priori P-power by claiming that the three assumptions of Model 6.4.1 'seemed reasonable (in a wide variety of modeling situations)'.[45] However, we shall argue that if office-seeking voting behaviour is posited — as it must be in the present

[45] [22, p. 114].

context — then the three assumptions of the model are severally dubious and jointly implausible.

It will be helpful to discuss the model with reference to a specific example. Consider the WVG

$$\mathcal{W} \cong [15; 8, 7, \underbrace{1, 1, \ldots, 1}_{7 \text{ times}}],$$

with nine voters: b (Mr Big, weight 8), c (Ms Cunning, weight 7), and seven dwarfs, $d_1, \ldots, d_7$. $\mathcal{W}$ has two MWCs: $\{b, c\}$ and $\{b, d_1, \ldots, d_7\}$.

In this example, Model 6.4.1 is sharply at odds with standard bargaining theory, which predicts that the grand coalition of all nine voters will be formed; b will receive the whole, or virtually the whole, of the purse; and the other voters will get nothing or next to nothing. However, in criticizing Model 6.4.1 we need not adopt the standpoint of the standard theory; instead, we shall consider the assumptions of the model on their own merit.

According to assumption (1) of Model 6.4.1, a coalition of the type $\{b, c, d_i\}$ will never be formed. The common-sense argument in support of this is that b and c together can win without the help of the dwarf d_i, so before concluding a pact they will either kick him out of $\{b, c, d_i\}$ or refuse to offer him any share in the purse, thus provoking him to leave, banging the door behind him. But, pushing this common-sense approach to its logical conclusion, a possible response is that while b has no reason to cut a dwarf in, c certainly has: she knows that, given half a chance, b (who is a blocker) may ditch her and form a winning coalition with the seven dwarfs. Note that this threat is real if the alternative minimal coalition has positive probability (it is not even necessary to assume that the two MWCs are *equally* probable). So according to *this* logic c is right to feel threatened. Fortunately, during the bargaining process she can forestall b's betrayal by inducing one of the dwarfs to join $\{b, c\}$ in exchange for part of *her* anticipated payoff (b should not object, because he loses nothing).

Of course, in this scenario c has no reason to invite *more* than one dwarf to join $\{b, c\}$: as we pointed out in Com. 2.2.2, if voting is motivated by office seeking then, *ceteris paribus*, a 'yes' voter should want to win with the help of few rather than many partners, because the spoils of office must be shared with them. But the *ceteris paribus* proviso is crucial; by dropping it, assumption (1) of Model 6.4.1 turns a reasonable conditional statement into an unreasonable absolute precept.

Assumption (2) would perhaps be acceptable as a 'null hypothesis', which is adopted provisionally, perhaps for lack of any stronger alternative or simply because one has to take *some* assumption as a benchmark for testing alternative hypotheses. But here it is offered as a *postulate*, which ought to be compelling. However, far from being compelling, it is rather unpersuasive: the two minimal coalitions of our WVG are very unlike each other, so there seems to be no prima-facie reason to suppose that they are equally likely to form.

Similarly, assumption (3) seems quite arbitrary. The members of each of the two MWCs are not equally powerful (according to *any* reasonable measure of power); so why should the powerful b agree to share equally with his weaker partners?

Whatever the force of the arguments for or against each of the three assumptions separately, the conjunction of all three is totally incongruous. In our nine-voter example, the blocker b gets a payoff of $\frac{1}{2}$ if he joins forces with c, but only $\frac{1}{8}$ if he teams up with the seven dwarfs. Why should he *ever* agree to the latter? Yet the model assigns probability $\frac{1}{2}$ to each of these options. And what if instead of seven dwarfs there were seventy, weighing $\frac{1}{10}$ each? Would Mr Big be as likely to form a coalition giving him only $\frac{1}{71}$ of the purse as a coalition that gives him half of the purse?

Of course, debunking Model 6.4.1 does not automatically invalidate the D-P index; it leaves open the possibility that the latter may be justified in other ways. But in Ch. 7 we shall show that this index is discredited by its behaviour.

Meantime let us turn to the Js index. After stating its definition, we shall comment on the idea behind it and the circumstances that led to its conception.

6.4.5 Definition If S is a coalition of an SVG $\mathcal{W}$, we shall refer to the number of voters who are $\mathcal{W}$-critical in S as the *critical number of* S [in $\mathcal{W}$] and denote it by '$\mathrm{Cr}(S;\mathcal{W})$'. We say that S is *vulnerable* [in $\mathcal{W}$] if $\mathrm{Cr}(S;\mathcal{W}) > 0$.

The *Js score* is the function JS that assigns to any SVG $\mathcal{W}$ and any voter a of $\mathcal{W}$ a value

$$\mathrm{JS}_a[\mathcal{W}] =_{\mathrm{def}} \sum_{X\,:\,a \text{ critical in } X} \frac{1}{\mathrm{Cr}(X;\mathcal{W})}.$$

The *Js index* is the function JI defined by

$$\mathrm{JI}_a[\mathcal{W}] =_{\mathrm{def}} \frac{\mathrm{JS}_a[\mathcal{W}]}{\sum_{x\in N} \mathrm{JS}_x[\mathcal{W}]},$$

where N is the assembly of $\mathcal{W}$.

6.4.6 Remarks (i) As usual, we shall often omit reference to $\mathcal{W}$ where there is no risk of ambiguity.

(ii) Comparison of the definition of the Js index with that of the Bz index (Def. 3.2.2) shows a clear formal similarity between the two. This is no accident: the Js index was invented deliberately as a 'modification' of the Bz index.[46] The difference is in the definition of the score. In the Bz case voter a's score, η_a, is notched up a whole unit for each coalition X in which a is critical; in the Js case the score JS_a is incremented by $1/\mathrm{Cr}(X)$. In both cases the score is normalized to produce the index, so that the index values of all voters of an SVG add up to 1.

(iii) It is easy to see that $\sum_{x\in N} \mathrm{JS}_x[\mathcal{W}]$, the denominator in the *definiens* of $\mathrm{JI}_a[\mathcal{W}]$, is equal to the number of vulnerable coalitions in $\mathcal{W}$. This reveals a formal analogy between the D-P and Js indices. We shall say a bit more on this at the end of Com. 6.4.7. See also Ex. A.4, where we illustrate in detail how these two indices are calculated.

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[46] See Johnston [55, p. 909].

6.4.7 Comment The Js index was invented by R J Johnston in response to a critique directed by Laver [62] against the former's approach to the measurement of voting power. Johnston had published a series of papers about the distribution of voting power in decision-making bodies of the EC,[47] in which he used the Bz index β as a measure of relative a priori ('latent') voting power. He did not refer to the Bz index by name, but introduced it with the following words:

> The crucial axiom used here, which is taken from a variety of game theory applications to the study of political power, is that power in a committee or parliament is the ability of a member or group of members to destroy an otherwise winning coalition of members ... by withdrawing support from it. The more potential coalitions that a member (or group) could destroy in this way, the greater his (its) bargaining power in terms of the support obtained for his policies in return for continued membership of the coalition. If the threat of unilateral withdrawal is irrelevant to a coalition's viability, then it brings no bargaining power.[48]

This is followed by detailed explanations and illustrations, leaving no room for doubt that the (unnamed) index being used is in fact β. As we explained on p. 160 in connection with our own use of β for similar purposes in Ch. 5, this index is appropriate for the task at hand, which is concerned primarily with I-power.

Two statements included in the passage just quoted call for special comment. First, the reference to game theory is somewhat misleading, an echo of a common misconception.[49]

Second, the middle sentence of the quoted passage, claiming in effect that a member's Bz index is proportional to his or her bargaining power, is perhaps open to misinterpretation, especially in conjunction with the mention of game theory. It all depends on what is meant by 'bargaining power' and what kind of bargaining

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[47] Only the first of these, [54], need concern us here; for a full list see [55].

[48] [54, p. 571].

[49] See fn. 34, p. 206 and text to that footnote.

is envisaged. If bargaining is understood in the sense of cooperative game theory, as leading to a the formation of a coalition and division of a fixed purse among its members — then there is no warrant for the claim that the Bz index is proportional to bargaining power. Yet there is *another* sense of 'bargaining' in which this claim is perfectly correct: as we saw in Com. 3.2.15, the supremal price that an outsider would be prepared to pay for voter a's vote is proportional to β_a. Thus a's bargaining power *in bargaining with a vote buyer* can indeed be said to be proportional to β_a.

In fact, the sense in which Johnston understood 'bargaining' was neither of the two we have just mentioned. He makes it quite clear that the kind of transaction he does have in mind is a respectable cousin of vote buying: *log-rolling*, in which two voters barter votes with each other.[50] So his claim regarding the relation between β and bargaining power is, as it stands, unsupported. However, it may perhaps be argued that the bargaining power of a voter in relation to log-rolling is not unlike his or her bargaining power in relation to vote selling.

Johnston's choice of β as power index aroused the fierce criticism of Laver [62].[51] The main brunt of his attack was directed against the passage we have quoted from [54], and in particular against the claim regarding bargaining power. However, detailed reading of Laver's arguments makes it pretty clear that what he meant by voting power is (in our terminology) *P-power*; and he understood 'bargaining' and 'bargaining power' in the sense of cooperative game theory, as aimed at the formation of a winning coalition and division of a fixed prize. This was at cross purposes with Johnston's intention, which was (quite appropriately, in our view) concerned with I-power; and it misinterpreted the latter's reference to bargaining.

Compounding the misunderstanding, Laver produced a specific argument to show the superiority of the S-S index to the Bz. His

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[50] He states this explicitly at [54, p. 570].

[51] The latter was evidently unacquainted with this index, and thought that Johnston had made it up himself; so throughout [62] β is mistakenly referred to as 'Johnston's index'.

advocacy of the former is based on what we have called the queue bargaining model in its more sophisticated version, which he seems to take at face value and regard as realistic. As our discussion in Com. 6.3.9 showed, in that model a coalition S can form only if it has a critical member (in other words, is a vulnerable coalition); and subject to the condition that S is formed, all critical members of S have equal conditional expected payoff. This can be stated as follows: if a is critical in S, then (in this model) the contribution of S to the S-S score of a is $1/\mathrm{Cr}(S)$.[52] On the other hand, the contribution of S to the Bz score of a is a full unit. Laver claims that this militates against the Bz index and in favour of the S-S index:

> If one assumes that all winning coalitions are of equal value (both Johnston and Shapley and Shubik do this, either explicitly or implicitly) then, presumably, if two parties can each destroy a particular winning coalition, they have equal power with respect to that coalition, and share the profits. Similarly, if three parties can each destroy a winning coalition, they also divide the spoils equally, and get less than in the former case. Despite this, Johnston's index registers one point every time a party can destroy a coalition, regardless of how many others can do the same thing.[53]

This argument is fallacious, not only because it takes the 'sophisticated' queue bargaining model at face value as underlying the S-S index, but mainly because it evidently treats the Bz index (wrongly attributed to Johnston) as an index of P-power. It tacitly assumes that the unit of Bz score that a voter derives from critical membership in a coalition is the voter's 'share' in the 'profits' or 'spoils' that the coalition gains when victorious. But of course the Bz score does not have this meaning at all; it is simply a numerical score proportional to the probability that the voter will be in a position to exert *influence*.

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[52]This contribution must however be weighted: that is, multiplied by the probability $p(k; m, n)$ that S will be formed, which we calculated in Com. 6.3.9. Laver does not discuss, let alone justify, this weighting.

[53][62, p. 902]. The references to Johnston are mistaken and should have been to Banzhaf; cf. fn. 51 above.

In his reply, Johnston [55] rebuts Laver's critique by pointing out, in effect, that they were addressing two different notions of voting power, and had two different kinds of bargaining in mind. But the rebuttal is rather vague and fails to come to grips with the basic issue. Apparently, Johnston himself was diffident about his own position, for he bowed to Laver's critique on one crucial point: he proposed 'altering the [Bz] index slightly' ([55, p. 909]), by re-defining the score, so that instead of the Bz score η we now have the Js score JS. This is how the Js index came into being.

Of course, the alteration was not slight at all. A mathematical definition or formula is not a cooking recipe, where alteration, such as adding seasoning according to taste, can often (not always!) be made with impunity. The Bz index β is not an ad hoc expedient; as we pointed out in Com. 3.2.5, its meaning and justification as an index of relative I-power derive entirely from its being obtained by normalization from the Bz measure β' of absolute I-power. The latter has a clear probabilistic meaning as a voter's a priori ability to affect the outcome of a division, as shown by Thm. 3.2.4.

The Js index was produced by doctoring the Bz index of I-power, grafting onto it alien notions of P-power. The result is neither fish nor fowl but a chimera. The Js index cannot be justified as an index of relative I-power, because, unlike the Bz index, it is not derived by normalization from any recognizable measure of absolute I-power.

Formally speaking, it may be regarded as an index of P-power, derived from a bargaining model based on three assumptions:

(1′) Only *vulnerable* coalitions will emerge victorious.

(2′) Each vulnerable coalition has an equal probability of forming.

(3′) The critical voters in a victorious vulnerable coalition divide the 'spoils' equally.

This is clearly a variant of Model 6.4.1; and like the latter it can be shown, by similar arguments, to be untenable. To this end we can use the same nine-voter SVG $\mathcal{W}$ as in Com. 6.4.4. The main difference is that the present model assigns positive probability to the formation of a coalition of the form $\{b, c, d_i\}$. But as the dwarf d_i is not critical in this coalition, assumption (3′) implies that he cannot receive any payoff for his participation in it. Since the seven dwarfs are known to be every bit as avaricious ('rational') as any character in the fairy tales of game theory, there does not seem to be any reason why d_i should agree to participate gratis in forming a coalition.

As in the case of the D-P index, these prima-facie reservations against the Js index will be reinforced in Ch. 7 by evidence of its unacceptable behaviour.

7. Paradoxes and Postulates[1]

7.1 Preliminaries

Voting power, however you measure it, seems to be a strange beast, often displaying behaviour that ranges from the slightly surprising to the bizarre. This chapter is devoted to the description, explanation and classification of the so-called *paradoxes of voting power*.

By *paradox* we mean a true proposition that appears to be absurd.[2] By extension, a real phenomenon that seems to be contrary to common sense is also referred to as a *paradox*. Paradoxicality is a matter of degree: a true proposition may be slightly surprising or barely believable. It is also largely subjective: an experienced well-informed prudent observer may be unimpressed by a phenomenon that astonishes a naïve novice.

More often than not, authors on voting power are content to point at some apparently strange piece of behaviour of, say, the S-S and Bz indices, and declare it to be a voting-power paradox. But this begs the question as to whether the alleged paradox is inherent in the very notion of voting power or merely an artefact of

[1]This chapter is largely based on [29], [36] and [33].

[2]In his book *Paradoxes* [91, p. 1], Sainsbury defines *paradox* somewhat differently, as 'an apparently unacceptable conclusion derived by apparently acceptable reasoning from apparently acceptable premises. Appearances have to deceive, since the acceptable cannot lead by acceptable steps to the unacceptable. So generally we have a choice: either the conclusion is not really unacceptable, or else the starting point, or the reasoning, has some non-obvious flaw.' His definition is geared to the subject matter of [91]: the so-called logical and semantic paradoxes or antinomies; but it is not suited to our present purpose. Note however that both definitions agree that paradoxicality lies in appearances, and may therefore be dissolved by analysis.

some particular way of measuring that power. In order to facilitate an analysis that does make such distinctions, let us introduce the following general definition of *measure* and *index* of voting power, in which we impose very minimal *adequacy postulates* that any reasonable measure (or index) of voting power must satisfy.

7.1.1 Definition By a *measure* [*of voting power*] we mean a mapping ξ that assigns to any SVG $\mathcal{W}$ and any voter a of $\mathcal{W}$ a non-negative real value $\xi_a[\mathcal{W}]$, satisfying the following three conditions:

(1) *Iso-invariance*: if there is an isomorphism of SVGs from $\mathcal{W}$ to $\mathcal{W}'$ that maps a voter a to a', then $\xi_a[\mathcal{W}] = \xi_{a'}[\mathcal{W}']$;

(2) *Ignoring dummies*: If $\mathcal{W}$ and $\mathcal{W}'$ are SVGs that have exactly the same MWCs, then $\xi_a[\mathcal{W}] = \xi_a[\mathcal{W}']$ for any voter a common to both;

(3) *Vanishing just for dummies*: $\xi_a[\mathcal{W}] = 0$ iff a is a dummy in $\mathcal{W}$.

We say that ξ is an *index* [*of voting power*] if, in addition, it satisfies the condition

(4) *Normalization*: $\sum_{x \in N} \xi_x[\mathcal{W}] = 1$ for any SVG $\mathcal{W}$.

7.1.2 Remark For the justification of condition (1) in this definition, see Rem. 2.1.8.

Condition (2) means that the value ξ_a for any voter of $\mathcal{W}$ is unchanged if $\mathcal{W}$ is extended to $\mathcal{W}|\&_\emptyset$ by adding a new dummy voter $\&_\emptyset$ (see Rem. 2.3.24(iii)). Equivalently, if d is a dummy in an SVG $\mathcal{W}$ and $\mathcal{W}'$ is the subgame obtained from $\mathcal{W}$ by removing d — that is, $\mathcal{W}' = \{X \in \mathcal{W} : d \notin X\}$ — then the value of ξ_a for any remaining voter is unchanged. For a justification of this condition, see Rem. 2.3.5(i).

Condition (3) is obvious: the powerless voters of an SVG are just its dummies.

Condition (4) says that we reserve the term 'index' for a normalized measure. Note, however, that other authors use this term also for non-normalized measures.

7.1.3 Comments (i) The conditions of Def. 7.1.1 are merely necessary but by no means sufficient for making any measure (in the sense of this definition) a valid or acceptable yardstick of voting power. The point of considering such a weak notion of measure is that it helps to classify the paradoxes into three broad types.

First, any paradox that is displayed by all measures, or perhaps by all measures satisfying some additional weak conditions, must be due to very general and fairly superficial reasons. To be sure, even such a relatively superficial paradox should not be dismissed out of hand as trivial: although when properly explained it sheds its paradoxical appearance, it may still tell us something useful and interesting about the quantitative aspect of voting power.

Second, at the other extreme is the type of alleged paradox displayed by some measures, but which in effect constitutes unacceptable behaviour for any reasonable measure of voting power. An alleged paradox of this type is not a true paradox at all; it reveals little about the nature of voting power, but is rather a pathology of those measures afflicted by it, which disqualifies them from serving as valid yardsticks of voting power.

Third, in between these two extremes there is the type of paradox that is neither superficial nor pathological, but is capable of a non-trivial explanation that reveals something about both the nature of voting power as such and the measure concerned. For example, a paradox of this type may be a piece of behaviour that is tolerable for an index of I-power but not of P-power (or vice versa). This tells us something important about voting power, by highlighting the distinction between its different aspects. At the same time we gain valuable information about particular indices, which are shown to be unsuitable for measuring one aspect of voting power, but may still be eligible for measuring another aspect.

Of course, the borderline between the second and third type of phenomena cannot be drawn with objective precision. Opinions may well differ as to whether a given piece of behaviour is a disqualifying pathology or a tolerable, albeit surprising, peculiarity.

(ii) Closely related to the issue of paradoxes is that of postulates that ought to be imposed on measures of voting power. The negation of a mildly paradoxical proposition is a proposition that is at least apparently plausible. The negation of an (alleged) paradox of the second type (a disqualifying pathology) is an intuitively compelling proposition. Thus, an unacceptable paradox is a failure to satisfy a wholly justifiable postulate. Conversely, the purpose of imposing such a postulate is to rule out a particular kind of pathological behaviour.

The rest of this chapter is organized as follows. In §§ 7.2, 7.3, 7.4, 7.5 and 7.7 we analyse paradoxes of the first, superficial type. In § 7.6 we deal with a paradox of the second type, which in our view is merely an anomaly of some inappropriate measures. In §§ 7.8 and 7.9 we discuss some paradoxes of the third type. Some concluding comments are made in § 7.10.

7.2 The Paradox of Large Size

Suppose a coalition S in an SVG $\mathcal{W}$ amalgamates into a bloc $\&_S$, thus creating a new SVG, $\mathcal{W}|\&_S$ (see Def. 2.3.23). How should the power of the bloc $\&_S$ in $\mathcal{W}|\&_S$ compare with the powers of the individual members of S in $\mathcal{W}$? At first sight it may perhaps seem reasonable to assume that a measure ξ ought to satisfy the following

7.2.1 Condition (superadditivity)

$$\xi_{\&_S}[\mathcal{W}|\&_S] \geq \sum_{x \in S} \xi_x[\mathcal{W}],$$

for any coalition S in any SVG $\mathcal{W}$. In particular,

$$\xi_{a\&b}[\mathcal{W}|a\&b] \geq \xi_a[\mathcal{W}] + \xi_b[\mathcal{W}],$$

for any two distinct voters a and b in $\mathcal{W}$.

7.2.2 Remarks (i) This simply means that whenever voters form a bloc, ξ assigns to the bloc in the new SVG (created by the formation of the bloc) a value that is at least as great as the sum of the values assigned to these voters in the original SVG.

(ii) If $S = \emptyset$, the bloc $\&_\emptyset$ is a dummy in $\mathcal{W}|\&_\emptyset$ (see Rem. 2.3.24(iii)); so in this case the superadditivity condition reduces to the trivially true statement that ξ assigns a non-negative value to this dummy.

If $a \in S$ but all the other members of S (if any) are dummies, then $\mathcal{W}|\&_S$ can be obtained from $\mathcal{W}$ by first removing these dummies and then renaming a as '$\&_S$'; so in this case the superadditivity condition holds automatically because the measure ξ ignores dummies and is iso-invariant.

Thus the superadditivity condition is of interest only in cases where S has at least two non-dummy members.

What possible reason is there for expecting superadditivity to hold in all cases? As Brams [11, p. 178] puts it, this is suggested by 'the conventional wisdom that the whole is greater than — or at least equal to — the sum of its parts'. If ξ is an index, measuring relative rather than absolute voting power, there is another, apparently more robust, argument in favour of superadditivity. It seems natural to expect that if two or more voters form a bloc, this should not increase the relative power of any rival voter; in other words, for any voter y of $\mathcal{W}$ who is not in S we should have $\xi_y[\mathcal{W}|\&_S] \leq \xi_y[\mathcal{W}]$. Then Condition 7.2.1 would follow at once from the fact that ξ is normalized.

However, it is easy to see that although superadditivity must hold in some cases for *any* index, there are also some cases where it must fail. Shapley [96, pp. 57–58] illustrates these facts by the following very simple example, in which ξ can be any index.

7.2.3 Example Consider $\mathcal{B}_3$, the unanimity SVG with three voters, $1, 2, 3$. Since $\mathcal{B}_3$ is symmetric and ξ, being an index, is iso-invariant and normalized, it follows that $\xi_i[\mathcal{B}_3] = \frac{1}{3}$ for $i = 1, 2, 3$.

Now, $\mathcal{B}_3|1\&2$ is a unanimity SVG with two voters, $1\&2$ and 3. Hence $\xi_{1\&2}[\mathcal{B}_3|1\&2] = \xi_3[\mathcal{B}_3|1\&2] = \frac{1}{2}$. This violates superadditivity, according to which $\xi_{1\&2}[\mathcal{B}_3|1\&2]$ ought to be at least $\frac{2}{3}$.

For contrast, consider $\mathcal{M}_3$, the majority SVG with the same three voters. Again, by symmetry and normalization we have

$\xi_i[\mathcal{M}_3] = \frac{1}{3}$ for $i = 1, 2, 3$. But in $\mathcal{M}_3|1\&2$ the bloc 1&2 is a dictator and 3 is a dummy; hence $\xi_{1\&2}[\mathcal{M}_3|1\&2] = 1 > \frac{2}{3}$, so here superadditivity does hold.

Failure of superadditivity was not regarded by Shapley as paradoxical, but later it came to be known as the *paradox of large size.* Brams [11, p. 176] mentions in a footnote Shapley's example of such failure ($\mathcal{B}_3$, Ex. 7.2.3), but half-dismisses it as 'trivial'. He presents more elaborate examples of SVGs in which both the S-S and Bz indices violate superadditivity. However, his treatment of the paradox tends to overplay its 'strangeness' by creating the impression that one needs to be committed to these two particular indices, or at least know about them, in order to realize that superadditivity is untenable as a general rule.

7.2.4 Comments (i) According to the classification we have proposed in Com. 7.1.3(i), this paradox — if indeed it is a paradox at all — is of the most superficial type. It seems to us that the arguments in favour of superadditivity are quite weak in the first place, and can be rebutted using intuitive common-sense reasoning, without any need for detailed technicalities or complex calculations.

The 'conventional wisdom' that *the whole is greater than — or at least equal to — the sum of its parts* is no argument at all but a mere saying. It has no greater persuasive power than any other trite pearl of wisdom, such as 'small is beautiful' or 'barking dogs do not bite'.

There are indeed very good common-sense arguments suggesting that the power of a bloc ought to be at least as great as the power of *the most powerful* of its component parts — we shall return to this important point in § 7.8 — but there is no convincing reason for supposing that in all cases it should be at least as great as their *sum.* As we shall see in a moment, it is only to be expected that in some circumstances the opposite will be true.

At this point of our discussion we must consider separately the two concepts of voting power distinguished in earlier chapters.

(ii) We begin by looking at superadditivity from the perspective of P-power. As explained in § 6.1, the notion of P-power assumes office-seeking voting behaviour: a process of bargaining leads to the formation of a winning coalition, whose members then divide among themselves the fixed prize of victory. The creation of a bloc $\&_S$ means that the members of S forgo their right to act as separate individuals; instead, the bloc bargains and votes collectively as one person. This of course changes the original SVG $\mathcal{W}$, which is now replaced by $\mathcal{W}|\&_S$.

As pointed out in Com. 6.1.5, P-power is essentially relative; so we shall now assume that ξ is some reasonable *index* of P-power. Then $\xi_a[\mathcal{W}]$ is interpreted as the expected share of voter a in the fixed prize of victory, whose total is taken to be 1 unit.

If we rule out coercion and irrational behaviour, then cases in which superadditivity fails, that is $\xi_{\&_S}[\mathcal{W}|\&_S] < \sum_{x \in S} \xi_x[\mathcal{W}]$, are precisely those in which the bloc $\&_S$ will not be created, because in these cases the expected payoff of $\&_S$ is less than the total expected payoffs of the members of S acting as separate individuals. Thus at least one of these voters must lose by participating in the bloc.

Under what circumstances may ξ be expected to violate superadditivity?

In [96, pp. 57–58], Shapley proposes the following 'general rule of thumb'. Extrapolating from the two contrasting cases in his example ($\mathcal{B}_3$ and $\mathcal{M}_3$, Ex. 7.2.3), he suggests that in SVGs in which 'there is a good deal of blocking, and relatively few winning coalitions, it pays not to contract players' into a bloc, because in such SVGs superadditivity tends to be violated; conversely, in SVGs that 'have few blocking coalitions, it pays to combine players' into a bloc because superadditivity tends to hold.

We find Shapley's rule of thumb untenable. Whether superadditivity holds or fails to hold for a given bloc $\&_S$ must surely depend not only on the global structure of the SVG concerned but also on the nature of the coalition S that turns itself into this bloc. Besides, if an SVG $\mathcal{W}$ has relatively few winning coalitions and many blocking ones, then in its dual, $\mathcal{W}^*$, the position is reversed. Yet, as we saw in Rem. 6.2.4(iii), the S-S index — the index proposed by Shapley himself — is self-dual; hence it satisfies superadditivity with respect to S in $\mathcal{W}$ iff it does so with respect to S in $\mathcal{W}^*$.

In proposing our own analysis, let us, for the sake of simplicity, deal with a two-voter bloc $a\&b$.

Consider a winning coalition S of $\mathcal{W}$, in which both a and b are critical. Presumably, a and b derive part of their expected payoffs — that is, part of their P-powers — in $\mathcal{W}$ from their membership in S and their ability to threaten to defect from it, leaving behind them a losing coalition. If S contains other critical voters aside from a and b, then we may assume that those too derive part of their P-powers in the same way. If now a and b form a bloc $a\&b$, then S is replaced by a winning coalition $S' = S - \{a, b\} \cup \{a\&b\}$ of $\mathcal{W}|a\&b$. The critical members of S' are the same as those of S, except that instead of the two critical members a and b there is now a single critical member $a\&b$. It seems reasonable to expect that $a\&b$ should be able to extract from the coalition S' at least as much payoff as either a or b were each able to extract from S — but not quite as much as these two payoffs put together. After all, $a\&b$ cannot threaten to destroy S' twice over. (You may be coerced at gunpoint to pay ransom to a gunman; but it is unlikely that he could extract twice that ransom by pointing at you a double-barrelled gun.) Therefore, if sufficiently large parts of the P-powers of a and b in $\mathcal{W}$ derive from their membership in winning coalitions of this kind, in which *both* of them are critical, we cannot reasonably expect superadditivity to hold in this case.

A similar consideration shows that it is not reasonable to expect that in all cases the inequality $\xi_y[\mathcal{W}|a\&b] \leq \xi_y[\mathcal{W}]$ should hold for voters $y \in N$ other than a and b. Certainly, from the point of view of a third voter y the bloc $a\&b$ is at least as formidable a rival as either a or b separately. But it need not always be as formidable as both of them put together. In $\mathcal{W}$, if y wishes to form a winning coalition in which both a and b are critical, then she has to negotiate with each of them separately, and presumably offer each a separate reward. But in $\mathcal{W}|a\&b$, in order to form the corresponding winning coalition, y needs to negotiate with a single voter $a\&b$, which may be easier and less costly than making separate deals with a and b.

(iii) Let us now take up the perspective of I-power. Here *absolute* power is not only meaningful but primary; whereas relative I-power is a derived concept. So now we shall first assume that

ξ is a measure of absolute I-power. Under what circumstances is it reasonable to expect ξ to violate superadditivity? As we shall see in a moment, the case of a two-voter bloc is quite atypical for the behaviour of absolute I-power. So we shall first deal with the simplest typical case, that of a three-voter bloc $a\&b\&c$.

Consider the position in $\mathcal{W}$ of one of the three voters, say a. The situations in which a is able to exert influence in $\mathcal{W}$ may be of two kinds: first, there may be divisions in which the other two voters, b and c are opposed to each other and a is able to decide the outcome by siding with one or the other. Second, there may be divisions in which b and c are on the same side, and a is able to decide the outcome by siding with both of them or opposing them. The same classification can of course be made from the viewpoint of b and c. Now, if a large proportion of the situations in which each of the three voters is able to exert influence in $\mathcal{W}$ are of the first kind (that in which the other two are opposed to each other), then it is reasonable to expect that the I-power of the bloc $a\&b\&c$ should be *less* than the sum of the I-powers of the three separate voters, *because in* $\mathcal{W}|a\&b\&c$ *these situations no longer exist*: they have disappeared without trace. So in such cases we may expect superadditivity to be violated. In contrast, if a large proportion of the situations in which the three voters exert influence in $\mathcal{W}$ are of the second kind, in which all three can pull together, then it is reasonable to expect that I-power of the bloc $a\&b\&c$ should be *greater* than the sum of the I-powers of the three separate voters.

The same reasoning clearly applies, *mutatis mutandis*, also to blocs of more than three voters. But the case of a two-voter bloc $a\&b$ is different. In any situation where a can exert influence, he can do so by choosing to pull together with b or in the opposite direction. The same holds of course with the roles of a and b reversed. So we should not be surprised to find that the creation of a two-voter bloc is neutral with respect to I-power: the I-power of $a\&b$ being equal to the sum of the I-powers of a and b.

The above analysis is corroborated by, and throws light on, the behaviour of the Bz power β', the only serious candidate for the role of valid measure of absolute I-power.

For example, in $\mathcal{M}_7$, the canonical majority SVG with seven voters, the Bz power of each voter is $\frac{5}{16}$. If three of the voters form a bloc, then in the resulting SVG the Bz power of the bloc is $\frac{14}{16}$. So here superadditivity is violated.

Contrast this with $\mathcal{B}_7$, the canonical unanimity SVG with seven voters. Here the Bz power of each voter is $\frac{1}{64}$. And if three voters form a bloc, the Bz power of the bloc in the resulting SVG is $\frac{1}{16}$, which is greater than $\frac{3}{64}$. So here the whole is indeed greater than its parts.

Further, according to Thm. 3.2.18 the Bz power of a two-voter bloc is in fact always equal to the sum of the original Bz powers of its two components.

(iv) When it comes to *relative* I-power, the analysis of the problem of superadditivity becomes quite complicated because when a bloc $\&_S$ is created a voter who is not in S may lose or gain absolute I-power. For example, if a and b form a bloc and c is a third voter whose influence in the original SVG $\mathcal{W}$ is exerted for the most part in divisions in which a and b oppose each other, then the formation of $a\&b$ may be expected to cause c to lose absolute I-power, because now a and b are hitched together. If there are relatively many voters in a position similar to c's, then the relative I-power of $a\&b$ will be greater than the sum of the original relative I-powers of a and b, so that superadditivy will hold in this case. The opposite may be expected to happen if in $\mathcal{W}$ relatively many voters exert their influences in situations in which a and b are on the same side.

For this reason we should not be surprised to see cases in which superadditivity fails for absolute I-power while holding for relative I-power, or vice versa.

This is borne out by the behaviour of the Bz measure β' and the Bz index β. We have seen a moment ago that in $\mathcal{M}_7$ Bz power violates superadditivity when a three-voter bloc is formed. But the Bz index of each voter in $\mathcal{M}_7$ is of course $\frac{1}{7}$, whereas the Bz index of the three-voter bloc 1&2&3 in $\mathcal{M}_7|1\&2\&3$ is $\frac{7}{11}$, which is considerably more than $\frac{3}{7}$; so here the Bz index behaves superadditively. On the other hand, we saw that in $\mathcal{B}_7$ Bz power satisfies superadditivity when a three-voter bloc is formed. But in

$\mathcal{B}_7$ the Bz index of each voter is again $\frac{1}{7}$, whereas the Bz index of the three-voter bloc 1&2&3 in $\mathcal{B}_7|1\&2\&3$ is $\frac{1}{5}$, much less than $\frac{3}{7}$; so here the Bz index violates superadditivity.

(v) The foregoing analysis of the behaviour of β with respect to superadditivity depends on the validity of β as an index of a priori I-power. This validity is securely grounded on the derivation of β as the normalized form of β' and the latter's probabilistic interpretation (see Com. 3.2.5). The same analysis does not necessarily apply to other indices. It should therefore come as no surprise that in some cases β and ϕ behave quite differently with respect to superadditivity. A simple example of this kind is $\mathcal{M}_{8,6} = [6; 1, 1, 1, 1, 1, 1, 1, 1]$. Here we have $\beta_i = \phi_i = \frac{1}{8}$ for $i = 1, \ldots, 8$ by symmetry. Now, if two voters, say 1 and 2, form a bloc, the resulting SVG is $\mathcal{W} \cong [6; 2, 1, 1, 1, 1, 1, 1]$. Here $\beta_{1\&2} = \frac{7}{29} < \frac{2}{8}$, whereas $\phi_{1\&2} = \frac{2}{7} > \frac{2}{8}$. Thus in this case superadditivity is violated by β but satisfied by ϕ. On the other hand, as the reader can verify, the opposite happens if the last three voters in the WVG $[14; 6, 5, 4, 1, 1, 1]$ form a bloc.

Examples of this sort show that whatever ϕ measures, it cannot be relative a priori I-power: two voters cannot both lose and gain the same kind of joint power by forming a bloc. By the same token, on the hypothesis that ϕ is a valid measure of P-power — and as we argued in Com. 6.3.10 this is no more than a hypothesis — β cannot also be a measure of P-power.

To sum up: we believe that a detailed analysis of the so-called paradox of large size dispels its apparent paradoxicality, while throwing some useful light on voting power.

7.3 The Paradox of Redistribution

Schotter [92] — drawing on Fischer and Schotter [38] and Dreyer and Schotter [25] — presents and discusses cases of WVGs in which 'it is possible to decrease a voter's voting weight within a voting body and [at the same time] increase his power' ([92, p. 324]). This phenomenon has been regarded as paradoxical; indeed, Fischer and Schotter [38] termed it the *paradox of redistribution*.

To define the phenomenon formally, consider a pair $\langle \mathcal{U}, \mathcal{W} \rangle$ of WVGs — which for simplicity we take to be canonical — having the same assembly, the same quota q and the same total weight:

$$\begin{aligned} &\mathcal{U} = [q; u_1, \ldots, u_n], \quad \mathcal{W} = [q; w_1, \ldots, w_n], \\ &\text{where} \quad \sum_{i=1}^{n} u_i = \sum_{i=1}^{n} w_i. \end{aligned} \tag{7.3.1}$$

7.3.2 Definition In the configuration (7.3.1), voter i is said to be a *donor* if $u_i > w_i$ and a *recipient* if $u_i < w_i$.

A measure ξ is said to display the *redistribution paradox* in the configuration (7.3.1), if some i is a donor but $\xi_i[\mathcal{U}] < \xi_i[\mathcal{W}]$; or if some j is a recipient but $\xi_j[\mathcal{U}] > \xi_j[\mathcal{W}]$.

7.3.3 Remark Intuitively, the idea is that $\mathcal{U}$ represents the initial distribution of weights and $\mathcal{W}$ arises from it by redistribution, the donor(s) 'donating' some weight to the recipient(s); so each donor loses weight and each recipient gains some, while the total weight is unchanged. Thus, the paradox of redistribution is the phenomenon of a donor gaining power or a recipient losing power.

Schotter [92] considers only the S-S and Bz indices in connection with the paradox of redistribution. This is somewhat too restrictive and may foster the false impression that the paradox is a peculiarity of these two indices. In fact, as the following example shows, there are configurations of the form (7.3.1) in which *any* index must display the paradox.

7.3.4 Example Let $\mathcal{U} = [8; 3, 3, 3]$ and $\mathcal{W} = [8; 2, 1, 6]$ and let ξ be any index. Then, by symmetry and normalization, $\xi_i[\mathcal{U}] = \frac{1}{3}$ for $i = 1, 2, 3$.

In $\mathcal{W}$ voters 1 and 3 are mutually symmetric while voter 2 is a dummy. So from Def. 7.1.1 it follows directly that $\xi_1[\mathcal{W}] = \xi_3[\mathcal{W}] = \frac{1}{2}$ and $\xi_2[\mathcal{W}] = 0$.

Here voter 1 has donated one unit of weight but (paradoxically?) increased his relative power (irrespective of the index used!) from $\frac{1}{3}$ to $\frac{1}{2}$.

We cannot show that every *measure* must display the redistribution paradox in the present case: conditions (1)–(3) of Def. 7.1.1, without condition (4), are insufficient for this. But there are measures that do so. In particular, for the Bz measure β' we have $\beta'_1[\mathcal{U}] = \frac{1}{4}$ whereas $\beta'_1[\mathcal{W}] = \frac{1}{2}$; so voter 1, a donor, gains Bz power.

7.3.5 Comment Closer examination soon dispels the puzzling impression created by Ex. 7.3.4 or the examples presented in [92].

First, an important observation that is one of the keys to the resolution of this and several other paradoxes: it is only to be expected that any transaction, such as transfer of weight, between two voters may not only affect the powers of these two, but have a global effect involving other voters as well.

Thus, if voter i donates weight to voter j, it is quite possible — and intuitively plausible — that this may affect the power of a third voter k, who is a mere bystander in this transaction. In this kind of game *no man is an island*, as John Donne put it.

Now, note that in our Ex. 7.3.4 as well as in the examples of Schotter [92] the donor who gains power is not the *sole* donor. It would indeed be paradoxical if a donor gained power *purely as a result of his own donation*: we do not really expect that in matters of power *it is better to give than to receive*. We surely ought to expect that donating weight may, if anything, cause a reduction in the donor's power. But if our donor is not the only one, then we need not be astonished if any loss of power due to his own donation may be more than offset by his gain (as a bystander) due to the donation of another donor.

With this in mind, let us look more closely at Ex. 7.3.4. This time, instead of going directly from $\mathcal{U}$ to $\mathcal{W}$, let us make the transition in two steps. First, let us go from $\mathcal{U} = [8; 3, 3, 3]$ to $\mathcal{V} = [8; 2, 3, 4]$. Here voter 1 has made his donation of one unit of weight to voter 3. However, it is easy to see that $\mathcal{V}$ is isomorphic to $\mathcal{U}$, so by the iso-invariance postulate the voters have retained their old powers, $\frac{1}{3}$ each. Next, let us take the second step and go from $\mathcal{V}$ to $\mathcal{W} = [8; 2, 1, 6]$. Now voter 1 has indeed gained relative power — not because of his *own* donation, which was made earlier, but as a bystander benefiting from the self-sacrifice of voter 2, who has made herself a dummy!

A similar analysis also applies to the Bz measure and explains why it displays the redistribution paradox in Ex. 7.3.4.

Our analysis shows that the so-called paradox of redistribution is hardly earth-shattering. At the same time, it leads naturally to the question whether there are configurations of the form (7.3.1) with *just one* donor and *just one* recipient in which the donor gains, or the recipient loses, absolute or relative power according to any of the measures discussed in Chapters 3 and 6. We shall take up this question in § 7.8.

7.4 The Paradox of New Members

This paradox was first presented by Brams in [11]. See also Brams and Affuso [14]. Like the redistribution paradox, it concerns WVGs only, rather than more general SVGs. In order to present the paradox formally, it will be convenient to confine ourselves to canonical WVGs with a normalized weighting system. Recall that by Def. 2.3.14 such a WVG has the form $\mathcal{U} = [q; u_1, \ldots, u_n]$ where the u_i are non-negative reals such that $\sum_{i=1}^{n} u_i = 1$ and q is a positive real such that $q \leq 1$.

Now let us assume that a new voter, $n + 1$, joins the assembly, while the quota q and the proportions of weights among the old voters are left unchanged. This results in a new WVG $\mathcal{V}$, with normalized representation $[q; v_1, \ldots, v_n, v_{n+1}]$, where $v_{n+1} \in [0, 1]$ and $v_i = (1 - v_{n+1})u_i$ for $i = 1, \ldots, n$. Here v_{n+1} is the weight of the new voter. The other v_i are the new weights of the n old voters: these must now add up to $1 - v_{n+1}$ instead of 1. Note that they are proportional to the old weights u_i and that the quota q has been left unchanged.

In this configuration, a measure ξ displays the *paradox of new* *members* if $\xi_i[\mathcal{V}] > \xi_i[\mathcal{U}]$ for some $i \in I_n$: that is, if at least one old voter has greater voting power, as measured by ξ, in $\mathcal{V}$ than in $\mathcal{U}$. At first sight, it seems paradoxical that when the old voters share their total voting weight with a newcomer, other things being equal, one or more of them should benefit thereby.[3]

Brams [11] discusses the paradox of new members in relation to the S-S and Bz indices, while Brams and Affuso [14] consider the Coleman measures as well. However, as they note [14, p. 37], adding a new voter to a WVG may cause a voter who was previously a dummy to be 'empowered' (that is, to become a non-dummy). This shows that the paradox in question is not a peculiarity of this or that measure, but is displayed by *any* measure of voting power. Let us illustrate this by a simple example.

7.4.1 Example Let

$$\mathcal{U} = \left[\tfrac{51}{100}; \tfrac{30}{100}, \tfrac{30}{100}, \tfrac{30}{100}, \tfrac{10}{100}\right], \quad \mathcal{V} = \left[\tfrac{51}{100}; \tfrac{15}{100}, \tfrac{15}{100}, \tfrac{15}{100}, \tfrac{5}{100}, \tfrac{50}{100}\right].$$

It is easy to see that in $\mathcal{U}$ voter 4 (with weight $\frac{10}{100}$) is a dummy, since here the MWCs are $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$. On the other hand, in $\mathcal{V}$ this fourth voter is no longer a dummy, since now $\{4, 5\}$ is a [minimal] winning coalition. So if ξ is any measure it follows from postulate (3) of Def. 7.1.1 that $\xi_4[\mathcal{U}] = 0$ and $\xi_4[\mathcal{V}] > 0$. Note that this rather extreme manifestation of the paradox of new members is entailed by just one of the postulates of Def. 7.1.1.

There are also cases in which the advent of a new member does not empower a dummy but adds power to a voter that was already among the most powerful in the original WVG. Before we

[3] Note that our formulation here is somewhat stricter than that in [11] and [14], where the WVGs are not presented in normalized form, leading to some confusion as to whether the 'rule of decision' — given by the quota q — has changed when the new voter is added. If WVGs are not presented in normalized form, their rules of decision are not directly comparable. We insist that $\mathcal{U}$ and $\mathcal{V}$ should have the same quota in normalized form, because allowing this quota to change tends to obscure the issue. The essential question is whether adding a new voter can lead, *ceteris paribus*, to an increase in the absolute or relative power of an old voter. We interpret the *ceteris paribus* clause as requiring that the quota (as well as the ratios between the old voters' weights) be left unchanged.

present such an example we must introduce a new postulate that a reasonable measure must satisfy.

7.4.2 Definition We shall say that a measure ξ *prefers blockers* if whenever b is a blocker and c is a non-blocker in the same SVG $\mathcal{W}$, then $\xi_b[\mathcal{W}] > \xi_c[\mathcal{W}]$.

7.4.3 Remark Recall (Def. 2.3.4) that a blocker is a voter who can veto any bill. Intuitively speaking, this makes a blocker extremely powerful (whether one is thinking in terms of I-power or of P-power); indeed, more powerful than any non-blocker. It is therefore reasonable to postulate that a measure should reflect this intuition. In fact, all the measures defined in Chapters 3 and 6 — the Bz measure, the Bz index, the two Coleman measures, the S-S index, the D-P and Js indices — are easily seen to satisfy this condition.

7.4.4 Example Let

$$\mathcal{U} = \left[\tfrac{55}{60}; \tfrac{35}{60}, \tfrac{15}{60}, \tfrac{5}{60}, \tfrac{5}{60}\right], \quad \mathcal{V} = \left[\tfrac{55}{60}; \tfrac{7}{60}, \tfrac{3}{60}, \tfrac{1}{60}, \tfrac{1}{60}, \tfrac{48}{60}\right].$$

Let ξ be any index that prefers blockers. In $\mathcal{U}$ voters 1 and 2 are blockers and are mutually symmetric (as any two blockers must always be). The other two voters, 3 and 4, who are also mutually symmetric, are non-dummies. Therefore $\frac{1}{4} < \xi_1[\mathcal{U}] < \frac{1}{2}$.

But in $\mathcal{V}$ voters 2, 3 and 4 have become dummies and voters 1 and 5 are mutually symmetric non-dummies (in fact, they are blockers). So $\xi_1[\mathcal{V}] = \frac{1}{2}$. Voter 1's relative power has increased.

Note, by the way, that although the Bz measure is not an index, it too displays the paradox in the present case, because $\beta'_1[\mathcal{U}] = \frac{3}{8}$ and $\beta'_1[\mathcal{V}] = \frac{1}{2}$.

7.4.5 Comment Once again, upon reflection this phenomenon no longer seems so very paradoxical. As pointed out in Com. 7.3.5, the essential point is that the voting power of a voter in an SVG is not a localized property of that voter in isolation, but a global property that involves the whole structure of the SVG, and hence the situations of other voters. The power of a given voter a must clearly depend on the availability of other voters with whom a can form winning coalitions. In particular, the power of a voter a in a WVG cannot in general depend solely on the quota and on a's own weight — indeed, if this were the case then the whole problem of measuring voting power would have been quite simple, which evidently it is not. Rather, the power of each voter in general depends on the whole distribution of weights among all voters. It is therefore hardly astonishing if the introduction of a new voter, while reducing the weights of the old voters, may at the same time present some of them with greater or easier opportunities of forming winning coalitions, thereby helping to increase their absolute or relative power.

7.5 The Quarrelling Paradoxes

Let a and b be two distinct voters of an SVG $\mathcal{W}$. Let $\widetilde{\mathcal{W}}$ be obtained from $\mathcal{W}$ by removing from the latter all winning coalitions in which both a and b are members:

$$\widetilde{\mathcal{W}} =_{\text{def}} \{X \in \mathcal{W} : a \notin X \text{ or } b \notin X\}. \tag{7.5.1}$$

$\widetilde{\mathcal{W}}$ is said to arise from $\mathcal{W}$ through a *quarrel* between a and b. The intuitive idea is that a and b, having 'quarrelled', no longer collaborate; and therefore any previously winning coalition which contains both must now be ruled out.

If it turns out that the voting power of one, and *a fortiori* both, of a and b in $\widetilde{\mathcal{W}}$ is greater than in $\mathcal{W}$, this is regarded as an instance of the *paradox of quarrelling members*. Kilgour [57] studied instances of this paradox displayed by the S-S index. Brams [11, p. 181], who gave the paradox its name, produced an example in which the paradox occurs under both the S-S and Bz indices.[4] Deegan and Packel [23] give an example where their D-P index

[4]Take $\mathcal{W} = [3; 2, 1, 1]$. Here $\phi_1 = 2/3$ and $\phi_2 = \phi_3 = 1/6$; $\beta_1 = 3/5$ and $\beta_2 = \beta_3 = 1/5$. Now suppose voters 2 and 3 quarrel, so the grand coalition cannot be formed. For natural extensions of ϕ and β to the resulting $\widetilde{\mathcal{W}}$, we have $\phi_1 = 1/2$ and $\phi_2 = \phi_3 = 1/4$; and also $\beta_1 = 1/2$ and $\beta_2 = \beta_3 = 1/4$.

displays the paradox. Straffin [100, pp. 278–282] contains a useful discussion of the phenomenon.

7.5.2 Comments (i) We are unable to analyse this paradox formally in the same generality as the paradoxes discussed earlier — that is, with reference to a 'reasonable' but otherwise unspecified measure or index ξ of voting power. This is because $\widetilde{\mathcal{W}}$ as defined in (7.5.1) does not, in general, satisfy conditions (1) and (3) of Def. 2.1.1, and thus may not be an SVG. Since in Def. 7.1.1 ξ is only required to be defined for SVGs, the expressions '$\xi_a[\widetilde{\mathcal{W}}]$' and '$\xi_b[\widetilde{\mathcal{W}}]$' may be meaningless. In the case of the S-S, Bz, D-P and Js indices there is a straightforward natural way of extending their definition so as to assign meaning to these expressions; but there does not seem to be a general way of doing this for an arbitrary ξ. Indeed, as Straffin [100, p. 278] hints, the paradox of quarrelling members lies outside the formal framework we set up in § 2.1: the concept of *quarrelling* oversteps the limits of the aprioristic terrain staked out in Com. 2.2.3, and essentially requires a more general and elaborate framework of the kind to which we alluded there.

However, leaving aside such formal considerations, we can still discuss the paradox at an intuitive level.

(ii) The very concept of *quarrelling*, particularly as formalized in (7.5.1), makes no sense in the context of I-power.

In office-seeking voting behaviour, which underlies the notion of P-power, collaboration is indeed essential because a winning coalition can only be formed if its members conclude a binding pact, as explained in § 6.1 (see especially Com. 6.1.3). If two voters 'quarrel' and refuse to collaborate, then a winning coalition to which both belong cannot form — hence (7.5.1).

But policy-seeking voting behaviour is affected only by how voters feel about the proposed bill, not about each other. No collaboration or coalition-forming pact is required, so quarrelling is irrelevant. Of course, if we suspend our aprioristic viewpoint and consider actual (a posteriori) I-power, then we must envisage boards in which two voters have diametrically opposed *interests*, which cause them to be at odds with each other on every issue.

But this kind of antagonism is not captured correctly by (7.5.1): we must rule out not only winning coalitions which contain both a and b, but any bipartition of N that has a and b on the same side.[5]

For this reason, in the rest of our discussion we confine ourselves to P-power, which is inherently a *relative* magnitude (see Com. 6.1.5). In any case, if two voters get into a quarrel, this clearly cannot alter the total prize available to a winning coalition.

(iii) What is paradoxical about the phenomenon in question? According to Straffin [100, p. 279],

> We normally think that we maximize our power by keeping as many options open as possible, and that restricting our freedom to act lessens our influence. Quarreling, of course, restricts our freedom to act, and hence ... the paradox of quarreling members does seem paradoxical.

This feeling of paradoxicality is no doubt enhanced by the rhetorical device of focusing the discussion of the phenomenon — and the very terminology used to refer to it! — upon the two quarrelsome voters, thus helping to divert attention from the effect of their quarrel on others.

In the first place, it must be pointed out that there are well-known examples of conflict situations in which restricting a protagonist's freedom to act may improve his bargaining position. A real-life example is the strategy of burning one's bridges.

Moreover, quarrelling restricts the 'freedom to act' not only of the two voters directly involved but of other voters as well. A voter c may have derived some of her power in $\mathcal{W}$ from her ability to form a winning coalition that includes her as well as both a and b. In $\widetilde{\mathcal{W}}$, after a and b have quarrelled, c is no longer able to form that winning coalition. The point is that we are dealing

[5] If we re-define $\widetilde{\mathcal{W}}$ in this way, we are left with half of the space $\mathbf{B}_N$, but at the same time we have eliminated half of the bipartitions for which a was $\mathcal{W}$-critical (see Def. 2.3.6); so the probability that a is critical remains unchanged. Thus it may be argued that under a reasonable extension of β' we should have $\beta'_a[\widetilde{\mathcal{W}}] = \beta'_a[\mathcal{W}]$, and similarly for b. But here the very notion of criticality is problematic, because once b's vote is given, a can only vote one way.

with *relative* power; so if the quarrel between two voters may hurt not only themselves but also others, it does not seem to us so very paradoxical that *in relative terms a* and *b* may actually gain power. After all, in real life it often happens that the main sufferers in a fight are innocent bystanders rather than the protagonists.

We are therefore not too astonished that in some cases the move from $\mathcal{W}$ to $\widetilde{\mathcal{W}}$ should increase the relative power of the 'quarrelling' voters, irrespective of how precisely such relative power is measured. In other words, the (alleged) paradox belongs to the superficial first type in the classification of Com. 7.1.3(i). Thus we share Straffin's [100, p. 281] feeling that the quarrelling paradox is *not* 'a peculiarity of the [particular] power indices [that display it], showing that they have strange properties that should make us wary of where and how we use them'.

A related paradox presented in Straffin [100] is the paradox of *quarrelling with a dummy.* This is said to occur if one of the two quarrelling voters, say b, is a dummy, and the other voter, a, loses some power as a result of the quarrel. Straffin [100, p. 280] gives an example in which both the S-S and Bz indices display this paradox.[6] On the other hand, the D-P index is clearly immune to the paradox because this index considers only MWCs, which are not affected by the quarrel.

7.5.3 Comment Why is this phenomenon, in which a voter loses power by quarrelling with a dummy, felt to be paradoxical? Presumably, this is because a dummy — a powerless voter — nevertheless appears to have the power to harm other voters by quarrelling with them. Here again the air of paradoxicality is enhanced (if not created) by the rhetoric used in describing the phenomenon. However, this air is largely dispelled upon closer examination.

If b is a dummy in the SVG $\mathcal{W}$, then the winning coalitions (that is, the members $\mathcal{W}$) can be paired off as follows: each $S \in \mathcal{W}$ such

[6] Take $\mathcal{W} = [4; 2, 2, 2, 1]$. Here voter 4 is a dummy and the other three are symmetric to each other, so for any index ξ we have $\xi_i = 1/3$ for $i = 1, 2, 3$. If voter 1 quarrels with the dummy, then for natural extensions of ϕ and β to the resulting $\widetilde{\mathcal{W}}$ we have $\phi_1 = \beta_1 = 1/4$.

that $b \notin S$ is paired with $S \cup \{b\}$. If we now modify $\mathcal{W}$ by excluding all winning coalitions that contain b, this amounts to throwing out the second member of each pair of winning coalitions. This act merely removes duplication, while treating all remaining voters equally. It is then intuitively clear that the relative powers of these remaining voters should not be affected as a result of this operation. Indeed, this is the intuitive justification for condition (2) in Def. 7.1.1, the postulate of ignoring dummies. (Note that by removing all winning coalitions that contain b we always obtain an SVG, in which b is no longer a voter.)

But if we remove only *some* winning coalitions that contain the dummy — namely, those that contain a as well — then we are not treating all voters equally: this operation is blatantly asymmetric. Using Straffin's terminology quoted earlier, we may say that the *freedom of action* of a (and perhaps also of other quite innocent voters) has been restricted more than that of some other voters. Ought we to be astonished if the overall result is that the relative power of a is reduced? In our opinion, then, the paradox of quarrelling with a dummy, like the paradox of quarrelling members, is paradoxical in a rather weak sense, belonging to the first type in the classification of Com. 7.1.3(i).

7.6 Dominance; The Paradox of Weighted Voting

In this section we will introduce a new postulate that — as we shall argue — any decent measure must satisfy. First we need some new notation and terminology.

7.6.1 Definition Let N be any set and let $a \in N$ and $b \in N$. We shall denote by '$\updownarrow^a_b$' the *transposition* that interchanges a and b and leaves all other members of N fixed. We shall write '$\updownarrow^a_b$' to the right of its argument. Thus $a\updownarrow^a_b = b$, $b\updownarrow^a_b = a$ and $x\updownarrow^a_b = x$ for any other $x \in N$.

Also, if $S \subseteq N$, then $S\updownarrow^a_b = \{x\updownarrow^a_b : x \in S\}$. And if $\mathcal{W}$ is a set of subsets of N then $\mathcal{W}\updownarrow^a_b = \{X\updownarrow^a_b : X \in \mathcal{W}\}$.

If $\mathcal{W}$ is an SVG and $\updownarrow^a_b$ is an automorphism of $\mathcal{W}$, we shall say that the voters a and b are *interchangeable* — briefly: $a \leftrightarrow b$ — *in* $\mathcal{W}$.

7.6.2 Remarks (i) We shall often omit reference to $\mathcal{W}$ and write simply '$a \leftrightarrow b$', when the context allows us to do so without ambiguity.

(ii) $\leftrightarrow$ is an equivalence relation: it is reflexive (because $\updownarrow^a_a$ is the identity mapping), symmetric (because $\updownarrow^a_b = \updownarrow^b_a$) and transitive (because if c is distinct from a and b then $\updownarrow^a_c = \updownarrow^a_b \updownarrow^b_c \updownarrow^a_b$).

(iii) We have already used the term 'interchangeable' informally, in Rem. 2.3.11(ii). Recall that an SVG is (fully) symmetric iff any two of its voters are interchangeable.

Interchangeable voters are obviously mutually symmetric, but the converse is not generally true, as is shown by Ex. 2.3.18 and Ex. 2.3.19. Another simple example is the SVG whose MWCs are $\{a, c, e\}$ and $\{b, d, e\}$. Here a and b are mutually symmetric, but not interchangeable.

7.6.3 Definition Let $\mathcal{W}$ be an SVG. We say that voter a *dominates* voter b — briefly: $a \succeq b$ — *in* $\mathcal{W}$ if whenever $b \in S \in \mathcal{W}$ then also $S\updownarrow^a_b \in \mathcal{W}$.

Moreover, if $a \succeq b$ but not $b \succeq a$ in $\mathcal{W}$, we say that a dominates b *strictly* — briefly: $a \succ b$ — *in* $\mathcal{W}$.

7.6.4 Remarks (i) Again, when the context allows it, we shall omit the reference to $\mathcal{W}$ and, for example, write simply '$a \succeq b$'.

(ii) The meaning of $a \succeq b$ is clear: any winning coalition S that contains b remains a winning coalition if a and b are interchanged. Of course, this condition is non-trivial only in case $a \notin S$, because if S contains both a and b then it is unaltered when they are interchanged.

(iii) The [strict] dominance relation is also known as [*strict*] *desirability*. This terminology emphasizes the fact that if $a \succeq b$ (or $a \succ b$) then a's vote for a bill is at least as useful as (or more useful than) b's for passing that bill.

(iv) It is easy to see that $a \leftrightarrow b$ iff both $a \succeq b$ and $b \succeq a$. Hence $a \succ b$ iff $a \succeq b$ but $a \not\leftrightarrow b$.

(v) It is also easy to see that dominance and strict dominance are invariant under duality of SVGs: $\succeq$ and $\succ$ in $\mathcal{W}$ coincide with $\succeq$ and $\succ$ in $\mathcal{W}^*$ respectively.

(vi) Another easily verified fact is that a blocker in an SVG dominates every voter, and strictly dominates every non-blocker.

(vii) Let $\mathcal{W}$ be a WVG, obtained from a weighting system $\langle q, w \rangle$. If $w_a \geq w_b$ then clearly $a \succeq b$. Thus in this case the relation of dominance is *total*: for any voters a and b we have $a \succeq b$ or $b \succeq a$.

If $w_a > w_b$ then it does not necessarily follow that $a \succ b$, because voters having unequal weights may be interchangeable (as, for example, in $[3; 1, 2]$). However, recall (Rem. 2.3.15(ii)) that the same WVG $\mathcal{W}$ is obtained from infinitely many weighting systems. Now, it is not difficult to prove that $a \succ b$ iff $w_a > w_b$ holds in *all* these weighting systems.[7]

In this sense, strict dominance is a natural extension to arbitrary SVGs of the relation ... *is necessarily heavier than* ... in WVGs.

(viii) As we have just seen, in a WVG dominance is a total relation. But in an unweighted SVG this need not be so. Thus, for example, in the SVG whose MWCs are $\{a, c, e\}$ and $\{b, d, e\}$ voters a and b are incomparable with respect to dominance: neither dominates the other (although as we pointed out in Rem. 7.6.2(iii) they are mutually symmetric).

However, dominance is transitive: if $a \succeq b$ and $b \succeq c$ then, as can easily be seen, $a \succeq c$. Also, it is reflexive: clearly, $a \succeq a$ always.

While in a WVG the dominance relation is total, the converse is not generally true, as the following example illustrates.

7.6.5 Example Let $\mathcal{W}$ be the SVG with assembly

$$N = \{a, b_1, b_2, c_1, c_2, c_3, c_4\},$$

[7] For a proof, see Taylor and Zwicker [105, Proposition 3.2.7]. That book contains a wealth of information about dominance and related subjects.

whose winning coalitions are all $S \subseteq N$ satisfying three conditions: first, $|S| \geq 4$; second, $a \in S$; third, $b_1 \in S$ or $b_2 \in S$. Then a, who is a blocker, strictly dominates all the other voters. Also each of the bs, who are clearly interchangeable, strictly dominates each of the cs, who are likewise mutually interchangeable. So

$$a \succ b_1 \leftrightarrow b_2 \succ c_1 \leftrightarrow c_2 \leftrightarrow c_3 \leftrightarrow c_4.$$

Now suppose this $\mathcal{W}$ were a WVG, with weighting system $\langle q, w \rangle$. First consider the two coalitions $\{a, b_1, c_1, c_2\}$ and $\{a, b_2, c_3, c_4\}$. As these are winning coalitions, each must have weight $\geq q$. Note that the blocker a belongs to both coalitions, whereas each of the other voters belongs to just one. Therefore adding up the weights of these two coalitions we get

$$wN + w_a \geq 2q.$$

Now consider the two coalitions $\{a, b_1, b_2\}$ and $\{a, c_1, c_2, c_3, c_4\}$. As these are losing coalitions, each must have weight $< q$. But again a belongs to both coalitions, whereas each of the other voters belongs to just one. So adding up the weights of *these* two coalitions we get

$$wN + w_a < 2q.$$

This contradiction shows that $\mathcal{W}$ cannot be weighted.

We are now ready to introduce our new postulate.

7.6.6 Definition We shall say that a measure ξ *respects dominance* if whenever $a \succ b$ in $\mathcal{W}$ then $\xi_a[\mathcal{W}] > \xi_b[\mathcal{W}]$.

Also, we shall say that a measure ξ is *monotone* if whenever $w_a \geq w_b$ in a weighting system $\langle q, w \rangle$, then $\xi_a \geq \xi_b$ in the resulting WVG.

7.6.7 Remarks (i) If ξ is any measure, then $a \leftrightarrow b$ implies $\xi_a = \xi_b$ by symmetry. Hence if ξ respects dominance, then $a \succeq b$ implies $\xi_a \geq \xi_b$; in particular, ξ is monotone by Rem. 7.6.4(vii).

(ii) Monotonicity is much weaker than respect for dominance. For example, an index that assigns the value 0 to all dummies and equal positive values to all non-dummies is evidently monotone but does not respect dominance.

(iii) From Rem. 7.6.4(vi) it follows at once that if ξ respects dominance then it prefers blockers (see Def. 7.4.2).

The converse, however, is not generally true: as we shall see in a moment (Thm 7.6.9), a measure that prefers blockers may not even be monotone.

7.6.8 Comment In our view, any reasonable measure of a priori voting power — be it I-power or P-power — must respect dominance. The case for this postulate is so strong that it hardly needs spelling out: if $a \succ b$ then anything that b can do to help pass a bill, a can do as well, and in some cases better. This must give a greater influence and bargaining power than b.

7.6.9 Theorem *The Bz measure and the Bz, S-S and Js indices respect dominance. The D-P index is not monotone.*

Proof The positive first part of the theorem is left to the reader for straightforward verification.

To show that the D-P index is not monotone, consider a WVG $\mathcal{W}$ with assembly $\{a, b, c, d, e, f, g, h\}$ such that

$$\mathcal{W} \cong [23; 10, 9, 8, 5, 4, 3, 2, 1]$$

in alphabetic order. There are 21 MWCs, which we list in lexicographic order:

$$\begin{gathered}
\{a, b, c\},\ \{a, b, d\},\ \{a, b, e\},\\
\{a, b, f, g\},\ \{a, b, f, h\},\\
\{a, c, d\},\\
\{a, c, e, f\},\ \{a, c, e, g\},\ \{a, c, e, h\},\ \{a, c, f, g\},\\
\{a, d, e, f, g\},\ \{a, d, e, f, h\},\\
\{b, c, d, e\},\ \{b, c, d, f\},\ \{b, c, d, g\},\ \{b, c, d, h\},
\end{gathered}$$

$$\{b,c,e,f\},\ \{b,c,e,g\},$$
$$\{b,c,f,g,h\},\ \{b,d,e,f,g\},$$
$$\{c,d,e,f,g,h\}.$$

From Def. 6.4.2 we get:

$$\begin{array}{llll} \text{D-P}_a = \frac{194}{1260}, & \text{D-P}_b = \frac{204}{1260}, & \text{D-P}_c = \frac{212}{1260}, & \text{D-P}_d = \frac{146}{1260} \\ \text{D-P}_e = \frac{156}{1260}, & \text{D-P}_f = \frac{148}{1260}, & \text{D-P}_g = \frac{121}{1260}, & \text{D-P}_h = \frac{79}{1260}. \end{array}$$

Listing the voters in descending order of their D-P values we have:

$$c, b, a, e, f, d, g, h.$$

Comparing this list to the alphabetic order of decreasing weight, we see that the heaviest and third heaviest voters have exchanged places, whereas the fourth heaviest has been pushed down to sixth place.[8] ∎

7.6.10 Comment In [23], Deegan and Packel describe the failure of their index to be monotone as a paradox — the 'paradox of weighted voting'. However, the non-monotonicity of the D-P index can be so regarded only if this index is taken to be a reasonable measure of relative voting power. As far as we know, the only attempt to justify this presupposition is based on Model 6.4.1. But as we argued in Com. 6.4.4, this model itself is implausible.

In our view, the fact that an index based on an implausible model fails to satisfy an eminently reasonable postulate is not in the least paradoxical. Rather, we regard the non-monotonicity of the D-P index as a serious pathology of this index, which makes it highly unlikely that some plausible alternative justification can be found for it.

[8]In [23, p. 253, Excercise 16c], Deegan and Packel, who by that time were aware that their index is not monotone, left open the question whether it could assign to the *heaviest* voter a smaller value than to some other voter(s). Our example shows that this can indeed happen. The first such example, where the heaviest voter is assigned a smaller value than the *second heaviest*, is in [29, p. 213].

7.7 The Meet Paradox

We called attention to this phenomenon in [33], where we categorized it as belonging to a group of 'mild peculiarities or apparent paradoxes'.[9] It concerns a meet

$$\mathcal{W} = \mathcal{W}_1 \wedge \mathcal{W}_2, \tag{7.7.1}$$

of two SVGs whose assemblies N_1 and N_2 have at least two members in common. As mentioned in Rem. 2.3.13(iv), such a meet can be used to model a legislature in which a bill needs the approval of each of two chambers — provided the 'voters' are not taken to be individual legislators but party blocs, each of which is assumed to vote as one person; so the two chambers can have voters in common. Also, (7.7.1) can be used to model two interlocking boards (say a finance committee and a board of governors) of some organization, each board having its own rule of decision, where the approval of both boards is required in order for a resolution to bind the organization as a whole.

7.7.2 Definition A measure ξ is said to display the *meet paradox* in (7.7.1) if there are voters a and b in $N_1 \cap N_2$ such that $\xi_a[\mathcal{W}_i] > \xi_b[\mathcal{W}_i]$ for $i = 1, 2$ but $\xi_a[\mathcal{W}] \leq \xi_b[\mathcal{W}]$.

Moreover, under these conditions we shall say that the paradox is displayed *sharply* if $\xi_a[\mathcal{W}] < \xi_b[\mathcal{W}]$.

At first blush it seems impossible that a reasonable measure of voting power should behave in this way: how can a have more voting power than b in each of the component chambers $\mathcal{W}_1$ and $\mathcal{W}_2$ separately, yet have no more — let alone less — voting power than b in the bicameral composite $\mathcal{W}$? But we shall show by means of examples that this must happen sometimes.

We shall assume throughout that (7.7.1) holds. In each example, we shall present $\mathcal{W}_1$ and $\mathcal{W}_2$ by means of their MWCs: this is all we need to know about these SVGs, because by Def. 7.1.1 a measure must ignore dummies (cf. also Rem. 2.3.5(i).). Recall

[9] There it is called the '*product* paradox'. However, we now feel that 'meet' is the more accurate term; cf. Rem. 2.3.13(v).

(Rem. 2.3.13(iii)) that $\mathcal{W} = \{X_1 \cup X_2 : X_1 \in \mathcal{W}_1,\ X_2 \in \mathcal{W}_2\}$; hence a list of the MWCs of $\mathcal{W}$ can be obtained by first listing all sets of the form $X_1 \cup X_2$ where $X_1 \in {\mathcal{W}_1}^{\min}$ and $X_2 \in {\mathcal{W}_2}^{\min}$, and then weeding out duplications as well as any set that includes another set in the list.

7.7.3 Example Let

$$
{\mathcal{W}_1}^{\min} = \big\{\{a,c\},\ \{b,c,d\}\big\},\ {\mathcal{W}_2}^{\min} = \big\{\{a,d\},\ \{b,c,d\}\big\};
$$

hence $\mathcal{W}^{\min} = \big\{\{a,c,d\},\ \{b,c,d\}\big\}$.

It can be seen at a glance that $a \succ b$ in both $\mathcal{W}_1$ and $\mathcal{W}_2$, whereas in $\mathcal{W}$ a and b are interchangeable. This can be made even plainer, because all these three SVGs are in fact WVGs; assuming that there are no dummies, we have

$$
\mathcal{W}_1 \cong [5; 2, 1, 3, 1],\ \mathcal{W}_2 \cong [5; 2, 1, 1, 3],\ \mathcal{W} \cong [5; 1, 1, 2, 2],
$$

in alphabetic order. Therefore any measure that respects dominance must display here the meet paradox, albeit not sharply.

Although by Thm. 7.6.9 the D-P index does not respect dominance, it too displays the meet paradox in this example. Indeed, $\text{D-P}_a = \frac{1}{4}$ and $\text{D-P}_b = \frac{1}{6}$ in both $\mathcal{W}_1$ and $\mathcal{W}_2$, whereas in $\mathcal{W}$ $\text{D-P}_a = \text{D-P}_b = \frac{1}{6}$.

7.7.4 Remark The fact that the D-P index, which does not respect dominance, nevertheless displays in Ex. 7.7.3 the meet paradox suggests that this paradox may be due to a more rudimentary property of measure than respect for dominance.

On the other hand, an argument of the kind we used in Ex. 7.7.3, relying purely on respect for dominance, cannot deliver a sharp version of the meet paradox: it is easy to see that if $a \succeq b$ holds in both $\mathcal{W}_1$ and $\mathcal{W}_2$, then it also holds in $\mathcal{W}_1 \wedge \mathcal{W}_2$.

The following example uses a somewhat different argument to deliver a sharp version of the paradox.

7.7.5 Example Let

$$\mathcal{W}_1{}^{\min} = \big\{\{a,c,d,j\},\ \{b,e,f,g,j\}\big\},$$
$$\mathcal{W}_2{}^{\min} = \big\{\{a,h,i,j\},\ \{b,e,f,g,j\}\big\};$$

hence $\mathcal{W}^{\min} = \big\{\{a,c,d,h,i,j\},\ \{b,e,f,g,j\}\big\}$.

In all three SVGs j is a (sole) blocker, and therefore must be assigned greater value than all the other voters by any measure that prefers blockers (Def. 7.4.2). But we are not concerned with j, which we inserted only in order to make these SVGs proper: the example would work just as well (albeit with improper SVGs) were j to be removed. Rather, let us focus on a and b.

First, let us look at $\mathcal{W}_1$. Here a, c and d are mutually interchangeable, and must therefore have equal power according to any measure. The same applies to b, e, f and g. But which is more powerful: a or b? Intuitively, it is clear that the former must be more powerful, because *ceteris paribus* it takes only a and his *two* clones to make the same contribution to passing a bill that b can make together with her *three* clones.[10]

In $\mathcal{W}_2$ the positions of a and b are exactly the same as in $\mathcal{W}_1$; in fact, by iso-invariance, any measure must assign to these voters the same respective values in both SVGs; hence any reasonable measure must assign a more power than b in $\mathcal{W}_2$ as well.

But an entirely similar argument shows that in $\mathcal{W}$ b and her three clones must be more powerful than a and his four clones. Hence any reasonable measure must display here the meet paradox in its sharp form.

This intuitive analysis is borne out by the behaviour of the measures defined in Chapters 3 and 6.

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[10] We could perhaps make this argument more formal by defining generally the sense in which two disjoint coalitions, each consisting of 'clones', are 'interchangeable' in a given SVG — as $\{a,c,d\}$ and $\{b,e,f,g\}$ are in $\mathcal{W}_1$ — and postulating that if S and T are interchangeable in this sense, and $|S| < |T|$ then a reasonable measure ought to assign to each member of S greater power than to each member of T. But we feel that the argument in the present example is clear enough even without such abstract formal generalities.

Indeed, we find that in both $\mathcal{W}_1$ and $\mathcal{W}_2$

$$\begin{array}{llll} {\beta'}_a = \frac{15}{128}, & \phi_a = \frac{5}{40}, & \text{D-P}_a = \frac{1}{8}, & \text{JI}_a = \frac{75}{460}, \\ {\beta'}_b = \frac{7}{128}, & \phi_b = \frac{3}{40}, & \text{D-P}_b = \frac{1}{10}, & \text{JI}_b = \frac{28}{460}; \end{array}$$

whereas in $\mathcal{W}$ we get

$$\begin{array}{llll} {\beta'}_a = \frac{15}{512}, & \phi_a = \frac{1}{15}, & \text{D-P}_a = \frac{1}{12}, & \text{JI}_a = \frac{25}{470}, \\ {\beta'}_b = \frac{31}{512}, & \phi_b = \frac{1}{10}, & \text{D-P}_b = \frac{1}{10}, & \text{JI}_b = \frac{62}{470}. \end{array}$$

We have omitted the figures for the Bz index β and the Coleman measures γ and γ^*, which can be obtained from those for β' by appropriate re-scalings.

7.7.6 Remark The meet paradox has a dual: the *join paradox*, which is presented in [33] under the name 'sum paradox'. The description of this paradox is obtained from that of the meet paradox upon replacing '$\wedge$' in (7.7.1) by '$\vee$'.

Now, the operations $\wedge$, $\vee$ and * are easily seen to obey the following analogues of De Morgan's laws:

$$(\mathcal{W}_1 \wedge \mathcal{W}_2)^* = {\mathcal{W}_1}^* \vee {\mathcal{W}_2}^*, \quad (\mathcal{W}_1 \vee \mathcal{W}_2)^* = {\mathcal{W}_1}^* \wedge {\mathcal{W}_2}^*.$$

Therefore if ξ is a *self-dual* measure, in the sense that $\xi_a[\mathcal{W}] = \xi_a[\mathcal{W}^*]$ for any voter a of any SVG $\mathcal{W}$, and if ξ displays the meet paradox in the configuration (7.7.1), then ξ automatically displays the join paradox in the dual configuration. This applies, in particular, to β' and ϕ by Thm. 3.2.7 and Rem. 6.2.4(iii).

The D-P and Js indices are not self-dual, as can be seen quite easily. Nevertheless, they also display the join paradox. For an example in which the S-S, Bz, D-P and Js indices all display the join paradox sharply, see [33, Example 4.1]; in fact, an argument broadly similar to that in Ex. 7.7.5 shows that any reasonable measure must display the join paradox there.

7.7.7 Comment In the classification of Com. 7.1.3(i), the meet paradox is of the superficial type and — like other paradoxes of this type — is quite easily explicable. A close look at our examples reveals how the apparently paradoxical phenomenon arises. We direct our attention to Ex. 7.7.5, which presents the paradox in its sharp form. Let us ignore the blocker, j, who does not affect the argument. The reason why a is more powerful than b in $\mathcal{W}_1$ is that in order to help pass a bill a requires the collaboration of *two* allies-clones, c and d, whereas b needs the collaboration of *three* allies-clones, e, f and g. In $\mathcal{W}_2$ the situation is similar; but whereas the allies-clones of b in $\mathcal{W}_2$ are the same three as in $\mathcal{W}_1$, a now has a *new* pair of allies-clones. Therefore in order to help pass a bill in *both* $\mathcal{W}_1$ *and* $\mathcal{W}_2$ — that is, in their meet $\mathcal{W}$ — a needs altogether the collaboration of *four* allies-clones, whereas b only needs the same three as before. So here b is, quite reasonably, more powerful.

A broadly similar analysis applies also to Ex. 7.7.3. Here a has no clones in any of the three SVGs; b has one clone in each of $\mathcal{W}_1$ and $\mathcal{W}_2$, but her clone in the former is a blocker in the latter, and vice versa.

All the same, the meet paradox is in our view an interesting phenomenon. Like other superficial paradoxes, when properly analysed it no longer seems paradoxical; but it does provide an insight into the inherent complexities of voting power.

7.8 The Transfer Paradoxes: Donation and Bloc

7.8.1 Preview The donation and bloc paradoxes discussed in this section are of the intermediate type in the classification of Com. 7.1.3: they are not superficial; nor, as we shall argue, are they pathologies as far as relative I-power is concerned. They share a similar general form: a given SVG $\mathcal{U}$ is modified, yielding a new SVG $\mathcal{V}$ with the same assembly as $\mathcal{U}$, in a way that appears to transfer some additional power to a particular voter a. Reasonably, we expect a's voting power to be greater in $\mathcal{V}$ than in $\mathcal{U}$. So a measure ξ displays a *transfer paradox* if $\xi_a[\mathcal{V}] < \xi_a[\mathcal{U}]$.

The close connection between these two paradoxes can be expressed more formally: they are variant ways of violating a fairly powerful but reasonable postulate. However, instead of stating this *transfer postulate* straight away, we shall first look at the transfer paradoxes individually, and approach the postulate by way of generalization (see Obs. 7.8.19 and Def. 7.8.20).

We start by addressing the question posed at the end of § 7.3. To this end, we consider a configuration like (7.3.1), but with voter 1 as sole recipient and voter 2 as sole donor:

$$\begin{aligned} &\mathcal{U} = [q; u_1, u_2, u_3, \ldots, u_n], \\ &\widetilde{\mathcal{U}} = [q; u_1 + \delta, u_2 - \delta, u_3, \ldots, u_n], \\ &\text{where} \quad 0 < \delta \leq u_2. \end{aligned} \tag{7.8.2}$$

Here δ is the weight that voter 2 has 'donated' to voter 1, and as a result of this single change the original WVG $\mathcal{U}$ has changed into $\widetilde{\mathcal{U}}$.

7.8.3 Definition A measure ξ is said to display the *donation paradox* in the configuration (7.8.2), if $\xi_1[\widetilde{\mathcal{U}}] < \xi_1[\mathcal{U}]$.

7.8.4 Remarks (i) From the definitions of the Bz measure and the S-S index (Defs. 3.2.2 and 6.3.1) it is easy to see that they can never display the donation paradox. In fact, it will transpire (Thm. 7.8.26) that β' and ϕ are immune to all transfer paradoxes.

(ii) In [29] we presented the donation paradox in equivalent but reversed form, as a single donor apparently gaining power (rather than a single recipient losing it). In that reversed form the paradox may be named after St Francis of Assisi, who reportedly preached: 'in giving, we receive'.

(iii) Any instance where a measure ξ is not monotone can always be put in the form of a WVG $\mathcal{U} = [q; u_1, u_2, \ldots, u_n]$ for which $u_1 < u_2$ but $\xi_1[\mathcal{U}] > \xi_2[\mathcal{U}]$. If we take this as the $\mathcal{U}$ in (7.8.2) with $\delta = u_2 - u_1$, we obtain an instance in which ξ displays the donation paradox. Thus the paradox of weighted voting may be regarded as an acute special case of the donation paradox.

Since the D-P index is not monotone (Thm. 7.6.9) it must also display the donation paradox. In the following two examples it displays the donation paradox but (as the reader can easily verify) *without* violating monotonicity. The first of these examples catches two birds with one stone: the donation paradox is displayed also by the Bz index (as distinct from the Bz measure!). In the second example *three* birds are caught with one stone: the paradox is displayed by the Bz index and the Js index as well.

7.8.5 Example In (7.8.2) take $\mathcal{U} = [8; 4, 4, 1, 1, 1]$. Here the first two voters are interchangeable, while the remaining small fry are dummies; hence $\xi_1[\mathcal{U}] = \frac{1}{2}$ for any index ξ.

Now let $\delta = 1$; so $\widetilde{\mathcal{U}} = [8; 5, 3, 1, 1, 1]$. The small fry are no longer dummies: together with voter 1 they make up an MWC.

We get $\beta_1[\widetilde{\mathcal{U}}] = \frac{9}{19} < \frac{1}{2}$ and D-P$_1[\widetilde{\mathcal{U}}] = \frac{3}{8} < \frac{1}{2}$.

7.8.6 Example In (7.8.2) take $\mathcal{U} = [24; 9, 7, 9, 1, 1, 1, 1, 1]$. Here the first three voters are mutually interchangeable and the remaining small fry are dummies. So for any index ξ we must have $\xi_1[\mathcal{U}] = \frac{1}{3}$.

Now let $\delta = 1$; so $\widetilde{\mathcal{U}} = [24; 10, 6, 9, 1, 1, 1, 1, 1]$. Here again the small fry are no longer dummies.

We get $\beta_1[\widetilde{\mathcal{U}}] = \frac{11}{34}$, D-P$_1[\widetilde{\mathcal{U}}] = \frac{5}{21}$ and JI$_1[\widetilde{\mathcal{U}}] = \frac{461}{1386}$. All three values are $< \frac{1}{3}$.

7.8.7 Remarks (i) The presentation of the foregoing examples can be modified slightly so as to yield another paradox. The $\widetilde{\mathcal{U}}$ of Ex. 7.8.5 can also be presented as $[8; 5, 4, 1, 1, 1]$. (This is the *same* WVG, merely presented by a slightly different weighting system.) Now the transition from $\mathcal{U}$ to $\widetilde{\mathcal{U}}$ does not involve any 'donation', merely an increase in the weight of voter 1. Similarly, the $\widetilde{\mathcal{U}}$ of Ex. 7.8.6 can be presented as $[24; 10, 7, 9, 1, 1, 1, 1, 1]$.

Thus we have here what may be called the *fattening paradox*: a voter who puts on extra weight loses power according to some measure. Our examples now show that the Bz, D-P and Js indices display the fattening paradox.

(ii) A measure ξ displaying the fattening paradox violates thereby Young's [114, p. 69] *strong monotonicity* condition. This postulates for a value assignment ξ (in the sense of Def. 6.2.3) that if $\mathbf{u}$ and $\tilde{\mathbf{u}}$ are games having the same grand coalition, and if a is a player such that

$$\tilde{\mathbf{u}}X - \tilde{\mathbf{u}}(X - \{a\}) \geq \mathbf{u}X - \mathbf{u}(X - \{a\})$$

for every coalition X, then $\xi_a(\tilde{\mathbf{u}}) \geq \xi_a(\mathbf{u})$.

This clearly implies the marginality condition of Def. 6.2.9(v). It is also immediate from Def. 6.2.3 that β' and ϕ satisfy Young's strong monotonicity condition; hence the Bz measure and the S-S index are immune to the fattening paradox.

Summarizing the observations made so far, from Rem. 7.8.4 on, we have:

7.8.8 Theorem *The Bz measure and S-S index are immune to the donation and fattening paradoxes. The Bz, D-P and Js indices are vulnerable to them.* ∎

We shall make our comments on the significance of these results after we introduce and illustrate the bloc paradox.

This concerns the operation of formation of a two-voter bloc $a\&b$, which we first recast as a transfer operation in the sense of Preview 7.8.1. As originally defined (Def. 2.3.23) $\mathcal{W}|a\&b$ does not fit this mould; so we first introduce a new SVG, a variant of $\mathcal{W}|a\&b$, which does.

7.8.9 Definition Let a and b be two distinct voters of an SVG $\mathcal{W}$. Let g be the mapping of the assembly N of $\mathcal{W}$ into itself such that $gb = a$ and $gx = x$ for all $x \in N$ other than b. We put

$$\mathcal{W}|\overleftarrow{a\&b} =_{\text{def}} \{X \subseteq N : g^{-1}[X] \in \mathcal{W}\},$$

and say that $\mathcal{W}|\overleftarrow{a\&b}$ arises from $\mathcal{W}$ through a's *annexation* of b.

7.8.10 Remarks (i) $\mathcal{W}|\overleftarrow{a\&b}$ can be obtained from $\mathcal{W}$ in three steps: first, form the bloc $a\&b$, obtaining $\mathcal{W}|a\&b$; next, rename

$a\&b$ as 'a' (thus obtaining an SVG isomorphic to $\mathcal{W}|a\&b$, in which a replaces $a\&b$); finally, introduce b as a dummy.

(ii) Intuitively, you can think of $\mathcal{W}|\overleftarrow{a\&b}$ as arising from $\mathcal{W}$ by a takeover: a takes over b's voting mandate and adds it to his own, leaving b as a dummy.

Thus, if $\mathcal{W}$ is a WVG with weighting system $\langle q, w\rangle$ then $\mathcal{W}|\overleftarrow{a\&b}$ can be obtained by b donating her entire weight to a: so w_a is replaced by $w_a + w_b$, w_b is replaced by 0, and all other weights and the quota remain unchanged.

However, $\mathcal{W}|\overleftarrow{a\&b}$ is defined also when $\mathcal{W}$ is not weighted.

(iii) It is easy to see that the winning coalitions of $\mathcal{W}|\overleftarrow{a\&b}$ are of two kinds: first, any $S \in \mathcal{W}$ such that $a \notin S$ and b is not critical in S; second, any $S \subseteq N$ such that $a \in S$ and $S \cup \{b\} \in \mathcal{W}$.

(iv) It is also easy to verify that for any measure ξ,

$$\begin{aligned} \xi_a[\mathcal{W}|\overleftarrow{a\&b}] &= \xi_{a\&b}[\mathcal{W}|a\&b], \\ \xi_b[\mathcal{W}|\overleftarrow{a\&b}] &= 0, \\ \text{and} \quad \xi_x[\mathcal{W}|\overleftarrow{a\&b}] &= \xi_x[\mathcal{W}|a\&b] \end{aligned}$$

for all other voters x of $\mathcal{W}$.

7.8.11 Definition A measure ξ satisfies the *bloc postulate* if whenever a and b are distinct voters of an SVG $\mathcal{W}$ and b is not a dummy in $\mathcal{W}$, then

$$\xi_a[\mathcal{W}|\overleftarrow{a\&b}] > \xi_a[\mathcal{W}].$$

By the *bloc paradox* we mean any violation of the bloc postulate. Moreover, in any instance where

$$\xi_a[\mathcal{W}|\overleftarrow{a\&b}] < \xi_a[\mathcal{W}]$$

we shall say that ξ displays the bloc paradox *sharply.*

7.8.12 Remarks (i) Since $\xi_a[\mathcal{W}|\overleftarrow{a\&b}] = \xi_{a\&b}[\mathcal{W}|a\&b]$, the bloc paradox may be regarded as a particularly acute form of the paradox of large size.

(ii) By Rem. 7.8.10(ii), when the sharp version of the bloc paradox occurs in a WVG, this can be regarded as an extreme case of the donation paradox, in which the donor donates her *entire* weight and becomes a dummy. When the bloc paradox occurs in an unweighted SVG it can therefore be regarded as an *analogue* of this extreme form of the donation paradox. In Obs. 7.8.19 and Def. 7.8.20 we shall put this analogy on a firm formal basis.

The following theorem is a consequence of a more general result, Thm. 7.8.26, which we shall prove a bit later. But the direct proof we give here is instructive.

7.8.13 Theorem *The Bz measure and S-S index satisfy the bloc postulate and are therefore immune to the bloc paradox.*

Proof First consider the Bz measure β'. Thm. 3.2.18 can now be restated as

$$\beta'_a[\mathcal{W}|\overleftarrow{a\&b}] = \beta'_a[\mathcal{W}] + \beta'_b[\mathcal{W}]. \qquad (*)$$

If b is not a dummy in $\mathcal{W}$ then $\beta'_b[\mathcal{W}] > 0$; therefore $(*)$ implies $\beta'_a[\mathcal{W}|\overleftarrow{a\&b}] > \beta'_a[\mathcal{W}]$, as claimed.

From Rem. 7.8.10(iii) it is clear that if a is critical inside a coalition S in $\mathcal{W}$ then a is, *a fortiori*, critical inside S in $\mathcal{W}|\overleftarrow{a\&b}$. Moreover, assuming that b is not a dummy in $\mathcal{W}$, it now follows from $(*)$ that there must exist at least one coalition S such that a is critical inside S in $\mathcal{W}|\overleftarrow{a\&b}$ but not in $\mathcal{W}$.

From this fact and Def. 6.3.1 we infer that $\phi_a[\mathcal{W}|\overleftarrow{a\&b}] > \phi_a[\mathcal{W}]$, as claimed. ∎

In the following examples the Bz, D-P and Js indices display the sharp form of the bloc paradox.

7.8.14 Example Let

$$\mathcal{W} = [11; 6, 5, 1, 1, 1, 1, 1].$$

Then $\beta_1[\mathcal{W}] = \frac{11}{23}$. Now let voter 1 annex voter 3:

$$\mathcal{W}|\overleftarrow{1\&3} = [11; 7, 5, 0, 1, 1, 1, 1].$$

Then $\beta_1[\mathcal{W}|\overleftarrow{1\&3}] = \frac{17}{36} < \frac{11}{23}$, a sharp violation of the bloc postulate by the Bz index.

7.8.15 Example Let

$$\mathcal{W} = [6; 4, 1, 1, 1, 1, 1, 1].$$

Then D-P$_2[\mathcal{W}] = \frac{11}{96}$. Now let voter 2 annex voter 3:

$$\mathcal{W}|\overleftarrow{2\&3} = [6; 4, 2, 0, 1, 1, 1, 1].$$

Then D-P$_2[\mathcal{W}|\overleftarrow{2\&3}] = \frac{7}{80} < \frac{11}{96}$, a sharp violation of the bloc postulate by the D-P index.

7.8.16 Example Let

$$\mathcal{W} = [23; 8, 8, 7, 1, 1, 1, 1, 1, 1, 1].$$

Then $\beta_1[\mathcal{W}] = \frac{129}{392}$ and JI$_1[\mathcal{W}] = \frac{773}{2322}$. Now let voter 1 annex voter 4:

$$\mathcal{W}|\overleftarrow{1\&4} = [23; 9, 8, 7, 0, 1, 1, 1, 1, 1, 1].$$

Then $\beta_1[\mathcal{W}|\overleftarrow{1\&4}] = \frac{65}{199} < \frac{129}{392}$ and JI$_1[\mathcal{W}|\overleftarrow{1\&4}] = \frac{173}{520} < \frac{773}{2322}$, sharp violations of the bloc postulate by the Bz and Js indices.

7.8.17 Remark Exs. 7.8.15 and 7.8.16 are borrowed from [29, pp. 209 f.], where the bloc paradox was first presented. Ex. 7.8.14 is new: it is the simplest case we could find in which the Bz index displays this paradox.

Examples of violation of the bloc postulate by the D-P index are fairly easy to find. For the Bz and Js indices it is considerably harder; but after discovering one or two examples you begin to develop a nose for it. The ones presented here were chosen for their simplicity. All three involve WVGs, but unweighted ones could also be given (they are somewhat more complicated).

The Bz index also displays the bloc paradox in configurations that, unlike Exs. 7.8.14 and 7.8.16, involve no vetoers. (The simplest example of this sort we have been able to find is the WVG $\mathcal{W} = [22; 13, 12, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$, with voter 1 annexing one of the small fry.)

7.8.18 Comment What are we to make of the transfer paradoxes? In our view it is totally unreasonable that a voter should be worse off as a result of being the sole recipient of a weight donation, still less of having annexed another voter. Any measure of voting power that has such unreasonable implications is therefore unacceptable. Thus, the fact that the D-P and Js indices display the transfer paradoxes is a severe pathology of these indices; rather than being genuinely paradoxical, it provides further reason for rejecting these indices.

The case of the Bz index is quite different.[11] As pointed out in Com. 3.2.5, while β' is a measure of a voter's absolute I-power, the derived index β measures a voter's *relative share* in the total I-power of the given SVG $\mathcal{W}$ — a total *which is not a fixed quantity* but depends on $\mathcal{W}$. Thus, if the value of β for a voter a goes down as a result of some change in the decision rule, it does not necessarily follow that a is worse off.

Indeed, Thms. 7.8.8 and 7.8.13 tell us that a voter can never lose Bz power as a result of being sole recipient of a weight donation, and will always gain Bz power as a result of annexing a non-dummy. What examples such as Exs. 7.8.5, 7.8.6 and 7.8.14 illustrate is that in some cases a change in the decision rule that increases the Bz power of the direct beneficiary a may, as a by-product, be of greater relative benefit to other voters. If the increase in the sensitivity Σ of the SVG (see Def. 3.3.1) is proportionally greater than the increase in β'_a, then a's *relative* share in Σ will go down, although a will be absolutely better off.

In our view, this phenomenon is genuinely paradoxical. It is not a pathology of β but a surprising fact about the nature of I-power.

Note that the foregoing analysis that exculpates β is inapplicable to the D-P and Js indices. These two are not derived relative forms of some absolute measures. And in any case it would not make sense to assume that total P-power — the prize to be divided among the members of a winning coalition — can change merely due to a donation or annexation.

[11] Our failure to appreciate this point in [29] was a grave error, leading to an unjustified rejection of the Bz index. This error is rectified in [36, pp. 103 f.].

We conclude that the bloc postulate is compelling for a measure of absolute I-power (though not for an index of relative I-power) as well as for an index of P-power.

We would similarly like to advocate a postulate banning the donation paradox for a measure of absolute I-power as well as for an index of P-power. But instead of stating a condition that applies only to WVGs, we shall take a more general approach, leading to a postulate that applies to any SVG and bans both the donation and the bloc paradoxes.

7.8.19 Observations (i) Let us take a close look at the configuration (7.8.2) and see how the shift of weight from voter 2 to voter 1 can only strengthen the position of the latter in $\widetilde{\mathcal{U}}$ as compared with $\mathcal{U}$.

The most convenient way of doing this is in terms of bipartitions of the assembly and the respective BPRs U and $\widetilde{\mathsf{U}}$ of $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ (see Def. 2.1.5).

Clearly, if voters 1 and 2 are on the same side of a bipartition B (that is, if both of these voters are in B^+ or both are in B^-) then the outcome of B is identical in $\mathcal{U}$ and $\widetilde{\mathcal{U}}$, because in this case the shift of weight takes place *within* B^+ or B^-, so the total weights of these two complementary coalitions are the same under both weight functions, u and $\tilde{u}$.

Equally clearly, if voters 1 and 2 are on opposite sides of a bipartition B (that is, if one of these voters is in B^+ and the other in B^-) then the outcome of B in $\widetilde{\mathcal{U}}$ cannot be worse from voter 1's viewpoint than the outcome of B according to $\mathcal{U}$, because the side of B to which voter 1 belongs is heavier under $\tilde{u}$ than under u. Therefore in this case if voter 1 agrees with the outcome of B in $\mathcal{U}$ she must *a fortiori* agree with the outcome of B in $\widetilde{\mathcal{U}}$.

Of course, it may happen that despite the donation $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ as presented in (7.8.2) actually coincide. For example, if we take $\mathcal{U} = [5; 3, 3, 3]$ and $\delta = 1$ then $\widetilde{\mathcal{U}} = [5; 4, 2, 3]$, which is the same WVG as $\mathcal{U}$, merely represented by a different weighting system.

However, if $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ are not the same then there must exist at least one bipartition B whose outcome in $\widetilde{\mathcal{U}}$ differs from that in $\mathcal{U}$. In view of what we have established a moment ago, voters 1 and 2 must be on opposite sides of B, and voter 1 must agree with the outcome of B in $\widetilde{\mathcal{U}}$ but not in $\mathcal{U}$.

(ii) Now let us take a close look at how annexation of voter b by voter a can only strengthen the position of the latter in $\mathcal{W}|\overleftarrow{a\&b}$ as compared with $\mathcal{W}$. Using Rem. 7.8.10, we reach virtually the same conclusions as before.

First, if a and b are on the same side of a bipartition B then the outcome of B is identical in $\mathcal{W}$ and $\mathcal{W}|\overleftarrow{a\&b}$.

Second, if a and b are on opposite sides of a bipartition B and if a agrees with the outcome of B in $\mathcal{W}$ he must *a fortiori* agree with the outcome of B in $\mathcal{W}|\overleftarrow{a\&b}$.

Third, if $\mathcal{W}$ and $\mathcal{W}|\overleftarrow{a\&b}$ are not the same, there must exist at least one bipartition B such that a agrees with the outcome of B in $\mathcal{W}|\overleftarrow{a\&b}$ but not in $\mathcal{W}$.

Let us add here that if b is a non-dummy in $\mathcal{W}$ then $\mathcal{W}$ and $\mathcal{W}|\overleftarrow{a\&b}$ cannot be the same because b *is* a dummy in the latter. Hence there exists a bipartition B as described in the preceding paragraph.

These observations reveal the structural similarity between weight donation and annexation and lead us to the following definition of a more abstract relation that subsumes both donation and annexation.

7.8.20 Definition Let $\mathcal{U}$ and $\mathcal{V}$ be SVGs with the same assembly and let a and b be two distinct voters such that the following three conditions hold:

(1) whenever a and b are on the same side of a bipartition B then the outcome of B is identical in $\mathcal{U}$ and $\mathcal{V}$;

(2) whenever a and b are on opposite sides of a bipartition B and a agrees with the outcome of B in $\mathcal{U}$ then a also agrees with the outcome of B in $\mathcal{V}$;

(3) there exists at least one bipartition B such that a agrees with the outcome of B in $\mathcal{V}$ but not in $\mathcal{U}$.

Then we shall say that $\mathcal{V}$ *arises from* $\mathcal{U}$ *by transfer from* b *to* a, and write briefly '$\mathfrak{Tr}(\mathcal{U},\mathcal{V};\overleftarrow{a,b})$'.

Further, we shall say that a measure ξ satisfies the *transfer postulate* if whenever $\mathfrak{Tr}(\mathcal{U},\mathcal{V};\overleftarrow{a,b})$ holds then $\xi_a[\mathcal{V}] > \xi_a[\mathcal{U}]$.

7.8.21 Remark The transfer relation is symmetric in the sense that $\mathfrak{Tr}(\mathcal{U},\mathcal{V};\overleftarrow{a,b})$ implies $\mathfrak{Tr}(\mathcal{V},\mathcal{U};\overleftarrow{b,a})$. Indeed, conditions (1)–(3) of Def. 7.8.20 are essentially unchanged when $\mathcal{U}$ is interchanged with $\mathcal{V}$ and a with b. For condition (1) this is obvious. But condition (2) is also symmetric in this sense, because if a and b are on opposite sides of a bipartition B, then in a given SVG a agrees with the outcome of B iff b does not agree with the outcome of B. Finally, in the presence of condition (1), condition (3) implies that there exists a bipartition B such that b agrees with the outcome of B in $\mathcal{U}$ but not in $\mathcal{V}$.

From Obs. 7.8.19 we have at once:

7.8.22 Theorem *A measure that satisfies the transfer postulate is immune to the donation paradox. Moreover, the transfer postulate implies the bloc postulate; hence a measure satisfying the transfer postulate is immune to the bloc paradox.* ∎

As an extra bonus we have:

7.8.23 Theorem *A measure that satisfies the transfer postulate respects dominance.*

Proof Suppose that $b \succ a$ in an SVG $\mathcal{U}$. Put $\mathcal{V} = \mathcal{U}\updownarrow^a_b$. Thus $\mathcal{V}$ is obtained from $\mathcal{U}$ by interchanging a and b (see Def. 7.6.1). It is easy to verify that $\mathfrak{Tr}(\mathcal{U},\mathcal{V};\overleftarrow{a,b})$. Hence, if ξ satisfies the transfer postulate, it follows that $\xi_a[\mathcal{V}] > \xi_a[\mathcal{U}]$.

But $\updownarrow^a_b$ is an isomorphism from $\mathcal{U}$ to $\mathcal{V}$; hence $\xi_b[\mathcal{U}] = \xi_a[\mathcal{V}]$ by the iso-invariance of ξ. Thus $\xi_b[\mathcal{U}] > \xi_a[\mathcal{U}]$. ∎

The following diagram summarizes the logical relationships among the various properties of measures discussed so far in this chapter.

7.8.24 Diagram

$$
\begin{array}{l}
\text{Monotonicity} \\
\Uparrow \\
\text{Respecting dominance} \Longrightarrow \text{Preferring blockers} \\
\Uparrow \\
\text{Transfer postulate} \Longrightarrow \text{Bloc postulate} \\
\Downarrow \\
\text{Immunity to donation paradox}
\end{array}
$$

7.8.25 Remark From the last two theorems it follows that a measure violates the transfer postulate whenever it displays the donation paradox or the bloc paradox, and whenever it fails to respect dominance. But there are other ways in which the transfer postulate can be violated. For example, the $\mathcal{U}$ of Ex. 7.8.6 can also be presented in the form

$$\mathcal{U} = [25; 10, 6, 9, 1, 1, 1, 1, 1].$$

Now $\widetilde{\mathcal{U}} = [24; 10, 6, 9, 1, 1, 1, 1, 1]$ is obtained from $\mathcal{U}$ (as presented here) not by donation or annexation, but by lowering the quota from 25 to 24. Yet the fact remains that $\mathfrak{Tr}(\mathcal{U}, \widetilde{\mathcal{U}}; \overleftarrow{1, 2})$, and the Bz, D-P and Js indices all violate the transfer postulate here.

7.8.26 Theorem *The Bz measure and S-S index satisfy the transfer postulate.*

Proof Suppose that $\mathfrak{Tr}(\mathcal{U}, \mathcal{V}; \overleftarrow{a, b})$. From Def. 7.8.20 it is easy to verify that whenever B is a bipartition of the assembly such that a is $\mathcal{U}$-critical for B then a is also $\mathcal{V}$-critical for B; but there exist bipartitions for which a is $\mathcal{V}$-critical but not $\mathcal{U}$-critical.

This clearly implies that whenever S is a coalition such that a is $\mathcal{U}$-critical in S then a is also $\mathcal{V}$-critical in S; but there exists a coalition in which a is $\mathcal{V}$-critical but not $\mathcal{U}$-critical.

The claim of our theorem now follows at once from the definitions of β' and ϕ. ∎

7.8.27 Comments (i) In our view, the transfer postulate is compelling for a measure of absolute I-power (but not for an index of relative I-power), as well as for an index of P-power. The argument for this are essentially the same as that offered in Com. 7.8.18, except that now the matter is placed in a more general setting.

If $\mathfrak{Tr}(\mathcal{U}, \mathcal{V}; \overleftarrow{a, b})$ holds, conditions (1)–(3) of Def. 7.8.20 make it obvious that the move from $\mathcal{U}$ to $\mathcal{V}$ is favourable to a; and by the symmetry of the transfer relation (Rem. 7.8.21) they also imply that this move is unfavourable to b. Moreover, in this transaction only a and b are involved directly: this is due to condition (1), which ensures that when a and b are on the same side the outcome in $\mathcal{V}$ is identical to that in $\mathcal{U}$. Therefore the move from $\mathcal{U}$ to $\mathcal{V}$ must involve a transfer of power from b to a: the latter gets stronger at the former's expense. Other voters may of course be affected, but only as bystanders, not as protagonists.

If under these circumstances a measure of absolute I-power fails to register an increase in a's influence then in our view this measure shows itself to be inadequate. The same applies to an index of P-power that fails to register an increase in a's bargaining power, leading to a higher expected payoff.

(ii) This provides an added argument for β' as a measure of absolute I-power. The argument is a negative one: by Thm. 7.8.26 β' *does not fail* the test of the transfer postulate. However, β' is in any case amply justified by a persuasive positive argument.

At the same time, the Bz, D-P and Js indices fail the test as indices of P-power. This leaves in the field the S-S index, whose claim as a valid index of P-power now gains negative support — much-needed because, as we saw in Com. 6.3.10, the positive arguments in favour of the S-S index are not all that compelling.

7.9 The Blocker's Share and Added Blocker Postulates

In this section we consider two postulates that address the behaviour of relative voting power in the presence of blockers. The first sets a lower bound to the relative power of a blocker; the second concerns the effect of introducing a new blocker to an SVG on the relative voting powers of the old voters.

7.9.1 Definition We say that an index ξ satisfies the *blocker's share postulate* — briefly, *BSP* — if whenever v is a blocker and S is a winning coalition in an SVG $\mathcal{W}$, then

$$\xi_v[\mathcal{W}] \geq \frac{1}{|S|}.$$

7.9.2 Remarks (i) We have stated the BSP for an index, rather than more generally for a measure, because in any case we intend to apply it to *relative* voting power.

(ii) The BSP obviously yields the sharpest result when S is a winning coalition of least size, which must of course be an MWC. Thus the BSP is equivalent to the condition that if an MWC of $\mathcal{W}$ of least size has k members, then $\xi_v[\mathcal{W}] \geq 1/k$.

7.9.3 Comment In our view the BSP is compelling for an index of P-power. Let us assume office-seeking voting behaviour, as described in § 6.1, and suppose that a winning coalition S is being formed. A blocker, by definition, is an indispensable member of any winning coalition; he is the boy who owns the ball (or, if there are other blockers, he owns part of the ball). Therefore our blocker v cannot be fobbed off by a smaller share of the loot than any other member of S. Surely, if S does form, v will get at least $1/|S|$ of the total payoff. Thus v's conditional expected payoff, subject to the condition that S will be formed, is at least $1/|S|$. But what if another winning coalition, say T, is about to be formed? Well, if this does not guarantee v as much payoff as he expects to get from S, why should he give a hand to forming T? Since T cannot win without v, it must also grant him at least $1/|S|$. Therefore v's *prior* (unconditional) expected payoff is also at least $1/|S|$.

This argument only applies to P-power. We can offer no plausible reason why an index of I-power should satisfy the BSP.

7.9.4 Theorem *The S-S index satisfies the BSP. The Bz, D-P and Js indices violate it.*

Proof For the S-S index, suppose v is a blocker and S a winning coalition in an SVG $\mathcal{W}$ with assembly N. By Thm. 6.3.7, $\phi_v[\mathcal{W}]$ is the probability, in the queue space $\mathbf{Q}_N$, that v is the pivot of a queue. Now, each of S's members has equal probability of coming after all the others; hence the probability that v is preceded in a queue by all the other members of S is exactly $1/|S|$. But whenever v is preceded by the other members of S then v, being a blocker, is clearly the pivot. Therefore

$$\phi_v[\mathcal{W}] \geq \frac{1}{|S|}.$$

For the Bz and D-P indices, look at $\widetilde{\mathcal{U}}$ of Ex. 7.8.5. There voter 1 is a (sole) blocker and $\{1, 2\}$ is a winning coalition; so according to the BSP voter 1 should get at least half. But we saw that $\beta_1[\widetilde{\mathcal{U}}] = \frac{9}{19} < \frac{1}{2}$ and D-P$_1[\widetilde{\mathcal{U}}] = \frac{3}{8} < \frac{1}{2}$.

For the Js index look at $\widetilde{\mathcal{U}}$ of Ex. 7.8.6. Here voters 1 and 3 are blockers and $\{1, 2, 3\}$ is a winning coalition; so the BSP grants at least $\frac{1}{3}$ to each blocker. But we saw that JI$_1[\widetilde{\mathcal{U}}] < \frac{1}{3}$. ∎

7.9.5 Comments (i) The proof of Thm. 7.9.4 illustrates an empirically observed fact: violation of the BSP is often associated with the donation paradox. (However, the logical relationship between these two phenomena is by no means straightforward.)

(ii) Thm. 7.9.4 lends support — albeit negative — to the claim of ϕ as a well-behaved index of P-power. On the other hand, it provides further evidence against the plausibility of the Bz, D-P and Js indices as P-power indices. The validity of the Bz index as an index of I-power is of course unaffected.

The second postulate we wish to propose in this section addresses a configuration of the form

$$\begin{aligned} &\mathcal{W} = \big\{X \cup \{v\} : X \in \mathcal{U}\big\}, \\ &\text{where } \mathcal{U} \text{ is an SVG and } v \text{ is not a voter of } \mathcal{U}. \end{aligned} \tag{7.9.6}$$

In this connection we say that $\mathcal{W}$ is obtained from $\mathcal{U}$ by *adding v as blocker* (or *vetoer*).

7.9.7 Remarks (i) An SVG $\mathcal{W}$ of the form (7.9.6) can be used to model a constitution in which any bill adopted by a legislature $\mathcal{U}$ must be ratified by an individual v, who is not a member of the legislature, before becoming law. For example, v may be a monarch wielding the prerogative of royal assent, or a president who has the right to veto any legislation. (Of course, we are thinking here of royal assent that is not a mere formality as in constitutional monarchies; and of presidential veto that, unlike that in the US Constitution, cannot be overridden.)

(ii) Adding a blocker is a limiting special case of forming the product of $\mathcal{U}$ with an arbitrary SVG $\mathcal{V}$, whose assembly must be disjoint from that of $\mathcal{U}$ (see Def. 2.3.12):

$$\mathcal{W} = \mathcal{U} \times \mathcal{V}.$$

The special case (7.9.6) is obtained when $\mathcal{V}$ is the SVG having v as sole voter. As pointed out in Rem. 2.3.13(iv), a product $\mathcal{U} \times \mathcal{V}$ may be used to model a bicameral legislature, in which a new law needs the approval of each of two chambers, and no voter can belong to both chambers.

7.9.8 Definition An index ξ satisfies the *added blocker postulate* — briefly, *ABP* — if whenever $\mathcal{U}$ and $\mathcal{W}$ are as in (7.9.6) and a and b are non-dummy voters of $\mathcal{U}$ then

$$\frac{\xi_a[\mathcal{W}]}{\xi_b[\mathcal{W}]} = \frac{\xi_a[\mathcal{U}]}{\xi_b[\mathcal{U}]}.$$

We shall say that a *flagrant* violation of this postulate by an index ξ occurs in any instance where

$$\xi_a[\mathcal{U}] > \xi_b[\mathcal{U}] \text{ and } \xi_a[\mathcal{W}] < \xi_b[\mathcal{W}]$$

or

$$\xi_a[\mathcal{U}] < \xi_b[\mathcal{U}] \text{ and } \xi_a[\mathcal{W}] > \xi_b[\mathcal{W}].$$

7.9.9 Remarks (i) We have stated the ABP for an index rather than, more generally, for a measure, because in any case it concerns *relative* voting power.

(ii) The *bicameral postulate* proposed in [36] is the same as the ABP, except that instead of applying just to configurations of the form (7.9.6) it applies to all products $\mathcal{W} = \mathcal{U} \times \mathcal{V}$.

However, the arguments for embracing the bicameral postulate are essentially the same (*mutatis mutandis*) as those we shall offer in a moment in favour of the ABP. Moreover, the examples given in [36] of violations of the bicameral postulate are in fact violations of the ABP. For this reason we have chosen to focus the present discussion on the latter postulate.

7.9.10 Comments (i) In our view a reasonable index of a priori voting power ought to satisfy the ABP. In its most general form, the argument in favour of the ABP is that in the configuration (7.9.6) there is nothing at all to imply that the addition of the new blocker v is of greater relative advantage to some of the voters of $\mathcal{U}$ than to others. Of course, as we noted in Rem. 7.4.3, a blocker is a very powerful voter; so v's advent will certainly mean that the powers of all the non-dummies of $\mathcal{U}$ must be reduced compared to what it was in $\mathcal{U}$, because v will now take a share of power. However, we can see no reasonable mechanism that would create a *differential* effect, skewing the distribution of the total power that remains to be shared by $\mathcal{U}$'s voters. After all, the winning coalitions of $\mathcal{W}$ are just those of $\mathcal{U}$, except that v must now be added to each of the latter. If S and T are two winning coalitions of $\mathcal{U}$, then each of them, in order to transform itself into a winning coalition of $\mathcal{W}$, must do exactly the same thing: get v to join. But no *losing* coalition of $\mathcal{U}$ can become a winning coalition of $\mathcal{W}$ in this way.

A comparison with the paradox of new members (§ 7.4) is instructive in this connection. As we noted in Com. 7.4.5, the introduction of the new member has a very uneven effect on the old voters. Thus in Ex. 7.4.1 voter 4, who was originally a dummy, can form a winning coalition with the new member, and is thereby empowered. In Ex. 7.4.4, on the contrary, voters 3 and 4 become dummies due to the advent of the new member. The mechanism by which the new member affects old voters differentially is clear, and the 'paradox' is revealed as an intelligible behaviour of voting power (independent of which index is used to measure it) which, in retrospect, should not have surprised us very much.

Similarly, in the case of the meet paradox (§ 7.7), the formation of the meet $\mathcal{W} = \mathcal{W}_1 \wedge \mathcal{W}_2$ has a differential effect on the powers of the voters of each of the components, $\mathcal{W}_1$ and $\mathcal{W}_2$, precisely because the assemblies of these two SVGs overlap, and those voters who belong to both do not all interact in the same way with voters who belong to only one component.

Nothing analogous seems to be happening in the case of (7.9.6). In fact, the ABP can, loosely speaking, be seen as preventing a special case of the paradox of new members in which the structure of the configuration (7.9.6) rules out the possibility of any mechanism similar to that which operates in the standard examples of the paradox of new members; and the fact that v is new (not a voter of $\mathcal{U}$) rules out any mechanism that operates in examples of the meet paradox.

So far our discussion applies to both I-power and P-power. For a more precise argumentation we must now separate the two cases.

(ii) First let us consider I-power. From the point of view of the voters of $\mathcal{U}$, the addition of the new blocker v imposes a selection on the class of bills in relation to which I-power within $\mathcal{U}$ is distributed. When $\mathcal{U}$ is considered as a free-standing sovereign legislature, all bills brought before it for a vote are relevant for judging the distribution of power among its voters; but when it is made subject to v's veto, only those bills which are approved by v count for the distribution of power among $\mathcal{U}$'s voters. Indeed, if it is known that a given bill has been (or will be) vetoed by v then it is a waste of time for the old voters to consider it. Therefore, if the ratios of the voting powers of $\mathcal{U}$'s voters are altered by the advent of v, this can only mean one thing: that the selection imposed by v is systematically biased compared to the unrestricted situation that prevails when $\mathcal{U}$ is a stand-alone legislature, and this bias favours some voters of $\mathcal{U}$ and disfavours others. A bias of this sort is to be expected in some real-life situations, where voters and groups of voters have, say, particular political inclinations and affinities. Thus, if v were a president with markedly conservative leanings, v's veto would perhaps affect the more conservative voters of $\mathcal{U}$ differently from the more liberal ones. But such selective bias is explicitly ruled out by the very notion of a priori voting power, as explained in Com. 2.2.3. Since we are considering *a priori* I-power, we must take v's veto to have a neutral effect.

The ABP may be viewed as imposing a requirement of independence — 'independence' not in the constitutional or juridical sense of the term, but in a sense akin to that which is formalized in the theory of probability. In the absence of any a priori suppositions about the political or other inclinations of the voters and the particular nature of the issues to be voted upon, it should be assumed that the 'unconditional' ratios of power within the legislature $\mathcal{U}$ are the same as the ratios conditional upon bills being approved by v, who is not one of $\mathcal{U}$'s voters. In fact, when I-power is explicated in probabilistic terms (which, as far as we can see is the only reasonable way to explicate it) we end up with the Bz index β as the yardstick of relative a priori voting power. And the ABP, as applied to β, amounts precisely to the fact that in the relevant Bernoulli space the vote of each voter of $\mathcal{U}$ is probabilistically independent of v's vote.

(iii) Now let us turn to P-power. Here the argument is analogous, but not identical, to the foregoing. The point is that in the configuration 7.9.6 forming a winning coalition of $\mathcal{W}$ is tantamount to forming a winning coalition of $\mathcal{U}$ and adding v to it. Being a blocker, v is in a powerful position; and this position is precisely the same *vis-à-vis* all winning coalitions of $\mathcal{U}$: all of them are in equal need of v's cooperation in order to win in $\mathcal{W}$, and v is clearly indifferent between them, so long as she gets what she regards as her due cut of the (fixed) total loot. In fact, v can sit back and tell the voters of $\mathcal{U}$: 'Such and such is my minimum price; I will consider joining any winning coalition of $\mathcal{U}$ provided it promises to give me at least this much — and the more, the better'. Clearly, the old voters are now in exactly the same relative positions with respect to the remainder of the loot (after deducting v's cut) as they were originally in $\mathcal{U}$ with respect to the whole loot. Note that by condition (3) of Def. 7.1.1 this remainder is presumably still positive. To put it briefly: we cannot see any reason why the bargaining positions of $\mathcal{U}$'s voters *in relation to each other* should be affected in any way by the advent of v. Therefore the ratios of their P-powers ought to remain unchanged.

From Com. 7.9.10(ii) it is clear that the Bz index satisfies the ABP — and it is also obvious why it does so. In fact, we have the following more general result.

7.9.11 Theorem *The Bz index satisfies the bicameral postulate.*

Proof If $\mathcal{W} = \mathcal{U} \times \mathcal{V}$, then $\eta_x[\mathcal{W}] = \eta_x[\mathcal{U}] \cdot |\mathcal{V}|$ for every voter x of $\mathcal{U}$. ∎

We now present several examples showing violation of the ABP by the S-S, D-P and Js indices. For the S-S and Js indices we state the values of the scores κ and JS respectively (see Defs. 6.3.1 and 6.4.5) rather than those of the indices themselves; the relevant index ratios are of course the same as the corresponding score ratios. We leave out the detailed calculations, which are quite simple, although in some cases rather laborious.

Let us start with non-flagrant violations.

7.9.12 Example Take

$$\mathcal{U} = [12; 4, 4, 4, 2, 1, 1, 1], \quad \mathcal{W} = [18; 4, 4, 4, 2, 1, 1, 1, 6].$$

Here voter 8 (with weight 6) is the added blocker. For the S-S scores of voters 1 and 4 we have:

$$\kappa_1[\mathcal{U}] = 1128, \quad \kappa_4[\mathcal{U}] = 792; \quad \kappa_1[\mathcal{W}] = 5760, \quad \kappa_4[\mathcal{W}] = 4320.$$

Hence

$$\frac{\phi_1[\mathcal{U}]}{\phi_4[\mathcal{U}]} = \frac{1128}{792} \approx 1.424, \quad \frac{\phi_1[\mathcal{W}]}{\phi_4[\mathcal{W}]} = \frac{5760}{4320} \approx 1.333.$$

For the D-P index we obtain:

$$\begin{aligned} \text{D-P}_1[\mathcal{U}] &= \tfrac{23}{150}, \quad \text{D-P}_4[\mathcal{U}] = \tfrac{27}{150};\\ \text{D-P}_1[\mathcal{W}] &= \tfrac{15}{120}, \quad \text{D-P}_4[\mathcal{W}] = \tfrac{18}{120}. \end{aligned}$$

Hence

$$\frac{\text{D-P}_1[\mathcal{U}]}{\text{D-P}_4[\mathcal{U}]} \approx 0.852, \quad \frac{\text{D-P}_1[\mathcal{W}]}{\text{D-P}_4[\mathcal{W}]} \approx 0.833.$$

And for the Js score we get:

$$\text{JS}_1[\mathcal{U}] = \tfrac{88}{15}, \quad \text{JS}_4[\mathcal{U}] = \tfrac{42}{15}; \quad \text{JS}_1[\mathcal{W}] = \tfrac{18}{4}, \quad \text{JS}_4[\mathcal{W}] = \tfrac{9}{4}.$$

Hence

$$\frac{\text{JI}_1[\mathcal{U}]}{\text{JI}_4[\mathcal{U}]} = \frac{88}{42} \approx 2.095, \quad \frac{\text{JI}_1[\mathcal{W}]}{\text{JI}_4[\mathcal{W}]} = \frac{18}{9} = 2.$$

Thus all three indices violate the ABP in this example.

7.9.13 Remark There are simpler examples in which the ABP is violated by *two* of these three indices. Thus, the reader is invited to verify that in $\mathcal{U} = [3; 2, 1, 1]$ and $\mathcal{W} = [8; 2, 1, 1, 5]$ the ABP is violated by the S-S and Js indices (but not by the D-P index); whereas in $\mathcal{U} = [6; 5, 4, 1, 1]$ and $\mathcal{W} = [12; 5, 4, 1, 1, 6]$ the ABP is violated by the D-P and Js indices (but not by the S-S index).

We now proceed to examples where the ABP is violated flagrantly. For the D-P index we can still use WVGs.

7.9.14 Example Take

$$\begin{aligned} \mathcal{U} &= [51; 4, 21, 4, 4, 4, 4, 1, 1, 1, 30],\\ \mathcal{W} &= [81; 4, 21, 4, 4, 4, 4, 1, 1, 1, 30, 30]. \end{aligned}$$

Here the added blocker is voter 11. We get:

$$\text{D-P}_1[\mathcal{U}] = \tfrac{3}{28}, \quad \text{D-P}_2[\mathcal{U}] = \tfrac{1}{8}; \quad \text{D-P}_1[\mathcal{W}] = \tfrac{3}{32}, \quad \text{D-P}_2[\mathcal{W}] = \tfrac{1}{12}.$$

Thus

$$\text{D-P}_1[\mathcal{U}] < \text{D-P}_2[\mathcal{U}], \quad \text{D-P}_1[\mathcal{W}] > \text{D-P}_2[\mathcal{W}],$$

in flagrant violation of the ABP.

7.9.15 Remark All the examples given so far in this section have been WVGs, in which the dominance relation $\succeq$ is total, as we saw in Rem. 7.6.4(vii). However, the S-S and Js indices respect dominance, and dominances among the old voters are preserved when a new blocker is added; therefore flagrant violations of the ABP by these indices must involve voters who are incomparable with respect to dominance. In particular, the SVGs cannot be weighted.

7.9.16 Example Let $\mathcal{U}$ be the SVG whose MWCs are

$$\begin{gathered} \{a, c, i\}, \\ \{a, d, e, f, i\}, \quad \{a, d, e, g, i\}, \\ \{b, c, d, i\}, \quad \{b, c, e, i\}, \quad \{b, c, f, i\}, \quad \{b, d, f, i\} \\ \{b, e, f, g, h, i\}, \end{gathered}$$

and let $\mathcal{W}$ be obtained as in (7.9.6) by adding a new blocker v. For the S-S scores of voters a and b we have:

$$\begin{aligned} \kappa_a[\mathcal{U}] &= 39\,744, \quad \kappa_b[\mathcal{U}] = 38\,448; \\ \kappa_a[\mathcal{W}] &= 237\,600, \quad \kappa_b[\mathcal{W}] = 239\,040. \end{aligned}$$

Hence

$$\phi_a[\mathcal{U}] > \phi_b[\mathcal{U}], \quad \phi_a[\mathcal{W}] < \phi_b[\mathcal{W}],$$

in flagrant violation of the ABP.

7.9.17 Example Let $\mathcal{U}$ be the SVG whose MWCs are

$$\begin{gathered}
\{a,c,d,e,f,g,k,l,m\}, \quad \{a,c,d,e,f,h,k,l,m\}, \\
\{a,c,d,e,f,i,k,l,m\}, \quad \{a,c,d,e,f,j,k,l,m\}, \\
\{a,c,d,e,g,h,k,l,m\}, \quad \{a,c,d,e,g,i,k,l,m\}, \\
\{a,c,d,e,g,j,k,l,m\}, \quad \{a,c,d,e,h,i,k,l,m\}, \\
\{a,c,d,e,h,j,k,l,m\}, \quad \{a,c,d,e,i,j,k,l,m\}, \\
\{a,c,d,f,g,i,k,l,m\}, \quad \{a,c,d,f,h,i,k,l,m\}, \\
\{a,c,d,g,h,i,j,k,l,m\}, \quad \{a,d,e,f,g,h,i,j,k,l,m\}, \\
\{b,c,d,j,k,l,m\},
\end{gathered}$$

and again let $\mathcal{W}$ be obtained as in (7.9.6) by adding a new blocker v. For the Js scores of voters a and b we have:

$$\begin{aligned}
\mathrm{JS}_a[\mathcal{U}] &= \tfrac{88573}{13860} \approx 6.391, \quad \mathrm{JS}_b[\mathcal{U}] = \tfrac{45}{7} \approx 6.429; \\
\mathrm{JS}_a[\mathcal{W}] &= \tfrac{39031}{6930} \approx 5.632, \quad \mathrm{JS}_b[\mathcal{W}] = \tfrac{45}{8} = 5.625.
\end{aligned}$$

Hence

$$\mathrm{JI}_a[\mathcal{U}] < \mathrm{JI}_b[\mathcal{U}], \quad \mathrm{JI}_a[\mathcal{W}] > \mathrm{JI}_b[\mathcal{W}],$$

in flagrant violation of the ABP.

7.9.18 Analysis The *technical-arithmetical* reason why the S-S, D-P and Js indices violate the ABP can be readily explained. To this end, let us look closely at (7.9.6) and first observe how the addition of the new blocker v affects the S-S score of a voter a of $\mathcal{U}$.

Let S be a coalition of $\mathcal{U}$ in which a is critical, let $|S| = k$ and let n be the number of voters of $\mathcal{U}$. Then by Def. 6.3.1 the contribution of S to the S-S score of a in $\mathcal{U}$ is $(k-1)!(n-k)!$. This number is relatively large when k has extreme values — near 1 or near n — and is relatively small when k is near $n/2$. In fact, as n increases, the ratio between the highest and lowest values of $(k-1)!(n-k)!$ grows at a roughly exponential rate with n, so the difference between the contributions of very small or very large coalitions and middle-sized ones can be very spectacular indeed. Thus voters can achieve approximately equal S-S scores by being critical in just a few coalitions of extreme size (very small or very large), or by being critical in many middle-sized coalitions.

However, when the new blocker v is added in, this has quite different effects on the contributions of coalitions of different sizes. In $\mathcal{W}$, S is replaced by $S \cup \{v\}$, whose contribution to the S-S score of a is $k!(n-k)!$. This means that the contribution of S to the score of a in $\mathcal{W}$ is increased k-fold compared to that in $\mathcal{U}$. Thus, the large contributions of large coalitions (those with k near n) are boosted by a large factor; the large contributions of small coalitions are boosted by a small factor; and the small contributions of middle-sized coalitions are boosted by a middle-sized factor.

So the effect of the advent of v on the S-S score of voter a depends on the precise mix of sizes of coalitions from which the score of a in $\mathcal{U}$ was derived. It is therefore only to be expected that the S-S scores of non-symmetric voters of $\mathcal{U}$ — scores derived perhaps in very different ways, from coalitions of different sizes — may be affected in different ways.

Broadly similar analysis applies also to the D-P and Js indices, except that here the differential effect of v on the contribution of S depends on S in other ways. In particular, a large minimal coalition S, containing many critical members, tends to make a relatively small contribution to the Js and D-P indices of its members, whereas its contribution to their S-S index is relatively large. These considerations have guided us in discovering the examples presented above.

The Bz index alone is unaffected in any way, because the contribution of S to the Bz score of a is just 1, irrespective of $|S|$ or of the number of other critical members of S.

7.9.19 Comments (i) Since the Bz index β satisfies the ABP, any flagrant violation of the ABP by another index ξ provides an instance in which β and ξ are not co-monotone; that is, an SVG for two of whose voters, say a and b, we have $\beta_a < \beta_b$ while $\xi_a > \xi_b$. It follows that whatever real quantity (if any) is measured by ξ, it cannot possibly be the same — or even of the same kind — as that measured by β, which, as we know, is relative a priori I-power.

Thus Ex. 7.9.16 shows that ϕ and β are not co-monotone.[12] Therefore, whatever ϕ may be measuring, it cannot be the same sort of thing as a priori I-power. (The same also applies to the D-P and Js indices, but these indices in any case stand on much shakier ground, and are therefore of no great interest.)

(ii) If ϕ can still be regarded as a valid index of P-power — and we shall question this in a moment — then we are faced with the following somewhat paradoxical phenomenon. Recall the interpretation of $\beta'_x[\mathcal{W}]$ as the price that voter x of $\mathcal{W}$ can *individually* obtain from a vote-buying outsider (Com. 3.2.15). Recall also that $\phi_x[\mathcal{W}]$ is the Shapley value of x in the *cooperative* vote-selling game $\mathbf{w}^\beta$ (Com. 6.2.24).

It now follows that in some SVGs $\mathcal{W}$ there are voters, say a and b, such that under policy-seeking voting a's vote is worth less than b's vote to an outside buyer, yet under office-seeking voting a's expected payoff is greater than b's, and a's Shapley value in the cooperative vote-selling game $\mathbf{w}^\beta$ is likewise higher than b's.

(iii) In our opinion, the validity of ϕ as an index of a priori P-power must now be questioned. While the S-S index has passed all previous tests required from a reasonable index of P-power, it is now seen to fail the test of the ABP. Therefore those who wish to vindicate ϕ as an index of a priori P-power must offer a convincing refutation of the arguments we have put forward in Com. 7.9.10(i) and (iii).

7.10 Concluding Comments

We end this chapter with comments in which we amplify several conclusions reached earlier, and state an open problem.

[12] The failure of ϕ to be co-monotone with β was known before the issue of the ABP had been raised. In [101, p. 73], Straffin gives an example of an SVG with eight voters, in which this failure occurs (but the ABP is not violated).

7.10.1 Comment First let us return to the classification outlined in Com. 7.1.3, and in its light survey the paradoxes analysed in this chapter.

Our analysis has shown that the paradoxes of large size (§ 7.2), redistribution (§ 7.3), new members (§ 7.4), quarrelling (§ 7.5) and meet (§ 7.7) are rather superficial. By saying this we do not mean to trivialize these phenomena. In fact, studying them and trying to understand their underlying reasons is a very good way for gaining a deeper insight into the true nature of voting power. But once such insight is achieved, the phenomena no longer seem paradoxical.

At the other extreme, the so-called paradox of weighted voting (§ 7.6) is in our view not a real paradox at all, but simply a piece of unreasonable behaviour of the D-P index. It is one among several symptoms that show this index — which lacks plausible justification in the first place — to be unfit for measuring voting power.

The transfer paradoxes (§ 7.8) and violations of the BSP (first part of § 7.9) are phenomena of a third, more subtle kind: they help to discriminate between indices of I-power and P-power. The transfer paradoxes are in our view intolerable for an index of P-power (as well as for a measure of *absolute* I-power). Similarly, violation of the BSP is unacceptable for an index of P-power. This disqualifies the Bz, D-P and Js indices as reliable yardsticks of P-power. The D-P and Js indices, lacking any credibility for measuring I-power, are thus ruled out altogether.

The Bz index survives as an index of I-power. In any case, it has a strong *positive* justification in this capacity. The transfer paradoxes displayed by the Bz index are truly surprising but explicable real phenomena of relative I-power. But it now follows that whatever P-power may be, it is not measured by the Bz index and is not at all the same kind of thing as I-power.

The violation of the ABP by the S-S index poses the most tantalizing dilemma. The D-P and Js indices also violate this postulate, but they suffer from other serious defects as well, so they would have to be ruled out even if our arguments in Com. 7.9.10(i) and (iii) were to be successfully rebutted.

The position of the S-S index is different. If P-power can be measured by any index at all,[13] then surely the S-S index is by far the most serious known candidate. It passes all the tests of this chapter, except for the ABP. Therefore, as we pointed out in Com. 7.9.19(iii), those who wish to vindicate the S-S index as a yardstick of P-power are faced with the challenge of showing that its failure to satisfy the ABP is tolerable. Failing that, the S-S index would have to be rejected. This, in turn, would raise the following alternative: either a priori P-power must be measured by some other index, yet to be discovered, or the very notion of P-power must be condemned as incoherent, or at least unquantifiable by an index.

7.10.2 Comment Whether or not the S-S index is accepted as a valid measure of P-power — indeed, whether or not P-power is a quantifiable concept — it is clear that whatever the S-S index does measure is qualitatively different from what is measured by the Bz index (which is relative a priori I-power).

Perhaps the clearest evidence for this is the fact, to which we called attention in Com. 7.9.19(i), that in some cases the two indices are not co-monotone: in one and the same SVG $\beta_a < \beta_b$ while $\phi_a > \phi_b$.

But there are many other qualitative differences. Thus, as noted in Com. 7.2.4(v), there are instances where ϕ is superadditive while β is not. An even more extreme divergence in behaviour was noted in § 7.8: ϕ is immune to the transfer paradoxes whereas β displays them. Thus it may happen that when voter a annexes voter b, the former loses relative voting power (as measured by β); but he must gain whatever ϕ is a measure of. Similarly, by Thm. 7.9.4, ϕ satisfies the BSP, while β violates it.

On the other hand, in Ex. 7.9.16 we witnessed flagrant violation

[13] This assumption may perhaps be challenged on very general grounds. For example, some approaches to general bargaining theory may insist that in the presence of a blocker all non-blockers' expected payoffs must vanish. This would contradict condition (3) of Def. 7.1.1 and imply that P-power cannot be measured by *any* index in the sense of that definition. In our discussion we put aside such root-and-branch critique, at least for the sake of argument.

of the ABP by ϕ, whereas by Thm. 7.9.11 β is extremely well-behaved in this respect.

These are by no means the only marked differences in behaviour between the two indices.[14] Thus we must conclude that the widespread view that 'there are no significant qualitative differences between the two measures'[15] is erroneous.

7.10.3 Comment Let us recall the postulates we have proposed in this chapter. First, there was the transfer postulate and its logical consequences (see Diagram 7.8.24). Then there were the BSP and ABP, which is a special case of the bicameral postulate. In our view, all these postulates are compelling for a valid index of P-power. Although we have yet to find an index that does satisfy all these postulates, we conjecture that such indices do exist. Our hunch is that a likely candidate is an index $\beta^{(k)}$ obtained by normalizing the k-th powers of the Bz scores of the voters, where k is sufficiently large. Formally, let us put

$$\beta^{(k)}{}_a[\mathcal{W}] =_{\text{def}} \frac{(\eta_a[\mathcal{W}])^k}{\sum_{x \in N}(\eta_x[\mathcal{W}])^k}.$$

For any k, $\beta^{(k)}$ clearly satisfies the bicameral postulate; and it seems likely that if k is chosen sufficiently large, the BSP and the transfer postulate will also be satisfied. Perhaps it is enough to take $k = 2$.

For $k > 0$, $\beta^{(k)}$ has another property that a reasonable index of P-power should perhaps be required to possess: it is co-monotone with β. Although we cannot expect P-power to be strictly proportional to I-power, it is arguable that the two ought to be positively correlated.

Of course, even if an index having all these nice properties can be found, it does not necessarily mean that it is a reasonable index of P-power. Some further justification — ideally, a positive one — would still be required.

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[14] For additional differences see Straffin [102, pp. 1133 f.].

[15] Cf. p. 95, fn. 51.

8. Taking Abstention Seriously[1]

8.1 Why Bother?

As we observed in Com. 2.2.4, the mainstream literature on voting power has confined itself almost exclusively to the SVG model, which does not admit abstention as a *tertium quid* that can affect the outcome of a division differently from both a 'yes' and 'no' vote.[2]

Moreover, real-life decision rules that do treat abstention as a distinct option — whose effect is not always the same as 'yes' or always the same as 'no' — are often mis-reported as though they conform to the binary, SVG model. Thus, decisions in each of the two Houses of the US Congress require the 'yes' vote of a simple majority or, in some cases, two-thirds of the members *present* (provided the members present constitute the needed quorum of a simple majority of the entire membership);[3] but some of the best writers on voting power erroneously substitute the *total* membership for those present.[4] Also, in the UNSC a permanent member's abstention, as distinct from a 'no' vote, does not count as a veto;[5] but the voting-

[1]This chapter is largely based on our [32] and [35].

[2]In the mainstream literature, Fishburn [39, pp. 53–55] is an isolated and brief exception.

[3]See [35] for details, including quotation from US Supreme Court opinion in the case of *Missouri Pacific Railway Co. v State of Kansas*, 248 U.S. 276.

[4]For example, [11, p. 192], [16, p. 62], [59, p. 235], [66, p. 212], [95, p. 59], [97, p. 789] and [103, p. 46].

[5]For an authoritative discussion of this see Simma [99, pp. 447 ff.], summa-

power literature mostly gets this fact wrong.[6] In [32] and [35] we hypothesize that the mis-reporting is due to what philosophers of science have called *theory-laden* (or *theory-biased*) *observation* — a common occurrence, akin to optical illusion, whereby an observer's perception is unconsciously distorted so as to fit a preconception.

How do writers on voting power justify their practice of confining themselves to binary theoretical models, which do not admit abstention? Banzhaf devotes to this just two sentences in a footnote:

> This analysis has also assumed that all legislators are voting because this is the most effective way for each legislator to exercise his power. Naturally, some may choose to exercise their power in a less effective manner by abstaining or by being absent from the legislative chamber.[7]

Banzhaf's argument for disregarding abstentions seems to us inadequate, as we shall explain below. But at least he does not ignore the whole matter, as do other writers.

In the published literature, as far as it is known to us, we have not found any other attempt to provide theoretical justifications for disregarding abstentions. But when we raised this issue at an inter-disciplinary seminar, some of the game theorists in the audience reacted rather heatedly with a somewhat more elaborate form of Banzhaf's argument, which may be paraphrased as follows.

> The study of voting power belongs to game theory; more specifically, it is a branch of the theory of cooperative games. Game theory is a theory of rational behaviour. Abstaining voters are not behaving rationally, because they are not using their powers to the full. Therefore such behaviour ought to be disregarded by the theory.

As we admitted in Com. 2.2.4, this argument is plausible — but only as regards P-power. It does not apply at all to I-power, which

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rized by us in [35].

[6] For example, [11, pp. 182 ff.], [16, p. 58], [20, pp. 274, 283], [59, p. 230], [66, p. 196], [83, pp. 218 f.], [85, p. 52] and [95, p. 65].

[7] [5, fn. 34].

has little to do with cooperative game theory. In fact, this argument is a revealing instance of the common failure to distinguish between the two different notions of voting power. A consequence of this failure is the widespread misconception that the *whole* of the theory of a priori voting power is a branch of cooperative game theory. This goes hand in hand with the mistaken view, accepted even by Banzhaf himself, that 'there are no significant qualitative differences' between the Bz and S-S indices.[8]

From a policy-seeking perspective on voting, the argument for disregarding abstention in theory loses most of its force. There may be several reasons why a voter would prefer to abstain on a given bill. One reason can be the wish to use abstention as a way of making a public statement: the voter expects to derive some benefit from *being seen to abstain.* Such abstention should perhaps be disregarded by the theory of voting power, because it depends on the propaganda advantage of abstention itself, as a kind of side payment. Note that abstention for propaganda cannot operate if voting is secret. But in our view there are also other reasons for abstaining, which can operate even when voting is secret. A voter may be indifferent to the bill, because his or her interests are not affected by it in any way. (This is a reasonable motive for abstention by absence, particularly if participation in the division involves some cost.) Or the arguments for and against the bill — the estimates of the payoff to the given voter in case the bill is adopted or rejected — may be so finely balanced that the voter is unable to decide one way or the other. Is it so irrational to abstain for these reasons? It is a bizarre kind of rationality that would require you to cast a 'yes' or 'no' vote even when you couldn't care less, or when you were not sure whether passage of the bill would serve your interests better than its defeat!

The study of voting power is a branch of social-choice theory. In other branches of the theory — for example, in the study of social choice functions — it is usual practice to admit individual preference rankings that are not totally ordered but rank two or more outcomes (or candidates) as coequal. It is not that ques-

[8]Cf. p. 95, fn. 51 and Com. 7.10.2.

tions of individual rationality are ignored: for example, it is often argued and widely accepted that non-transitive individual preference rankings ought to be disallowed, precisely on the ground that they are not rational. But to the best of our knowledge there are not many social-choice theorists who would condemn as irrational an individual voter who does not wish or is unable to choose between Tweedledum and Tweedledee, or even between the Walrus and the Carpenter. Social-choice theorists explicitly recognize that voters may prefer to abstain rather than select arbitrarily one of the alternatives among which they are indifferent. Why should the theory of voting power be different in this respect?

In this chapter we will outline the rudiments of a theory of a priori voting power that takes abstention seriously.

8.2 Ternary Voting Rules

In § 2.1 we defined an SVG as a collection of winning coalitions, which are certain subsets of an assembly N (Def. 2.1.1). We saw that such a structure may also be described by means of two other mathematical constructs: to each SVG there corresponds a unique characteristic function (Def. 2.1.3) and a unique BPR (Def. 2.1.5).

Each of these three mutually equivalent notions has an analogue that can be used to model decision rules that admit abstention as a distinct option. Of these three equivalent approaches, the simplest one technically is the third. So we take our cue from Def. 2.1.5 and Rem. 2.1.6(iii).

8.2.1 Definition (i) A *tripartition* of a set N is a map T from N to $\{-1, 0, 1\}$. We denote by 'T^-', 'T^0' and 'T^+' the inverse images of $\{-1\}$, $\{0\}$, and $\{1\}$ respectively under T:

$$T^0 = \{x \in N : Tx = 0\},$$
$$T^- = \{x \in N : Tx = -1\}, \quad T^+ = \{x \in N : Tx = 1\}.$$

We define a partial ordering $\leq$ among tripartitions: if T_1 and T_2

are two tripartitions of N, we put:

$$T_1 \leq T_2 \Leftrightarrow_{\text{def}} T_1 x \leq T_2 x \text{ for all } x \in N.$$

(ii) Let N be nonempty finite set. By a *ternary voting rule*—briefly, *TVR*—with N as *assembly* we mean any map W from the set ${}^N\{-1, 0, 1\}$ of all tripartitions of N to $\{-1, 1\}$, satisfying the following three conditions:

(1) $T^+ = N \Rightarrow \mathsf{W}T = 1$;

(2) $T^- = N \Rightarrow \mathsf{W}T = -1$;

(3) *Monotonicity*: $T_1 \leq T_2 \Rightarrow \mathsf{W}T_1 \leq \mathsf{W}T_2$.

We say that members of N are *voters* of W; and a set of voters is a *coalition*. We call $\mathsf{W}T$ the *outcome* of T (under W).

(iii) By the *dual* of a TVR W we mean the TVR W^* with the same assembly N as W, such that $\mathsf{W}^*T = -\mathsf{W}(-T)$ for every tripartition T of N. Here $-T$ is the tripartition obtained from T in the obvious way: $(-T)x = -(Tx)$ for all $x \in N$.

8.2.2 Remarks (i) Like a bipartition, a tripartition T models a division of a board; but now, in addition to the sets T^- and T^+ representing the sets of 'no' and 'yes' voters, respectively, we have a third set T^0 of abstainers. Thus tripartitions are *three* valued: the value Ta may be -1, 0 or 1—standing for 'no', abstention and 'yes', respectively.

(ii) On the other hand, TVRs are still *two* valued: the outcome $\mathsf{W}T$ must be either -1 (the bill is defeated) or 1 (the bill is passed); here *tertium non datur*.

However, for some purposes one needs to admit 'para-TVRs', which are like TVRs except that they are allowed to take a third value, 0, representing a tie. Thus, if we wish to define a composite TVR $\mathsf{W} = \mathsf{V}[\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_m]$ to model a two-tier decision-making system, then we can allow the components W_i to be para-TVRs, because they only model interim decisions, at the bottom tier, where a tie is admissible. But the top, V, and hence the composite

W as a whole, must be TVRs proper, because the final outcome has to be a definite failure or success.

In the present account we shall not discuss composites, so we do not need para-TVRs.

(iii) In connection with TVRs we shall use terminological and notational conventions similar to those we have used for SVGs. So, for example, where the context allows it we shall refer to voters of a particular TVR W simply as 'voters', suppressing the reference to W. Also, we always assume that $|N| = n$.

8.2.3 Definition (i) The *sum* and *margin* of a tripartition T of a finite set N are defined in exactly the same way as for bipartitions (see Def. 2.3.8):

$$\mathsf{S}T =_{\text{def}} \sum_{x \in N} Tx, \quad \mathsf{M}T =_{\text{def}} |\mathsf{S}T|.$$

(ii) A TVR W is said to be a *majority* TVR if $\mathsf{W}T = 1 \Leftrightarrow \mathsf{S}T > 0$ for every bipartition T of the assembly.

8.2.4 Remark The meanings of S and M are exactly the same as for bipartitions (see Rem. 2.3.9), with the proviso that now the terms *majority*, *minority* and *margin* refer only to the decided voters, ignoring the abstainers. Thus, under a majority TVR a bill is passed iff more vote for it than against it.

8.2.5 Definition Let a be a voter of a TVR W and let T be a tripartition of the assembly N.

(i) If $Ta = \mathsf{W}T$, we say that a *agrees* with the outcome of T under W. Moreover, we say that a agrees with the outcome of T *negatively* or *positively*, according as the common value of Ta and $\mathsf{W}T$ is -1 or 1. If $Ta = -\mathsf{W}T$, we say that a *disagrees* with the outcome of T under W.

(ii) If $Ta \leq 0$, we denote by '$T_{a\uparrow}$' the tripartition of N such that $T_{a\uparrow}a = Ta + 1$ but $T_{a\uparrow}x = Tx$ for all other $x \in N$. If, moreover,

$\mathsf{W}T = -1$ and $\mathsf{W}T_{a\uparrow} = 1$ we say that a is *negatively* W-*critical for* T.

Similarly, if $Ta \geq 0$, we denote by '$T_{a\downarrow}$' the tripartition such that $T_{a\downarrow}a = Ta - 1$ but $T_{a\downarrow}x = Tx$ for all other $x \in N$. And if, moreover, $\mathsf{W}T = 1$ and $\mathsf{W}T_{a\downarrow} = -1$ we say that a is *positively* W-*critical for* T.

We say that a is W-*critical for* T if a is negatively or positively W-critical for T.

8.2.6 Remarks (i) Speaking less formally, a agrees with the outcome of T if the outcome goes the way a has voted; and a disagrees with that outcome if it goes the opposite way. Note that if $Ta = 0$ — that is, if a abstains in the tripartition T — then a neither agrees nor disagrees with the outcome: in a sense, this is what abstention is all about.

(ii) The tripartition $T_{a\uparrow}$ is obtained from T by a upgrading his attitude to the bill from 'no' to abstention, or from abstention to 'yes'. And a is negatively critical for T iff such upgrading would change the outcome from failure to success.

The mirror image of this applies to $T_{a\downarrow}$ and to a being positively critical for T.

(iii) If we were to admit 'para-TVRs' in the sense of Rem. 8.2.2(ii), then we would have to distinguish two degrees of criticality. For example, suppose $\mathsf{W}T = -1$. If $\mathsf{W}T_{a\uparrow} = 0$, then a should count as *singly* critical for T; but if $\mathsf{W}T_{a\uparrow} = 1$, then a should count as *doubly* critical for T. This complication is avoided in the present outline.

To conclude this section, we introduce the quantity Z in the same way, *mutatis mutandis*, as in Def. 3.3.4.

8.2.7 Definition For any TVR W we define the random variable $\mathsf{Z}[\mathsf{W}]$ by stipulating that for any tripartition T of the assembly N of W, the value of $\mathsf{Z}[\mathsf{W}]$ at T is the number of voters who agree with the outcome of T under W minus the number of voters who disagree with the outcome of T under W.

8.2.8 Remark Clearly, if W is a majority TVR then $\mathsf{Z}[\mathsf{W}]$ coincides with the margin M.

8.3 Voting Power under TVRs

A reasonable Bz measure for TVRs ought to have probabilistic meaning similar to that of the Bz measure for SVGs. So we must start by defining a ternary analogue of the Bernoulli space $\mathbf{B}_N$ (see Def. 3.1.1).

8.3.1 Definition Let N be a finite set, with $|N| = n$. The *ternary space* $\mathbf{T}_N$ is the probability space consisting of the set ${}^N\{-1, 0, 1\}$ of all tripartitions of N, with each tripartition assigned the same probability: $1/3^n$.

Where there is no risk of confusion, we omit the subscript 'N' in '$\mathbf{T}_N$' and write simply '$\mathbf{T}$'.

8.3.2 Comment The intended interpretation of $\mathbf{T}$ is clearly analogous to that $\mathbf{B}$. But in the present case justification of this probabilistic model is more problematic.

The assumption of independence between voters can be justified on the same aprioristic grounds we put forward in Com. 3.1.3 in the case of $\mathbf{B}$. This holds as well for the assignment of equal a priori probabilities to 'no' and 'yes' votes, because clearly these are a priori symmetric to each other. But the assignment of the same a priori probability to abstention as to each of the other two options is much less compelling, because it is not at all self-evident that abstention is symmetric to 'no' and 'yes'.

A more cautious approach would be to take the probability p of abstention as an undetermined parameter in the general theory, leaving $1 - p$ to be shared equally between 'no' and 'yes'. In any application of the theory, the value of p might be determined on the basis of specific arguments, including perhaps empirical data.

However, in the present account, which is intended primarily for illustration, we take the simplest course of assuming $p = \frac{1}{3}$.

8.3.3 Convention In this section, when we use probabilistic terms, unless stated otherwise the space we have in mind is $\mathbf{T}_N$, where N is the assembly of the TVR under consideration. In particular, 'P' will denote the probability function (of events) in $\mathbf{T}_N$; and 'E' will denote the mathematical expectation (expected value) operator in this space.

8.3.4 Definition Let W be a TVR with assembly N and let $a \in N$. We define the *Bz score* $\eta[\mathsf{W}]$ by stipulating that $\eta_a[\mathsf{W}]$ is the number of tripartitions of N for which a is positively W-critical.

We define the *Bz index of voting power* $\beta[\mathsf{W}]$ by putting

$$\beta_a[\mathsf{W}] =_{\text{def}} \frac{\eta_a[\mathsf{W}]}{\sum_{x \in N} \eta_x[\mathsf{W}]}.$$

We define the *Bz measure of voting power* $\beta'[\mathsf{W}]$ by putting

$$\beta'_a[\mathsf{W}] =_{\text{def}} \frac{\eta_a[\mathsf{W}]}{3^{n-1}}.$$

Here, as usual, $n = |N|$ is the number of voters of W.

8.3.5 Theorem $\mathrm{P}(a \text{ is } \mathsf{W}\text{-critical}) = \frac{2}{3}\beta'_a[\mathsf{W}]$.

Proof Clearly, $\eta_a[\mathsf{W}]$ is also the number of tripartitions of N for which a is negatively W-critical. ∎

8.3.6 Remarks (i) This result is not quite the obvious analogue of Thm. 3.2.4. However, in strict analogy to the case of SVGs, it is true that $\beta'_a[\mathsf{W}]$ exactly equals the probability that in a randomly chosen tripartition a will be in a position to reverse the outcome by altering his vote.

The reason for this apparent discrepancy is that if $Ta = -1$ and $\mathsf{W}T = -1$, and if moreover $\mathsf{W}T_{a\uparrow a\uparrow} = 1$ (that is, by changing his vote from 'no' to 'yes' a reverses the outcome) then a is critical for only two of the three tripartitions T, $T_{a\uparrow}$ and $T_{a\uparrow a\uparrow}$ (negatively critical for one and positively for another). Note that in this case a must be (negatively or positively) critical for $T_{a\uparrow}$, the middle one of the three tripartitions: the one in which a abstains.

(ii) From what we have just seen it follows at once that a is (negatively or positively) critical for exactly $\eta_a[\mathsf{W}]$ tripartitions in which a abstains.

8.3.7 Example Let W be the TVR with assembly $\{a, b, c\}$, where a bill is passed iff a votes for it and at least one of the other two does not oppose it. Thus $\mathsf{W}T = 1$ iff $Ta = 1$, while Tb and Tc are not both -1. There are eight such tripartitions, and a is positively critical in all of them. Voter b is positively critical in just one tripartition (that in which $Ta = 1$, $Tb = 0$ and $Tc = -1$). Similarly for c. Thus $\beta'_a = \frac{8}{9}$ and $\beta'_b = \beta'_c = \frac{1}{9}$.

The values of β are obtained by normalization: $\beta_a = \frac{4}{5}$ and $\beta_b = \beta_c = \frac{1}{10}$. Note that no SVG with three voters yields these values of β.

8.3.8 Example In an illustrative exercise Straffin [100, pp. 314–315] finds the values of the Bz index for the UNSC, modelled as an SVG. (Since 1966, there are 15 members, yielding $2^{15} = 32\,768$ bipartitions.) The resulting values of β are approximately 0.1669 for each of the five permanent members and 0.0165 for each of the 10 ordinary members. However, this modelling is based on the incorrect assumption that abstention by a permanent member is tantamount to a 'no' vote, having the effect of a veto.

Using the more realistic TVR model (with $3^{15} = 14\,348\,907$ tripartitions), we get as the values of β approximately 0.1009 and 0.0495 for a permanent member and ordinary member, respectively.

It could be argued that since abstention by an ordinary member has exactly the same effect as a 'no' vote, these members have in effect just two voting options — 'no' and 'yes' — whereas only for the permanent members is abstention a distinct *tertium quid*. One can extend the definition of β to a hybrid ternary/binary model, which in the case of the UNSC has $3^5 2^{10} = 248\,832$ divisions. Using this hybrid model we get as the values of β approximately 0.1038 and 0.0481 for a permanent member and ordinary member, respectively. Note that these values are much closer to those obtained with the pure TVR model than with the pure SVG model.

Several theorems proved in Ch. 3 have analogues for TVRs. We present here a small selection, outlining the proofs and leaving it to the reader to fill in some simple details. As an analogue of Thm. 3.2.16 we have

8.3.9 Theorem

$$\mathrm{P}(a \text{ agrees with outcome under } \mathsf{W}) = \frac{1 + \beta'_a[\mathsf{W}]}{3}.$$

Proof We have just seen (Rem. 8.3.6(ii)) that a is (negatively or positively) critical for exactly $\eta_a[\mathsf{W}]$ tripartitions in which a abstains.

If T is one of these $\eta_a[\mathsf{W}]$ tripartitions in which a abstains and for which he is critical, then a agrees with the outcomes of both $T_{a\downarrow}$ and $T_{a\uparrow}$.

On the other hand, if T is one of the remaining $3^{n-1} - \eta_a[\mathsf{W}]$ tripartitions in which he abstains (but for which he is not critical), then a agrees with the outcomes of just one of the tripartitions $T_{a\downarrow}$ and $T_{a\uparrow}$.

All in all, a agrees with the outcomes of $3^{n-1} + \eta_a[\mathsf{W}]$ tripartitions. ∎

8.3.10 Definition For any TVR W we put:

$$\Sigma[\mathsf{W}] \ =_{\mathrm{def}} \ \sum_{x \in N} \beta'_x[\mathsf{W}].$$

We call $\Sigma[\mathsf{W}]$ the *sensitivity* of W. Further, we denote by '$\widehat{\Sigma}_n$' the sensitivity of a majority TVR with n voters.

8.3.11 Theorem *For any TVR* W*,*

$$\mathrm{E}(\mathsf{Z}[\mathsf{W}]) = \frac{2\Sigma[\mathsf{W}]}{3}.$$

Proof Argue as in the proof of Thm. 3.3.5. Use Thm. 8.3.9. ∎

By virtue of Rem. 8.2.8 we have:

8.3.12 Corollary

$$\mathrm{E}(\mathsf{M}) = \frac{2\widehat{\Sigma}_n}{3},$$

where n is the number of voters. ∎

8.3.13 Definition For any TVR W with assembly N, we define the random variable $\mathsf{D}[\mathsf{W}]$, called the *majority deficit* [of W], on the space $\mathbf{T}_N$ by stipulating that the value of $\mathsf{D}[\mathsf{W}]$ at any tripartition T of N equals the size of the majority camp among the non-abstainers in T minus the number of voters who agree with the outcome of T under W.

Further, we put $\Delta[\mathsf{W}] =_{\text{def}} \mathrm{E}(\mathsf{D}[\mathsf{W}])$. We call $\Delta[\mathsf{W}]$ the *mean majority deficit* of W.

8.3.14 Theorem *For any TVR W with n voters,*

$$\Delta[\mathsf{W}] = \frac{\widehat{\Sigma}_n - \Sigma[\mathsf{W}]}{3}.$$

Proof The size of the majority camp among the non-abstainers is

$$\frac{\mathsf{X} + \mathsf{M}}{2},$$

where X is the number of non-abstainers (which is a random variable). Similarly, the number of voters who agree with the outcome is

$$\frac{\mathsf{X} + \mathsf{Z}}{2}.$$

Hence, as in the proof of Thm. 3.3.17,

$$\mathsf{D} = \frac{\mathsf{M} - \mathsf{Z}}{2}.$$

The claim of our theorem now follows by applying the operator E to both sides and using Thm. 8.3.11 and Cor. 8.3.12. ∎

Since the majority deficit D is always non-negative, we obtain:

8.3.15 Theorem *Let N be a finite set with $|N| = n \geq 1$. Among all TVRs W that have N as assembly, $\Sigma[\mathsf{W}]$ attains its maximal value iff $\Delta[\mathsf{W}] = 0$. This is the case iff W satisfies the following two conditions:*

(1) $\mathsf{S}T < 0 \Rightarrow \mathsf{W}T = -1$;

(2) $\mathsf{S}T > 0 \Rightarrow \mathsf{W}T = 1$.

Moreover, the maximal value of $\Sigma[\mathsf{W}]$ is $\widehat{\Sigma}_n$. ∎

8.3.16 Remark Stated more simply, the theorem says that of all TVRs with a given assembly, the ones that maximize sensitivity are those for which the MMD vanishes: namely, those under which the outcome is negative whenever more vote 'no' than 'yes' and positive whenever more vote 'yes' than 'no'. Among the TVRs satisfying these conditions is the majority TVR; but it is not the only one, because the theorem imposes no condition on tripartitions in which there are the same number of 'no' and 'yes' voters.

Let us now turn to the S-S index.

8.3.17 Comment In § 8.1 we admitted that as far as office-seeking voting behaviour is concerned the argument for ignoring abstention has some merit. Therefore it would seem that extending ϕ — which is supposed to be an index of a priori P-power — to TVRs serves little purpose. However, the problem of defining such an extension is at the very least of some technical mathematical interest. Besides, we saw in Com. 6.3.16 that ϕ makes good sense as a measure of I-power, albeit not a priori but subject to the condition that the voters act as clones.

We therefore outline very briefly how the S-S index can be extended to TVRs.

8.3.18 Outline Def. 6.3.1 and Thm. 6.3.7 do not lend themselves to emulation for TVRs. However, Thm. 6.3.8, which characterizes the S-S index using probability in the roll-call space $\mathbf{R}_N$, suggests a very natural analogue.

First, we define the *ternary roll-call space* $\widehat{\mathbf{R}}_N$ in the same way as $\mathbf{R}_N$ (Def. 6.2.16), but replacing $\mathbf{B}_N$ by $\mathbf{T}_N$:

$$\widehat{\mathbf{R}}_N =_{\text{def}} \mathbf{Q}_N \times \mathbf{T}_N.$$

Formally, each ternary roll-call R is an ordered pair $\langle \mathrm{q}R, \mathrm{t}R\rangle$ where $\mathrm{q}R$ is a queue of N and $\mathrm{t}R$ a tripartition of N. We can think of R as the voters queuing up to vote; $\mathrm{q}R$ tells us how the voters are ordered, and $\mathrm{t}R$ how each of them votes (-1, 0 or 1 for 'no', 'abstain' and 'yes' respectively). All $n!3^n$ ternary roll-calls are assigned equal probability.

If W is a TVR with assembly N and R is a ternary roll-call of N then the W-*pivot of* R, denoted by '$\mathrm{piv}(R;\mathsf{W})$' is uniquely defined as the voter whose vote in R clinches the outcome of $\mathrm{t}R$ under W. In other words, $\mathrm{piv}(R;\mathsf{W})$ is the earliest voter in $\mathrm{q}R$ such that no matter how all subsequent voters were to change their votes, the outcome would still be the same as that of $\mathrm{t}R$.

We can now define:

$$\phi_a[\mathsf{W}] =_{\text{def}} \frac{1}{n!3^n}\left|\{R \in \widehat{\mathbf{R}}_N : \mathrm{piv}(R;\mathsf{W}) = a\}\right|.$$

So

$$\phi_a[\mathsf{W}] = \mathrm{P}(a \text{ is the } \mathsf{W}\text{-pivot}),$$

where P is the probability distribution in the ternary roll-call space $\widehat{\mathbf{R}}_N$.

8.3.19 Example Consider the TVR of Ex. 8.3.7. Voter a is pivotal in the following events.

(1) a votes first and does not vote 'yes'. The probability of this is $\frac{2}{9}$.

(2) a votes second, the first voter voted 'no' and a does not vote 'yes'. This has probability $\frac{2}{27}$.

(3) a votes second and the first voter did not vote 'no'. The probability of this is $\frac{6}{27}$.

(4) a votes last, and the other two did not both vote 'no'. This has probability $\frac{8}{27}$.

Thus $\phi_a = \frac{22}{27}$; hence $\phi_b = \phi_c = \frac{5}{54}$. Again, note that no SVG with three voters can yield such values of ϕ (because they are not integral multiples of $\frac{1}{6}$).

8.3.20 Example In [100, p. 314 f.], using the (inappropriate) SVG model, Straffin finds the following values of ϕ for the UNSC: 0.1963 and 0.0019 for a permanent and ordinary member, respectively.

Modelling the UNSC as a TVR, we obtain 0.1636 and 0.0182 respectively. And using the hybrid model (see Ex. 8.3.8) we get 0.1762 and 0.0119 respectively. Again, the values for the hybrid model are much closer to those for the pure TVR model.

We shall go no further into the theory of the S-S index for TVRs, except to note the following interesting result.

8.3.21 Remark With each SVG $\mathcal{W}$ with assembly N we can associate in a rather simple-minded way a TVR $\overline{\mathsf{W}}$ with the same assembly: put

$$\overline{\mathsf{W}}T = 1 \Leftrightarrow_{\text{def}} T^+ \in \mathcal{W}$$

for any tripartition T of N. Thus $\overline{\mathsf{W}}$ is a somewhat degenerate TVR that treats abstention in the same way as a 'no'. Its outcome is positive iff the set T^+ of 'yes' voters is a winning coalition of $\mathcal{W}$.

Now, it turns out that $\phi_a[\mathcal{W}] = \phi_a[\overline{\mathcal{W}}]$ for every $a \in N$. This follows quite easily from Thm. 6.3.8 by taking $p = \frac{1}{3}$.

Of course, instead of using $\overline{\mathsf{W}}$, we could similarly associate with $\mathcal{W}$ a TVR that treats abstention as a 'yes'. But this would again yield the same S-S values: this time we use Thm. 6.3.8 with $p = \frac{2}{3}$.

Note, by the way, that there is no similar result for the Bz index.

This concludes our cursory introduction to the theory of voting power in TVRs. This theory is in its infancy, and much remains to be discovered.

A. Appendix: Numerical Examples

In Ex. A.1 we illustrate the calculation of the Bz measure and Bz index, using a fairly simple but non-trivial WVG with four voters. In Ex. A.2 we illustrate the calculation of the Shapley value, using a game with four players. In Ex. A.3 we return to the WVG of Ex. A.1 and calculate the S-S index for it. In Ex. A.4 we calculate the D-P and Js indices for the same WVG.

A.1 Example (Bz Measure and Index) Let $\mathcal{W}$ be a WVG with assembly $N = \{a, b, c, d\}$ that is isomorphic, in alphabetic order, to $[5; 3, 2, 1, 1]$.

To compute the Bz power of each of the voters, you must first list the coalitions of $\mathcal{W}$ and their critical members. Fortunately, you do not have to list all 16 coalitions: a coalition need be listed only if it has at least one critical member. Such a coalition is said to be *vulnerable*. Here are the five vulnerable coalitions of $\mathcal{W}$, listed in order of size, with their critical members underlined:

$$\{\underline{a}, \underline{b}\},\ \{\underline{a}, \underline{b}, c\},\ \{\underline{a}, \underline{b}, d\},\ \{\underline{a}, \underline{c}, \underline{d}\},\ \{\underline{a}, b, c, d\}.$$

By Def. 3.2.2, the Bz power, β'_x, of voter x is equal to the number of [vulnerable] coalitions in which x is critical, divided by 2^{n-1}. Here $n = 4$, so in our WVG we have:

$$\beta'_a = \tfrac{5}{8},\ \beta'_b = \tfrac{3}{8},\ \beta'_c = \beta'_d = \tfrac{1}{8}.$$

Note that you need not repeat the calculation separately for c and d: these voters are symmetric to each other, so automatically $\beta'_c = \beta'_d$.

You obtain the values of the Bz index, β_x, by normalizing the Bz power values: dividing each Bz power by the sum of all Bz powers. In the present case the sum is $\frac{10}{8}$; so we get:

$$\beta_a = \tfrac{1}{2},\ \beta_b = \tfrac{3}{10},\ \beta_c = \beta_d = \tfrac{1}{10}.$$

Of course, these values add up to 1.

A.2 Example (Shapley value) Let $\mathbf{v}$ be a game (see Def. 6.2.1) with grand coalition $N = \{a, b, c, d\}$, such that

$$\begin{gathered}
\mathbf{v}\{a\} = \tfrac{5}{8},\ \mathbf{v}\{b\} = \tfrac{3}{8},\ \mathbf{v}\{c\} = \mathbf{v}\{d\} = \tfrac{1}{8},\\
\mathbf{v}\{a,b\} = 1,\ \mathbf{v}\{a,c\} = \mathbf{v}\{a,d\} = \tfrac{3}{4},\\
\mathbf{v}\{b,c\} = \mathbf{v}\{b,d\} = \tfrac{1}{2},\ \mathbf{v}\{c,d\} = \tfrac{1}{4},\\
\mathbf{v}\{a,b,c\} = \mathbf{v}\{a,b,d\} = \mathbf{v}\{a,c,d\} = 1,\ \mathbf{v}\{b,c,d\} = \tfrac{1}{2},\\
\mathbf{v}\{a,b,c,d\} = 1.
\end{gathered}$$

We have listed only the worths of the 15 nonempty coalitions; the worth of $\emptyset$ is automatically 0 by Def. 6.2.1.

There is a very long way and a somewhat less long way for computing the Shapley value $\phi_x(\mathbf{v})$ for each player x of $\mathbf{v}$. The very long way is to use Thm. 6.2.6. This requires you to list all 24 queues of N, and to find the contribution of x to the worth of x's head in each queue; the average of these contributions is $\phi_x(\mathbf{v})$. You can do it this way if you like.[1]

The somewhat less long way is to use Def. 6.2.3 directly. In order to calculate $\phi_x(\mathbf{v})$, you need to find the contribution of x to the worth of each coalition X containing x and multiply it by the coefficient $\big(|X| - 1\big)!\big(4 - |X|\big)!$ (because here $n = 4$); then add up all these products and divide by 24 (because $4! = 24$).

Let us do it for a. The contribution of a to the worth of $\{a\}$ is $\frac{5}{8}$, the entire worth of $\{a\}$; because if you take a out, what is left is the empty coalition, worth 0. The coefficient here is $0!3! = 6$. The contribution of a to the worth of $\{a, b\}$ is also $\frac{5}{8}$; because this

[1] If you wish to do penance for some sin, you can even do it the very-very long way, using Thm. 6.2.18, which requires you to list all 384 roll-calls of N.

coalition is worth 1, and if you take a out the remaining coalition, $\{b\}$, is worth $\frac{3}{8}$. The coefficient here is $1!2! = 2$. Let us look at $\{a, b, c\}$. This is worth 1, and if you take a out you get $\{b, c\}$, which is worth $\frac{1}{2}$; so a's contribution is $\frac{1}{2}$. The coefficient is $2!1! = 2$.

Going through all the coalitions to which a belongs, in the order in which they are listed above, we get the sum

$$\tfrac{5}{8} \cdot 6 + \tfrac{5}{8} \cdot 2 + \tfrac{5}{8} \cdot 2 + \tfrac{5}{8} \cdot 2 + \tfrac{1}{2} \cdot 2 + \tfrac{1}{2} \cdot 2 + \tfrac{3}{4} \cdot 2 + \tfrac{1}{2} \cdot 6 = 14.$$

Hence $\phi_a(\mathbf{v}) = \frac{14}{24} = \frac{7}{12}$. In the same way we obtain for b the sum

$$\tfrac{3}{8} \cdot 6 + \tfrac{3}{8} \cdot 2 + \tfrac{3}{8} \cdot 2 + \tfrac{3}{8} \cdot 2 + \tfrac{1}{4} \cdot 2 + \tfrac{1}{4} \cdot 2 + \tfrac{1}{4} \cdot 2 + 0 \cdot 6 = 6.$$

Hence $\phi_b(\mathbf{v}) = \frac{6}{24} = \frac{1}{4}$. For c we obtain

$$\tfrac{1}{8} \cdot 6 + \tfrac{1}{8} \cdot 2 + \tfrac{1}{8} \cdot 2 + \tfrac{1}{8} \cdot 2 + 0 \cdot 2 + \tfrac{1}{4} \cdot 2 + 0 \cdot 2 + 0 \cdot 6 = 2.$$

Hence $\phi_c(\mathbf{v}) = \frac{2}{24} = \frac{1}{12}$. There is no need to do the calculation for d, because d is symmetric to c, so automatically $\phi_d(\mathbf{v}) = \frac{1}{12}$ as well. To sum up:

$$\phi_a(\mathbf{v}) = \tfrac{7}{12},\ \phi_b(\mathbf{v}) = \tfrac{1}{4},\ \phi_c(\mathbf{v}) = \phi_d(\mathbf{v}) = \tfrac{1}{12}.$$

Note that the Shapley values for the four players add up to 1 — as they should by the efficiency of ϕ (Thm. 6.2.11.)

A.3 Example (S-S Index) We now return to $\mathcal{W}$ of Ex. A.1 and calculate the S-S index. By Rem. 6.3.2(ii), $\phi_a[\mathcal{W}] = \phi_a(\mathbf{w})$, where $\mathbf{w}$ is the CF of $\mathcal{W}$. Therefore the procedure here is essentially the same as in Ex. A.2. But here life is a bit easier, because $\mathbf{w}$ takes just two values: 1 for a winning coalition and 0 for a losing one. It follows that a voter x can only make a contribution to the worth of a coalition X if x is critical in X. So you only need to consider the five vulnerable coalitions:

$$\{\underline{a}, \underline{b}\},\ \{\underline{a}, \underline{b}, c\},\ \{\underline{a}, \underline{b}, d\},\ \{\underline{a}, \underline{c}, \underline{d}\},\ \{\underline{a}, b, c, d\}.$$

Moreover, the contribution of a critical member x of a coalition X to its worth is always 1. The coefficient by which you need to multiply this contribution is $\big(|X|-1\big)!\big(4-|X|\big)!$, as in Ex. A.2. Taking the five coalitions in the order in which they are listed, we get for a the sum

$$2+2+2+2+6=14.$$

Hence $\phi_a = \frac{14}{24} = \frac{7}{12}$. For b we get the sum

$$2+2+2=6.$$

So $\phi_b = \frac{1}{4}$. Voter c is critical only in $\{a,c,d\}$, and this single contribution of 1 must be multiplied by the coefficient $2!1! = 2$, so $\phi_c = \frac{1}{12}$. Similarly (or by symmetry), $\phi_d = \frac{1}{12}$. Thus

$$\phi_a = \tfrac{7}{12},\ \phi_b = \tfrac{1}{4},\ \phi_c = \phi_d = \tfrac{1}{12}.$$

The S-S index values we have got here turn out to be identical with the Shapley values obtained in Ex. A.2. This is no accident: the underlying connection between our present $\mathcal{W}$ and the $\mathbf{v}$ of Ex. A.2 is revealed in Thm. 6.2.23 and Ex. 6.2.25.

A.4 Example (D-P and Js indices) We use the same $\mathcal{W}$ as in Ex. A.1. Of the five winning coalitions listed there, two are minimal — they are immediately recognizable in that listing as those having *all* their members underlined: $\{\underline{a},\underline{b}\}$ and $\{\underline{a},\underline{c},\underline{d}\}$.

To calculate the D-P index of a voter (Def. 6.4.2), we add up $1/|S|$ for each MWC S containing this voter, and then divide the sum by the number of MWCs (which in the present case is 2). For a we have to add up $\frac{1}{2}$ and $\frac{1}{3}$, because a belongs to both MWCs, whose sizes are 2 and 3 respectively. Dividing the sum by 2, we get D-P$_a = \frac{5}{12}$.

Since b belongs to a single MWC, whose size is 2, we obtain D-P$_b = \frac{1}{4}$. Voter c belongs to one MWC, of size 3, so D-P$_c = \frac{1}{6}$; and similarly D-P$_d = \frac{1}{6}$. Thus

$$\text{D-P}_a = \tfrac{5}{12},\ \text{D-P}_b = \tfrac{1}{4},\ \text{D-P}_c = \text{D-P}_d = \tfrac{1}{6}.$$

The calculation of the Js index (Def. 6.4.5) is quite similar, with the following two differences: first, *all* vulnerable coalitions must be taken into consideration, not only the MWCs; second, the contribution of a vulnerable coalition S to the score of each of its members is not $1/|S|$ but $1/\mathrm{Cr}(S)$, where $\mathrm{Cr}(S)$ is the number of critical members of S.

In our example, it so happens that all five winning coalitions are vulnerable; this can be seen from their listing in Ex. A.1, in which each of the five has at least one member underlined. The critical numbers of these five coalitions, in the order in which they are listed, are $2, 2, 2, 3, 1$. Voter a belongs to all five coalitions, so her Js score is

$$\tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{3} + 1 = \tfrac{17}{6}.$$

To obtain the Js index, we must divide the Js score by the number of vulnerable coalitions (see Rem. 6.4.6(iii)), which in this case is 5. Thus $\mathrm{JI}_a = \frac{17}{30}$.

Voter b belongs to three vulnerable coalitions, all having critical number 2; so his score is $\frac{3}{2}$ and $\mathrm{JI}_b = \frac{3}{10}$. Finally, c belongs to a single vulnerable coalition, whose critical number is 3, so his score is $\frac{1}{3}$ and $\mathrm{JI}_c = \frac{1}{15}$; and the same holds for d. Thus

$$\mathrm{JI}_a = \tfrac{17}{30},\ \mathrm{JI}_b = \tfrac{3}{10},\ \mathrm{JI}_c = \mathrm{JI}_d = \tfrac{1}{15}.$$

B. Appendix: Axiomatic Characterizations

B.1 Convention In this Appendix we shall use the terminology and notation of § 6.2.

By 'CF' we shall always mean CF of some SVG (rather than a more general cooperative game). We shall attribute various properties of SVGs to their respective CFs. Thus, for example, when we say that a CF is *proper* we mean that its SVG is proper.[1] Similarly, we shall attribute various pieces of furniture of an SVG to its CF. Thus, for example, by the *assembly* of a CF we mean the assembly of its SVG.

We shall always denote a CF and its SVG by corresponding letters, possibly with affixes. So, for example, if we denote a CF by '$\mathbf{w}_M$', the reader should take it for granted that '$\mathcal{W}_M$' denotes its SVG.

The characterization Thms. 6.2.14 and 6.2.15 fail if we replace the class of all games (or all superadditive games) on N by the class of CFs (or proper CFs) with assembly N. We shall illustrate this for the case $n = 3$, although counter-examples can be found for any $n > 2$.

B.2 Example Let $\boldsymbol{\mathcal{G}}$ be the class of all CFs with assembly $I_3 = \{1, 2, 3\}$. Let ξ be any value assignment such that $\xi_a(\mathbf{w}) = \beta_a[\mathcal{W}]$ for all $\mathbf{w} \in \boldsymbol{\mathcal{G}}$.

[1] As we noted in Rem. 6.2.2(ii), this amounts to saying that the CF is superadditive.

It is easy to see that ξ is iso-invariant, efficient and vanishes for dummies on $\boldsymbol{\mathcal{G}}$ (cf. Rem. 6.2.12). It is also additive on $\boldsymbol{\mathcal{G}}$ — vacuously, because the sum of two members of $\boldsymbol{\mathcal{G}}$ is never in $\boldsymbol{\mathcal{G}}$. As there are only a small number (in fact 8, including 3 improper) isomorphism types of CFs in $\boldsymbol{\mathcal{G}}$, it is easy to check that ξ is marginal on $\boldsymbol{\mathcal{G}}$. Moreover, ξ satisfies on $\boldsymbol{\mathcal{G}}$ Young's [114, p. 69] stricter condition of *strong monotonicity*: whenever $\mathbf{w} \in \boldsymbol{\mathcal{G}}$ and $\mathbf{w}' \in \boldsymbol{\mathcal{G}}$ and $i \in I_3$ is a player such that

$$\mathbf{w}X - \mathbf{w}(X - \{i\}) \geq \mathbf{w}'X - \mathbf{w}'(X - \{i\})$$

for every coalition X, then $\xi_i(\mathbf{w}) \geq \xi_i(\mathbf{w}')$. (Cf. Rem. 7.8.7(ii).)

Nevertheless, if we take $\mathbf{w}$ as the CF of $[3; 2, 1, 1]$ (which is a proper WVG) we have $\xi_1(\mathbf{w}) = \frac{3}{5}$ whereas $\phi_1(\mathbf{w}) = \frac{2}{3}$.

However, there is a variant of the additivity postulate that can be used instead of the latter to obtain characterizations of ϕ (as well as β') on the class of CFs (or proper CFs) with given assembly N. We shall state that variant in a moment (Def. B.5); but first we need to define two operations on games.

B.3 Definition Let $\mathbf{u}$ and $\mathbf{v}$ be games on the same set N. Their *supremum* $\mathbf{u} \vee \mathbf{v}$ and *infimum* $\mathbf{u} \wedge \mathbf{v}$ are the games on N such that

$$(\mathbf{u} \vee \mathbf{v})X = \max(\mathbf{u}X, \mathbf{v}X),$$
$$(\mathbf{u} \wedge \mathbf{v})X = \min(\mathbf{u}X, \mathbf{v}X),$$

for every $X \subseteq N$.

B.4 Remarks (i) The operations $\vee$ and $\wedge$ are evidently associative. So, for example, if $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are games on the same set, the expression '$\mathbf{u} \vee \mathbf{v} \vee \mathbf{w}$', without brackets, is unambiguous.

(ii) The operations $\vee$ and $\wedge$ on games are connected with the operation of game addition by the easily verified identity

$$(\mathbf{u} \vee \mathbf{v}) + (\mathbf{u} \wedge \mathbf{v}) = \mathbf{u} + \mathbf{v}.$$

(iii) In the special case where $\mathbf{u}$ and $\mathbf{v}$ are CFs (with common assembly), it is easy to see that $\mathbf{u} \vee \mathbf{v}$ and $\mathbf{u} \wedge \mathbf{v}$ are CFs, whose respective SVGs are $\mathcal{U} \cup \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V}$.

By the way, because here $\mathcal{U}$ and $\mathcal{V}$ have the same assembly, $\mathcal{U} \cup \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V}$ coincide with the composites $\mathcal{U} \vee \mathcal{V}$ and $\mathcal{U} \wedge \mathcal{V}$ respectively, as pointed out in Rem. 2.3.13(iii).

(iv) Note that if $\mathbf{u}$ and $\mathbf{v}$ are proper CFs (with common assembly), then $\mathbf{u} \wedge \mathbf{v}$ must be proper as well; but $\mathbf{u} \vee \mathbf{v}$ may not be proper, because some $S \in \mathcal{U}$ may be disjoint from some $T \in \mathcal{V}$.

B.5 Definition We say that a value assignment ξ is *measure-additive* on a class $\boldsymbol{\mathcal{G}}$ of games if, whenever $\mathbf{u}$ and $\mathbf{v}$ are games with a common grand coalition and the games $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} \vee \mathbf{v}$ and $\mathbf{u} \wedge \mathbf{v}$ are all in $\boldsymbol{\mathcal{G}}$, then

$$\xi(\mathbf{u} \vee \mathbf{v}) = \xi(\mathbf{u}) + \xi(\mathbf{v}) - \xi(\mathbf{u} \wedge \mathbf{v}).$$

(Here $+$ and $-$ are vector addition and subtraction, in the sense of Def. 6.2.3.)

B.6 Remarks (i) We call this condition 'measure-additivity' because it is closely related to the condition characterizing additive measures (in the sense of measure theory), such as probabilities. Recall the identity familiar from probability theory:

$$\mathrm{P}(A \vee B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \wedge B),$$

where A and B are events in a probability space and $\vee$ and $\wedge$ are the logical operations of disjunction and conjunction, respectively.

(ii) The condition is often referred to as *Dubey's axiom*, because it was Dubey [26] who proposed using it (instead of ordinary additivity) to characterize the S-S index.

B.7 Theorem *Any value assignment that is additive on the class of all games is also measure-additive on that class. In particular, the Shapley value and the Bz value are measure-additive on the class of all games.*

Proof The first claim of the theorem is a direct consequence of Rem. B.4(ii). The rest follows by virtue of Thm. 6.2.11 and Rem. 6.2.12. ∎

The following result is proved in [26].

B.8 Theorem (Dubey) *For any nonempty finite set N, ϕ is characterized on the class of all CFs with assembly N, as well as on the class of proper CFs with assembly N, by the conditions of iso-invariance, efficiency, vanishing for dummies and measure-additivity.*

Proof We prove the characterization for the class of proper CFs on N, which we shall denote here by '$\boldsymbol{\mathcal{G}}$'; the other case is very similar, albeit slightly simpler.

We know from Thms. 6.2.11 and B.7 that ϕ is iso-invariant, efficient, vanishes for dummies, and is measure-additive on the class of *all* games; hence *a fortiori* on $\boldsymbol{\mathcal{G}}$.

Now let ξ be a value assignment that satisfies on $\boldsymbol{\mathcal{G}}$ the four conditions of the theorem. We must show that $\xi(\mathbf{w}) = \phi(\mathbf{w})$ for all $\mathbf{w} \in \boldsymbol{\mathcal{G}}$. We proceed by induction on $|\mathcal{W}|$, the number of winning coalitions of $\mathbf{w}$.

First consider the special case of a CF having exactly one MWC M, which may be any nonempty $\subseteq N$. We denote this CF by '$\mathbf{w}_M$'. Thus $\mathcal{W}_M = \{X \subseteq N : M \subseteq X\}$.

Clearly, here all the the members of M are symmetric to each other, and all the voters outside M are dummies. Since ξ is iso-invariant, efficient and vanishes for dummies on $\boldsymbol{\mathcal{G}}$, it follows that

$$\xi_a(\mathbf{w}_M) = \begin{cases} \frac{1}{|M|} & \text{if } a \in M, \\ 0 & \text{if } a \in N - M. \end{cases}$$

So $\xi(\mathbf{w}_M) = \phi(\mathbf{w}_M)$ — which proves the theorem in this special case.

Now consider an arbitrary $\mathbf{w} \in \boldsymbol{\mathcal{G}}$. Let $M_1, M_2, \ldots, M_k$ be all the distinct MWCs of $\mathcal{W}$. Then

$$\mathcal{W} = \mathcal{W}_{M_1} \cup \mathcal{W}_{M_2} \cup \cdots \cup \mathcal{W}_{M_k}.$$

(If you cannot see this at once, recall Rem. 2.3.3(i).) We have already dealt with the case $k = 1$, so we may assume $k > 1$. Therefore

$$\mathcal{W} = \mathcal{W}_{M_1} \cup \mathcal{V},$$

where

$$\mathcal{V} = \mathcal{W}_{M_2} \cup \cdots \cup \mathcal{W}_{M_k}.$$

Note that both $\mathcal{V}$ and $\mathcal{W}_{M_1} \cap \mathcal{V}$ are proper SVGs, because they are included in $\mathcal{W}$, which is proper by assumption.[2]

Now, by Rem. B.3(iii) the CFs of $\mathcal{W} = \mathcal{W}_{M_1} \cup \mathcal{V}$ and $\mathcal{W}_{M_1} \cap \mathcal{V}$ are $\mathbf{w}_{M_1} \vee \mathbf{v}$ and $\mathbf{w}_{M_1} \wedge \mathbf{v}$ respectively. By the measure-additivity of ξ on $\boldsymbol{\mathcal{G}}$ we have:

$$\begin{aligned}\xi(\mathbf{w}) &= \xi(\mathbf{w}_{M_1} \vee \mathbf{v}) \\ &= \xi(\mathbf{w}_{M_1}) + \xi(\mathbf{v}) - \xi(\mathbf{w}_{M_1} \wedge \mathbf{v}). \qquad (*)\end{aligned}$$

Next, observe that in the union $\mathcal{W} = \mathcal{W}_{M_1} \cup \mathcal{V}$ neither of the SVGs $\mathcal{W}_{M_1}$ and $\mathcal{V}$ is included in the other. For example, M_1 belongs to the former but not to the latter, whereas M_2 belongs to the latter but not to the former. Therefore both $|\mathcal{W}_{M_1}|$ and $|\mathcal{V}|$ must be smaller than $|\mathcal{W}|$. And $|\mathcal{W}_{M_1} \cap \mathcal{V}|$ is *a fortiori* smaller than $|\mathcal{W}|$. Hence we can apply the induction hypothesis to $\mathbf{v}$ and $\mathbf{w}_{M_1} \wedge \mathbf{v}$. As for $\mathbf{w}_{M_1}$, we can apply the induction hypothesis to it as well, or recall that we have already proved the theorem for such special CFs. Therefore, using the induction hypothesis and the measure-additivity of ϕ, we obtain from $(*)$

$$\begin{aligned}\xi(\mathbf{w}) &= \phi(\mathbf{w}_{M_1}) + \phi(\mathbf{v}) - \phi(\mathbf{w}_{M_1} \wedge \mathbf{v}) \\ &= \phi(\mathbf{w}_{M_1} \vee \mathbf{v}) \\ &= \phi(\mathbf{w}),\end{aligned}$$

as claimed. ∎

For the characterization of the Bz measure β', we shall need the following definition, in which we use CFs of the form $\mathbf{w}_M$ introduced in the proof of Thm. B.8.

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[2] This sentence would be omitted if $\boldsymbol{\mathcal{G}}$ were the class of *all* CFs on N. This is the only difference between the proofs in the two cases.

B.9 Definition We shall say that a value assignment ξ satisfies the *Bz normalization condition* on a class $\boldsymbol{\mathcal{G}}$ of CFs if, for every $\mathbf{w} \in \boldsymbol{\mathcal{G}}$ that has exactly one MWC M,

$$\sum_{a \in M} \xi_a(\mathbf{w}) = \frac{|M|}{2^{|M|-1}}.$$

B.10 Remark It is a simple matter to verify that β' satisfies the Bz normalization condition on the class of all CFs.

The argument we have used in the proof of Thm. B.8 is borrowed from Dubey and Shapley [27, pp. 104–105], where it is used in exactly the same way to prove the following result.

B.11 Theorem (Dubey and Shapley) *For any nonempty finite set N, β' is characterized on the class of all CFs with assembly N, as well as on the class of proper CFs with assembly N, by the conditions of iso-invariance, Bz normalization, vanishing for dummies and measure-additivity.* ∎

B.12 Remarks (i) In [27] the statement of the characterization theorem for β' imposes a condition that is apparently stronger than Bz normalization, but the proof uses no more than Bz normalization.

(ii) The argument used in the proof of Thm. B.8 is entirely constructive: in principle it could be used as a recursive procedure for computing the numerical value of $\phi_a(\mathbf{w})$ for any given CF $\mathbf{w}$ and voter a. The same applies, *mutatis mutandis*, to the proofs of other characterization theorems: B.11, 6.2.14 and 6.2.15.

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