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The geometry and topology of steady Euler flows, integrability and singular geometric structures

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Abstract

In this thesis, we make a deep investigation of the geometry and dynamics of several objects (singular or not) appearing in nature. The main goal is to study rigidity versus flexibility dynamical behavior of the objects considered. In particular, we inspect normal forms, h-principles, classifications, and existence theorems. These concern a series of objects which are either close or far away from what we call "integrable situations" in the sense of Frobenius theorem and the existence of first integrals. Such dynamical systems arise in the context of symplectic and contact geometry (and their singular counterparts), as well as in the Euler equations on Riemannian manifolds.

As integral objects, we consider integrable systems appearing in symplectic manifolds but also on singular symplectic manifolds. Singularities show up naturally on these phase spaces by considering spaces with cylindrical ends and studying b-symplectic forms as initiated by Guillemin-Miranda-Pires. Other types of singularities are folded structures originally considered by Martinet and then by Cannas da Silva, Guillemin, Woodward for geometrical purposes. We give classification results of steady Euler flows which admit a Morse-Bott first integral using techniques coming from the symplectic world, and study obstructions arising from the ambient topology.

Our analysis includes the existence of action-angle coordinates on folded symplectic manifolds, and a correspondence between the recently introduced b-contact forms and Beltrami fields on b-manifolds. As examples of systems that are "far from integral" we consider the case of contact manifolds and their close allies in the study of Euler flows (Beltrami vector fields). This gives us the possibility to extend the h-principles from the contact realm to that of Beltrami vector fields. This last observation enables us to consider universality properties, as introduced by Tao, of steady Euler flows by analyzing those of high-dimensional Reeb flows in contact geometry. In the same spirit, we address the construction of steady Euler flows in dimension 3 which simulate a universal Turing machine, using tools coming from symbolic dynamics. In particular, these solutions have undecidable trajectories. In all these discussions, a key role is played by different classes of vector fields such as geodesible, Beltrami, and Eulerisable fields. We set up the study of the relations between such classes in higher odd-dimensions, showing that new phe-

nomena arise as soon as one leaves the realm of three-dimensional manifolds. For these high dimensional Euler flows (or more generally, flows admitting a strongly adapted one-form), we show that they satisfy the periodic orbit conjecture, which was known to be satisfied with the stronger assumption of geodesibility.

Resum en Català

En aquesta tesi, duem a terme una investigació en profunditat de la geometria i la dinàmica de diferents objectes (singulars o no) que apareixen a la natura. El principal objectiu és l'estudi del comportament dinàmic rígid vs. flexible dels objectes considerats. En particular, inspeccionem formes normals, h-principis, classificacions, i teoremes d'existència. Aquests es refereixen a una sèrie d'objectes que estan a prop o lluny del que denominem "situacions integrables" en el sentit del teorema de Frobenius i l'existència de primeres integrals. Aquests sistemes dinàmics sorgeixen en el context de la geometria simplèctica i de contacte (i les seves contrapartides singulars), a més a més d'en les equacions d'Euler en varietats Riemannianes.

Com a objectes integrals, considerem sistemes integrables que apareixen en varietats simplèctiques però també en varietats simplèctiques singulars. Les singularitats apareixen naturalment en espais de fase quan considerem espais amb finals cilíndrics i estudiant formes b-simplèctiques, estudi que va ser iniciat per Guillemin-Miranda-Pires. Altres tipus de singularitats són les estructures plegades, originalment considerades per Martinet i després per Cannas da Silva, Guillemin i Woodward amb motius geomètrics. Donem resultats de classificació per a fluxos d'Euler estacionaris que admeten una integral de tipus Morse-Bott, usant tècniques del món simplèctic, i estudiem obstruccions que sorgeixen de la topologia de la varietat ambient.

La nostra anàlisi inclou l'existència de coordenades acció-angle en varietats simplèctiques plegades, i una correspondència entre les recentment introduïdes formes de b-contacte i camps de Beltrami en b-varietats. Com a exemples d'objectes que estan "lluny de la situació integrable", considerem el cas de les varietats de contacte i els seus aliats propers en l'estudi de fluxos d'Euler (els camps de Beltrami). Això ens permet estendre els h-principis del regne de la geometria de contacte al dels camps de Beltrami. Aquesta última observació ens permet considerar propietats d'universalitat, introduïdes per Tao, de fluxos estacionaris d'Euler analitzant les dels camps de Reeb en geometria de contacte. Des d'aquest mateix punt de vista, encarem la construcció de fluxos estacionaris d'Euler en dimensió 3 capaços de simular una màquina de Turing universal, usant tècniques de dinàmica simbòlica. Aquestes solucions tenen, en particular, trajectòries indecidibles. En

totes aquestes discussions, diferents classes de camps tenen un rol clau, com ara els camps geodesibles, de Beltrami o Euleritzables. Posem en marxa l'estudi de les relacions entre aquestes classes en dimensions senars arbitràries, demostrant que nous fenòmens apareixen quan abandonem el regne de les 3-varietats. Per aquests fluxos d'Euler (o més en general fluxos que admeten una uno-forma fortament adaptada), demostrem que se satisfà la conjectura d'òrbita periòdica, la qual es sabia certa amb la suposició més forta de geodesibilitat.

Contents

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Preamble

In this thesis, we use geometrical and topological methods to study dynamical systems arising in mechanical systems such as steady Euler flows and Hamiltonian systems on singular symplectic manifolds. We will address questions that arise naturally such as existence, classifications, dynamical and topological properties and other related aspects.

Ideal fluids in Riemannian manifolds

It was Leonhard Euler, in 1757, who introduced the hydrodynamic equations for ideal fluids. They were among the first partial differential equations to appear in mathematical literature. These equations, originally formulated in the Euclidean space, model the velocity field of a fluid assumed to be inviscid and incompressible. When the fluid has some viscosity, it is described by the so-called Navier-Stokes equations. In $\mathbb{R}^3$ with the standard metric, the Euler equations read

$$\begin{cases} \partial_t u + \nabla_u u & = -\nabla p \\ \operatorname{div} u = 0 \end{cases}.$$

Here u denotes the velocity field of the fluid, ∇ denotes the covariant derivative, div is the divergence operator and p is a scalar function called the pressure. The first equation, called the Euler momentum equation, can be interpreted as a continuous version of Newton's second law. On the other hand, the second equation is simply imposing that the fluid is incompressible. For a smooth and compactly supported initial condition u_0, the existence of a solution for a certain short time is well known. However, the global existence of solutions is a long standing open problem.

More generally, given an arbitrary Riemannian manifold (M, g) of any dimension, the corresponding Euler equations have the same expression, but considering the covariant derivative with respect to the metric g and the operator div with respect to the Riemannian volume form. If the manifold has a boundary, it is customary to assume that u is tangent to it. In this thesis, we are interested in stationary solutions: we assume that $\partial_t u = 0$. The fact that stationary solutions are of special interest to geometers will become clear when introducing its geometric wealth in Sections 1.1 and 2.1. A dual geometric formulation of the stationary

equations is

$$\begin{cases} \iota_X d\alpha = -dB \\ d\iota_X \mu = 0 \end{cases},$$

where $\alpha = g(X, \cdot)$, μ is the Riemannian volume and $B = \frac{1}{2}g(X, X) + p$ is the Bernoulli function. If the metric g is *fixed*, it is in general hard to find explicit stationary solutions: this is due to the rigidity imposed by fixing the partial differential equation. However, one can take another approach, allowing the metric to *vary*. Instead of a fixed set of equations, we find now an infinite family of equations. As we will see through this thesis, we can take advantage of this flexibility and use highly geometrical arguments to prove the existence of solutions with certain dynamical or topological properties. This is a very natural setting, since it allows to understand which dynamical phenomena can occur in Riemannian hydrodynamics, and study properties which are inherent to flows that are steady Euler solutions for some metric (rather than a fixed one).The relation between topology and ideal fluids started in Lord Kelvin's works on knotted vortex tubes in stationary solutions. In 1969, Moffatt [151] analyzed the very important concept of helicity and its topological interpretation. The geometric point of view of fluids in Riemannian manifolds has its origins in pioneering works by Arnold [3, 4, 5], followed by the classical paper by Ebin and Marsden [53]. More concrete properties of stationary solutions in Riemannian manifolds were studied in the 90's by Ginzburg and Khesin [81], Etnyre and Ghrist [67, 66, 68] and a series of lectures by Sullivan at CUNY in 1994.

A remarkable theorem by Etnyre and Ghrist establishes a connection between rotational Beltrami type steady solutions to the Euler equations (for some metric) and Reeb fields in contact geometry. A Reeb field R is a vector field associated to a contact form α. A one-form α is contact when it is maximally non-integrable, in the sense that $\alpha \wedge (d\alpha)^n \neq 0$ where the dimension of the ambient manifold is $2n + 1$. The vector field R is given by the set of equations

$$\begin{cases} \alpha(R) = 1 \\ \iota_R d\alpha = 0 \end{cases}.$$

The correspondence of R with some steady Euler flows allows us to import techniques from contact topology to exhibit properties of Beltrami type solutions to the steady Euler equations. Within the applications of this correspondence, we can find the Seifert conjecture for analytic Euler flows [67] or the existence of a steady Euler flow in a Riemannian sphere which contains all possible knots and links as orbits [68]. The latter result remains an open question for stationary solutions in the round sphere, but was settled for the Euclidean $\mathbb{R}^3$ using sophisticated analytical tools [60]. In the line of the works by Etnyre and Ghrist, in the first chapter, we will develop techniques in Reeb dynamics to prove existence of Reeb flows (and their Euler counterpart) with various interesting dynamical properties. We will

address the problem of embedding arbitrary dynamics into Reeb flows, and use a combination of contact topology and tools in symbolic dynamics to construct a steady Euler flow in three dimensions which simulates a universal Turing machine.

This idea of incorporating a "variable" Riemannian metric in the steady Euler equations was later generalized by Rechtman [170] who noticed that in fact there is a larger class of non-vanishing flows which are steady Euler flows for some metric. These are geodesible volume preserving vector fields (or, equivalently in dimension three, Reeb fields of stable Hamiltonian structures), which correspond to Beltrami type steady Euler flows. More recently, Peralta-Salas, Rechtman and Torres de Lizaur [164] fully characterized, both geometrically and topologically, non-vanishing vector fields that satisfy the Euler equations for some metric. This is captured by the notion of "Eulerisable field", which further unveils the geometric wealth of this point of view.

Definition. *Let M be manifold with a volume form μ. A volume-preserving vector field X is **Eulerisable** if there is a metric g on M for which X satisfies the Euler equations for some Bernoulli function $B : M \to \mathbb{R}$.*

In terms of one-forms, a non-vanishing volume-preserving vector field is Eulerisable if and only if there is some one-form α such that $\iota_X d\alpha = -dB$, i.e. $\iota_X d\alpha$ is exact. When there is some one-form for which B is constant, we recover geodesible volume-preserving vector fields. As shown by Cieliebak-Volkov [43], there exist Eulerisable flows which are not geodesible, at least in some three-manifold. The notion of Eulerisable field will be capital in Chapter 2. We will address questions such as: How are these classes of vector fields related in higher dimensions? Do they always exist in any odd dimensional manifold? Which functions B can arise from a three dimensional steady Euler flow? Can the topology of the ambient manifold have an influence on the admissible solutions for any metric?

Singular symplectic geometry and dynamics

The third and fourth chapters focus on geometrical and dynamical properties of singular symplectic manifolds. Symplectic forms arise from the geometry of classical mechanical systems, such as the famous three body problem in celestial mechanics. Nowadays, symplectic geometry and topology have become a very active field of research on its own. This geometric structure provides a framework for the study of the so called Hamiltonian systems, which are vector fields associated to a certain function via the symplectic form.

Several generalizations of symplectic geometry have been defined, arising from physical models, and started to be studied. Examples of such generalizations include Poisson manifolds, Jacobi manifolds and their related geometries. When seen as a Poisson manifold, a symplectic manifold is just a Poisson manifold of

constant maximal rank. Even if topological methods of global character have proved to be of capital use for the study of symplectic manifolds, it is much more complicated to study global aspects of more general Poisson manifolds. This is why several particular subclasses of Poisson manifolds are studied with their own applicable methods. A specially interesting subclass of Poisson manifolds are *b*-Poisson manifolds: their associated Poisson structure is symplectic everywhere except along a hypersurface where some non degeneracy condition is imposed.

Definition. *A b-Poisson manifold is a manifold M of dimension $2n$ equipped with a Poisson bivector field Π such that $\Pi^n \pitchfork 0$.*

For this subclass of Poisson manifolds, global methods seemed within reach: the global classification of *b*-Poisson surfaces was achieved by Radko [169]. They attracted the interest of the scientific community, specially after the seminal paper by Guillemin-Miranda-Pires [92]. In that work, the authors unveil a key tool to import techniques from the symplectic geometry world. A *b*-Poisson structure can be interpreted as a symplectic form in a modified tangent bundle over the manifold called the *b*-tangent bundle. This bundle was introduced by Melrose [139] to study the Atiyah-Patodi-Singer theorem for manifolds with boundary. The symplectic formulation of *b*-Poisson manifolds (also *b*-symplectic manifolds) was used to extend some of the main results in symplectic geometry to *b*-symplectic geometry. For example, Moser's path method works in this context and proves a Darboux type normal form for the bivector field Π (or its dual singular *b*-form ω). Locally, any *b*-symplectic form can be written in suitable coordinates $(t, q, x_2, y_2, ..., x_n, y_n)$ of $\mathbb{R}^{2n}$ as

$$\frac{dt}{t} \wedge dq + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

More results whose inspiration comes from symplectic geometry were extended to this singular setting: study of group actions [17], classifications of toric *b*-symplectic manifolds [93], existence and obstructions [133, 76], etc... The appearance of b^m-symplectic forms (a generalization of *b*-symplectic forms with higher order poles) in the study of singularities of physical systems, such as the restricted three body problem [46], motivated the development of the dynamical theory of these geometric structures. The first steps were introducing *b*-integrable systems and proving the existence of action angle coordinates [123]. As we will see through chapter 3, this kind of Poisson structures (as well as *b*-contact structures, which are related to them) arise in the study of steady ideal fluid flows.

Previously to the development of *b*-symplectic geometry, another class of geometric structures with singularities had started to be studied: folded symplectic forms. These are smooth two forms (instead of bivector fields as *b*-Poisson structures) which degenerate in a controlled way along a hypersurface. Even though *b*-symplectic and folded symplectic structures are related [94], they are not exactly

dual to each other. Folded symplectic forms also admit a Darboux type normal form obtained by Martinet:

$$tdt \wedge dq + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

Classification of toric folded symplectic actions has also been developed in the literature [165, 23, 104], but the geometrical theory is in general less developed than the *b*-symplectic one. In these two last chapters of the thesis, we will push forward the theory of folded symplectic forms. We will cover a gap in the literature by giving an isotopic classification of folded symplectic surfaces, and from the dynamical point of view, we will introduce the corresponding notion of Hamiltonian dynamics and integrable systems.

Integrable systems, which are originally defined on symplectic manifolds, are those Hamiltonian systems which admit the maximal amount of symmetries: $n-1$ additional first integrals, where $2n$ is the dimension of the ambient symplectic manifold.

Definition. *Let (M, ω) be a symplectic manifold of dimension $2n$. An integrable system is a set of n functions $F = (f_1, ..., f_n)$, functionally independent almost everywhere (i.e. $df_1 \wedge ... \wedge df_n \neq 0$ on a dense subset of M) and which Poisson commute with respect to ω.*

The existence of action-angle coordinates, known as Arnold-Liouville-Mineur's theorem, captures this symmetry near regular values. Both in the symplectic and *b*-symplectic case, the action angle coordinates can be identified with a cotangent model of a neighborhood of a regular torus [121]. Some questions that we will address include: Can we find a differential topology proof Liouville's theorem? What is the semi-local structure of folded integrable systems? What can be said about the existence of *b*-integrable systems on four dimensional *b*-symplectic manifolds?

Results and guide to content

To a large extent, the content of this thesis can be found in the different articles produced, together with my coauthors, during the last three years. The core of each article is contained in the thesis with some changes in the way of presenting it. I have modified the exposition, collected common preliminaries of each chapter into a single section, and extended some discussions. In particular, I added more examples, remarks, and alternative arguments for some of the proofs.

List of articles

This thesis is based on the following list of articles. It includes a paper produced, together with my advisor Eva Miranda, during my master's thesis.

[26] R. Cardona. *A note on the periodic orbit conjecture.* Preprint (2021) arXiv:2103.02481 (see [27] for an updated version).

[34] R. Cardona, E. Miranda, D. Peralta-Salas and F. Presas. *Constructing Turing complete Euler flows in dimension 3.* Proceedings of the National Academy of Sciences, 118 (2021), DOI: 10.1073/pnas.2026818118.

[30] R. Cardona, E. Miranda. *Integrable systems in singular symplectic manifolds: From local to global.* Preprint (2020) arXiv: 2007.10314.

[25] R. Cardona. *The topology of Bott integrable fluids.* Preprint (2020) arXiv: 2006.16880.

[24] R. Cardona. *Steady Euler flows and Beltrami fields in high dimensions.* Ergodic Theory and Dynamical Systems (2020) 1-24. doi:10.1017/etds.2020.124.

[33] R. Cardona, E. Miranda, D. Peralta-Salas, F. Presas. *Universality of Euler flows and flexibility of Reeb embeddings.* Preprint (2019) arXiv: 1911.01963.

[31] R. Cardona, E. Miranda, D. Peralta-Salas. *Euler flows and singular geometric structures.* Phil. Trans. R. Soc. A. (2019) 377:20190034.

[29] R. Cardona, E. Miranda. *On the volume elements of a manifold with transverse zeroes.* Regular and Chaotic Dynamics (2019) vol. 24, no. 2, pp. 187-197.

[28] R. Cardona, E. Miranda. *Integrable systems and closed one forms.* Journal of Geometry and Physics 131 (2018), 204-209.

Chapter 1

The first chapter of this thesis deals with universality properties of ideal fluids in Riemannian manifolds obtained in [33, 34]. As we explained in the preamble, the notion of Eulerisable fields allows us to formalize the study of the stationary Euler equations with a variable metric. This unveils the geometric wealth of these vector fields and allows us to use geometrical and topological techniques for dynamical systems.

The flexibility/rigidity dichotomy is central in the development of the fields of symplectic and contact geometry. One approach of this dichotomic principle is studying which properties can, cannot, or must be satisfied by our object of study. In view of the tight connection between Eulerisable fields and symplectic geometry, it is natural to approach the study of steady ideal fluids with this point of view. This will be our leitmotiv both in Chapters 1 and 2. The first chapter deals with exhibiting flexibility phenomena in the form of universality properties. The concept of *Universality* comes from a series of papers by Terence Tao [183, 184, 185, 186], where Tao exhibits flexible behavior of the Euler equations with arbitrary Riemannian metrics and arbitrary dimensions. Motivating questions in this direction include:

- Which dynamical behavior is allowed by the Euler equations? Concretely, which flows can be embedded in a high dimensional Euler flow?

- Can undecidable phenomena arise in ideal fluids? Is there a fluid flow capable of universal computation?

The program of Tao is motivated by trying to use Turing complete dynamical systems modeled by ideal fluids to construct a blow-up solution to the Euler equations, at least in certain Riemannian manifold of a certain dimension. This very last goal belongs to the realm of partial differential equations and their regularity. However, the question "is hydrodynamics capable of universal computation?" is of interest on its own, and was already asked in the pioneering works of Moore [153] on undecidability in dynamical systems. Taking into account the connection between steady Euler flows and contact topology, these very same questions can be asked for Reeb vector fields. Hence, Chapter 1 can be seen as the concrete study of universality properties (and computational power) of Reeb flows in contact geometry and their corresponding steady Euler flows.

We will test the flexibility of dynamical behavior appearing in Reeb vector fields. We already know that rigid dynamical phenomena have a large history in contact geometry: there is a vast literature concerning Weinstein's conjecture that

we introduced in the preamble. However, contact geometry also has its flexible side, mainly captured using the language of homotopy principles.

After introducing some necessary background in Section 1.1, in Section 1.2 we follow [33] and use the flexibility captured by h-principle techniques in contact geometry to exhibit universality properties of Reeb and Euler flows. Let us introduce a very concrete universal property.

Definition. *A non-autonomous time-periodic vector field $u_0(\cdot, t)$ on a compact manifold N is* Euler-extendible *if there exists an embedding $e : N \times \mathbb{S}^1 \to \mathbb{S}^n$ for some dimension $n > \dim N + 1$ (that only depends on the dimension of N), and an Eulerisable flow u on $\mathbb{S}^n$, such that $e(N \times \mathbb{S}^1)$ is an invariant submanifold of u and $e_*(u_0(\cdot, \theta) + \partial_\theta) = u$, $\theta \in \mathbb{S}^1$. If the non-autonomous field $u_0(\cdot, t)$ is not time-periodic, we say it is Euler-extendible if there exists a proper embedding $e : N \times \mathbb{R} \to \mathbb{R}^n$ for some dimension $n > \dim N + 1$ (that only depends on the dimension of N), and an Eulerisable flow u on $\mathbb{R}^n$, such that $e(N \times \mathbb{R})$ is an invariant submanifold of u and $e_*(u_0(\cdot, \theta) + \partial_\theta) = u$, $\theta \in \mathbb{R}$. If any non-autonomous dynamics $u_0(\cdot, t)$ is Euler-extendible, we say that the Euler flows are* universal.

The extendibility of non-autonomous dynamics implies that u_0 describes the "horizontal" behavior of the integral curves of the extended vector field u. Note that u_0 is completely arbitrary, hence if Euler flows are universal then this horizontal behavior can exhibit any dynamical behavior.

In this sense, the main result in Section 1.2 is that Reeb flows are universal, and hence also their hydrodynamic counterpart:

Theorem. *Steady Euler flows are universal. Moreover, the dimension of the ambient manifold $\mathbb{S}^n$ or $\mathbb{R}^n$ is the smallest odd integer $n \in \{3 \dim N + 5, 3 \dim N + 6\}$. In the time-periodic case, the extended field u is a steady Euler flow with a metric $g = g_0 + \delta_P$, where g_0 is the canonical metric on $\mathbb{S}^n$ and δ_P is supported in a ball that contains the invariant submanifold $e(N \times \mathbb{S}^1)$.*

In order to reduce as much as possible the dimension of the ambient manifold, and study the topological behavior of such embeddings, we prove a theorem that is of independent interest in contact geometry. This theorem exhibits the flexibility of what we call "Reeb embeddings" in the context of contact topology: confer Section 1.2.5 for the details. A corollary of the theorem above, which illustrates the striking implications of the universality, is the embeddability of diffeomorphisms. An orientation-preserving diffeomorphism $\phi : N \to N$ is *Euler-embeddable* if there exists an Eulerisable field u on $\mathbb{S}^n$ (for some n that only depends on the dimension of N) with an invariant submanifold exhibiting a cross-section diffeomorphic to N such that the first return map of u at this cross-section is conjugate to ϕ.

Corollary. *Let N be a compact manifold and ϕ an orientation-preserving diffeomorphism on N. Then ϕ is Euler-embeddable in dimension n, where n is the smallest odd integer $n \in \{3 \dim N + 5, 3 \dim N + 6\}$.*

Both results exhibit a flexible behavior of Euler flows in arbitrary dimensions. The classical connection between symbolic dynamics and smooth dynamics leads, as a byproduct, to the existence of an Euler flow (of Reeb type) which encodes a universal Turing machine (i.e. it is *Turing complete*). The details of this property will be explained through Section 1.2.

Corollary. *There is a steady Euler flow in S^{17}, equipped with some Riemannian metric, which encodes a universal Turing machine.*

A concrete property of a Turing complete flow is that it contains undecidable trajectories: there is an explicitly constructible point $p \in S^{17}$ and some open set U such that it is undecidable (in the computational sense: there is no algorithm answering yes or no in a finite amount of time) whether the trajectory of the flow through p intersects U or not. This follows from the classical undecidability of the halting problem for Turing machines, as established by Turing himself [191].

This is a classical problem that lies in the intersection of mathematics and computation complexity: which equations and physical systems are capable of universal computation? In [34], a sequel of [33], we focus on this question for stationary Euler flows of Reeb type. Can we reduce the dimension of the ambient Riemannian manifold down to three? New techniques have to be applied to achieve this goal. In Section 1.3 we combine the computational power of symbolic dynamics, inspired by the works of Moore [153], with recent constructions in symplectic dynamics [19] to prove the Turing completeness of Euler flows in dimension three.

Theorem. *There is a steady Euler flow in some Riemannian S^3 which encodes a universal Turing machine.*

We settle the problem in the most interesting dimension, and raises the natural rigid version of this question: can this happen in the Euclidean space or the sphere with the round metric? In a sequel to these works [32] (which was not included in this document) we address the Turing completeness of the time-dependent Euler equations on a high dimensional Riemannian manifold, which was the original question raised by Tao.

Chapter 2

The second chapter corresponds to the results obtained in [24, 25] related to different classes of non-vanishing vector fields such as Eulerisable, Beltrami (fields parallel to their curl), and geodesible vector fields (fields whose orbits are geodesics for some metric). As introduced in the preamble, the notion of Eulerisable vector field and its geometric characterization [164] provides a nice working framework to study steady solutions to the Euler equations for a variable Riemannian metric. In

three dimensions, we have the following relations between classes of non-vanishing volume-preserving vector fields.

$$\text{Reeb} \subset \text{Geodesible} = \text{Beltrami} = \text{Stable Hamiltonian Reeb} \subset \text{Eulerisable}$$

We refer here to Beltrami fields as vector fields that are parallel to their curl for some Riemannian metric (one could also call them Beltramisable). In 3D, they correspond to those steady solutions to the Euler equations with a constant Bernoulli function. Recall that the Bernoulli functions is the function B such that $\iota_X d\alpha = -dB$, where $\alpha = g(X, \cdot)$ is the form dual to the vector field by the metric. Several questions remain open concerning the inclusions above: there is a single construction, in certain closed three-manifolds, of vector fields which are Eulerisable but not geodesible, introduced by Cieliebak and Volkov [43]. The goal of Section 2.2 is to understand the properties of the higher dimensional analogs of these classes of flows. In three dimensional closed manifolds, the existence of a Reeb field in every homotopy class of non-vanishing vector fields is guaranteed by the existence of contact structures in its orthogonal plane field homotopy class. However, it is well known that in higher dimensions there are obstructions to the existence of a contact structure in a given homotopy class or even in a given closed manifold. We show that for the wider class of geodesible volume-preserving vector fields, the existence result generalizes to higher dimensions.

Theorem. *Let M be a closed odd-dimensional manifold. Then any homotopy class of non-vanishing vector fields admits a volume-preserving geodesible field.*

These flows are furthermore Eulerisable, and can be seen as the Reeb field of a geometric structure generalizing stable Hamiltonian structures that we call stable Eulerisable structure. The properties of Beltrami fields in higher dimensions are in strong contrast with the three dimensional context.

Theorem. *Let M be any closed odd-dimensional manifold of dimension higher than three. Then*

- *there exist volume-preserving Beltrami fields which are neither geodesible nor Eulerisable,*

- *there exist (non necessarily volume-preserving) Beltrami fields without closed orbits.*

We already discussed how the first statement was not possible in three dimensions. On the other hand, it is still an open problem to prove or disprove that (non volume-preserving) smooth Beltrami fields always admit a periodic orbit on the three sphere. It is known that this is true in the analytic setting [171], and that Beltrami fields do not admit plugs [170]. The construction of aperiodic Beltrami fields in the higher dimensional setting uses the fact that this obstruction is no

longer valid: it is possible to have plugs in Beltrami fields. Consequently, the relations between these classes of volume-preserving vector fields is the following:

$$\text{Reeb} \subset \text{geodesible} = \text{stable Eulerisable} \quad \text{Reeb} \subset \text{Eulerisable}$$

$$\text{geodesible} \subset \text{Beltrami} \not\subset \text{Eulerisable}$$

The understanding of these kinds of linear foliations which admit adapted one-forms is far from being complete in three dimensions and is merely at an embryonic stage in higher odd dimensions. We end up the analysis of high dimensional Eulerisable flows by proving that they satisfy the periodic orbit conjecture: a non-vanishing Eulerisable flow all whose orbits are closed has a bounded set of lengths. The periodic orbit conjecture was shown to be false in general by Sullivan [180], who provided a very beautiful counterexample. The conjecture is known to be true with the assumption of geodesibility, which is weaker than Eulerisability: we show that admitting a strongly adapted one-form (as in the case of Eulerisable flows) is already enough for the periodic orbit conjecture to be satisfied.

Theorem. *Eulerisable flows (or more generally flows which admit a strongly adapted one-form) satisfy the periodic orbit conjecture.*

After analyzing steady solutions with constant Bernoulli function in high dimensions, we come back to a three dimensional setting in Section 2.3. In the study of those solutions with a non-constant Bernoulli function, Arnold set a milestone by proving his structure theorem [3]. This theorem provides a precise description of those steady Euler flows whose Bernoulli function is analytic and non-constant (or Morse-Bott). However, it does not provide hints about the existence of solutions in these hypotheses. One can ask in particular, for which analytic or Morse-Bott functions $B \in C^{\infty}(M)$ can we find a steady Euler flow for which $\iota_X d\alpha = -dB$? This can be thought of as an inverse problem to Arnold's theorem and was already suggested by Dennis Sullivan in series of lectures at CUNY in 1994. In the very same spirit of this point of view, Peralta-Salas asked in [162] if there exist three-manifolds where certain solutions cannot exist. We address these two problems for Morse-Bott Bernoulli functions.

If we restrict to non-vanishing steady solutions, it follows from different arguments that a Morse-Bott Bernoulli function can only occur on a graph three-manifold. We show that any Morse-Bott function, which is not topologically obstructed to be the integral of a volume-preserving vector field, is the Bernoulli function of some non-vanishing steady Euler flow.

Theorem. *Let M be a graph closed three-manifold. Any admissible Morse-Bott function can be realized as the Bernoulli function of a non-vanishing steady Euler flow for some metric.*

The proof combines the geometry of the Euler equations with a variable metric and the study of Bott integrable systems developed by Fomenko and his collaborators. A corollary is that the invariants developed for Hamiltonian systems give

a topological classification of non-vanishing steady solutions with a Morse-Bott Bernoulli function. Surprisingly, even if we drop the non-vanishing assumption, the topological obstruction on the ambient manifold still holds. This answers the question by Peralta-Salas in the Morse-Bott case.

Theorem. *Let M be a non graph closed three-manifold. Then M does not admit a steady Euler flow with a Morse-Bott Bernoulli function for any Riemannian metric g.*

This is the first result in the literature showing that the ambient space can be an obstruction to the existence of a certain type of stationary fluids in three dimensions.

Chapters 3 and 4

The last two chapters are highly interlaced since they have a common background: singular symplectic structures. In the third chapter we focus on finding examples and classifications of geometric structures with singularities. We start by giving some background about these two topics. In Section 3.2, we fill a gap in the literature of folded symplectic manifolds by giving an isotopic classification of folded symplectic surfaces [29]. The classification follows from the more general isotopic classification of top degree forms which cut transversally the zero section (forms that we call folded volume forms). In the statement below, Ω_1 and Ω_2 are top degree forms transverse to the zero section at the critical locus Z_1 and Z_2 respectively.

Theorem. *Let M be an oriented closed manifold. Two folded volume forms Ω_1 and Ω_2 with critical sets Z_1 and Z_2 are isotopic if and only if:*

- *The critical sets are isotopically equivalent with orientation*

- *their De Rham cohomology class coincide,*

- *their relative De Rham cohomology class (with respect to the critical set) coincide.*

We compare this classification with the classification of b-volume forms, proved by Martinez-Torres [136] after generalizing works of Radko [169], via the desingularization procedure introduced by Guillemin-Miranda-Pires [94].

In Section 3.3, we analyze singular geometric structures arising in stationary Euler flows. We start by giving an alternative proof of Arnold's theorem which uses a dual approach (via differential forms) and a Tischler's theorem for manifolds with boundary. This classical theorem is studied and used more in depth in the fourth chapter. We analyze b-symplectic forms that are induced in the critical level sets of a Morse-Bott Bernoulli function: they capture the singular area form that

is preserved by the fluid on those level sets. Lastly, we extend the correspondence between contact geometry and hydrodynamics to the singular setting.

Theorem. *Let M be a b-manifold of dimension three. Any rotational Beltrami field and non-vanishing as a section of ^bTM on M is a Reeb vector field (up to rescaling) for some b-contact form on M. Conversely given a b-contact form α with Reeb vector field X then any nonzero rescaling of X is a rotational Beltrami field for some b-metric and b-volume form on M.*

This provides a fruitful source of examples of *b*-contact forms, which were recently introduced and studied by Miranda and Oms [142, 143]. The classical contact-hydrodynamics developed by Etnyre and Ghrist has found strong applications for Riemannian hydrodynamics. One uses tools from contact topology to investigate the properties of steady Euler flows in Riemannian manifolds, as we did in Chapter 1. On the other hand, the theorem above for *b*-contact forms was recently used in the opposite direction. The properties of *b*-Beltrami fields were used to prove some cases of the so-called singular Weinstein conjecture in a work by Miranda, Oms and Peralta-Salas [144].

In the last chapter, we study integrable systems, and the core of the chapter is devoted to the development of the theory for folded and *b*-symplectic forms. In Section 4.2, following [28], we generalize a theorem by Tischler and use it to give an alternative proof of the classical Liouville's theorem, both in the symplectic and in the Poisson context. For *b*-symplectic manifolds, the notion of integrable system was introduced and studied in [123]. On the other hand, for folded symplectic manifolds, no discussion of Hamiltonian dynamics exists in the literature with the exception of Hamiltonian group actions.

In Section 4.3, based in [30], we introduce "folded functions" on folded symplectic manifolds, which are those functions which admit a solution to the Hamiltonian equation. We characterize them, and define integrable systems in this singular context, generalizing the notion of folded toric action as studied by several authors [23, 104, 128]. We generalize several results for folded integrable systems, and prove as one of the main results an analog of the Arnold-Liouville theorem.

Theorem. *Let $F = (f_1, ..., f_n)$ be a folded integrable system on a folded symplectic manifold (M, ω) and $p \in Z$ a regular point in the folding hypersurface. We assume the integral manifold $\mathcal{F}_p$ containing p is compact. Then there exist an open neighborhood U of the torus $\mathcal{F}_p$ and a diffeomorphism*

$$(\theta_1, ..., \theta_n, t, b_2, ..., b_n) : U \to \mathbb{T}^n \times B^n,$$

where t is a defining function of Z and such that

$$\omega_U = \sum_{i=1}^n d\theta_i \wedge dp_i.$$

where the p_i are folded functions which depend only on $(t, b_2, ..., b_n)$ (and so do the f_i).

The S^1-valued functions

$$\theta_1, ..., \theta_n$$

are called angle coordinates and the $\mathbb{R}$-valued folded functions

$$p_1, p_2, ..., p_n$$

are called folded action functions.

In symplectic and *b*-symplectic geometry, the respective action-angle coordinates can be interpreted in terms of a cotangent model [121]. Surprisingly, we analyze how the rigidity imposed by the existence of a null line bundle obstructs, in general, the existence of such a model for folded integrable systems.

We devote the last part of Section 4.3 to the study of *b*-integrable systems. Using the recent *b*-symplectic slice theorem [17] we obtain the following existence result.

Theorem. *Let (M, ω) be a b-symplectic manifold of dimension 4 whose critical set Z has an induced symplectic foliation given by a mapping torus of a periodic symplectomorphism. Then (M, ω) admits a b-integrable system.*

We finish by exhibiting semi-local obstructions to the existence of global action-angle coordinates arising from the topology of the critical level set of the *b*-symplectic form.

Chapter 1

Universality properties of steady Euler flows

In this chapter, we will address some properties of steady solutions to the Euler equations using techniques coming from contact topology. This properties include embedding arbitrary dynamics into high dimensional stationary solutions in compact Riemannian manifolds, and the appearance of undecidable trajectories via universal computing models. This chapter is based on the contents of [33] and [34]. The organization of the results and introduction to the topic is different in this thesis than in the articles, some alternative arguments are presented, and there is an extended discussion of the literature on undecidability and additional details in the construction of 3D steady flows which are Turing complete.

1.1   Steady Euler flows and contact geometry

Steady solutions to the Euler equations have a very rich geometry associated to them. This wealthness arises from a geometric formulation of the stationary equations. In this section, we present this formulation, analyze it, and discuss its connection to contact geometry.

1.1.1   Geometric formulation of the Euler equations

The Euler equations model the dynamics of an inviscid and incompressible fluid. They were originally formulated in the Euclidean space, but can be generalized to any Riemannian manifold (M, g) of any dimension. In this case, the equations read

$$\begin{cases} \partial_t X + \nabla_X X & = -\nabla p \\ \operatorname{div} X = 0 \end{cases}, \tag{1.1}$$

where X is the velocity field of the fluid, the scalar function p is the pressure function, and the differential operators are taken with respect to the metric g.

A topological and geometric approach happens to be very enriching for the study of steady Euler solutions (cf. [162] for an introduction to a geometric formulation of steady Euler flows and [9] for a monograph in topological hydrodynamics). The Euler equations have a dual formulation which is more appealing for geometers.

The dual form to the vector field $\nabla_X X$ is $\mathcal{L}_X \alpha - \frac{1}{2} d(g(X,X)^2) = \mathcal{L}_X \alpha - \frac{1}{2} d\iota_X \alpha$. Using Cartan's formula, we have that $\mathcal{L}_X \alpha = \iota_X d\alpha + d\iota_x \alpha$ and so the first Euler equation, dualized, is equivalent to

$$\frac{\partial \alpha}{\partial t} + \iota_X d\alpha = -\frac{1}{2} d\iota_X \alpha - dp. \tag{1.2}$$

Observe that the second Euler equation is equivalently written as $\mathcal{L}_X \mu = 0$ where μ is the induced Riemannian volume form. By Cartan's formula, this is equivalent to $d\iota_X \mu = 0$. We now introduce the function $B := \frac{1}{2} g(X,X) + p$, called the **Bernoulli function**. This allows to write the first Euler equation as $\frac{\partial \alpha}{\partial t} + \iota_X d\alpha = -dB$. In the autonomous case, we get

$$\begin{cases} \iota_X d\alpha = -dB \\ d\iota_X \mu = 0 \end{cases} . \tag{1.3}$$

Observe that first equation is just saying that $\iota_X d\alpha$ is an exact form.

1.1.2 Beltrami type solutions and geodesible vector fields

For the discussion of Beltrami type solutions, we assume that M is a three dimensional manifold. However, with some minor changes, analogous definitions and statements can be given for higher odd dimensions. We will come back to higher dimensional fluids in Chapter 2.

A vector field which is very important in hydrodynamics is the **vorticity field**. It is usually denoted[1] by ω, and corresponds to the curl of X. It is defined as the only vector field satisfying the equation

$$\iota_\omega \mu = d\alpha. \tag{1.4}$$

In the time-dependent case, Helmholtz's transport of vorticity theorem tells us that the vorticity is transported by the fluid. This is expressed by the equation $\frac{\partial \omega}{\partial t} = -\mathcal{L}_X \omega$. In the stationary case, which is that of our interest in this thesis, the transport of vorticity is just telling us that X commutes with ω, which is an autonomous vector field. The Bernoulli function B can lead to very different dynamical behavior of a steady solution to the Euler equations. The main property of B is contained in the next lemma.

---

[1]In some sections of this thesis, we will introduce an alternative notation for the vorticity field. We will write Y instead of ω, since ω will refer to some symplectic form.

Lemma 1.1.1. *The function B is a first integral both of X and ω. The vector fields X and ω are independent at a point $p \in M$ if and only if it is a regular point of B.*

Proof. For X this is immediate since the first Euler equation is $\iota_X d\alpha$. By contracting again with X, we get $\iota_X dB = 0$ which is the condition of being a first integral.

On the other hand, the vector field ω is defined by Equation (1.4). Contracting the first Euler equation by ω, we get

$$\iota_\omega \iota_X d\alpha = -\iota_\omega dB. \tag{1.5}$$

We claim that this quantity is zero. Indeed, the left hand side is equal to $-\iota_\omega \iota_Y d\alpha$, and by the definition of vorticity we deduce that $\iota_\omega d\alpha = 0$. Hence $\iota_\omega dB = 0$ and B is also an integral of ω.

The second statement can be deduced also from the first Euler equation and the definition of ω. If $dB|_p \neq 0$, then $\iota_X d\alpha|_p = \iota_X \iota_\omega \mu|_p \neq 0$ and so X and ω are independent at p. If $dB|_p = 0$, then $\iota_X \iota_\omega \mu|_p = 0$ and X is parallel to its vorticity. $\qquad\square$

In sections 2.1.1 and 2.3 we will discuss the case where B is nonconstant analytic or Morse-Bott. This leads to steady fluids which look like integrable Hamiltonian systems of two degrees of freedom in an energy level set.

When the function B is everywhere constant, then X is every parallel to its vorticity.

Definition 1.1.2. *A vector field X in (M, g) is a Beltrami field if it is volume-preserving and everywhere parallel to its vorticity. Equivalently, X satisfies the equations*

$$\begin{cases} \omega & = \lambda X, \ \ \lambda \in C^\infty(M) \\ d\iota_X \mu & = 0 \end{cases}.$$

Only in Section 2.2 we will omit from this definition the volume-preserving condition. It is clear from Lemma 1.1.1 that any solution with constant Bernoulli function is a Beltrami field. On the other hand, a Beltrami field satisfies the Euler equations. Since X and ω are parallel, we have $\iota_\omega \iota_X \mu = 0$ which implies that $-\iota_X \iota_\omega \mu = 0$ and by the definition of ω we get $\iota_X d\alpha = 0$.

Example 1.1.3 (ABC fields)**.** *The ABC flows is a very famous family of examples of Beltrami fields defined in the torus T^3 with the flat metric. Take coordinates (x, y, z) in the torus, seen as the quotient of $\mathbb{R}^3$ by $\mathbb{Z}^3$. The parametric family, named after Arnold Beltrami and Childress, is given by*

$$X = [A \sin z + C \cos y]\frac{\partial}{\partial x} + [B \sin x + A \cos z]\frac{\partial}{\partial y} + [C \sin y + B \cos x]\frac{\partial}{\partial z}.$$

The parameters are A, B, C which are real non-negative numbers. For generic values of such parameters, this flow is typically chaotic.

Beltrami fields have been widely studied: they represent a very interesting class of solutions since no rigidity is imposed by the presence of a first integral B. Recent breakthroughs include the existence of Beltrami fields in R^3 with the Euclidean metric which contain any knot and link as a closed stream line or as a vortex invariant tube [60, 61]. Even more recently, it was proved in [62] that in fact a "typical" Beltrami field in the Euclidean space contains every knot and link, vortex tubes of every knot and link type, and chaotic regions associated to horse shoes. These constructions use sophisticated analytical methods, which are necessary since the metric of our manifold is fixed. We will next look at an alternative approach: if we let the metric vary, geometrical and topological methods become very useful.

Geodesible vector fields

A class of vector fields which is closely related to Beltrami fields is the class of geodesible flows.

Definition 1.1.4. *A non-vanishing vector field X is **geodesible** if there exists a metric g such that its orbits are geodesics.*

Gluck proved in [83] (see also [181]) that the geodesibility condition is equivalent to the existence of a 1-form β such that $\beta(X) > 0$ and $\iota_X d\beta = 0$. If we also assume that the 1-form can be taken so that $\beta(X) = 1$, we say that X is of unit length. Unless otherwise stated, all along this paper we shall assume that a geodesible field has unit length. A similar characterization for Eulerisable flows was introduced in [162].

Another characterization that we shall use later is that X is geodesible of unit length if and only if it preserves a transverse hyperplane distribution. The necessity is immediate from the aforementioned Gluck's theorem. To prove that it is sufficient, let η be the hyperplane distribution and β a defining 1-form such that $\ker \beta = \eta$ and $\beta(X) > 0$. Dividing β by the function $\beta(X)$ we can safely assume that $\beta(X) = 1$. The condition that X preserves $\ker \beta$ is tantamount to saying that

$$\mathcal{L}_X \beta = f\beta \,,$$

for some function $f \in C^\infty(M)$. Cartan's formula implies that $\iota_X d\beta = f\beta$, and contracting with the vector field X we conclude that $f = 0$.

In the next sections we shall usually denote a unit geodesible field by (N, X) or (N, X, η), where N is the ambient manifold, X is the field and η is the transverse hyperplane distribution preserved by X. In particular we might fix the 1-form β, and hence the hyperplane distribution $\eta = \ker \beta$ preserved by X.

Remark 1.1.5. It will be important in Section 1.2 to fix η or β. A unit geodesible vector field can preserve more than a single plane distribution. An easy example of this situation is given in Example 1.1.17 in the next subsection.

The relationship between Beltrami and geodesible vector fields in dimension three was unveiled in [170].

Proposition 1.1.6. *A vector field X is geodesible if and only if there is some metric for which X is parallel to its curl.*

We will come back to this relation in Chapter 2. The prototype example of a geodesible vector field is a flow with section. A characterization by Tischler [190] shows that a flow X has a global section if and only if there is some closed one form α such that $\alpha(X) > 0$.

Example 1.1.7. *A very natural way to construct flows with section is via the suspension of a diffeomorphism. Let N be a manifold and $\varphi : N \to N$ an orientation-preserving diffeomorphism of N. We can construct the manifold*

$$M = N \times [0,1] / \sim,$$

where we identified $(p, 0)$ with $(\varphi(p), 0)$. If t denotes a coordinate in $[0, 1]$, the vector field $\frac{\partial}{\partial t}$ descends to some vector field $\tilde{X}$ in M with a global hypersurface of section given by $N \times \{0\}$. The first-return map of $\tilde{X}$ is conjugated to φ by construction.

On the other hand, not every geodesible vector field is a flow with section: for example, every Reeb field (cf. the next subsection) is geodesible.

1.1.3 Contact geometry and hydrodynamics

It was in the early 90s that the geometric structures appearing in steady Euler flows began to unravel. Ginzburg and Khesin introduced in [81] a symplectic interpretation of generic steady fluids in four dimensional manifolds. In the three dimensional setting, Dennis Sullivan, in a series of lectures at CUNY in 1994, investigated the geometric structures arising from certain steady Euler flows. He suggested a possible connection with Reeb flows, which appear in the context of contact geometry, and their dynamics are of crucial importance to the fields of contact and symplectic topology. This connection was proved by Etnyre and Ghrist, and takes the form of a correspondence that we will introduce in this subsection.

Definition 1.1.8. *A one form α in a manifold M of dimension $2n+1$ is a contact form if $\alpha \wedge (d\alpha)^n \neq 0$.*

Contact forms define, through their kernel, a **contact structure**. This hyperplane distribution $\xi = \ker \alpha$ can be interpreted in terms of Frobenius integrability as a maximally non-integrable distribution. The existence of a globally defined α defining ξ is a very common assumption, which ensures that ξ is coorientable.

Observe that for a given contact structure ξ, there are several contact forms which define it. Any positive rescaling of α will be another contact form defining ξ. For a fixed contact form, there is a naturally defined vector field R which is called the Reeb field of α. It is defined by the set of equations

$$\begin{cases} \alpha(R) = 1 \\ \iota_R d\alpha = 0 \end{cases}.$$

Since $d\alpha$ is a two form of rank $2n$, its kernel defines uniquely a non-vanishing line field L, while the first equation is just a normalizing condition.

Example 1.1.9. *The canonical example of contact structure in $\mathbb{R}^{2n+1}$ with coordinates $(z, x_1, y_1, ..., x_n, y_n)$ is the kernel of*

$$\alpha_{std} = dz + \sum_{i=1}^{n} x_i dy_i.$$

The plane field defined as $\xi = \ker \alpha$ is represented in Figure 1.1.
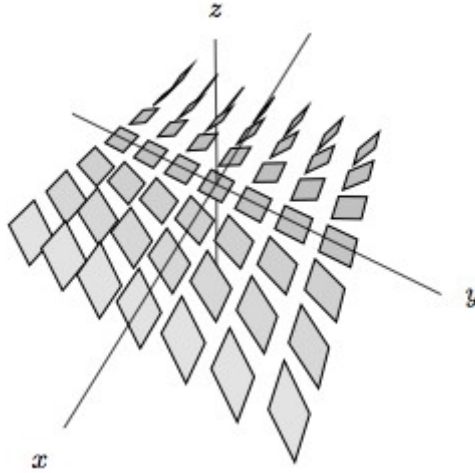
Figure 1.1: $\ker(dz + xdy)$ in $\mathbb{R}^3$

In fact, as it happens in symplectic geometry, any contact structure is locally contactomorphic to the previous example. This is known as the contact Darboux theorem, and follows from applying a Moser's path method known as Gray stability.

Theorem 1.1.10. *Any contact structure is locally contactomorphic to $\ker(\alpha_{std})$.*

It follows that any interesting geometrical property will be of semi-local or global nature. The existence of contact structures is a problem with a lot of

history, with great difficulty for closed manifolds. In 1969 Gromov [88] completely solved the problem for open manifolds using *h*-principle techniques. For closed manifolds, Martinet [134] proved the existence of a contact structure in every three-manifold, and together with the works of Lutz [132] we obtain the existence in every homotopy class of planes. In higher dimensions, there are topological obstructions to the existence of a contact structure in a given homotopy class of hyperplane fields: concretely, it needs to admit some almost complex structure at least. In [36, 65], this theorem was generalized to dimension 5, and finally in [15] the result (together with the classification of a subclass of contact structures called "overtwisted") was obtained in full generality.

Theorem 1.1.11. *Let M be a closed odd dimensional manifold of dimension $2n+1$. Then any homotopy class of hyperplanes which admits an almost complex structure is homotopic to a contact structure.*

The study of contact manifolds is interlaced with the study of its associated Reeb dynamics. In this direction, a classical conjecture known as Weinstein's conjecture states that a Reeb field in a closed manifold always has a closed orbit. The seminal paper by Hofer [105] proved the conjecture for overtwisted manifolds, and in 2008 Taubes [188] proved it in dimension three.

Theorem 1.1.12. *Let R be the Reeb field of a contact form α in a closed three dimensional manifold M. Then R has a closed orbit.*

The class of Beltrami fields which happen to correspond to reparametrized Reeb fields are those which are non-vanishing and rotational.

Definition 1.1.13. *A Beltrami field is called rotational if $\omega = \lambda X$ for an everywhere positive function $\lambda \in C^\infty(M)$.*

The correspondence between Reeb and Beltrami fields was proved by Etnyre and Ghrist [67], originally for manifolds of dimension three. However, the proof readily adapts to any odd dimension as detailed in [33].

Remark 1.1.14. The vorticity vector field ω of X in higher odd dimensions is defined by the only vector field satisfying

$$\iota_\omega \mu = (d\alpha)^n,$$

where α is the form dual to X by the metric.

Theorem 1.1.15. *Any nonsingular rotational Beltrami field is a reparametrization of a Reeb vector field for some contact form. Any reparametrization of a Reeb vector field of a contact structure is a nonsingular rotational Beltrami field for some metric and volume form.*

Proof. Let X be a Beltrami field for some volume form μ and metric g. Denote $\alpha = \iota_X g$ the dual form to X. The vorticity field satisfies $\omega = \lambda X$ for $\lambda \neq 0$ and is defined by $\iota_\omega \mu = (d\alpha)^n$. Combining the equations we obtain

$$(d\alpha)^n = \lambda \iota_X \mu.$$

It is clear then that α satisfies the contact condition

$$\alpha \wedge (d\alpha)^n = \lambda \iota_X g \wedge \iota_X \mu \neq 0.$$

Also X satisfies $\iota_X (d\alpha)^n = \iota_X \iota_X \mu = 0$ so $X \in \ker d\alpha$. Hence it is a reparametrization of the Reeb vector field, in fact by the function $\alpha(X) = g(X, X)$.

Consider now a contact form α and its Reeb vector field R. Let $Y = fR$ with $f > 0$ be a reparametrization of it. Take an almost-complex structure J on $\ker \alpha = \xi$ adapted to $d\alpha$ i.e. $d\alpha(\cdot, J\cdot)$ is definite positive. Define the metric

$$g(u, v) = \frac{1}{f}(\alpha(u) \otimes \alpha(v)) + d\alpha(u, Jv).$$

It is satisfied $\iota_Y g = \alpha$ by definition, and hence $d\iota_Y g = d\alpha$. Taking as volume form $\mu = \frac{1}{f}\alpha \wedge (d\alpha)^n$, Y is a Beltrami field for g and μ since $\iota_Y \mu = (d\alpha)^n$ and so $d\iota_Y \mu = 0$. $\qquad\square$

This correspondence theorem introduces contact geometry techniques to the study of steady Euler solutions.

Example 1.1.16. *We can explicitely construct the contact forms associated to the ABC fields introduced in Example 1.1.3. The dual form α with respect to the flat metric $g = dx^2 + dy^2 + dz^2$ is given by*

$$\alpha = [A \sin z + C \cos y]dx + [B \sin x + A \cos z]dy + [C \sin y + B \cos x]dz.$$

Computing its exterior differential we get

$$d\alpha = [+C \sin y + B \cos x]dx \wedge dy + [-A \cos z - B \sin x]dx \wedge dz + [A \sin z + C \cos y]dy \wedge dz.$$

Computing $\alpha \wedge d\alpha$ we get

$$\alpha \wedge d\alpha = [A^2 + B^2 + C^2]dx \wedge dy \wedge dz.$$

Since $A^2 + B^2 + C^2 > 0$ we have that α is a contact form.

Having introduced some background in contact geometry, we can now provide an example of unit geodesible field which preserves a family of transverse plane distributions.

Example 1.1.17. *Consider (M, ξ) a contact manifold. Let γ be a transverse knot in M and U a neighborhood of γ, it is modelled by*

$$(S^1 \times \mathbb{R}^2, \xi = \ker(d\theta + xdy - ydx)),$$

where θ the usual S^1-coordinate. The form $\alpha = d\theta + xdy - ydx$ extends to a contact form in M and its Reeb vector field R satisfies $R|_U = \partial_\theta$. The Reeb vector field preserves ξ, which is generated in U by $\langle -x\partial_\theta + \partial_y, -y\partial_\theta + \partial_x \rangle$.

Let $\varphi(x, y)$ be a bump function of a ball B of radius ε in the (x, y)-plane. Let $f(x)$ be any function and consider

$$g(x, y) = \varphi(x, y)f(x) + (1 - \varphi(x, y))x,$$

that satisfies $g(x, y) = x$ outside an open neighborhood V of the ball B, and $g(x, y) = f(x)$ in B. The one form $\bar{\alpha} = d\theta + g(x, y)dy - ydx$ coincides with α outside in $S^1 \times \mathbb{R} \setminus V \subset S^1 \times \mathbb{R}^2$. It generates the hyperplane distribution $\bar{\xi} = \langle g(x, y)\partial_\theta + \partial_y, y\partial_\theta + \partial_x \rangle$. This distribution, that extends to M and coincides with ξ outside of $S^1 \times V$, is also preserved by R. This is clear since $\iota_R \bar{\alpha} = 1$ and

$$d\bar{\alpha} = \frac{\partial g}{\partial x}dx \wedge dy - dy \wedge dx,$$

which implies that $\iota_R d\bar{\alpha} = 0$. In fact, one can construct a family of distributions preserved by R. By taking the path of functions $f_t = (1 - t)x + tg(x, y)$, we construct one-forms $\alpha_t = d\theta + f_t dy - ydx$ whose kernels are preserved by R.

1.2 Universality of Euler flows and flexibility of Reeb embeddings

When studying the Euler equations (1.1), the analysis of the evolution $X(\cdot, t)$ of a smooth initial condition $X(\cdot, 0) := X_0(\cdot)$ is a notoriously difficult problem where even the existence of a global-time solution is a challenging open question (the celebrated blow-up problem for the Euler equations). Recently, Terry Tao launched a programme to address the global existence problem, not only for the Euler equations, but also for their viscid counterpart, i.e. the Navier-Stokes equations, based on the concept of *universality* [184, 185, 186]. This notion concerns the Euler equations without fixing neither the ambient manifold M nor the metric g, and roughly speaking can be defined as the property that any smooth non-autonomous flow on a manifold N may be "extended" to a solution of the Euler equations for some (M, g), where the dimension of M is usually much bigger than the dimension of N. In [186], Tao introduced a particular way of extending a smooth (non-autonomous) flow on N to a solution of the Euler equations on a manifold M which is a product $M = N \times \mathbb{T}^m$ endowed with a warped product metric g. In particular, he showed that the set of flows that are extendible in the aforementioned sense is the countable union of nowhere dense sets (in the smooth topology), and that there exists

a somewhere dense set of flows that can be extended provided that N is diffeomorphic to the n-torus, $n \geq 2$. This interesting result provides further evidence of the universality of the Euler dynamics, but leaves open the problem whether the Euler equations on some high-dimensional Riemannian manifold can encode the behavior of a universal Turing machine [184, 185]. Tao discussed in [183, 187] that the "Turing completeness" of the Euler equations could be used as a route to construct solutions of the Navier-Stokes equations that blow-up in finite time, by creating an initial datum that is "programmed" to evolve to a rescaled version of itself (as a Von Neumann self-replicating machine).

Our goal in this section is to address the study of the universality of the Euler equations using stationary solutions, which model fluid flows in equilibrium. While at first glance it seems that the steady Euler flows are too restrictive to encode arbitrarily complicated dynamics, we shall see that the surprising connection between the Euler equations and contact topology, allows us to use the flexibility provided by the existence of h-principles in the contact realm to show that the stationary solutions exhibit universality features, and in particular they are Turing complete.

To this end, we introduce the concept of *Eulerisable flow* [162]: a volume-preserving (autonomous) vector field u on M is Eulerisable if there exists a Riemannian metric g on M compatible with the volume form, such that u satisfies the stationary Euler equations on (M, g)

$$\nabla_u u = -\nabla p \,, \qquad \operatorname{div} u = 0 \,. \tag{1.6}$$

The concept of Eulerisable flow will be further studied in the next Chapter of this thesis. As we know by Section 1.1.2, when the dimension of M is odd, a particularly relevant class of Eulerisable fields are those which are proportional to their curl through a not necessarily constant factor (a definition of the curl of a vector field in dimension $n > 3$, which is a nonlinear differential operator which assigns to a vector field another vector field, will be introduced in Section 1.2.2). These vector fields are the *Beltrami flows*, and in recent years they have found application as powerful tools to analyze different features of fluid flows, including anomalous weak solutions [129], complicated vortex structures [60, 61] and reconnections in Navier-Stokes [59]. The remarkable connection with contact geometry introduced in Section 1.1.3, which we will exploit in this part of this thesis, allows one to bring tools from (high dimensional) contact topology to the analysis of the stationary Euler equations provided that the Riemannian metric is not fixed, which is precisely the context where Tao introduced the notion of universality.

1.2.1   Main results

To state our main theorems, we need to provide a geometric definition of extendibility. The following captures the key ingredients of Tao's definition in [186]

but it is weaker in the sense that the ambient manifold M does not need to be a product $N \times \mathbb{T}^m$ and the metric is not forced to be a warped product.

Definition 1.2.1. *A non-autonomous time-periodic vector field $u_0(\cdot, t)$ on a compact manifold N is* Euler-extendible *if there exists an embedding $e : N \times \mathbb{S}^1 \to \mathbb{S}^n$ for some dimension $n > \dim N + 1$ (that only depends on the dimension of N), and an Eulerisable flow u on $\mathbb{S}^n$, such that $e(N \times \mathbb{S}^1)$ is an invariant submanifold of u and $e_*(u_0(\cdot, \theta) + \partial_\theta) = u$, $\theta \in \mathbb{S}^1$. If the non-autonomous field $u_0(\cdot, t)$ is not time-periodic, we say it is Euler-extendible if there exists a proper embedding $e : N \times \mathbb{R} \to \mathbb{R}^n$ for some dimension $n > \dim N + 1$ (that only depends on the dimension of N), and an Eulerisable flow u on $\mathbb{R}^n$, such that $e(N \times \mathbb{R})$ is an invariant submanifold of u and $e_*(u_0(\cdot, \theta) + \partial_\theta) = u$, $\theta \in \mathbb{R}$. If any non-autonomous dynamics $u_0(\cdot, t)$ is Euler-extendible, we say that the Euler flows are* universal.

Remark 1.2.2. In the time-periodic case, the choice of the ambient manifold $\mathbb{S}^n$, where the Eulerisable flow u is defined, is made for the sake of concreteness, but all the results we state in this paper hold for any other manifold. However, for general non-autonomous dynamics, the ambient space where u is defined does not need to be $\mathbb{R}^n$, but must be noncompact (because we need to embed properly $N \times \mathbb{R}$).

Roughly speaking, the extendibility of a non-autonomous dynamics implies that, in the appropriate local coordinates, u_0 describes the "horizontal" behavior of the integral curves of the extended vector field u. We want to emphasize that u_0 is not assumed to be volume-preserving, although certainly u will be.

We are now ready to present our first main result, which shows that the Eulerisable flows are flexible enough to encode any non-autonomous dynamics as above. Since these fields are stationary solutions of the Euler equations on some (M, g), they exist for all time.

Theorem 1.2.3. *The Euler flows are universal. Moreover, the dimension of the ambient manifold $\mathbb{S}^n$ or $\mathbb{R}^n$ is the smallest odd integer $n \in \{3 \dim N + 5, 3 \dim N + 6\}$. In the time-periodic case, the extended field u is a steady Euler flow with a metric $g = g_0 + \delta_P$, where g_0 is the canonical metric on $\mathbb{S}^n$ and δ_P is supported in a ball that contains the invariant submanifold $e(N \times \mathbb{S}^1)$.*

Remark 1.2.4. The extension of the non-autonomous flow u_0 to an Eulerisable flow on, say, $\mathbb{S}^n$ is not unique. In fact, we prove that given any embedding $\tilde{e} : N \times \mathbb{S}^1 \to \mathbb{S}^n$, there exists a smooth embedding e isotopic to $\tilde{e}$ and C^0-close to it which gives the Euler extension of u_0 introduced in Definition 1.2.1.

A striking corollary of this result, which illustrates the implications of the universality, is the embeddability of diffeomorphisms. We say that a (orientation-preserving) diffeomorphism $\phi : N \to N$ is *Euler-embeddable* if there exists an Eulerisable field u on $\mathbb{S}^n$ (for some n that only depends on the dimension of N) with an invariant submanifold exhibiting a cross-section diffeomorphic to N such that the first return map of u at this cross section is conjugate to ϕ.

Corollary 1.2.5. *Let N be a compact manifold and ϕ an orientation-preserving diffeomorphism on N. Then ϕ is Euler-embeddable in dimension n, where n is the smallest odd integer $n \in \{3 \dim N + 5, 3 \dim N + 6\}$.*

Let us mention a few words on the ideas of the proof of Theorem 1.2.3. The Eulerisable field u that we construct on $\mathbb{S}^n$ (or $\mathbb{R}^n$) is nonvanishing and of Beltrami type with constant proportionality factor (notice that n is an odd number). Using the correspondence between these fields and contact forms, the universality problem is then tantamount to studying the universality features of high-dimensional Reeb flows. A first difficulty is that the Reeb flows are geodesible, so their restriction to any invariant submanifold must be geodesible as well. Introducing the concept of *Reeb embedding* of a compact manifold into a contact manifold, and using the flexibility (existence of an h-principle) of the isocontact embeddings, we prove that in fact geodesibility is the only obstruction for a vector field to be extendable to a Reeb flow on some contact manifold. A second difficulty is that the field u_0 that we want to extend is not generally geodesible, a problem that we address considering the suspension of the field.

A consequence of our methods of proof, which is of interest in itself, is an almost sharp novel embedding theorem for manifolds endowed with a geodesible flow into a contact manifold, so that the Reeb field of the ambient manifold for some contact form extends the geodesible field on the submanifold. In view of the connection between Reeb and Beltrami fields which will be stated in Section 1.1.3, this theorem shows the flexible character of the steady Euler flows.

Theorem 1.2.6. *Let $e : (N, X) \to (M, \xi)$ be a embedding of N into a contact manifold (M, ξ) with X a geodesible vector field on N. Then:*

- *If $\dim M \geq 3 \dim N + 2$, then e is isotopic to a (small) Reeb embedding $\tilde{e}$, and $\tilde{e}$ can be taken C^0-close to e.*

- *If $\dim M \geq 3 \dim N$ and M is overtwisted, then e is isotopic to a Reeb embedding.*

The notion of *small Reeb embedding* in this statement will be introduced in Section 1.2.5. Moreover, we also obtain a full h-principle for what we call iso-Reeb embeddings (Reeb embeddings with certain fixed data) into overtwisted manifolds (Theorem 1.2.37) and into general contact manifolds (Theorem 1.2.39). We believe that these ideas may be useful to attack some purely geometric problems in Contact Topology.

Since Tao introduced the concept of universality to analyze the Turing completeness of the Euler equations [183, 184], we want to finish this introduction with an application of Theorem 1.2.3 in this setting. We say that an Eulerisable flow on $\mathbb{S}^n$ is *Turing complete* if the halting of any Turing machine with a given input is equivalent to a certain bounded trajectory of the flow entering a certain open set of $\mathbb{S}^n$ (what is known as the "reachability problem", see Section 1.2.4 for more

details). This implies, in particular, that the flow has undecidable trajectories. Our second main result is the following.

Theorem 1.2.7. *There exists an Eulerisable flow on $\mathbb{S}^{17}$ which is Turing complete.*

The solution of the Euler equations that encodes a universal Turing machine provided by this theorem is stationary. We do not know if it gives rise to a global-time solution when it is considered as the initial condition for the Navier-Stokes equations on $\mathbb{S}^{17}$ with the corresponding Riemannian metric.

This part of the thesis is organized as follows. In Section 1.2.2 we review some classical results on contact geometry and h-principles that will be instrumental in the next sections. In Section 1.2.3 we study the extendibility properties of the geodesible fields to Reeb fields for some contact manifold and state several Reeb embedding theorems that will be used in the proof of the theorems above. For the benefit of the reader, the proof of the most technically demanding Reeb embedding theorem, which gives an "almost optimal" dimension for the ambient manifold, is postponed to Section 1.2.5. In Section 1.2.4 we apply the previous results to the Euler equations to prove Theorems 1.2.3 and 1.2.7, and Corollary 1.2.5. Combining the results in Section 1.2.3 with a number of h-principles for embeddings into contact manifolds [58, 15], in Section 1.2.5 we establish a fairly general h-principle for iso-Reeb embeddings. Finally, in Section 1.2.6 we provide some examples and generalizations of the iso-Reeb embedding theorems proved in Section 1.2.3, in particular depicting the space of iso-Reeb embeddings. Unless otherwise stated, all the manifolds and submanifolds in this paper are orientable, connected and have no boundary.

1.2.2   An excursion to contact topology

We first review some concepts and results that will be instrumental in the forthcoming discussion. We state some classical flexibility theorems for embeddings in contact topology and introduce some basic facts about geodesible vector fields.

Contact geometry and h-principle for isocontact embeddings

In the first part of this chapter we introduced contact manifolds. The world of contact geometry exhibits a lot of flexibility which usually enables to use arguments from differential topology to prove geometric properties. The pioneering work of Gromov [88] shows that there exists a parametric h-principle for contact structures on open manifolds. For general manifolds, a parametric and relative h-principle was proved in [15] using overtwisted disks, see also [57] and [36] for previous results.

Grosso modo, the general philosophy of the h-principle leans on the idea of deforming *formal* solutions into *honest solutions* of an equation (PDE or, more generally, a partial differential relation). When this is possible, finding a solution is simplified to a homotopic-theoretical problem. A reincarnation of this principle

in the contact set-up requires a fine inspection of the notion of formal contact structure. Specifically, the topological information given by the contact distribution consists of the codimension one distribution ξ and the symplectic structure on it induced by $d\alpha$. In fact, only the conformal class is determined because a rescaling $\alpha' = f\alpha$ is a contact form for the same contact structure. This allows one to introduce the concept of a formal contact structure that is defined as a cooriented hyperplane distribution and a conformally symplectic class on it. In the literature this structure has been usually called almost contact structure, however in the last few years the term formal has become standard since it implements the formal condition for the h-principle. We can find a 2-form ω on M such that $(\xi, \omega|_\xi)$ is a conformal symplectic vector bundle: we just say that two formal contact structures defined as $(\xi, \omega|_\xi)$ and $(\xi, \hat{\omega}|_\xi)$ if ω and $\hat{\omega}$ are conformally equivalent. So a formal contact structure is described by a codimension one conformal symplectic vector bundle.

The flexibility statements that we need in this paper concern isocontact embeddings. Recall that a map $f : (N, \xi_N) \to (M, \xi_M)$ between contact manifolds is called isocontact if $f_* \xi_N = \xi_M$. In the formal level, a monomorphism $F : TN \to TM$ is called isocontact if $\xi_N = F^{-1}(\xi_M)$ and F induces a conformally symplectic map with respect to the conformal symplectic structures $CS(\xi_N)$ and $CS(\xi_M)$. The following h-principle was proved in [58, Section 12.3.1]. We recall that N_0 is called a core of an open manifold N if for an arbitrarily small neighborhood U of N_0, there is an isotopy which brings U to the whole N.

Theorem 1.2.8. *Let (N, ξ_N) and (M, ξ_M) be contact manifolds of dimension $2n+1$ and $2m+1$ respectively. Let $f_0 =: (N, \xi_N) \to (M, \xi_M)$ be an embedding such that its differential $F_0 := df_0$ is homotopic (via monomorphisms $F_t : TN \to TM$, with projections onto the base given by f_0) to a conformal symplectic monomorphism F_1. Then*

- *If N is open and $n \leq m - 1$ then there is an isotopy $f_t : N \to M$ such that the embedding f_1 is isocontact and df_1 is homotopic to F_1 through conformal symplectic monomorphisms. Given a core N_0 of N, f_t can be chosen arbitrarily C^0-close to f_0 near N_0.*

- *If N is closed and $n \leq m - 2$ then the above f_t exists. Moreover, one can choose f_t to be arbitrarily C^0-close to f_0.*

In [15], the authors showed that every formal contact structure is deformable to a genuine contact structure, thus proving the long standing conjecture of the existence of contact structures in every formal contact manifold. Restricting to a particular class of contact structures called *overtwisted*, a full h-principle was proved, thus implying a result stronger than Theorem 1.2.8 for isocontact embeddings into overtwisted manifolds. This result, which holds for codimension 0 isocontact embeddings of open manifolds, can be summarized as follows:

Theorem 1.2.9. *Let (M^{2m+1}, ξ) be a connected overtwisted contact manifold and (N^{2m+1}, ζ) an open contact manifold of the same dimension. Let $f : N \to M$ be a smooth embedding covered by an isocontact bundle homomorphism $\varphi : TN \to TM$, that is such that $\varphi(\zeta_x) = \xi_{f(x)}$ and φ preserves the conformal symplectic structures on the distributions. If df and φ are homotopic as injective bundle homomorphisms then f is isotopic to an isocontact embedding $\tilde{f}$.*

1.2.3 Reeb-embeddability and geodesible fields

The goal of this section is to prove the following theorem:

Theorem 1.2.10. *Let (N, X) be a compact manifold endowed with a geodesible field X. Then there is a smooth embedding $e : N \to \mathbb{S}^n$ with $n = 4 \dim N - 1$ and a 1-form α defining the standard contact structure ξ_{std} on $\mathbb{S}^n$ such that $e(N)$ is an invariant submanifold of the Reeb field R defined by α and $e_* X = R$. Moreover, α is equal to the standard contact form α_{std} in the complement of a ball that contains $e(N)$.*

To prove this result, we first recall (Subsection 1.2.3) Inaba's characterization of the vector fields on a submanifold of a contact manifold (M, ξ) that can be extended as Reeb flows for some contact form defining the contact structure ξ. In Subsection 1.2.3 we introduce the concept of *Reeb embedding* and prove Theorem 1.2.10 using an *h*-principle for isocontact embeddings. Finally, in Subsection 1.2.3 we state a stronger Reeb embedding result (Theorem 1.2.6) which substantially improves the dimension n in Theorem 1.2.10 and shows that, roughly speaking, any embedding of high enough codimension can be deformed into a Reeb embedding. The proof of this result is more involved and will be postponed to Section 1.2.5.

We shall see in Section 1.2.4 how these results can be used, in combination with the correspondence theorem in Subsection 1.1.3, to prove the universality results stated in the introduction of Section 1.2. As an immediate corollary we obtain:

Corollary 1.2.11. *Let (N, X) be a compact manifold endowed with a geodesible field X which is not necessarily of unit length. Then there exists an embedding $e : N \to \mathbb{S}^n$ with $n = 4 \dim N - 1$ and a non-vanishing Beltrami field u on $\mathbb{S}^n$ with constant proportionality factor such that $e_* X = u$. The Riemannian metric for which u is a Beltrami field is the canonical metric of $\mathbb{S}^n$ in the complement of a ball containing $e(N)$.*

Proof. Reparametrizing X we obtain another geodesible vector field $\tilde{X}$ of unit length. Theorem 1.2.10 implies that $(N, \tilde{X})$ admits an embedding into $(\mathbb{S}^n, \xi_{std})$, $n = 4 \dim N - 1$, such that there is a defining contact form α whose Reeb vector field satisfies $R|_{e(N)} = \tilde{X}$. Obviously, we can now reparametrize R by a function f such that $fR|_{e(N)} = X$ and $f = 1$ in the complement of a ball B that contains

$e(N)$. By Theorem 1.1.15, the vector field fR, which is no longer Reeb in general, is a Beltrami field with constant proportionality factor for some Riemannian metric on $\mathbb{S}^n$ that can be taken to be the canonical metric in the complement of B. $\square$

Extension of Reeb flows

We recall a simple characterization due to Inaba [114] of the vector fields on a submanifold of a contact manifold that can be extended to a Reeb vector field. For the sake of completeness, we include a concise proof.

Lemma 1.2.12. *Let (M, ξ) be a (cooriented) contact manifold and (N, X) a compact submanifold of M endowed with a tangent (non-vanishing) vector field X which is positively transverse to ξ on N. Then there is a contact form α defining ξ such that its Reeb vector field R satisfies $R|_N = X$ if and only if X preserves $TN \cap \xi$.*

Proof. The necessity is trivial because a Reeb vector field preserves the contact distribution. To prove the sufficiency, assume that the vector field X on N preserves the tangent distribution $\eta := TN \cap \xi$. It is useful to denote the embedding of N into M by $e : N \to M$, where with a slight abuse of notation we are identifying N with its embedded image.

Let α_0 be a defining contact form of ξ. Fix the strictly positive smooth function h_N on N, given as $h_N := \frac{1}{e^*\alpha_0(X)}$. By using partitions of unity, we can find a strictly positive function $h : M \to \mathbb{R}^+$ such that $h|_N = h_N$. Define a new 1-form $\alpha_1 := h\alpha_0$, still associated to the contact structure ξ, which by construction satisfies the first condition in the defining Reeb equations

$$\iota_X \alpha_1 = 1 \,.$$

Since X preserves $\ker \alpha_1$, it preserves $\ker e^*\alpha_1$. Hence this reads as,

$$\mathcal{L}_X e^*\alpha_1 = f e^*\alpha_1,$$

where f is a smooth function. By Cartan formula this implies that $\iota_X e^* d\alpha_1 = f e^*\alpha_1$. Contracting this equation (in 1-forms) with the vector field X, we immediately obtain that $f = 0$. Thus, we have

$$\iota_X d e^*\alpha_1 = 0. \tag{1.7}$$

Now we want to find a new associated contact form multiplying by a strictly positive smooth function λ on M such that the vector field X satisfies the Reeb equations when applied to the 1-form $\alpha := \lambda\alpha_1$. Taking a function λ such that $\lambda|_N = 1$, by the uniqueness of the Reeb vector field, this is tantamount to saying that X verfies $\iota_X d(\lambda\alpha_1) = 0$, since this new contact form still satisfies $e^*(\alpha)(X) = 1$. Thus, we just need to find a function λ such that the 1-form $\lambda\alpha_1$ satisfies the second Reeb equation

$$\iota_X d(\lambda\alpha_1) = 0 \,,$$

on $TM|_N$. Expanding it we obtain

$$0 = \iota_X d(\lambda \alpha_1) = \iota_X(d\lambda \wedge \alpha_1 + \lambda d\alpha_1) \,,$$

that restricted to N reads as

$$0 = -d\lambda + d\alpha_1(X) \,, \tag{1.8}$$

where we have used that X is tangent to N and $\lambda|_N = 1$. Accordingly, the condition that λ must satisfy reads as $d\lambda = d\alpha_1(X)$ on N (over the whole $TM|_N$).

Since we proved above that $\iota_X e^* d\alpha_1 = 0$, and $\lambda|_N = 1$, the equality of 1-forms (1.8) holds on $TN \subset TM$. For the normal directions, just find a smooth function such that the partial derivatives for any $v \in T_p M$ with $p \in N$ satisfy $\frac{\partial \lambda}{\partial v} = d\alpha_1(X, v)$. This determines the whole 1-jet of the function λ on N. Again, by a standard argument taking partitions of the unity, this implies the existence of a positive smooth function λ on M that extends this given 1-jet. The lemma then follows. □

Remark 1.2.13. We remark that the vector field X in Lemma 1.2.12 is geodesible. Indeed, following the notation of the proof of the lemma, the 1-form $\beta := e^* \alpha$ satisfies that $\iota_X \beta = 1$ and $\iota_X d\beta = 0$, which implies the geodesibility according to Subsection 1.1.2.

Remark 1.2.14. It follows from the proof of the lemma, that if α_0 is an associated contact form for ξ, then the 1-form α can be taken to be equal to α_0 in the complement of a neighborhood of $N \subset M$ (just take extensions of the functions h and λ in the proof so that $h = \lambda = 1$ in the complement of the neighborhood).

Existence of Reeb embeddings

The characterization of vector fields extendible to Reeb flows presented in the previous subsection suggests the following definition:

Definition 1.2.15. *Let* (N, X) *be a geodesible field on a compact manifold. An embedding* $e : (N, X) \to (M, \xi)$ *of N into a contact manifold M is called a* Reeb embedding *if there is a contact form α defining ξ such that its Reeb vector field R satisfies $e_* X = R$ (in particular $e(N)$ is an invariant submanifold of R). If we further assume that the geodesible vector field comes with a fixed preserved distribution* $\ker \beta = \eta$, *then an embedding is called an* iso-Reeb embedding *if* $e^* \xi = \eta$.

Observe that a Reeb embedding $e : (N, X) \to (M, \xi)$ clearly induces an iso-Reeb embedding just by declaring $\eta := e^* \xi$. As noticed before, any Reeb vector field tangent to a submanifold is geodesible on it. Theorem 1.2.10 then claims that the converse also holds, i.e. that for any geodesible field (N, X) there exists a Reeb embedding into a high-dimensional sphere endowed with the standard contact structure. The following technical lemma is key to prove the main result of this section. For the proof, we follow [58, Section 16.2.2].

Lemma 1.2.16. *Let (N, X, η) be a geodesible field on a compact manifold N of dimension n_0, and β a defining 1-form of the hyperplane distribution η. Assume that there exists an embedding $e : (N, X, \eta) \to M$ into a manifold of dimension $2m - 1$ endowed with a hyperplane distribution ξ defined on $e(N)$ such that*

1. *$\eta = e^*\xi$.*

2. *There is a nondegenerate 2-form ω on $\xi|_N$.*

3. *$\omega|_{TN} = d\beta$.*

Then there is a small neighborhood U of $e(N)$ in M and a contact form α on U such that $e^*\alpha = \beta$.

Proof. For notational simplicity, we shall identify N with its embedded image $e(N)$. Consider a small neighborhood $U \subset M$ of N, which can be identified with a normal disk bundle $\pi : U \to N$. Fix a covering by small open sets $V_j \subset N$ where $U = V_j \times \mathbb{R}^{m'-1}$, $m' := 2m - n_0$. Since $e^*\xi = \eta$, then the hyperplane distribution ξ on N can be split as $\xi|_N = \eta \oplus \mathbb{R}^{m'-1}$. In terms of this splitting, the assumption $\omega|_{TN} = d\beta$ implies that the 2-form ω can be written as

$$\omega = d(\pi^*\beta) + \omega'$$

where ω' is a 2-form that satisfies that $\omega'|_\eta = 0$.

Let us introduce coordinates $(y_{1,j}, \cdots, y_{m'-1,j})$ in the second factor of $V_j \times \mathbb{R}^{m'-1}$. In these coordinates, we can assume that ω' has the form

$$\omega' = \sum_{k=1}^{m'-1} dy_{k,j} \wedge \beta_{k,j}$$

where $\beta_{k,j}$ are suitable 1-forms on N. Now we can define on U the 1-form

$$\alpha_j := \pi^*\beta + \sum_{k=1}^{m'-1} y_{k,j}\beta_{k,j} \, .$$

Notice that $e^*\alpha_j = \beta$ and that $\ker \alpha_j|_N = \xi$. Moreover, since

$$
\begin{aligned}
(d\alpha_j)|_N &= d(\pi^*\beta) + \sum_{k=1}^{m'-1} (dy_{k,j} \wedge \beta_{k,j}|_N + y_{k,j} \wedge d\beta_{k,j}|_N) \\
&= d(\pi^*\beta) + \sum_{k=1}^{m'-1} dy_{k,j} \wedge \beta_{k,j}|_N \\
&= \omega|_N,
\end{aligned}
$$

is nondegenerate on $\xi|_N$, and X is transverse to ξ. Observe that α_j is a contact form over $V_j \times \mathbb{R}^{m'-1}$. Choose a partition of the unity χ_j subordinated to the open cover V_j, and define $\alpha = \sum_j \chi_j \alpha_j$. Expanding this expression we get that:

$$(d\alpha)|_N = d(\pi^* \beta) + \sum_j \chi_j \sum_{k=1}^{m'-1} dy_{k,j} \wedge \beta_{k,j}|_N$$
$$= \omega|_N.$$

This concludes the proof. $\square$

Proof of Theorem 1.2.10 : Following the notation introduced in Lemma 1.2.16, η is the hyperplane distribution on N preserved by X, and β is a defining 1-form. Consider the vector bundle M defined by the dual distribution η^* over N, and denote the bundle projection as $\pi : M \to N$. Observe that $\dim M = 2n_0 - 1$ and that the tangent bundle of M at the zero section (which is the manifold N) splits as $TM|_N = TN \oplus \eta^* = \langle X \rangle \oplus \eta \oplus \eta^*$. This distribution $\eta \oplus \eta^*$ over N has dimension $2n_0 - 2$ and is equipped with the canonical symplectic form ω_0 defined by

$$\omega_0((v_1 \oplus \alpha_1), (v_2 \oplus \alpha_2)) := \alpha_2(v_1) - \alpha_1(v_2),$$

where v_k is a section of η and α_k is a section of η^*. Observe that with this symplectic structure η on N is an isotropic subspace, i.e. denoting by $j : \eta \to \eta \oplus \eta^*$ the natural inclusion, we have that $j^* \omega_0 = 0$.

Let us now perturb the symplectic structure ω_0 by lifting a 2–form on N to $TM|_N$. For every point $p \in N$ we define the 2-form

$$\omega_N|_p = A(\omega_0)|_p + (d\beta)|_p,$$

where $A > 0$ is a constant large enough so that ω_N defined on N is still nondegenerate. It follows from the previous construction that, if $e_0 : (N, X, \eta) \to M$ denotes the natural inclusion then we can apply Lemma 1.2.16 to conclude that there is a contact form α in a neighborhood U of N in M such that $e_0^* \alpha = \beta$. Notice that the contact distribution $\xi := \ker \alpha$ coincides with $\eta \oplus \eta^*$ on N.

Summarizing, we have constructed an open contact manifold U of dimension $2n_0 - 1$ with a submanifold N endowed with a vector field X that is positively transverse to the contact distribution ξ and preserves $TN \cap \xi = \eta$. The h-principle for isocontact embeddings [58, 88] (see Lemma 1.2.17 below) implies that U can be isocontact embedded into $(\mathbb{S}^n, \xi_{std})$ for $n = 4n_0 - 1$. Denoting this embedding by $e : U \to \mathbb{S}^n$ it obviously satisfies $e_0^* e^* \xi_{std} = \eta$. Identifying N with its embedded image in $\mathbb{S}^n$ (via the embedding $e \circ e_0$), we have that the field X preserves $TN \cap \xi_{std} = \eta$, so we can apply Lemma 1.2.12 to conclude that there is a contact form $\tilde{\alpha}$ whose Reeb field R coincides with X on N, and $\tilde{\alpha} = \alpha_{std}$ in the complement of a neighborhood of N. The theorem then follows. $\square$

Lemma 1.2.17. *Any smooth embedding of a contact manifold (N^{2n_0-1}, η) into $(\mathbb{S}^{4n_0-1}, \xi_{std})$ is a formal isocontact embedding.*

Proof. Let $f : N^{2n_0-1} \to \mathbb{S}^{4n_0-1}$ be a smooth embedding. Let us construct a family of monomorphisms $F_t : TN^{2n_0-1} \to T\mathbb{S}^{4n_0-1}|_N$ such that $F_0 = df$, $F_1(R_\eta) = R_{std}$ (the corresponding Reeb fields), $F_1(\eta) \subset \xi_{std}$ and F_1 is a complex monomorphism. To this end, we first find a family of vector fields R_t over $T\mathbb{S}^{4n_0-1}|_N$ such that $R_0 = R_\eta$ and $R_1 = R_{std}$. This family exists because the connectedness of the sphere is higher than the dimension of N, i.e., any two sections of $T\mathbb{S}^{4n_0-1}|_N$ are homotopic through non-vanishing sections since $4n_0 > 2n_0 - 1$. Now fix ξ_t to be any complementary of R_t which satisfies $f_*\eta \subset \xi_1$. This automatically provides a family, canonical up to homotopy of (real) monomorphisms, $F_t : TN \to T\mathbb{S}^{4n_0-1}|_N$ such that $F_0 = df$. It remains to show that $F_1 : \eta \to \xi_{std}|_N$ is homotopic to a complex monomorphism, but this is an easy consequence of the connectedness of the inclusion map of the space of complex monomorphisms into the space of real monomorphisms, i.e., the rank of connectedness is bigger than the dimension of N. $\square$

Corollary 1.2.18. *Let X be a nonvanishing vector field on a compact manifold N. Then N embeds into some contact manifold (M, ξ) such that $X = R|_N$ for a Reeb vector field R of some contact form if and only if X is geodesible.*

Remark 1.2.19. The isocontact embedding theorem used in the proof of Theorem 1.2.10 works for any ambient contact manifold of dimension $n = 4 \dim N - 1$ (because it gives an embedding into a Darboux neighborhood of any contact manifold of dimension bigger or equal than $4 \dim N - 1$). This implies, in particular, that the ambient manifold in Theorem 1.2.10 can be taken to be $(\mathbb{R}^n, \xi_{std})$.

Remark 1.2.20. When the manifold N is non-compact the following observation allows one to prove a result analogous to Theorem 1.2.10. Indeed, Lemma 1.2.12 works if N is a properly embedded submanifold, and the embedding provided by Whitney embedding theorem can be taken proper [127]. Accordingly, Theorem 1.2.10 provides a Reeb embedding of any pair (N, X) with N non-compact and X geodesible into $(\mathbb{R}^n, \xi_{std})$, $n = 4 \dim N - 1$.

An improved Reeb embedding theorem

Theorem 1.2.10 shows the existence of a Reeb embedding of (N, X) into $\mathbb{S}^n$ for $n = 4 \dim N - 1$. This suggests two questions:

1. Can we improve the bound on the dimension n of the target space?

2. Can an embedding $e : (N, X) \to (M, \xi)$ be deformed into a Reeb embedding via an isotopy which is C^0-close to the identity?

Let us finish this section by stating Theorem 1.2.6, which is a generalization of Theorem 1.2.10, and answers these questions. Its proof, which makes use of some non trivial modern h–principle results in contact topology, is technically much more involved than the proof of Theorem 1.2.10, and will be presented in

Section 1.2.5 together with a few corollaries that can be useful for other applications in Contact Geometry. This theorem is key for the proofs of Theorems 1.2.3 and 1.2.7 stated in the Introduction.

Theorem (Theorem 1.2.6). *Let $e : (N, X) \to (M, \xi)$ be a embedding of N into a contact manifold (M, ξ) with X a geodesible vector field on N. Then:*

- *If $\dim M \geq 3 \dim N + 2$, then e is isotopic to a (small) Reeb embedding $\tilde{e}$, and $\tilde{e}$ can be taken C^0-close to e.*

- *If $\dim M \geq 3 \dim N$ and M is overtwisted, then e is isotopic to a Reeb embedding.*

The notions in the statement will be introduced in Section 1.2.5. For the proofs of Theorems 1.2.3 and 1.2.7 the (weaker) statement that provides a general Reeb embedding is sufficient. Roughly speaking, Theorem 1.2.6 shows that Reeb embeddings are completely determined by differential topology invariants. This fact can be easily encoded in the *h*-principle philosophy (see [58, 88]), details will be provided in Section 1.2.5. As a Corollary we obtain the following improved version of Corollary 1.2.11; the proof is analogous so we omit it.

Corollary 1.2.21. *Let (N, X) be a compact manifold endowed with a geodesible field X which is not necessarily of unit length. Then there exists an embedding $e : N \to \mathbb{S}^n$ with n the smallest odd integer $n \in \{3 \dim N + 2, 3 \dim N + 3\}$, and a non-vanishing Beltrami field u on $\mathbb{S}^n$ with constant proportionality factor such that $u|_{e(N)} = X$. The Riemannian metric for which u is a Beltrami field is the canonical metric of $\mathbb{S}^n$ in the complement of a ball containing $e(N)$.*

1.2.4 Applications: proof of Theorems 1.2.3 and 1.2.7

Our goal in this section is to apply the results on Reeb embeddings in Section 1.2.3 to prove the main theorems stated in the Introduction on the universality of the Euler flows.

Non-autonomous dynamics and universality

Let $u_0(\cdot, t)$ be a non-autonomous vector field on a compact manifold N, and assume that it is 2π periodic in t. The suspension of u_0 on the manifold $N \times \mathbb{S}^1$ (with $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$) is another vector field defined as

$$X(x, \theta) := u_0(x, \theta) + \partial_\theta,$$

with $x \in N$ and $\theta \in \mathbb{S}^1$.

The vector field X on $N \times \mathbb{S}^1$ is geodesible. Indeed, the closed 1-form $\beta := d\theta$ obviously satisfies that $\beta(X) = 1$ and $\iota_X d\beta = 0$, so Gluck's characterization implies that X is geodesible, c.f. Subsection 1.1.2. Now, applying Theorem 1.2.6

to the pair $(N \times \mathbb{S}^1, X)$, we conclude that there exists a Reeb embedding $e : (N \times \mathbb{S}^1, X) \to (\mathbb{S}^n, \xi_{std})$ for the smallest odd integer $n \in \{3 \dim N + 5, 3 \dim N + 6\}$. This means, c.f. Definition 1.2.15, that there is a defining 1-form α of ξ_{std} whose Reeb field R satisfies that $R|_{e(N \times \mathbb{S}^1)} = X$, and $\alpha = \alpha_{std}$ in the complement of a ball B that contains $e(N \times \mathbb{S}^1)$.

It follows from the Beltrami-Reeb correspondence Theorem 1.1.15, that R is a Beltrami field (and hence a steady Euler flow) for some metric g on $\mathbb{S}^n$. Moreover, since the adapted metric to the standard contact form on the sphere is the round metric g_0, it turns out that $g = g_0$ in the complement of $B \subset \mathbb{S}^n$.

Setting $u := R$, the previous construction shows that any (time-periodic) non-autonomous dynamics u_0 is Euler-extendible, recall Definition 1.2.1.

The general case of a non-autonomous flow $u_0(\cdot, t)$ is analogous. The suspension manifold is $N \times \mathbb{R}$ and X is defined as above with $\theta \in \mathbb{R}$. Gluck's theorem also implies that it is geodesible, so proceeding as before we conclude that u_0 is Euler-extendible to $\mathbb{R}^n$, for the smallest odd integer $n \in \{3 \dim N + 5, 3 \dim N + 6\}$. Note that in this case the adapted metric to the standard contact form on $\mathbb{R}^n$ is not the Euclidean one. This completes the proof of Theorem 1.2.3.

Remark 1.2.22. When the extended manifold is $\mathbb{S}^n$, the steady Euler flow u is equal to the Hopf field in the complement of B (because the Hopf field is the Reeb field associated to the standard contact form). In the case that the extension is in $\mathbb{R}^n$, the vector field u is the vertical field ∂_{x_n} in the complement of a neighborhood of the non-compact manifold $e(N \times \mathbb{R})$.

Remark 1.2.23. When the vector field u_0 is autonomous and geodesible (not necessarily of unit length) we do not need to take the suspension of u_0. In this case we can directly apply Corollary 1.2.21 to conclude that (N, u_0) can be embedded into $\mathbb{S}^n$, where n is the smallest odd integer $n \in \{3 \dim N + 2, 3 \dim N + 3\}$, so that $e_* u_0$ extends as a Beltrami field with constant proportionality factor on $\mathbb{S}^n$.

We conclude this subsection by proving Corollary 1.2.5. The main idea is again a suspension construction, depicted in Figure 1.2.

Indeed, let $\tilde{N}$ be the manifold defined as $\tilde{N} := N \times [0,1]/\sim$ where we identify $(x, 0)$ with $(\phi(x), 1)$. Consider the horizontal vector field ∂_θ on $N \times [0,1]$, where $\theta \in [0,1]$. This vector field immediately pushes down to another field on $\tilde{N}$ that we call X. Observe that N is a cross section of X and its (time-one) return map is conjugate to ϕ. Arguing as before, we show that X is geodesible and can be extended to a steady Euler flow on $\mathbb{S}^n$, $n \in \{3 \dim N + 5, 3 \dim N + 6\}$, thus showing that ϕ is Euler-embeddable.

Turing completeness

We prove now Theorem 1.2.7, i.e. that there exists a steady Euler flow on $\mathbb{S}^{17}$ that is Turing complete. To this end, let us first recall that a Turing machine is a 5-tuple $(Q, q_0, F, \Sigma, \delta)$ where

$$\phi(N) \cong N$$

x

$\phi(x)$
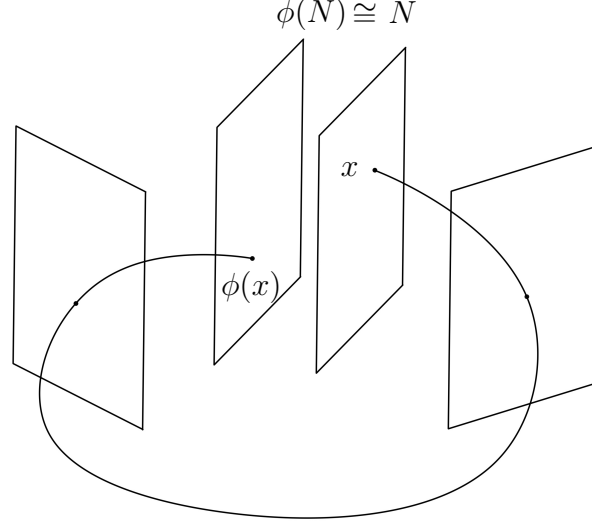
Figure 1.2: Suspended diffeomorphism

- Q is a finite non-empty set, the set of "states".

- $q_0 \in Q$ is the initial state.

- $F \in Q$ is the halting state.

- Σ is the alphabet, a finite set of cardinality at least two.

- $\delta : (Q \backslash F) \times \Sigma \to Q \times \Sigma \times \{L, N, R\}$ is a partial function called a transition function. We denote by L the left shift, R is the right shift and N represents a "no shift".

Following Tao [184], we consider a Turing machine with a single tape that is infinite in both directions and a single halting state, with the machine shifting the tape rather than a tape head; in particular we do not need to isolate a blank symbol character in the alphabet (anyway, all the results here apply to other variants of a Turing machine). We denote by q the current state, and $t = (t_n)_{n \in \mathbb{Z}}$ the current tape. For a given Turing machine $(Q, q_0, F, \Sigma, \delta)$ and an input tape $s = (s_n)_{n \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}}$ the machine runs applying the following algorithm:

1. Set the current state q as the initial state and the current tape t as the input tape.

2. If the current state is F then halt the algorithm and return t as output. Otherwise compute $\delta(q, t_0) = (q', t_0', \varepsilon)$, with $\varepsilon \in \{L, R, N\}$.

3. Replace q with q' and t_0 with t_0'.

4. Replace t by the ε-shifted tape, then return to step (2).

For any input the machine will halt at some point and return an output or run indefinitely. The Turing completeness of a dynamical system can be understood in terms of the concept of a *universal Turing machine*, which is a machine that can simulate all Turing machines.

Tao showed in [184, Proposition 1.10] the existence of an orientation-preserving diffeomorphism ϕ of the torus $\mathbb{T}^4$ that encodes a universal Turing machine in the following sense:

Proposition 1.2.24. *There exists an explicitly constructible diffeomorphism ϕ : $\mathbb{T}^4 \to \mathbb{T}^4$ such that for any Turing machine $(Q, q_0, F, \Sigma, \delta)$ there is an explicitly constructible open set $U_{t_{-n},...,t_n} \subset \mathbb{T}^4$ attached to each finite string $t_{-n}, ..., t_n \in \Sigma$, and an explicitly constructible point $y_s \in \mathbb{T}^4$ attached to each $s \in \Sigma^{\mathbb{Z}}$ such that the Turing machine with input tape s halts with output $t_{-n}, ..., t_n$ in positions $-n, ..., n$, respectively, if and only if the orbit $y_s, \phi(y_s), \phi^2(y_s), ...$ enters $U_{t_{-n},...,t_n}$.*

Let us now prove Theorem 1.2.7 using that the diffeomorphism $\phi : \mathbb{T}^4 \to \mathbb{T}^4$ that encodes a universal Turing machine constructed in Proposition 1.2.24 can be Euler-embedded in $\mathbb{S}^{17}$, c.f. Corollary 1.2.5. More precisely, calling N the 5-dimensional manifold defined as $N := \mathbb{T}^4 \times [0, 1]/ \sim$, where we identify $(x, 0)$ with $(\phi(x), 1)$, and X the vector field on N obtained by pushing down the horizontal vector field ∂_θ on $\mathbb{T}^4 \times [0, 1]$ ($\theta \in [0, 1]$), there exists an embedding $e : (N, X) \to \mathbb{S}^{17}$ and an Eulerisable field u on $\mathbb{S}^{17}$ of Beltrami type such that $u|_{e(N)} = X$. Notice that X is geodesible and e is a Reeb embedding (then by Theorem 1.2.6, given any embedding of N in $\mathbb{S}^{17}$, it can be deformed into e by an isotopy that is C^0-close to the identity). The diffeomorphism e is explicitly constructible because the applied h-principle is algorithmic; also the vector field u is constructible, because it is the Reeb field of a defining contact form α of ξ_{std}, which is also algorithmic (see the proof of Theorem 1.2.6).

In view of the previous discussion, let us take a point $\tilde{y}_s \in \mathbb{S}^{17}$ as the image of the point $y_s \times \{0\} \in N$ under the embedding e, and a neighborhood $\tilde{U}_{t_{-n},...,t_n} \subset \mathbb{S}^{17}$ as a neighborhood in $\mathbb{S}^{17}$ of the image of the open set $U_{t_{-n},...,t_n} \times \{0\} \subset N$ under the embedding e. Then, Theorem 1.2.7 can be restated in a more precise way as follows:

Theorem (Beltrami fields are Turing complete)**.** *There exists a Beltrami field u on $\mathbb{S}^{17}$ for some Riemannian metric g such that for any Turing machine $(Q, q_0, F, \Sigma, \delta)$ there is an explicitly constructible open set $U_{t_{-n},...,t_n} \subset \mathbb{T}^4$ attached to each finite string $t_{-n}, ..., t_n \in \Sigma$ and an explicitly constructible point $y_s \in \mathbb{T}^4$ attached to each $s \in \Sigma^{\mathbb{Z}}$ such that the Turing machine with input tape s halts with output $t_{-n}, ..., t_n$ in positions $-n, ..., n$ respectively if and only if the trajectory of u with initial datum $\tilde{y}_s$ enters $\tilde{U}_{t_{-n},...,t_n}$.*

Remark 1.2.25. The metric g in this theorem is the canonical metric of $\mathbb{S}^{17}$ in the complement of a neighborhood of $e(N)$. Observe that in fact we may reduce

the dimension of the target sphere by 2 in all the applications of this section by considering a sphere with an overtwisted contact structure. In this case we would obtain a Turing complete Euler flow in $\mathbb{S}^{15}$. In that case we cannot longer guarantee that the metric g is the canonical metric in the complement of a neighborhood of $e(N)$.

The existence of a universal solution in $\mathbb{R}^n$

Using the ideas developed in this work, we can show that there exists an Eulerisable flow in $\mathbb{R}^n$ which, in some sense, exhibits all possible lower-dimensional dynamics. To be more precise, let us introduce the following definitions:

Definition 1.2.26. *Given two vector fields X_1 and X_2 in $N \times \mathbb{S}^1$, where N is a compact manifold, we say that X_1 is (ε, k)-conjugate to X_2 if there is a diffeomorphism $\varphi : N \times \mathbb{S}^1 \to N \times \mathbb{S}^1$ such that*

$$||\varphi_* X_1 - X_2||_{C^k(N \times \mathbb{S}^1)} < \varepsilon.$$

Definition 1.2.27. *Fix a positive integer k. A vector field u in $\mathbb{R}^n$ is N-universal if for any ε and any vector field X on N there is an invariant submanifold $\tilde{N}$ of u diffeomorphic to $N \times \mathbb{S}^1$ such that $u|_{\tilde{N}}$ is (ε, k)-conjugate to $X + \partial_\theta$ with $\theta \in \mathbb{S}^1$.*

Theorem 1.2.28. *Let N be a compact manifold. There exists an N-universal Eulerisable flow of Beltrami type in $\mathbb{R}^n$, where the dimension is the smallest odd integer $n \in \{3 \dim N + 5, 3 \dim N + 6\}$.*

Proof. We first recall that the space $\mathfrak{X}(N)$ of smooth vector fields on N is second countable with the Whitney topology [103, Chapter 2.1]. In particular, it is separable and hence there is a countable set of vector fields $\{X_j\}_{j \in \mathbb{Z}}$ which is dense in $\mathfrak{X}(N)$. For every pair (N, X_j), we can take the suspension of the vector field X_j as in Subsection 1.2.4 to obtain a countable set of pairs (N_j, Y_j) where N_j is diffeomorphic to $N \times \mathbb{S}^1$ and $Y_j := X_j + \partial_\theta$ is a geodesible flow. Now take a countable collection of contact balls $(U_j, \xi_{std}) \subset \mathbb{R}^n$ with pairwise disjoint closures, where n is the smallest odd integer $n \in \{3 \dim N + 5, 3 \dim N + 6\}$. By Theorem 1.2.6 there exists an embedding e_j of (N_j, Y_j) for each $j \in \mathbb{Z}$ into (U_j, ξ_{std}) such that there is a defining contact form α_j whose Reeb field R_j on $e_j(N_j)$ restricts to Y_j. Observe that we can take $\alpha_j = \alpha_{std}$, the standard contact form, in a neighborhood of the boundary ∂U_j. This allows us to define a smooth global contact form α on $\mathbb{R}^n$ by setting $\alpha := \alpha_j$ on each U_j and $\alpha := \alpha_{std}$ on $\mathbb{R}^n \backslash \bigcup_{j \in \mathbb{Z}} U_j$; it is obvious that the Reeb field R associated to α satisfies $R|_{e_j(N_j)} = Y_j$ for all j.

Fixing an integer k, it follows from the previous construction that for any vector field $X \in \mathfrak{X}(N)$ and any $\varepsilon > 0$, there exists $j_0 \in \mathbb{Z}$ so that $e_{j_0}(N_{j_0})$ is an invariant submanifold of R, and $R|_{e_{j_0}(N_{j_0})} = Y_{j_0}$. Moreover, the density of the sequence $\{X_j\}_{j \in \mathbb{Z}}$ allows us to take j_0 such that X_{j_0} is ε-close (in the Whitney topology) to X; accordingly, Y_{j_0} is (ε, k)-conjugate to $X + \partial_\theta$. Since any Reeb field is an Eulerisable flow of Beltrami type (c.f. Section 1.1.3), the theorem follows.

$\square$

The method of proof of Theorem 1.2.28 allows us to provide a different proof of a theorem of Etnyre and Ghrist in [68]. Specifically, we can show that there exists an Eulerisable flow in $\mathbb{R}^3$ exhibiting periodic integral curves of all possible knot and link types; when the Riemannian metric of $\mathbb{R}^3$ is fixed and analytic, this result was proved in [60].

Corollary 1.2.29. *There exists an Eulerisable flow of Beltrami type in $\mathbb{R}^3$ exhibiting stream lines of all possible knot and link types.*

Proof. The set of all knot and link types of smoothly embedded circles in $\mathbb{R}^3$ is known to be countable. Let us now embed a representative L_j of each knot and link type in pairwise disjoint Darboux balls $(U_j, \xi_{std}) \subset \mathbb{R}^3$ as in the proof of Theorem 1.2.28. Then, for all $j \in \mathbb{Z}$, there is an isotopy of the link L_j, C^0-close to the identity, which makes it positively transverse to ξ_{std}, see e.g. [77]. For the ease of notation, we still denote the deformed link by L_j. Applying Lemma 1.2.12 to each L_j endowed with the vector field $X_j := \partial_\theta$, where $\theta \in \mathbb{S}^1$ parametrizes L_j, we conclude that there is a contact form α in $\mathbb{R}^3$ whose Reeb vector field contains periodic orbits of all possible knot and link types. (Note that the condition that X_j preserves $TL_j \cap \xi_{std}$ is trivially satisfied in this case.) The statement then follows using the correspondence between Reeb flows and Beltrami fields. $\square$

Even dimensional Euler flows

In all the constructions that we have used to prove Theorems 1.2.3 and 1.2.7, the ambient manifold is odd dimensional because we exploit the connection between hydrodynamics and contact geometry. We finish this section with a result that allows us to establish the universality of the Euler flows also for even dimensional ambient manifolds. The main observation, which is the even dimensional analog of Theorem 1.2.10, is the following proposition:

Proposition 1.2.30. *Let N be a compact manifold endowed with a geodesible flow X. Then there exists an embedding $e : (N, X) \to \mathbb{S}^n \times \mathbb{S}^1$ with n the smallest odd integer $n \in \{3 \dim N + 2, 3 \dim N + 3\}$, and an Eulerisable field u on $\mathbb{S}^n \times \mathbb{S}^1$ such that $u|_{e(N)} = X$.*

Proof. Applying Theorem 1.2.6 we obtain an embedding $\tilde{e} : (N, X) \to (S^n, \xi_{std})$ and a defining contact form $\tilde{\alpha}$ whose Reeb vector field R restricts to X on $\tilde{e}(N)$. By the correspondence Theorem 1.1.15, the field R is a Beltrami field with constant proportionality factor for some Riemannian metric $\tilde{g}$ on $\mathbb{S}^n$. Consider now the $(n+1)$-manifold $S^n \times S^1$ endowed with the Riemannian metric $g := \tilde{g} + d\theta^2$, $\theta \in \mathbb{S}^1$, and define the trivial extension of the Reeb flow R as the vector field $u := (R, 0)$ on $\mathbb{S}^n \times \mathbb{S}^1$. The dual 1-form of u using the metric g is

$$\alpha = \iota_u g = \pi^* \tilde{\alpha} \,,$$

where π is the canonical projection $\pi : \mathbb{S}^n \times \mathbb{S}^1 \to \mathbb{S}^n$. Accordingly, $\iota_u d\alpha = \pi^*(i_R d\tilde{\alpha}) = 0$ and u preserves the (Riemannian) volume form $\mu = \mu_{\tilde{g}} \wedge d\theta$. Defining the embedding $e : N \to \mathbb{S}^n \times \mathbb{S}^1$ of N as $e := i \circ \tilde{e}$, where i is the natural inclusion of $\mathbb{S}^n$ into $\mathbb{S}^n \times \mathbb{S}^1$, we conclude that u is a steady Euler flow on $\mathbb{S}^n \times \mathbb{S}^1$ such that $u|_{e(N)} = X$. $\qquad\qquad\square$

The proof of Theorems 1.2.3 and 1.2.7 for even dimensional ambient manifolds is then the same, mutatis mutandis, as in Subsections 1.2.4 and 1.2.4, but invoking Proposition 1.2.30 instead of Theorem 1.2.6.

1.2.5 Flexibility of Reeb embeddings

The goal of this section is to prove the Reeb embedding Theorem 1.2.6 and provide some generalizations that can be useful for further applications in Contact Geometry. The proof of this result follows the usual pattern in the h-principle theory:

1. We first define a purely topological condition that an embedding needs to satisfy and introduce the concept of *formal iso-Reeb embedding* (Definition 1.2.36).

2. As it is customary in the h-principle theory (see e.g. [158, 15, 39]), we restrict ourselves to a particular subclass of formal iso-Reeb embeddings called *small formal iso-Reeb embeddings* (Definition 1.2.38), and prove that any small formal iso-Reeb embedding can be deformed into a genuine (small) iso-Reeb embedding (Theorem 1.2.39).

3. Finally, we check under which conditions a given embedding can be equipped with a small formal iso-Reeb embedding structure and show that for embeddings of high enough codimension we can always find such a formal structure, see Lemma 1.2.42. These dimensional restrictions account for the bounds in Theorem 1.2.6.

These results are presented as follows. In Subsection 1.2.5 we introduce some basic notions of the h-principle that are used along this section. A technical stability lemma for vector bundles which is instrumental for the next subsections is presented in Subsection 1.2.5. In Subsection 1.2.5 we introduce the definitions of formal iso-Reeb embedding and small iso-Reeb embedding, and prove a full h-principle in this context (Theorems 1.2.37 and 1.2.39). The key lemma to establish the existence of formal iso-Reeb embeddings of high codimension is presented in Subsection 1.2.5. Finally, using this machinery we prove Theorem 1.2.6 in Subsection 1.2.5.

Basic notions of the h-principle

Let us introduce some basic notions in the h-principle theory which are key to provide precise statements.

Fix a smooth fibration $\pi : X \to V$. Denote by $\pi^r : J^r(X) \to V$ the associated r-jet fibration. There is also a natural projection $p_r : J^r(X) \to X$. Given a section $\sigma : V \to X$, denote by $j^r(\sigma) : V \to J^r(X)$ the canonical r-jet extension. Thus, we have a natural inclusion $\mathrm{Sec}(V, X) \to \mathrm{Sec}(V, J^r(X))$ where $\mathrm{Sec}(V, X)$ and $\mathrm{Sec}(V, J^r(X))$ are the spaces of sections from V to X and $J^r(X)$ respectively.

A subset $\mathcal{R} \subset J^r(X)$ is called a partial differential relation of order r. Define $\mathrm{Sec}_{\mathcal{R}}(V, J^r(X)) \subset \mathrm{Sec}(V, J^r(X))$ as the space of formal solutions. It is defined as the space of sections satisfying that the image of the section lies in $\mathcal{R}$. Moreover, define the space of solutions, and denote it by $\mathrm{Sec}_{\mathcal{R}}(V, X) \subset \mathrm{Sec}(V, X)$, to be the space of sections whose r-jet extension is a formal solution. A solution in $\mathrm{Sec}_{\mathcal{R}}(V, X)$ is called a holonomic solution.

Definition 1.2.31. *We say that a partial differential relation $\mathcal{R}$ obeys the rank k h-principle if the inclusion $e : \mathrm{Sec}_{\mathcal{R}}(V, X) \to \mathrm{Sec}_{\mathcal{R}}(V, J^r(X))$ of the space of solutions into the space of formal solutions, which induces morphisms $\pi_j(e) : \pi_j(\mathrm{Sec}_{\mathcal{R}}(V, X)) \to \pi_j(\mathrm{Sec}_{\mathcal{R}}(V, J^r(X)))$, satisfies that $\pi_j(e)$ is an isomorphism for $j \leq k$. If $k = \infty$ we say that $\mathcal{R}$ satisfies the full h-principle.*

The following terminology is standard. We say that $\mathcal{R}$ satisfies a:

- *parametric h-principle* if we can deform formal solutions by holonomic solutions parametrically.

- *relative parametric h-principle* if the following holds: Fix a closed subset $C \subset K$, where K is any compact parameter space. Assume we have a family of formal solutions $\sigma_k, k \in K$ such that σ_k with $k \in C$ is a holonomic solution. Then there exists a parametric family of formal solutions $\tilde{\sigma}_{k,t}, t \in [0, 1]$ such that $\tilde{\sigma}_{k,0} = \sigma_k$, $\tilde{\sigma}_{k,1}$ are holonomic solutions and moreover $\tilde{\sigma}_{k,t} = \sigma_k$ for $k \in C$ and all t.

- *relative to the domain h-principle* if the following is satisfied: For any closed subset $D \subset V$, assume we have a formal solution σ that is holonomic in an open neighborhood U of D. Then there exists a family of formal solutions $\sigma_t, t \in I$ such that $\sigma_0 = \sigma$, σ_1 is holonomic and $\sigma_t|_U = \sigma|_U$ for all t.

- C^0-*dense h-principle* if any formal solution $s : V \to J^r(X)$ can be approximated by a holonomic solution $j^r(\tilde{\sigma})$ such that $p_r(s)$ is C^0-close to $\tilde{\sigma}$.

It is known (see for instance [58, Chapter 6]) that any partial differential relation that satisfies an h-principle: parametric, relative to the parameter, relative to the domain, actually satisfies a full h-principle.

Classical stability lemmas for vector bundles

The following technical results will be used in the next subsections. Proofs are provided for the sake of completeness though they are well known to experts (see e.g. [110, Corollary 4.6]).

Lemma 1.2.32. *Let $V_{k,t}$ be a parametric family of complex bundles over a fixed smooth manifold M with parameters given by $(k,t) \in K \times [0,1]$. Then, there exists a family of complex isomorphisms $\phi_{k,t} : V_{k,0} \to V_{k,t}$.*

Proof. Take a finite number of sections $\sigma_r^{k,t} : M \to V_{k,t}$, $r \in \{1,\dots,n\}$ varying continuously with the parameters such that for any point $p \in M$ and any parameter value (k,t), there are $l := \operatorname{rank} V_{k,t}$ sections $\sigma_r^{k,t}, 1 \le r \le l$ (relabeling the index r if necessary) defining a framing of the fiber over p. Then the bundle map:

$$
\begin{aligned}
p_{k.t} : \mathbb{C}^n &\to V_{k,t} \\
(\lambda_1,\dots,\lambda_n,p) &\to (\Sigma_{r=0}^n \lambda_r \sigma_r^{k,t}(p), p)
\end{aligned}
$$

is an epimorphism of vector bundles. By choosing a metric on each bundle, we find the adjoint map $p_{k,t}^* : V_{k,t} \to \mathbb{C}^n$ that is a monomorphism of vector bundles.

So we may assume that $V_{k,t} \subset \mathbb{C}^n$. Now, fix an hermitian metric on $\mathbb{C}^n$. Denote by $H_{k,t}$ the orthogonal to $V_{k,t}$ with respect to the fixed metric. Define a map

$$
f_{k,t}^\varepsilon : V_{k,t} \to V_{k,t+\varepsilon}
$$

in the following way. Choose for each $p \in M$ and $v \in V_{k,t}$ the unique intersection point in $\mathbb{C}^n$ of the affine subspaces $v + H_{k,t}$ and $V_{k,t+\varepsilon}$ and denote it by v_ε. We define then the map as $f_{k,t}^\varepsilon(v) = v_\varepsilon$. Finally, for each $p \in M$ define $X_{k,t} := \lim_{\varepsilon \to 0} \frac{f_{k,t}^\varepsilon(v) - v}{\varepsilon}$. This defines a time dependent vector field over each fiber $\{p\} \times \mathbb{C}^n$. Clearly, its associated flow $\phi_{k,s}$ satisfies that $\phi_{k,s}(V_{k,0}) = V_{k,s}$ by the construction of $X_{k,t}$ and moreover, it is an isomorphism of complex bundles. $\square$

Corollary 1.2.33. *Let $(V_{k,t}, [\omega_{k,t}])$ be a parametric family of conformal symplectic bundles over a fixed smooth manifold M with parameter given by $(k,t) \in K \times [0,1]$. Then, there exists a family of isomorphisms $\phi_{k,t} : V_{k,0} \to V_{k,t}$ which furthermore are conformal symplectomorphisms.*

Proof. Since in this paper we only consider conformal symplectic structures induced on contact distributions that are cooriented, we restrict to this case (the general case can be easily reduced to this one by a finite covering argument). In particular, we may assume that the conformal symplectic structure is induced by a symplectic structure $\omega_{k,t}$. Then, fix compatible complex structures $J_{k,t}$. This can be done continuously in families since the space of complex structures which are compatible with a fixed symplectic structure is contractible and thus, we can always find global sections: i.e. almost complex structures in the bundle, also in parametric families. This produces an hermitian metric $h_{k,t}$ on $V_{k,t}$.

Extend $h_{k,t}$ to a global hermitian structure $\tilde{h}_{k,t}$ in $\mathbb{C}^n$. We can then mimic the proof of Lemma 1.2.32 to obtain a family of hermitian preserving isomorphisms, which are in addition conformal symplectomorphisms (and in fact symplectomorphisms for the chosen $\omega_{k,t}$). $\qquad\square$

Adapting the proof for the real case, we obtain:

Lemma 1.2.34. *Let $V_{k,t}$ be a parametric family of real bundles over a fixed smooth manifold M with parameters given by $(k,t) \in K \times [0,1]$. Then, there exists a family of real isomorphisms $\phi_{k,t} : V_{k,0} \to V_{k,t}$.*

An h-principle for iso-Reeb embeddings

Following previous notation, let X be a geodesible vector field on N, and denote by β the 1-form such that $\eta = \ker \beta$ and $\beta(X) = 1$. Let (M,ξ) be a contact manifold with defining contact form α, i.e. $\ker \alpha = \xi$.

Remark 1.2.35. As in previous sections we either assume that N is compact or N is properly embedded into M.

With a slight abuse of notation, we will denote $\alpha \circ F_1$ for $\alpha(F_1(\cdot))$ and $d\alpha \circ F_1$ for $d\alpha(F_1(\cdot), F_1(\cdot))$. This is also denoted by $F_1^*\alpha$ and $F_1^*d\alpha$ in similar discussions in [58].

Definition 1.2.36. *An embedding $f : (N, X, \eta) \to (M, \xi)$ is a* formal iso-Reeb *embedding if there exists a homotopy of monomorphisms*

$$F_t : TN \longrightarrow TM,$$

such that F_t covers f, $F_0 = df$, $h_1\alpha \circ F_1 = \beta$ and $d\beta|_\eta = h_2 d\alpha \circ F_1|_\eta$ for some strictly positive functions h_1 and h_2 on N.

It is clear that any genuine iso-Reeb embedding is a formal iso-Reeb embedding. Indeed, take an iso-Reeb embedding $e : (N, X, \eta) \to (M, \xi)$, so by hypothesis we have $e^*\alpha = \beta$, which reads as $\alpha \circ de = \beta$. Thus, we also obtain $e^*d\alpha = d\beta$ that restricted to η can be written as $d\beta|_\eta = d\alpha \circ F_1|_\eta$, and it is clear that $(e, F_t = de)$ is a formal iso-Reeb embedding.

Both conditions $h_1\alpha \circ F_1 = \beta$ and $d\beta|_\eta = h_2 d\alpha \circ F_1|_\eta$ are required to fix the definition of formal iso-Reeb embedding. One may be tempted to say that the first condition naturally implies the second one, but this is tantamount to saying that F_1 commutes with the exterior differential. This only holds when F_1 is holonomic, i.e. the pull-back (through the differential of a morphism) commutes with the exterior differential.

The first main result of this subsection is a full h-principle for iso-Reeb embeddings into overtwisted contact manifolds. The general case is more elaborated because it involves the introduction of a particularly appropriate subclass of iso-Reeb embeddings, and will be discussed later.

Theorem 1.2.37 (*h*-principle for iso-Reeb embeddings into overtwisted mani-folds)**.** *Let* $f : (N, X, \eta) \to (M, \xi)$ *be a formal iso-Reeb embedding with formal differential* F_t *such that* $\dim N < \dim M$*. Furthermore, assume that* ξ *is an over-twisted contact structure. Then, there exists a homotopy* (f^s, F_t^s) *of formal iso-Reeb embeddings such that* $(f^0, F_t^0) = (f, F_t)$ *and such that* $(f^1, F_t^1) = (f^1, df^1)$ *is a genuine iso-Reeb embedding. Moreover, the natural inclusion of the space of iso-Reeb embeddings whose image does not intersect a fixed overtwisted disk* Δ *into the space of formal iso-Reeb embeddings whose image does not intersect* Δ *on a fixed overtwisted contact manifold is a homotopy equivalence.*

Proof. All the bundles in the next paragraph are bundles over N, i.e. TM, TN, ξ, etc. are to be understood as the restriction over N of these bundles, but we shall omit notations like $TM|_N$ for the sake of simplicity.

Step 1: Deform ξ to a homotopic formal contact structure $\bar{\xi}_1$ on N for which $F_0(\eta) \subset \bar{\xi}_1$. It is standard to find a family of isomorphisms $G_t : TM \to TM$ such that $G_0 = id$ and $G_t \circ F_0 = F_t$. Denote $\bar{\xi}_t := G_t^{-1}(\xi)$, so we have $\bar{\xi}_0 = \xi$. Define $\omega_t := d\alpha \circ G_t$ that equips $\bar{\xi}_t$ with a symplectic vector bundle structure $(\bar{\xi}_t, \omega_t)$ such that, for $t = 1$ we obtain $F_0(\eta) \subset \bar{\xi}_1$. Denote by β a defining 1-form for η. Then

$$(\omega_1)|_\eta = d\alpha \circ G_1|_\eta = d\alpha \circ F_1 = (h_2)^{-1} d\beta,$$

where the last equality comes from the definition of formal iso-Reeb embedding. Up to conformal transformation, we can assume that $(\omega_1)|_\eta = h_2(d\alpha \circ F_1) = d\beta$. Therefore, $\bar{\xi}_1$ is a formal contact structure, homotopic to ξ, such that $F_0(\eta) \subset \bar{\xi}_1$.

Step 2: Extend $\bar{\xi}_1$ to a contact structure on a neighborhood of N and make the inclusion an iso-Reeb embedding. Extend the family of distri-butions $\bar{\xi}_t$ that are defined over N to a family of distributions $\tilde{\xi}_t$ defined over a neighborhood $\mathcal{O}p(N)$. A possible way to do this is just to extend the isomor-phisms $G_t : TM \to TM$ over N to a new family $\tilde{G}_t : TM \to TM$ over $\mathcal{O}p(N)$ that can be used to define $\tilde{\xi}_t := \tilde{G}_t(\tilde{\xi}_0)$. Then, using Lemma 1.2.16 where (M, ξ) is the neighborhood $\mathcal{O}p(N)$ and $\tilde{\xi}_1$, we obtain a contact structure $\hat{\xi}_1$ that is defined on $\mathcal{O}p(N) \supset N$ inside M and is homotopic to $\tilde{\xi}_1$. Also, we obtain an iso-Reeb embedding of (N, X, η) with respect to a contact form $\tilde{\alpha}_1$ defining the contact structure $\hat{\xi}_1$.

Step 3: Reduce to formal isocontact embeddings. Summarizing, we have obtained that $\tilde{\xi}_0 = \xi$ and $\hat{\xi}_1$ are homotopic as formal contact structures in the neighborhood of N. By Corollary 1.2.33, we can find a family of bundle isomor-phisms $\phi_t : \hat{\xi}_1 \to \tilde{\xi}_t$ that preserves the conformal symplectic structures on a small neighborhood of N. Extend ϕ_t to $TM|_{\mathrm{Op}(N)}$ and define the family $(e = id, H_t = \phi_t)$ with $e = id : \mathrm{Op}(N) \to \mathrm{Op}(N)$. It is a codimension 0 formal isocontact embed-ding.

Step 4: Conclusion. Applying the h-principle for isocontact embeddings in codimension 0 with overtwisted target, c.f. Theorem 1.2.9, we obtain the first part of Theorem 1.2.37.

Now observe that the previous arguments work parametrically. Also, it is simple to check that the proof is relative to any closed subdomain of the domain N. It is left to check that it works relative to the parameter, however this is not true in general. It is simple to realize that a sufficient condition to reproduce the proof making it relative to the parameter, see [15], is restricting to the class of embeddings which do not intersect a fixed overtwisted disk. This is because in the previous construction we naturally obtain genuine iso-Reeb embeddings which do not intersect a fixed overtwisted disk. It is clear that for this subclass the previous three properties, parametric, relative to the domain and relative to the parameter, imply a full h-principle. The theorem then follows. □

Let us consider now a specific subclass of iso-Reeb embeddings, what we call *small iso-Reeb embedding*. While it imposes an extra condition on the iso-Reeb embedding, the advantage is that it will allow us to prove a full h-principle.

Definition 1.2.38. *Assume that there is a decomposition* $(\xi|_N, d\alpha|_N) = (\xi' \oplus V, d\alpha|_{\xi'} + d\alpha|_V)$ *as orthogonal conformal symplectic subbundles, and we further assume that V is a proper subbundle[2] of ξ.*

An embedding $f : (N, X, \eta) \to (M, \xi = \ker \alpha)$ is a small formal iso-Reeb embedding *if there exists a homotopy of monomorphisms*

$$F_t : TN \longrightarrow TM,$$

such that F_t covers f, $F_0 = df_0$, $F_1(\eta) \subsetneq \xi'$ and $d\beta|_\eta = h_2 d\alpha \circ F_1|_\eta$, for some strictly positive function h_2 on N.

Likewise we say that $f : (N, X, \eta) \to (M, \xi = \ker \alpha)$ is a small iso-Reeb *embedding if $df(\eta) = TN \cap \xi$ and $df(\eta) \subsetneq \xi'$, where $\xi = \xi' \oplus V$ is an orthogonal conformal symplectic decomposition and V is a non trivial subbundle.*

Clearly, any small iso-Reeb embedding is in particular an iso-Reeb embedding. The embedding satisfies that $\xi \cap TN = \eta$, and hence by Lemma 1.2.12 there is a contact form such that its Reeb vector field satisfies $R|_N = X$. If X is negatively transverse to ξ, one can consider $-X$ instead. Otherwise, the contact form such that a negatively transverse X is Reeb is a negative contact form.

Theorem 1.2.39 (*h-principle for small iso-Reeb embeddings*)**.** *Let $f : (N, X, \eta) \to (M, \xi)$ be a small formal iso-Reeb embedding into a contact manifold with formal derivative F_t. Then there is a homotopy (f^s, F_t^s) such that $(f^1, F_t^1 = df^1)$ is a genuine (small) iso-Reeb embedding and one can take f^s to be arbitrarily C^0-close to f.*

---

[2]We do not allow V to be $\{0\}$

Moreover the natural inclusion of the space of small iso-Reeb embeddings into the space of small formal iso-Reeb embeddings on a fixed contact manifold is a homotopy equivalence.

Remark 1.2.40. Observe that an *h*-principle in general cannot be satisfied: if we take (N, X) with X a Reeb vector field and associated hyperplane distribution a contact structure ξ', then an iso-Reeb embedding is equivalent to an isocontact embedding. It is well known that codimension-2 isocontact embeddings do not satisfy the *h*-principle. The inclusion of formal isocontact embeddings into genuine isocontact embeddings is not injective [35].

Proof of Theorem 1.2.39.

Step 1: Deform the pair $\xi' \subset \xi$ to a new pair of formal contact structures $\bar{\xi}'_1 \subset \bar{\xi}_1$ such that $F_0(\eta) \subset \bar{\xi}'_1$. We start by fixing the small formal iso-Reeb embedding f. Find $G_t : TM \to TM$ a family of isomorphisms such that $G_0 = id$ and $G_t \circ F_0 = F_t$. Denote $\bar{\xi}'_t := G_t^{-1}(\xi')$ and $\bar{\xi}_t := G_t^{-1}(\xi)$, so we have $\bar{\xi}_0 = \xi$. Define $\omega_t := d\alpha \circ G_t$ that equips $\bar{\xi}_t$ with a conformal symplectic vector bundle structure $(\bar{\xi}_t, \omega_t)$ such that, for $t = 1$ we obtain $F_0(\eta) \subset \bar{\xi}_1$. Likewise we obtain a conformal symplectic vector subbundle structure $\omega'_t = (d\alpha)|_{\xi'} \circ G_t$. Denote by β the defining 1-form for η, i.e. $\ker \beta = \eta$. We have

$$(\omega_1)|_\eta = d\alpha|_{\xi'} \circ G_1|_\eta = d\alpha|_{\xi'} \circ F_1 = (h_2)^{-1} d\beta \,,$$

where the last equality comes from the definition of formal small iso-Reeb embedding. Up to conformal transformation, we may assume that $(\omega_1)|_\eta = h_2(d\alpha \circ F_1) = d\beta$. We also obtain $h_1(\alpha \circ F_1) = \beta$.

Step 2: Find a positive codimension contact submanifold on a neighborhood of N that contains it. Since, by hypothesis, there is a conformal symplectic orthogonal decomposition $(\bar{\xi}_1, \omega) = (\bar{\xi}'_1, \omega'_1) \oplus (\bar{\xi}'_1)^{\perp \omega_1}$, consider a small neighborhood of the zero section of the bundle $\bar{\xi}'_t \to N$ (that exists because $F_0(\eta)$ is included but not equal to $\bar{\xi}'_1$), and denote it by E_t. Build an embedding of codimension (greater or equal than) 2, $E_t \supset N$, by fixing a metric and applying the exponential map. Extending the exponential map to $(\bar{\xi}'_1)^{\perp \omega_1}$, we obtain a local fibration of a neighborhood of N as $\mathcal{O}p(N) \to E_t$, with linear conformal symplectic fiber given by $\bar{\xi}'_1$. Thus, the conclusion is that the neighborhood $\mathcal{O}p(N)$ can be understood as a small tubular neighborhood of the formal contact submanifold E_t.

Step 3: Mimic the proof of Theorem 1.2.37. We apply steps 2 and 3 as in Theorem 1.2.37 to obtain a contact structure $\tilde{\xi}'_1$ in E_1 and by Lemma 1.2.16 an iso-Reeb embedding of (N, X, η) into $(E_1, \tilde{\xi}')$. Using that $(E_1, \tilde{\xi}'_1) \overset{id}{\to} (M, \xi)$ is a positive codimension formal isocontact embedding with open source manifold, we can apply the *h*-principle Theorem 1.2.8 to obtain an isocontact embedding, whose

restriction to N is C^0-close to the original embedding. To obtain the C^0-closeness we use the fact that we are just obtaining C^0-closeness on a positive codimension core, i.e. N, of the manifold E_1. All the previous constructions can be done parametrically, relative to the parameter and relative to the domain. Accordingly, we obtain a full C^0-dense h-principle. $\qquad\square$

Note that the data of a formal (small) iso-Reeb embedding include the choice of a distribution η invariant under the flow of X. It is important to realize that this choice is not unique. In particular, the space of invariant distributions for a fixed geodesible vector field is a vector space, the transverse ones conforming a cone inside it. Moreover, Theorems 1.2.37 and 1.2.39 depend on the invariant distribution chosen, as the following result illustrates.

Proposition 1.2.41. *For an isocontact embedding $e : (N^{2n_0+1}, \xi_N) \to (M, \xi_M)$ of codimension 2 (which is clearly an iso-Reeb embedding for any Reeb vector field on N), there is a Reeb field R and a distribution η' invariant under R, which is C^0-close to ξ_N, such that (N, R, η') does not admit an iso-Reeb embedding into M, if $n_0 \geq 2$.*

Proof. It is standard that one can take a Reeb field R on N with a standard neighborhood around a periodic orbit of type $\mathbb{S}^1 \times D^{2n_0}$ endowed with a contact form $\alpha = d\theta + r^2 \alpha_{std}$, where r is the radial coordinate on D^{2n_0} and α_{std} is the standard contact form on $\mathbb{S}^{2n_0-1}$. In particular, the Reeb field has the form ∂_θ in this neighborhood. Now choose function $f : [0,1] \to \mathbb{R}^+$ satisfying the following conditions:

- $f(r) = 0$ for $r \leq \frac{1}{2}$,

- $f(r)$ is r^2 for $r \in [\frac{3}{4}, 1]$.

The form $\beta := d\theta + f(r)\alpha_{std}$ extends to the whole manifold since it coincides with α on the boundary of the neighborhood. Moreover, it defines a transverse distribution $\eta' := \ker\beta$ that is invariant under the flow of R. Assume that the triple (N, R, η') admits an iso-Reeb embedding e' in (M, ξ). Then the submanifold $\{0\} \times D^{2n_0} \subset \mathbb{S}^1 \times D^{2n_0} \subset N \xrightarrow{e'} (M, \xi)$ is clearly a submanifold tangent to ξ_M, which leads to a contradiction. $\qquad\square$

A technical lemma: the j-connectedness of the space of isotropic subbundles inside a symplectic bundle

The main result of this subsection is Lemma 1.2.42 below. It is an instrumental lemma that will be our main tool to check that any smooth embedding with high enough codimension is a small formal iso-Reeb embedding. This is the most delicate point of the proof of Theorem 1.2.6. Throughout this subsection, the dimension of N is denoted by n and the dimension of M is denoted by $2m + 1$.

Lemma 1.2.42. *Let $(N, X, \eta) \hookrightarrow (M, \xi)$ be an embedding such that $X \pitchfork \xi$, and (ξ, ω) is a symplectic hyperplane bundle of real rank $2m$. Denote by β a defining 1-form of η in N. If $2m \geq 3n - 1$ then there exists a family (ξ_t, ω_t) of symplectic distributions such that $(\xi_1, \omega_1) = (\xi, \omega)$ and (ξ_0, ω_0) satisfies $\eta = \xi_0 \cap TN$ and η is an isotropic subspace of ξ_0. Furthermore (ξ_t, ω_t) coincides with (ξ, ω) away from a neighborhood of N.*

Proof. It is clear by assumption that $TM|_N = \langle X \rangle \oplus \xi$. This implies that ξ and TN are transverse subspaces in $TM|_N$ and thus we can define a new bundle $\eta_1 := \xi \cap TN$. The linear interpolation between $\eta = \eta_0$ and η_1, which is well defined since η_0 and η_1 are contained in TN and are transverse to X, provides a homotopy of subbundles between these two subbundles inside $TM|_N$. Denote this homotopy by $e_t : \eta_t \to TM$. Fix an auxiliary metric on TM satisfying that ξ is orthogonal to X. Define $\xi_t = \eta_t \oplus TN^\perp$, so clearly $\xi_1 = \xi$. We apply Lemma 1.2.34 to obtain $G_t : \xi_1 \to \xi_{1-t}$, chosen to satisfy $G_0 = id$, which is symplectic by taking the symplectic structure $\omega_{1-t} = \omega \circ G_t^{-1}$. Hence (ξ_t, ω_t) is a family of symplectic hyperplane bundles such that $\eta \subset \xi_0$. The situation before the first homotopy is pictured in Figure 1.3.
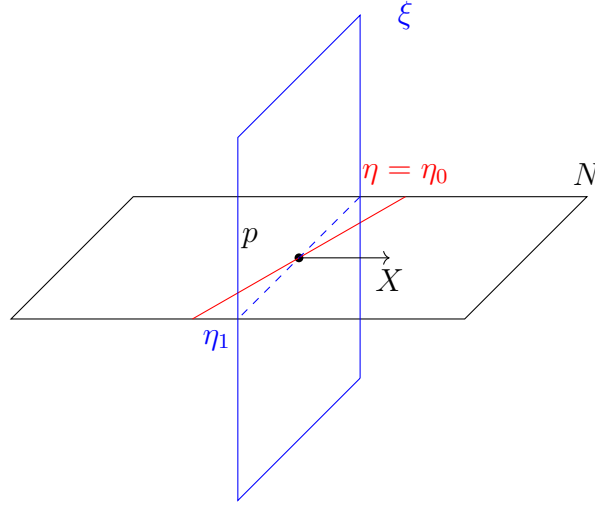
Figure 1.3: Picture before first homotopy

Assume that, if $2m \geq 3n - 1$, any subbundle $\eta \subset (\xi_0, \omega_0)$ can be homotoped onto an isotropic one, i.e. any rank $n-1$ subbundle of a $2m$ dimensional symplectic bundle, over an n-dimensional manifold, is homotopic to an isotropic subbundle. This statement is the content of Lemma 1.2.43 below. In other words, we have a family of monomorphisms $F_t : \eta \to \xi_0$ such that $\eta_{is} := F_1(\eta)$ is isotropic. We extend the monomorphisms F_t into isomorphisms $H_t : \xi_0 \to \xi_0$ satisfying $H_0 = id$, $F_t = H_t \circ F_0$. Clearly, the family of symplectic hyperplane bundles $(\xi_0, \omega_0 \circ H_t)$ composed with the homotopy constructed in the previous paragraph, gives the

required homotopy. The situation before this second homotopy is pictured in Figure 1.4.
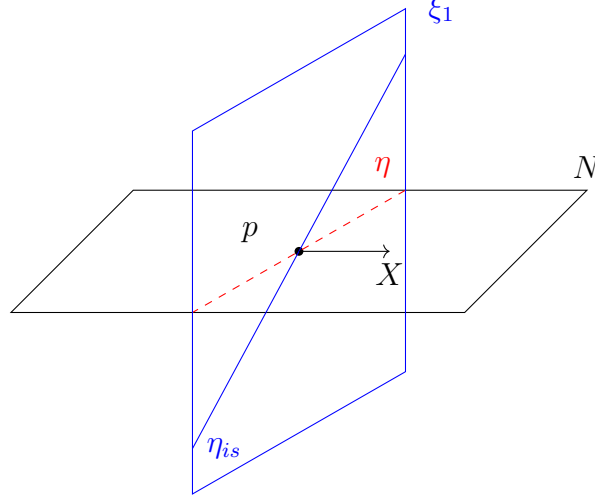
Figure 1.4: Picture before second homotopy

□

Lemma 1.2.43. *Similarly to the previous lemma, let ξ be a symplectic bundle of rank $2m$ over N and denote $\eta = TN \cap \xi$, where η has rank $n-1$ and ξ is a symplectic bundle of rank $2m$. Then if $2m \geq 3n-1$, η is homotopic to an isotropic subbundle.*

Proof. Observe that we need to find a section of a bundle $E \to N$ whose fiber is $P = \text{Path}(\text{Grass}(n-1, \mathbb{R}^{2m}), \text{Grass}_{is}(n-1, \mathbb{R}^{2m}))$, i.e. the space of paths connecting a fixed base point in $\text{Grass}(n-1, \mathbb{R}^{2m})$ with end point in the Grassmanian of isotropic subspaces of dimension $n-1$ in $\mathbb{R}^{2m}$. This is a homotopy fibration with fiber homotopic to the space of loops in the $\text{Gr} := \text{Grass}(n-1, \mathbb{R}^{2m})$ based on the subspace $\text{Gr}_{is} := \text{Grass}_{is}(n-1, \mathbb{R}^{2m})$.

As explained in [102, Section 4.3, Proposition 4.64] and the subsequent discussion, we have the identification $\pi_j(P) \cong \pi_{j+1}(\text{Gr}, \text{Gr}_{is})$. By standard obstruction theory, a sufficient condition for the existence of such a section is to assume that the fiber P is $(n-1)$-connected.

Recall that,

$$
\begin{aligned}
\text{Gr} &\cong \frac{SO(2m)}{SO(n-1) \times SO(2m-(n-1))}, \\
\text{Gr}_{is} &\cong \frac{U(m)}{SO(n-1) \times U(m-(n-1))}.
\end{aligned}
$$

We have the following commutative diagram, given vertically by the relative exact sequences, and horizontally by quotients.

$$\begin{array}{ccccc}
\pi_j(SO_{2m-n+2}, U_{m-n+1}) & \longrightarrow & \pi_j(SO_{2m}, U_m) & \longrightarrow & \pi_j(\mathrm{Gr}, \mathrm{Gr}_{is}) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\pi_j(SO_{n-1} \times SO_{2m-n+1}) & \longrightarrow & \pi_j(SO_{2m}) & \longrightarrow & \pi_j(\mathrm{Gr}) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\pi_j(SO_{n-1} \times U_{m-n+1}) & \longrightarrow & \pi_j(U_m) & \longrightarrow & \pi_j(\mathrm{Gr}_{is}) & \longrightarrow & \cdots
\end{array}$$

$$\begin{array}{ccc}
\cdots \longrightarrow \pi_{j-1}(SO_{2m-n+2}, U_{m-n+1}) & \longrightarrow & \pi_{j-1}(SO_{2m}, U_m) \\
\uparrow & & \uparrow \\
\cdots \longrightarrow \pi_{j-1}(SO_{n-1} \times SO_{2m-n+1}) & \longrightarrow & \pi_{j-1}(SO_{2m}) \\
\uparrow & & \uparrow \\
\cdots \longrightarrow \pi_{j-1}(SO_{n-1} \times U_{m-n+1}) & \longrightarrow & \pi_{j-1}(U_m)
\end{array}$$

We made the identification

$$\pi_j(SO_{n-1} \times SO_{2m-n+1}, SO_{n-1} \times SO_{2m-n+1}) \cong \pi_j(SO_{2m-n+2}, U_{m-n+1}),$$

hence $\pi_j(SO_{n-1} \times SO_{2m-n+1}, SO_{n-1} \times SO_{2m-n+1}) \cong \pi_j(SO_{2m-n+2}, U_{m-n+1})$ by using in the last isomorphism that we are in the stable range of $SO(2m - n + 1)$.

Denote the gamma spaces $SO(2n)/U(n)$ by Γ_n. Observe that $SO(2n)$ is a fibration over Γ_n with fiber $U(n)$. It is standard that the relative homotopy groups of a fibration with respect to the fiber are isomorphic to the homotopy groups of the base, see for instance [102, Theorem 4.41]. Hence we have the identification $\pi_j(SO(2n), U(n)) \cong \pi_j(\Gamma_n)$. In conclusion, the previous diagram is equivalent to the following one.

$$\begin{array}{ccccc}
\pi_j(\Gamma_{m-n+1}) & \xrightarrow{a_j} & \pi_j(\Gamma_m) & \longrightarrow & \pi_{j-1}(P) & \longrightarrow & \cdots \\
\uparrow & & c_j \uparrow & & \uparrow & & \\
\pi_j(SO_{n-1} \times SO_{2m-n+1}) & \xrightarrow{b_j} & \pi_j(SO_{2m}) & \longrightarrow & \pi_j(\mathrm{Gr}) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\pi_j(SO_{n-1} \times U_{m-n+1}) & \longrightarrow & \pi_j(U_m) & \longrightarrow & \pi_j(\mathrm{Gr}_{is}) & \longrightarrow & \cdots
\end{array}$$

$$\begin{array}{ccc}
\cdots \longrightarrow \pi_{j-1}(\Gamma_{m-n+1}) & \xrightarrow{a_{j-1}} & \pi_{j-1}(\Gamma_m) \\
\uparrow & & \uparrow \\
\cdots \longrightarrow \pi_{j-1}(SO_{n-1} \times SO_{2m-n+1}) & \longrightarrow & \pi_{j-1}(SO_{2m}) \\
\uparrow & & \uparrow \\
\cdots \longrightarrow \pi_{j-1}(SO_{n-1} \times U_{m-n+1}) & \longrightarrow & \pi_{j-1}(U_m)
\end{array}$$

We want to prove that $\pi_{j-1}(P)$ is trivial up to $j - 1 = n - 1$. To this end, let us show that we are in the stable range of Γ_{m-n+1} up to rank $n - 1$, and prove that a_n is an epimorphism. The stable range of Γ_{m-n+1} is $2(m - n + 1) - 2$, hence imposing that $n - 1$ is in the stable range we obtain $n - 1 \leq 2(m - n + 1) - 2$ which implies $3n - 1 \leq 2m$, our dimensional hypothesis. Hence a_j is an isomorphism for $j \leq n - 1$ and we deduce $\pi_r(P) = 0$ for $r \leq n - 2$.

To conclude, observe that $\pi_n(\Gamma_{m-n+1})$ is in general no longer in the stable range. Let us check that a_n is always at least an epimorphism, which will imply that $\pi_{n-1}(P) = 0$. If $2m \geq 3n$, then we are in the stable range and a_n is an isomorphism. If not, then $2m = 3n - 1$ and n is odd. But the exact sequence induced by $\Gamma_k \to \Gamma_{k+1} \to \mathbb{S}^{2k}$, see [87], at rank $n = 2m - 2n + 1$ is the following.

$$\pi_{2m-2n+1}(\Gamma_{m-n+1}) \to \pi_{2m-2n+1}(\Gamma_{m-n+2}) \to \pi_{2m-2n+1}(\mathbb{S}^{2m-2n+2})$$

Since $\pi_{2m-2n+1}(\mathbb{S}^{2m-2n+2}) = 0$, the first arrow is an epimorphism. This implies that a_n is always an epimorphism and the proof is complete. $\square$

Proof and discussion of Theorem 1.2.6

We proceed with the proof of Theorem 1.2.6, and a discussion of the result.

Proof of Theorem 1.2.6. Let N be a compact manifold endowed with a geodesible field X. Denote by $e : (N, X) \to (M, \xi)$ an embedding into a contact manifold (M, ξ). Let us assume that M is overtwisted. Because of the codimension hypothesis, there is an homotopy $F_t : TN \to TM$ such that $F_0 = de$, $F_1(X) \pitchfork \xi$ and ξ is positively transverse, i.e. by genericity it is needed $\dim(N) < 2\dim(M)$, which is clearly satisfied under our assumption $2\dim(M) + 1 \geq 3\dim(N)$. Find isomorphisms $G_t : TM \to TM$ satisfying $F_t = G_t \circ F_0$. Define $\xi_t = G_t^{-1}(\xi)$ and define $\omega_t = d\alpha \circ G_t^{-1}$. It deforms ξ to a formal contact structure ξ_1 satisfying that $F_0(X) \pitchfork \xi_1$.

Denote by $\eta = \ker \beta$ a transverse hyperplane distribution preserved by X. We can now apply Lemma 1.2.42 to obtain a formal contact structure $(\tilde{\xi}, \tilde{\omega})$ satisfying that $\tilde{\xi} \cap TN = \eta$ and η is isotropic. So we have a family $(\tilde{\xi}_t, \tilde{\omega}_t)$, $t \in [0, 1]$, such that for $(\tilde{\xi}_0, \tilde{\omega}_0) = (\xi, d\alpha)$ and $(\tilde{\xi}_1, \tilde{\omega}_1) = (\tilde{\xi}, \tilde{\omega})$.

In particular $\eta_{\mathbb{C}}$ is a complex subbundle of $\tilde{\xi}_1$, and hence η^* naturally lies, over N, in the normal bundle of N. The formal contact structure splits as $\tilde{\xi}_1 = \eta_{\mathbb{C}} \oplus (\eta_{\mathbb{C}})^\perp$ on a small tubular neighborhood of N, denoted as $\mathcal{O}p(N) \xrightarrow{pr} N$. For a real constant A, take the homotopy of symplectic structures

$$\omega_t = ((t - 1)A + (2 - t))\tilde{\omega} + (t - 1)pr^*d\beta, \ \ t \in [1, 2],$$

which will be a path of symplectic structures for a big enough $A > 0$, as being symplectic is an open condition. We define $\tilde{\xi}_t = \tilde{\xi}_1$ for $t \in [1, 2]$. We obtain naturally $(\tilde{\xi}_t, \tilde{\omega}_t)$ for $t \in [0, 2]$, a family of formal contact structures obtained by concatenating both homotopies. Clearly, we have that

$$\tilde{\omega}_2 \circ de = d\beta \tag{1.9}$$

Now, as usual we undo the homotopy of contact structures by deforming the formal embedding. In order to do it, apply Corollary 1.2.33 to find a family of isomorphisms $\tilde{G}_t : TM \to TM$, $t \in [0,2]$, such that

- $\tilde{G}_t = G_t$ for $t \in [0,1]$,

- $(\tilde{G}_t^{-1}(\xi), d\alpha \circ \tilde{G}_t) = (\tilde{\xi}_t, \tilde{\omega}_t)$ for $t \in [0,2]$.

Thus, we define a family of monomorphisms $\tilde{F}_t = \tilde{G}_t \circ F_0$ that satisfy $d\alpha \circ \tilde{F}_t = d\alpha \circ \tilde{G}_t \circ F_0 = \tilde{\omega}_t \circ F_0$. For $t = 2$, using equation (1.9), we have $\omega_2 \circ \tilde{F}_0 = \omega_2 \circ de = d\beta$ and therefore it is a formal iso-Reeb embedding. We conclude applying Theorem 1.2.37.

Assume now that M is not overtwisted, and hence $\dim M \geq 3 \dim N + 2$. Because of the dimension condition we can find an orthogonal symplectic decomposition $\xi|_N = \xi' \oplus L$, with rank $L = 2$ and for every $p \in N$ we have $L_p \cap TN_p = \{0\}_p$. We can assume this as long as $\dim M \geq 2 \dim N + 4$, which is true for $\dim N \geq 2$. Hence we can formally make X transversal to ξ', and the proof applies verbatim by projecting η into ξ', which is a symplectic bundle of rank $2 \dim M - 2 \geq 3 \dim N - 1$. $\qquad\square$

Observe that, in fact, in Theorem 1.2.6 we proved that for high enough codimension, any smooth embedding is isotopic to a (small) iso-Reeb embedding for any geodesible field and any invariant distribution. If we were to prove that our Theorem is sharp, we should find a geodesible field with a fixed invariant distribution on a manifold N that does not admit an iso-Reeb embedding into a carefully fixed contact manifold of dimension $3 \dim N - 2$ or $3 \dim N - 1$ (depending on the parity of $\dim N$).

What we can prove is that there is a manifold of dimension $4k_0 + 1$ with a geodesible field and a fixed invariant distribution, and a smooth embedding of such a manifold into $\mathbb{S}^n$, where $n = 3 \dim N - 4$, which is not deformable into an iso-Reeb embedding. So we are two dimensions away from the perfect sharpness.

Proposition 1.2.44. *There is a sequence of triples (N_k, X_k, η_k) of geodesible vector fields on a k-dimensional compact manifold N_k with $k = 4k_0 + 1$ such that there is no iso-Reeb embedding of (N_k, X_k, η_k) into $(\mathbb{S}^n, \xi)$ with $n < 3k - 2$ and ξ any contact structure.*

Proof. Let W be a compact manifold such that $\dim W = 4k_0$. Assume its Pontryagin classes $p_j(W)$ are all vanishing except the top one $p_{k_0}(W)$, which is non trivial (such as the manifolds constructed in [118]). Consider the manifold endowed with a geodesible vector field $(N = W \times \mathbb{S}^1, \partial_\theta)$ of dimension $k = 4k_0 + 1$, with invariant 1-form $d\theta$. The distribution $\eta = \ker d\theta$ is given by TW seen as a distribution. If it admits a Reeb embedding into $(\mathbb{S}^n, \xi)$, we would have that TW is an isotropic subspace of ξ. This follows from the fact that $d\theta$ is closed. Indeed, if we have a Reeb embedding e, there is a contact form α such that $e^*\alpha = d\theta$, so $e^*d\alpha = 0$ and hence $d\alpha|_{TW} = 0$.

Therefore, we have the decomposition $\xi|_N = TW^{\mathbb{C}} \oplus V$, where V is the symplectic orthogonal to $TW^{\mathbb{C}}$. Using the Whitney sum formula for the total Chern class, we obtain that $0 = c_{2k_0}(TW^{\mathbb{C}}) + c_{2k_0}(V)$. Hence V is of rank at least $4k_0$. This implies that $n \geq 8k_0 + 4k_0 + 1 = 3k - 2$. For instance, $(\mathbb{C}P^2 \times \mathbb{S}^1, \partial_\theta)$ does not admit a Reeb embedding into $(\mathbb{S}^{11}, \xi)$. $\qquad\square$

1.2.6 Final remarks

To conclude, let us make a few observations about the results in Section 1.2.5 that are of independent interest. In Subsection 1.2.6 we provide some natural examples of iso-Reeb embeddings that appear in Contact Geometry, and in Subsection 1.2.6 we analyze the topology of the moduli space of iso-Reeb embeddings, thus illustrating the wide range of iso-Reeb embeddings that our construction yields.

Examples

In this section we give some additional examples of formal iso-Reeb embeddings:

Formal isotropic η. Let X be a geodesible vector field on N with associated 1-form β, and denote $\ker \beta = \eta$. Fix an embedding $e : N \to (M, \xi)$. Assume we can formally deform the embedding in such a way that X is transverse to ξ and η is an isotropic subspace. Then perturbing the symplectic form as done in the proof of Theorem 1.2.6, we prove that it is a small formal iso-Reeb embedding.

Totally isotropic embeddings. Consider an embedding $e : N \to (M, \xi)$ that is formal isotropic, we can actually make it isotropic by the h–principle for isotropic / Legendrian immersions [58, Sections 12.4 and 16.1]. So we assume that it is isotropic. Take any geodesible vector field X on N that preserves $\ker \beta = \eta$. We have the decomposition $TN = \langle X \rangle \oplus \eta$. Then, by the Weinstein neighborhood theorem $TM|_N = TN^{\mathbb{C}} \oplus V \oplus \langle R \rangle = \eta^{\mathbb{C}} \oplus \langle X, JX \rangle \oplus V \oplus \langle R \rangle$, where V is the symplectic orthogonal to $TN^{\mathbb{C}}$ inside ξ. We claim that there is an arbitrarily small C^∞-perturbation of the isotropic embedding that makes X transverse and η remains isotropic. The way of producing it is to flow the image $e(N)$ through the flow associated to JX. Do note that

$$\alpha(\mathcal{L}_{JX}X) = \alpha([JX, X]) = d\alpha(JX, X) - X(\alpha(JX)) - JX(\alpha(X)) = d\alpha(X, JX) > 0.$$

This shows that the image of X through the flow becomes transverse to ξ. On the other hand, we obtain for any $Y \in \eta$ that, $\alpha(\mathcal{L}_{JX}Y) = 0$ and thus η remains tangent to ξ. By perturbing the symplectic structure in $\eta^{\mathbb{C}}$ as in the proof of Theorem 1.2.6, it is clear that it is a small formal iso-Reeb embedding.

Remark 1.2.45. An alternative explanation of the last example was suggested to us by Emmy Murphy: apply the h-principle for isotropic immersions to make the embedding into an isotropic immersion that, by genericity, is an embedding.

Realize that there is a neighborhood of the embedding $\mathcal{O}p(N)$ contactomorphic to a neighborhood of the zero section of the bundle $T^*N \times \mathbb{R} \times S$, where S is a conformal symplectic bundle orthogonal to T^*N. The contactomorphism is provided by fixing the standard contact form $\alpha_{std} = dt - \lambda_{Liou}$ over $T^*N \times \mathbb{R}$, where $t \in \mathbb{R}$ and λ_{Liou} is the canonical Liouville form in the cotangent bundle. Fix your geodesible vector field $(N, X, \eta = \ker \beta)$. There is a canonical embedding $\tilde{\beta} : N \to \mathbb{S}(T^*N) \subset T^*N \times \mathbb{R}$. By the universal property of the Liouville form, we have $\tilde{\beta}^* \lambda_{Liou} = \beta$. This implies, just by definition, that $\tilde{\beta}$ is a iso-Reeb embedding. In other words, if the vector field X is geodesible on N, then it can be understood as the restriction of the geodesic flow on $\mathbb{S}(T^*N)$ and the geodesic flow is just the Reeb flow.

An alternative argument for Lemma 1.2.43

In the proof of Theorem 1.2.6, the key step is to prove that a smooth embedding is a formal iso-Reeb embedding consists in showing that η is formally isotropic. This is the content of Lemma 1.2.42, which basically reduces to Lemma 1.2.43. There is a less algebraic alternative argument to prove it.

Alternative proof of Lemma 1.2.43. Consider the complex vector bundle $\eta_{\mathbb{C}} = \eta \oplus \eta^*$, of complex rank $n-1$. We will apply general position arguments to find $\eta_{\mathbb{C}}$ as a subbundle of $\xi \to N$. Consider, chart by chart, a map from $\eta_{\mathbb{C}}$ over a neighborhood U of N to the bundle ξ over U. A generic bundle map from $U \times \mathbb{C}^{n-1}$ to $U \times \mathbb{C}^m$ will not cut the zero section $\mathcal{O}$ as long as $\dim U \times \mathbb{C}^m > \dim \mathcal{O} + \dim U \times C^{n-1}$ which imposes $2m + n > n + 2(n-1) + n$, or equivalently $\dim M = 2m + 1 \geq 3n$.

Hence the situation is the following: there is $\eta = TN \cap \xi$ that we denote η_1, and another $\eta_2 \subset \xi$ isomorphic to ξ but that lies in $\eta_{\mathbb{C}}$ and hence is an isotropic subspace.

$$
\begin{array}{ccc}
\eta^{\mathbb{C}} & \longrightarrow & \xi \\
\uparrow & \nearrow & \\
\eta_2 & &
\end{array}
$$

Let us show that we can set η_2 to be the one tangent to N. Again using a general position argument, we can assume that η_2 and η_1 do not intersect and also that η_2 does not intersect with TN. For the generic bundle map from η to ξ to avoid η_1, we need that $\dim U \times \mathbb{C}^m > \dim \operatorname{rank}_{\mathbb{R}} \eta + \dim U \times C^{n-1}$, which implies $n + 2m > 2n - 2 + n + 2n - 2$, or equivalently $\dim M = 2m + 1 > 3n - 3$ which is satisfied.

In particular ξ splits as $\xi = \eta_1 \oplus \eta_2 \oplus V$. For some real subbundle V. The fact that the two subbundles η_i are in direct sum inside our big vector bundle allows to define an homotopy between η_1 and η_2 . Since we have $\eta_1 \cong_{\mathbb{R}} \eta_2$, we can take the

canonical complex structure J on $\eta_1 \oplus \eta_2$ then $\eta_2 = J\eta_1$. The homotopy between both subspaces is given by

$$L_t = \{(1-t)v + tJv \mid \forall v \in \eta_1\}.$$

This is an homotopy between the two subbundles and by the same reasoning as above it provides an homotopy of the orthogonal complement of $\eta_1 \oplus \langle X \rangle$. Denote ψ_t this second homotopy. We can lift the homotopy to a small neighborhood of N and then to all M to obtain a global homotopy which is identically equal to ξ outside a neighborhood of N. $\qquad\square$

The topology of the space of (small) iso-Reeb embeddings

Finally, in this subsection, we compare the topology of the moduli space of iso-Reeb embeddings with the topology of the moduli space of smooth embeddings. To this end, we introduce some notation. For a compact manifold N endowed with a geodesible field X, and target contact manifold (M, ξ), we denote the space of iso-Reeb embeddings of (N, X, η) into (M, ξ) as $\mathfrak{Reeb}(N, M)$ and the space of formal iso-Reeb embeddings as $\mathcal{F}\mathfrak{Reeb}(N, M)$. Similarly we denote the space of small iso-Reeb embeddings as $\mathfrak{Reeb}_s(N, M)$ and the space of small formal iso-Reeb embeddings as $\mathcal{F}\mathfrak{Reeb}_s(N, M)$. In these last spaces we have made the notation minimal, since we should refer to (N, X, η, M, ξ) instead of (N, M). Finally, denote by $\mathcal{S}(N, M)$ the space of smooth embeddings of N in M. We have the following commutative diagram, where the maps are given by the natural inclusions.

$$\mathfrak{Reeb}(N, M) \overset{i}{\hookrightarrow} \mathcal{F}\mathfrak{Reeb}(N, M)$$
$$\Big\downarrow{\scriptstyle j}$$
$$\mathcal{S}(N, M)$$

$$\mathfrak{Reeb}_s(N, M) \overset{i_s}{\hookrightarrow} \mathcal{F}\mathfrak{Reeb}_s(N, M)$$
$$\Big\downarrow{\scriptstyle j_s}$$
$$\mathcal{S}(N, M)$$

Using this notation, Theorem 1.2.37 implies that if (M, ξ) is overtwisted, and we only consider embeddings that do not intersect a fixed overtwisted disk, then i is a homotopy equivalence. Theorem 1.2.39 implies that i_s is always a homotopy equivalence. In both cases we assume $\dim N < \dim M$. A parametric discussion of Theorem 1.2.6 implies that adding codimension is translated into isomorphisms in higher homotopy groups induced by j and j_s.

Corollary 1.2.46. *Let N be a compact manifold endowed with a geodesible vector field X, and (M, ξ) a contact manifold.*

- *If* $\dim M > 3 \dim N + 2 + k$ *then*

$$j_s^r : \pi_r(\mathcal{F}\mathfrak{Reeb}_s(N, M)) \to \pi_r(\mathcal{S}(N, M))$$

  is an isomorphism for $r \leq k$.

- *If* M *is overtwisted,* $\dim M > 3 \dim N + k$ *and we consider embeddings not intersecting a fixed overtwisted disk, then*

$$j^r : \pi_r(\mathcal{F}\mathfrak{Reeb}(N, M)) \to \pi_r(\mathcal{S}(N, M))$$

  is an isomorphism for $r \leq k$.

Remark 1.2.47. Observe that for $k = 0$, we are increasing by 1 the minimum codimension with respect to Theorem 1.2.6, this is because we are getting more. Theorem 1.2.6 gives surjectivity of j^0 and here we obtain an isomorphism at the π_0 level.

Proof. Let us discuss the case where M is overtwisted, the other case is analogous. The result follows from the proof of Theorem 1.2.6, which works parametrically by adding codimension. We want to prove that j induces an isomorphism in homotopy groups up to rank k. To achieve this we only need that in the key step of Theorem 1.2.6, which is Lemma 1.2.42, the fiber $P = \mathrm{Path}(\mathrm{Grass}(n - 1, \mathbb{R}^{2m}), \mathrm{Grass}_{is}(n-1, \mathbb{R}^{2m}))$ is $n+k$ connected. Following the notation and computations of Lemma 1.2.43, we need that the space $\Gamma(m-n+1)$ is in the stable range up to rank $n+k$. Since the stable range is up to $2(m-n+1)-2$, we need to impose that $n+k \leq 2(m-n+1)-2$. This implies that $\dim M = 2m+1 > 3 \dim N + k$. $\square$

By combining Corollary 1.2.46 with Theorem 1.2.39, we deduce that we can replace $\mathcal{F}\mathfrak{Reeb}_s$ by $\mathfrak{Reeb}_s$ in Corollary 1.2.46, i.e. the isomorphisms of homotopy groups are between the spaces of genuine small iso-Reeb embeddings and smooth embeddings.

1.3 Turing completeness of 3D Euler flows

In the book *The Emperor's new mind* [161] Roger Penrose returns to the artificial intelligence debate to convince us that creativity cannot be presented as the output of a "mind" representable as a Turing machine. This idea, which is platonic in nature and highly philosophical, evolves into more tangible questions such as: *What kind of physics might be non-computational?.*

The ideas of the book are a source of inspiration and can be taken to several landscapes and levels of complexity: Is hydrodynamics capable of performing computations? (Moore [153]). Given the Hamiltonian of a quantum many-body system, is there an algorithm to check if it has a spectral gap? (this is known as *the spectral gap problem,* recently proved to be undecidable [45]). And last but

not least, can a mechanical system (including a fluid flow) simulate a universal Turing machine (*universality*)? (Tao [184, 185, 186]).

This last question has been analyzed related to the conjecture of the regularity of the Navier-Stokes equations [183], which is one of the unsolved problems in the Clay's millennium list. In [187] Tao suggests a connection between a potential blow-up of the Navier-Stokes equations and Turing completeness and fluid computation. It is interesting to mention that another of the one million dollars problem on the same list whose resolution is still pending is the *P versus NP problem*, which concerns the complexity of systems. Grosso modo, the question is if any problem whose solution can be *verified* by an algorithm polynomial in time ("of type *NP*") can also be *solved* by another algorithm polynomial in time ("of type *P*"). The delicate distinction between verification and solution has opened up an intricate scenery combining research in theoretical computer science, physics and mathematics. Although there is no apparent relation between these two celebrated problems, understanding a fluid flow as a Turing machine may shed some light on their connection.

On the other hand, undecidability of systems is everywhere and also on the invisible fine line between geometry and physics: As proven by Freedman [75] non-abelian topological quantum field theories exhibit the mathematical features (combinatorics) necessary to support an NP-hard model. This relates topological quantum field theory and the Jones polynomial (as described by Witten [201]) to the $P \neq NP$ problem. Other undecidable problems on the crossroads of geometry and physics are the stability of an *n*-body system [152], the problem of finding an Einstein metric for a fixed 4-fold as observed by Wolfram [202], ray tracing problems in 3D optical systems [172], or neural networks [178]. Fundamental questions at the heart of low dimensional geometry and topology such as verifying the equivalence of two finitely specified 4–manifolds [202] or the problem of computing the genus of a knot [1] have also been proven to be undecidable and NP-hard problems, respectively.

In this section, we address the appearance of undecidable phenomena in fluid dynamics proving the existence of Turing complete fluid flows on a Riemannian 3-dimensional sphere. In the previous Section, as a byproduct of the universality properties that we proved, we established the Turing completeness of certain Euler flow in a sphere of dimension 17. We will now focus on this property, and combine the computational power of symbolic dynamics and the connection between contact topology and hydrodynamics introduced in Section 1.1.3 to reduce the dimension down to three. Associated to the Euler equations, we have the Navier-Stokes equations, which describe the dynamics of the viscid case. We end up this work discussing an application to these equations in the mentioned Riemannian 3-sphere.

1.3.1   Another excursion to contact topology

Recall that a contact structure on an odd dimensional manifold M^{2n+1} is determined by a hyperplane distribution ξ given (at least locally) by the kernel of a one form α such that $\alpha \wedge (d\alpha)^n \neq 0$ (condition known as *maximal non-integrability*). We will assume that the distribution is co-oriented (i.e., its normal bundle is oriented). This condition is equivalent to having a global one form defining the contact structure (called a defining contact form). For a fixed contact form we define its associated Reeb field R by the equations $\alpha(R) = 1$, $\iota_R d\alpha = 0$. Contact geometry is often seen as the odd dimensional analogue of symplectic geometry. Indeed, symplectic and contact manifolds are related by several constructions. In particular, the contactization of an exact symplectic manifold $(M, d\lambda)$ is defined as the manifold $\mathbb{R} \times M$ equipped with the contact structure $\xi_\lambda = \ker(dt + \lambda)$. A key result in contact geometry is the existence of a Darboux theorem: the only local invariant of a contact structure is the dimension. The most simple proof is given by the following path method result in the contact realm:

Theorem 1.3.1 (Gray stability theorem [87])**.** *Let ξ_t, $t \in [0, 1]$, be a smooth homotopy of contact structures on a closed contact manifold M. Then there is an isotopy ψ_t of M such that $\psi_t^* \xi_0 = \xi_t$ for each $t \in [0, 1]$.*
Moreover, if the family is constant in the complement of a compact set K, then the diffeomorphisms ψ_t are the identity away from K.

The goal of this section is to prove the following result in contact topology, which is a key ingredient for the proof of the main result in this section. All along this section D_ρ is a 2-dimensional disk of radius ρ. If $\rho = 1$ we just omit it to write D.

Theorem 1.3.2. *Let (M, ξ) be a contact 3-manifold and $\varphi : D \to D$ an area-preserving diffeomorphism of the disk which is the identity (in a neighborhood of) the boundary. Then there exists a defining contact form α whose associated Reeb vector field R exhibits a Poincaré section with first return map conjugated to φ.*

Combining this result with Theorem 1.1.15, we obtain a metric g on M for which R is a Beltrami field. This yields the following corollary:

Corollary 1.3.3. *Let M be a 3-manifold. Then, given any area-preserving diffeomorphism $\varphi : D \to D$ of the disk which is the identity (in a neighborhood of) the boundary, there exists a metric g on M such that φ can be realized as the first return map of some Beltrami field X on (M, g), up to conjugation.*

Remark 1.3.4. To prove Corollary 1.3.3 we need to use the well known fact (since the works of Martinet) that any 3-manifold admits a contact structure. In higher dimensions only almost contact manifolds admit contact structures and thus the existence of contact structures on a given manifold is topologically obstructed [15].

Proof of Theorem 1.3.2.
We divide the proof in two steps. The first one realizes the diffeomorphism φ as the first-return map of a Reeb vector field on a solid torus. This result is not new, but we provide an alternative (and simpler) argument to the proof presented by Bramham in [19, Chapter 4]. In the second step we globalize the previous construction to obtain a Reeb field on any 3-manifold.

Step 1: Constructing a Reeb mapping torus.

Let us denote by λ the one form $r^2 d\phi$, where (r, ϕ) are polar coordinates on the disk D. In particular, the form $dt + \lambda$ on $D \times [0, 1]$ (t is the coordinate on $[0, 1]$) defines a contact form. Since the diffeomorphism φ is area-preserving and the identity in a neighborhood of ∂D, it is isotopic to the identity and the time-1 flow of a Hamiltonian (non-autonomous) vector field. More precisely, there is a family of diffeomorphisms φ_t, which are the identity in a neighborhood of ∂D for all $t \in [0, 1]$, such that $\varphi_1 = \varphi$, $\varphi_0 = \mathrm{id}$, and this family is generated by a family of compactly supported vector fields X_t so that

$$\iota_{X_t} d\lambda = dH_t \,, \tag{1.10}$$

where H_t is a family of functions (Hamiltonians) of the disk. For each $t \in [0, 1]$, H_t is obviously constant on a neighborhood of ∂D. Additionally, we can safely assume that φ_t is the identity for $t < \delta$ and $t > 1 - \delta$, which implies that H_t is constant on D (t-dependent) for $t < \delta$ and $t > 1 - \delta$. Accordingly, redefining H_t if necessary, we can assume that $H_t = 0$ in a neighborhood of ∂D for all $t \in [0, 1]$, and $H_t = 0$ on the whole D for $t < \delta$ and $t > 1 - \delta$. Let us now define the function

$$\widetilde{H} : D \times [0, 1] \longrightarrow \mathbb{R}$$
$$(a, z) \longmapsto \tilde{H}(a, z) := H_z(a) \,,$$

and the one form

$$\widetilde{\alpha} := (\widetilde{H} + C)dz + \lambda$$

on the cylindrical set $D \times [0, 1]$, where C is a positive constant. We claim that for a large enough constant C, $\widetilde{\alpha}$ is a contact form. Indeed, a straightforward computation shows that

$$\widetilde{\alpha} \wedge d\widetilde{\alpha} = Cd\lambda \wedge dz + \widetilde{H}d\lambda \wedge dz - d\widetilde{H} \wedge \lambda \wedge dz \,, \tag{1.11}$$

which is obviously positive if $C > C_0$, a constant that only depends on the C^1-norm of $\widetilde{H}$ on D. Additionally, the Reeb field of this contact form is a multiple of $\frac{\partial}{\partial z} + X$, where X is defined as $X(a, z) := X_z(a)$. This is equivalent to checking the condition $\iota_{\frac{\partial}{\partial z} + X} d\widetilde{\alpha} = 0$. Indeed, we have

$$\iota_{\frac{\partial}{\partial z} + X} d\widetilde{\alpha} = \iota_{\frac{\partial}{\partial z} + X}(d\lambda + dH_z \wedge dz)$$
$$= \iota_X d\lambda - dH_z + (\iota_X dH_z)dz = 0 \,,$$

where we have used Equation (1.10) to cancel out the first two terms in the second equality, and the third summand in the equality vanishes by contracting the same equation with X. In particular, the flow of the Reeb vector field of $\widetilde{\alpha}$ is a reparametrization of the flow of $\frac{\partial}{\partial z} + X$, whose time-one map is given by φ.

By the construction of the Hamiltonian family H_t, we conclude that the contact form $\widetilde{\alpha}$ is equal to $Cdz + \lambda$ on a neighborhood of the boundary of the set $D \times [0, 1]$, and it descends to the quotient (the solid torus $D \times \mathbb{S}^1$, where the coordinate z goes to a coordinate θ in $\mathbb{S}^1$). Still denoting the contact form in the quotient as $\widetilde{\alpha}$, it is obvious that near the boundary of the solid torus, $\widetilde{\alpha}$ is $Cd\theta + r^2 d\phi$.

Step 2: Global extension.

Let (M, ξ) be a contact 3-manifold and take a circle γ transverse to the contact structure (which always exists and can be chosen C^0 close to any given closed curve). It is standard that there are coordinates (r', ϕ, θ) in a neighborhood $U = D_\rho \times S^1$ of the circle $\gamma = \{0\} \times S^1$ such that ξ is defined by the kernel of the contact form

$$\beta_0 = C(d\theta + r'^2 d\phi) \,.$$

Here D_ρ is a 2-dimensional disk of small enough radius ρ, the coordinates are the standard angle θ on $\mathbb{S}^1$ and polar coordinates (r', ϕ) on D_ρ, and C is the large constant introduced in Step 1. In particular, multiplying by a suitable positive factor if necessary, we can take a global contact form β defining ξ such that $\beta|_U = \beta_0$. Now we observe that the contact form $\widetilde{\alpha}$ obtained in Step 1 can be constructed on a disk D_ρ of arbitrary radius using a ρ-rescaling of D (a conjugation): $\Phi_\rho : D \times \mathbb{S}^1 \to D_\rho \times \mathbb{S}^1$, with $(r', \phi, \theta) := \Phi(r, \phi, \theta) = (\rho r, \phi, \theta)$. Since $C > 0$ is any large enough constant, it is clear that we can take the radius ρ to be

$$\rho = C^{-1/2} \,.$$

The (large) constant C is fixed in what follows.

Using the conjugation Φ_ρ we define the contact form $\widetilde{\alpha}' := \Phi_{\rho*}\widetilde{\alpha}$ on U. Specifically,

$$\widetilde{\alpha}' = (\widetilde{H}' + C)d\theta + Cr'^2 d\phi \,,$$

where $\widetilde{H}' \equiv \widetilde{H}'(r', \phi, \theta) = \widetilde{H}(C^{1/2}r', \phi, \theta)$. Obviously the associated Reeb field is a multiple of $\Phi_{\rho*}\left(\frac{\partial}{\partial z} + X\right)$; in particular, its first return map on the section $D_\rho \times \{0\}$ is given by $\Phi_\rho \circ \varphi \circ \Phi_\rho^{-1}$. Additionally, by construction,

$$\widetilde{\alpha}' = C(d\theta + r'^2 d\phi)$$

in a neighborhood of the boundary of the solid torus U. $\widetilde{\alpha}'$ can then be extended to the complement $M \backslash U$ as β because $\widetilde{\alpha}' = \beta_0$ on ∂U. In summary, denoting by α this globally defined contact form on M, we conclude that it coincides with

$\widetilde{\alpha}'$ in U, and its associated Reeb field in this set has a first return map that is conjugated to φ.

It remains to show that the contact structure defined by $\ker \alpha$ is homotopic through contact structures to ξ. Indeed, let us define the family of one forms

$$a_t := ((1-t)\widetilde{H}' + C)d\theta + Cr'^2 d\phi$$

in the toroidal set U. A straightforward computation shows that a_t is a contact form for all $t \in [0,1]$. Moreover, $a_0 = \alpha$ and $a_1 = C(d\theta + r'^2 d\phi) = \beta_0$. Noticing that $\alpha = \beta$ in $M \backslash U$, this yields a global homotopy of contact forms that interpolates α with β, which immediately implies an homotopy of contact structures ξ_t such that $\xi_0 = \ker \alpha$ and $\xi_1 = \xi$. Applying Theorem 1.3.1, we deduce that $\ker \alpha$ is contactomorphic to ξ. The theorem then follows.

$\square$

Remark 1.3.5. An easy modification of the proof of Step 2 allows us to choose the defining contact form β in the complement of the toroidal set U. More precisely, given any contact form β defining the contact structure ξ, there is another defining contact form α such that $\alpha = \widetilde{\alpha}'$ on U and $\alpha = \beta$ in the complement of a neighborhood of U.

A similar proof provides an equivalent statement for higher dimensional contact manifolds. It reads as follows:

Theorem 1.3.6. *Let (M, ξ) be a contact $(2n + 1)$-manifold and $\varphi : D \to D$ a symplectomorphism of the $2n$-ball which is the identity (in a neighborhood of) the boundary. Then there exists a defining contact form α whose associated Reeb vector field R exhibits a Poincaré section with first return map conjugated to φ.*

1.3.2   Turing machines and symbolic dynamics

The key tool to construct a dynamical system that simulates a Turing machine is symbolic dynamics. Our goal in this subsection is to recall some basic properties of Turing machines and to introduce Moore's theory [153] on the connection between diffeomorphisms of manifolds and computation. In particular, we shall show that suitable generalizations of the shift map are enough to simulate universal Turing machines. This paves the way to construct a Turing complete area-preserving diffeomorphism of the disk, as we shall see in Section 1.3.3.

Remark 1.3.7. As discussed in several parts of the literature [49, 48], there are misleading intuitions that lead to conclude that the shift map can simulate a universal Turing machine. This happens when we accept to take initial points that are not constructible, i.e., they contain as initial information all the computations of the Turing machine instead of just its initial tape. See Section 1.3.2 for a detailed explanation.

Turing machines

A Turing machine is defined via the following data:

- A finite set Q of "states" including an initial state q_0 and a halting state q_{halt}.

- A finite set Σ which is the "alphabet" with cardinality at least two.

- A transition function $\delta : (Q \times \Sigma) \longrightarrow (Q \times \Sigma \times \{-1, 0, 1\})$.

Let us denote by $q \in Q$ the current state, and by $t = (t_n)_{n \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}}$ the current tape. For a given Turing machine $(Q, q_0, q_{halt}, \Sigma, \delta)$ and an input tape $s = (s_n)_{n \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}}$ the machine runs applying the following algorithm:

1. Set the current state q as the initial state and the current tape t as the input tape.

2. If the current state is q_{halt} then halt the algorithm and return t as output. Otherwise compute $\delta(q, t_0) = (q', t_0', \varepsilon)$, with $\varepsilon \in \{-1, 0, 1\}$.

3. Replace q with q' and t_0 with t_0'.

4. Replace t by the ε-shifted tape, then return to step (2). Following Moore [153], our convention is that $\varepsilon = 1$ (resp. $\varepsilon = -1$) corresponds to the left shift (resp. the right shift).

In particular, the space of all possible internal states of a Turing machine is given by $\mathcal{P} := Q \times \Sigma^{\mathbb{Z}}$. The transition function δ induces a global transition function $\Delta : Q \setminus \{q_{halt}\} \times \Sigma^{\mathbb{Z}} \to \mathcal{P}$, which sends an internal state in $\mathcal{P}$ to the internal state obtained after applying a step of the algorithm.

It is convenient to decompose the transition function δ in three components that we can denote by F_1, F_2, F_3:

$$F_1 : Q \times \Sigma \longrightarrow Q$$
$$F_2 : Q \times \Sigma \longrightarrow \Sigma$$
$$F_3 : Q \times \Sigma \longrightarrow \{-1, 0, 1\}.$$

The first component F_1 tells you the new state q' in terms of the current state q and the tape value t_0. The second component F_2 computes the new value of the tape cell t_0' in terms of the state q and the tape value t_0. Finally, the last component F_3 tells you if the tape should stay, shift to the left or shift to the right in terms of the current state and the tape value t_0.

As we shall see in Section 1.3.2, the reversible condition plays a crucial role to construct well behaved dynamical systems that simulate universal computation. Reversibility of a Turing machine can be defined in several equivalent ways. The definition in terms of the global transition function Δ is suitable for our discussion.

Definition 1.3.8. *A Turing machine $T = (Q, q_0, q_{halt}, \Sigma, \delta)$ is* reversible *if the global transition function Δ is injective.*

Generalized shifts and Turing machine simulation

In this section we introduce Moore's theory of generalized shifts [153], which will be instrumental to construct a Turing complete area-preserving diffeomorphism of the disk in Section 1.3.3.

First, an important remark on the necessity of this theory is in order: let us elaborate on the reason why the shift map is not suitable to perform universal computation. Indeed, consider a Turing machine with some given initial input. We can associate to it a sequence $(q_i)_{i \in \mathbb{N}}$ of states, where q_i is the state of the machine at the step i. If the machine reaches the halting state at a step j, we define $q_i = q_{halt}$ for all $i \geq j$. Using this sequence, we can construct another sequence $(p_i)_{i \in \mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ by setting $p_i = 0$ if $i < 0$, $p_i = 0$ if $i \geq 0$ and $q_i \neq q_{halt}$ and $p_i = 1$ if $q_i = q_{halt}$. Iterating this sequence by the standard (left) shift map, it is clear that the Turing machine halts if and only if the shift map finds a digit 1 in position 0 at some iteration. The problem is, however, that the initial sequence is not constructible; it is a priori an undecidable problem to construct the whole sequence $(q_i)_{i \in \mathbb{N}}$.

To introduce the notion of generalized shift, let A be an alphabet and $S \in A^{\mathbb{Z}}$ an infinite sequence. A generalized shift is specified by two maps F and G which depend on *finitely many* positions of S. Denote by $D_F = \{i, ..., i + r - 1\}$ and $D_G = \{j, ..., j + l - 1\}$ the sets of positions on which F and G depend, respectively. They have cardinality $r \geq 1$ and $l \geq 1$, respectively. Obviously, these functions take a finite number of different values since they depend on a finite number of positions. The function G modifies the sequence only at the positions indicated by D_G:

$$G : A^l \longrightarrow A^l$$
$$(s_j...s_{j+l-1}) \longmapsto (s'_j...s'_{j+l-1})$$

Here $s_j...s_{j+l-1}$ are the symbols at the positions $j, ..., j+l-1$ of an infinite sequence $S \in A^{\mathbb{Z}}$.

On the other hand, the function F assigns to the subsequence $(s_i, ..., s_{i+r-1})$ of the infinite sequence $S \in A^{\mathbb{Z}}$ an integer:

$$F : A^r \longrightarrow \mathbb{Z}$$

A generalized shift $\phi : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is then defined as follows:

- Compute $F(S)$ and $G(S)$.

- Modify S changing the positions in D_G by the function $G(S)$, obtaining a new sequence S'.

- Shift S' by $F(S)$ positions. That is, we obtain a new sequence $s''_n = s'_{n+F(S)}$ for all $n \in \mathbb{Z}$.

The sequence S'' is then $\phi(S)$. For example, the standard shift is obtained by taking G to be the identity and $F \equiv 1$. For later convenience, when taking a sequence in $A^{\mathbb{Z}}$, we will write a point to denote that the symbol at the right of that point is the symbol at position 0. In particular, the sequence (s_n) can be denoted by $(\ldots s_{-1}.s_0 s_1 \ldots)$.

The remarkable property of generalized shifts is that they can simulate any Turing machine in the following sense:

Definition 1.3.9. *We say that a generalized shift ϕ with alphabet A is conjugated to a Turing machine $T = (Q, q_0, q_{halt}, \Sigma, \delta)$ if there is an injective map*

$$\varphi : \mathcal{P} \to A^{\mathbb{Z}}$$

such that the global transition function of the Turing machine is given by

$$\Delta = \varphi^{-1} \phi \varphi \,. \tag{1.12}$$

We recall that $\mathcal{P} = Q \times \Sigma^{\mathbb{Z}}$ is the space of all possible internal states of T.

The following result, which was first proved in [153], establishes that any Turing machine is conjugated to a generalized shift. Although this result is relatively standard, we include a proof because it helps to elucidate the connection between Turing machines and Generalized Shifts.

Lemma 1.3.10. *Given a Turing machine $T = (Q, q_0, q_{halt}, \Sigma, \delta)$, there is a generalized shift conjugated to it.*

Proof. Recall that F_1, F_2, F_3 denote the three components of the transition function δ of the Turing machine T. Let us construct a generalized shift whose alphabet is given by $A := \Sigma \cup Q$, i.e., both the alphabet and the set of states of the Turing machine T. First, notice that to every internal state $(q, (t_i)_{i \in \mathbb{Z}}) \in \mathcal{P}$ of T, we can assign the sequence $(\ldots t_{-1}.qt_0 t_1 \ldots)$ in $A^{\mathbb{Z}}$.

Now let us define the maps F and G, which will depend only on the three positions $-1, 0, 1$, i.e., $D_F = D_G = \{-1, 0, 1\}$. For a sequence $(s_n) \in A^{\mathbb{Z}}$, denote by $a := (s_{-1}s_0 s_1)$ the subsequence of symbols in positions $-1, 0, 1$. If this subsequence a is not of the form $(t_{-1}qt_0)$ with $t_{-1}, t_0 \in \Sigma$ and $q \in Q$, then we define $F(a) := 0$ and $G(a) := a$. Otherwise, we have $a = t_{-1}qt_0$ for some symbols t_{-1}, t_0 in Σ and $q \in Q$. Setting $q' := F_1(q, t_0)$ and $t_0' := F_2(q, t_0)$, we can define F and G as:

$$F(a) := F_3(q, t_0) \,,$$

and

$$G(a) := \begin{cases} t_{-1}.t_0'q' & \text{if } F(a) = 1 \,, \\ q'.t_{-1}t_0' & \text{if } F(a) = -1 \,, \\ t_{-1}.q't_0' & \text{if } F(a) = 0 \,. \end{cases} \tag{1.13}$$

Although the function F only depends on the positions a_0 and a_1 of a, we have chosen $D_F = D_G = \{-1, 0, 1\}$ so that both domains are the same. These maps F and G define a generalized shift ϕ as explained above.

Finally, given an internal state $(q, (t_i)_{i \in \mathbb{Z}}) \in \mathcal{P}$, it is straightforward to check that a step of the Turing machine T corresponds to $\varphi^{-1} \phi \varphi$, where

$$\varphi : \mathcal{P} \longrightarrow A^{\mathbb{Z}}$$
$$(q, (t_i)_{i \in \mathbb{Z}}) \longmapsto s = \ldots t_{-1}.q t_0 t_1 \ldots$$

is an injective map. By definition, this proves that T is conjugated to ϕ, and the lemma follows. □

The main property that we introduce in this section is that a reversible Turing machine is conjugated to a bijective generalized shift. This fact was stated in [153]. Observe that the global transition function of a Turing machine is not defined for the halting state q_{halt}, which prevents to extend the generalized shift when the symbol in position 0 is q_{halt}. In order to fix this issue, and to obtain a globally defined injective map Δ, we extend δ as

$$\delta(q_{halt}, t_0) := (q_0, t_0, 0). \tag{1.14}$$

This way, δ is defined in all the domain of states Q. For Δ to be injective, we need to assume that the original transition function δ satisfied that $F_1(q, t) \neq q_0$ for all $(q, t) \in Q \times \Sigma$. Otherwise, an internal state with q_0 could be achieved from two different internal states in $\mathcal{P}$ and Δ would not be injective. As explained in [155, Section 6.1.2], given any reversible Turing machine it is possible, even though not trivial, to construct an equivalent one which satisfies this condition. When necessary, we shall assume that the global transition function has been extended this way without further mention. Moreover, Equation (1.12) is satisfied on the whole domain $\mathcal{P}$.

Lemma 1.3.11. *A reversible Turing machine T, whose transition function has been extended as above, is conjugated to a bijective generalized shift.*

As shown by Moore, the relevance of generalized shifts comes from the fact that they are conjugated to maps of the square Cantor set, which allows one to use the machinery of symbolic dynamics (compare with the particular case of the standard shift map). We recall the following:

Definition 1.3.12. *The square Cantor set is the product set $C^2 := C \times C \subset I^2$, where C is the (standard) Cantor ternary set in the unit interval $I = [0, 1]$. Additionally, we say that a Cantor block is a block of the form $B = [\frac{a}{3^i}, \frac{a+1}{3^i}] \times [\frac{b}{3^j}, \frac{b+1}{3^j}]$, where i, j are nonnegative integers and $a < 3^i$, $b < 3^j$ are nonnegative integers such that there are points of C^2 in the interior of B.*

It is clear that for given i, j we can find a finite amount of disjoint Cantor blocks whose union contains all the points of the square Cantor set. In what follows, we shall consider generalized shifts with alphabet $A = \{0, 1\}$. Actually, as proved in [153, Lemma 1], this can always be assumed. Given an infinite sequence $s = (...s_{-1}.s_0 s_1...) \in A^{\mathbb{Z}}$, we can associate to it an explicitly constructible point in the square Cantor set. The usual way to do this is to express the coordinates of the assigned point in base 3: the coordinate y corresponds to the expansion $(y_1, y_2, ...)$ where $y_i = 0$ if $s_i = 0$ and $y_i = 2$ if $s_i = 1$. Analogously, the coordinate x corresponds to the expansion $(x_1, x_2, ...)$ in base 3 where $x_i = 0$ if $s_{-i} = 0$ and $x_i = 2$ if $s_{-i} = 1$.

The aforementioned assignment between infinite sequences and points in the square Cantor set is key to prove a fundamental lemma that we borrow from Moore's work [153]. Combined with Lemma 1.3.11 it will be key to construct a Turing complete area-preserving diffeomorphism of the disk in Section 1.3.3.

Lemma 1.3.13 (Moore). *Any generalized shift is conjugated to the restriction to the square Cantor set of a piecewise linear map of I^2. This map consists of k finitely many area-preserving linear components defined on Cantor blocks, with k bounded as:*

$$k \leq n^{|D_F \cup D_G| + max|F|}.$$

Here $n := |A|$. If the generalized shift is bijective, then the image blocks are pairwise disjoint.

Remark 1.3.14. In the literature, there have been other attempts to simulate a reversible Turing machine by means of selecting the space of states of the machine as a constructible choice of coordinates in the square I^2 and extending the global transition function from that set of points to a bijective map of I^2. However, these other models do not provide a continuous [172] or a compactly supported extension [86] or they increase the dimension [184]. In Section 1.3.3 we will show that Moore's approach has the advantage that it can be used to promote the map constructed in Lemma 1.3.13 to a smooth (area-preserving) diffeomorphism of the disk.

Let us briefly explain the main ideas of the construction of the map that we used in Lemma 1.3.13. If we fix our attention on a single Cantor block, the piecewise linear map is constructed as the composition of two linear maps. The first one is a translation (depending on the function G of the generalized shift), which sends a block onto another one. Next, using the function F, we get an integer which tells us how many shifts have to be applied to the block. The action of the shift map on a block can be obtained by restriction of a positive or negative power of the horseshoe map. This second well known map, is a composition of a translation, a rotation and a rescaling in each coordinate. We finish with the following example, which illustrates Lemma 1.3.13.

Example 1.3.15. *A simple example of a generalized shift and its associated piecewise linear map can be constructed as follows. Consider a generalized shift with alphabet $\{0, 1\}$, and such that $D_F = D_G = \{-1, 0\}$. We define the functions F and G as: $G(0.1) = 0.1$, $G(1.1) = 0.0$, $G(0.0) = 0.1$, $G(1.0) = 1.1$ and $F(0.1) = F(0.0) = -1$, $F(1.1) = F(1.0) = 0$. By assigning letters to the Cantor blocks corresponding to each possible finite string of two elements, the associated map can be represented by blocks. Denote by A, B, C and D the Cantor blocks whose corresponding sequences have in positions $-1, 0$ respectively the pairs $(0.1), (1.1), (0.0)$ and (1.0); the position of these blocks in the square is computed following the assignment that we introduced before, thus obtaining Figure 1.5 (in the same figure we also represent the images of the blocks). The piecewise linear map can be explicitly written as:*

$$(x, y) \longmapsto \begin{cases} (3x, y/3) \text{ if } (x, y) \in A \\ (3(x - 2/3), 1/3(y - 2/3)) \text{ if } (x, y) \in B \\ (x, y + 2/3) \text{ if } (x, y) \in C \cup D \end{cases}$$
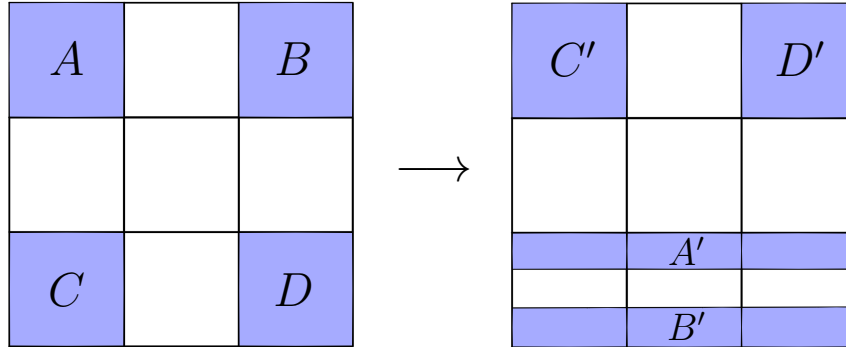
Figure 1.5: Blocks map in the unit square

1.3.3 An area-preserving diffeomorphism of the disk that is Turing complete

The goal of this section is to construct an area-preserving diffeomorphism of the disk that simulates a universal Turing machine. The main tool is the generalized shifts introduced in Section 1.3.2 and their connection with piecewise linear maps of Cantor blocks. In this direction, a first simple observation is that if we choose a set of disjoint blocks containing all the Cantor set, they lie in the unit square with some gaps in between that do not contain points of the square Cantor set. We will

use these gaps to extend the piecewise linear map constructed in Lemma 1.3.13 to an area-preserving diffeomorphism of the disk, provided that the generalized shift is bijective.

Smoothing the map

For any generalized shift, Lemma 1.3.13 establishes the existence of a piecewise linear map defined on finitely many Cantor blocks whose action on the square Cantor set is conjugated to the generalized shift. In [153, Theorem 12], Moore sketches an argument to extend this map to a diffeomorphism of the disk. In the following proposition, using standard arguments, we show that this map can be done area-preserving as long as the generalized shift is bijective.

Proposition 1.3.16. *For each bijective generalized shift and its associated map of the square Cantor set ϕ, there exists an area-preserving diffeomorphism of the disk $\varphi : D \to D$ which is the identity in a neighborhood of ∂D and whose restriction to the square Cantor set is conjugated to ϕ.*

Proof. For simplicity we use the same notation ϕ for the generalized shift and its associated map of the square Cantor set obtained via Lemma 1.3.13. This map is defined on a finite disjoint union of Cantor blocks (that contain the whole square Cantor set), and the images of these blocks are pairwise disjoint because the generalized shift is bijective. Taking an open neighborhood D (diffeomorphic to a disk) of the square I^2, our goal is to extend ϕ to the whole D.

To this end, we start by choosing a set contained in D of disjoint (small enough) open neighborhoods B_i of each Cantor block. Since the map ϕ, which is piecewise linear, is obviously defined on each neighborhood, it maps each B_i onto a neighborhood V_i of the images of the Cantor blocks. Obviously, V_i are pairwise disjoint and have the same area as B_i.

This immediately yields a diffeomorphism $F : \bigcup B_i \to \bigcup V_i$ that preserves the standard area form $\omega_{std} = dx \wedge dy$. We claim that this map extends to a diffeomorphism of the disk that is isotopic to the identity. To prove this, we construct a family of maps $F_t : \bigcup B_i \to D$ such that $F_1 = F$, $F_0 = \mathrm{id}$, and F_t is a diffeomorphism into its image for each $t \in [0, 1]$. To construct this family we first define $F_t^{(1)}$ for $t \in [0, 1/3]$ to be an homothety in each open ball B_i, which contracts each ball to a ball of small enough area δ. Specifically, taking a point $p_i \in B_i$ such that B_i is star shaped with respect to p_i (this is possible because B_i is a neighborhood of a Cantor block), for each $x \in B_i$

$$F_t^{(1)}(x) := p_i + \lambda_t(x - p_i)\,,$$

where λ_t is a smooth function on B_i such that $\lambda_0 = 1$ and $\lambda_{1/3} < \delta$. Next we choose different points q_i inside the image balls V_i and construct $F_t^{(2)} : \bigcup F_{1/3}^{(1)}(B_i) \to D$ for $t \in [1/3, 2/3]$ to be a map such that $F_{1/3}^{(2)} = \mathrm{id}$ and $F_{2/3}^{(2)}$ sends each ball
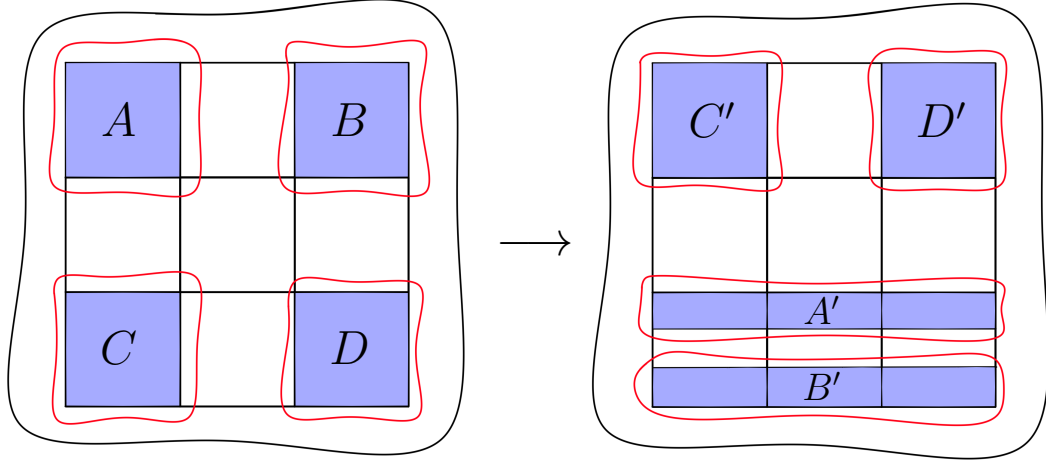
Figure 1.6: Blocks map by open balls

$F_{1/3}^{(1)}(B_i)$ inside a δ-neighborhood of q_i (contained in V_i). Finally, we define an expansion $F_t^{(3)} : \bigcup F_{2/3}^{(2)}(F_{1/3}^{(1)}(B_i)) \to D$ for $t \in [2/3, 1]$, analogous to $F_t^{(1)}$, so that $F_{2/3}^{(3)} = \mathrm{id}$ and $F_1^{(3)}(F_{2/3}^{(2)}(F_{1/3}^{(1)}(B_i))) = V_i$. The map F_t is then obtained as:

$$F_t := \begin{cases} F_t^{(1)} \text{ for } t \in [0, 1/3] \,, \\ F_t^{(2)} \circ F_{1/3}^{(1)} \text{ for } t \in [1/3, 2/3] \,, \\ F_t^{(3)} \circ F_{2/3}^{(2)} \circ F_{1/3}^{(1)} \text{ for } t \in [2/3, 1] \,. \end{cases}$$

Now, if we set F_t to be the identity in a neighborhood of the boundary of D, the homotopy extension property allows us to extend F_t to a family of diffeomorphisms φ_t of the disk such that $\varphi_1|_{\bigcup B_i} = F$, $\varphi_0 = \mathrm{id}$ and φ_t is the identity near the boundary of D for all t.

The standard area form ω_{std} is sent to another area form that we denote by $\omega_1 := \varphi_1^* \omega_{std}$. Notice that $\omega_1 = \omega_{std}$ on $\bigcup B_i$ because F is area-preserving. Additionally, we can interpolate linearly between these two area forms:

$$\omega_t := t\omega_1 + (1 - t)\omega_{std} \,.$$

Of course, ω_t is nondegenerate for all $t \in [0, 1]$. Noticing that both forms have the same area since $\int_D \omega_{std} = \int_{\varphi^* D} \omega_{std} = \int_D \varphi^* \omega_{std} = \int_D \omega_1$, it follows that $\omega_1 - \omega_{std}$ is an exact 2-form. Applying Moser's path method we then obtain a family of diffeomorphisms $G_t : D \to D$, $G_0 = \mathrm{id}$, such that $G_t^* \omega_t = \omega_{std}$ for all $t \in [0, 1]$. Moreover, we can assume that $G_t|_{\bigcup B_i} = id$ because $\omega_t = \omega_{std}$ for all $t \in [0, 1]$. Finally, the diffeomorphism $\varphi := \varphi_1 \circ G_1$ satisfies the required conditions, i.e., $\varphi|_{\bigcup B_i} = F$ and $\varphi^* \omega_{std} = \omega_{std}$. The proposition follows noticing that D can be identified with the unit disk in $\mathbb{R}^2$, after applying a suitable diffeomorphism. $\square$

A Turing complete area-preserving diffeomorphism of the disk

We are now ready to establish the existence of a Turing complete area-preserving diffeomorphism of the disk that is the identity on the boundary. We remark that our notion of Turing completeness is slightly different from the one used in [184, 33], see Remark 1.3.18 below, but it has the same computational power. Key to the proof are Proposition 1.3.16 and the constructibility of the unique point in the square Cantor set associated to an infinite sequence in $\{0,1\}^{\mathbb{Z}}$, cf. Section 1.3.2. In the proof we also make use of an instrumental result (Lemma 1.3.19) allowing us to show that our area-preserving diffeomorphism can check a finite substring of the output of a Turing machine.

Theorem 1.3.17. *There exists a Turing complete area-preserving diffeomorphism φ of the disk that is the identity in a neighborhood of the boundary. Specifically, for any integer $k \geq 0$, given a Turing machine $T = (Q, q_0, q_{halt}, \Sigma, \delta)$, an input tape $(t_n) \in \Sigma^{\mathbb{Z}}$ and a finite string $(t^*_{-k}, ..., t^*_k) \in \Sigma^{2k+1}$, there is an explicitly constructible point $p \in D$ and an explicitly constructible open set $U \subset D$ such that the orbit[3] of φ through p intersects U if and only if T halts with an output tape whose positions $-k, ..., k$ correspond to $t^*_{-k}, ..., t^*_k$.*

Proof. The first observation is that there are several constructions of reversible universal Turing machines. For instance, in [154] there is an explicit construction with 17 states and an alphabet of 5 symbols. In fact, it is known [12] that for any Turing machine (and in particular for a universal one) there is a reversible Turing machine doing the same computations. Hence, let us denote by T_{un} some reversible universal Turing machine. By Lemma 1.3.10, we can associate to T_{un} a conjugated generalized shift ϕ, which is, in fact, bijective in view of Lemma 1.3.11. Applying Proposition 1.3.16, we can construct an area-preserving diffeomorphism φ of the disk D which is the identity in a neighborhood of ∂D and whose restriction to the square Cantor set is conjugated to ϕ.

We claim that the map φ is Turing complete. Indeed, given a Turing machine $T = (Q, q_0, q_{halt}, \Sigma, \delta)$ and a finite part of the output tape $(t^*_{-k}, ..., t^*_k) \in \Sigma^{2k+1}$, Lemma 1.3.19 below allows us to construct another Turing machine T' which reads the output of T. Since T_{un} is universal, it can simulate the evolution of T'. In particular, given an input (q_0, t) of T' there is an explicit input $(\widehat{q}, \widehat{t})$ of T_{un}, with $\widehat{q} \in Q_{un}$ (in particular $\widehat{q}$ is the initial state of T_{un}) and $\widehat{t} \in \Sigma^{\mathbb{Z}}_{un}$ (here, Q_{un} and Σ_{un} are the space of states and the alphabet of T_{un}, respectively), such that T' halts with the aforementioned input if and only if T_{un} halts with input $(\widehat{q}, \widehat{t})$.

As explained in the proof of Lemma 1.3.10 we obtain an explicit sequence $s = (...s_{-1}.s_0 s_1...) \in A^{\mathbb{Z}}$, where $A = \Sigma_{un} \cup Q_{un}$, from the input $(\widehat{q}, \widehat{t})$ of T_{un}, which defines a unique point $p \in D$ via the correspondence introduced in Section 1.3.2.

---

[3]Here by orbit we mean the set of non-negative powers of the area-preserving map applied to the point.

For example, taking $\{0, 1\}$ to be the alphabet A of the generalized shift ϕ (which can always be done as mentioned before), then the coordinates of p are given by

$$p = (x, y) = \left(\sum_{i=1}^{\infty} \frac{2s_{-i}}{3^i}, \sum_{i=1}^{\infty} \frac{2s_{i-1}}{3^i}\right).$$

(In general, it is an expansion in base $2r + 1$ where r is the number of symbols of the alphabet.) Finally, we can take the open set U to be a neighborhood of the Cantor blocks (and hence a disjoint set) that contain all the points associated to sequences in $A^{\mathbb{Z}}$ whose symbols in positions $-1, 0, 1$ are of the form $(a_{-1}q_{halt}^{un}a_1)$ for some $a_{\pm 1} \in A$. Here q_{halt}^{un} is the halting state of T_{un}. Of course, this set U exists because the Cantor blocks are pairwise disjoint, and it is explicitly constructible. This completes the proof of the theorem. $\qquad\square$

Remark 1.3.18. There is a key technical difference between the diffeomorphism φ we construct in Theorem 1.3.17 and the Turing complete diffeomorphism of $\mathbb{T}^4$ constructed in [184]. In Tao's construction, the point p depends only on the Turing machine T and the input (q_0, t). Then, for any given finite string $t^* := (t^*_{-k}, ..., t^*_k)$ there is some open set U_{t^*} such that the orbit through p intersects U_{t^*} if and only if T halts with input (q_0, t) and output whose positions $-k, ..., k$ correspond to t^*. In contrast, in the diffeomorphism φ we construct in Theorem 1.3.17, the point p depends on all the information: the Turing machine T, the input (q_0, t) and the finite string $t^* = (t^*_{-k}, ..., t^*_k)$. In particular, if we pick another finite string t^*_2, the point p will be different. Additionally, U is always the same, i.e., a neighborhood of those blocks associated to the halting state of T_{un}.

Finally, we prove the lemma that is used in the proof of Theorem 1.3.17: given a Turing machine T and a finite string $(t^*_{-k}, ..., t^*_k)$, one can construct a Turing machine T' which halts with a given input if and only if T halts with the same input and with the output tape having in positions $(-k, ..., k)$ the fixed symbols $(t^*_{-k}, ..., t^*_k)$. This is intuitively clear, one simply needs to construct a machine T' that works exactly as T, but when T reaches the halting state, T' reads the positions $-k, ...k$ to compare with $(t^*_{-k}, ..., t^*_k)$. This is formalized in the following lemma (which is probably standard in the theory of Turing machines).

Lemma 1.3.19. *Let $T = (Q, q_0, q_{halt}, \Sigma, \delta)$ be a Turing machine. For any $k \geq 0$ and finite string $(t^*_{-k}, .., t^*_k) \in \Sigma^{2k+1}$, there is a Turing machine T' which halts with input (q_0, t) if and only if the machine T with input (q_0, t) halts with coefficients $t^*_{-k}, ..., t^*_k$ in positions $-k, ..., k$ in the output tape.*

Proof. Fix a Turing machine $T = (Q, q_0, q_{halt}, \Sigma, \delta)$ and a finite string $(t^*_{-k}, ..., t^*_k)$. As before, F_1, F_2, F_3 denote the three components of the transition function $\delta :$ $Q \times \Sigma \to Q \times \Sigma \times \{-1, 0, 1\}$. To define the Turing machine T', take as alphabet $\Sigma' := \Sigma$ and as set of states $Q' := Q \sqcup \{r_0, ..., r_{3k}, q_{nohalt}\}$, where r_j and q_{nohalt} simply denote new states we include in the space. The initial and halting states

of T' are the same as for T. The idea is to use the states $r_0, ..., r_{3k}$ as "reading states" that will check if the final output is the desired one.

Let us denote the current state of the Turing machine by (q, t) and by t_0 the symbol in the central position. The transition function δ' can be defined as follows. For $q \in Q \setminus \{q_{halt}\}$, if $F_1(q, t_0) \in Q \setminus \{q_{halt}\}$ then we set $\delta'(q, t_0) := \delta(q, t_0)$ and if $F_1(q, t_0) = q_{halt}$ we define $\delta'(q, \tilde{t}) := (r_0, F_2(q, t_0), F_3(q, t_0))$. This way, when T reaches a halting state, T' will reach the state r_0.

Now we define the transition function δ' for $q \in \{r_0, ..., r_{3k}, q_{nohalt}\}$ as:

$$\delta'(r_i, t_0) := \begin{cases} (r_{i+1}, t_0, -1) \text{ if } t_0 = t^*_{-i} \\ (q_{nohalt}, t_0, 0) \text{ otherwise} \end{cases}, \text{ for } i = 0, ..., k-1. \qquad (1.15)$$

$$\delta'(r_i, t_0) := \begin{cases} (r_{i+1}, t_0, +1) \text{ if } t_0 = t^*_{i-2k} \\ (q_{nohalt}, t_0, 0) \text{ otherwise} \end{cases}, \text{ for } i = k, ..., 3k-1, \qquad (1.16)$$

and $\delta'(r_{3k}, t_0) := (q_{halt}, t_0, 0)$ if $t_0 = t^*_k$ and $(q_{nohalt}, t_0, 0)$ otherwise. Finally, we define δ' for q_{nohalt} so that the machine gets trapped in a loop, e.g. we can set $\delta'(q_{nohalt}, t_0) := (q_{nohalt}, t_0, 0)$ for any symbol t_0.

Let us check that T' satisfies the required property. Suppose that T halts with a given input (q_0, t). Denote by $t^h := (...t^h_{-1}.t^h_0 t^h_1...)$ the output tape of T, i.e., the tape when T reaches the halting state. By the construction, the machine T' with input (q_0, t) will reach the state r_0 with tape t^h instead of halting. By Equation (1.15), if $t^h_0 = t^*_0$ the machine will shift the tape to the right and change the current state to r_1. If the symbol t^h_0 does not correspond to t^*_0, then T enters a loop through the state q_{nohalt} and will never halt.

After shifting to the right, the current tape is now $(...t^h_{-2}.t^h_{-1}t^h_0...)$ and the current state is r_1. Again by Equation 1.15, the machine enters a loop unless $t^h_{-1} = t^*_{-1}$, in which case we shift to the right and change to state r_2. Iterating this process, the machine reaches the state r_k if and only if $t^h_{-i} = t^*_{-i}$ for each $i = 0, 1, ..., k-1$. The current tape is then $(...t^h_{-(k+1)}.t^h_{-k}...)$. Similarly, by Equation (1.16) for states r_j with $j = k, ..., 3k-1$, at each step the machine is at the state r_j with current tape $(...t^h_{j-2k-1}.t^h_{j-2k}...)$, and it checks if $t^h_{j-2k} = t^*_{j-2k}$, in which case it shifts to the left with new state r_{j+1}. Finally, the machine reaches the state r_{3k} if and only if $(t^h_{-k}....t^h_{k-1}) = (t^*_{-k}...t^*_{k-1})$, and the current tape becomes $(...t^h_{k-1}.t^h_k...)$. By the definition of δ' at r_{3k}, the machine halts if and only if $t^h_k = t^*_k$ or else enters a loop. It is then obvious that T' halts with input (q_0, t) if and only if T halts and its output satisfies that $(t^h_{-k}....t^h_k) = (t^*_{-k}...t^*_k)$, which completes the proof of the lemma. $\square$

1.3.4 A Turing complete Euler flow of Reeb type

In this last section we use the Turing complete area-preserving diffeomorphism constructed in Theorem 1.3.17 to establish the existence of an Eulerisable field in

$\mathbb{S}^3$ which is Turing complete. In the proof we use Etnyre-Ghrist's contact mirror, cf. Theorem 1.1.15, and the realization Theorem 1.3.2 which allows one to embed a diffeomorphism of the disk as the return map of a Reeb flow.

Embedding diffeomorphisms as cross sections of Beltrami flows

In [33] we constructed a Turing complete Eulerisable flow on $\mathbb{S}^{17}$ using a new h-principle for Reeb embeddings; the dimension 17 is essentially sharp with this approach. In contrast, the ideas we introduced in Sections 1.3.1, 1.3.2 and 1.3.3 allow us to reduce the dimension to 3, as shown in the following theorem, which is the main result of this work.

Theorem 1.3.20. *There exists an Eulerisable flow X in $\mathbb{S}^3$ that is Turing complete in the following sense. For any integer $k \geq 0$, given a Turing machine T, an input tape t, and a finite string $(t^*_{-k}, ..., t^*_k)$ of symbols of the alphabet, there exist an explicitly constructible point $p \in \mathbb{S}^3$ and an open set $U \subset \mathbb{S}^3$ such that the orbit[4] of X through p intersects U if and only if T halts with an output tape whose positions $-k, ..., k$ correspond to the symbols $t^*_{-k}, ..., t^*_k$. The metric g that makes X a stationary solution of the Euler equations can be assumed to be the round metric in the complement of an embedded solid torus.*

Proof. By Theorem 1.3.17, there exists a Turing complete area-preserving diffeomorphism φ of the disk which is the identity in a neighborhood of the boundary. Take the standard contact sphere $(\mathbb{S}^3, \xi_{std})$ and apply Theorem 1.3.2 to obtain a defining contact form α whose Reeb field X exhibits an invariant solid torus $\mathcal{T}$ where the first return map on a disk section is conjugated to φ via a diffeomorphism $\Phi : D \to D$. By Remark 1.3.5 we can assume that in the complement of a neighborhood V of $\overline{\mathcal{T}}$, the one form α coincides with the standard contact form α_{std} of $\mathbb{S}^3$. In particular, X coincides with a Hopf field in $\mathbb{S}^3 \backslash V$. When applying the contact/Beltrami correspondence in Lemma 1.1.15, we obtain a Riemannian metric g which coincides with the round one (as done also in [33]) on $\mathbb{S}^3 \backslash V$. By construction of the metric, X satisfies the equation $\mathrm{curl}_g X = X$, so it is a stationary solution of the Euler equations on $(\mathbb{S}^3, g)$.

Finally, let us check that X satisfies the stated Turing completeness property. Take a Turing machine T with an input t and a finite string $(t^*_{-k}, ..., t^*_k)$ of symbols. Denoting by $D_0 = \{0\} \times D \subset \mathbb{S}^3$ the transverse section in $\mathcal{T}$ where the first return map of X is conjugated to φ, we can find the point $p \in D_0$ and the set $U_0 \subset D_0$ (open as a subset of D) defined as $p := \Phi(p_*)$ and $U_0 := \Phi(U_*)$, where p_* and U_* are, respectively, the point and open set given by Theorem 1.3.17. We then take the open set

$$U := \bigcup_{t \in (-\varepsilon_0, \varepsilon_0)} \phi_t(U_0),$$

---

[4]Here by orbit we mean the trajectory of the flow through p with non-negative times.

where $\varepsilon_0 > 0$ is a small enough constant, and ϕ_t is the flow defined by X. It is then clear that the point $p \in \mathbb{S}^3$ and the set $U \subset \mathbb{S}^3$ satisfy that T will halt with the given output positions if and only if the orbit of X through p intersects the open set U, thus completing the proof of the theorem. $\qquad\square$

About the Navier-Stokes equations

The Navier-Stokes equations describe the dynamics of an incompressible fluid flow with viscosity. On a Riemannian 3-manifold (M, g) they read as

$$\begin{cases} \frac{\partial u}{\partial t} + \nabla_u u - \nu \Delta u = -\nabla p \,, \\ \operatorname{div} u = 0 \,, \\ u(t = 0) = u_0 \,, \end{cases} \qquad (1.17)$$

where $\nu > 0$ is the viscosity. Here all the differential operators are computed with respect to the metric g, and Δ is the Hodge Laplacian (whose action on a vector field is defined as $\Delta u := (\Delta u^\flat)^\sharp$).

In this final section we analyze what happens with the vector field X constructed in Theorem 1.3.20 when taken as initial condition for the Navier-Stokes equations with the metric g that makes X a steady Euler flow. Specifically, using that $\operatorname{curl}_g(X) = X$, the solution to Equation (1.17) with $u_0 = MX$, $M > 0$ a real constant, is easily seen to be

$$\begin{cases} u(\cdot, t) = MX(\cdot)e^{-\nu t} \,, \\ p(\cdot, t) = c_0 - \frac{1}{2}M^2 e^{-2\nu t}\|X\|_g^2 \,, \end{cases} \qquad (1.18)$$

for any constant c_0. The integral curves (fluid particle paths) of the non-autonomous field u solve the ODE

$$\frac{dx(t)}{dt} = Me^{-\nu t}X(x(t)) \,.$$

Accordingly, reparametrizing the time as

$$\tau(t) := \frac{M}{\nu}(1 - e^{-\nu t}) \,,$$

we show that the solution $x(t)$ can be written in terms of the solution $y(\tau)$ of the ODE

$$\frac{dy(\tau)}{d\tau} = X(y(\tau)) \,,$$

as

$$x(t) = y(\tau(t)) \,.$$

When $t \to \infty$ the new "time" τ tends to $\frac{M}{\nu}$, and hence the integral curve $x(t)$ of the Navier-Stokes equations travels the orbit of X just for the time interval $\tau \in [0, \frac{M}{\nu})$. In particular, the flow of the solution u only simulates a finite number

of steps of a given Turing machine, so we cannot deduce the Turing completeness of the Navier-Stokes equations using the vector field MX as initial condition. More number of steps of a Turing machine can be simulated if $\nu \to 0$ (the vanishing viscosity limit) or $M \to \infty$ (the L^2 norm of the initial datum blows up). For example, to obtain a universal Turing simulation we can take a family $\{M_k X\}_{k \in \mathbb{N}}$ of initial data for the Navier-Stokes equations, where $M_k \to \infty$ is a sequence of positive numbers. The energy (L^2 norm) of this family is not uniformly bounded, thus raising the challenging question of whether there exists an initial datum of finite energy that gives rise to a Turing complete solution of the Navier-Stokes equations.

Final remark: an alternative proof of Corollary 1.3.3

In Section 1.3.1, we used tools from contact topology to prove that any area-preserving compactly supported map of the disk can be realized as the first return map of a Reeb flow in any 3-contact manifold. We needed this result to deduce that the same property holds for flows which are Beltrami for a certain Riemannian metric. The fact that this property is satisfied by Reeb flows has its own interest, but if we were only interested about Corollary 1.3.3, there is an alternative proof which avoids having to realize the mapping torus as a Reeb flow. We will use the following lemma (that will be used in several parts of this thesis, confer Lemma 2.1.7 in Section 2.1.2 for a proof of this fact).

Lemma 1.3.21. *Let X be a non-vanishing vector field on a manifold M and α a one form such that $\alpha(X) > 0$. Let μ be a volume form in M. Then, there exist a Riemannian metric g such that $g(X, \cdot) = \alpha$ and μ is the induced Riemannian volume.*

Let f denote the area-preserving map of the disk. Consider in $D_1 \times [0, 1]$ the vector field $X = \frac{\partial}{\partial t}$ and the one form $\beta = dt$, where t is the canonical coordinate in $[0, 1]$. Construct the mapping torus $D_1 \times [0, 1] \backslash \sim$, where we identified $(p, 0)$ with $(f(p), 1)$. We obtain a solid torus S since f is isotopic to the identity, and a one form α such that $\alpha(X) = 1$, $\iota_X d\alpha = 0$ and X has f as first return map. Furthermore $\alpha = d\theta$ in the boundary, since f is the identity near the boundary. Here θ defines the angle coordinate of the first component in $S^1 \times D^2$. Now one can interpolate between $d\theta$ and $d\theta + r^2 d\varphi$, where (r, φ) are coordinates in D^2. This is easily done by taking $d\theta + h(r)d\varphi$ such that $h(1) = 0$, $h(1 + \varepsilon) = r^2$. The resulting one-form can be glued with a contact form, and X can be extended as the Reeb field of the contact form in $M \backslash S$ as done in Step two of the proof of Theorem 1.3.2.

We obtain a globally defined pair (X, β), which satisfy $\beta(X) > 0$ and $\iota_X d\beta = 0$. By Lemma 1.3.21, there is a metric for which X is a Beltrami field. However, this is not a Reeb flow: observe that even if β is of contact type in $M \backslash S$, it is closed in parts of the solid torus S. In particular, the resulting Beltrami field does not have a non-vanishing proportionality factor for its adapted metric.

Chapter 2

The steady Euler equations with a variable metric

In this chapter, we analyze the steady solutions to the Euler equations for some metric in manifolds of dimension three and higher odd dimensions. The variability metric is captured by the geometric formulation of the stationary equations, which leads to the natural definition of Eulerisable flow introduced in [164]. This generalizes the correspondence between Reeb flows and certain Beltrami type steady Euler flows that we used in the previous Chapter. The content of this chapter is based on [24, 26] and [25].

2.1 Integrability and adapted one forms in fluids

Let us recall the steady Euler equations in their geometric formulation, which were introduced in Section 1.1. Given a Riemannian manifold (M, g) of any dimension, the steady Euler equations are

$$\begin{cases} \iota_X d\alpha = -dB \\ d\iota_X \mu = 0 \end{cases}. \tag{2.1}$$

Here X denotes the velocity of the fluid, α is one form dual to X by g, B is the Bernoulli function and μ denotes the induced Riemannian volume form.

2.1.1 Arnold's theorem: integrable steady solutions

In Chapter 1, we mainly worked and discussed those solutions for which B is a constant function. This leads to a flexible (both for a variable and for a fixed metric) class of solutions called Beltrami fields. The "opposite" case is when the Bernoulli function is non-constant and generic in some sense. Then Lemma 1.1.1 tells us that B is a first integral of the vector field X. When B is, say, non-constant and analytic or Morse-Bott, we can deduce the very rigid behavior of these type of

solutions. This is captured by Arnold's structure theorem [3], which inaugurated the field of topological hydrodynamics.

Theorem 2.1.1 (Arnold's structure theorem). *Let X be an analytic stationary solution to the Euler equations on an analytic compact manifold M of dimension three with non-constant Bernoulli function. The flow is assumed to be tangent to the boundary if there is one. Then there is an analytic set C of codimension at least 1 such that $M \backslash C$ consists of finitely many domains M_i such that either*

1. *M_i is trivially fibered by invariant tori of X and on each torus the flow is conjugated to the linear flow,*

2. *or M_i is trivially fibered by invariant cylinders of X whose boundaries lie on the boundary of M, and all stream lines are periodic.*

This was the original statement of Arnold, but there are other combinations of hypotheses that lead to the same conclusion. For instance, if M is a closed manifold, then it is enough to assume that B has a critical set of zero measure which is nicely stratified. This happens for example when B is analytic or Morse-Bott. After removing the set C, which is the union of critical level sets of B, we get some domains M_i fibered by invariant tori where the flow is conjugated to a linear flow.
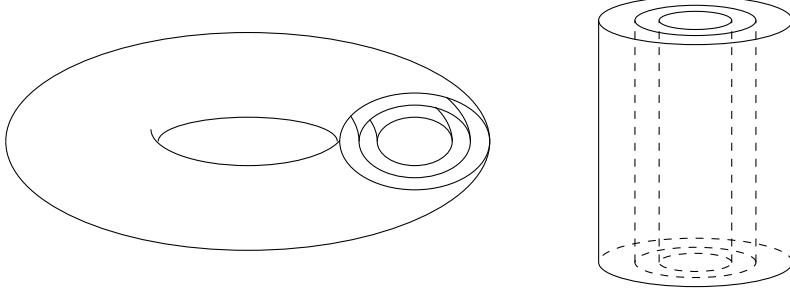
Figure 2.1: Domains M_i

Examples of fluids with analytic Bernoulli function in S^3 and T^3 with respectively the round and the flat metric were constructed in [119].

Example 2.1.2. *Consider the three sphere S^3 seen inside $\mathbb{R}^4$ with coordinates (x, y, z, t). Endow it with the round metric, which is induced by the Euclidean metric $g_0 = dx^2 + dy^2 + dz^2 + dt^2$ in $\mathbb{R}^4$.*

$$S^3 = \{(x, y, z, t) \in \mathbb{R}^4 |\ x^2 + y^2 + z^2 + t^2 = 1\}$$

In these coordinates, we can introduce the Hopf fields $X_1 = (-y, x, t, -z)|_{S^3}$ and $X_2 = (-y, x, -t, z)|_{S^3}$. These vector fields are tangent to the level sets of the

function $u = x^2 + y^2$, whose regular level sets are tori, and are also divergence free. For any two analytic function $f, g : \mathbb{R} \to \mathbb{R}$ we consider the vector field

$$X = f(u)X_1 + g(u)X_2.$$

The fact that u is an integral of X_1 and X_2 ensures that X is divergence free. Indeed:

$$
\begin{aligned}
d(\iota_X \mu) &= d(f(u)\iota_{X_1}\mu + g(u)\iota_{X_2}\mu) \\
&= f'du \wedge \iota_{X_1}\mu + f(u)d\iota_{X_1}\mu + g'du \wedge \iota_{X_2}\mu + g(u)d\iota_{X_2}\mu
\end{aligned}
$$

Since X_1, X_2 are divergence free, we deduce that $d\iota_{X_1}\mu = d\iota_{X_2}\mu = 0$. In the other hand, we have that $\iota_{X_1}du = \iota_{X_2}du = 0$ since u is an integral of X_1 and X_2. This readily implies that the two other terms $du \wedge \iota_{X_i}\mu = 0$ and hence that X is volume preserving. A few more computations show that

$$\iota_X d\alpha = [ff' + gg' + 4fg + (2u - 1)(fg' + gf')]du,$$

where α is the dual form to X. In particular, X is an Euler flow whose Bernoulli function is $B := \int_0^u ff' + gg' + 4fg + (2u - 1)(fg' + gf')$, which is in general a non-constant analytic function.

Example 2.1.3. *Another explicit example can be given in the flat torus T^3, with coodinates x, y, z. The vector field*

$$X = f(z)\frac{\partial}{\partial x} + g(z)\frac{\partial}{\partial y},$$

where f and g are analytic 2π-periodic functions, is a volume preserving vector field. In this case, the Bernoulli function is given by $B = \frac{1}{2}(f^2 + g^2)$.

2.1.2 Eulerisable, geodesible and Beltrami fields

Note that one usually fixes the Riemannian metric, fixing the partial differential equation and then looking for solutions. However it is also possible to study vector fields which are solutions to the Euler equations for some metric. This is the approach that we took in Chapter 1, focusing on rotational Beltrami fields.

This idea of being a solution for some metric is framed by the notion of Eulerisable field, introduced in [164].

Definition 2.1.4. *Let M be manifold with a volume form μ. A volume-preserving vector field X is **Eulerisable** if there is a metric g on M for which X satisfies the Euler equations for some Bernoulli function $B : M \to \mathbb{R}$.*

When the preserved volume is not the Riemannian one, the equations describe the behavior of an ideal barotropic fluid. However given an Eulerisable field one can always construct a metric such that the Riemannian volume is μ.

This is in fact a generalization of geodesible vector fields as introduced in Section 1.1.2.

Definition 2.1.5. *A vector field X is **geodesible** if there exists a metric g such that its orbits are geodesics.*

Recall that by a characterization of Gluck [83], a vector field is geodesible if and only if there is a one form α such that $\alpha(X) > 0$ and $\iota_X d\alpha = 0$. When we further have $\alpha(X) = 1$, then the vector field is geodesible of unit length. We might refer to α as the connection one form.

Observe that a geodesible field is a particular case of an Eulerisable field for which there is some one form such that $\alpha(X) > 0$ and $\iota_X d\alpha$ vanishes (instead of simply being exact).

In [164] a characterization *à la Sullivan* is given for the more general class of Eulerisable fields. However, we will only use the following simpler characterization.

Lemma 2.1.6 ([164])**.** *A non-vanishing volume-preserving vector field X in M is Eulerisable if and only if there exists a one form α such that $\alpha(X) > 0$ and $\iota_X d\alpha$ is exact.*

This follows from the following standard lemma, which is used in several points of the literature (see [83] for example).

Lemma 2.1.7. *Let X be a non-vanishing vector field on a manifold M of any dimension and α a one form such that $\alpha(X) > 0$. Let μ be a volume form in M. Then, there exist a Riemannian metric g such that $g(X, \cdot) = \alpha$ and μ is the induced Riemannian volume.*

Proof. Construct a metric g by requiring that

- $g(X, \cdot) = \alpha$,

- X is orthogonal to $\ker \alpha$,

- arbitrary metric on $\ker \alpha$.

For such a metric, we have $g(X, Y) = \alpha(Y)$ for any Y, hence $\iota_X g = \alpha$. By taking an appropiate conformal factor in the arbitrary metric on $\ker \alpha$, we can ensure that μ is the induced Riemannian volume. $\square$

The last kind of vector fields that we are interested in are Beltrami fields. Recall that for a given vector field X in a Riemannian manifold (M, g) of odd dimension $2n + 1$, we define its curl as the only vector field Y satisfying the equation

$$\iota_Y \mu = (d\alpha)^n,$$

where μ is the Riemannian volume and $\alpha := \iota_X g$. In this chapter, we will drop the assumption that X is volume-preserving in the definition of a Beltrami field.

Definition 2.1.8. *A vector field X in a Riemannian manifold of odd dimension (M, g) is **Beltrami** for that metric if it is everywhere parallel to its curl, i.e. we have $Y = fX$ for some function $f \in C^\infty(M)$.*

Abusing notation, we will say that a vector field is Beltrami if it is Beltrami for some metric g (one can also speak of a vector field being Beltramisable). The interactions between geodesible, Beltrami and Eulerisable vector fields in three dimensions are very well understood. As we explained in 1.1.2, Beltrami and geodesible fields are equivalent in dimension three. This correspondence leads to another one, covered partially or totally at different points of the literature.

Proposition 2.1.9. *The following classes of vector fields are equivalent in three dimensions:*

1. *Vector fields such that for some metric they satisfy the Euler equations with constant Bernoulli function,*

2. *reparametrizations of Reeb fields of stable Hamiltonian structures.*

3. *volume-preserving geodesible fields,*

4. *volume-preserving Beltrami fields.*

In the next section, we will study geodesible and Beltrami fields in higher odd dimensions, and understand which of the previous equivalences remain true and which ones are broken.

2.2   Beltrami and Eulerisable fields in high odd dimensions

The aim of this section is to study three kinds of non-vanishing vector fields and its interactions in high dimensions: steady solutions to the Euler equations, geodesible fields and Beltrami fields.

We start introducing a geometric structure, that we call stable Eulerisable structure, which uniquely determines a vector field, and coincides with stable Hamiltonian structures in dimension three. This vector field is a unit length geodesible volume-preserving field and we can check that such structures provide an equivalent geometric formulation (in terms of differential forms) for the study of such fields. Volume-preserving geodesible fields are in correspondence with vector fields that are solutions to the steady Euler equations for some metric and constant Bernoulli function. It follows that vector fields defined by stable Eulerisable structures are steady Euler flows of this type. This viewpoint unveils their geometric wealth and allows us to naturally import topological techniques from the contact and stable Hamiltonian world. We show that one can construct stable Eulerisable structures supported by open books, and use it to prove the existence of such structures in every homotopy class of non-vanishing vector fields of any odd dimensional manifold. In particular, by the mentioned relation with steady Euler flows, we deduce the following result.

Theorem 2.2.1. *Given an odd dimensional manifold and a homotopy class of non-vanishing vector fields, there exist a metric and a vector field in the given class that is a steady solution to the Euler equations with constant Bernoulli function.*

The constructed solutions are geodesible and hence of Beltrami type, but are not a reparametrized Reeb field of a contact form. A more geometric interpretation of this existence theorem by saying that an odd dimensional manifold can be foliated by geodesics of some metric in any homotopy class, and furthermore the vector field whose orbits are the geodesics preserves the Riemannian volume. By means of a local modification of the constructed solutions we exhibit Euler flows that are chaotic, in the sense that there is a compact invariant set with positive topological entropy. Another consequence, which follows from results in [33], is the fact that any homotopy class of vector fields of every odd dimensional manifold can be realized in the invariant submanifold of a Reeb field in a standard contact sphere of higher dimension (see the last paragraph of Section 2.2.1 for a precise statement).

In three dimensions, a very fruitful source of examples of steady Euler flows are volume-preserving Beltrami fields: volume-preserving vector fields which are parallel to their curl (for some metric). The correspondence between geodesible and Beltrami fields in three-manifolds shows that steady Euler flows with constant Bernoulli function are equivalent to volume-preserving Beltrami fields. The study of Beltrami flows and its properties in high odd dimensions was already proposed in [81], where it is proved that non integrable analytic examples of steady flows are always Beltrami fields. It is mentioned that it would be interesting to construct examples and compare its properties to the three dimensional case, we do so in this work. In fact, the constructions that lead to Theorem 2.2.1, provide a lot of examples of such Beltrami type steady Euler flows (cf. Section 2.2.1).

In the high dimensional setting the correspondence between geodesible and Beltrami fields is broken: any geodesible field is Beltrami but the converse is not true. We give a characterization of Beltrami fields and provide a construction, which uses plugs and can be done volume-preserving, of vector fields that are parallel to their curl for some metric but are not geodesible. This yields also examples of volume-preserving Beltrami fields which are not solutions to the Euler equations for any metric (i.e. it is not Eulerisable): this is highly in contrast with the situation for 3-manifolds.

Theorem 2.2.2. *In every manifold of dimension $2n + 1 > 3$ and every homotopy class of non-vanishing vector fields, there is a volume-preserving Beltrami field which is not geodesible nor a solution to the Euler equations for any metric.*

It was proved [113, 171] that, except in a torus bundle over S^1, every Beltrami field in a three-manifold which is either analytic or volume-preserving has a periodic orbit. This shows that Reeb fields of stable Eulerisable structures satisfy the Weinstein conjecture in dimension three except in torus bundles over the circle. We give examples of manifolds in every dimension that admit aperiodic Reeb

fields of stable Eulerisable structures, which generalize the torus bundle over S^1 counterexample. Even if the construction in Theorem 2.2.2 is done using plugs, it does not directly imply that volume-preserving Beltrami fields can be aperiodic. This is because the plug cannot be used arbitrarily: one needs to find points where the geometric information of the Beltrami field has a specific normal form.

It is natural to ask if for the class of Beltrami fields one can find examples without periodic orbits. Taking into account the mentioned constrains to insert plugs, we construct aperiodic Beltrami fields (not necessarily volume-preserving) in high dimensions. Hence the existence of periodic orbits remains open only for three dimensional smooth Beltrami fields.

Theorem 2.2.3. *Let M be a closed manifold of dimensions $2n + 1 > 3$. Then M admits a (not necessarily volume-preserving) Beltrami vector field without periodic orbits.*

2.2.1 Steady Euler flows of Beltrami type

We will first introduce an alternative approach to the study of geodesible volume-preserving vector fields. This will be useful to import techniques from contact topology and stable Hamiltonian topology, and prove an existence theorem.

Stable Eulerisable structures

A formulation in terms of differential forms can be given for the study of geodesible volume-preserving vector fields. This formulation opens the possibility to study these structures from a topological perspective, as done for stable Hamiltonian structure [42].

Definition 2.2.4. *A stable Eulerisable structure is a pair (α, ν) in a manifold M^{m+1}, where α is a one form and ν a m-form such that*

- $\alpha \wedge \nu > 0$,

- $d\nu = 0$,

- $\ker \nu \subset \ker d\alpha$.

A stable Eulerisable structure defines a unit length geodesible and volume-preserving vector field R, defined by the equations $\iota_R \nu = 0$ and $\alpha(R) = 1$. We will call this vector field the Reeb vector field of (α, ν), as it is done for stable Hamiltonian structures. In fact, stable Eulerisable structures are exactly the same as geodesible volume-preserving vector fields, but with some extra information fixed: the preserved volume and the connection one form (as in Gluck's characterization). Note that in three dimensions the definition coincides with the one of stable Hamiltonian structure.

In order to justify the name of a "stable Eulerisable structure", we prove a correspondence between Reeb fields of these structures and some solutions to the Euler equations for some metric. This generalizes the correspondences proved in [67], extended in [42] and studied in more settings [31].

Proposition 2.2.5. *In a manifold M of any dimension, there is a correspondence between reparametrizations of Reeb fields of stable Eulerisable structures and solutions to the Euler equations for some metric and constant Bernoulli function.*

Proof. Suppose $X = fR$ is a reparametrization of a Reeb field of a stable Eulerisable structure (α, ν) for some positive function $f > 0$ in a manifold M. Using Lemma 2.1.7 construct a metric g such that $\iota_X g = \alpha$ and such that the Riemannian volume is $\mu = \frac{1}{f}\alpha \wedge \nu$.

Using that $R \in \ker \nu$ we deduce that X preserves the volume μ. In particular, it is a solution to the Euler equations for the metric g, with constant Bernoulli function.

Conversely, suppose that X is a vector field satisfying the Euler equations for some metric g, with constant Bernoulli function. Denoting $\alpha = g(X, \cdot)$ and μ the Riemannian volume, X satisfies the equations

$$\begin{cases} \iota_X d\alpha &= 0 \\ d\iota_X \mu &= 0 \end{cases}.$$

Take $\nu := \iota_X \mu$ and we have that $\alpha(X) = 1$, $\iota_X \nu = 0$, $d\nu = 0$ and $\alpha \wedge \nu > 0$. Hence X is the Reeb field of the stable Eulerisable structure (α, ν). $\square$

The terminology becomes now clear, since the Reeb field of a stable Eulerisable structure is a solution to the Euler equations for some metric. The notion of being stabilized by a one form α comes from the world of stable Hamiltonian structures. Concretely, following [42], it is said that a one dimensional foliation $\mathcal{L}$ is *stabilizable* if there is some vector field X generating $\mathcal{L}$ and some one form α such that $\alpha(X) = 1$ and $\iota_X d\alpha = 0$.

We will restrict ourselves to odd dimensions in the next sections, where the curl operator works similarly to the three dimensional case. Then the dimension of M will be $2n + 1$ and ν is a form of degree $2n$. In this case, the Reeb field of a stable Eulerisable structure is Beltrami for some metric.

Proposition 2.2.6. *Let R be the Reeb field of a stable Eulerisable structure (α, ν) in a manifold M of odd dimension. Then any reparametrization of R is Beltrami for some metric g and preserves the induced Riemannian volume.*

Proof. The same metric and volume that we constructed in the first implication of Proposition 2.2.5 work. Let $X = fR$ be a reparametrization of R by a positive function f. The curl vector field of X is defined as the only vector field Y such that

$$\iota_Y \mu = (d\alpha)^n.$$

Then $\iota_X \iota_Y \mu = f \iota_R (d\alpha)^n = 0$ since $R \in \ker d\alpha$. Hence X is parallel to its curl, and again preserves μ. $\qquad\square$

We will prove an existence theorem for stable Eulerisable structures in odd dimensions, Theorem 2.2.11, implying Theorem 2.2.1. The relation between geodesible vector fields and open book decompositions was already suggested by Gluck [84], and used in [99] to construct a geodesible vector field in any odd dimensional manifold. By using techniques coming from contact and stable Hamiltonian structures, we improve the construction to obtain the existence theorem for stable Eulerisable structures.

Open book decompositions

We first discuss some results on open book decompositions proved in [65]. These decompositions have played a key role in contact topology since the works of Giroux [82].

Definition 2.2.7. *An open book decomposition for a $(2n + 1)$-manifold M is a pair (B, π) such that*

- *B is a codimension 2 submanifold that admits a trivial neighborhood $U = B \times D^2$,*

- *$\pi : M \backslash B \to S^1$ is a fibration which, when restricted to U, corresponds to the projection $(p, r, \theta) \mapsto \theta$ where (r, θ) are polar coordinates in D^2.*

We call B the binding and $P := \pi^{-1}(\theta)$ a page of the open book. If the page satisfies that it is a handlebody with handles of maximum index n, we say that the page is almost canonical.

Given a hyperplane field ξ on the binding B we can construct an hyperplane field in the whole manifold M. Denote by α a one form defining ξ, i.e. $\ker \alpha = \xi$. Restricting ourselves to the neighborhood $U = B \times D^2$ with polar coordinates (r, θ) in D^2. We define

$$\beta = \tilde{f}(r)d\theta + \tilde{g}(r)\alpha,$$

where $\tilde{f}$ and $\tilde{g}$ are smooth functions satisfying the following conditions.

$$\begin{cases} \tilde{f}(r) = r^2 \text{ near } 0, \quad \tilde{f}(r) = 1 \text{ near } 1 \\ \tilde{g}(r) = 1 \text{ near } 0, \quad \tilde{g}(r) = 0 \text{ near } 1 \end{cases} . \tag{2.2}$$

The one form β can be extended as $\pi^* d\theta$ to the whole manifold. The kernel of β defines an hyperplane field in M, whose restriction to B is ξ. If we denote $\mathcal{H}(M)$ the hyperplane fields of a manifold, we just constructed a map

$$H : \mathcal{H}(B) \longrightarrow \mathcal{H}(M)$$
$$\xi = \ker \alpha \longmapsto \ker \beta.$$

Under the assumption that the pages of the open book are almost canonical, the map has a useful homotopic property.

Theorem 2.2.8 ([65])**.** *Let M be any manifold of dimension $2n + 1 > 3$ and (B, π) any open book decomposition of M. Then we have*

1. *The map H is well defined up to homotopy.*

2. *If the pages of (B, π) are almost canonical, then the map H is surjective at the homotopic level. More specifically, for any hyperplane η in M there is some hyperplane ξ in B such that $H(\xi)$ is homotopic to η.*

We will modify a little bit the map H while keeping its properties. This modification will be useful in the construction of the stable Eulerisable structures of Theorem 2.2.11. To do so, take a smooth function $h : [0, 1] \to \mathbb{R}$ such that $h(0) = 0$, $h(1) = 1$ and $h'(x) \neq 0$ only in $[\frac{1}{3}, \frac{2}{3}]$.

Lemma 2.2.9. *Consider the map $\tilde{H} : \mathcal{H}(B) \to \mathcal{H}(M)$ that sends ξ to the hyperplane $\ker(h(r)d\theta + (1 - h(r))\alpha)$ in a neighborhood of the binding and extended in its complement as $\ker d\theta$. Then $\tilde{H}$ also satisfies Theorem 2.2.8.*

Proof. Clearly, the function $\tilde{f}$ is homotopic to h by an homotopy $\tilde{f}_t = (1-t)\tilde{f}+th$, which satisfies $\tilde{f}_t(0) = 0$ and $\tilde{f}_t(1) = 1$. The function $\tilde{g}$ is homotopic to $1 - h$ by an homotopy $\tilde{g}_t = (1 - t)\tilde{g} + t(1 - h)$, which satisfies $\tilde{g}_t(0) = 1$ and $\tilde{g}_t(1) = 0$. In particular, in the neighborhood of the binding the kernel of the one form

$$\beta_t = \tilde{f}_t d\theta + \tilde{h}_t \alpha$$

can be extended to the rest of the manifold as being equal to $\pi^* d\theta$. We get a family of one-forms whose kernels define an homotopy between $H(\xi)$ and $\tilde{H}(\xi)$. $\qquad\square$

An existence result

As in the case of stable Hamiltonian structures, we can relate stable Eulerisable structures to open books.

Definition 2.2.10. *A stable Eulerisable structure (α, ν) is supported by the open book (B, π) if ν restricts as a positive volume form to each page of the open book. If the restriction of (α, ν) to any connected component of B is a positive stable Eulerisable structure, we say that it is positively supported by the open book. Analogously, if the restriction to each connected component of B is a negative stable Eulerisable structure, we say that it is negatively supported by the open book.*

In the three dimensional case, it was proved in [42] that any open book supports a stable Hamiltonian structure, and that any two stable Hamiltonian structures in the same cohomology class and same signs (induced orientations) in the binding circle components are connected by a stable homotopy supported by the open

book. In constrast with these results, we prove that in a fixed open book decomponsition with almost cannonical pages, we can have stable Eulerisable structure positively supported by the open book in every homotopy class of hyperplane fields. Maybe asking that both stable Eulerisable structures induce the same stable Eulerisable homotopy class in the binding, or at least the same hyperplane field homotopy class, is enough to prove that they are connected by a stable homotopy supported by the open book. In the three dimensional case, this would correspond to the fact of inducing the same signs in the circle binding components and to Theorem 4.2 in [42].

Theorem 2.2.11. *Let M be a manifold of dimension $2n + 1 > 3$ with a fixed open book decomposition (B, π) with almost canonical pages. Then for every homotopy class of hyperplane field $[\eta]$ and every cohomology class $\gamma \in H^{2n}(M)$ there is a stable Eulerisable structure (α, ν) positively supported by (B, π) such that $\ker \alpha$ is homotopic to η and $[\nu] = \gamma$.*

Proof. Let us first prove there is some stable Eulerisable structure supported by (B, π) in every homotopy class of hyperplane fields, delaying the discussion about the cohomology class of ν.

We will prove it by induction. As a first step, Martinet-Lutz [132, 134] proved that in any 3 manifold every plane field is homotopic to a contact structure α, which is also a stable Eulerisable structure given by $(\alpha, d\alpha)$.

Assume now that for every manifold up to dimension $2n - 1$ there is a stable Eulerisable structure in every homotopy class of non-vanishing vector fields. Let M be an arbitrary compact manifold of dimension $2n + 1$. In [168], Quinn shows that any odd dimensional manifold M of dimension at least 5 admits an open book decomposition (B, π) such that its pages are handlebodies with handles of index less or equal than n.

We are in the hypothesis of Theorem 2.2.8, and by Lemma 2.2.9 any homotopy class of hyperplane fields of M is in the image of $\tilde{H}$. Hence given a class $[\eta]$, there is a hyperplane field in B such that $[\tilde{H}(\xi)] = [\eta]$. By hypothesis there exist a stable Eulerisable structure (β, ν') with connection form β on B in the homotopy class of ξ. Let us denote by X the Reeb field of (β, ν'). The form β defines an hyperplane field $\xi' = \ker \beta$ homotopic to ξ. Hence $[\tilde{H}(\xi')] = [\tilde{H}(\xi)]$ and the form β extends in the trivial neighborhood of the binding $U = B \times D^2$ as

$$\alpha = h(r)d\theta + (1 - h(r))\beta,$$

where $h(r)$ was such that $h(0) = 0$, $h(1) = 1$ and $h' \neq 0$ only in $[\frac{1}{3}, \frac{2}{3}]$. Its derivative is $d\alpha = h'dr \wedge d\theta - h'dr \wedge \beta + (1 - h)d\beta$. Let us consider a vector field of the form

$$Y = f(r)\frac{\partial}{\partial \theta} + (1 - f(r))\pi^* X, \tag{2.3}$$

with f being a smooth function $f : [0, 1] \to \mathbb{R}$ satisfying that $f(0) = 0$ and $f(1) = 1$. To simplify notation, we will keep denoting X the vector field $\pi^* X$ defined in U.

$$B$$
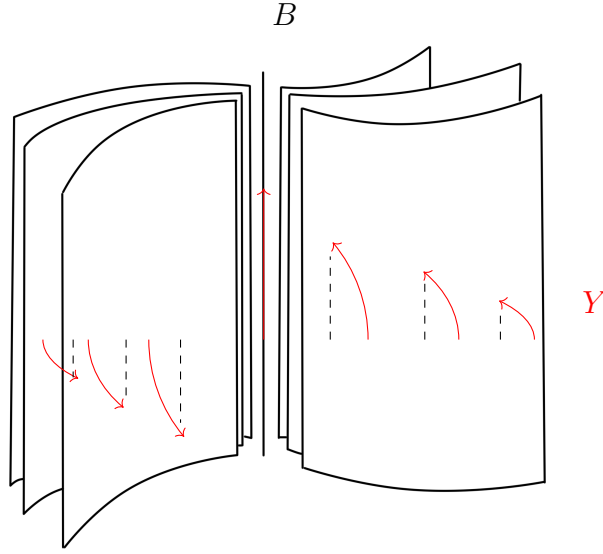
Figure 2.2: Vector field supported by an open book

Imposing the geodesibility conditions $\iota_Y \alpha > 0$ and $\iota_Y d\alpha = 0$ we obtain the equations:

$$\begin{cases} hf + (1-h)(1-f) > 0 \\ h'(1-2f) = 0 \end{cases}$$

Take the function f such that $f \neq 0$ in $(0,1]$ and $f = \frac{1}{2}$ in $[\frac{1}{3}, \frac{2}{3}]$. Possible choices are depicted in Figure 2.3.
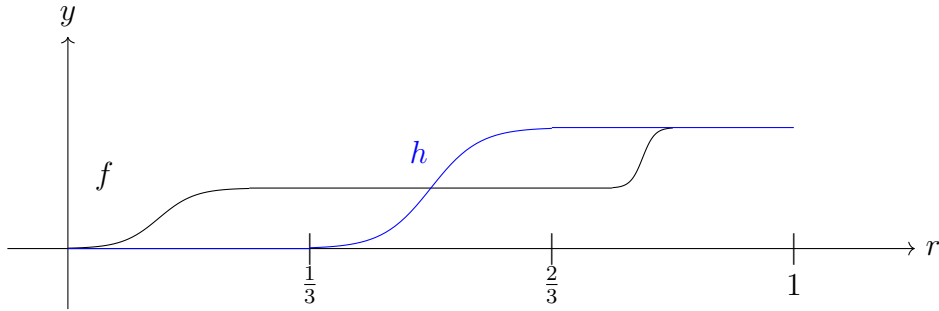
Figure 2.3: Possible choice of f and h

The vector field Y can be extended as $\pi^* \frac{\partial}{\partial \theta}$ to the whole manifold M and α as $\pi^* d\theta$. Hence we obtained a vector field Y which is geodesible with connection form α, and its orthogonal hyperplane field is in the class $[H(\xi')]$.

In the trivial neighborhood U, there is a volume form

$$\Theta = r dr \wedge d\theta \wedge \mu,$$

where $\mu = \beta \wedge \nu'$ is the volume form in B preserved by X. This is a volume form in U and $r dr \wedge \mu$ is a volume form in any fixed page $\{\theta = \theta_0\}$ of the trivial neighborhood U. Hence it can be extended on some page P to a volume form μ'_P in the whole page. Up to applying Moser's path method, we can assume that the monodromy of the open book preserves μ'_P. Hence μ'_P can be glued with the pullback of the monodromy to give a globally defined $2n$-form that we denote μ_P. In particular Θ can be extended to M such that away from U it has the form $\pi^* d\theta \wedge \mu_P$.

Let us check that Y preserves Θ. Away from U this is clear, since $Y = \pi^* \frac{\partial}{\partial \theta}$ and the flow preserves μ_P. In U we compute using the local expression (2.3) of Y:

$$\begin{aligned}
\mathcal{L}_Y \Theta = d(\iota_Y \Theta) &= d(f(r) r dr \wedge \mu + (1 - f(r)) r \iota_X \mu \wedge dr \wedge d\theta) \\
&= 0 + (1 - f(r)) r (d\iota_X \mu) \wedge dr \wedge d\theta \\
&= 0,
\end{aligned}$$

where we used that $d\iota_X \mu = d\nu' = 0$, since X is the Reeb field of (β, ν'). Taking $\nu = \iota_Y \Theta$, the pair (α, ν) defines a stable Eulerisable structure supported by the open book (B, π). Since η was arbitrary, this proves that any homotopy class of hyperplane fields contains a stable Eulerisable structure supported by the open book.

It remains to check that ν can be modified to obtain any homology class in $H^{2n}(M)$. This works as the three dimensional case for stable Hamiltonian structures. The open book decomposition (B, π) can be seen as a mapping torus of a $2n$-manifold with boundary, the page P, with diffeomorphism given by the monodromy map of the open book. Denote $(P, \partial P)$ the page with its boundary and $(P_\varphi, \partial P_\varphi)$ the associated mapping torus. The exact sequence of the pair (M, B) is

$$H^{2n-1}(B) \xrightarrow{d*} H^{2n}(M, B) \xrightarrow{j*} H^{2n}(M) \to H^{2n}(B) = 0.$$

The space $H^{2n}(M, B)$ is the same as $H^{2n}(P_\varphi, \partial P_\varphi)$. As in [43, Lemma 4.4], we have the following lemma in any dimension with the same proof. As before, we denote by (r, θ) coordinates in the disk component of the trivial neighborhood $U = B \times D^2$.

Lemma 2.2.12. *Any De Rham cohomology class $\eta \in H^{2n}(M)$ has a representative of the form $\pi^* d\theta \wedge \beta$ where β is a $(2n - 1)$-form with compact support in $M \backslash B$.*

The cohomology class $\eta - [\nu]$ can be represented by $\pi^* d\theta \wedge \beta$, and we can assume β is with support in $M \backslash U$ i.e. vanishing in the trivial neighborhood $U = B \times D^2$. Defining the closed $2n$-form

$$\tilde{\nu} = \nu + \pi^* d\theta \wedge \beta,$$

it satisfies that its restriction to the pages coincides with ν. Hence $\tilde{\nu}$ is a positive volume form in the pages and represents the cohomology class η. Furthermore,

in the support of β the form α is given by $\pi^* d\theta$ by construction, since it is away from the neighborhood U. In particular $d\alpha = 0$, and we have necessarily that $\ker \nu \subset \ker d\alpha$. Furthermore $\alpha \wedge \tilde{\nu} = \pi^* d\theta \wedge (\nu + \pi^* d\theta \wedge \beta) = \alpha \wedge \nu > 0$ and so $(\alpha, \tilde{\nu})$ defines a stable Eulerisable structure, positively supported by (B, π). This concludes the proof. $\square$

The homotopy classes of hyperplane fields defined by the kernel of the one form of a stable Eulerisable structure are in correspondence with the homotopy classes of non-vanishing vector fields defined by the Reeb field. From a dynamical viewpoint, we can interpret the result: every non-vanishing vector field is homotopic through non-vanishing vector fields to a geodesible and volume-preserving field.

Combining Theorem 2.2.11 with Proposition 2.2.5, we obtain the existence result for Euler steady flows Theorem 2.2.1, generalizing results in [67] to higher dimensions. Another interpretation, in terms of foliation theory is the following.

Corollary 2.2.13. *Let M be an odd dimensional manifold. In an arbitrary homotopy class of non-vanishing vector fields, there exists a metric such that M can be foliated by geodesics.*

It follows from the construction of the proof of Theorem 2.2.11 that given a stable Eulerisable structure in the binding of any open book, we can construct one in the ambient manifold supported by the open book. Observe this holds for any open book: the hypothesis on the pages of the open book in the last Theorem is only used to show that there is a stable Eulerisable structure in every homotopy class of hyperplane fields.

Corollary 2.2.14. *Given an open book decomposition (B, π) of M, a manifold of odd dimension, and a stable Eulerisable structure (α_B, ν_B) on the binding, there is a stable Eulerisable structure positively supported by (B, π) inducing (α_B, ν_B) on the binding.*

This provides even more examples of steady solutions to the Euler equations, and also of geodesible and Beltrami volume-preserving fields.

Remark 2.2.15. Corollary 2.2.14 holds also for even dimensional manifolds, so any open book decomposition of an even dimensional manifold, whose binding admits a stable Eulerisable structure, also admits a stable Eulerisable structure. Since the 2-torus admits trivially such a structure, we deduce that any four manifold admitting an open book decomposition with torus binding components admits also a stable Eulerisable structure. These type of open book decompositions were considered for example in [44].

In [33] the authors prove that given a geodesible field in a manifold M, it can be "Reeb embedded" in the standard sphere of dimension $3 \dim M + 2$. This means that for a given geodesible field X in M, there exists an embedding $e : M \rightarrow (S^{3 \dim M + 2}, \xi_{std})$ such that there is a contact form whose Reeb field R satisfies $e_* X = R$. Since the constructed steady flows are geodesible, we deduce the following corollary.

Corollary 2.2.16. *Any homotopy class of non-vanishing vector fields of a manifold M of dimension $2n + 1$ can be realized as an invariant submanifold of the Reeb field of a contact form defining the standard contact structure in the sphere (S^{6n+5}, ξ_{std}).*

Chaotic steady Euler flows

In [81] it is proved that in the analytic case, chaotic solutions to the Euler equations are always of Beltrami type. However, the construction of such chaotic Euler flows in high dimensions is left as a question. In [78], Ghrist gives the first smooth examples: if one takes a Reeb field of a contact form, which is an Eulerisable flow, one can locally modify it to obtain another chaotic Reeb field of a contact form and hence a steady Euler flow. By chaotic we mean such that there is a compact invariant set of the Reeb field possessing positive topological entropy. This proves that any contact manifold in any dimension admits a non integrable steady Euler flow. Adapting Ghrist's result to the solutions constructed in Theorem 2.2.1, we can generalize it to prove that any odd dimensional manifold admits such non integrable flows. Let us recall the contact case.

Given a contact form α in M of dimensions $2n + 1$, in the neighborhood $U = D^{2n+1}$ of any point p the contact form has the expression

$$\alpha = dz + \sum_{i=1}^{n} x_i dy_i,$$

where $(z, x_1, y_1, ..., x_n, y_n)$ are coordinates in the neighborhood U. The following theorem shows the existence of a contact chaotic Reeb field whose contact form coincides with the standard one away of a compact subset of $\mathbb{R}^{2n+1}$.

Theorem ([68, 78]). *There is function $F(x_i, y_i, z)$ which is equal to 1 away of a neighborhood of $0 \in D^n$ such that*

$$\lambda = F.(dz + \sum_{i=1}^{n} x_i dy_i),$$

is a contact form in $\mathbb{R}^{2n+1}$ and a compact invariant set of the Reeb field possessing positive topological entropy i.e. the Reeb flow is "chaotic".

We obtain the corollary below by combining this "inserted field" with the construction in Theorem 2.2.11.

Corollary 2.2.17. *Every odd dimensional manifold admits a chaotic non-vanishing solution to the Euler equations for some metric.*

Proof. The three dimensional case is covered, since any three-manifold is contact. Let M be a manifold of dimension $2n + 1 > 3$ and an almost canonical open book decomposition (B, π) on it. Using Theorem 2.2.11, we construct a stable

Eulerisable structure (α, ν) supported by (B, π). By construction, in the trivial neighborhood of the binding $U = B \times D^2$ the Reeb vector field X and the form α have the expressions

$$\begin{cases} X &= f(r)\frac{\partial}{\partial \theta} + (1 - f(r))\pi^* X \\ \alpha &= h(r)d\theta + (1 - h(r))\beta \end{cases}.$$

In particular, when $r > 2/3$ we have $h = 0$ and we can pick f such that in a neighborhood $r \in (1 - \varepsilon, 1)$ we have $f = 1$ (as the one depicted in Figure 2.3). In particular there is a neighborhood of the form $V = S^1 \times D^{2n}$ with coordinates $(\theta, x_1, ..., x_{2n})$ where $X|_V = \frac{\partial}{\partial \theta}$ and $\alpha|_V = d\theta$. Consider the function $r = \sqrt{\sum_{i=1}^{2n} x_i^2}$ and a function $\varphi(r)$ which is r^2 in a neighborhood of $r = 0$ and vanishes in a neighborhood of 1. Then we can change α by

$$\tilde{\alpha} = d\theta + \varphi(r)\alpha_{std},$$

where α_{std} is the standard contact structure in the sphere S^{2n-1} seen in D^{2n}. In a neighborhood of $r = 0$ we have that $\tilde{\alpha}$ is a contact form with Reeb field equal to X. It is easy to check that the vector field $X = \frac{\partial}{\partial \theta}$ still satisfies $\iota_X d\tilde{\alpha} = 0$. Hence X is also the Reeb field of the stable Eulerisable structure defined by $(\tilde{\alpha}, \nu)$. However, using Darboux theorem we can now find coordinates (z, x_i, y_i) at a neighborhood D of a point where $\tilde{\alpha}$ is a contact form such that $X|_D = \partial_z$ and $\tilde{\alpha}|_D = dz + \sum_{i=1}^n x_i dy_i$. Inserting a contact form as the one in the previous Theorem yields a one form λ which is contact, defines a Reeb field R such that λ coincides with $\tilde{\alpha}$ and R with X away of a small neighborhood of the point p. Extending λ as $\tilde{\alpha}$ and R as X in the rest of the manifold we obtain a vector field R which preserves a volume μ and such that $\iota_R \lambda = 1$ and $\iota_R d\lambda = 0$. Hence R is the Reeb field of the stable Eulerisable structure $(\lambda, \iota_R \mu)$ and defines a steady solution to the Euler equations for some metric. Furthermore, the vector field is chaotic. $\square$

The geometric formulation of geodesible volume-preserving vector fields was key to naturally import techniques coming from contact topology.

2.2.2 High dimensional Beltrami fields

We study in this section the interactions between Beltrami, geodesible and Eulerisable fields. We construct vector fields which are Beltrami for some metric but that are not geodesible, in any odd dimensional manifold of dimension at least 5. The construction, which uses plugs, can be done volume-preserving and yields examples of volume-preserving Beltrami fields which are not Eulerisable.

One can characterize vector fields which are Beltrami in a similar way as Gluck's characterization for geodesible fields.

Lemma 2.2.18. *Let M be a manifold of dimension $2n + 1$. A vector field X is a Beltrami field for some metric g if and only if there is a one form α such that*

$\alpha(X) > 0$ and $\iota_X(d\alpha)^n = 0$. *If furthermore X preserves a volume μ, one can construct a metric g such that μ is the Riemannian volume.*

Proof. Suppose there is such a one form. Using Lemma 2.1.7, construct a metric g such that $\iota_X g = \alpha$ and μ is the Riemannian volume. Then the vorticity field of X, denoted Y, satisfies

$$\iota_Y \mu = (d\alpha)^n,$$

where μ is the Riemannian volume. Contracting with X and using the hypothesis we obtain that $\iota_X \iota_Y \mu = 0$, which implies that X is parallel to its curl and hence is Beltrami.

Conversely, if X is parallel to its curl Y we have that $\iota_X(\iota_Y \mu) = 0$ and $\iota_X(\iota_Y \mu) = \iota_X(d\alpha)^n$ where $\alpha = \iota_X g$. $\qquad \square$

Wilson plugs and obstructions

Let us start by recalling Wilson's plug [200], used to prove the existence of non-vanishing vector fields without periodic orbits in S^{2n+1} with $n > 1$. We will follow the description in [163].

Standard Wilson's plug. We consider the manifold $W = [-2,2] \times \mathbb{T}^2 \times [-2,2] \times D^{n-4}$, and put coordinates $(z, \varphi_1, \varphi_2, r, x_1, ..., x_{n-4})$. The manifold W is embedded into $\mathbb{R}^n$ by a map $i : W \longrightarrow \mathbb{R}^n$ sending a point p to

$$(z, \cos\varphi_1(6 + (3 + r)\cos\varphi_2), \sin\varphi_1(6 + (3 + r)\cos\varphi_2), (3 + r)\sin\varphi_2, x_1, ..., x_{n-4}).$$

Let us denote $\mathbf{x} = (x_1, ..., x_{n-4})$. We consider a vector field X_W in W with expression

$$X_W = f(z, r, \mathbf{x})\left(\frac{\partial}{\partial\varphi_1} + b\frac{\partial}{\partial\varphi_2}\right) + g(z, r, \mathbf{x})\frac{\partial}{\partial z}.$$

Choosing b an irrational number and f, g satisfying the following properties ensures that W is a plug trapping the orbits entering through $\{z = -2, |r| \leq 1, |\mathbf{x}| \leq 1/2\}$. The properties satisfied by f and g are

- f is skewsymmetric and g is symmetric with respect to the z coordinate,

- $g \equiv 1$, $f \equiv 0$ close to the boundary of W,

- $g \geq 0$ everywhere and vanishes only in $\{|z| = 1, |r| \leq 1, |\mathbf{x}| \leq 1/2\}$,

- $f \equiv 1$ in $\{z \in [-3/2, -1/2], |r| \leq 1, |\mathbf{x}| \leq 1/2\}$.

The same plug can be done using a manifold of the form $\tilde{W} = [-2,2] \times T^{n-2} \times [-2,2]$, and the trapped orbits wind around some components of the torus. The

plug exist also in dimension three, however the invariant set is a circle that creates a new periodic orbit.

It is well known [80] that Wilson's plug can be done volume-preserving, providing volume-preserving counterexamples to the generalized Seifert conjecture in S^{2n+1} for $n \geq 2$. The first construction of this volume-preserving plug is due to G. Kuperberg [124].

Volume-preserving Wilson's plug.   In [170, Section 2.3.1], the explicit construction is done for three dimensions. Omitting details, let us recall the construction and give a explicit coordinate description for the case of any dimension.

Consider the manifold $P = T^{n-2} \times [1, 2] \times [-1, 1]$, endowed with coordinates $(\theta_1, ..., \theta_{n-2}, r, z)$. The first step is constructing a vector field of the form

$$X_P = H_1(r, z) + f(r, z)(\frac{\partial}{\partial \theta_1} + b\frac{\partial}{\partial \theta_2})$$

with b an irrational constant number. Taking suitable functions makes P a volume-preserving semi-plug (meaning that the entry and exit region do not coincide), which traps a set of zero measure isomorphic to T^{n-2}. This is done by taking the vector field $H_1 = h_1(r, z)\frac{\partial}{\partial z} + h_2(r, z)\frac{\partial}{\partial r}$ such that $\iota_{H_1}\mu = dh$ for some volume μ of $[1, 2] \times [-1, 1]$ and function h. Taking a suitable h, the flow lines of H_1 look like in the Figure 2.4, with a single singularity.
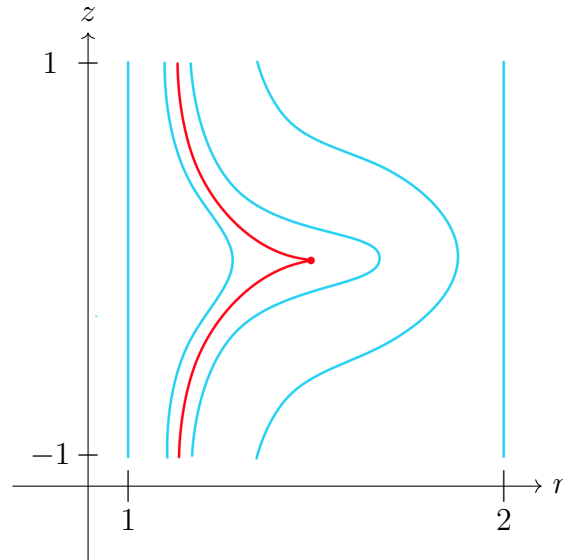

Figure 2.4: Flow lines of H

Choosing the function $f : [1, 2] \times [-1, 1] \to \mathbb{R}^+$ such that it is zero on the boundary and positive at the singularity of H is enough to make P a semi-plug. Taking the mirror-image to exchange the entry and exit regions and rescaling

so that it fits P yields a volume-preserving plug in P. Observe that again in coordinates $(r, z, \theta_1, ..., \theta_{n-2})$ the constructed vector field in the plug (that we denote again X_P) is of the form

$$X_P = \tilde{h}_1(r,z)\frac{\partial}{\partial z} + \tilde{h}(r,z)\frac{\partial}{\partial r} + \tilde{f}(r,z)(\frac{\partial}{\partial \theta_1} + b\frac{\partial}{\partial \theta_2}), \qquad (2.4)$$

and in the boundary we have $X_P|_{\partial P} = \frac{\partial}{\partial z}$. The function $\tilde{h}_1$ is positive everywhere except in the singularity, where it vanishes, but then $\tilde{f}$ is non-vanishing. The preserved volume is $\mu \wedge d\theta_1 \wedge ... \wedge d\theta_{n-2}$.

Obstructions to plugs   Sullivan's characterization of geodesible vector fields [181] was used to prove that a vector field admitting a plug is not geodesible.

Theorem ([170, 164]). *Plugs are not geodesible in any dimension.*

In [164] the result was obtained for the class of Eulerisable fields.

Theorem ([164]). *Plugs are not Eulerisable in any dimension.*

Beltrami fields admitting plugs

We proceed to construct a volume-preserving Beltrami field which admits the Wilson volume-preserving plug, and hence cannot be geodesible nor Eulerisable.

Theorem 2.2.19. *There are volume-preserving Beltrami fields in any manifold of dimension $2n + 1 > 3$ and any homotopy class of non-vanishing vector fields which are not geodesible nor Eulerisable.*

Proof. Consider M any odd dimensional manifold of dimension $2n + 1 \geq 5$.

Applying Theorem 2.2.11, we know it admits a stable Eulerisable structure (α, ν) with geodesible Reeb field X in an arbitrary homotopy class of non-vanishing vector fields. If we denote (B, π) the open book decomposition we used to construct the structure, by construction there are points p outside the trivial neighborhood of B where the vector field and its connection form are $\frac{\partial}{\partial \theta}$ and $d\theta$.

In a small neighborhood $U \cong \mathbb{R}^n$ we can take coordinates $(z, y_1, ..., y_{2n})$ such that $X|_U = \frac{\partial}{\partial z}$ and $\alpha|_U = dz$.

Consider the manifold $P = T^{n-2} \times [1, 2] \times [-1, 1]$ of the previous section with its coordinates $(z, r, \theta_1, ..., \theta_{n-2})$ and vector field of equation (2.4), of the form $X_P = h_1(r,z)\frac{\partial}{\partial z} + h_2(r,z)\frac{\partial}{\partial r} + f(r,z)(\frac{\partial}{\partial \theta_1} + b\frac{\partial}{\partial \theta_2})$, where we have omitted the tildes of the functions to simplify the notation. Take the form

$$\alpha_P = f(z,r)d\theta_1 + h_1(z,r)dz.$$

It satisfies $\alpha_P(X_P) = f^2 + h_1^2 > 0$ at every point since the only points where h_1 vanishes, f does not. In a neighborhood of the boundary of P, the form α_P

coincides with dz. This implies that once P is embedded in the neighborhood U, both the field X_P and the form α_P can be extended as X and α outside the embedded copy of P. By standard arguments (cf. [170, page 78]) one can make sure that the volume preserved by X_P coincides in the boundary of P with $\alpha \wedge \nu$. Denote $\tilde{X}$ and $\tilde{\alpha}$ the vector field and one form of the plug extended as X and α outside of it. If we further denote μ the volume given by extending the volume in the plug as $\alpha \wedge \nu$ outside of it, it is clear that $\tilde{X}$ preserves μ.

Let us check that $\tilde{X}$ is, in addition to volume-preserving, a Beltrami field. Outside the embedded copy of P, we have $\tilde{X} = X$ and $\tilde{\alpha} = \alpha$. Hence we have

$$\iota_X d\alpha = 0, \tag{2.5}$$

which trivially implies $\iota_X (d\alpha)^n = 0$. Inside P, we have that $\tilde{\alpha} = \alpha_P$. By looking the coordinate description of α_P, we have

$$d\alpha_P = \frac{\partial f}{\partial z} dz \wedge d\theta_1 + \frac{\partial f}{\partial r} dr \wedge d\theta_1 + \frac{\partial h_1}{\partial r} dr \wedge dz.$$

Hence $(d\alpha_P)^2 = 0$. Using Lemma 2.2.18 we deduce that $\tilde{X}$ is a volume-preserving Beltrami field for some metric. Crearly in P the vector field cannot be geodesible, since we know plugs are not geodesible. The fact that $(d\alpha_P)^2 = 0$ is not a contradiction with the fact that $\tilde{X}$ is not geodesible. Computing the contraction of $\tilde{X}$ with $d\alpha_P$ we obtain

$$\begin{aligned}
\iota_{\tilde{X}} d\alpha_P = &\left(h_1(r,z)\frac{\partial f}{\partial z} + h_2(r,z)\frac{\partial f}{\partial r}\right)d\theta_1 \\
&+ \left(h_2(r,z)\frac{\partial h_1}{\partial r} - f\frac{\partial f}{\partial z}\right)dz + \left(-f\frac{\partial f}{\partial r} - h_1(r,z)\frac{\partial h_1}{\partial r}\right)dr,
\end{aligned}$$

which is clearly not constantly zero. $\qquad\square$

Remark 2.2.20. One can also use the standard Wilson's plug in Theorem 2.2.19. It is not volume-preserving and so one constructs only an example of a Beltrami field (not volume-preserving) admitting a plug. It is not geodesible, and traps a set of orbits of positive measure (in the boundary of the plug).

Combining it with the obstruction to the existence of plugs, we deduce Theorem 2.2.2. Furthermore, these vector fields cannot be "Reeb-embedded" in the sense of [33] into any contact manifold.

Remark 2.2.21. In fact one can say even more. The vector fields produced by Theorem 2.2.19 cannot be embedded in any other manifold such that X extends to an Eulerisable vector field. Let M be an odd dimensional manifold and X a non geodesible Beltrami volume-preserving vector field. Assume that M is embedded in a manifold N, where there is an Eulerisable vector field Y such that $Y|_M = X$. Since Y is Eulerisable, there is a one form α such that $\iota_Y \alpha > 0$ and $\iota_Y d\alpha$ is exact. If we denote $e : M \to N$ the embedding, we have that the one form $e^*\alpha \in \Omega^1(M)$ satisfies that $e^*\alpha(X) > 0$ and $\iota_X d(e^*\alpha)$ is exact. But then X is Eulerisable in M by Lemma 2.1.6, which is a contradiction.

Other remarks

As additional observations, we present another source of examples of Beltrami fields and a property concerning the relation with geodesibility.

Example 2.2.22. *Let M be an odd dimensional manifold. If $\mathcal{F}$ is a codimension one foliation, M admits a Beltrami field transverse to it. If furthermore the foliation was minimal, the Beltrami field is volume-preserving. Let us just explain the case when $\mathcal{F}$ is minimal. Denote α the defining form of $\mathcal{F}$. By a result of Sullivan [182], there is a $2n$-form ω which is closed and positive in the leaves. The vector field defined by $\iota_X \omega = 0$ and $\alpha(X) = 1$ preserves the volume form $\alpha \wedge \omega$. Observe that since α defines a foliation, we have $\alpha \wedge d\alpha = 0$ implying $(d\alpha)^2 = 0$. By Lemma 2.1.7, one can construct a metric such that X is parallel to its curl and the Riemannian volume is $\alpha \wedge \omega$. These examples are irrotational, since their curl is vanishing. This follows from the fact that $(d\alpha)^2 = 0$.*

Volume-preserving examples that arise from minimal foliations exist in every odd-dimensional manifold as shown in [138]. The following observation was suggested by Daniel Peralta-Salas.

Proposition 2.2.23. *Let X be a Beltrami field in a manifold M of dimension $2n + 1 > 3$. If $\alpha = \iota_X g$ is generic, in the sense that $d\alpha$ is of maximal rank almost everywhere, then X is geodesible.*

Proof. The curl of X satisfies $\iota_Y \mu = (d\alpha)^n$ and since X is Beltrami we know that $Y = fX$ for $f \in C^\infty(M)$. In particular, we deduce that $f\iota_X \mu = (d\alpha)^n$ and f is non-vanishing almost everywhere by the genericity assumption. By contracting this equation with X, it follows that X is in the kernel of $(d\alpha)^n$. Since $(d\alpha)^n$ is of maximal rank almost everywhere, it follows that X is in the kernel of $d\alpha$ almost everywhere. Hence $\iota_X d\alpha$ vanishes almost everywhere and by continuity $\iota_X d\alpha \equiv 0$. By Gluck's characterization, we have that X is geodesible. $\square$

2.2.3 Aperiodic Beltrami fields

The constructed plug in Theorem 2.2.19 cannot be immediately used to prove the existence of Beltrami fields (volume-preserving or not) without periodic orbits in arbitrary manifolds. This is because the plug requires a specific expression of the connection form α in the neighborhood of the point where the plug is inserted. In this section we present the state of art on the existence of periodic orbits and prove that every manifold of dimension $2n + 1 > 3$ admits a Beltrami field (not necessarily volume-preserving) without periodic orbits.

The Weinstein conjecture

The Weinstein conjecture states that any Reeb vector field in a closed contact manifold has at least one periodic orbit. The conjecture is known to be true in dimension three [188], as well as for overtwisted contact structures in any dimension

[2]. Concerning stable Eulerisable structures, it is known to be true in dimension three (where they coincide with stable Hamiltonian structures) in the following form.

Theorem ([113])**.** *Let M be a closed oriented three-manifold with a stable Hamiltonian structure. If M is not a T^2-bundle over S^1, then its Reeb field has a closed orbit.*

A counterexample in the T^2-bundle over S^1 is provided by taking the mapping torus of an irrational rotation in T^2. This counter example generalizes to any dimension for stable Eulerisable structures. Even if we defined stable Eulerisable structures in odd dimensions, since it is the natural set for the study of Beltrami fields, the definition makes sense also in even dimensions.

Claim. *Let N be a closed manifold of dimensions $n \geq 2$ such that $\chi(N) = 0$. Then there is a N-bundle over S^1 endowed with a stable Eulerisable structure such that its Reeb field has no periodic orbits.*

Proof. Following [167] and [196], any manifold such that $\chi(N) = 0$ admits a volume-preserving diffeomorphism $\varphi : N \to N$ without periodic points. Consider the suspension of this diffeomorphism, i.e. the manifold $M = N \times I / \sim$ where we identified $(p, 0)$ with $(\varphi(p), 1)$. If we denote t a coordinate in I, it induces a coordinate θ in M. The vector field $X = \frac{\partial}{\partial \theta}$ has no periodic orbits, and preserves a volume form μ since φ was volume-preserving. It is the Reeb field of the stable Eulerisable structure $(d\theta, \iota_X \mu)$.   □

The fact that geodesible fields do not admit plugs, as well as the Weinstein conjecture for stable Hamiltonian structures, motivates the idea that some version of the Weinstein conjecture could be true for stable Eulerisable structures in high dimensions.

In the non volume-preserving case, it was proved in [171] the following positive result, with the assumption that both the vector field and the metric making its orbits geodesics are real analytic.

Theorem. *Let M be a closed oriented 3-manifold which is not a torus bundle over the circle. Then any real analytic geodesible (or equivalently Beltrami) field has a periodic orbit.*

The smooth case is still open. We will prove in the next subsection that, in the high dimensional setting, there always exist Beltrami fields without periodic orbits.

Aperiodic Beltrami fields using round Morse functions

In [10], Asimov introduced round handle decompositions and proved that every manifold of dimension at least 4 satisfying $\chi(M) = 0$ admits such a decomposition. This concept was later related to the existence of round Morse functions, introduced by Thurston [189].

Definition 2.2.24. *A **round Morse function** on a smooth manifold M is a function $f : M \to \mathbb{R}$ such that:*

- *the set of critical points of f is a disjoint union of circles,*

- *the corank of f in a critical point is* 1.

The existence of such a function is equivalent to the existence of a round handle decomposition, a fact that was rigorously proved by Miyoshi in [150]. Miyoshi obtained a round Morse lemma, where one can have standard Morse coordinates or twisted Morse coordinates. However, it is always possible to find a round Morse function without twisted critical circles, and hence we only state the untwisted case.

Lemma 2.2.25 (Untwisted Round Morse Lemma)**.** *Let $f : M^{n+1} \to \mathbb{R}$ be a round Morse function without twisted singular circles. Then there exist global coordinates $(\theta, x_1, ..., x_n)$ in a neighborhood $U = S^1 \times D^n$ near any critical circle C such that*

$$f = -x_1^2 - ... - x_r^2 + x_{r+1}^2 + ... + x_n^2,$$

where r is the index of the critical circle.

The well known relation between round Morse functions and Morse-Smale flows provides a starting point to construct aperiodic Beltrami flows.

Proof of Theorem 2.2.3. Taking into the account previous discussions, we only need to construct a vector field X satisfying the following three properties:

1. There is a one form α such that $\alpha(X) > 0$ and $\iota_X (d\alpha)^n = 0$,

2. X has a finite number of periodic orbits,

3. for every periodic orbit γ, there is a point $p \in \gamma$ and a coordinate z in a neighborhood U of p such that $X|_U = \frac{\partial}{\partial z}$ and $\alpha|_U = dz$.

If we achieve this, inserting a plug in each neighborhood of the point p of each periodic orbit yields a vector field without periodic orbits and a one form β such that $\beta(X) > 0$ and $\iota_X (d\beta)^n = 0$ and hence by Lemma 2.2.18 the vector field X is a Beltrami field.

Construction of a vector field satisfying (1)-(3).

Take a round Morse function f without twisted components and a metric which is "nice" in the neighborhood of the finite amount of critical circles: i.e. it looks like the standard metric in $S^1 \times D^{2n}$ with the round Morse Lemma coordinates. Then the gradient defined by f is the vector field satisfying $g(X, \cdot) = df$, and is a vector field without periodic orbits but with the set of fixed points being the critical circles of f. We will do a modification of this vector field around the

critical circles to obtain a vector field with a finite amount of periodic orbits and construct a one form satisfying (1) and (3). Consider one of the critical circles γ of f, let us assume that it is a maximum, since everything works analogously on each critical circle.

Step 1: around the orbit.

In the neighborhood $U = S^1 \times D^{2n}$ with coordinates $(\theta, x_1, ..., x_{2n})$, we assume that the metric is the standard $g = d\theta^2 + \sum_{i=1}^{2n} dx_i^2$. Hence the gradient of f has the following expression.

$$\text{grad}(f) = \sum_{i=1}^{2n} x_i \frac{\partial}{\partial x_i},$$

where $f = \sum_{i=1}^{2n} x_i^2$. Take the function $\rho = \sum_{i=1}^{2n} x_i^2$ (independently of the index of the critical circle of f), and $\varphi(\rho)$ is a bump function which is constantly equal to 1 around $\rho = 0$ and 0 around $\rho = 1$. We can now modify the gradient of f, taking instead

$$X = \varphi(\rho)\frac{\partial}{\partial \theta} + \text{grad}(f),$$

which has a single periodic orbit at $S^1 \times \{0\}$. Construct the one form

$$\alpha_\gamma = \varphi(\rho)d\theta + df,$$

which satisfies $\alpha_\gamma(X) > 0$, $\alpha_\gamma|_{\partial U} = df$. Computing its exterior derivative we have

$$d\alpha_\gamma = \varphi'(\rho)d\rho \wedge d\theta,$$

which satisfies $(d\alpha_\gamma)^2 = 0$. In a small neighborhood V of the orbit where $\varphi(\rho) \equiv 1$ we have $\alpha_\gamma = d\theta + df$. The form α_γ extends outside of U as df. Denote α the one-form which is df outside the neighborhoods of the critical circles and the constructed α_γ on them. Doing this at every orbit, we obtain a vector field X with a finite number of periodic orbits and a one form α satisfying $\alpha(X) > 0$ and $(d\alpha)^2 = 0$. Only condition (3) is left to check. Figure 2.5 depicts schematically the modification for a critical circle with arbitrary index.
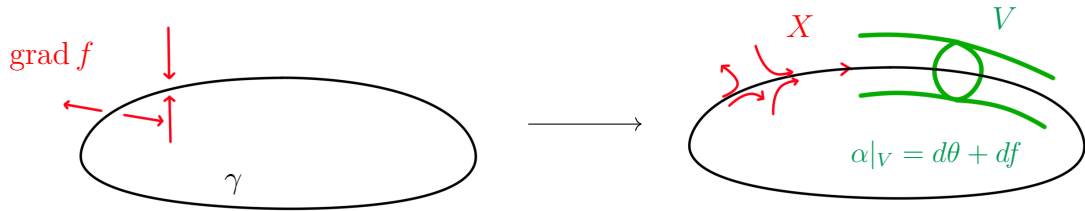

Figure 2.5: Modification around circle

Step 2: around a point.

As mentioned in the previous step, in the neighborhood V of the orbit we can now assume $X = \frac{\partial}{\partial \theta} + \text{grad} f$ and $\alpha = d\theta + df$. Around a point p in the orbit

$S^1 \times D^{2n}$, the S^1-coordinate θ defines a function z. Hence there are coordinates $(z, x_1, ..., x_{2n})$ on a small neighborhood U such that $X|_U = \frac{\partial}{\partial z} + \sum_{i=1}^{2n} x_i \frac{\partial}{\partial x_i}$ and $\alpha|_U = dz + df$. By taking the neighborhood small enough, we can assume that there is a function h such that $X = \frac{\partial}{\partial h}$ by the flow box theorem. Denote $r = z^2 + x_1^2 + ... + x_{2n}^2$. Take $\varphi(r)$ a bump function which is 1 around 0 and vanishes around 1. Construct the one form

$$\beta = \varphi(r)dh + (1 - \varphi(r))[dz + df].$$

Contracting it with X we have that

$$\beta(X) = \varphi(r)dh(X) + (1 - \varphi(r))[dz + df](X)$$
$$= \varphi(r)dh(\frac{\partial}{\partial h}) + (1 - \varphi(r))dz(\frac{\partial}{\partial z} + \operatorname{grad} f) + (1 - \varphi(r))df(\frac{\partial}{\partial z} + \operatorname{grad} f)$$
$$= \varphi(r) + (1 - \varphi(r))[1 + df(\operatorname{grad} f)].$$

Since $df(\operatorname{grad} f)$ is positive except at $r = 0$, we deduce that $\beta(X) > 0$. Furthermore, we have $d\beta = \varphi' dr \wedge dh - \varphi' dr \wedge dz - \varphi' dr \wedge df$ which implies $(d\beta)^2 = 0$ and coincides with $dz + df$ on $\{r = 1\}$. Denote again α the form β extended as α outside the neighborhood U. Hence in a neighborhood of $r = 0$ where $\varphi(r) \equiv 1$ we have that $X|_U = \frac{\partial}{\partial h}$ and $\beta|_U = dh$. The orbit through $r = 0$ is the isolated periodic orbit. Figure 2.6 depicts schematically the modification for a critical circle with arbitrary index.

$$\alpha|_U = dh$$
$$X|_U = \partial_h$$
$$U$$

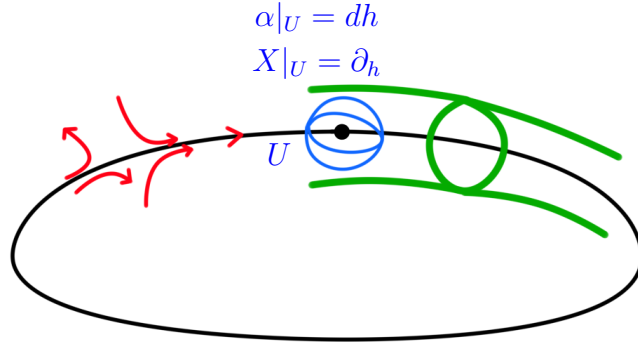Figure 2.6: Modification around point

In particular, condition (3) is satisfied for the closed orbit γ.

Doing this at every critical circle, we prove that the pair (X, α) satisfies the conditions (1)-(3), which proves the theorem. □

Remark 2.2.26. Note that the constructed aperiodic Beltrami fields are furthermore irrotational. Since $(d\beta)^n \equiv 0$, their curl vanishes everywhere.

2.2.4   The periodic orbit conjecture for Eulerisable fields

We end up the discussion of higher dimensional Eulerisable fields by adressing a question [1] related to foliations by compact leaves. The content of this subsection is contained in [26] (see also [27] for an updated version). This classical question asks about the existence of an upper bound on the volume of the leaves of a compact foliation on a compact manifold. For one-dimensional foliations, this was known as the periodic orbit conjecture.

Periodic orbit conjecture: Let X be a vector field in a manifold M such that every orbit of X is closed. Then there is an upper bound on the lengths of the orbits of X.

This conjecture was proved in dimension three by Epstein [63]. In higher dimensions, however, Sullivan constructed a beautiful counterexample on a five-dimensional compact manifold [180]. A counterexample in the sharpest case of dimension four was settled by Epstein and Vogt [64] a couple of years later. A theorem by Wadsley [192] shows that a necessary and sufficient geometric condition for the conjecture to hold is that the vector field X is geodesible, i.e. there is some metric making its orbits geodesics. In this section we show that even if we allow X to be Eulerisable (or to admit a strongly adapted one-form), which is a larger class of vector fields, the conjecture is still satisfied.

Theorem 2.2.27. *Eulerisable fields (or more generally flows with a strongly adapted one-form) on closed manifolds satisfy the periodic orbit conjecture.*

Currents and Eulerisable fields

Let M be a closed smooth manifold and consider $\Omega^k(M)$ the space of differential k-forms on M. This space is endowed with a natural C^∞-topology. A k-current is a continuous linear function over $\Omega^k(M)$. The space of k-currents is denoted by $\mathcal{Z}^k(M)$. A concise introduction to currents is contained in [84], we will give here only some basic background.

Currents are equipped with a "boundary" operator $\partial : \mathcal{Z}^k(M) \to \mathcal{Z}^{k-1}(M)$ defined by

$$\partial c(\omega) = c(d\omega),$$

where c is a k-current, ω is a k-form and d denotes the usual exterior derivative. A current c which has no boundary is called a "cycle". A classical theorem by Schwartz establishes that the dual space to $\mathcal{Z}^k(M)$ is $\Omega^k(M)$.

Example 2.2.28. *An example of a k-current is given by integration along a k-chain, i.e. $c(\omega) = \int_c \omega$ where c is a k-chain. This example will be of our interest as it is used for the characterization of geodesible and Eulerisable vector fields.*

---

[1] I am grateful to Daniel Peralta-Salas for proposing this question when I was in Madrid for the Workshop on Geometric Methods in Symplectic Topology in December 2019.

Let us now fix some non-vanishing vector field X in M. To each point $p \in M$, there is an associated Dirac 1-current

$$\delta_p : \Omega^1(M) \longrightarrow \mathbb{R}$$
$$\alpha \longmapsto \alpha_p(X_p).$$

The closed cone in $\mathcal{Z}^1(M)$ generated by Dirac 1-currents is called the space of foliation currents. A property of such currents is that for a one-form α such that $\alpha(X) > 0$, any foliation current z satisfies $z(\alpha) > 0$. We denote the space of foliation cycles by $\mathcal{C}_X$. The theory of currents and cycles was used to study foliations in a broad sense: for one dimensional foliations they can be used to characterize geodesible vector fields, as introduced in Section 2.1.1.

Definition 2.2.29. *A vector field X is **geodesible** if there exist a Riemannian metric such that the orbits of X are geodesics.*

Recall that a geometric characterization of geodesible fields, due to Gluck, states that a vector field is geodesible if and only if there is a one-form α such that $\alpha(X) > 0$ and $\iota_X d\alpha = 0$. The topological characterization by Sullivan [181] is given in terms of currents and cycles.

Theorem 2.2.30. *A non-vanishing vector field is geodesible if and only if there is no sequence of tangent two chains whose boundary approximates a foliation cycle.*

We introduced earlier in this thesis a wider class of non-vanishing vector fields, which was defined and characterized using this very same language of currents in [164].

Definition 2.2.31. *Let M be manifold with a volume form μ. A volume-preserving vector field X is **Eulerisable** if there is a metric g on M for which X satisfies the Euler equations for some Bernoulli function $B : M \to \mathbb{R}$.*

Non-vanishing Eulerisable fields have both a geometric and a topological characterization. Consider the following linear subspace of 1-currents:

$$\mathcal{F}_{d\alpha} = \{\partial c \mid c \text{ is a 2-chain s.t. } \int_c d\alpha = 0\}.$$

We state below a partial statement of Theorem 5.2 in [164].

Theorem 2.2.32 ([164])**.** *Let X be a non-vanishing volume-preserving vector field on a closed manifold of dimension $n \geq 3$. Then X is Eulerisable if and only if there is some α so that $\alpha(X) > 0$ and $\iota_X d\alpha$ is exact. Furthermore, if X is Eulerisable with one-form α, no sequence of elements in $\mathcal{F}_{d\alpha}$ can approximate a non-trivial foliation cycle of X (i.e. $\overline{\mathcal{F}_{d\alpha}} \cap \mathcal{C}_X = \{0\}$).*

Remark 2.2.33. As introduced in [184], a non-vanishing vector field X admits a strongly adapted one-form α if $\alpha(X) > 0$ and $\iota_X d\alpha$ is exact. The previous theorem applies to this class of vector fields simply by dropping the condition of volume preservation.

Sullivan-Thurston's example

We first introduce a counterexample to the periodic orbit conjecture given by Thurston and show that it is volume-preserving (and even Beltrami for some metric). We follow [157] to give an explicit description of Thurston's analytic counterexample to the periodic orbit conjecture. This example was inspired by Sullivan's paper [180]. We will see how X is volume-preserving and even a Beltrami field for some metric. However, as proved in [24], not every volume-preserving Beltrami field is Eulerisable.

Let H be the Heisenberg group with parameters $(x, y, z) \in \mathbb{R}^3$:

$$H = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, (x, y, z) \in \mathbb{R}^3.$$

We can consider the action of the following lattice on it (via left matrix multiplication).

$$\Lambda = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, (a, b, c) \in \mathbb{Z}^3$$

We denote by π the projection from $H \times \mathbb{R}^2$, which is equipped with coordinates (x, y, z, t, u), to $M = H/\Lambda \times S^1 \times S^1$. The vector fields $V_1 = \cos t \partial_x + \sin t (\partial_y + x \partial_z)$ and $V_2 = -\sin t \partial_x + \cos t (\partial_y + x \partial_z)$ form together with $\partial_z, \partial_t, \partial_u$ a global set of independent vector fields.

Thurston's example is given by the following vector field in the quotient space M.

$$X = \sin(2u) V_1 + 2 \sin^2 u \partial_t - \cos^2 u \partial_z$$
$$= \sin(2u) \cos t \partial_x + \sin(2u) \sin t (\partial_y + x \partial_z) + 2 \sin^2 u \partial_t - \cos^2 u \partial_z$$

On $U = \{u \neq 0 \mod \pi\}$, the orbits of $W = \frac{1}{2\sin^2 u} X$ are all closed of period 2π. This can be checked by direct computation of the flow of W, which is 2π-periodic. In particular, as u approaches 0 or π, the orbits of X have arbitrarily large periods. Furthermore, X extends as $\frac{\partial}{\partial z}$ along $u = 0 \mod 2\pi$ and as $-\frac{\partial}{\partial z}$ along $u = \pi \mod 2\pi$. Since there is an element in Γ acting by translation along z, we deduce that the orbits of X at U are also closed. The vector field X preserves the volume-form $\mu = dx \wedge dy \wedge dz \wedge dt \wedge du$, that descends to M, since

$$d\iota_X \mu = d \Big[\sin(2u) \cos t \, dy \wedge dz \wedge dt \wedge du - \sin(2u) \sin t \, dx \wedge dz \wedge dt \wedge du$$
$$+ (x \sin t \sin(2u) - \cos^2 u) dx \wedge dy \wedge dt \wedge du - 2 \sin^2 u \, dx \wedge dy \wedge dz \wedge du \Big]$$
$$= 0.$$

This shows that this counterexample is volume-preserving. In fact it is even Beltrami (in the sense of Section 2.2) for some Riemannian metric. The one-form

$\beta = \frac{1}{2}dt - dz + xdy$ is well defined in the quotient and satisfies

$$\begin{cases} \beta(X) = \sin^2 u + \cos^2 u = 1 \\ (d\beta)^2 = 0. \end{cases}$$

It is now standard to construct a metric (cf. the proof of Corollary 2.2.39) such that the vector field X has a vanishing curl and preserves the Riemannian volume: it is an irrotational volume-preserving Beltrami field.

Proof of the main theorem

We proceed with the proof of Theorem 3.2.7. The strategy is to find a suitable family of two chains whose boundary approximates a foliation cycle and make a few technical modifications to the argument used in [164] (inspired by the geodesible case [170]) to prove that plugs are not Eulerisable.

Let M be a compact manifold (that we can assume to be of dimension at least four), and let X be a non-vanishing vector field all whose orbits are closed. These orbits define a one-dimensional foliation by compact leaves. We denote by B_1 the set of points where the length of orbits is not locally bounded. This set is referred to as the "bad set" following Epstein [63]. It is a compact nowhere dense subset of M, and we assume that $B_1 \neq \emptyset$. This is obviously satisfied by any counterexample to the periodic orbit conjecture. Even if the length depends on a fixed Riemannian metric, the fact that it has an upper bound does not depend on the choice of metric.

Assume further that X is Eulerisable (or admits a strongly-adapted one-form). Then there is some one-form α such that

$$\begin{cases} \alpha(X) > 0 \\ \iota_X d\alpha = -dB \end{cases} \tag{2.6}$$

for some function $B \in C^\infty(M)$. We can suppose that B is not constant (even though we do not need it in the proof), since otherwise X would be geodesible by Gluck's characterization. We know that this cannot be the case due to Wadsley's theorem.

Foliations all whose leaves are compact were studied in depth by Edwards-Millet-Sullivan. We state here a version of the "moving leaf proposition", confer [54, Section 5].

Proposition 2.2.34 (Moving leaf proposition). *Assume B_1 is compact and non empty (this is satisfied when M is compact). Then there is a embedded family of leaves with trivial holonomy L_t, $t \in [0,\infty)$ such that L_t approaches B_1 when $t \to \infty$ and* length$(L_t) \to \infty$.

Given a sequence L_i of leaves such that length$(L_i) \to \infty$, Section 3 in [54] shows how to find a subsequence (that we still denote L_i) and a sequence of natural number n_i such that $\langle \frac{1}{n_i} L_i, \cdot \rangle$ converges to a positive foliation current $\eta : \Omega^1(M) \to \mathbb{R}$ supported in B_1. This current is in fact a cycle, as explained at the end of Section 2 in [54]. Since the length of L_i tends to ∞, the integers n_i satisfy $n_i \to \infty$.

We will now choose the initial sequence L_i of leaves from the moving leaf L_t, i.e. each L_i corresponds to L_{t_i} for some $t_i \in [0, \infty)$. We want the sequence L_{t_i} to satisfy the property that for each i

$$\text{length}(L_{t_i}) \geq \text{length}(L_t) \text{ for all } t \in [0, t_i]. \tag{2.7}$$

This can be done for the following reason. By Proposition 2.2.34, L_t is a locally compact invariant subset of M with trivial holonomy. Then by Proposition 4.1 in [54], the length function is continuous along L_t. We can now choose each t_i so that the above condition is satisfied: simply consider the intervals $t \in [0, i]$, and pick the value t_i for which the length of L_{t_i} is a maximum in $[0, i]$. Such maximum always exists by the extreme value theorem.

From the discussion above, we can find a subsequence of leaves, which we still denote by L_{t_i} to simplify the notation, and some positive integers n_i so that $\lim \langle \frac{1}{n_i} L_{t_i}, \cdot \rangle$ converges to a positive foliation cycle η with support in B_1.

Construct the sequence of tangent two chains

$$T_i = \{L_t \mid t \in [0, t_i]\},$$

which are increasingly larger parts of the moving leaf family. The sequence of tangent two chains $A_i = \frac{1}{n_i} T_i$ satisfies

$$\lim_{i \to \infty} \partial A_i = \lim_{i \to \infty} \frac{1}{n_i} (L_{t_i} - L_0) = \eta,$$

since the initial leaf L_0 has finite length and the n_i go to infinity.

Remark 2.2.35. As suggested by Gluck [83], a sequence of two chains such as the $(A_i)_{i \in \mathbb{N}}$ readily contradicts Sullivan's characterization (Theorem 2.2.30). This shows that X is not geodesible and gives an alternative argument to prove Wadsley's theorem. In our discussion, we carefully chose the family A_i with the additional property (2.7) which will be used to show that X is not Eulerisable.

Each T_i is just an embedded family of closed curves, hence diffeomorphic to a cylinder. Denoting $\lambda = \frac{\alpha}{\alpha(X)}$, the exterior derivative of α decomposes as $d\alpha = -\lambda \wedge dB + \omega$ for some two form ω such that $\iota_X \omega = 0$. Observe that the function B is constant along each curve L_t: this follows from the fact that $\iota_X d\alpha = -dB$, which implies that $\iota_X dB = 0$. In particular, we can write $B(t)$ when restricting

B to the subset $L_t \subset M$. The integral $A_i(d\alpha)$ can be computed as

$$
\begin{aligned}
A_i(d\alpha) &= \frac{1}{n_i} \int_{T_i} d\alpha \\
&= \frac{1}{n_i} \int_{T_i} \omega + \frac{1}{n_i} \int_{T_i} -\frac{1}{\alpha(X)} \alpha \wedge dB \\
&= \frac{1}{n_i} \int_{T_i} -\frac{1}{\alpha(X)} \alpha \wedge dB,
\end{aligned}
$$

where we used that $\iota_X \omega = 0$. We want to prove that there is some subsequence A_{i_r} of A_i such that $A_{i_r}(d\alpha) \to 0$. This would lead to a contradiction because $A_{i_r}(d\alpha) = \partial A_{i_r}(\alpha) \to \eta(\alpha)$, but η is a foliation cycle and α is positive on X implying that $\eta(\alpha) > 0$.

We will use the motonicity property (2.7) and the lemmas below to prove the existence of the subsequence A_{i_r}.

Lemma 2.2.36. *For a given arbitrary $\varepsilon > 0$, there is some $t_l \in [0, \infty)$ and a sequence $(t_{i_k})_{k \in \mathbb{N}} \subset (t_l, \infty)$ such that $i_k \to \infty$ and $|B(t_{i_k}) - B(t_l)| < \varepsilon$ for all k. Furthermore, we can choose the sequence $(t_{i_k})_{k \in \mathbb{N}}$ such that for all $k \in \mathbb{N}$ we have*

$$
\frac{\operatorname{length}(L_{t_l})}{n_{i_k}} < \varepsilon.
$$

Proof. Fix some $\varepsilon > 0$. The function B is a smooth function on the compact manifold M and hence is bounded. In particular there is some constant K such that $|B(t)| < K$ for all $t \in [0, \infty)$. The sequence of numbers $B(t_i)$ is a bounded sequence, so there is a subsequence which is convergent to $y \in \mathbb{R}$. By continuity of B, $B(t) \to y$ and we can find a $t_l \in (t_i)_{i \in \mathbb{N}}$ such that $|B(t_l) - y| < \frac{\varepsilon}{2}$. Similarly, we can find a subsequence $t_{i_k} \to \infty$ such that $|B(t_{i_k}) - y| < \frac{\varepsilon}{2}$ for each k. In particular $|B(t_{i_k}) - B(t_l)| \le |B(t_{i_k}) - y| + |y - B(t_l)| < \varepsilon$ as we wanted.

Recall that the sequence n_i satisfies that $n_i \to \infty$. Since t_l is fixed for the given ε, taking k greater than some k_0 we can ensure that

$$
\frac{\operatorname{length}(L_{t_l})}{n_{i_k}} < \varepsilon.
$$

The value of ε was arbitrary, so this concludes the proof of the lemma. $\qquad \square$

Lemma 2.2.37. *There exist positive constants C_1 and C_2 such that*

- $\frac{1}{\min_M |X|} |B(t) - B(0)| < C_1$ *for any $t \in [0, \infty)$,*

- $\frac{1}{\min_M |X|} \frac{\operatorname{length}(L_{t_i})}{n_i} < C_2$ *for every i big enough.*

Proof. The vector field X is non-vanishing, so the value of $\frac{1}{\min_M |X|}$ is a well defined positive number. Let K be an upper bound of $|B|$ over M. Then

$$\frac{1}{\min_M |X|}|B(t) - B(0)| < C_1 = \frac{1}{\min_M |X|}2K,$$

for any value of t.

For the second inequality, we simply use the fact that $\lim \langle \frac{1}{n_i}L_{t_i}, \cdot \rangle$ tends to a smooth positive foliation cycle. This implies that $\frac{\text{length}(L_{t_i})}{n_i}$ is a bounded sequence, let K_2 be some upper bound. Then $C_2 = \frac{1}{\min_M |X|}K_2$ satisfies the second inequality.
□

Given an arbitrary ε, construct a sequence t_{i_k} using Lemma 2.2.36. The cylinder $T_{i_k} = \{L_t \mid t \in [0, t_{i_k}]\}$, can be parametrized by $s \in [0, t_{i_k}]$ and θ some parameter such that $X = \partial_\theta$. We can now explicitly write the integral $A_{i_k}(d\alpha)$.

$$\begin{aligned}
|A_{i_k}(d\alpha)| &= \frac{1}{n_{i_k}}\left| \int_0^{t_l} \int_{L_t} -\frac{1}{\alpha(X)}\alpha \wedge dB + \int_{t_l}^{t_{i_k}} \int_{L_t} -\frac{1}{\alpha(X)}\alpha \wedge dB \right| \\
&\leq \frac{1}{n_{i_k}}\left| \int_0^{t_l} \int_{L_t} -\frac{1}{\alpha(X)}\alpha \wedge dB \right| + \frac{1}{n_{i_k}}\left| \int_{t_l}^{t_{i_k}} \int_{L_t} -\frac{1}{\alpha(X)}\alpha \wedge dB \right| \\
&= \frac{1}{n_{i_k}}\left| \int_0^{t_l} \int_{L_t} -\frac{\alpha(\partial_\theta)}{\alpha(X)}dB(\partial_s)d\theta ds \right| + \frac{1}{n_{i_k}}\left| \int_{t_l}^{t_{i_k}} \int_{L_t} -\frac{\alpha(\partial_\theta)}{\alpha(X)}dB(\partial_s)d\theta ds \right| \\
&\leq \frac{\text{length}(L_{t_l})}{n_{i_k}\min_M |X|}|B(t_l) - B(0)| + \frac{\text{length}(L_{t_{i_k}})}{n_{i_k}\min_M |X|}|B(t_k) - B(t_l)| \\
&< C_1 \varepsilon + C_2 \varepsilon.
\end{aligned}$$

We used the triangle inequality, the fact that $dB(\partial_\theta) = 0$, Lemmas 2.2.36 and 2.2.37 and equation (2.7) in this computation. The initial ε was arbitrary while C_1, C_2 are fixed and do not depend on ε.

Take a sequence of positive numbers $\varepsilon_r \to 0$. For each ε_r, we can use Lemma 2.2.36 to construct a subsequence r_k satisfying $A_{r_k}(d\alpha) < C_1 \varepsilon_r + C_2 \varepsilon_r$ for all $k \in \mathbb{N}$. Using a diagonal argument, we choose for each r an element $i_r \in (r_k)_{k \in \mathbb{N}}$ such that $i_r \to \infty$ as $r \to \infty$. The resulting subsequence i_r satisfies that $A_{i_r}(d\alpha) < C_1 \varepsilon_r + C_2 \varepsilon_r$, since each i_r is in (r_k). We deduce that $A_{i_r}(d\alpha) \to 0$ when $r \to \infty$, leading to a contradiction with the fact that $A_{i_r}(d\alpha) = \partial A_{i_r}(\alpha) \to \eta(\alpha) > 0$ when $r \to \infty$. This proves Theorem 3.2.7.

Remark 2.2.38. As done in [164, Theorem 4.6], it is possible to construct a sequence of zero-flux 2-chains $\tilde{A}_i \in \mathcal{F}_{d\alpha}$ such that $\partial \tilde{A}_i \to \eta$. This given an explicit contradiction with Theorem .

There is a simple corollary of Theorem 3.2.7 that follows from Wadsley's theorem.

Corollary 2.2.39. *Let X be a non-vanishing steady solution to the Euler equations all whose orbits are closed. Then there is some other metric g for which X is a Beltrami type steady solution to the Euler equations.*

Proof. Denote by μ the volume form which is preserved by X. Applying Wadsley's theorem, X is geodesible: there is one form such that $\alpha(X) > 0$ and $\iota_X d\alpha = 0$. We apply Lemma 2.1.7 to construct a metric g such that $g(X, \cdot) = \alpha$ and μ is the induced Riemannian volume. Then X is a steady solution to the Euler equations for the metric g with constant Bernoulli function, i.e. it is a Beltrami type solution. □

2.3 Steady ideal fluids with a Bott Bernoulli function

In this section, we follow [25] to study steady Euler flows which are integrable by means of a Morse-Bott function. In the analysis of stationary fluids, Arnold's celebrated structure theorem, introduced in Section 2.1.1, provides an almost complete description of the rigid behavior of the flow if the Bernoulli function, which depends on the pressure, is non-constant and analytic (or C^2 Morse-Bott). Except on an analytic stratified subset of positive codimension, the manifold is trivially fibered by invariant tori where the flow is conjugate to a linear field. However, Arnold's theorem is an *a posteriori* conclusion: it gives the structure of such solutions but says nothing about their existence. When the Bernoulli function is constant and Arnold's theorem does not apply, we know that the solutions are Beltrami fields: vector fields parallel to their curl. The existence of non-vanishing Beltrami fields for some metric has been extensively studied. For instance, it was proved in [67] that solutions of these type exist in every homotopy class of vector fields (each solution for some particular metric) of any three-manifold. In the previous Section, we extended this result in Theorem 2.2.1 to every odd dimensional manifold in [24].

Some of the few examples of flows that satisfy the hypotheses of the structure theorem are in the round sphere and the flat torus [119]. A natural question is to ask in which other manifolds such solutions exist. Motivated by this problem, it is left as an open question in [162] to study which manifolds admit Euler flows of this type for some metric. We will adress it in two different contexts: when the steady flow is non-vanishing, and when it can vanish but the Bernoulli function is of Morse-Bott type.

With the extra assumption that the vector field X is non-vanishing, it was shown in [43] that it forces the manifold to be of a certain topological type: it has to be the union of Seifert manifolds glued along their torus boundaries, or equivalently a graph manifold. A more general case was observed before in [66], and which follows from previous works [71]: a non-singular flow with a stratified integral can only exist in a graph manifold. We will refer to a non-vanishing

solution to the Euler equations for some metric with non-constant analytic (or Morse-Bott) Bernoulli function as an *Arnold fluid*.

In the first part of this note, we show that every possible three-manifold, which we know to be of graph type, admits an Arnold fluid with analytic Bernoulli function.

Theorem 2.3.1. *Any closed, oriented graph three-manifold M admits a non-vanishing steady solution to the Euler equations for some metric and non-constant analytic Bernoulli function.*

By the discussion above, the statement is in fact an if and only if. We prove this using the standard construction of Seifert manifolds. However, adapting the arguments in the proof of the theorem above, one can use another decomposition of graph manifolds developed by Fomenko et alli to obtain a richer source of examples of Arnold fluids. In particular, one can produce an Arnold fluid realizing as Bernoulli function, up to diffeomorphism, any possible Morse-Bott function. In the language developped in [13], the topological characterization of a graph three-manifold and the foliation by level sets of a Bott integral is given by a molecule with gluing matrices. We will show that any such configuration with or without critical Klein bottles can be realized by an Arnold fluid. A concrete way to reformulate the result is the following.

Theorem 2.3.2. *Given an admissible Morse-Bott function B in a graph manifold M, and any volume form μ, there exist a steady solution to the Euler equations for some metric g such that the Bernoulli function is B (modulo diffeomorphism of M) and the Riemannian volume μ.*

By admissible we mean a Morse-Bott function which can topologically be the integral of a fluid. Observe that the volume form can also be prescribed. This theorem shows that when we allow the metric to vary, the apparent difficulty to construct fluids with such a rigid behavior is overcomed, and one can realize flows in all allowed Morse-Bott topological configurations. An immediate corollary, explained in the Appendix, is that the invariants developed for Bott integrable systems lead to a topological classification of Arnold fluids with Morse-Bott Bernoulli function. This classification can be compared with the classification of vorticity functions of Morse type studied in [115] for the 2D Euler equation in surfaces.

The ressemblance between Arnold's structure theorem and the Arnold-Liouville theorem in the theory of integrable systems already suggest that there might be some connection between these two worlds. By taking a careful look at the Arnold fluids constructed in Theorem 2.3.2, we can relate them to Bott integrable systems by means of the symplectization of an appropiate stable Hamiltonian structure. This yields an alternative proof that any molecule with gluing matrices can be realized by an integrable system, with the additional property that the isoenergy hypersurface is of stable Hamiltonian type. Without this extra property, this was originally proved in [14].

Theorem 2.3.3. *The constructed steady Euler flows together with their vorticity in Theorem 2.3.2 can be realized as the isoenergy hypersurface of a Hamiltonian system with a Bott integral (the Bernoulli function) in a symplectic manifold with boundary. Additionally, the Hamiltonian vector field is up to rescaling a Reeb field of a stable Hamiltonian structure.*

This materializes the intuition that Arnold's structure theorem and the classical Arnold-Liouville theorem are closely related. However, the construction is *ad hoc* and we can find examples of Arnold fluids that cannot reasonably be though as energy levels set of an integrable system.

In the last part of this work, we will drop the assumption that the solution is non-vanishing. In this general case, no obstruction on the topology of M is known. Hence, we analyze the topology of Euler flows with Morse-Bott Bernoulli function that can have stagnation points. We show that the Bernoulli function cannot have a non-degenerate critical point, and use it to prove that the topology of the ambient manifold can prevent the existence of Bott integrable fluids.

Theorem 2.3.4. *Let M be a three-manifold that is not of graph type. Then M does not admit a solution to the Euler equations (for any metric) with a Morse-Bott Bernoulli function.*

For example, the class of hyperbolic manifolds satisfies the hypotheses of this theorem. This is the first example of a topological obstruction to the existence of an integrable fluid, and answers the question raised in [162] in the Morse-Bott case.

2.3.1 About graph manifolds

In this section, we introduce the basic background about Seifert manifolds, required to prove Theorem 2.3.1. We will eventually introduce other preliminaries when required through the discussion.

The ambient topology of a non-vanishing integrable fluid

Let us recall the steady Euler equations in terms of the dual one form $\alpha = g(X, \cdot)$ and the so called Bernoulli function.

$$\begin{cases} \iota_X d\alpha = -dB \\ d\iota_X \mu = 0 \end{cases},$$

where the Bernoulli function is $B = p + \frac{1}{2}g(X, X)$ and μ is the induced Riemannian volume. A key property of this function is that we already mentioned is that it is an integral of the field X. We aim to study vector fields that are solutions to such equations for some metric, i.e. Eulerisable fields as introduced in Section 2.1.

In [43] the authors study the case of **non-vanishing** solutions in the hypotheses of Arnold's structure theorem (cf. Theorem 2.1.1). It is proved that for a non-constant analytic Bernoulli function B, one can always find some other metric in M such that X is a solution to the Euler equations with constant Bernoulli function. In particular, it is the Reeb field of some stable Hamiltonian structure. As a corollary of the methods in the proof, topological obstructions to the existence of solutions with non-constant analytic Bernoulli function are obtained. The manifold has to be a union of Seifert manifolds glued along their torus boundary components, i.e. a graph manifold. This result was obtained in [66] for a more general context: it holds for the existence of a non-vanishing vector field with a stratified integral. It follows from a previous work in [71].

Theorem 2.3.5 ([43, Corollary 2.10], [66, Theorem 5.1]). *If a three-manifold admits steady solution to the Euler equations for some metric and non-constant analytic (or in general stratified) Bernoulli function, then the manifold is of graph type.*

To simplify the notation, we might refer to a vector field which is a non-vanishing steady solution to the Euler equations for some metric and non-constant analytic Bernoulli function as an **Arnold fluid**. This is reminiscent of the nomenclature introduced in [119], where one speaks of a divergence free vector field which is *Arnold integrable* when it is almost everywhere fibered by invariant tori.

Seifert and graph manifolds

Let us recall the definition and construction of Seifert manifolds, introduced and classified by Seifert [175].

Definition 2.3.6. *A **Seifert fiber space** is a three-manifold together with a decomposition as a disjoint union of circles.*

Equivalently, a Seifert fiber space admits a circle bundle over a two dimensional orbifold. Denote by $\pi : M \to B$ the bundle map over the compact base B. When the manifold M is oriented, Seifert fiberings are classified up to bundle equivalence by the following invariants (up to some operations that yield isomorphic fiberings):

$$M = \{g; (\alpha_1, \beta_1), ..., (\alpha_m, \beta_m)\}.$$

The only thing we will need in this work is the way to reconstruct the manifold M when given a collection of invariants. The integer g denotes the genus of the base space B, and (α_i, β_i) are pairs of relatively prime positive integers $0 < \beta_i < \alpha_i$. The integer g can be negative, and then B is connected sum of g copies of $\mathbb{RP}^2$. The integer m represents the amount of orbifold singularities. Given such a collection of invariants, there is a precise construction to obtain the manifold M that we describe following [116].
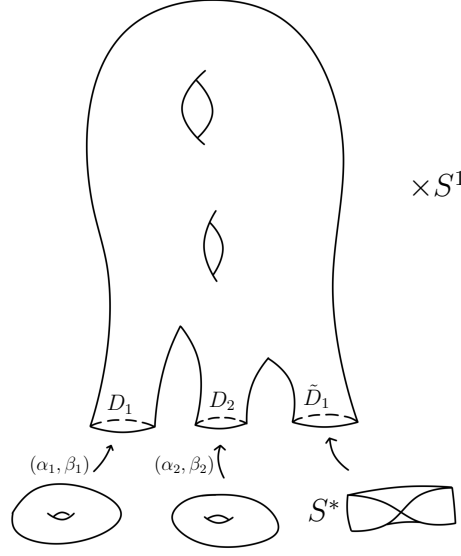
Figure 2.7: Example for $(\Sigma_2 \# \mathbb{RP}^2; (\alpha_1, \beta_1), (\alpha_2, \beta_2))$

The base space B is either an orientable surface Σ_g of genus g, or decomposes as $\Sigma_{g'} \# \mathbb{RP}^2$ or $\Sigma_{g'} \# \mathbb{RP}^2 \# \mathbb{RP}^2$ for some g'. To simplify, denote in any case by Σ_g the orientable surface with genus g or g' respectively. Remove m open disks of Σ_g, that we denote by $D_i, i = 1, ..., m$. If B has some non-orientable part, one or two additional open disks $\tilde{D}_1, \tilde{D}_2$ are removed depending on the amount of $\mathbb{RP}^2$ components that have to be added to Σ_g to obtain B. We will denote by Σ_0 the resulting surface with boundary.

Over Σ_0, we take the trivial S^1 bundle and denote it by $M_0 = \Sigma_0 \times S^1$. At each component of the boundary, which is of the form $\partial D_i \times S^1$, we glue a solid torus $V_i = \bar{D}^2 \times S^1$ by means of a Dehn surgery with coefficients (α_i, β_i).

If B was orientable, this concludes the construction: we obtain a closed three-manifold. If B was non orientable, we denote by M_1 the resulting three-manifold with boundary and we need to fill one or two holes $\partial \tilde{D}_1 \times S^1$ and $\partial \tilde{D}_2 \times S^1$ if there was respectively one or two copies of $\mathbb{RP}^2$ attached to the orientable part of B. To do so, consider the only orientable S^1-bundle over the Möebius band: namely, the twisted I-bundle over the Klein bottle fibered meridianally. Denote two copies of such a space as M_2 and M_3. The torus boundary of both M_2 and M_3 can be framed longitudinally by a fiber and meridianally by a section to the bundle. The boundary components $\partial \tilde{D}_1 \times S^1$ and $\partial \tilde{D}_2 \times S^1$ in M_1 can be framed also longitudinally by a fiber and meridianally by a section to the bundle. Then glue M_2 and M_3 respectively to the boundary components of M_1 according to the identity between first homology groups represented by the given framings. This concludes the construction of M in the most general case. We depict an example in Figure 2.7.

For a Seifert manifold with boundary, we take the base surface B with some

boundary circles. The boundary of M is then a collection of tori.

Definition 2.3.7. *A **graph manifold** is a manifold obtained by gluing Seifert spaces with boundary along their torus boundary components.*

These manifolds were introduced and classified by Waldhausen [193, 194]. Observe that this is a larger class of manifolds, since the gluing of the boundary tori might not match the fibering in each piece. Hence the resulting total space might not admit a foliation by circles.

2.3.2 Arnold fluids in Seifert and graph manifolds

In this section we will prove Theorem 2.3.1. To do so, we first show that any Seifert manifold M admits an Arnold fluid. In view of Lemma 2.1.7, we only need to prove that there is vector field X in M and a one form α such that

- X is volume preserving,

- $\alpha(X) > 0$,

- $\iota_X d\alpha = -dB$ for some analytical function B.

The strategy of the proof is to break the manifold M into pieces according to the construction detailed above, construct a vector field and a one form satisfying some conditions in each piece and glue them together in an appropriate way. Concretely, we will show that in a neighborhood of the gluing locus, one can interpolate between the Arnold fluids in each piece to obtain a globally defined steady Euler flow. Once we have a globally defined pair (X, α), we show it is a steady Euler flow for some analytic Bernoulli function and hence an Arnold fluid. Finally, we adapt the interpolation to glue Seifert manifolds along their boundary to deduce the general case of graph manifolds.

Building pieces

Let M be a three dimensional manifold which is a Seifert fibered space. Hence M is an S^1-fibration over an orientable surface Σ. To cover the most general case, we shall assume that the base space decomposes as $\Sigma_g \# \mathbb{RP}^2 \# \mathbb{RP}^2$. As in Subsection 2.3.1, Σ_0 is an orientable surface of genus g and with some removed open disks $D_i, i = 1, ..., m$ and $\tilde{D}_i, i = 1, 2$. It is a surface with boundary $M_0 = \Sigma_0 \times S^1$. We think of Σ_0 as being embedded in $\mathbb{R}^3$ so that there is a naturally defined height h function, whose minimum is equal to 1 and is reached at the boundary components of Σ_0. This height function lifts trivially to M_0. See Figure 2.8 for an example with genus 2, two singular fibers and one $\mathbb{RP}^2$ component.

In M_0 denote by θ the coordinate in the S^1 component. Consider the vector field $X = \frac{\partial}{\partial \theta}$, and as connection form $\alpha = h d\theta$. We have $\alpha(X) > 0$ and $\iota_X d\alpha = -dh$.
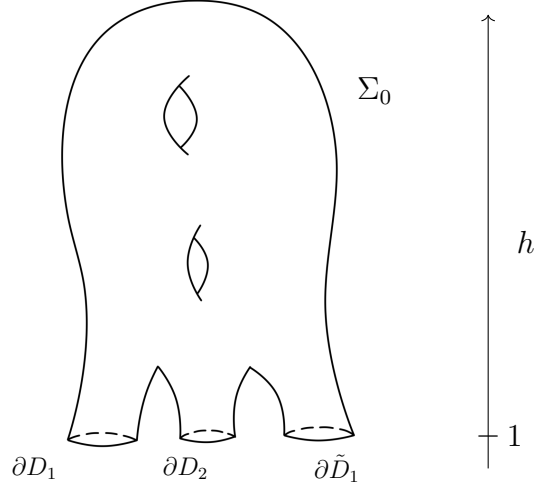
Figure 2.8: Height function in Σ_0

For each solid torus $V_i \cong \bar{D}^2 \times S^1$, we take coordinates (r, φ', θ'). Consider in V_i the vector field $Y = \frac{\partial}{\partial \theta'}$ and as one form $\beta = v(r)d\theta'$, where $v' > 0$ in $(0, 1]$, close to $r = 1$ the function is r and close to $r = 0$ the function is $\varepsilon + r^2$. We have $\beta(Y) > 0$ and $\iota_Y d\beta = -v'(r)dr$. Hence taking as Bernoulli function $B = \int v'(r)dr$ such that $B(0) = 0$, we have that $\iota_Y d\beta = -dB$.

The remaining blocks M_1 and M_2 are twisted I-bundles over a Klein bottle. This space $M_1 \cong M_2 \cong K^2 \tilde{\times} I$ can be seen as the mapping torus of

$$\phi : S^1 \times [-1, 1] \longrightarrow S^1 \times [-1, 1]$$
$$(\theta, r) \longmapsto (-\theta, -r).$$

Hence we have $K^2 \tilde{\times} I = \frac{(S^1 \times [-1,1]) \times [0,1]}{((p,0) \sim (\phi(p),1))}$. Such manifold is foliated by tori parallel to the boundary, together with a Klein bottle at the core. If we denote by r the coordinate in $[-1, 1]$, the function r^2 is well defined in the mapping torus total space. Furthermore, we have a natural defined vector field $Z = \frac{\partial}{\partial \theta'}$, where θ' is in $[0, 1]$, induced by the flow of the mapping torus and that is 2-periodic. Construct a function $v(r^2)$ such that close to $r = 0$ it is $\varepsilon + r^2$ and close to $r = 1$ it is r^2. By taking $\gamma = v(r)d\theta'$, we have $\gamma(Z) > 0$ and $\iota_Y d\gamma = -dG$ where $G = \int v'(r)dr$ with $G(0) = 0$.

Interpolation Lemma and gluing

In order to reconstruct the whole manifold, we need first to paste the sets $V_i = \overline{D_i} \times S^1$ to the boundary components $\partial D_i \times S^1$ of M_0 by means of a Dehn twist. Denote $U_i \cong [1, 2) \times S^1 \times S^1$ the boundary components at D_i of M_0 and fix one of them that we denote by U with coordinates (t, φ, θ). We can assume the first

component t corresponds to the restriction of h to U, and the coordinate θ is on the S^1 fibre over Σ_0. Denote by $\lambda = \{1\} \times \{\varphi\} \times S^1$ and by $\mu = \{1\} \times S^1 \times \{\theta\}$ the longitude and the meridian of the boundary component $\{1\} \times T^2$ of U. We will glue a solid torus $V = D^2 \times S^1$ to the boundary of U. Again take $\lambda_1 = \{p \in \partial \bar{D}^2\} \times S^1$ and $\mu_1 = \partial \bar{D}^2 \times \{\theta'\}$ the longitude and the meridian of V. The Dehn surgery is described by gluing in a way that

$$\varphi : \partial V_i \longrightarrow \partial U_i \times S^1$$
$$\mu_1 \longrightarrow p\mu + q\lambda$$
$$\lambda_1 \longrightarrow m\mu + n\lambda$$

Coordinate wise we have that $\varphi' = p\varphi + q\theta$ and $\theta' = m\varphi + n\theta$. We can assume that the surgery is such that the radial coordinate r in V is sent to t in the neighborhood of the boundaries. The vector field $Y = \frac{\partial}{\partial \theta'}$ which generates the longitude, is sent to $Y|_{\partial U} = \frac{q}{qm - pn} \frac{\partial}{\partial \varphi} + \frac{p}{pn - qm} \frac{\partial}{\partial \theta}$ in the surgered target coordinates, the boundary of U. This can be easily deduced by the fact the coordinate wise we have

$$\begin{cases} \varphi & = \frac{n}{pn - qm} \varphi' + \frac{q}{qm - pn} \theta' \\ \theta & = \frac{m}{qm - pn} \varphi' + \frac{p}{pn - qm} \theta' \end{cases}.$$

Analogously, the one form β is sent to $\beta|_{\partial U} = t(md\varphi + nd\theta)$. This follows from the fact that near the boundary of B, the one form is $\beta = rd\theta$. It satisfies $\beta(Y) = t(\frac{qm}{qm - pn} + \frac{pn}{pn - qm}) > 0$.

Once we have our building blocks of an Euler flow, we need to glue the flows in a smooth way: both the vector field and the one form, and such that the critical set of the Bernoulli function is controlled. More precisely, we can do this interpolation by keeping the Bernoulli function regular.

Lemma 2.3.8 (Interpolation Lemma). *Suppose we are given a vector field $Y = A\frac{\partial}{\partial \theta} + B\frac{\partial}{\partial \varphi}$ in the torus T^2 and a one form $\gamma = Cd\theta + Dd\varphi$, for some constants A, B, C, D, such that $\gamma(Y) = AC + DB = 1 > 0$. Denote t the coordinate in $[1, 2]$. Then there is a volume perserving vector field X and a one form α in $T^2 \times [1, 2]$ such that*

- $X|_{\{t=1\}} = \frac{\partial}{\partial \theta}$, $\alpha|_{\{t=1\}} = d\theta$,

- $X|_{\{t=2\}} = Y$, $\alpha|_{\{t=2\}} = \gamma$,

- $\alpha(X) > 0$ *everywhere,*

- $\iota_X d(t\alpha) = -dh$ *where $h(t)$ is a function without critical points and equal to t at the boundary.*

Proof. Break the interval $[1, 2]$ into seven disjoint intervals $I_1, ..., I_7$, for instance $I_i = [1 + \frac{i-1}{7}, 1 + \frac{i}{7}]$. Denote by $H_i(t)$ a cutoff function with support in I_i such that $H_i' \geq 0$, with $H_i = 0$ at $1 + \frac{i-1}{7}$ and $H_i = 1$ at $1 + \frac{i}{7}$.

Since we have $AC + DB > 0$, we might assume that A and C are of the same sign. Otherwise, the constants D and B are of the same sign and an analogous interpolation is done.

1. In the first interval, take $X = \frac{\partial}{\partial\theta}$ and $\alpha = d\theta + H_1(t)d\varphi$. We have $\alpha(X) = 1 > 0$.

2. In the second interval, take $X = \frac{\partial}{\partial\theta} + H_2(t)\frac{\partial}{\partial\varphi}$ and $\alpha = d\theta + d\varphi$. We have $\alpha(X) = 1 + H_2 > 0$.

3. In the third interval, take $X = (1 - H_3(t))\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\varphi}$ and $\alpha = d\theta + d\varphi$. We have $\alpha(X) = (1 - H_3) + 1 > 0$.

4. In the fourth interval, take $X = \frac{\partial}{\partial\varphi}$ and $\alpha = (1 + H_4(t)(C - 1))d\theta + d\varphi$. We have $\alpha(X) = 1 > 0$.

5. In the fifth interval, take $X = H_5(t)A\frac{\partial}{\partial\theta} + (1 - H_5(t))\frac{\partial}{\partial\varphi}$ and $\alpha = Cd\theta + d\varphi$. We have $\alpha(X) = ACH_5 + (1 - H_5) > 0$ since $AC > 0$.

6. In the sixth interval, take $X = A\frac{\partial}{\partial\theta}$ and $\alpha = Cd\theta + (1 + H_6(t)(D - 1))d\varphi$. We have $\alpha(X) = AC > 0$.

7. In the last interval, take $X = A\frac{\partial}{\partial\theta} + H_7(t)B\frac{\partial}{\partial\varphi}$ and $\alpha = Cd\theta + Dd\varphi$. We have $\alpha(X) = AC + H_7BD > 0$. It is positive because if BD is positive, then $AC + H_7BD > AC > 0$ and if BD is negative then $AC + H_7BD > AC + BD > 0$.

This a priori arbitrary way of interpolating by steps is done to achieve both that h has no critical points and that it suits an application in the last part of this section. The key point in achieving that h has no critical points is to avoid decreasing a component in α which is non-vanishing when evaluated at X.

We constructed (X, α) satisfying the first three conditions. To check the fourth condition, we will do it interval by interval. If we compute $\iota_X d(t\alpha)$ we obtain:

$$\iota_X d(t\alpha) = \begin{cases} -dt & t \in I_1 \\ -[1 + H_2]dt & t \in I_2 \\ -[2 - H_3]dt & t \in I_3 \\ -dt & t \in I_4 \\ -[H_5AC + (1 - H_5)]dt & t \in I_5 \\ -ACdt & t \in I_6 \\ -[AC + H_7BD]dt & t \in I_7 \end{cases}$$

It is clear from the case by case description that the function h, which is the indefinite integral with respect to t of the dt-term, has no critical points in $[1, 2]$. At the boundary we have that $\iota_X d(t\alpha) = -dt$.
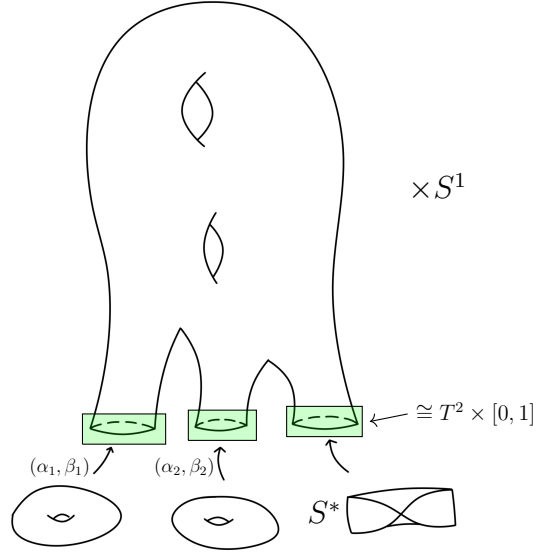
Figure 2.9: Interpolation lemma is applied in the green areas

The vector field preserves the volume form $\mu = dt \wedge d\varphi \wedge d\theta$, since at any point the vector field is of the form $X = A(t)\frac{\partial}{\partial\theta} + B(t)\frac{\partial}{\partial\varphi}$ which implies that $\mathcal{L}_X \mu = d\iota_X \mu = 0$. $\qquad\square$

We can now apply this lemma in $U = [1,2] \times T^2$ to interpolate between (Y, β) and (X, α). The interpolation areas are depicted in Figure 2.9 for the previous example. By Lemma 2.3.8, we can do so in such a way that the function h extends in the building blocks as B and has no critical points in U. Hence we constructed a pair $(\tilde{X}, \tilde{\alpha})$ in M_1 with $\tilde{\alpha}(\tilde{X}) > 0$ and $\iota_{\tilde{X}} d\tilde{\alpha} = -dB'$ for some function $B' \in C^\infty(M)$.

Finally, it only remains to glue the M_1 and M_2 to $W_1 = \partial\tilde{D}_1 \times S^1$ and $W_2 = \partial\tilde{D}_2 \times S^1$. Take for example M_1, and M_2 is glued analogously. The vector field Z is tangent to the leaves of the foliation (by torus and core Klein bottle) in M_1, and on the torus leaves Z is linear and periodic. Hence at the surgered boundary, the vector field Z and the one form γ are sent to $Z = C_1\frac{\partial}{\partial\theta} + C_2\frac{\partial}{\partial\varphi}$ and to $t\gamma = t(D_1 d\theta + D_2\varphi)$ for some constants C_1, C_2, D_1, D_2 with $\gamma(Z) = C_1 D_1 + C_2 D_2 = 1$.

We are under the hypotheses of Lemma 2.3.8 in a neighborhood of the surgery locus, so we can obtain a vector field and one form (X', α') in W_1 that can be extended in M_0 as $(\tilde{X}, \tilde{\alpha})$ and in M_1 as (Z, γ). Doing this to both pieces M_1 and M_2, we constructed a globally defined function $B \in C^\infty(M)$ such that $\alpha'(X') > 0$, the equation $\iota_{X'}\alpha' = -dB$ satisfied and B coincides with $\tilde{B}$ in $M_1 \setminus U_1 \cup U_2$ and with G in M_1 and M_2.

Volume preservation and critical set

To prove that X' is volume preserving, we will prove that it preserves some volume in each part of the manifold.

1. Denote by A a neighborhoods of the boundary circles of M_0 where we applied the interpolation lemma. Then in $M_0 \setminus A$ we have that $X' = X = \frac{\partial}{\partial \theta}$ and X clearly preserves $\mu = \mu_{\Sigma_0} \wedge d\theta$, where μ_{Σ_0} is any area form of Σ_0. This follows from the fact that $\iota_X \mu = \mu_{\Sigma_0}$, which is a closed form.

2. In any interpolation area, it is by construction volume preserving. The vector field is of the form $X' = A(t)\frac{\partial}{\partial \theta} + B(t)\frac{\partial}{\partial \varphi}$. Taking as volume $\mu_U = dt \wedge d\varphi \wedge d\theta$, we have that
$$\iota_{\tilde{X}} \mu_U = A(t)dt \wedge d\varphi - B(t)dt \wedge d\theta,$$
which is again closed.

3. In every solid torus V_i, we have $Y = \frac{\partial}{\partial \theta'}$ which preserves the volume $rdr \wedge d\varphi' \wedge d\theta'$.

4. In the mapping tori M_1 and M_2, the vector field $X' = \frac{\partial}{\partial \theta'}$ preserves the volume form $\mu = \mu_S \wedge d\theta'$, where $\mu_S = d\varphi \wedge dt$ is a volume of $S^1 \times [-1, 1]$. This follows from the fact that diffeomorphism φ that we used for the mapping torus preserves the area form $d\varphi \wedge dt$

It is now standard to construct a globally defined volume μ on M preserved by X'. We just proved that X' is volume preserving and admits a one form α' such that $\alpha'(X') > 0$ and $\iota_{X'}\alpha' = -dB$ for some function $B \in C^\infty(M)$. By applying Lemma 2.1.7, we deduce that X' satisfies the Euler equations for some metric with Bernoulli function B.

Analiticity The function B is a priori only smooth. However, we have the following theorem of equivalence between smooth and analytic functions. We state a particular case which is enough for our purposes.

Theorem 2.3.9 ([177, Theorem 7.1])**.** *Let f be a smooth function on a manifold M. Suppose that at every point we have locally that:*

1. *f is regular,*

2. *f is the sum of a constant and a power of a regular function,*

3. *f is $\pm x_1^2 \pm ... \pm x_k^2 + \text{const}$ for suitable coordinates $(x_1, ..., x_n)$.*

Then f is equivalent to an analytic function.

We only need to prove that B satisfies the conditions of this theorem, then B is analytic taking some suitable charts. In $M_0 \setminus A$, we have that $B = h$. We initially defined h in Σ_0 where it is a Morse function, hence h is a Morse-Bott function in the considered three dimensional space. In the solid tori V_i attached via Dehn surgery, we have that $B = r^2$ and the only singularity is of Morse-Bott type (the critical core circle). In the pieces M_1 and M_2, the Bernoulli function has a Klein bottle as singular set, since at the core of the mapping torus we have $B = r^2$. The singularity is again of Morse-Bott type. Finally, on the regions U_i and W_i, we know that B is regular.

Hence, the only singular points of B admit an expression of type (2) or (3). We just proved that every Seifert manifold with orientable base admits a steady Euler flow with a Morse-Bott Bernoulli function, and by Theorem 2.3.9 equivalent to a non-constant analytic Bernoulli function.

Theorem 2.3.10. *Every Seifert manifold admits a non-vanishing steady solution to the Euler equations (for some metric) with non-constant analytic Bernoulli function. Analogously for a smooth Morse-Bott function.*

To prove the general case of graph manifolds, it only remains to glue Seifert manifolds with boundary and obtain globally defined Arnold fluids.

Proof of Theorem 2.3.1

Let M_1, M_2 be two Seifert manifolds with boundary. We shall assume that there is only a single torus component in the boundary. The manifold M_1 is glued to M_2 by a diffeomorphism

$$\varphi : \partial M_1 \longrightarrow \partial M_2,$$

between both torus boundaries. By Theorem 2.3.10, we can construct steady Euler flows with non-constant analytic Bernoulli function in both manifold M_1, M_2. The idea is to introduce a singular torus to the Bernoulli function and then apply the interpolation lemma. It follows from the construction in the previous section that we can assume that in the neighborhood of the boundaries $\partial M_i = U_i = T^2 \times [1, 2]$ we have that $X_i = \frac{\partial}{\partial \theta_i}$ and $\alpha = t_i d\theta_i$ where $(\theta_i, \varphi_i, t_i)$ are coordinates in U_i. Once we glue the boundaries, we can construct a coordinate t in $U = T^2 \times [-1, 1]$ such that

$$t^2 = t_1 \text{ for } t \in [-1, -1 + \varepsilon] \tag{2.8}$$

$$t^2 = t_2 \text{ for } t \in [1 - \varepsilon, 1] \tag{2.9}$$

We can assume that $T^2 \times \{-1\}$ is the boundary of M_1 and $T^2 \times \{+1\}$ is the boundary of M_2, and we thickened the gluing torus. Consider the coordinates (θ_2, φ_2, t) of U, obtained by extending θ_2, φ_2 from the boundary of M_2 to U.

The gluing diffeomorphism, which is in the mapping class of a two torus, can be assumed to be a Dehn twist (up to isotopy). This implies that the vector field X_1 and the one form α_1, which are defined $\{-1\} \times T^2$ are of the form $X_1 = C_1 \frac{\partial}{\partial \theta_2} + C_2 \frac{\partial}{\partial \varphi_2}$ and $\alpha_1 = t^2 (D_1 d\theta_2 + D_2 d\varphi_2)$ for some constants C_1, C_2, D_1, D_2. In the other hand, we have $X_2 = \frac{\partial}{\partial \theta_2}$ and $\alpha_2 = t^2 d\theta_2$ defined in $T^2 \times \{1\}$. Using Lemma 2.3.8 in $[-1, -1+\varepsilon]$, we obtain a vector field X' and a one form β satisfying $\iota_{X'}\beta = -dH$ for some function H without critical points that extends as t^2 in the boundary of $[-1, -1+\varepsilon]$. Furthermore, in $\{-1\} \times T^2$, the pair is equal to (X_1, α_1) and in $\{-1+\varepsilon\}$ is equal to $(\frac{\partial}{\partial \theta}, t^2 d\theta)$. Finally, the vector field $\frac{\partial}{\partial \theta}$ and the one form $t^2 d\theta$ extend trivially to $[-1+\varepsilon, 1]$ and hence to all $M_1 \cup_\varphi M_2$ by conditions (2.8) and (2.9).

We obtained a global volume preserving vector field X' and a one form β such that $\beta(X) > 0$ and $\iota_X \beta = -dB$. The only new critical level set of the Bernoulli function is given by $\{0\} \times T^2$, a non degenerate critical torus, since there $\alpha = t^2 d\theta$.

This concludes the proof of the Theorem 2.3.1, since the real analiticity of B follows from Theorem 2.3.9.

2.3.3 Fomenko's theory for Arnold fluids

In this section, we use the theory of Bott integrable systems studied by Fomenko et alli to realize Arnold fluids with any possible Morse-Bott Bernoulli function. The connection between these steady flows and the theory of Morse-Bott integrable systems was already observed in [66]. We will show that any topological configuration (in the sense of a graph manifold and a given admissible Morse-Bott function) can be realized by an Arnold fluid. To simplify the discussion, we will treat in this section the case where the Morse-Bott integral contains a single connected critical submanifold in each connected component of the critical level set. In the language of atoms introduced in [18], this means that we assume that the Bott integral has only simple atoms. We leave for the Appendix the case of arbitrary 3-atoms, where we also discuss the topological classification of the moduli of Arnold fluids with Morse-Bott Bernoulli function. Taking into acccount this appendix, we get a proof of Theorem 2.3.2.

Topology of Bott integrable systems

We will describe in this subsection some aspects of the topological classification of integrable systems with Bott integrals in isoenergy surfaces of dimension three. For a more detailed introduction to integrable systems in symplectic manifolds, confer Section 4.1.1 in Chapter 3. We ommit it here since the theory is used only instrumentally in this part of the thesis, but the interested reader may simply read that section and come back to this discussion. For more details on the theory of Bott integrable systems in four dimensional symplectic manifolds, we confer the reader to [13] which we will mainly follow below.

Consider a symplectic manifold (M, ω) of dimension four with an integrable Hamiltonian system $F = (H, f)$. That is, they satisfy $dH \wedge df \neq 0$ almost everywhere and they commute with respect to the Poisson bracket $\{f, H\} = 0$. Denote by Q a three dimensional regular isoenergy hypersurface of H, and assume that f restricts to Q as a Morse-Bott function.

Let us denote by (H) the class of orientable closed three-manifolds that are isoenergy hypersurfaces of some integrable Hamiltonian system with the properties described above, in some four dimensional symplectic manifold with boundary. Similarly, we denote by (G) the class of orientable graph manifolds and (Q) the class of three-manifolds that can be decomposed into the sum of "elementary bricks" which are solid tori $D^2 \times S^1$, a torus times an interval $T^2 \times I$ or $N^2 \times S^1$. Here N^2 denotes a disk with two holes. In a series of papers [18, 70, 73, 74], it was proved that all three classes coincide.

Theorem 2.3.11 (Brailov-Fomenko, Fomenko, Fomenko-Zieschang). *The three classes coincide, i.e. we have $(H) = (Q) = (G)$.*

Even better, with the assumption we took on the critical level set of f, up to five types of blocks describe the topology of the foliation induced by the Bott integral. These five blocks are:

- Type I: The solid torus $S^1 \times D^2$.

- Type II: The thick torus $T^2 \times [1, 2]$.

- Type III: The space $N^2 \times S^1$, where N^2 is a 2-dimensional disk with two holes.

- Type IV: The mapping torus of N^2, with a rotation of angle π that we will denote $N^2 \tilde{\times} S^1$.

- Type V: The mapping torus of $S^1 \times [-1, 1]$ by the diffeomorphism $\varphi(\theta, t) = (-\theta, -t)$.

Denote by f the Bott integral and consider the following integers: m stable periodic orbits (minimum or maximum), p critical tori (minimum or maximum), q the number of unstable critical circles with orientable separatrix diagram, s the number of unstable critical circles with non-orientable separatrix diagram, and r the number of critical Klein bottles (minimum or maximum). Then the manifold M can be represented as $M = mI + pII + qIII + sIV + rV$, gluing the elementary blocks by certain diffeomorphisms of the torus boundary components.

If we further indicate how the blocks are connected by means of edges (if we want, oriented with respect to the increase direction of the function), we obtain a complete topological description of the level sets of the Bott integral.

The topology of the function is then determined by some graph, as similarly discussed in [40]. The graph satisfies that each vertex has one, two or three edges.

That is because blocks I and V have one torus in the boundary, blocks II and IV have two boundary tori and blocks III has three boundary tori. In order to fix the topology of the ambient manifold, one needs to specify the mapping class of each gluing diffeomorphism: the coefficient of a Dehn twist. As described in [13, Section 4.1], there is a family of natural choices of framings (all equivalent) in each boundary torus, and hence the coefficient of a Dehn twist determines the gluing isotopy class. Fixing a graph with Dehn coefficients in each edge determines both the topology of the manifold and the topology of the function.

The whole theory applies also to Arnold fluids with a Morse-Bott Bernoulli function in the following sense. The topological properties originally follow from the analysis of the neighborhood of the critical set of the Morse-Bott integral of some non-vanishing vector field, and hence works analogously in this context. Another way to formalize this connection is the following lemma.

Lemma 2.3.12. *Let B denote the Bernoulli function of a non-vanishing steady Euler flow X on a Riemannian three-manifold (M, g). Then there is a symplectic form ω in $M \times [-\varepsilon, \varepsilon]$ such that X is the Hamiltonian vector field of the coordinate t in $(-\varepsilon, \varepsilon)$. In particular, if $dB \neq 0$ on a dense of M, we obtain that (t, B) define an integrable system in $(M \times [-\varepsilon, \varepsilon], \omega)$.*

Proof. By assumption, the vector field X satisfies

$$\begin{cases} \iota_X d\alpha = -dB \\ d\iota_X \mu = 0 \end{cases}$$

where $\alpha = g(X, \cdot)$ and μ is the Riemannian volume form. Denote by t the coodinate in the second component of $M \times (-\varepsilon, \varepsilon)$, where ε will be taken small enough. Consider the one-form $\beta = \frac{\alpha}{\alpha(X)}$ and construct the two-form

$$\omega = d(t\beta) + \iota_X \mu.$$

It is closed since $d\omega = 0 + d\iota_X \mu = 0$. On the other hand

$$\omega^2 = dt \wedge \beta \wedge \iota_X \mu + t d\beta \wedge \mu.$$

But $dt \wedge \beta \wedge \iota_X \mu$ is a volume form, which implies that for t small enough ω is non degenerate and hence a symplectic form in $M \times [-\varepsilon, \varepsilon]$. The vector field X (trivially extended to $M \times [-\varepsilon, \varepsilon]$) satisfies $\iota_X \omega = -dt$ and so is the Hamiltonian vector field of t. This shows that X on $M \times \{0\}$ is the restriction of a Hamiltonian vector field to a regular energy level set. Furthermore, since $\iota_X dB = 0$ by Lemma 1.1.1, we deduce that if $dB \neq 0$ on a dense set of M we have that $dt \wedge dB \neq 0$ on a dense set of $M \times [-\varepsilon, \varepsilon]$ and (t, B) define an integrable system. $\square$

In particular, if B is Morse-Bott, we deduce that it is the integral of the Hamiltonian vector field X. The pair (M, B) is topologically classified by the theory of Bott integrable systems.

Example 2.3.13. *Take for example the Arnold fluid constructed in Theorem 2.3.1 for Figure 2.8. Asssume we take a height function that only has a critical point in each value: assume for example that the critical value joining ∂D_2 and $\partial \tilde{D}_1$ is lower than the critical value joining ∂D_1 and ∂D_2. A representation of the graph associated to such topological decomposition would be Figure 2.10. We took a framing in the boundary of the Klein bottle neighborhood for which the gluing is trivial as described in Section 2.3.1. Whenever the Dehn coefficients are trivial in some gluing, nothing is indicated in the edge. The coefficients (α_i, β_i) are indicated by the Seifert invariants.*
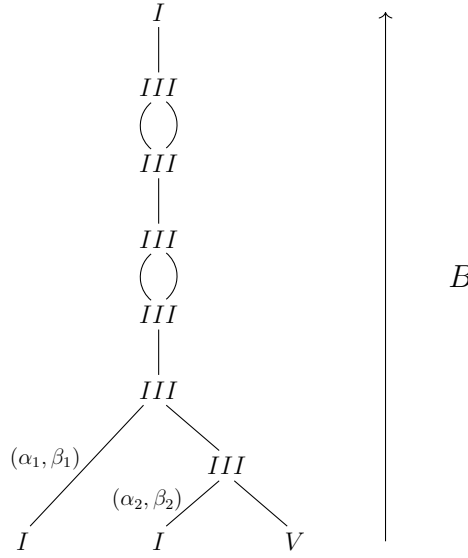
Figure 2.10: Example of graph representation

Topological realization of Arnold fluids

Using similar arguments as we did to prove Theorem 2.3.1, we can construct an Arnold fluid realizing each topogical configuration. Let us start by constructing an Arnold fluid in each of the "elementary blocks".

Proposition 2.3.14. *All blocks admit an Arnold fluid with the following properties. For type I, the longitudinal core circle is a minimum or maximum of the Bernoulli function. For type II, the torus $T^2 \times \{3/2\}$ is a minimum or maximum of the Bernoulli function. For type III, critical set is a figure eight times a circle: the central circle is of saddle type. For type IV, exactly as for type III but with a non orientable separatrix diagram for the critical circle. For type V, the core Klein bottle is a minimum or maximum of the Bernoulli function. In all cases, the boundary components are regular level sets of the Bernoulli function. For blocks III and IV, we can assume that the Bernoulli function decreases (or increases)*

outwards in the exterior of N^2 boundary component and respectively increases (or decreases) in the other boundary components.

Proof. We give construct in each block a vector field X and a one form α such that $\iota_X d\alpha = -dh$ for some function h satisfying the mentioned properties. In all cases, it is easy to check that the vector field is volume preserving as in Subsection 2.3.2.

Types I and V follow from the discussion in Section 2.3.2. In the first one, the vector field is the longitudinal flow $\frac{\partial}{\partial\theta}$ with one form $v(r)d\theta$, where θ is the longitudinal coordinate of the solid torus and r the radial coordinate in D^2. The function $v(r)$ is equal to $(\varepsilon + r^2)$ close to $r = 0$ and equal to r close to the boundary $\{r = 1\}$ if it is a minimum. If it a maximum we can take for example $v(r) = 1 + \varepsilon - r^2$ close to $r = 0$ and $v(r) = 1 + \varepsilon - r$ close to $r = 1$.

Type V is given by the mapping torus with core Klein bottle introduced in Subsection 2.3.2. The vector field is given by the mapping torus direction $\frac{\partial}{\partial\theta}$, and again the one form is $v(r^2)d\theta$, where the function v is equal to $\varepsilon + r^2$ close to $r = 0$ and r^2 close to $r = 1$. Similarly for a maximum, take $1 + \varepsilon - r^2$ close to $r = 0$ and $1 + \varepsilon - r^2$ close to $r = 1$. Recall that r is the coordinate in $[-1, 1]$, for the mapping torus obtained by the diffeomorphism $\varphi : S^1 \times [-1, 1] \to S^1 \times [-1, 1]$.

For the type II block, consider the standard coordinates (θ, φ, t) in $T^2 \times [1, 2]$. Take the vector field $X = \frac{\partial}{\partial\theta}$ and the one form $\alpha = v(t)d\theta$. If we want a minimum, we choose as function $v(t) = +\varepsilon + (t - 3/2)^2)$, which implies that the Bernoulli function is $h = \int_t v'(t) = t^2 - 3t$. If we want it as a maximum then $v(t) = 1 + \varepsilon - (t - 3/2)^2$ and $h = -t^2 + 3t$. We have $\iota_X d\alpha = -dh$ and $\alpha(X) > 0$.

For the type III and type IV, denote by θ the coordinate in the S^1 component. Take a one form $\alpha = v(x, y)d\theta$, where (x, y) are coordinates in N^2. Taking the function v such that it has a saddle point in $(0, 0)$ and two minima or maxima in $(\pm 1, 0)$ is enough. For example, take $v(x, y) = K \pm \frac{1}{4}(y^2 - x^2 + 1/2x^4)$, for some big enough constant $K > 0$ added to the Hamiltonian of the Duffing equation. Then as Bernoulli function take similarly $h(x, y) = C \pm \frac{1}{4}(y^2 - x^2 + 1/2x^4)$, for some constant C. We have

$$\iota_X d\alpha = \iota_X\left(\frac{\partial v}{\partial x}dx \wedge d\theta + \frac{\partial v}{\partial y}dy \wedge d\theta\right) = -\frac{\partial v}{\partial x}dx - \frac{\partial v}{\partial y}dy,$$

which is exactly equal to $-dh$. It is also satisfied that $\alpha(X) > 0$. Depending on the sign, we obtain that h is decreasing or increasing (outwards with respect to the boundary, that we take to be a level set of h) in the interior boundary components, and respectively increasing or decreasing in the exterior boundary components. Observe that the defined one form and function h are well defined in the mapping torus in the case of type IV blocks. This is because the Duffing potential is invariant with respect to the rotation of angle π, which is easily seen in polar coordinates. Hence if we denote by p the projection of $N^2\tilde{\times}S^1$ into S^1, the function p^*h is well defined. This covers the case of block IV.
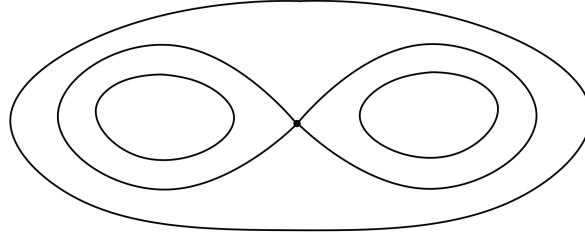
Figure 2.11: Level sets of h

The N^2 copy that we take is the one given by the level sets of the function h or v: i.e. the boundary and holes we take are given by some of the regular level sets of these functions. Figure 2.11 gives a representation of the critical level set and the boundary level sets given by the function.

In all cases, the vector field is volume preserving and Lemma 2.1.7 concludes.

□

Combining this result with the interpolation lemma, we can realize any configuration graph as Figure 2.10, and so any possible topology is realized by an Arnold fluid. We also state it in the general case of a molecule with gluing coefficients.

Theorem 2.3.15. *Given a graph with blocks $I - V$ and Dehn coefficients, there exist an Arnold fluid with Morse-Bott Bernoulli function realizing it. In general, given a molecule with gluing coefficients there is an Arnold fluid realizing it.*

Proof. As in the whole section, we restrict to graphs with simple atoms, i.e. blocks of the form $I - V$, and leave for the appendix the general case.

Take a graph with Dehn coefficients and oriented edges. Each vertex indicates the type of block (and hence of the neighborhood of some connected component of a critical level set) of the Bernoulli function. The amount of up-directed edges for type III blocks indicates if in the interior boundary components or in the exterior one the Bernoulli function is increasing. We start from the bottom and construct in the minima blocks an Arnold fluid using Proposition 2.3.14. We proceed by induction.

Assume we have an Arnold fluid in a manifold with boundary N realizing a subgraph of the given marked molecule. Denote by B the Bernoulli function in N. In a neighborhood of a torus boundary component of some of its blocks, there exist coordinates (t, θ) such that the one form is $\alpha = t d\theta$. We attach the following block, that we assume to be of type III or IV, via a Dehn twist with the coefficients indicated by the edge of the graph. Using Proposition 2.3.14, we endow the block with an Arnold fluid. Up to choosing well the constants K and

C in Proposition 2.3.14, we can make sure that the minimal value of the Bernoulli function is higher than the maximal value of B in N. Denote by A and B the maximal value of B in N and the minimal value of the Bernoulli function in the new block. Hence in a neighborhood of the gluing locus $U(T^2) \cong T^2 \times [A, B]$ we can assume that in each boundary component $T^2 \times \{A\}$ and $T^2 \times \{B\}$ we have the Arnold fluids respectively of N and the glued block. We are in the hypotheses of the interpolation Lemma 2.3.8, since the vector fields are always linear in the torus boundaries. In a neighborhood $U(T^2)$, we obtain a globally defined non-vanishing vector field X, and a one form α such that $\alpha(X) > 0$ and $\iota_X d\alpha = -dB'$. Here B' is a function which coincides with B in N except at the neighborhood where we applied the interpolation lemma.

The cases of attaching the last blocks of type I, II or V containing a maximum are done analogously.

At the end, the Bernoulli function realizes the given graph. Furthermore, by construction, the vector field is volume preserving in each block and hence globally preserves some volume form. This proves, by Lemma 2.1.7, that the construction yields a globally defined Arnold fluid realizing the initial graph with coefficients.

In the general case, when we can have more than one critical circle in the same connected component of the critical level set, the neighborhood of a singular leaf is a 3-atom as described in [13]. In the Appendix we explain how to construct an Arnold fluid in the case of an arbitrary 3-atom, in the sense of any possible foliation around a singular leaf. Then using the interpolation lemma as above proves that given any graph with arbitrary 3-atoms of any complexity as vertices, there is an Arnold fluid realizing it. $\qquad\square$

Remark 2.3.16. By the same arguments as in the previous section, we can in fact assume that the Bernoulli function is analytic.

Theorem 2.3.2 stated in the introduction is just a reformulation of the realization of any marked molecule. However, in the previous Theorem we didn't fix a volume-form a priori. A simple application of Moser's path mathod ensures that it can be chosen arbitrarily.

Lemma 2.3.17. *Let X be a steady Euler flow with Bernoulli function B on (M, g). Let μ' be any volume form. Then there is some other metric g' such that X is a steady solution to the Euler equations with Bernoulli function B (up to diffeomorphism) and induced Riemannian volume form μ'.*

Proof. Denote by μ the volume induced by g, it is preserved by X. Up to multiplying μ by a constant, Moser's path method shows that there is a diffeomorphism $\varphi : M \to M$ (actually an isotopy) such that $\varphi^*\mu = \mu'$. In particular, the vector field $Y = \varphi^*X$ preserves $\varphi^*\mu = \mu'$. On the other hand, if α denotes $g(X, \cdot)$, we know it satisfies $\iota_X d\alpha = -dB$. We deduce that $\beta = \varphi^*\alpha$ satisfies

$$\iota_{\varphi^*X} d\varphi^*\alpha = -d\varphi^*B.$$

To conclude, we construct a metric such that $g'(Y, \cdot) = \beta$ and μ' is the induced Riemannian volume using Lemma 2.1.7. We conclude that Y satisfies the stationary Euler equations in (M, g') with Bernoulli function $\varphi^* B$ and induced volume form μ'. $\qquad\square$

Symplectization of fluids

In this last subsection, we will show how the Arnold fluids we constructed can be symplectized in to give rise to a Bott integrable system.

In order to transform the constructed Arnold fluids into integrable systems, we will need a one form with some properties. Recall that a stable Hamiltonian structure is a pair (α, ω) of a one form and a two form such that $\alpha \wedge \omega > 0$ and $\ker \omega \subset \ker d\alpha$. The equations $\iota_R \omega = 0$ and $\alpha(R) = 1$ uniquely define the Reeb field of the stable Hamiltonian structure. It was already proved in [43] that any non-vanishing steady solution to the Euler equations with non-constant analytic Bernoulli function can be rescaled to the Reeb field of a stable Hamiltonian structure.

For some rescaling of the solutions we constructed (with B of Morse-Bott type and eventually analytic) we can explicitly construct a one form β satisfying the condition $\iota_X d\beta = 0$ and $\beta(X) > 0$, but with an additional property: that it vanishes when evaluated at the curl of X.

Lemma 2.3.18. *In Theorem 2.3.2 denote by Y the curl (for the constructed metric) of the steady flow X. Then, there exist a one form β such that*

- $\beta(X) > 0$,

- $\iota_X d\beta = 0$,

- $\beta(Y) = 0$.

Proof. Let us keep the simplifying assumption that the marked molecule has only simple atoms i.e. blocks of type $I - V$. We have a solution as constructed in Theorem 2.3.14: a given Arnold fluid in each block using Proposition 2.3.14, and interpolations in each gluing locus using Lemma 2.3.8.

For a type I block of the form $S^1 \times D^2$ with coordinates (θ, r, φ), the preserved volume form is $\mu = r dr \wedge d\varphi \wedge d\theta$. For the type II block, the volume form is $dt \wedge d\varphi \wedge d\theta$ for coordinates in $T^2 \times [1, 2]$. In type III and IV, the volume form is $\mu_N \wedge d\theta$, where μ_N is an area form in the disk with two holes and θ a coordinate of the mapping torus. Finally, type V block has as volume form $\mu_s \wedge d\theta$ where μ_S is an area form in $S^1 \times [-1, 1]$. We want to study the curl Y of the solutions constructed in Proposition 2.3.14. In cases I and II, the curl is of the form $Y = H(r)\frac{\partial}{\partial \varphi}$. In the three remaining cases, the curl equation writes

$$\iota_Y \mu = d\alpha \qquad (2.10)$$
$$= df \wedge d\theta, \qquad (2.11)$$

where f is some function in the base space of the mapping torus. We shall prove that the form $d\theta$ satisfies $d\theta(Y) = 0$.

Assume the converse, that is that $d\theta(Y)$ is not vanishing everywhere. Denote by B the base of the mapping torus and i the inclusion into the mapping torus total space. Then we have that $i^*\iota_Y\mu$ is not vanishing. But this contradicts equation (2.11). From the whole discussion, it follows that the one form $\beta = d\theta$ satisfies $\beta(X) > 0$, $\iota_X d\beta = 0$ and $\beta(Y) = 0$.

Finally, we only need to extend β in the interpolation areas. In any of the applications of the interpolation lemma we did in Theorem 2.3.2, observe that in the boundary $U = [1,2] \times T^2$ we have $\beta|_{t=1} = d\theta$ and $\beta|_{t=2} = Cd\theta + Dd\varphi$. Let us prove that in an arbitrary interpolation, we can find a β satisfying the boundary conditions and the required conditions inside. For the volume form $dt \wedge d\theta \wedge d\varphi$, which is preserved by X and extends as a globally preserved volume form, we can compute the curl of X. It satisfies the condition $\iota_Y\mu = d(t\alpha)$. By writing such an equation in every interval (1)-(7) of Lemma 2.3.8, we find the following expression of Y.

$$
Y = \begin{cases}
\frac{\partial}{\partial\varphi} - [tH_1' + H_1]\frac{\partial}{\partial\theta} & t \in I_1 \\
\frac{\partial}{\partial\varphi} - \frac{\partial}{\partial\theta} & t \in I_2 \\
\frac{\partial}{\partial\varphi} - \frac{\partial}{\partial\theta} & t \in I_3 \\
[1 + H_4(C-1) + tH_4'(C-1)]\frac{\partial}{\partial\varphi} - \frac{\partial}{\partial\theta} & t \in I_4 \\
C\frac{\partial}{\partial\varphi} - \frac{\partial}{\partial\theta} & t \in I_5 \\
C\frac{\partial}{\partial\varphi} - [1 + H_6(D-1) + tH_6'(D-1)]\frac{\partial}{\partial\theta} & t \in I_6 \\
C\frac{\partial}{\partial\varphi} - D\frac{\partial}{\partial\theta} & t \in I_7
\end{cases}
$$

Hence, we define β as the following one form.

$$
\beta = \begin{cases}
d\theta + [tH_1' + H_1]d\varphi & t \in I_1 \\
d\theta + d\varphi & t \in I_2 \\
d\theta + d\varphi & t \in I_3 \\
[1 + H_4(C-1) + tH_4'(C-1)]d\theta + d\varphi & t \in I_4 \\
Cd\theta + d\varphi & t \in I_5 \\
Cd\theta + [1 + H_6(D-1) + tH_6'(D-1)]d\varphi & t \in I_6 \\
Cd\theta + Dd\varphi & t \in I_7
\end{cases}
$$

Such one form clearly satisfies $\beta(Y) = 0$. Furthermore, looking at the expression of X in Lemma 2.3.8, we have that $\beta(X) > 0$ and $\iota_X d\beta = 0$. $\qquad\square$

The interpolation Lemma 2.3.8 was adjusted so that one can also find the one form β.

Take M to be a graph manifold and X an Arnold fluid constructed as in Theorem 2.3.2, so that for some metric g we have

$$\begin{cases} \iota_X d\alpha & = -dB, \\ d\iota_X \mu & = 0 \end{cases},$$

where $\alpha = g(X, \cdot)$ and μ is the Riemannian volume. Denote Y the curl of X with respect to g. By Lemma 2.3.18 we know there is a one form β such that $\beta(X) > 0$, $\beta(Y) = 0$ and $\iota_X d\beta = 0$.

Remark 2.3.19. Note that for this β, the vector field X is a rescaling of the Reeb field of the stable Hamiltonian structure $(\beta, \iota_X \mu)$.

Consider in $M \times \mathbb{R}$, with coordinate t in the second component, equipped with the two form

$$\omega = dt \wedge \beta + t d\beta + \iota_X \mu.$$

For t small enough, it is clearly a symplectic form. This is in fact the symplectization (cf. [42]) of the stable Hamiltonian structure $(\beta, \iota_X \mu)$. The Euler flow X and its curl can be seen as some Hamiltonian system with Bott integral in the symplectic manifold $M \times (-\varepsilon, \varepsilon)$ equipped with ω as symplectic form.

Proposition 2.3.20. *The pair $F = (t, -B)$ defines an integrable system in $M \times [-\varepsilon, \varepsilon]$. The Hamiltonian vector fields of t and B in $M \times \{0\}$ are respectively X and its curl.*

Proof. The vector field X satisfies that

$$\iota_X \omega = -dt + t\iota_X d\beta = -dt,$$

which implies that X is the Hamiltonian vector field of the function $H = t$. Furthermore, contracting Y with the symplectic form we obtain

$$\iota_Y \omega = \iota_Y \iota_X \mu + t\iota_Y d\beta.$$

Recall that Y satisfies that $\iota_Y \mu = d\alpha$, so we have $\iota_Y \iota_X \mu = -\iota_X d\alpha = dB$. If X_B denotes the Hamiltonian vector field of the function $-B$, we have that $X_B|_{t=0} = Y$.

It remains to check that $F = (t, B)$ define an integrable system. Clearly, $dt \wedge dB \neq 0$ almost everywhere, since dB vanishes in zero measure stratified sets. Additionally, we have

$$\begin{aligned} \omega(X_t, X_B) &= -\omega(X_B, X_t) \\ &= -\iota_{X_B}\omega(X_t) \\ &= -dB(X) \\ &= 0. \end{aligned}$$

The last equality follows from the first Euler equation: the fact that $\iota_X d\alpha = -dB$. $\qquad \square$

We obtain an alternative proof that any topological configuration of a Bott integrable system can be realized, with the additional property that the Hamiltonian vector field is, up to rescaling, the Reeb field of a stable Hamiltonian structure. The result is stated in the introduction as Theorem 2.3.3. The realization theorem for Bott integrable systems was originally proved by Bolsinov-Fomenko-Matveev [14].

Proposition 2.3.20 unveils an example of an explicit (and expected) relation between Arnold's structure theorem and the classical Arnold-Liouville theorem in the theory of integrable systems. However the symplectization procedure to obtain integrable systems is *ad hoc*. In general, for a non-vanishing flow with an analytic or even Morse-Bott Bernoulli function, it is not possible to find a one form as in Lemma 2.3.18.

In a point of the critical set of the Bernoulli function, we have that $\iota_X d\alpha = 0$ and $\iota_Y d\alpha = 0$. This implies that either $d\alpha$ vanishes and so does Y, or Y is non-vanishing and parallel to X. It is clear that in the second case one cannot find a one form such that $\beta(X) > 0$ and $\beta(Y) = 0$. It is possible to find examples where this happens, using Example 4.4 in [119].

Example 2.3.21. *Consider the three torus T^3 with the standard metric on it $g = d\theta_1^2 + d\theta_2^2 + d\theta_3^2$. We take the volume preserving vector field*

$$X = \sin^2 \theta_3 \frac{\partial}{\partial \theta_1} + \cos \theta_3 \frac{\partial}{\partial \theta_2},$$

which is tangent to the tori obtained by fixing the third coordinate. The curl of X is given by $Y = \sin \theta_3 \frac{\partial}{\partial \theta_1} + 2 \sin \theta_3 \cos \theta_3 \frac{\partial}{\partial \theta_2}$. The dual form to X is $\alpha = \sin^2 \theta_3 d\theta_1 + \cos \theta_3 d\theta_2$, from which we can deduce that the analytic Bernoulli function is $B = \frac{1}{2}(\sin^4 \theta_3 + \cos^2 \theta_3)$. Along the torus $\theta_3 = \pi/2$, we have that the derivative of the Bernoulli function vanishes. However, both X and Y are non-vanishing and parallel. Hence in such example one cannot find a one form as in Lemma 2.3.18.

We can also produce an example with a Morse-Bott Bernoulli function.

Example 2.3.22. *Consider the solid torus as in the block of Section 2.3.2. Take coordinates (θ, x, y) in $S^1 \times D^2$, and denote by (r, φ) polar coordinates in D^2. We consider the one form*
$$\alpha = (r^2 + \varepsilon)d\theta + \varphi(r)xdy,$$
where $r = x^2 + y^2$, the function $\varphi(r)$ is constantly equal to 1 close to 0 and equal to 0 for $r \geq \delta$. The vector field will still be $X = \frac{\partial}{\partial \theta}$ and the volume form is $\mu = rdr \wedge d\varphi \wedge d\theta = dx \wedge dy \wedge d\theta$. We have

$$d\alpha = 2rdr \wedge d\theta + (\frac{\partial \varphi}{\partial x}2x^3 + \varphi)dx \wedge dy.$$

As before, we have that $\iota_X d\alpha = d(r^2)$, so the Bernoulli function is Morse-Bott $B = r^2$. However, constructing a metric with Lemma 2.1.7, the curl of X is no

longer $\frac{\partial}{\partial\varphi}$. For $r > \delta$, we have $Y = \frac{\partial}{\partial\varphi}$. For r very close to 0 we have that $\varphi(r) = 1$ and hence $d\alpha = 2rdr \wedge d\theta + dx \wedge dy$. For such a form, the curl of X is

$$Y = 2\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial\theta},$$

which doesn't vanish at $r = 0$. The construction in the solid torus can be completed to a compact manifold as done in Section 2.3.2.

In both cases the one form β cannot be constructed, and in fact these fluids cannot be seen as integrable systems, as long as we ask the natural compatibility conditions that X and Y are respectively the Hamiltonian vector fields of the integrals t and B. Indeed if Y was the Hamiltonian vector field of B, it should always vanish at the critical points of B, since it would be defined by the equation $\iota_Y\omega = dB$ for some symplectic form ω. Such condition is not satisfied in the previous examples.

2.3.4 Obstructions to Morse-Bott integrability

In this last section we study the most general case of Euler flows with a Morse-Bott Bernoulli function. Those are singular Arnold fluids, and by singular we mean that we allow the vector field X to have stagnation points. We will prove that these fluids do not exist in non graph manifolds.

Critical sets of the Bernoulli function

Let us first analyze the level sets of a Morse-Bott Bernoulli function. A first lemma is the non-existence of non-degenerate critical points.

Lemma 2.3.23. *Let X be a steady Euler flow with smooth Bernoulli function B. Then B does not have any non-degenerate critical point.*

Proof. Assume there is a critical point p which is not of saddle type. By the Morse lemma there is a local chart $(U, (x, y, z))$ around c such that the functon is $B = x^2 + y^2 + z^2$ or $B = -x^2 - y^2 - z^2$. But then either $B^{-1}(\varepsilon)$ or $B^{-1}(-\varepsilon)$ is a regular level set diffeomorphic to a sphere. This is a contradiction with Arnold's theorem, which ensures that all regular level sets are tori. It only remains the case of a saddle point.

Now let p be a saddle point of B. Again by the Morse lemma, there are coordinates (x, y, z) such that

$$B = x^2 + y^2 - z^2.$$

In these coordinates, we can write

$$X = X_1\frac{\partial}{\partial x} + X_2\frac{\partial}{\partial y} + X_3\frac{\partial}{\partial z}$$

and

$$d\alpha = a\,dy \wedge dz + b\,dx \wedge dz + c\,dx \wedge dy,$$

where X_1, X_2, X_3, a, b, c are functions depending on x, y, z. The first Euler equation $\iota_X d\alpha = -dB$ implies

$$\begin{cases} X_1 b - X_2 a = 2z \\ X_1 c - X_3 a = -2y \\ -X_3 b - X_2 c = -2x \end{cases} \tag{2.12}$$

We claim that $d\alpha|_p \neq 0$. Assume the converse, that $d\alpha|_p = 0$. Because B is an integral of X, we have that X vanishes at p necessarily. There are several ways to see this. Take coordinates (x, y, z) around p and denote by X_1, X_2, X_3 the components in each coordinates of X. Similarly, denote B_1, B_2, B_3 the derivatives of B with respect to x, y and z. The fact that B is an integral of X implies that $X_i B_i = 0$. Deriving this equation and restricting to the critical point we get

$$(X_1, X_2, X_3) D^2 B (X_1, X_2, X_3)^T|_p = 0,$$

where $D^2 B$ is the Hessian of B. This matrix is non-degenerate at p so we deduce that $(X_1, X_2, X_3)|_p = 0$.

Taking the Taylor expansions of the functions X_1, X_2, X_3 and a, b, c, they all have a vanishing coefficient of order 0. In particular, the combinations of the system (2.12) yield functions that vanish at order two at least. This contradicts the system of equations.

We deduce that $d\alpha|_p \neq 0$. However, we know that the vorticity Y is determined by the equation

$$\iota_Y \mu = d\alpha,$$

which implies that $Y|_p \neq 0$. But B is also an integral of Y because $\iota_Y dB = -\iota_Y \iota_X d\alpha = \iota_X \iota_Y d\alpha = 0$ and by the previous argument this implies that $Y|_p = 0$. We reacg a contradiction and conclude that a saddle point cannot exist. $\square$

The previous lemma applies in full generality for any manifold and Riemannian metric. We will see that in the case of a non graph manifold this is enough to prove the non-existence of Bott integrable fluids. The critical level sets of B can now be described in general.

Lemma 2.3.24. *Let X be an Euler flow with Morse-Bott Bernoulli function. Let c be a critical submanifold. Then c is either a circle, a torus or a Klein bottle. If c is a circle of saddle type, denote by Z a regular component of the critical level set containing c. Then Z is an orientable finitely punctured surface with finite genus.*

Proof. By Lemma 2.3.23, each critical submanifold c is of dimension one or two. If c is two dimensional, it has to be a compact surface. Furthermore, the regular level sets in a trivial neighborhood of c must be tori because of Arnold's theorem.

This implies that c is either a torus or a Klein bottle. If c is one dimensional, it is compact and hence a circle.

For the second part of the lemma, denote by Z a 2-dimensional strata of a critical level set: it is an open embedded surface. The fact that Z is orientable follows from the fact that $dB \neq 0$ everywhere in Z and is transverse to it. Then the gradient of B, which satisfies $g(\text{grad } B, \cdot) = dB$, is a vector field everywhere transverse to Z. This implies that Z is an open orientable surface. By compactness, it has a finite amount of punctures (approaching the critical points or circles) and has finite genus. □

Non existence of Bott integrable fluids

We proceed with the proof of Theorem 2.3.4.

Proof of Theorem 2.3.4. We will first show that a stratified Bernoulli function has necessarily a non-empty 0-strata if M is not of graph type. A function is stratified [66] if its critical values are isolated and the critical level sets are Whitney stratified sets of codimension greater than zero. This includes both analytic and Morse-Bott functions. The claim follows easily from the theory of tame functions introduced in [71].

Assume that there are no 0-strata. The 1-strata are necessarily critical circles, by compactness. By Arnold's theorem, every regular level set is a torus. Then the function B is a tame function in the sense of [71], and M has to be a graph manifold: this is a contradiction.

Hence, if B is a Morse-Bott Bernoulli function of some steady Euler flow in a non-graph manifold, it necessarily has an isolated critical point. By Lemma 2.3.23, this is not possible and we conclude that such steady flow cannot exist. □

The Morse-Bott assumption was key in the proof, and so for the case of an analytical Bernoulli function the problem of existence of integrable steady fluids in non graph manifolds remains open.

2.3.5 Appendix: 3-atoms and topological classification

In this appendix we will introduce the notion of 3-atom as in [13], show how to construct an Arnold fluid in an arbitrary 3-atom and discuss the topological classification of the moduli of Morse-Bott Arnold fluids.

Given a non-vanishing vector field with a Morse-Bott integral F, we denote by L a critical level set of F. We are now in the general case and a single critical level set can have more than one critical circle. An example is given by the level set of the height function in Figure 2.8, where the cyllinders of the boundary components of Σ_0 merge. The level set in the total space $\Sigma_0 \times S^1$ is Figure 2.12 times a circle.

We consider a neighborhood $U(L)$ of L foliated by the function: that is $F^{-1}(c - \varepsilon, c + \varepsilon)$ where $f(L) = c$. We call the topological resulting foliation a three atom.
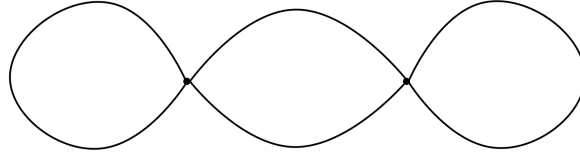
Figure 2.12: Non simple 2-atom

Originally, these are considered up to diffeomorphism preserving the foliation and the orientation induced by the flow in the possibly existing critical circles.

In turns out that the topological classification of three atoms depends on the classification of two atoms. A two atom is the neighborhood of a singular level set of a Morse function in a surface. That is, again, $U(L') = f^{-1}(c - \varepsilon, c + \varepsilon)$ where f is a Morse function in a surface and L' a critical level set of f. The classification of three atoms is then the following. A three atom is always of the form $P^2 \times S^1$ or $P^2 \tilde{\times} S^1$. Here P^2 denotes some two atom, and the second case is a twisted product that denotes the mapping torus by certain involution $\tau : P^2 \to P^2$ which preserves the Morse function f inducing the foliation in P^2. Denote by $\pi : U(L) \to P^2$ the projection to the zero section P^2. It follows from the description of an arbitrary three atom that $\pi^* f$ is always a well defined Morse-Bott function. The blocks I, III and IV presented in 2.3.3 are the 3-atoms in the case where the Bernoulli function only has a single critical circle in the critical level set. Blocks II and V are introduced to take into the account the case of critical surfaces.

One can construct, similarly to type III and IV blocks, an Arnold fluid in a given 3-atom using its structure of mapping torus. If we denote by θ the coordinate in the S^1 component, we take as vector field $X = \frac{\partial}{\partial \theta}$. As one form we take $\alpha = (K + \pi^* f)d\theta$, where K is a constant such that $\alpha(X) > 0$ everywhere. Finally, take $B = C + \pi^* f$ as Bernoulli function for some other constant C. We clearly have that $\iota_X d\alpha = -dB$. Given any area form ω in P^2, the area form $\omega + \tau^* \omega$ is invariant by the mapping torus and hence X is volume preserving for some volume. Lemma 2.1.7 concludes that it is an Arnold fluid. The torus boundary components are regular level sets of the Bernoulli function. Hence, one can apply the arguments of the proof of Theorem 2.3.2 that we used for simple atoms in this more general setting. Instead of a graph whose vertices are blocks of type $I - V$, one can have blocks of type II, III and any other possible 3-atom. It is also immediate to check that the proof of Theorem 2.3.3 also applies for Morse-Bott function with atoms of arbitrary complexity. The one form β in Lemma 2.3.18 can be constructed in a given 3-atom analogously to how it is done for blocks of type III and IV.

In [13], the study of equivalence classes of such more general graphs gives rise to the notion of marked molecule. Marked molecules classify topologically stable

Bott integrable systems. In our setting, we were just interested in the topology of B, i.e. the foliation by level sets, and not in the orientations at the critical circles. When we forget about the orientation of the critical circles and drop the topologically stable condition, the classification is also possible in terms of equivalence classes of these graphs (molecules with gluing coefficients). In that case however, it becomes more technical that with the simplifying assumptions taken in [13].

If we follow the orientation of the critical circles and take the simplifying assumptions that the orientation induced by the fluid on the critical circles is compatible in each critical level set, then the marked molecule is a complete topological invariant of Morse-Bott Arnold fluids.

Corollary 2.3.25. *Marked molecules classify topologically the moduli of non-vanishing Euler flows with Morse-Bott Bernoulli function.*

This classification can be compared to [115], where vorticity functions of Morse type are topologically classified in the context of the Euler equations in surfaces.

Chapter 3

Singular geometric structures

In this chapter, we will analyze singular geometric structures such as b-symplectic forms, folded symplectic forms and b-contact forms. We give a classification result for top degree forms transverse to the zero section, compare it to the classification of certain Nambu structures, and study the appearance of b-symplectic and b-contact forms in steady Euler flows. The contents of this chapter is based in [29] and [31].

3.1 Singular symplectic geometry

In this section we introduce a basic background on singular symplectic structures.

3.1.1 b-symplectic and b-contact geometry

In this section we follow closely [92] and [142] to introduce singular symplectic and contact structures that will be of utter relevance in this chapter.

b-symplectic manifolds

The language of b-forms was introduced by Melrose [139] in order to study manifolds with boundary. The subject gained attention in the realm of Poisson geometry as a special class of Poisson manifolds can be studied using b-calculus [92]. Most definitions can be used replacing the boundary by any given hypersurface of a manifold without boundary:

Definition 3.1.1. *A b-manifold (M, Z) is an oriented manifold M with an oriented hypersurface Z.*

Remark 3.1.2. It is possible to extend this definition to consider non-orientable manifolds. See for instance [89] and [146].

In order to have the b-category we introduce the notion of b-map.

Definition 3.1.3. *A b-map is a map*

$$f : (M_1, Z_1) \longrightarrow (M_2, Z_2)$$

so that f is transverse to Z_2 and $f^{-1}(Z_2) = Z_1$.

Vector fields and differential forms have to be redefined also.

Definition 3.1.4. *A b-vector field on a b-manifold (M, Z) is a vector field which is tangent to Z at every point $p \in Z$.*

Observe, in particular, that a *b*-vector field is tangent to the hypersurface Z, so from a dynamical point of view Z is an *invariant manifold* by the flow of these vector fields. These *b*-vector fields form a Lie subalgebra of vector fields on M. Let t be a defining function of Z in a neighborhood U and $(t, x_2, ..., x_n)$ be a chart on it. Then the set of *b*-vector fields on U is a free $C^\infty(U)$-module with basis

$$\left(t\frac{\partial}{\partial t}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n} \right).$$

We deduce that the sheaf of *b*-vector fields on M is a locally free C^∞-module and therefore it is given by the sections of a vector bundle on M. This vector bundle is called the *b***-tangent bundle** and denoted by ${}^bTM$. Its dual bundle is called the *b***-cotangent bundle** and is denoted ${}^bT^*M$.

By considering sections of powers of this bundle we obtain *b***-forms**.

Definition 3.1.5. *Let (M^{2n}, Z) be a b-manifold and $\omega \in {}^b\Omega^2(M)$ a closed b-form. We say that ω is b-symplectic if ω_p is of maximal rank as an element of $\Lambda^2({}^bT_p^*M)$ for all $p \in M$.*

In the class of Poisson manifolds there is the distinguished subclass of *b*-Poisson manifolds which is indeed formed by *b*-symplectic manifolds together with a bivector field naturally associated to the *b*-symplectic forms.

Definition 3.1.6. *Let (M^{2n}, Π) be an oriented Poisson manifold. Let the map*

$$p \in M \mapsto (\Pi(p))^n \in \Lambda^{2n}(TM)$$

be transverse to the zero section. Then Π is called a b-Poisson structure on M. The hypersurface Z where the multivectorfield Π^n vanishes,

$$Z = \{p \in M | (\Pi(p))^n = 0\}$$

is called the critical hypersurface of Π. The pair (M, Π) is called a b-Poisson manifold.

The transversality condition is equivalent to saying that 0 is a regular value of the map $p \longrightarrow (\Pi(p))^n$. The hypersurface Z has a defining function obtained by dividing this map by a non-vanishing section of $\bigwedge^{2n}(TM)$.

The set of b-symplectic manifolds is in one-to-one correspondence with the set of b-Poisson manifolds. This correspondence, detailed in [92], can be formulated as

Proposition 3.1.7. *A two-form ω on a b-manifold (M, Z) is b-symplectic if and only if its dual bivector field Π is a b-Poisson structure.*

In this context we have a normal form theorem analogous to Darboux theorem for symplectic manifolds. This result is also proved in [92].

Theorem 3.1.8 (b-**Darboux theorem**). *Let (M, Z, ω) be a b-symplectic manifold. On a neighborhood of a point $p \in Z$, there exist coordinates $(x_1, y_1, ..., x_n, y_n)$ centered at p such that*

$$\omega = \frac{1}{x_1}\, dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

Note that with this chart, the symplectic foliation of (M, Π) has a specific form. It has two open subsets where the Poisson structure has maximal rank given by $\{x_1 > 0\}$ and $\{x_1 < 0\}$. The hyperplane $\{x_1 = 0\}$ contains leaves of dimension $2n - 2$ given by the level sets of y_1.

The critical hypersurface Z of a b-symplectic structure has an induced regular Poisson structure which can also be visualized as a cosymplectic manifold (see [91, 92]).

In [91] it was shown that if Z is compact and connected, then the critical set Z is the mapping torus of any of its symplectic leaves L by the flow of the any choice of modular vector field u:

$$Z = (L \times [0, k])/_{(x,0) \sim (\phi(x), k)},$$

where k is a certain positive real number and ϕ is the time-k flow of u. In particular, all the symplectic leaves inside Z are symplectomorphic. As in [91], we refer to a fixed symplectomorphism inducing the mapping torus as the **monodromy** of Z.

This yields the following definition:

Definition 3.1.9 (Modular period). *Taking any modular vector field u_{mod}^{Ω}, the* **modular period** *of Z is the number k such that Z is the mapping torus*

$$Z = (L \times [0, k])/_{(x,0) \sim (\phi(x), k)},$$

and the time-t flow of u_{mod}^{Ω} is translation by t in the $[0, k]$ factor above.

One of the research directions has been to generalize *b*-structures and consider more degenerate singularities of the Poisson structure. This is the case of b^m-Poisson structures, for which ω^n has a singularity of A_n-type in Arnold's list of simple singularities [7] [6]. A dual approach is also possible and interesting, working with forms instead of bivector fields.

Definition 3.1.10. *A symplectic b^m-manifold is a pair (M^{2n}, Z) with a closed b^m-two form ω which has maximal rank at every $p \in M$.*

Such as in the *b*-symplectic case, an analogous b^m-Darboux theorem holds. A decomposition for these forms is given in [174].

Definition 3.1.11. *A Laurent Series of a closed b^m-form ω is a decomposition of ω in a tubular neighborhood U of Z of the form*

$$\omega = \frac{dx}{x^m} \wedge \left(\sum_{i=0}^{m-1} \pi^*(\hat{\alpha}_i) x^i \right) + \beta,$$

where $\pi : U \to Z$ is the projection, where each $\hat{\alpha}_i$ is a closed form on Z, and β is form on U.

It is proved in [174] that every closed b^m-form admits in a tubular neighborhood U of Z a Laurent form of this type, when fixing a semi-local defining function.

Proposition 3.1.12. *In a tubular neighborhood of Z, every closed b^m-form ω can be written in a Laurent form and the restriction of $\sum_{i=0}^{m-1} \pi^*(\hat{\alpha}_i) x^i$ and β to Z are well-defined closed 1 and 2-forms respectively.*

b-contact manifolds

Following these ideas and in analogy with contact structures, *b*-contact structures are developed in [142].

Definition 3.1.13. *Let (M, Z) be a (2n+1)-dimensional b-manifold. A b-contact structure is the distribution given by the kernel of a one b-form $\xi = \ker \alpha \subset {}^bTM$, $\alpha \in {}^b\Omega^1(M)$, that satisfies $\alpha \wedge (d\alpha)^n \neq 0$ as a section of $\Lambda^{2n+1}({}^bT^*M)$. We say that α is a b-contact form and the pair (M, ξ) a b-contact manifold.*

As in contact geometry one can define the Reeb vector field that satisfies

$$\begin{cases} i_{R_\alpha} d\alpha = 0 \\ \alpha(R_\alpha) = 1. \end{cases}$$

A Darboux type theorem can be proved, providing a normal local form for these structures.

Theorem 3.1.14. *Let α be a b-contact form inducing a b-contact structure ξ on a b-manifold (M, Z) of dimension $(2n + 1)$ and $p \in Z$. We can find a local chart $(\mathcal{U}, z, x_1, y_1, \ldots, x_n, y_n)$ centered at p such that on $\mathcal{U}$ the hypersurface Z is locally defined by $z = 0$ and*

1. *if $R_p \neq 0$*

   (a) ξ_p is singular, then

   $$\alpha|_{\mathcal{U}} = dx_1 + y_1 \frac{dz}{z} + \sum_{i=2}^{n} x_i dy_i,$$

   (b) ξ_p is regular, then

   $$\alpha|_{\mathcal{U}} = dx_1 + y_1 \frac{dz}{z} + \frac{dz}{z} + \sum_{i=2}^{n} x_i dy_i,$$

2. *if $R_p = 0$, then $\tilde{\alpha} = f\alpha$ for $f(p) \neq 0$, where*

 $$\tilde{\alpha}_p = \frac{dz}{z} + \sum_{i=1}^{n} x_i dy_i.$$

Remark 3.1.15. There is also a dual correspondence between *b*-contact structures and other structures that play the role of Poisson in the contact context: Jacobi manifolds. The particular subclass is the one of *b*-Jacobi manifolds that satisfy also a transversality condition. For more details you may consult [142].

Desingularizing b^m-forms

In [94] a desingularization procedure for b^m-symplectic manifolds associates a family of folded symplectic or symplectic forms to a given b^m-symplectic structure depending on the parity of m. Namely,

Theorem 3.1.16 (**Guillemin-Miranda-Weitsman**, [94])**.** *Let ω be a b^m-symplectic structure on a compact orientable manifold M and let Z be its critical hypersurface.*

- *If $m = 2k$, then there exists a family of symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighborhood of Z and for which the family of bivector fields $(\omega_\epsilon)^{-1}$ converges in the C^{2k-1}-topology to the Poisson structure ω^{-1} as $\epsilon \to 0$.*

- *If $m = 2k + 1$, then there exists a family of folded symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighborhood of Z.*
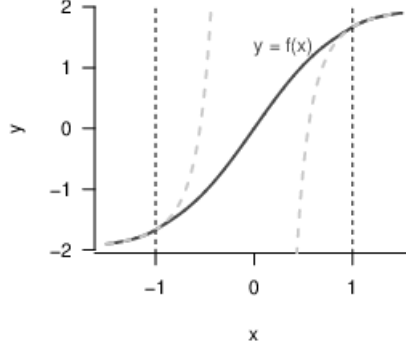
This desingularization can be applied to any b^m-form as we detail in [29]. Let us describe how the desingularization works in the even and odd case.

Case I: even m .

Assume $m = 2k$ and let $f \in \mathcal{C}^\infty(\mathbb{R})$ be an odd smooth function such that $f'(x) > 0$ for all $x \in [-1, 1]$ as shown below, and satisfying

$$f(x) = \begin{cases} \frac{-1}{(2k-1)x^{2k-1}} - 2 & \text{for} \quad x < -1 \\ \frac{-1}{(2k-1)x^{2k-1}} + 2 & \text{for} \quad x > 1 \end{cases}$$

outside the interval $[-1, 1]$. Scaling the function consider the function

$$f_\epsilon(x) := \frac{1}{\epsilon^{2k-1}} f\left(\frac{x}{\epsilon}\right).$$

And outside the interval,

$$f_\epsilon(x) = \begin{cases} \frac{-1}{(2k-1)x^{2k-1}} - \frac{2}{\epsilon^{2k-1}} & \text{for} \quad x < -\epsilon \\ \frac{-1}{(2k-1)x^{2k-1}} + \frac{2}{\epsilon^{2k-1}} & \text{for} \quad x > \epsilon \end{cases}$$

Replacing $\frac{dx}{x^{2k}}$ by df_ϵ in the semi-local expression on U we obtain

$$\omega_\epsilon = df_\epsilon \wedge \alpha + \beta.$$
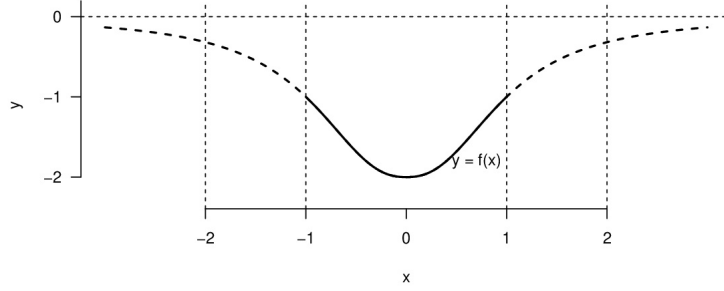
We call this form an f_ϵ-desingularization of ω.

Case II: odd m.

Consider $m = 2k + 1$, and consider a function $f \in C^\infty(\mathbb{R})$ satisfying

- $f(x) = f(-x)$

- $f'(x) > 0$ if $x > 0$

- $f(x) = x^2 - 2$ if $x \in [-1, 1]$

- $f(x) = \log(|x|)$ if $k = 0$, $x \in \mathbb{R} \setminus [-2, 2]$

- $f(x) = -\frac{1}{(2k+2)x^{2k+2}}$ if $k > 0$, $x \in \mathbb{R} \setminus [-2, 2]$.

Taking ϵ the width of a tubular neighborhood of Z define

$$f_\epsilon(x) := \frac{1}{\epsilon^{2k}} f\left(\frac{x}{\epsilon}\right)$$

and consider the form

$$\omega_\epsilon = df_\epsilon \wedge \alpha + \beta.$$

The f_ϵ-desingularization is again smooth and df_ϵ vanishes transversally at Z.

When ω is closed, its Laurent decomposition can be used as done in [94] to conclude that ω_ϵ is also closed. This yields the following theorem.

Theorem 3.1.17. *Let ω be a b-symplectic structure on a compact manifold M and let Z be its critical hypersurface. There exists a family of folded symplectic forms ω_ϵ which coincide with the b-symplectic form ω outside an ϵ-neighborhood of Z.*

As a consequence of this result any b-symplectic manifold admits a folded symplectic structure. However, it is well-known that the converse statement does not hold as not every folded symplectic form can be presented as a desingularization of a b-symplectic structures. In particular, as we will see, any compact orientable 4-dimensional manifold admits a folded symplectic form [21] but not every 4-dimensional compact manifold admits a b-symplectic manifold. For instance the 4-sphere S^4 does not admit a b-symplectic structure as it was proven in [92] that the class determined by the b-symplectic form is non-vanishing.

3.1.2 Folded symplectic manifolds

Folded symplectic structures are singular differential forms which are symplectic almost everywhere except in a hypersurface. As we will see one requires some transversality condition on this hypersurface, but also some condition on the rank of the form when restricted in the singular locus. We impose a transversality condition in the top wedge of the form instead of the bivector field as in the b-symplectic case.

Definition 3.1.18. *Let M be a $2n$-dimensional manifold. We say that $\omega \in \Omega^2(M)$ is folded-symplectic if*

1. *$d\omega = 0$,*

2. *$\omega^n \pitchfork \mathcal{O}$, where $\mathcal{O} \in \bigwedge^{2n}(T^*M)$ is the zero section, hence $Z = (\omega^n)^{-1}(\mathcal{O})$ is a codimension 1 submanifold,*

3. *$i_Z : Z \to M$ is the inclusion map, $i_Z{}^*\omega$ has maximal rank $2n - 2$.*

*We say that (M, ω) is a **folded-symplectic manifold** and we call $Z \subset M$ the folding hypersurface.*

Note that in contrast to a b-symplectic form, the condition of having maximal rank on Z has to be imposed and does not follow immediately from the transversality condition $\omega^n \pitchfork \mathcal{O}$. The property of being folded symplectic is an open property in the space of closed two forms. If ω_0 is folded, a closed 2-form ω that is C^1-close to it is also folded. Two subbundles appear naturally defined in the tangent space of M at Z.

Definition 3.1.19. *Let (M, ω) be a folded-symplectic manifold and Z the folding hypersurface with inclusion $i_Z : Z \to M$. Assume Z is nonempty.*

1. *$\ker(\omega) \to Z$ is a 2-plane bundle over Z whose fiber at a point $z \in Z$ is $ker(\omega_z) = \{X \in T_z Z \mid i_X \omega_z = 0\}$.*

2. *$\ker(i_Z{}^*\omega) \to Z$ the rank 1 vector bundle over Z, that can be defined also as the intersection $\ker(\omega) \cap TZ$.*

The line field $L = \ker(i_Z{}^*\omega)$ is sometimes refered as the null line bundle, and is generated by a vector field whose orbits define the the **null foliation**. Equivalence between folded symplectic manifold is given by the notion of folded symplectomorphism.

Definition 3.1.20. *Let (M, ω_1) and (N, ω_2) be two folded-symplectic manifolds. A smooth map $\phi : M \to N$ is folded-symplectic if $\phi^*\omega_2 = \omega_1$. If it is also a diffeomorphism, we say it is a folded-symplectomorphism.*

Folded symplectic structures have also a local model, first obtained by Martinet [135].

Theorem 3.1.21 (Martinet)**.** *Let (M, ω) be a $2n$-dimensional folded symplectic manifold and let z be a point in the folding hypersurface Z. Then there is a coordinate chart $(U; x_1, ..., x_n, y_1, ..., y_n)$ centered at z such that on U the set Z is given by $x_1 = 0$ and the folded symplectic form is*

$$\omega = x_1 dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

This normal form can also be obtained adapting Moser's path method, as detailed in [22]. In particular, it is a consequence of the following more general statement. If we denote v a section of the null line bundle, denote α a one form such that $\alpha(v) = 1$.

Proposition 3.1.22. *Assume Z is compact. Then there is a tubular neighborhood U of Z in M and an orientation preserving diffeomorphism $\varphi : Z \times (-\epsilon, \epsilon) \to U$ mapping $Z \times \{0\}$ onto Z such that*

$$\varphi^* \omega = p^* i^* \omega + d(t^2 p^* \alpha),$$

where $p : Z \times (-\epsilon, \epsilon) \to Z$ is the projection onto the first factor and t is the real coordinate in $(-\epsilon, \epsilon)$. When Z is not compact, replace ε by a continuous function.

A very simple example of compact folded symplectic manifold is S^{2n}. Let us consider the folding map $\pi : S^{2n} \to D^{2n}$ and the standard symplectic structure in $(D^{2n}, (x_i, y_i))$ given by the Darboux form $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$. Then $\pi^* \omega$ is a folded symplectic form in S^{2n}.

Notice that this is in contrast with the symplectic case, where only the two sphere admits a symplectic structre. What is more, one can have non orientable manifolds which admit folded symplectic structure.

Example 3.1.23. *Consider S^2 with the folded symplectic form $\omega = hdh \wedge d\theta$. The antipodal map*

$$\varphi : (h, \theta) \longmapsto (-h, \theta + \frac{\pi}{2})$$

leaves invariant ω. Hence the form descends by the quotient and we obtain a folded symplectic form in $\mathbb{RP}^2$. A neighborhood of Z is diffeomorphic to the Moebius band.

An interesting fact about folded symplectic manifolds, proved in [21], is that they admit an existence h-principle on closed manifolds. This shows that folded symplectic structures are more flexible than the symplectic ones, where an existence h-principle does not hold. The formal analogue of a folded symplectic structure happens to be a stable almost complex structure.

Definition 3.1.24. *A stable almost complex structure on a manifold M^{2n} is a complex vector bundle structure on $TM \oplus \mathbb{R}^2$.*

The existence h-principle can be stated in the following way.
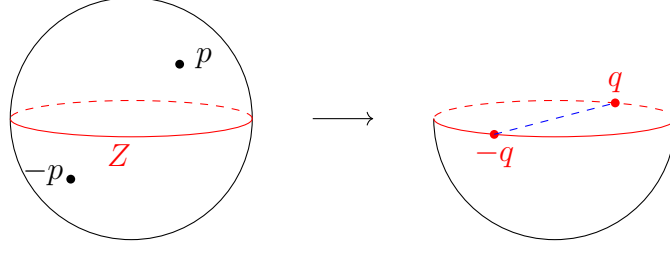
Figure 3.1: Folded symplectic structure in $\mathbb{RP}^2$

Theorem 3.1.25 ([21]). *Let M^{2n} be a manifold with a stable almost complex structure J. Then M admits a folded symplectic structure consistent with J in any degree 2 cohomology class.*

Furthermore, in the case of a 4-manifold, it can be proved that it always admits stable almost complex structure.

Corollary 3.1.26 ([21]). *Every four dimensional manifold admits a folded symplectic structure.*

A class of folded symplectic manifolds that has been largely studied in [165] are origami manifolds. Those are folded symplectic manifolds such that the null line bundle L on Z is a circle fibration instead of an arbitrary rank 1 foliation.

Definition 3.1.27. *An origami manifold is a folded symplectic manifold (M, ω) whose nullfoliation on Z integrates to a principal S^1-fibration, called the nullfibration, over a compact base B.*

$$S^1 \lhook\joinrel\longrightarrow Z$$
$$\downarrow \pi$$
$$B$$

The form ω is called an origami form.

We assume that the S^1-action matches the induced orientation of the nullfoliation V. Observe that if an origami manifold is folded symplectomorphic to another folded symplectic manifold, the latter is also of origami type.

Relation with b-symplectic structures   In the previous section we introduced b-symplectic structures, and shown how the existence of a b-symplectic form induces a folded symplectic form via a desingularization. However, one can already observe a fundamental difference at the structure induced in the critical hypersurface. Recall that b-symplectic structures induce on the critical hypersurface Z a cosymplectic structure: i.e. a foliation by symplectic leaves of dimension $2n - 2$.

This is not the case in general for folded symplectic structures. Using the folding method introduced in [22], it is possible to construct a folded symplectic structure inducing a contact structure on its critical hypersurface.

Example 3.1.28. *Take M to be a symplectic manifold with a boundary of contact type. An easy example of this is the standard closed ball B^{2n} in $\mathbb{R}^{2n}$ with its symplectic structure $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$. It is well known that convex hypersurfaces in the standard $\mathbb{R}^{2n}$ are of contact type. Denote ω_1 the symplectic structure in B^{2n}*

We consider now the manifold $M = B^{2n} \cup_\varphi \bar{B}^{2n}$, where $\bar{B}^{2n}$ has the opposite symplectic orientation, and the boundaries are indentified via φ, an orientation reversing diffeomorphism of a collar neighborhood of the boundary of B^{2n}. In a neighborhood of the boundary S^{2n-1}, the symplectic form can be written

$$\begin{cases} \omega_1 & = d(t_1 \wedge \pi^*\alpha), \\ \omega_2 & = d(t_2 \wedge \pi^*\alpha), \end{cases}$$

where α is the contact form induced in S^{2n-1} and t_i are coordinates in a trivial normal bundle of the boundary on each ball $U_i = S^{2n-1} \times [0,1)$. Take t a coordinate function on $(-1,1)$ such that

- *$t^2 = t_1$ for $t > 1/2$,*

- *$t^2 = t_2$ for $t < 1/2$.*

Then the folded symplectic form

$$\omega_f = d(t^2 \wedge \pi^*\alpha)$$

extends as ω_1 in both ball components of M. The induced structure in the hypersurface is clearly of contact type.

3.2 On the volume elements of a manifold with tranverse zeroes

Moser path method is one of the most commonly used methods in symplectic geometry and topology to prove that two given symplectic structures are equivalent. It first appeared in in Moser's celebrated article [156] where volume forms on a compact manifold are classified. In particular in dimension 2, a volume form determines a symplectic structure on a surface and Moser's theorem gives a classification of symplectic surfaces. Moser's classification is given in terms of De Rham Cohomology: two forms belong to the same cohomology class if and only if there exists a diffeomorphism conjugating them. Forms conjugated by a diffeomorphism are called *equivalent* for short in this paper.

Singular symplectic structures have lately attracted interest of the mathematical community. These structures are close to the symplectic world: they are symplectic everywhere except along a critical hypersurface. The so-called b^m-symplectic structures have a pole of order m along a critical hypersurface and are no longer symplectic as they go to infinity along it however the two-form induces a form of lower (the maximal possible) rank associated to the symplectic foliation of the Poisson structure on the critical hypersurface. In the dual case, folded symplectic structures lose rank by having a transverse zero along the critical hypersurface where the 2-form also restricts to a form of the possible maximal rank $(2n-2)$. As we will detail these structures are closely related by a process of desingularization which replaces b^m-symplectic forms by either symplectic or folded symplectic forms depending on the parity of m. The desingularization puts in the same picture singular symplectic structures (either b^m-symplectic or folded-symplectic) and symplectic ones so it is a natural to try to generalize symplectic methods in this new singular context. Moser's path method has already been used in the b^m-symplectic set-up and in its higher-dimensional analogues: b^m-Nambu structures. Those can be seen as volume forms in the realm of b^m-manifolds. However in the transversally vanishing case nothing has been said yet. If we allow the top degree form to have transverse zeroes, asking for the same cohomology class is not enough to apply Moser's path method. In this case relative cohomology captures the additional information needed.

Following [85] recall that given a smooth manifold M and a closed submanifold Z, with $i : Z \hookrightarrow M$ the inclusion. The **relative De Rham cohomology** groups of Z are given by the complex

$$\Omega^p(M, Z) = \{\alpha \in \textstyle\bigwedge^p T^*M \mid i^*\alpha = 0\}.$$

We will see that in this new scenario additionally having the same relative cohomology allows to apply the Moser's trick.

Even if the existence of transversal zeroes allows non-orientability in this picture, we will assume our manifolds to be orientable. For the sake of simplicity and mimicking the surface case we will call these volume forms **folded volume forms**.

In the last part of this section we study the compatibility between the classification of b^m-symplectic surfaces obtained by Geoff Scott in [174] and our classification scheme. This affinity is studied using the desingularization procedure developed in [94] for 2-forms. When m is odd, the desingularized structure is a folded-symplectic one. We will see that two equivalent b^{2k+1}-symplectic structures are sent to equivalent folded-symplectic forms. We extend this desingularization procedure to volume forms and prove an extension of this result for volume forms.

3.2.1   Isotopic critical sets

We will be studying top power forms that vanish satisfying a transversal condition[1]. Mimicking from the case of 2-forms [22, 21] we call these structures **folded volume forms.** As a consequence of transversality, the vanishing set for the top power will always be a closed hypersurface called **the critical set** and that may have several connected components. In order to have an equivalence relation between these singular forms, the following condition will be imposed on this critical set.

Definition 3.2.1. *Two sets of smooth disjoint oriented hypersurfaces* $(S_1, ..., S_n)$ *and* $(S_1', ..., S_n')$ *are* diffeomorphically equivalent *if there is a diffeomorphism* $\varphi : M \to M$ *mapping the first set to the second one preserving orientations.*

In the space of n disjoint oriented hypersurfaces on a manifold M this condition defines an equivalence relation. Then for a set of n disjoint oriented hypersurfaces $(S_1, ..., S_n)$ we denote $[(S_1, ..., S_n)]$ its class in the space of diffeomorphically equivalent classes.

Remark 3.2.2. When the hypersurfaces are the same we denote by $\mathrm{Diff}(M, Z)$ the set of diffemorphisms preserving the set of hypersurfaces Z.

3.2.2   A Moser trick for transversally vanishing volume forms

In order to apply the Moser's path method in this case, we need to prove a few auxiliary lemmas. Let Ω be a transversally vanishing volume form with critical set $\bar{Z}$. In what follows we will denote Z any of the connected components of the critical set and denote by t a defining function of it.

Observe that given a top degree form μ on U, a neighborhood of Z, the form $t\mu$ is a transversally vanishing volume form (in a possibly smaller neighborhood) having Z as critical set if and only if μ is non-vanishing along Z.

Let Ω_0 and Ω_1 stand for two transversally vanishing volume forms at $\bar{Z}$ which for simplicity will be denoted as **folded volume forms**. In what follows we assume that the orientation induced on each component of $\bar{Z}$ is the same for both forms.

Lemma 3.2.3. *For $0 \leq s \leq 1$, the form*

$$\Omega_s = (1 - s)\Omega_0 + s\Omega_1$$

is a folded volume form having Z as critical set.

---

[1] This condition can be generalized replacing standard tranversality by transversality à la Thom.

Proof. By the argument described above we may write $\Omega_0 = t\mu_0$ and $\Omega_1 = t\mu_1$ for μ_0 and μ_1 not vanishing at Z and positive (because of matching orientations). Consider the path $\mu_s = (1-s)\mu_0 + s\mu_1$ for $0 \le s \le 1$. Observe that $\Omega_s = t\mu_s$ and thus μ_s does not vanish at Z. $\qquad\square$

A consequence is that $\iota_v\Omega_s$ vanishes along Z, where v is any non-vanishing section of TM (or TU). By this lemma we deduce,

Claim. *Given $\alpha \in \Omega^{n-1}(U)$, there exists a vector field u such that*

$$\iota_u\Omega_s = \alpha$$

if and only if $\alpha|_Z = 0$.

Observe that since in $M \backslash \bar{Z}$ the form defines a volume, if the vector field exists it is unique.

Assume now that both the usual and relative cohomology class with respect to Z of Ω_0 and Ω_1 coincide. Then there is β such that $\Omega_0 - \Omega_1 = d\beta$. By definition we have that $i^*\beta = 0$, where $i : Z \hookrightarrow M$ is the inclusion of Z in M.

Lemma 3.2.4. *We can assume that β satisfies $\beta|_Z = 0$.*

Proof. For this we need to recall the relative Poincaré lemma for which we follow [197].

Theorem 3.2.5 (Relative Poincaré lemma). *Let $N \subset M$ be a closed submanifold of M, and ω a closed k-form of M whose pullback to N is zero. Then there is a $(p-1)$-form λ on a neighborhood of N such that $d\lambda = \omega$ and λ satisfies $i^*\lambda = 0$. If ω satisfies $\omega|_N = 0$ then λ can be chosen such that $\lambda|_N = 0$.*

Since the relative cohomology vanishes, we have β such that $i^*\beta = 0$. In a neighborhood $U(Z)$ of Z, we can apply the relative Poincaré lemma and there exist a 1-form λ in this neighborhood such that $\Omega_0 - \Omega_1 = d\lambda$ and $\lambda|_Z = 0$. In this neighborhood $d\beta = d\lambda$ and $i^*(\beta - \lambda) = 0$ so the relative Poincaré lemma yields the existence of a form α such that $\beta - \lambda = d\alpha$. Observe that in Z we have $d\alpha|_Z = \beta|_Z$.

Let φ be a bump function of a possibly smaller neighborhood of Z and consider $\varphi\alpha$ a global extension of α to M. Then the form $\gamma = \beta - d(\varphi\alpha)$ satisfies $\gamma|_Z = 0$ and $\Omega_0 - \Omega_1 = d\gamma$. This completes the proof of the lemma.

$\qquad\square$

We can improve this statement by having a more explicit expression for β. This will give some information about the isomorphism that we obtain via Moser's trick.

Lemma 3.2.6. *The form β can be written as $\beta|_U = t^2\alpha$ in a neighborhood of each connected component of $\bar{Z}$.*

Proof. The fact the relative cohomology of $\Omega_0 - \Omega_1$ is zero means that we can assume that β vanishes at $TM|_z$ for every point $z \in Z$ because of the previous lemma. In particular in a possibly smaller neighborhood U it is of the form $\beta = t\alpha$ for an $\alpha \in \Omega^{n-1}(U)$. Observe that $d\beta = dt \wedge \alpha - td\alpha$ but also $d\beta = \Omega_0 - \Omega_1 = t\mu$. Thus α needs to vanish at least linearly at Z; in particular β vanishes at least at order 2 in t.

$\square$

We can now state and prove a version of Moser's theorem for transversally vanishing volume forms.

Theorem 3.2.7. *Let Ω_0 and Ω_1 be two folded volume forms with critical set $\bar{Z} = Z_1 \cup ... \cup Z_n$. Assume that the cohomology classes of Ω_0 and Ω_1 coincide in both De Rham cohomology and relative cohomology (i.e., $[\Omega_0] = [\Omega_1]$ and $[\Omega_0]_r = [\Omega_1]_r$), then there exist a diffeomorphism φ such that $\varphi^*\Omega_1 = \Omega_0$ that restricts to the identity along $\bar{Z}$.*

Proof. Since the De Rham cohomology class of Ω_0 is the same as Ω_1, the following equality holds $\Omega_0 - \Omega_1 = d\beta$.

Let Z be one of the connected components of $\bar{Z}$ and let v be an oriented non-vanishing section of TM. Denoting by $U = U(Z)$, a neighborhood of Z, we may write $\Omega_i|_U = t\mu_i$ with μ_i is a non-vanishing form and t a defining function of Z, for $i = 1, 2$.

Consider now the path $\Omega_s = (1-s)\Omega_0 + s\Omega_1$ for $s \in [0, 1]$. By Lemma 3.2.3, Ω_s is vanishing transversally at the same critical set thus $\Omega_s|_Z = 0$. Because the relative cohomology class at Z of the two forms is the same, in a possibly smaller neighborhood we may apply Lemmas 3.2.4 and 3.2.6 and around Z the form is written as $\beta = t^2\alpha$ with t a defining function of Z. The same applies for any of the connected components in $\bar{Z}$. In order to apply Moser's trick we need to solve the equation

$$\mathcal{L}_{v_s}\Omega_s + \frac{d\Omega_s}{ds} = 0,$$

which may be written as $d\iota_{v_s}\Omega_s = \Omega_0 - \Omega_1 = d\beta$. This is equivalent to finding a vector field v_s satisfying

$$\iota_{v_s}\Omega_s = \beta.$$

Because Lemma 3.2.2 applies for any curve in $\bar{Z}$, there exist a unique solution to the equation. Now since β vanishes to second order, v_s vanishes to the first order in all the components of the critical set. The flow φ_s of v_s satisfies $\varphi_s^*\Omega_s = \Omega_0$, hence φ_1 is the desired diffeomorphism. Observe that this diffeomorphism restricts to identity in the critical set. $\square$

The theorem also applies if the critical sets of Ω_0 and Ω_1 are diffeomorphically equivalent by an orientation-preserving diffeomorphism. The fact that the relative cohomology is invariant for equivalent folded volume forms needs an extra assumption in the general setting.

Theorem 3.2.8. *Let φ be a diffeomorphism in the arc-connected component of the identity in $\mathrm{Diff}(M, Z)$ and Ω_0 and Ω_1 two folded volume forms such that $\varphi^*\Omega_1 = \Omega_0$ then the cohomology classes determined by Ω_0 and Ω_1 are the same in De Rham cohomology and in relative cohomology (i.e., $[\Omega_0] = [\Omega_1]$ and $[\Omega_0]_r = [\Omega_1]_r$).*

Proof. Since φ belongs to the arc-connected component of the identity, we can indeed construct an homotopy φ_t leaving Z invariant such that $\varphi_1 = \varphi$ and $\varphi_0 = id$. Denote $\Omega = \Omega_1 - \Omega_0$.

We can use this homotopy to define a de Rham homotopy operator:

$$Q\Omega = \int_0^1 \varphi_t^*(\iota_{v_t}\Omega)dt$$

where v_t is the t-dependent vector field defined by the isotopy φ_t.

Using this formula, we can prove (see for instance pages 110 and 111 in [98]) that $[\Omega_1] = [\Omega_0]$ as we can write $\Omega_1 = \Omega_0 + d\alpha$ for the 1-form $\alpha = Q\Omega$. From the formula above we can check that the relative cohomology class is also the same. Since Ω vanishes at Z, we deduce that $Q\Omega$ also vanishes at Z and in particular its pullback to Z is zero. $\square$

The main theorem that we deduce is a classification up to isotopy of folded volume forms. Assume that two folded volume forms Ω_1 and Ω_2 are isotopic by an isotopy φ_t, i.e. $\varphi_1^*\Omega_1 = \Omega_2$. Then the critical set of Ω_1 is sent by this isotopy to the critical set of Ω_2. If the critical set of Ω_i is denoted by Z_i, then the critical level set of $\varphi_1^*\Omega_1$ is Z_2 and it induces the same coorientation in Z_2 as Ω_2. We can say in this case that their critical level sets are *isotopically equivalent with orientation*. In particular, we can restrict to the case of two folded volume forms whose singular level set is the same and also the induced coorientation on it. The previous discussion implies:

Theorem 3.2.9. *Let M be an oriented closed manifold. Two folded volume forms Ω_1 and Ω_2 with critical sets Z_1 and Z_2 are isotopic if and only if:*

- *The critical sets are isotopically equivalent with orientation*

- *their De Rham cohomology class coincide,*

- *their relative De Rham cohomology class coincide.*

In dimension 2 this gives an isotopic classification of folded symplectic forms on closed surfaces.

3.2.3 Compatibility with the classification of b^m-structures

The aim of this section is to relate the classification of b^{2k+1}-symplectic surfaces and the classification of folded volume forms, using the desingularization formulas described in Section 3.1.1. Recall that b-symplectic structures were classified by

O.Radko in [169]. In [169] Radko uses the notion of diffeomorphism class of curves and uses cohomology and together with the modular period to classify *stable Poisson structures on surfaces*. Later on Scott classifies b^m-structures in surfaces (see theorem 6.7 in [174]).

Theorem 3.2.10 (Scott, Classification of b^m-surfaces)**.** *Let ω_0, ω_1 be two symplectic b^m-forms on a compact connected b^m-surface (M, Z, j_Z). The following are equivalent*

1. *The forms ω_0, ω_1 are b^m-symplectomorphic.*

2. *Their b^m-cohomology class is equal $[\omega_0] = [\omega_1]$.*

3. *The Liouville volumes of ω_0 and ω_1 agree, as do the numbers*

$$\int_{\gamma_r} \alpha_{-i},$$

   for all connected components $\gamma_r \subset Z$ and all $1 \leq i \leq k$, where α_{-i} are the terms appearing in the Laurent decomposition of the two forms.

We can also consider top degree volume forms in b^m-manifolds as studied in [145], [146] and introduced in [159]. These forms, called b^m-Nambu forms, satisfy also that if two of them have the same b^m-cohomology then they are isomorphic.

Theorem 3.2.11. *Let Θ_0 and Θ_1 be two b^m-Nambu structures of degree n on a compact orientable manifold M^n. If $[\Theta_0] = [\Theta_1]$ in b^m-cohomology then there exists a diffeomorphism φ such that $\varphi^* \Theta_1 = \Theta_0$.*

Remark 3.2.12. In fact two of these b^m-Nambu structures are equivalent if and only if their b^m-cohomology classes coincide. This can be proved as it is done for surfaces in [174] and the theorem can be stated as Theorem 3.2.10 replacing b^m-symplectic forms by b^m-Nambu structures. Since the b^m-Nambu structures of top degree are closed b^m-forms they admit a Laurent decomposition. It is detailed in section 5 of [174] where the class in b^m-cohomology $[\Theta]$ is identified with its Liouville-Laurent decomposition $([\Theta_{sm}], [\alpha_1], ..., [\alpha_m])$. This is in fact the b^m-Mazzeo-Melrose isomorphism for the top degree

$$^{b^m}H^n(M) \cong H^n(M) \oplus (H^{n-1}(Z))^m.$$

Using $\alpha_1, ..., \alpha_m$ the modular periods of $[\Theta]$ associated to each modular $(n-1)$-form can be determined and it can be proved that they are invariant as it is done in [136] for b-Nambu structures.

We can now state a compatibility theorem between this classification and its desingularized form.

Theorem 3.2.13. *Let Θ_0 and Θ_1 be two b^{2k+1}-Nambu structures in a b^{2k+1}-manifold that are equivalent then for all ϵ the f_ϵ-desingularized forms are also equivalent as folded volume forms (i.e., there exists a diffeomorphism conjugating them).*

Proof. Since the forms are equivalent the classes satisfy $[\Theta_0] = [\Theta_1]$ in b^{2k+1}-cohomology.

Denote t a defining function of Z. The forms Θ_0 and Θ_1 can be written close to any connected component of $\bar{Z}$ as:

$$\Theta_j = \alpha'_j + \beta_j \wedge \frac{dt}{t^{2k+1}},$$

where t is a defining function of the component of $\bar{Z}$. Since α'_j is a n form in M it can be written as $\alpha'_j = \gamma_j \wedge dt = \gamma_j t^{2k+1} \wedge \frac{dt}{t^{2k+1}}$. Hence denoting as $\alpha_j := \gamma_j t^{2k+1} + \beta_j$, as in section 6.4 of [92], the forms can be decomposed as

$$\Theta_j = \alpha_j \wedge \frac{dt}{t^{2k+1}}.$$

Then $\Theta_1 - \Theta_0 = (\alpha_0 - \alpha_1) \wedge \frac{dt}{t^{2k+1}} = d\mu \wedge \frac{dt}{t^{2k+1}}$ because they have the same b^{2k+1}-cohomology class. Once applying the desingularizing procedure, we obtain,

$$\Theta_{1,\epsilon} - \Theta_{0,\epsilon} = d\mu \wedge df_\epsilon,$$

and the right hand side looks locally as $\frac{2}{\epsilon^{2k+2}} t d\mu \wedge dt = d(\frac{2}{\epsilon^{2k+2}} t\mu \wedge dt)$.

We deduce that for any ϵ the forms $\Theta_{1,\epsilon}$ and $\Theta_{0,\epsilon}$ have the same cohomology class in $H^n(M)$ and same relative cohomology class in $H^n(M, Z)$, because they are exact with respect to a form $\beta = \frac{2}{\epsilon^{2k+2}} t\mu \wedge dt$ that vanishes at Z. Applying Theorem 3.2.7 we deduce that these two forms are isomorphic as folded volume forms. □

As an observation, note that the desingularized forms we consider depend on the decomposition in use. We obtain a compatibility theorem for the classification of b^{2k+1}-Nambu structures. Thus equivalent b^{2k+1}-Nambu structures are sent to equivalent folded volume forms. When the dimension of the manifold is 2, the compatibility is hence between b^{2k+1}-symplectic forms and folded symplectic forms.

Remark 3.2.14. The structures studied in this paper show up in the study of some particular relevant examples of dynamical systems in physical problems. For instance in [47, 16] some some particular cases of the three body problem both b^m-Nambu and volume forms vanishing at a certain order appear. In view of the results obtained in this section, the desingularization of these b^m-Nambu structures can be addressed as Theorem 3.2.13 shows that an equivalence class of those structures is desingularized to a unique class of folded volume forms (so, somehow, "canonical").

3.3 Euler flows and singular geometric structures

This section is at the crossroads of the two main topics of this thesis. We will analyze the appearance of singular geometric structures in steady Euler flows of different kinds. As we previously saw, the existence of closed one-forms on a manifold simplifies the topology of the manifold in a similar way in which the existence of first integrals of a dynamical system simplifies the topology of its invariant sets. In a dual language, the existence of first integrals also adds constraints on the topology of the invariant manifolds, and the classical Arnold-Liouville theorem shows that an integrable system on a symplectic manifold has tori as compact invariant submanifolds : in the previous section we studied an application of Tischler's ideas to provide a new proof of Arnold-Liouville theorem.

This same order of ideas can be applied to a more general picture in order to consider Fluid Dynamics and, more concretely, steady Euler flows on manifolds. In particular, we give a new proof of Arnold's structure theorem when the Bernoulli function is not constant, which is based on Tischler's theorem for manifolds with boundary. This starting point takes us to consider manifolds with boundary and b^{2k}-forms, thus providing a proof of the Tischler theorem for b^{2k}-manifolds. Additionally, we analyze the singular level sets of the Bernoulli function, which are not considered in Arnold's theorem, and prove that under some assumptions they can be described as b-symplectic manifolds. When the Bernoulli function is constant, we reconsider the correspondence between Beltrami fields and contact structures introduced in Section 1.1.3 and extend it to contact manifolds with cylindrical ends (compactified as b-manifolds) thus obtaining a new correspondence between Beltrami fields in this case with the b-contact manifolds that we introduced in Section 3.1.1. Several questions concerning the Hamiltonian and Reeb dynamics of b-contact manifolds, such as the existence of periodic orbits, can be extremely useful to understand some properties of the stream lines of Beltrami flows on manifolds with cylindrical ends.

3.3.1 A Tischler theorem for manifolds with boundary

Let us recall Tischler theorem [190] as presented in [28].

Theorem 3.3.1. *Let M^n be a closed manifold endowed with r linearly independent closed 1-forms $\beta_i, i = 1, \ldots, r$ which are nowhere vanishing. Then M^n fibers over a torus $\mathbb{T}^r$.*

As a remark in Tischler's original paper, the theorem also holds for compact manifolds with boundary with an extra assumption.

Theorem 3.3.2. *Let M^n be a compact connected manifold with boundary endowed with r linearly independent closed 1-forms $\beta_i, i = 1, \ldots, r$ which are nowhere vanishing and satisfy these conditions when restricted to the boundary. Then M^n fibers over a torus $\mathbb{T}^r$.*

Using the language of b^{2k}-forms and the deblogging procedure, one can state a Tischler theorem for manifolds with boundary. This theorem gives more information than the one we would get by simply applying the classical Tischler theorem restricted to the boundary.

Definition 3.3.3. *Let M be a manifold with boundary. Its double $\bar{M}$ is obtained by taking two copies of M and gluing along their boundary.*

$$\bar{M} = M \times \{0,1\}/ \sim,$$

where $(x,0) \sim (x,1)$ for all $x \in \partial M$.

Theorem 3.3.4. *Let $\alpha_1,...,\alpha_r$ be closed one b^{2k}-forms in a b^{2k}-manifold M such that $\alpha_1 \wedge \cdots \wedge \alpha_r \neq 0$ everywhere in M. If the pullback of the forms to the boundary are also independent then M fibers over $\mathbb{T}^r$. Otherwise the double $\bar{M}$ fibers over $\mathbb{T}^r$ and the glued boundary fibers over $\mathbb{T}^{r-1}$.*

Proof. If the forms are also independent when pullbacked to the boundary, we can apply the desingularization that we will detail for the second case in the manifold with boundary and apply Theorem 3.3.2.

Otherwise at least one of the forms has a singular part and one considers the extension of the forms α_i into $\bar{M}$ by symmetry. In this way we obtain a b^{2k}-manifold $\bar{M}$ with critical hypersurface Z where the boundaries have been glued. We can proceed to desingularize the 1-forms following [94]. Namely, the forms are closed and admit Laurent series in a neighborhood U of Z,

$$\alpha_i = \Big(\sum_{j=0}^{2k-1} \alpha_i^j t^j \Big) \frac{dt}{t^{2k}} + \beta_i,$$

for t a positively oriented defining function. Here each α_i^j is a constant function and β_i is smooth in Z. The term α_i^0 is constant and the only non-vanishing term of the singular part at the hypersurface Z. The rest of terms α_i^j for $j \neq 0$ are paired with powers of t that vanish at Z. The dividing term of $\frac{dt}{t^{2k}}$ does not cancel the powers of t because of the structure of the b^{2k}-cotangent bundle: one has to think of $\frac{dt}{t^{2k}}$ as if it was a $d\tilde{t}$ for a coordinate $\tilde{t}$.

Since at least one of these α_i^0 is non-vanishing, we can assume $\alpha_1^0 \neq 0$. Redefining

$$\alpha_i := \alpha_i - \frac{\alpha_i^0}{\alpha_1^0}\alpha_1, \text{ for } i = 2,...,n$$

we can assume that only the first form has a singular part at the hypersurface and independence of the forms still holds. Proceeding to the desingularization, one can take a suitable ϵ and the desingularized forms

$$\alpha_{i,\epsilon} = df_\epsilon \wedge \Big(\sum_{j=0}^{k} \alpha_i^j t^j \Big) + \beta_i.$$

Since we have $\alpha_1 \wedge ... \wedge \alpha_r \neq 0, df_\epsilon \neq 0$ and at least one singular form (for instance the first one $\alpha_1^0 \neq 0$) we deduce that $\alpha_{1,\epsilon} \wedge ... \wedge \alpha_{r,\epsilon} \neq 0$ using elementary linear algebra as $\alpha_{i,\epsilon}$ and α_i determine the same matrix of coefficients. One has simply changed the form $\frac{dt}{dt^{2k}}$ of the basis by df_ϵ. Applying Theorem 3.3.1 we deduce that $\bar{M}$ fibers over $\mathbb{T}^r$. Observe that in Z the form α_1 was the only one with a non-vanishing singular term. Hence its the only one with a non-vanishing term for df_ϵ: we deduce that $\alpha_{2,\epsilon}, ..., \alpha_{r,\epsilon}$ are independent when restricted to Z again by linear algebra. In particular, Z fibers over $\mathbb{T}^{r-1}$.

$\square$

Remark 3.3.5. The parity (evenness) of m comes from the desingularization procedure. The desingularized form obtained from a non-vanishing b^m-form is non-vanishing only when m is even. For odd m, as previously explained, the resulting form has a zero. This zero cannot be eliminated because the singular part of the form changes sign when crossing the hypersurface. This is why the conditions of the second statement of Theorem 3.4 cannot be met for odd m. However, the first part can be obtained by adding a constant to the desingularization formula to prevent the desingularized form from vanishing at the boundary.

Example 3.3.6. An easy example to consider is the compact cylinder C visualized as a subset of the torus $\mathbb{T}^2$ (as quotient of the plane $\mathbb{T}^2 \cong (\mathbb{R}/\mathbb{Z})^2$). Consider the b^{2k}-forms $\frac{1}{\sin(2\pi x)^{2k}} dx$ and dy on $\mathbb{R}^2$. The critical set is the boundary of a compact cylinder. The forms descend to the quotient, and in the compact cylinder they satisfy the hypotheses of the theorem.
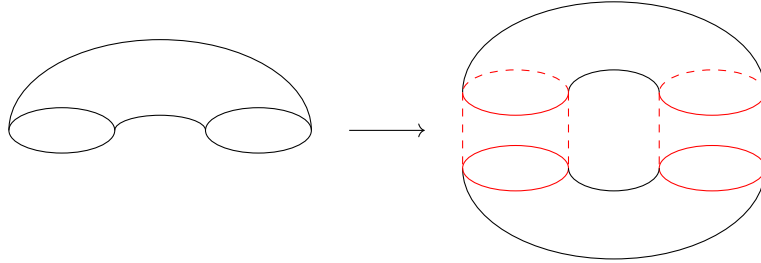

Figure 3.2: The double of a compact cylinder

Remark 3.3.7. The second statement can also be applied for honest De Rham forms with the following changes. Instead of one of the forms having a singular part, we ask one of the forms to be transversal to the boundary everywhere. Secondly we need that the forms can be extended to the doubling of the manifold by symmetry which might not be true in general.

As an easy corollary we obtain,

Corollary. *An n-dimensional manifold admitting n independent and closed b^{2k}-forms is a compact cylinder $\mathbb{T}^{n-1} \times [0,1]$.*

3.3.2 A proof of Arnold's theorem

Let us recall the Euer equations, introduced in the first two chapters of this thesis, which model the dynamics of an inviscid and incompressible fluid flow on a 3-dimensional manifold, see e.g. [9, 162]. For any Riemannian 3-manifold (M, g) one can write the Euler equations

$$\begin{cases} \frac{\partial X}{\partial t} + \nabla_X X & = -\nabla P \\ \operatorname{div} X & = 0 \end{cases}.$$

where ∇_X is the covariant derivative, and the operators ∇ and div are computed with the metric g.

Stationary solutions. For stationary solutions, we already know that the Bernoulli function is a first integral for both X and ω. In particular the stream lines are confined into the level sets of B. In the analytic setting, if B is not constant, denote its critical set $Cr(B) := \{p \in M | \nabla B(p) = 0\}$ which has a stratified structure and its codimension is at least 1. We are under the assumptions of Theorem 2.1.1, that we introduced in Section 2.1.1. Let us recall its statement and provide a new proof using the existence of certain closed one-forms as we did for integrable systems in Section 4.2.2.

Theorem 3.3.8 (Arnold's structure theorem)**.** *Let X be an analytic stationary solution of the Euler equations on an analytic compact manifold with non-constant Bernoulli function. The flow is assumed to be tangent to the boundary if there is one. Then there is an analytic set C of codimension at least 1 such that $M \backslash C$ consists of finitely many domains M_i such that either*

1. *M_i is trivially fibered by invariant tori of X and on each torus the flow is conjugated to the linear flow,*

2. *or M_i is trivially fibered by invariant cylinders of X whose boundaries lie on the boundary of M, and all stream lines are periodic.*

Proof. We define first the analytic set C. Consider $C_1 = \{B^{-1}(c) : c \in Cr(B)\}$ and C_2 the level sets which are tangent at some point to the boundary. Take

$$C = C_1 \cup C_2.$$

By compactness and analyticity [9], it is a finite union of level sets of the function B and hence it is an analytic set of codimension greater or equal to one.

Consider the following one-forms. On the one hand,

$$\beta = \iota_X \mu_2,$$

where $\mu_2 = \iota_{\frac{\nabla B}{|\nabla B|^2}} \mu$ and μ is the volume in M. The form μ_2 is sometimes called the Liouville form and satisfies $\mu = dB \wedge \mu_2$. On the other hand consider

$$\alpha(\cdot) = g(X, \cdot)$$

where g is the Riemannian metric in M. We claim that the pullback of these forms to a regular level set $i : N \hookrightarrow M$, $i^*\alpha$ and $i^*\beta$, are closed and independent. We recall that the 2-form $i^*\mu_2$ is an area-form on N.

To prove their independence, first notice that the velocity field X is tangent and non-vanishing on any regular level set of B, so the one-forms $i^*\alpha$ and $i^*\beta$ are non-degenerate on N. Since the kernel of $i^*\beta$ is given by $X|_N$, and the kernel of $i^*\alpha$ is transverse to $X|_N$ because $i^*\alpha(X|_N) > 0$, we conclude that $i^*\beta$ and $i^*\alpha$ are linearly independent at each point of N.

To prove that these one-forms are closed, recall the Euler stationary equations in terms of α:

$$\begin{cases} \iota_X d\alpha = -dB \\ d\iota_X \mu = 0 \end{cases}.$$

When restricted to a level set, the first equation implies that $\iota_X d(i^*\alpha) = 0$. Since it is a two dimensional submanifold and X is tangent to the level set this yields $d(i^*\alpha) = 0$. Observe now that μ_2 satisfies $dB \wedge \mu_2 = \mu$. Using that expression and the second Euler equation we obtain,

$$d\iota_X \mu = d(\iota_X(dB \wedge \mu_2)) = d(\iota_X dB \wedge \mu_2 - dB \wedge \iota_X \mu_2)$$
$$= -d(dB \wedge \iota_X \mu_2) = -dB \wedge d\iota_X \mu_2 = 0.$$

This equality stands everywhere. Since in the neighborhood of the regular level set we have that $dB \neq 0$, we infer that $d\iota_X \mu_2 = dB \wedge \gamma$ for some one-form γ. Accordingly, we obtain that $d(i^* \iota_X \mu_2) = 0$, thus proving that $i^*\beta$ is also closed.

Now, suppose that N has no boundary component. Then applying Theorem 3.3.1 we deduce that it is a torus. If N has a boundary, it must lie on ∂M and since it is invariant under a non-vanishing field X, the boundary consists of finitely many periodic orbits. The fact that X is non-vanishing and tangent to the boundary of the level set, implies that the pullback of α to the boundary is non-vanishing as well. By Remark 3.3.7 the first case of Theorem 3.3.4 can be applied. Hence N is an orientable surface with boundary that fibers over S^1, thus it is a cylinder. This determines the topology of the regular level sets of B. The rest of the proof is standard. Indeed, let ϕ_t be the flow of the vector field $S = \frac{\nabla B}{dB(\nabla B)}$, which satisfies $dB(S) = 1$. Then we have

$$\frac{\partial}{\partial t} B(\phi_t(x)) = dB(S) = 1, \quad B(\phi_0(x)) = B(x).$$

We deduce that $B(\phi_t(x)) = B(x) + t$ and hence the open set M_i is a trivial fibration $\mathbb{T}^2 \times I$ for a real interval I in the case that the level sets have no boundary. The same holds when the level sets are cylinders (due to the fact that in the complement of the set C the level sets of B intersecting the boundary have a transverse intersection). Since the vector field X commutes with $\mathrm{curl} X$ it follows that it is conjugated to a linear flow on each level set diffeomorphic to a torus. For the cylinder, all orbits are periodic as an easy consequence of the Poincaré-Bendixon theorem and the fact that X preserves the area form $i^*\mu_2$. $\qquad\square$

Remark 3.3.9. This proof also works to obtain the topology of the regular level sets in the four dimensional Euler equations studied in [81]. The way to obtain the closed and independent one forms is done as in [28].

Arnold's theorem shows that for non-constant Bernoulli functions, the situation is very similar to integrable systems. However for a constant B a contact structure appears and the situation is the opposite: a non integrable one. This case will be analyzed in Section 3.3.4.

3.3.3 Geometric structures on singular level sets

In this section we would like to understand the geometric structure induced on some of the singular level sets of the Bernoulli function. In the analytic setting a lot can be said about the structure of the level sets of B, both regular or singular with some assumptions, as studied in [3] and [43]. In Proposition 2.6 in [43] a topological classification of the singular level sets is obtained when B is analytic and X is assumed to be non-vanishing. Namely,

Proposition 3.3.10. *Let B be analytic and X a non-vanishing Euler flow on a closed 3-manifold M. Then, each singular level set of B (finitely many) is a finite union of embedded X-invariant sets that are periodic orbits, 2-tori, Klein bottles, open cylinders or open Möbius strips.*

In this section we shall assume that B is a Morse-Bott function instead of analytic and we shall not impose any assumptions on X. The standard Arnold's theorem is studied under analyticity assumptions. The next natural scenario would be to consider Morse-Bott functions as they are well-behaved at the critical set and are dense in the set of smooth functions. This assumption is not uncommon in our context: for instance Arnold's structure theorem is known to hold for Morse-Bott Bernoulli functions if the manifold has no boundary. In [68] the assumption considered is that of stratified singularities, which would go one step further. For Morse-Bott singular level sets we will see that some *b*-symplectic structures appear, which provides a new connection between these Poisson structures and physics. For these structures to appear, we need the existence of a singular submanifold in a level set thay might end up being the critical hypersurface of a *b*-symplectic manifold. For this to make sense we need level sets with a regular part and a singular one. Since the singularities are of Morse-Bott type, the only two options that admit this structure are the following two local forms for B:

1. An isolated critical point of saddle type: $B = x^2 + y^2 - z^2$.

2. A 1-dimensional critical set of saddle type: $B = x^2 - y^2$.

We are interested in the case where the level set is compact, which will have a topological singularity. In the first case the singularity is a point in a surface. In

the second case the singularity is a circle. In both cases there is a topological desingularization to obtain a manifold with a codimension one singular submanifold. The structure that we are interested in is the following. If μ is the Riemannian volume in the 3-manifold then the area form preserved by X in a level set of B is $i^*\mu_2$, as explained in the proof of Theorem 2.1.1, where

$$\mu_2 = \iota_{\frac{\nabla B}{|\nabla B|^2}}\mu,$$

and i is the inclusion of the level set of B into M. Let us answer to the following question: what kind of geometric structure is $i^*\mu_2$ in these desingularized singular level sets?

Case 1 Consider that B around the singularity looks like $B = x^2 + y^2 - z^2$. The volume form will be locally $\mu = dx \wedge dy \wedge dz$. The gradient of B is $(2x, 2y, -2z)$ and hence the vector field we are interested in is

$$X = \frac{\nabla B}{|\nabla B|^2} = \left(\frac{x}{x^2 + y^2 + z^2}\right)\frac{\partial}{\partial x} + \left(\frac{y}{x^2 + y^2 + z^2}\right)\frac{\partial}{\partial y} - \left(\frac{z}{x^2 + y^2 + z^2}\right)\frac{\partial}{\partial z}.$$

Denoting $r^2 = x^2 + y^2 + z^2$ and computing the two-form we obtain

$$\mu_2 = \frac{x}{r^2}dy \wedge dz - \frac{y}{r^2}dx \wedge dz - \frac{z}{r^2}dx \wedge dy.$$

Let $i : N \hookrightarrow M$ be the inclusion of the level set N into M, in coordinates $i : (\theta, \omega) \mapsto (\omega \cos\theta, \omega \sin\theta, \omega)$. A simple computation yields,

$$i^*\mu_2 = d\theta \wedge d\omega.$$

This already extends to an area form, one can think of it as a polar blow-up. However, to end up with a concrete smooth manifold we can also realize the topological singularity in a cylinder. Let σ be the desingularization

$$\begin{cases} x = u.w \\ y = v.w \\ z = w \end{cases}.$$

This desingularization sends the cylinder to the cone. Denote by σ the inverse of the previous transformation, defined in $w \neq 0$.

Letting $j : (\theta, \omega) \mapsto (\cos\theta, \sin\theta, \omega)$ be the inclusion of the cylinder, we obtain

$$j^*\sigma^*\mu_2 = d\theta \wedge d\omega,$$

which is a symplectic structure. This is a local model but using bump functions one obtains a symplectic surface globally defined.
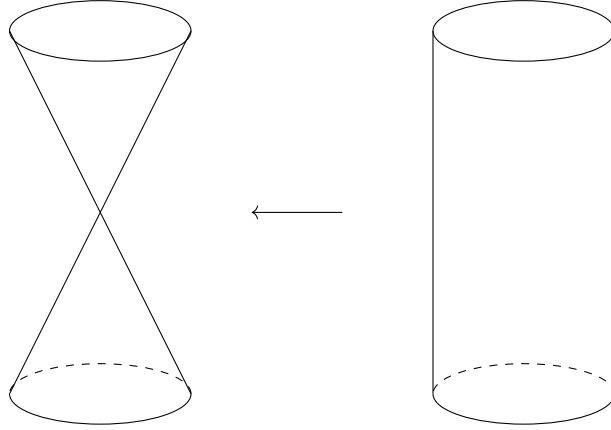
Figure 3.3: Case 1 desingularization

Case 2 Consider now a point in a 1-dimensional critical set of B of saddle type; hence the function looks locally as $B = x^2 - y^2$. Again the volume form will be written $\mu = dx \wedge dy \wedge dz$. The gradient of B is $(2x, -2y, 0)$ and the vector field is

$$X = \left(\frac{x}{x^2 + y^2}\right)\frac{\partial}{\partial x} - \left(\frac{y}{x^2 + y^2}\right)\frac{\partial}{\partial y}.$$

Denoting $r^2 = x^2 + y^2$ the two form is $\mu_2 = \frac{x}{r^2}dy \wedge dz + \frac{y}{r^2}dx \wedge dz$. The desingularization applied now is

$$\begin{cases} x = u.v \\ y = v \\ z = w \end{cases}.$$

It sends two separate planes to two intersecting planes. Denote by σ the inverse of the previous transformation, defined on $v \neq 0$. We are realizing the topological singularity of the level, and hence forgetting now about the function B. We analyze the structure of μ_2 after desingularization and restricted to the level set.

If j is the inclusion of any of the two planes, then we have

$$j^*\sigma^*\mu_2 = \frac{1}{v}dv \wedge d\omega,$$

where the change of sign depends on which one of the planes we consider. One could be confused by the change of induced orientations on the hyperplanes outside of the critical curve: However μ_2 was already well defined in the regular part of the level and there is not a problem of sign.

This model is local, but since the function is Morse-Bott, the desingularization is applied through a circle. Assuming that the negative (and positive) normal bundles of the singular set are orientable (to avoid problems as in [179]), the normal form $B = x^2 - y^2$ holds on a neighborhood of the critical circle (for
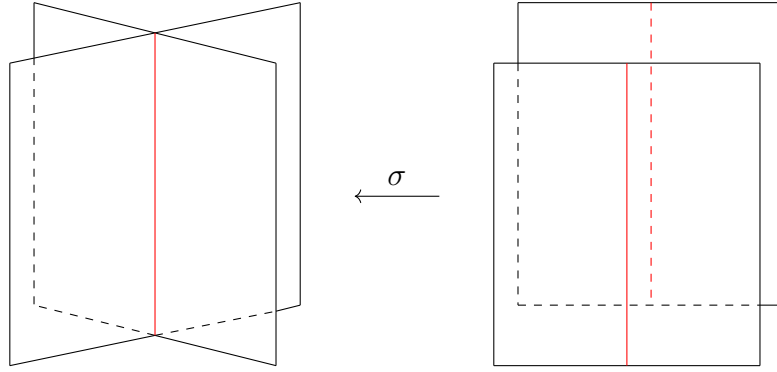
Figure 3.4: Case 2 desingularization

appropriate coordinates). Therefore one obtains globally a *b*-surface. Accordingly, we have produced a *b*-symplectic structure on each component, and globally a *b*-surface having two circles as critical set.

For the sake of simplicity in the analysis, we have used a model where the metric looks like the Euclidean one near the singular sets. This is true for *nice metrics* with respect to the Morse-Bott function B as the ones introduced in Hutching's thesis [111, 112]. Nevertheless, the qualitative picture described above is independent of this choice.

For our purposes, the most interesting situation is Case 2, where the singular locus is a whole curve. Observe that the area form μ_2 always satisfies the following identity:

$$dB \wedge \mu_2 = \mu. \tag{3.1}$$

By construction, when restricted to the planes obtained after the desingularization procedure, the form $i^*\mu_2$ is an area form that goes to infinity when approaching the critical curve Z. Letting $i : N \hookrightarrow M$ be the inclusion of any of the two planes $\{x = y\}$ or $\{x = -y\}$ we have

$$i^*\mu_2 = \frac{1}{f}\omega,$$

for a function $f \in N$ that vanishes along Z and an area form ω. Taking coordinates such that $\omega = du \wedge dv$ we can consider the dual vector field $\Pi_2 = f\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v}$. Observe also that equation (3.1) holds everywhere and $dB = 2xdx - 2ydy$ induces different orientations on each side of Z inside N.

Also dB vanishes when restricted to one of the planes $\{x = y\}$ or $\{x = -y\}$ in first order. This ensures that the pole of μ_2 is of order one since μ in M is a volume form. In the desingularized manifold, which is a surface, the bivector field defines a Poisson structure that vanishes along a curve in order 1 (which we call critical curve). Thus we have obtained a *b*-symplectic structure on the desingularized surface. Thus proving,
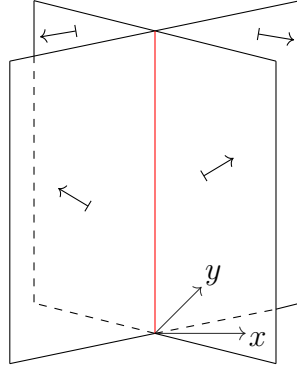
Figure 3.5: Orientation induced by dB

Proposition 3.3.11. *Singular sets of the second kind can be desingularized into surfaces with a b-symplectic structure that is preserved by the flow of the fluid.*

Remark 3.3.12. By using regularization-type techniques like in [142], one can produce artificially singularities or order $2k + 1$ for any $k \in \mathbb{N}$.

3.3.4 Beltrami fields in b-manifolds

When the Bernoulli function is constant, we know that we obtain Beltrami fields as described in 1.1.2. These are vector fields that are parallel to their vorticity i.e. $\operatorname{curl} X = f X$ for a function $f \in C^\infty(M)$. When f and X are non-vanishing we speak about nonsingular rotational Beltrami fields, and as described in Theorem 1.1.15 of Section 1.1.3, there is a correspondence between these fields and rescaled Reeb vector fields of contact structures is established.

We recall that the motivation for b-manifolds is studying manifolds with cylindrical ends. The critical surface captures the asymptotic behavior of geometric structures. We see a Riemannian manifold with a cylindrical end as the interior of a compact manifold with boundary.
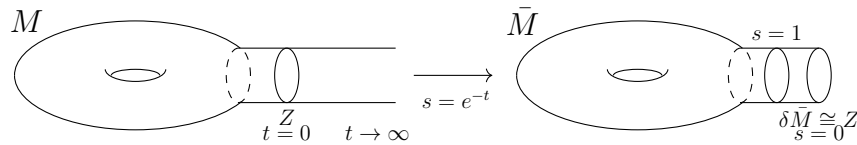
Figure 3.6: Compactification to a b-manifold

If we take the Euler equations in a manifold of this kind, one can consider them after the transformation to a b-manifold. The equations obtained are the same but with a resulting b-metric g and b-volume form μ that capture the asymptotical behavior of the geometric structures. Now working in the b-tangent and cotangent

bundles, in terms of the form $\alpha(\cdot) = g(X, \cdot)$ and the Bernoulli function, the Euler equations are still of the form:

$$\begin{cases} \iota_X d\alpha = -dB \\ d\iota_X \mu = 0, \end{cases}$$

The special case of non-vanishing rotational Beltrami fields is,

$$\begin{cases} d\alpha = f \iota_X \mu \\ d\iota_X \mu = 0, \end{cases}$$

for a non-vanishing function $f \in C^\infty(M)$. Following similar arguments as in the contact case, we can prove a correspondence between Beltrami fields in b-manifolds and b-contact structures. This is the formalization of the following idea: in the cylindrical manifold we can apply the usual correspondence of Beltrami fields with contact structures. When obtaining the b-manifold the interior admits again a contact structure, but the b-manifold looks only locally as the original manifold. Globally speaking the b-contact structure gives more information about the global asymptotic behavior close to the boundary.

Remark 3.3.13. For the discussion in this section let us put emphasis in a particularity of b-vector fields illustrating it with an example. Consider the b-manifold $(\mathbb{R}^2, Z = \{(x,y)|x=0\})$ with basis $\langle \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} \rangle$. The b-vector field $X = x\frac{\partial}{\partial x}$ vanishes in the usual sense of a vector field when $x = 0$. However as a section of the b-tangent bundle the term $(x\frac{\partial}{\partial x})$ is not vanishing. When paired with the dual form $\alpha = \frac{dx}{x}$ it satisfies $\alpha(X) = 1$ even in Z.

The statement of the b-Beltrami fields and b-contact correspondence is now presented.

Theorem 3.3.14. *Let M be a b-manifold of dimension three. Any rotational Beltrami field and non-vanishing as a section of ^bTM on M is a Reeb vector field (up to rescaling) for some b-contact form on M. Conversely given a b-contact form α with Reeb vector field X then any nonzero rescaling of X is a rotational Beltrami field for some b-metric and b-volume form on M.*

Proof. The proof is very similar to the one for usual Beltrami fields. One just needs to work with the b-tangent bundle ^bTM and its dual instead of the tangent bundle. Let X be a Beltrami field in (M, Z), a b-manifold of dimension three. For this implication we can follow [67]. Denote $e_1 = X/\|X\|$ which is globally defined as a b-vector field since X is a non-vanishing section of ^bTM and take a couple e_2, e_3 to have an orthonormal frame. Then consider $\alpha(\cdot) = g(X, \cdot) = \|X\|e^1$ where e^1 is the dual to e_1. This form is the dual of a b-vector field by a b-metric and hence defines a one b-form. Recall that $d\alpha = f\iota_X \mu$ which is also a b-form of degree 2 since it is a contraction of a b-vector field by a b-volume form.

The b-volume form has the form $\mu = he^1 \wedge e^2 \wedge e^3$ for a non-vanishing function $h \in C^\infty(M)$. Then it is clear that

$$\alpha \wedge d\alpha = g(X, \cdot) \wedge f\iota_x\mu = fh\|X\|^2 e^1 \wedge e^2 \wedge e^3 \neq 0.$$

Also $\iota_X d\alpha = f\iota_X\iota_X\mu = 0$ and X is a rescaled Reeb vector field of a b-contact structure given by α.

Conversely, consider a b-contact form α and a rescaling of its Reeb vector field $Y = hR$ for $h \in C^\infty(M)$ a non-vanishing function. We will follow the idea in [79] for this implication. Using the Darboux theorem for b-contact forms, Theorem 3.1.14, the subbundle $\ker\alpha$ is generated by $\xi_p = \langle z\frac{\partial}{\partial z}, \frac{\partial}{\partial y_1}\rangle$ or $\xi_p = \langle -\frac{\partial}{\partial x_1} + z\frac{\partial}{\partial z}, \frac{\partial}{\partial y_1}\rangle$ if the Reeb vector field does not vanish as a smooth vector field. When R vanishes as a smooth vector field (but not as a ${}^bTM$ section) then $R = z\frac{\partial}{\partial z}$ and $\xi_p = \langle \frac{\partial}{\partial x_1}, x_1z\frac{\partial}{\partial z} - \frac{\partial}{\partial y_1}\rangle$. Hence ξ is a vector bundle of constant rank 2 over M, it is indeed a subbundle of ${}^bTM$. Recall now from [137] the following proposition:

Proposition 3.3.15. *Let $E \to M$ be a $2n$-dimensional vector bundle with a non degenerate bilinear form ω_q in each fiber E_q which varies smoothly with $q \in M$. Then there exists an almost complex structure which is compatible with ω, i.e. such that $\omega(\cdot, J\cdot)$ is positive definite.*

Let g be the b-metric

$$g(u, v) = \frac{1}{h}(\alpha(u) \otimes \alpha(v)) + d\alpha(u, Jv).$$

The vector field Y satisfies $\iota_Y g = \alpha$. Take $\mu = \frac{1}{h}\alpha \wedge d\alpha$ as b-volume form in M. It obviously satisfies $\iota_Y\mu = d\alpha$. Hence Y is a Beltrami field (with constant proportionality factor) for this choice of g and μ.

$\square$

This correspondence was used, in a later work [144] by Miranda, Oms and Peralta-Salas, to prove the generic existence of escape orbits in the context of the singular Weinstein conjecture for b-contact manifolds.

Example 3.3.16 (ABC flows)**.** *A very well-known family of Beltrami flows in $\mathbb{T}^3$ are the ABC flows:*

$$X(x, y, z) = [A\sin z + C\cos y]\frac{\partial}{\partial x} + [B\sin x + A\cos z]\frac{\partial}{\partial y} + [C\sin y + B\cos x]\frac{\partial}{\partial z}.$$

Everything is computed in $\mathbb{R}^3$ and then quotiented depending on which hypersurface we consider. Taking as hypersurface $Z = \{z = 0\}$ one can check for which values of the parameters the b-vector field

$$X(x, y, z) = [A\sin z + C\cos y]\frac{\partial}{\partial x} + [B\sin x + A\cos z]\frac{\partial}{\partial y} + [C\sin y + B\cos x]z\frac{\partial}{\partial z}$$

is a Beltrami field in the corresponding b-manifold. The metric and volume forms are

$$g = dx^2 + dy^2 + (\frac{dz}{z})^2, \ \ \mu = dx \wedge dy \wedge \frac{dz}{z}.$$

We compute the one form

$$\alpha = g(X, \cdot) = [A \sin z + C \cos y] dx + [B \sin x + A \cos z] dy + [C \sin y + B \cos x] \frac{dz}{z},$$

and the contraction by the volume

$$\iota_X \mu = [-B \sin x - A \cos z] dx \wedge \frac{dz}{z} + [A \sin z + C \cos y] dy \wedge \frac{dz}{z} + [C \sin y + B \cos x] dx \wedge dy.$$

It is clear that $d\iota_X \mu = 0$, it remains to check the equation $d\alpha = f \iota_X \mu$. Computing the derivative of alpha

$$d\alpha = [-B \sin x - zA \cos z] dx \wedge \frac{dz}{z} + [zA \sin z + C \cos y] dy \wedge \frac{dz}{z} + [C \sin y + B \cos x] dx \wedge dy.$$

When differentiating with respect to z in the b-cotangent bundle, a z factor appears. For $d\alpha = f \iota_X \mu$ to be satisfied, we need $A = 0$ and $f = 1$. The two-parameter family of vector fields

$$X(x, y, z) = C \cos y \frac{\partial}{\partial x} + B \sin x \frac{\partial}{\partial y} + [C \sin y + B \cos x] z \frac{\partial}{\partial z}$$

is b-Beltrami on the b-manifold $\mathbb{T}^2 \times \mathbb{R}$ with a $\mathbb{T}^2$ as critical hypersurface. To obtain a vector field in a compact manifold, one can chose $\sin z$ instead of z as defining function of the critical set (which is now defined in the quotient to $\mathbb{T}^3$) and hence work with $\sin z \frac{\partial}{\partial z}$ and $\frac{dz}{\sin z}$. We obtain a Beltrami field on $\mathbb{T}^3$ with two $\mathbb{T}^2$ as critical hypersurfaces. It is an easy computation to check that the b-vector field X is non-vanishing as a section of ^bTM if and only if $|B| \neq |C|$.

Remark 3.3.17. For this *b*-manifold the corresponding original manifold can be thought as $M = \mathbb{R} \times \mathbb{T}^2$. When compactifying each of the cylindrical ends we obtain a manifold diffeomorphic to $\mathbb{T}^2 \times [0,1]$. When considering its double the resulting *b*-manifold is $\mathbb{T}^3$ with two $\mathbb{T}^2$ as critical hypersurfaces.

As an example of b-contact structure, let us compute it in the simple case $C = 0$ and $B > 0$ for ABC fields in $\mathbb{T}^3$, i.e. the defining function is $\sin z$. The one b-form α in this case is

$$\alpha = g(X, \cdot) = B \sin x \, dy + B \cos x \frac{dz}{\sin z}.$$

It is clearly a b-contact structure since

$$\alpha \wedge d\alpha = B^2 dx \wedge dy \wedge \frac{dz}{\sin z},$$

and its Reeb vector field is $R = \frac{1}{B} \sin x \frac{\partial}{\partial y} + \frac{1}{B} \cos x \sin z \frac{\partial}{\partial z}$, a rescaling of the original Beltrami field.

Remark 3.3.18. This correspondence holds true if we consider Beltrami fields on b^m-manifolds. The associated structure is then a b^m-contact structure.

The Weinstein conjecture [198] on periodic orbits of Reeb flows claims that any Reeb vector field admits a periodic orbit on a compact manifold. In [143] the authors conjecture a singular version of the Weinstein conjecture claiming that the Reeb vector field of any compact b-contact manifold possesses at least one periodic orbit which may be singular in the following sense.

Definition 3.3.19. *Let M be a manifold with hypersurface Z. A singular periodic orbit is either a periodic orbit in $M \backslash Z$ or an orbit γ such that $\lim_{t \to \pm\infty} \gamma(t) \in Z$.*

One could obtain information about the stream lines of a *b*-Beltrami flow depending on the possible casuistics that this conjecture opens. In particular, this would allow to establish the existence of either a periodic orbit or an unbounded orbit that escapes (in both directions) through a cylindrical end. In [144], the correspondence established in this work is used to prove some cases of this singular Weinstein conjecture.
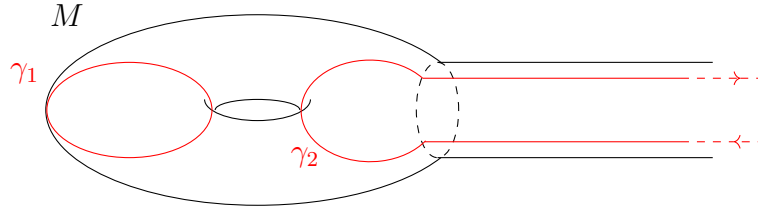

Figure 3.7: Two possible singular periodic orbits

Chapter 4

Integrability in Hamiltonian systems

In this chapter, we focus on integrable systems in symplectic and singular symplectic manifolds. We give a new point of view to Liouville's theorem for integrable systems in symplectic geometry, using as approach a classical theorem in differential topology proved by Tischler. We define the appropiate notion of Hamiltonian and integrable Hamiltonian dynamics on folded symplectic manifolds and prove an action-angle theorem in this context. We end up the discussion by studying the existence of b-integrable systems on b-symplectic manifolds, and address the problem of finding global action-angle coordinates in singular symplectic manifolds. The content of this chapter is based on [28] and [30].

4.1 Integrable systems

Integrable systems are Hamiltonian systems with the maximal number of additional first integral, $n-1$ where $2n$ is the dimension of the ambient manifold. From a geometrical point of view, Lie group theory helps capturing this symmetry and leads to very well known normal forms such as action-angle coordinates. In this section, we introduce these results both for symplectic forms and b-symplectic forms.

4.1.1 Integrable systems in regular symplectic manifolds

Let us recall the definition of integrable system on a symplectic manifold.

Definition 4.1.1. *An integrable system on a symplectic manifold (M^{2n}, ω) is a set of n functions $f_1, ..., f_n$ generically functionally independent (i.e. $df_1 \wedge ... \wedge df_n \neq 0$ on a dense set) which Poisson commute: $\{f_i, f_j\} = \omega(X_{f_i}, X_{f_j}) = 0, \forall i, j$.*

Here X_{f_i} denotes the Hamiltonian vector field of f_i, which is defined by the equation $\iota_{X_{f_i}} \omega = -df_i$. The integrability of the system is in fact related to the

fact that the equations of motion are integrable by quadratures. A way to think of integrable systems is as the "most symmetric" Hamiltonian systems.

The vector fields $X_{f_1}, ..., X_{f_n}$ are all tangent to $F^{-1}(p)$, where $F = (f_1, ..., f_n)$. Since $\omega(X_{f_i}, X_{f_j}) = 0 \ \forall i, j$, and the tangent space of $F^{-1}(p)$ is generated by the X_{f_i}, we deduce that ω vanishes when restricted to $L = F^{-1}(p)$.

Definition 4.1.2. *The particular case where the dimension of this submanifold is $1/2 \ dim(M)$ is called a Lagrangian submanifold. All the lagrangian submanifolds (the level sets) form a Lagrangian fibration.*

Example 4.1.3. An easy example of a mechanical system which is also an integrable system is the simple pendulum. Indeed, in dimension 2 a single Hamiltonian function can define an integrable system. The manifold where the pendulum moves is S^1 and we can look its cotangent bundle as $T^*S^1 \cong [0, 2\pi]_\sim \times \mathbb{R}$ knowing that points at $(0, \xi)$ are identified with $(2\pi, \xi)$. Taking coordinates (θ, ξ) with θ the oriented angle between the rod and the vertical direction and ξ the velocity induced by θ. The cotangent bundle comes equipped with the symplectic form $\omega = d\theta \wedge d\xi$.
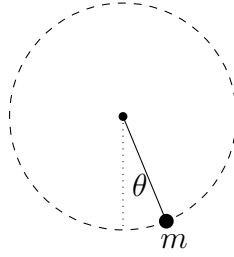
Figure 4.1: Simple pendulum

To simplify, let us assume that the mass and the length of the rod are 1. The Hamiltonian function for this system is

$$H(\theta, \xi) = \frac{\xi^2}{2} + 1 - \cos\theta.$$

Imposing $\iota_{X_H}\omega = -dH$ we get

$$X_H = -\xi\frac{\partial}{\partial\theta} + \sin\theta\frac{\partial}{\partial\xi}.$$

Some of the Lagrangian fibers in the plane (θ, ξ) are depicted in Figure 4.2.

As we can see, the lagrangian submanifolds are diffeomorphic to S^1, a 1-dimension torus. This is true only for regular values of the Hamiltonian, of course. If we consider the value 0, which is a singular point, the preimage is not an S^1 but a point.
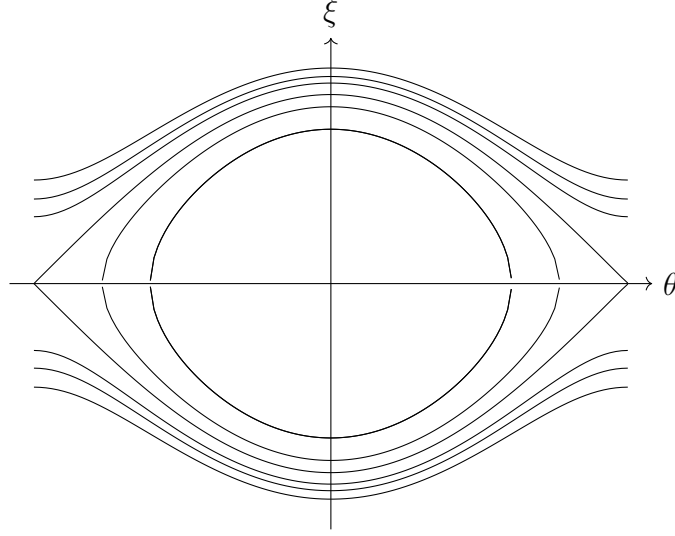
Figure 4.2: Level sets of H

The most important result about integrable systems in symplectic manifolds is the socalled Arnold-Liouville theorem [8]. This theorem describes semi-locally the integrable system around regular points.

Theorem 4.1.4 (Arnold-Liouville). *Let (M^{2n}, ω) be a symplectic manifold and $F = (f_1, ..., f_n)$ an integrable system. Let p be a regular point (i.e. $df_1 \wedge ... \wedge df_n(p) \neq 0$). Note $F(p) = c$ and $F^{-1}(c) = L_c$ (fibre associated to c). Assuming L_c is compact and connected, then*

1. *$L_c \cong T^n$*

2. *the Liouville foliation is trivial in some neighborhood of the Liouville torus, that is, a neighborhood U of the torus L_c is the direct product of T^n and the disc D^n.*

3. *In a neighborhood of L_c, $U(L_c)$, there exist coordinates $(\theta_1, ..., \theta_n, p_1, ..., p_n)$ such that ω is written $\omega = \sum dp_i \wedge d\theta_i$. F only depends on $p_1, ..., p_n$.*

We will explain a non-classical proof of the first statement in Section 4.2.

4.1.2 b-integrable systems

Hamiltonian dynamics can be defined for b-symplectic structures. The classical definition for general Poisson structure Π associates to a given smooth function $f \in C^\infty(M)$ a vector field X via the equation

$$X = \Pi(df, \cdot).$$

However, this imposes that the maximum amount of independent and commuting Hamiltonian vector fields at a point corresponds to half the rank of Π at that point. A natural extension of the set of functions allows to obtain up to n independent Hamiltonian vector fields at any point in a b-symplectic manifold of dimension $2n$.

Definition 4.1.5. *(b-functions) A b-function on a b-manifold (M, Z) is a function which is smooth away from the critical set Z, and near Z has the form*

$$c \log |t| + g,$$

where $c \in \mathbb{R}, g \in C^\infty$, and t is a local defining function. The sheaf of b-functions is denoted ${}^b C^\infty$.

For any b-function on a b-symplectic manifold (M, ω) the b-**Hamiltonian** vector field is the one X_{f_n} defined by $\iota_{X_{f_n}} \omega = -df_n$. The singular part of a b-function generates an extra direction on the tangent space via Hamiltonian vector fields, which is transverse to the symplectic foliation along Z. This allows to obtain b-Hamiltonian toric actions which span orbits of dimension n along points which lie in Z. In this spirit, a new definition of integrable system supersedes the definition for general Poisson manifolds.

Definition 4.1.6. *A b-**integrable system** on a $2n$-dimensional b-symplectic manifold (M^{2n}, ω) is a set of n b-functions which are pairwise Poisson commuting $F = (f_1, \ldots, f_{n-1}, f_n)$ with $df_1 \wedge \cdots \wedge df_n \neq 0$ as a section of $\wedge^n({}^b T^*(M))$ on a dense subset of M and on a dense subset of Z. A point in M is **regular** if the vector fields $X_{f_1}, \ldots, X_{f_n}$ are linearly independent (as* smooth *vector fields) at it.*

For these systems an action-angle coordinate, proved in [123], shows the existence of a semilocal invariant in the neighbourhood of Z (the modular period):

Theorem 4.1.7. *Let $(M, \omega, F = (f_1, \ldots, f_{n-1}, f_n = \log |t|))$ be a b-integrable system, and let $m \in Z$ be a regular point for which the integral manifold containing m is compact, i.e. a Liouville torus F_m. Then there exists an open neighborhood U of the torus F_m and coordinates $(\theta_1, \ldots, \theta_n, \sigma_1, \ldots, \sigma_n) : U \to \mathbb{T}^n \times B^n$ such that*

$$\omega|_U = \sum_{i=1}^{n-1} d\sigma_i \wedge d\theta_i + \frac{c}{\sigma_n} d\sigma_n \wedge d\theta_n, \qquad (4.1)$$

where the coordinates $\sigma_1, \ldots, \sigma_n$ depend only on F and the number c is the modular period of the component of Z containing m.

In [121] this normal form was identified as a cotangent model:

Theorem 4.1.8. *Let $F = (f_1, \ldots, f_n)$ be a b-integrable system on the b-symplectic manifold (M, ω). Then semilocally around a regular Liouville torus $\mathbb{T}$, which lies inside the exceptional hypersurface Z of M, the system is equivalent to the cotangent model $(T^* \mathbb{T}^n)_{tw,c}$ restricted to a neighbourhood of $(T^* \mathbb{T}^n)_0$. Here c is the modular period of the connected component of Z containing $\mathbb{T}$.*

This cotangent model uses a generalization of the cotangent lift to b-symplectic manifolds.

The (twisted) b-cotangent lift

The cotangent lift can also be defined on the b-cotangent bundle of a smooth manifold. In this case there are two different 1-forms that provide the same geometrical structure on the b-cotangent bundle (a b-symplectic form). These are the canonical (Liouville) b-form and the twisted b-form. Both forms of degree 1 have the same differential (a smooth b-symplectic form) but are indeed non-smooth forms. The b-cotangent lift in each of the cases is defined in a different manner. These were studied in detail in [121]. In this work we focus on the twisted b-cotangent lift as it gives the right model for the structure of a b-integrable system.

Definition 4.1.9. *Let $T^*\mathbb{T}^n$ be endowed with the standard coordinates (θ, a), $\theta \in \mathbb{T}^n$, $a \in \mathbb{R}^n$ and consider again the action on $T^*\mathbb{T}^n$ induced by lifting translations of the torus $\mathbb{T}^n$. Define the following non-smooth one-form away from the hypersurface $Z = \{a_1 = 0\}$:*

$$\lambda_{tw,c} \log |a_1| d\theta_1 + \sum_{i=2}^{n} a_i d\theta_i.$$

Then, the form $\omega := -d\lambda_{tw,c}$ is a b-symplectic form on $T^*\mathbb{T}^n$, called the twisted b-symplectic form on $T^*\mathbb{T}^n$. In coordinates:

$$\omega_{tw,c} := \frac{c}{a_1} d\theta_1 \wedge da_1 + \sum_{i=2}^{n} d\theta_i \wedge da_i. \tag{4.2}$$

Observe that this twisted forms comes endowed with a local invariant: The constant c. The interpretation of this invariant is that this gives the period of the modular vector field.

We call the lift together with the b-symplectic form (4.2) the **twisted b-cotangent lift** with modular period c on the cotangent space of a torus.

This was deeply studied in [121], where it is shown that the lifted action can be extended to groups of type $S^1 \times H$ which turns out to be b-Hamiltonian in general.

4.2   Closed one forms and regular fibers

Arnold-Liouville's theorem on integrable systems (cf. Theorem 4.1.4) asserts that a neighborhood of a compact invariant subset (Liouville torus) of an integrable system on a symplectic manifold (M^{2n}, ω) is fibred by other Liouville tori. Furthermore, the symplectic form can be described as the Liouville symplectic structure on $T^*(\mathbb{T}^n)$ in adapted coordinates to the fibration which can be described using the cotangent lift of translations of the base torus. In particular in adapted coordinates (action-angle) the moment map is indeed a moment map of a Hamiltonian toric action. This theorem admits generalizations to the Poisson setting [126, 123].

The fact that the fibers of the moment map are tori is a key point in the theory and it is a purely topological result. This topological result often attributed to Liouville [131] was indeed probably first observed by Einstein [56]. Probably, the best well-known proof of this fact (see for instance [52]) uses the existence of a toric action associated to the joint flow of the distribution of the Hamiltonian vector fields of the first integrals and the identification of the Liouville tori as orbits of this action. The classical proof is rich because it describes not only the Liouville torus but closeby tori but somehow diverts from the topological nature of Liouville fibers.

Given an integrable system the set of n 1-forms associated to the first integrals df_i defines a set of closed 1-forms. These closed 1-forms are constant on the fibers of the associated *moment map*. In this paper we pay attention to the following fact, the regular fibers of an integrable system are naturally endowed with n independent closed 1-forms defined using symplectic duality from the the constants of motion. This fact together with a generalization of a result of Tischler for compact manifolds admitting k closed forms yields a new proof of the classical Liouville theorem. This new topological proof of Liouville theorem works in the Poisson and non-commutative setting as well. In this paper, we concentrate on topological aspects of foliations associated to a set of k-closed one forms in the framework of integrable systems. We suspect that a detailed dual proof of the action-angle Theorem 4.1.4 may be obtained as a consequence of this dual viewpoint.

4.2.1   A topological result for manifolds with closed forms

In this first section we prove a generalization of a result of Tischler on manifolds with closed forms. We start recalling the following theorem by Tischler:

Theorem 4.2.1 (Tischler theorem). *Let M^n be a compact manifold admitting a nowhere vanishing closed 1-form ω, then M^n is a fibration over S^1.*

In this first section we prove the following generalization of Tischler's theorem which is stated without proof for foliations without holonomy in his paper [190]:

Theorem 4.2.2. *Let M^n be a compact connected manifold admitting k closed 1-forms $\beta_i, i = 1, \ldots, k$ which are linearly independent at every point of the manifold, then M^n fibers over a torus $\mathbb{T}^k$.*

We will need the following lemma:

Lemma 4.2.3 (Ehresmann lemma [55]). *A smooth mapping $f : M^m \longrightarrow N^n$ between smooth manifolds M^m and N^n such that:*

  1. f is a surjective submersion, and

  2. f is a proper map

is a locally trivial fibration.

Proof of Theorem 4.2.2. We start by proving that the cohomology classes $\{[\beta_i]\}_{i=1}^k$ in $H^1(M^n, \mathbb{R})$ are all different. Assume the opposite β_i and β_j with $i \neq j$ such that $[\beta_i] = [\beta_j]$. Then there exists $f \in C^\infty(M^n)$ such that

$$\beta_i = \beta_j + df. \tag{4.3}$$

Since the 1-forms β_i are linearly independent the k-form $\beta_1 \wedge ... \wedge \beta_k$ is nowhere vanishing. Using equation 4.3 we obtain

$$\beta_i \wedge \beta_j = \beta_i \wedge (\beta_i + df) = \beta_i \wedge df. \tag{4.4}$$

But note that due to Weierstrass theorem f has a maximum and a minimum on a compact manifold, thus $\beta_i \wedge \beta_j$ vanishes at these points (where $df = 0$). This contradicts the fact that $\beta_1 \wedge ... \wedge \beta_k$ is nowhere vanishing.

Denote b_1 the first Betti number of M^n and θ the usual angular coordinate in S^1. It is well known that there exist b_1 maps $g_j : M^n \to S^1$ such that the set of 1-forms $g_j^*(d\theta)$ define a set of cohomology classes $[g_j^*(d\theta)]$ which is a basis of $H_{DR}^1(M^n, \mathbb{R})$. With this basis, we can express β_i as:

$$\beta_i = \sum_{j=1}^{b_1} a_{ij}\nu_j + dF_i, \text{ for } i = 1, ..., k.$$

Using the argument on Tischler theorem proof [190], we can choose appropriate $q_{ij} \in \mathbb{Q}$ $\forall i, j$, such that $\tilde{\beta}_i = \sum_{j=1}^p q_{ij}\nu_j + dF_i$ are still non-singular and independent. Taking suitable $N_i \in \mathbb{Z}$ we obtain forms $\beta_i' = N_i\tilde{\beta}_i$ such that

$$\beta_i' = \sum_{j=1}^p k_{ij}\nu_j + dH_i,$$

where $k_{ij} = N_i q_{ij} \in \mathbb{Z}$ and $H_i = N_i F_i \in C^\infty(M)$. Of course, they are also non singular and independent.

Without loss of generality we can assume $dH_i = 0$. Indeed, the image $H_i \in C^\infty(M^n)$ is contained in a closed interval because M^n is compact. Functions H_i quotients to S^1 with a projection π, and we can redefine $g_i := g_i + \pi \circ H_i$ for $i = 1, ..., k$.

Recall that the basis ν_j is defined as $\nu_j = g_j^*(d\theta) = d(\tilde{g}_j)$, with $\tilde{g}_j = \theta \circ g_j$. Hence the forms β_i' can be written

$$\beta_i' = d(\sum_{j=1}^p p_{ij}\tilde{g}_j).$$

If we define the functions $\theta_i = \sum_{j=1}^p p_{ij}\tilde{g}_j$, then the induced mappings on the quotient $\tilde{\theta}_i : M^n \longrightarrow S^1$ are k submersions of M^n to S^1. Consider

$$\Theta : M^n \longrightarrow S^1 \times ... \times S^1 = \mathbb{T}^k$$
$$p \longmapsto (\tilde{\theta}_1(p), ..., \tilde{\theta}_k(p)).$$

Since the forms β_i' are independent in $H^1(M^n, \mathbb{R})$ this implies $d\theta_i$ are independent seen as one-forms from M^n to $\mathbb{R}^k$ and so $d\tilde{\theta}_i$ are also independent into $\mathbb{T}^k$, this implies that Θ is a surjective submersion. Since M^n is compact, we can apply Ehresmann lemma (Lemma 4.2.3) and Θ defines a locally trivial fibration. □

When $k = n$ we obtain the following as a corollary:

Corollary 4.2.4. *Let M^n be a compact connected manifold admitting n closed 1-forms $\beta_i, i = 1, \ldots, n$ which are which are linearly independent at every point of the manifold, then M^n is diffeomorphic to a torus $\mathbb{T}^n$.*

Proof. Applying Theorem 2.3, M^n fibers over a torus $\mathbb{T}^n$. From the invariance of domain theorem it is an immersion because the target space is n-dimensional too. Thus Θ defines a covering map but since M^n is connected it defines a diffeomorphism

$$M^n \cong \mathbb{T}^n.$$

□

4.2.2 Applications to regular integrable systems

One of the best well-known theorem of integrable systems is *Liouville-Mineur-Arnold theorem* which roughly speaking asserts that the fibers of the map F defined by the first integrals describe a fibration by tori (if the ambient manifold is compact and the fibers are regular) and also that there exists privileged coordinates (called action-angle coordinates) in which the symplectic form can be expressed in a unique Darboux chart in a neighborhood of one of these tori.

The first statement of Liouville-Mineur-Arnold theorem says that a compact connected regular set of an integrable system is in fact a torus of dimension n. This theorem has been attributed to Liouville for a long time but it was indeed probably first proved by Einstein [56]. The theorem remains valid when we consider an integrable system on a Poisson manifold but also for the so-called non-commutative systems.

In this section we apply Corollary 4.2.4 to reprove that the fibers are tori for integrable systems on symplectic and Poisson manifolds. The tools used for this new proof differ from the classical tools where a torus action is used [52, 126].

Liouville tori of integrable systems on symplectic manifolds

Recall the definition of an integrable system as well as the Liouville-Mineur-Arnold theorem.

Definition 4.2.5. *An **integrable system** on a symplectic manifold (M^{2n}, ω) is a set of n functions $f_1, ..., f_n$ generically functionally independent (i.e. $df_1 \wedge ... \wedge df_n \neq 0$ on a dense set) and pairwise commuting with respect to the Poisson bracket $\{f_i, f_j\} = \omega(X_{f_i}, X_{f_j}) = 0, \forall i, j.$*

A point p is called regular point for the integrable system if $df_1 \wedge ... \wedge df_n(p) \neq 0$.

Theorem 4.2.6. *Let (M^{2n}, ω) be a symplectic manifold and $F = (f_1, ..., f_n)$ an integrable system. Let p be a regular point denote $F(p) = c$ and assume $L_c = F^{-1}(c)$ is compact and connected, then*

1. *$L_c \cong \mathbb{T}^n$,*

2. *A neighborhood U of the torus L_c is the direct product of $\mathbb{T}^n$ and the disc D^n, and the fibration given by F coincides with the projection on the disc.*

3. *in a neighborhood of L_c, $U(L_c)$, there exist coordinates $(\theta_1, ..., \theta_n, p_1, ..., p_n)$ such that ω is written $\omega = \sum_{i=1}^n dp_i \wedge d\theta_i$ and F only depends of $p_1, ..., p_n$.*

Denote by L^n any connected component of $F^{-1}(c)$ (or all of it if assumed connected) and we also assume it is compact. Denote by X_i the Hamiltonian vector associated to f_i. Observe that

$$\begin{aligned}
0 &= \{f_i, f_j\} \\
&= \omega(X_i, X_j) \\
&= \iota_{X_i} \omega(X_j) \\
&= -df_i(X_j) = -X_j(f_i) \ \ \forall i, j = 1, ..., n.
\end{aligned}$$

and the vector fields $X_1, ..., X_n$ are tangent to L^n for all $p \in L^n$. So we can indeed write $T(L^n)_p = \langle X_{f_1}, ..., X_{f_n} \rangle_p$. Take now in $\mathbb{R}^n$ the canonical basis of vector fields $\{\partial_i = \frac{\partial}{\partial x_i}\}_{i=1}^n$ on $\mathbb{R}^n$ and consider their pullbacks by F, $S_i := F^*(\partial_i)$, which are vector fields on M^{2n} which are transverse to L^n. They satisfy:

$$S_i(f_j) = \delta_{ij}.$$

They are determined by this condition modulo $T_p L^n$.

Lemma 4.2.7. *Let $j : L^n \longrightarrow M^{2n}$ be the inclusion of the regular level set L^n into M^{2n}. Define the one-forms $\alpha_i = \iota_{S_i} \omega$. Then the one-forms $\beta_i = j^* \alpha_i$ are closed.*

Proof. By definition of S_i, we have $S_i(f_j) = \delta_{ij}$. Applying it $\forall i, j$:

$$\begin{aligned}
\alpha_i(X_j) &= \omega(S_i, X_j) \\
&= -\omega(X_j, S_i) \\
&= -\iota_{X_j} \omega(S_i) \\
&= df_j(S_i) = S_i(f_j) = \delta_{ij}.
\end{aligned}$$

To prove that β_i is closed, we just have to check that $d\alpha_i(X_i, X_j) = 0$ for all X_i, X_j sections of TL^n.

$$\begin{aligned}
d\alpha_i(X_j, X_k) &= X_k(\alpha_i(X_j)) - X_j(\alpha_i(X_k)) - \alpha_i([X_j, X_k]) \\
&= X_k(\delta_{ij}) - X_j(\delta_{ik}) - \alpha_i(0) = 0.
\end{aligned}$$

We conclude that $d(j^* \alpha_i) = d\beta_i = 0$ and so our forms β_i are closed in L^n. $\square$

Lemma 4.2.8. *The* 1*-forms* $\beta_1, ..., \beta_n$ *are linearly independent at all points of* L^n.

Proof. As seen in the previous lemma, we have that $\beta_i(X_j) = \delta_{ij}$. We deduce that $\beta_i = X_i^*$, by definition of dual basis. Since $X_1, ..., X_n$ form a basis of the tangent space at every point in L^n, we have that $\beta_1, ..., \beta_n$ form a basis of the cotangent space at every point in L^n. In particular all β_i are independent at all points of L^n. □

We now prove Liouville theorem using Corollary 4.2.4.

Theorem 4.2.9 (Liouville's theorem)**.** *The compact regular fiber of an integrable system on* (M^{2n}, ω) *is diffeomorphic to a torus* $\mathbb{T}^n$.

Proof. The forms β_i in the preceding lemma are n closed 1-forms which are independent at all points of L^n. Applying theorem 4.2.4, $L^n \cong \mathbb{T}^n$. □

Commutative and non-commutative integrable systems on Poisson manifolds

A Poisson manifold is a pair (M, Π) where Π is a bi-vector field with an associated bracket on functions

$$\{f, g\} := \Pi(df, dg), \quad f, g : M \to \mathbb{R}$$

satisfying the Jacobi identity. This is equivalent to the integrability equation $[\Pi, \Pi] = 0$. The Hamiltonian vector field of a function f in this context is defined as $X_f := \Pi(df, \cdot)$.

Poisson manifolds constitute a generalization of symplectic manifolds and it generalizes very natural structures such that of linear Poisson structures associated to the dual of a Lie algebra. Integrability of Hamiltonian systems in the Poisson setting is a rich field which is naturally connected to representation theory (Gelfand-Ceitlin systems on $\mathfrak{u}(n)^*$). We recall the notion of integrable system for Poisson manifolds.

Definition 4.2.10. *Let* (M, Π) *be a Poisson manifold of (maximal) rank* $2r$ *and of dimension* n. *An* s-*tuplet of functions* $\mathbf{F} = (f_1, \ldots, f_s)$ *on* M *is said to define a* Liouville integrable system *on* (M, Π) *if*

1. $f_1, \ldots, f_s$ *are independent (i.e., their differentials are independent on a dense open set),*

2. $f_1, \ldots, f_s$ *are pairwise in involution and* $r + s = n$.

The map $\mathbf{F} : M \to \mathbb{R}^s$ *is called the* moment map *of* $(M, \Pi, \mathbf{F})$.

A point m is called regular whenever $\mathrm{d}_m f_1 \wedge \ldots \wedge \mathrm{d}_m f_s \neq 0$. Observe that complete integrability in the Poisson context also implies that the distribution generated by $X_{f_1}, \ldots, X_{f_s}$ is integrable in a neighborhood of a regular point in the sense of Frobenius because $[X_{f_i}, X_{f_j}] = X_{\{f_i, f_j\}}$.

For integrable systems on Poisson manifolds it is possible to prove that the compact leaves of this distribution are tori and indeed to prove an action-angle theorem in a neighborhood of a regular torus (see [126]).

Theorem 4.2.11. *Let (M, Π) be a Poisson manifold of dimension n of maximal rank $2r$. Suppose that $\mathbf{F} = (f_1, \ldots, f_s)$ is an integrable system on (M, Π), i.e., $r + s = n$ and the components of $\mathbf{F}$ are independent and in involution. Suppose that $m \in M$ is a point such that*

(1) $\mathrm{d}_m f_1 \wedge \ldots \wedge \mathrm{d}_m f_s \neq 0$;

(2) The rank of Π at m is $2r$;

(3) The integral manifold $\mathcal{F}_m$ of the distribution generated by $X_{f_1}, \ldots, X_{f_s}$, passing through m, is compact.

Then there exists $\mathbb{R}$-valued smooth functions $(\sigma_1, \ldots, \sigma_s)$ and $\mathbb{R}/\mathbb{Z}$-valued smooth functions $(\theta_1, \ldots, \theta_r)$, defined in a neighborhood U of $\mathcal{F}_m$ such that

1. The manifold $\mathcal{F}_m$ is a torus $\mathbb{T}^r$.

2. The functions $(\theta_1, \ldots, \theta_r, \sigma_1, \ldots, \sigma_s)$ define an isomorphism $U \simeq \mathbb{T}^r \times B^s$;

3. The Poisson structure can be written in terms of these coordinates as

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial \sigma_i},$$

in particular the functions $\sigma_{r+1}, \ldots, \sigma_s$ are Casimirs of Π (restricted to U);

4. The leaves of the surjective submersion $\mathbf{F} = (f_1, \ldots, f_s)$ are given by the projection onto the second component $\mathbb{T}^r \times B^s$, in particular, the functions $\sigma_1, \ldots, \sigma_s$ depend on the functions $f_1, \ldots, f_s$ only.

The functions $\theta_1, \ldots, \theta_r$ are called **angle coordinates**, the functions $\sigma_1, \ldots, \sigma_r$ are called **action coordinates** and the remaining functions $\sigma_{r+1}, \ldots, \sigma_s$ are called **transverse coordinates**.

We can apply Tichler's trick to reprove the orbits of a non-commutative integrable systems are tori in the Poisson setting (and deduce the result for the commutative particular case). We start recalling some definitions from [126].

Definition 4.2.12. *Let (M, Π) be a Poisson manifold. An s-tuple of functions $F = (f_1, \ldots, f_s)$ on M is a **non-commutative (Liouville) integrable system of rank $r \leq s$ on (M, Π)** if*

1. $f_1, \ldots, f_s$ are independent (i.e. their differentials are independent on a dense open subset of M) and the Hamiltonian vector fields of the functions $f_1, \ldots, f_r$ are linearly independent at some point of M.

2. The functions $f_1, \ldots, f_r$ are in involution with the functions $f_1, \ldots, f_s$ and $r + s = \dim M$.

Remark 4.2.13. As a consequence the maximal rank of the Poisson structure is $2r$.

Remark 4.2.14. When $r = s$ and thus all the first integrals commute we obtain Liouville integrable systems as a particular case.

Some notation: We denote the subset of M where the differentials $df_1, \ldots, df_s$ are independent by U_F and the subset of M where the vector fields $X_{f_1}, \ldots, X_{f_r}$ are independent by $M_{F,r}$.

On the open subset $M_{F,r} \cap U_F$ of M, the Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_r}$ define an involutive distribution of rank r. Let us denote by $\mathcal{F}$ its foliation with r-dimensional leaves, see [126]. When $\mathcal{F}_m$ is a compact r-dimensional manifold, the action-angle coordinate theorem proved in [126] (Theorem 1.1) proves that $\mathcal{F}_m$ is a torus and gives a semilocal description of the Poisson structure in a neighborhood of a compact invariant set:

Theorem 4.2.15. *Let (M, Π, F) be a non-commutative integrable system of rank r, where $F = (f_1, \ldots, f_s)$ and suppose that $\mathcal{F}_m$ is compact, where $m \in M_{F,r} \cap U_F$. Then there exist $\mathbb{R}$-valued smooth functions $(p_1, \ldots, p_r, z_1, \ldots, z_{s-r})$ and $\mathbb{R}/\mathbb{Z}$-valued smooth functions $(\theta_1, \ldots, \theta_r)$, defined in a neighborhood U of $\mathcal{F}_m$, and functions $\phi_{kl} = -\phi_{lk}$, which are independent of $\theta_1, \ldots, \theta_r, p_1, \ldots, p_r$, such that*

1. $\mathcal{F}_m$ *is a torus* $\mathbb{T}^r$.

2. *The functions $(\theta_1, \ldots, \theta_r, p_1, \ldots, p_r, z_1, \ldots, z_{s-r})$ define a diffeomorphism $U \simeq \mathbb{T}^r \times B^s$;*

3. *The Poisson structure can be written in terms of these coordinates as,*

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial p_i} + \sum_{k,l=1}^{s-r} \phi_{kl}(z) \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_l};$$

4. *The leaves of the surjective submersion $F = (f_1, \ldots, f_s)$ are given by the projection onto the second component $\mathbb{T}^r \times B^s$ and as a consequence the functions $f_1, \ldots, f_s$ depend on $p_1, \ldots, p_r, z_1, \ldots, z_{s-r}$ only.*

Let us now prove the first part of the theorem above using Corollary 4.2.4.

Proof. (of first item above)

Consider $F = (f_1, \ldots, f_s)$ the set of first integrals of the non-commutative integrable system. Consider the span of the Hamiltonian vector fields $X_i :=$

$\Pi(df_i, \cdot)$. From definition of the non-commutative integrable system at each point on the regular set, dimension of the vector space is r. By assumption the first r Hamiltonian vector fields are independent. Denote by α_i the 1-forms such that $\alpha_i(X_j) = \delta_{ij}$ for $i = 1, ..., r$ and $j = 1, ..., s$. They are not uniquely determined, but if we consider the inclusion j of the orbit into the manifold, then the 1-forms $\beta_i = j^*\alpha_i$ are uniquely determined. We can easily check that the forms β_i are closed:

$$d\beta_i(X_j, X_k) = X_k(\beta_i(X_j)) - X_j(\beta_i(X_k)) - \beta_i([X_j, X_k])$$
$$= X_k(\delta_{ij}) - X_j(\delta_{ik}) - \beta_i(0) = 0$$

where in the last equality we have used that $[X_j, X_k] = X_{\{f_j, f_k\}}$ and from the definition of non-commutative integrable system $X_{\{f_j, f_k\}} = X_0 = 0$.

From the definition the dimension of the orbit is r and we have exactly r forms thus applying Corollary 4.2.4 we conclude that the orbit is a torus. From the regular value theorem, observe also that this orbit is the connected component through the point of the mapping given by $F = (f_1, \ldots, f_s)$.   □

Finally, when $r = s$ we obtain as corollary the first statement of Theorem 3.7 of the commutative case.

Corollary 4.2.16. *Given an integrable system* $\mathbf{F} = (f_1, \ldots, f_s)$ *on a Poisson manifold, the regular integral manifold* $\mathcal{F}_m$ *of the distribution generated by* $X_{f_1}, \ldots, X_{f_s}$, *passing through* m, *is a torus of dimension* r, $\mathbb{T}^r$.

4.3 Local and global features of integrable systems on singular symplectic manifolds

In this section we investigate the integrability of Hamiltonian systems on manifolds endowed with a smooth 2-form which is symplectic away from an hypersurface Z (called the *critical set*) and which degenerates in a controlled way (of order one) along it. Either this form lowers its rank at Z and it induces a form on Z with maximal rank or its associated symplectic volume blows-up with a singularity of order one. The manifolds endowed with the first type of singular structure are called *folded symplectic manifolds* and the ones endowed with the second one are called *b*-symplectic forms. Folded symplectic manifolds can be thought as symplectic manifolds *with a fold, Z* that "mirrors" the symplectic structure on both sides. The study of *folded symplectic manifolds* complements that of their *"duals"* to *b*-symplectic manifolds which have been largely investigated since [92] and [89] and are better described as Poisson manifolds whose Poisson bracket looses rank along an hypersurface keeping some transversality properties. This section is also an invitation to consider more degenerate cases (higher order singularities) which will be studied elsewhere and the models provided here can be considered as

a *toy model* for more complicated singularities. Integrable systems on singular manifolds show up naturally, for instance, in the study of the Toda systems when the particles in interaction collide or are far-away. Singular symplectic manifolds naturally model symplectic manifolds with boundary and, as such, the notion of integrable system is naturally extended to manifolds with boundary.

The research of integrability of Hamiltonian systems on these manifolds is of interest both from a Poisson and symplectic point of view. The existence of action-angle coordinates on symplectic manifolds has been of major importance as, other than integrating the system itself, it provides semilocal normal forms for integrable Hamiltonian systems which allow, for instance, a deep understanding perturbation theory of these systems (KAM theory). The existence of action-angle coordinates of integrable systems is also useful for quantization as already observed by Einstein when reformulating the Bohr-Sommerfeld quantization conditions [56].

Integrable systems on these singular symplectic manifolds define natural Lagrangian foliations on them and thus naturally yield real polarizations on these manifolds. In particular they are of interest to study geometric quantization of symplectic manifolds with boundary as one of the sources of examples for these singular structures. On symplectic manifolds with boundary deformation quantization is already well-understood [160] and formal geometric quantization has been object of recent study in [199] for non-compact manifolds and in [96, 97] and [20] for b^m-symplectic manifolds. More specifically, the existence of action-angle coordinates for these structures provides a *primitive* first model for geometric quantization by counting the integer fibers of the integrable system. As proved in [100, 98, 101, 148] this model has been tested to be successful in geometric quantization of toric symplectic manifolds and refines the idea of Bohr-Sommerfeld quantization. Understanding action-angle coordinates for integrable systems on singular symplectic manifolds can be a first good step in the study of geometric quantization of singular symplectic manifolds. Action-angle manifolds on singular symplectic manifolds also provide natural cotangent-type models that can be useful in understanding the notion of quantum integrable systems ([176], [11]) in the singular set-up.

The study of *folded symplectic manifolds* comprises the case of origami manifolds [23] where additional conditions are imposed on the critical set and a natural global toric action exists. Origami manifolds inherit their denomination from origami paper templates where a superposition of Delzant polytopes [50] gives rise to a toric action on a class of folded symplectic manifolds. Symplectic origami provides an example of integrable system on folded symplectic manifolds but there are other examples motivated by physical systems such as the folded spherical pendulum or the Toda system where the interacting particles are far-away.

In this section we show the existence of b-integrable systems on b-symplectic manifold of dimension 4 having as critical set a Seifert manifold, and via the desingularization technique we obtain folded integrable systems on the associated desingularized folded symplectic manifold. We prove the existence of action-angle

coordinates à la Liouville-Mineur-Arnold exploring the Hamiltonian actions by tori on folded symplectic manifolds. The action variable are not exactly coordinates, since these variables can degenerate in a certain way. We show that this action-angle theorem cannot always be interpreted in terms of a cotangent model as in the symplectic and b-symplectic case.

We end up this section investigating the obstruction theory of global existence of action-angle coordinates exhibiting a new topological obstruction for the singular symplectic manifolds that lives on the critical set of the singular symplectic form. This yields examples of integrable systems on b-symplectic manifolds and folded symplectic manifolds with critical set non diffeomorphic to a product of a symplectic leaf with a circle. For those systems the toric action does not even extend to a neighborhood of the critical set. We end up this section observing that the existence of finite isotropy for the transverse S^1-action given by the modular vector field obstructs the uniformization of periods of the associated torus action on the b-symplectic manifold and automatically yields the existence of singularities of the integrable system on the critical locus of the b-symplectic structure.

4.3.1 Hamiltonian dynamics on folded symplectic manifolds

Let (M, ω) be a folded symplectic manifold, with folding hypersurface Z. Consider p a point in Z, the folded-symplectic form ω can be written in a neighborhood of p as:

$$\omega = t dt \wedge dq + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

with $t = 0$ defining the folding hypersurface. The singularity in ω prevents the Hamiltonian equation $\iota_X \omega = -df$ from having a solution for every possible function f. So not every function $f \in C^\infty(U)$ defines locally a Hamiltonian vector field.

Example 4.3.1. Let $(U; t, q, ..., x_n, y_n)$ be a chart where ω takes the folded-Darboux form mentioned above. Take for example the function $f = t$. By imposing the Hamiltonian equation we get that $X = \frac{1}{t} \frac{\partial}{\partial q}$, which is not a well defined smooth vector field.

Fortunately, we can characterize the set of functions which define smooth Hamiltonian vector fields.

Lemma 4.3.2. *A function $f : M \to \mathbb{R}$ in a folded symplectic manifold (M, ω) has an associated smooth Hamiltonian vector field X_f if and only if $df|_Z(v) = 0$ for every $v \in V$. Furthermore X_f is tangent to Z.*

Proof. Assume that f has a well-defined smooth Hamiltonian vector field at a point p in Z. Take Darboux coordinates $(t, q, ..., x_n, y_n)$ at a neighborhood U of

p. In these coordinates, the form can be written as:

$$\omega = t \, dt \wedge dq + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

Any vector field can be written as $X = a_1 \frac{\partial}{\partial t} + b_1 \frac{\partial}{\partial q} + ... + a_n \frac{\partial}{\partial x_n} + b_n \frac{\partial}{\partial y_n}$. Imposing the Hamiltonian equation $\iota_X \omega = -df$ we obtain that

$$\begin{cases} a_1 & = -\frac{\partial f}{\partial q} \frac{1}{t} \\ b_1 & = \frac{\partial f}{\partial t} \frac{1}{t} \\ a_i & = -\frac{\partial f}{\partial y_i}, \quad i = 2, ..., n \\ b_i & = \frac{\partial f}{\partial x_i}, \quad i = 2, ..., n. \end{cases}$$

The coefficients a_1 and b_1 well defined, we need that $-\frac{\partial f}{\partial q} \frac{1}{t}$ and $\frac{\partial f}{\partial t} \frac{1}{t}$ to be smooth. For this to hold, we need that $\frac{\partial f}{\partial q} = tH$ and $\frac{\partial f}{\partial t} = tF$ for some smooth function H and G. From the second equation we get that $f = t^2 f_1 + f_2(q, x_2, ..., y_n)$ for some smooth functions f_1, f_2. The first equation implies that $\frac{\partial f_2}{\partial q} = 0$. Hence f has the form

$$f = t^2 f_1 + f_2(x_2, ..., y_n), \tag{4.5}$$

which implies that $df(\frac{\partial}{\partial t})|_Z = df(\frac{\partial}{\partial q})|_Z = 0$ since $Z = \{t = 0\}$. Since $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial q}$ space $V = \ker \omega|_Z$, we get that $df(v) = 0$ for each $v \in \ker \omega|_Z$. The converse is obviously true: if $df(v) = 0$ for each $v \in \ker \omega|_Z$ then it defines a smooth Hamiltonian vector field.

It follows from equation (4.5) that $\frac{\partial f}{\partial q} = t^2 \frac{\partial f_1}{\partial q}$ which implies that $a_1|_{\{t=0\}} = -t \frac{\partial f_1}{\partial q}|_{\{t=0\}} = 0$. We deduce that X_f, the Hamiltonian vector field of f, is tangent to Z. $\qquad \square$

We denote these functions as folded functions.

Definition 4.3.3. *A function $f : M \to \mathbb{R}$ in a folded symplectic manifold (M, ω) is a **folded function** if $df|_Z(v) = 0$ for every $v \in V = \ker \omega|_Z$.*

Note that even if a Hamiltonian vector field X_f is always tangent to Z, one can obtain non vanishing components of X_f in the null line bundle L. If one takes n folded functions, we will always have $df_1 \wedge ... \wedge df_n|_Z = 0$ when we look it as a section of $\Lambda^n T^* M$. However, the n functions can define n linearly independent Hamiltonian vector fields. This yields the following definition:

Definition 4.3.4. *An integrable system on a folded symplectic manifold (M, ω) with critical surface Z is a set of functions $F = (f_1, ..., f_n)$ such that they define Hamiltonian vector fields which are independent on a dense set of Z and M, and commute with respect to ω.*

Around the regular points of the integrable system, the expression of the functions can be simplified and as a consequence the Poisson bracket of the functions is well-defined:

Lemma 4.3.5. *Near a regular point of an integrable system, there exist coordinates $(t, q, x_2, ..., y_n)$ such that $\omega = tdt \wedge dq + \sum_{i=2}^{n} dx_i \wedge dy_i$, and the integrable system has the form*

$$f_1 = t^2/2$$
$$f_2 = g_2(t, q, x_2, ..., y_n)t^{k_2} + h_2(x_2, y_2, ..., x_n, y_n)$$
$$\vdots$$
$$f_n = g_n(t, q, x_2, ..., y_n)t^{k_n} + h_n(x_2, y_2, ..., x_n, y_n),$$

for $k_2, \ldots, k_n \in \mathbb{N}$ all of them ≥ 2 and t is a defining function of Z.

Proof. Denote the inclusion of Z in M by $i : Z \hookrightarrow M$. Since the pullback to Z of the folded symplectic form $i^*\omega$ has rank $2n-2$, there are at most $n-1$ independent Hamiltonian vector fields tangent to Z such that $\langle X_1, ..., X_{n-1} \rangle$ has no component in $\ker i^*\omega$. This implies that at any regular point $p \in Z$ of an integrable system, one of the n independent Hamiltonian vector fields $X_1, ..., X_n$ has a component in $\ker i^*\omega$. We might assume it is the first one X_1.

Let us show that in the points close to p in Z, this vector field X_1 can be written as $X_1 = v + X'$, where $v \in \ker i^*w$ and $X' \in \langle X_2, ..., X_n \rangle$.

Indeed if X_1 had a component in the complement of $\ker i^*\omega \cup \langle X_2, ..., X_n \rangle$, it would have a component either in the symplectic orthogonal space to $\langle X_2, ..., X_n \rangle$ with respect to $i^*\omega$ or in $TZ^\perp$. However, we know that the Hamiltonian vector fields with respect to ω are tangent to Z, and so X would have a component in the symplectic orthogonal to $\langle X_2, ..., X_n \rangle$ and would not satisfy $\omega(X, X_i) = 0$. In particular, we can take a new basis of Hamiltonian vector fields generating $\langle X_1, ..., X_n \rangle$ in a neighborhood of p such that X_1 lies exactly in the kernel of $i^*\omega$ in $U \cap Z$.

Take local coordinates in a neighborhood U of p such that $X_1 = \frac{\partial}{\partial q}$. Take symplectic coordinates $(x_2, y_2, ..., x_n, y_n)$ of $i^*\omega$, the existence of such coordinates follows from the Darboux theorem for closed two forms of constant rank [130, Proposition 13.7]. We can now use Theorem 3.1.22 with $\alpha = dq$ to conclude that

$$\omega = tdt \wedge dq + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

In these coordinates the vector field X_1 is the Hamiltonian vector field of $t^2/2$, hence $f_1 = t^2/2$. The remaining functions $f_2, ..., f_n$ are folded functions and can be expressed as in Equation (4.5). This concludes the proof of the lemma. □

Folded cotangent bundle

In this subsection we recall the construction of [104], a dual of the b-cotangent bundle for folded symplectic manifolds.

Definition 4.3.6. *Let M a manifold and Z a closed hypersurface. Let V a rank 1 subbundle of $i_Z^* TM$ so that for all $p \in Z$ the fiber V_p is transverse to $T_p Z$. We define for each open subset $U \subset M$*

$$\Omega_V^1(U) := \{\alpha \in \Omega^1(U) | \ \alpha|_V = 0\},$$

the space of 1-forms on U vanishing on V. If $U \cap Z = \emptyset$ then it is just $\Omega^1(U)$.

Following [104] there exists a vector bundle $T_V^* M$ called the **folded cotangent bundle**, of rank n whose global sections are isomorphic to $\Omega_V^1(M)$. This vector bundle is unique up to isomorphism, independently of the chosen V. For a small open neighborhood U of a point in Z, there exist suitable coordinates $(x_1, ..., x_{n-1})$ in $U \cap Z$ and a coordinate t such that $(x_1, ..., x_{n-1}, t)$ are coordinates in U and $T_V^*(U)$ is generated by $dx_1, ..., dx_{n-1}, tdt$. The dual bundle to $T_V^* M$ is denoted by $T_V M$ and called the folded tangent bundle.

In this bundle there is a canonical folded symplectic form which is obtained by taking a Liouville form λ_f which is canonical as it satisfies the *Liouville-type* equation $\langle \lambda_f|_p, v \rangle = \langle p, (\pi_p)_*(v) \rangle$ for every $v \in T_V(T_V^* M)$ and $p \in T_V^* M$. In coordinates $(x_1, ..., x_n, p_1, ..., p_n)$ we can write

$$\lambda = p_1 x_1 dx_1 + \sum_{i=2}^{n} p_i dx_i.$$

Its derivative gives rise to a folded symplectic structure

$$\omega_f = d\lambda = x_1 dp_1 \wedge dx_1 + \sum_{i=2}^{n} dp_i \wedge dx_i$$

which looks like the Darboux-type folded symplectic structure. The introduction of this bundle allows to restate the definition of a folded integrable system in terms of the folded cotangent bundle.

Definition 4.3.7. *An integrable system on a folded symplectic manifold (M, ω) is a set of folded functions $F = (f_1, ..., f_n)$ for which $df_1 \wedge ... \wedge df_n \neq 0$ as sections of $\Lambda^n T_V^* M$ on a dense set of M and Z, and whose Hamiltonian vector fields commute with respect to ω.*

Even if ω does not define a Poisson bracket in Z because the Hamiltonian vector fields are not defined for non-folded functions, the bracket is well defined for folded functions and the commutation condition $\omega(X_{f_i}, X_{f_j}) = 0$ for two Hamiltonian vector fields is still well-defined.

4.3.2   Motivating examples

In this section we present a series of examples of folded integrable systems. In particular, we exhibit examples of folded integrable systems whose dynamics cannot possibly be modeled by b-integrable systems. This will motivate the development of the theory of folded integrable systems, and in particular of the existence of action-angle coordinates.

Double collision in two particles system

In the literature of celestial mechanics like the restricted 3-body problem or the n-body problem several regularization transformation associated to ad-hoc changes (like time reparametrization) bring singularities into the symplectic structure. Below we describe a model of double collision in two particle systems where McGehee type changes are implemented. We model a system of two particles under the influence of a potential energy function of the form $U(x) = -|x|^{-\alpha}$, with $\alpha > 0$. In the phase space $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$ it is a Hamiltonian system with Hamiltonian function $F = \frac{1}{2}|y|^2 - |x|^{-\alpha}$. Let us introduce a notation for two constants: denote $\beta = \alpha/2$ and $\gamma = \frac{1}{\beta+1}$. By implementing the change of coordinates:

$$\begin{cases} x & = r^\gamma e^{i\theta} \\ y & = r^{-\beta\gamma}(v + iw)e^{i\theta} \end{cases}$$

and scaling with a new time parameter τ such that $dt = rd\tau$ we obtain the equations of motion

$$\begin{cases} r' & = (\beta + 1)rv \\ v' & = w^2 + \beta(v^2 - 2) \\ \theta' & = w \\ w' & = (\beta - 1)\omega v \end{cases}.$$

We will model the collision set $\{r = 0\}$ in the case $\beta = 1$ as the folding hypersurface of a folded symplectic manifold endowed with a folded integrable system. Let us consider the folded symplectic form $\omega = rdr \wedge dv + d\theta \wedge dw$ in the manifold $T^*(\mathbb{R} \times S^1) \cong \mathbb{R}^2 \times S^1 \times \mathbb{R}$ with coordinates (r, v, θ, w). We take the folded Hamiltonian function

$$H = -\frac{r^2}{2}(w^2 + (v^2 - 2)) + \frac{w^2}{2}.$$

Observe that $dH = -r^2 vdv + (w^2 + v^2 - 2)2rdr + (w - r^2 w)dw$, and the Hamiltonian vector field is

$$X_H = -rv\frac{\partial}{\partial r} + (w^2 + v^2 - 2)\frac{\partial}{\partial v} + (w + r^2 w)\frac{\partial}{\partial \theta}$$

The equations of motion in the critical hypersurface $\{r = 0\}$ coincide with the equations of motion in the collision manifold of the original problem, hence providing a folded Hamiltonian model for it. In fact, even the linear asymptotic behavior close to collision is captured by the model. Observe that X commutes with $\frac{\partial}{\partial \theta}$, which is a Hamiltonian vector field for the function $f_2 = w$. Hence the dynamics are modelled by a folded integrable system given by $F = (f_1 = H, f_2)$ in $T^*(\mathbb{R} \times S^1)$ with folded symplectic structure $r dr \wedge dv + d\theta \wedge dw$.

Folded integrable systems on toric origami manifolds

Not all integrable systems on folded symplectic manifolds come from standard systems on symplectic manifolds after singularization transformations or regularization techniques as in the example above. Take for instance $\mathbb{R}^4$ with the standard symplectic structure $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. The function $f_1 = x_1^2 + y_1^2 + x_2^2 + y_2^2$ and $f_2 = x_1 y_2 - x_2 y_1$ commute with respect to ω. There is a natural folding map from the sphere S^4 to $\bar{D}^4$, that we denote π. It is a standard fact that $\pi^* \omega$ is a folded symplectic structure in S^4, and in fact an origami symplectic structure. Taking $F = (\pi^* f_1, \pi^* f_2)$ yields an example of a folded integrable system in S^4 with its induced folded symplectic structure. Note that this is an example of an integrable system on a singular symplectic manifold which is not b-symplectic, as shown by the obstructions in [92] and [133].

Symplectic manifolds with fibrating boundary

Consider a symplectic manifold with boundary such that close to the boundary the symplectic form tends to degenerate and admits adapted Martinet-Darboux charts such that the boundary has local equation $x_1 = 0$ and the symplectic form degenerates on the boundary with the following local normal form:

$$\omega = x_1 dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \ldots + dx_n \wedge dy_n.$$

Let us take as starting point some integrable system naturally defined on a manifold with boundary. Assume the folding hypersurface fibrates by circles over a compact symplectic base (origami type). It would be enough to consider an integrable system on the $(2n-2)$-symplectic base $f_2, \ldots f_n$ and add t^2 as f_1. The set $(t^2, f_2, \ldots, f_n)$ defines a folded integrable system. Observe that complete integrability comes as a consequence of Theorem 3.1.22.

Product of folded surfaces with symplectic manifolds endowed with integrable systems

Take an orientable surface Σ, and ω a non-vanishing two form. Denote t any function in Σ which is transverse to the zero section. The critical set is a finite number of closed curves γ_j, $j = 1, ..., k$. Then the function t^2 defines a folded integrable system in $(\Sigma, t\omega)$, where $t\omega$ is a folded symplectic structure. Let $F =$

$(f_1, ..., f_n)$ be an integrable system in a symplectic manifold (M^{2n}, ω_1). Then $(t^2, f_1, ..., f_n)$ defines a folded integrable system in the manifold $M^{2n} \times \Sigma$ endowed with the folded symplectic form $\omega_f = t\omega + \omega_1$. In fact, taking any $(n+1)$-tuple of the form $(t^2 \sum_{i=1}^n \lambda_i f_i, f_1, ..., f_n)$ for some non trivial n-tuple of constants λ_i yields a folded integrable system. The critical set is of the form $Z = \sqcup_{j=1}^k \gamma_j \times M^{2n}$.

Origami templates

The study of toric folded symplectic manifolds was initiated in [23] in the origami case (see [104] for the general case).

Toric actions and integrable systems have always been hand-in-hand. In particular the action-angle coordinate theorem proof that we will provide in this work uses intensively this correspondence. So in particular a toric manifold provides examples of integrable systems which are described by a global Hamiltonian action of a torus. Indeed any integrable system can be semilocally described in these terms (as we will see in the next section).

The classical theory of toric symplectic manifolds is closely related to a theorem by Delzant [50] which gives a one-to-one correspondence between toric symplectic manifolds and a special type of convex polytopes (called Delzant polytopes) up to equivalence. Grosso modo, toric symplectic manifolds can be classified by their moment polytope, and their topology can be read directly from the polytope in terms of equivariant cohomology. In [108, 109] the authors examine the toric origami case and describe how toric origami manifolds can also be classified by their combinatorial moment data.

Origami templates form a visual way to describe toric origami manifolds and thus in particular integrable systems on folded symplectic manifolds.
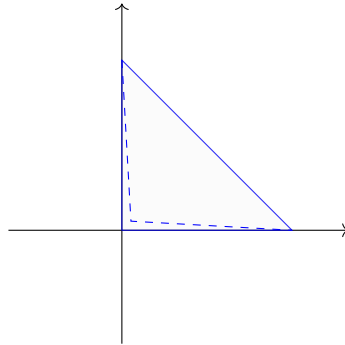

Figure 4.3: Origami template of Example 4.2

Toric origami manifolds are classified by combinatorial origami templates which overlap Delzant's polytopes in an special way providing pictorially beautiful examples of folded integrable systems.
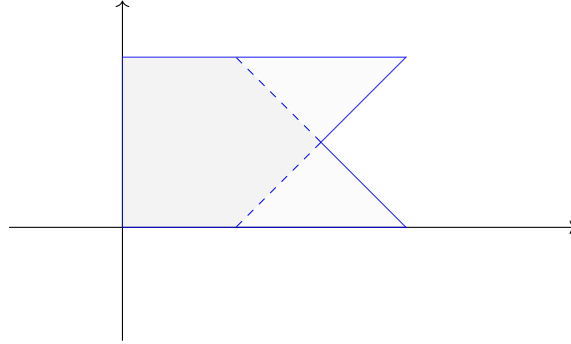
Figure 4.4: Origami template corresponding to the radial blow-up of two Hirze-bruch surfaces.

The folded spherical pendulum

Consider the spherical pendulum on S^2 defined as follows: Take spherical coordinates (θ, ϕ) with $\theta \in (0, \pi)$ and $\phi \in (0, 2\pi)$ if we denote each momentum as P_θ and P_ϕ respectively, the Hamiltonian function is

$$H = \frac{1}{2}(P_\theta^2 + \frac{1}{\sin^2 \theta} P_\phi^2) + \cos \theta.$$

Instead of taking the standard symplectic form in $T^* S^2$ we consider the folded symplectic form

$$\omega = P_\phi dP_\phi \wedge d\phi + dP_\theta \wedge d\theta.$$

Computing the Hamiltonian vector field associated to H we get

$$X_H = \frac{1}{\sin^2 \theta} \frac{\partial}{\partial \phi} + P_\theta \frac{\partial}{\partial \theta} + (\sin \theta + \frac{\cos \theta}{\sin^3 \theta} P_\phi^2) \frac{\partial}{\partial P_\theta}.$$

This vector field clearly commutes with $\frac{\partial}{\partial \phi}$, which is the Hamiltonian vector field of $f = P_\phi^2$. Observe furthermore that

$$dH \wedge dP_\phi^2 = -(\sin \theta + \frac{\cos \theta}{\sin^3 \theta} P_\phi^2) 2 P_\phi d\theta \wedge dP_\phi,$$

which is nondegenerate on a dense set of M and on a dense set of Z when seen as a section of the second exterior product of the folded cotangent bundle. The manifold is in fact $M = T^*(S^2 \setminus \{N, S\})$, i.e. we are taking out the poles of the sphere. In this sense M is equipped with an origami symplectic form: the critical set is $T^*(S^2 \setminus \{N, S\})$ and the null line bundle is an S^1 fibration generated by $\frac{\partial}{\partial \varphi}$.

Observe that dynamically this system is different from the standard spherical pendulum. When $P_\phi = 0$, the vector field can have a non vanishing $\frac{\partial}{\partial \phi}$ component.

A folded integrable system which cannot be modelled as a b-integrable system

Consider S^2 with the folded symplectic form $\omega = hdh \wedge d\theta$. A folded function whose exterior derivative is a non-vanishing one-form (when considered as section of the folded cotangent bundle) on a dense set of M and of Z defines a folded integrable system. Take for instance $f = \cos\theta h^2$, which satisfies this condition. Computing its Hamiltonian vector field we obtain

$$X_f = h\sin\theta \frac{\partial}{\partial h} + 2\cos\theta \frac{\partial}{\partial \theta}.$$

This vector field vanishes at some points in the critical locus $Z = \{h = 0\}$. A b-integrable system on a surface Σ is defined by a function $f = c\log h + g$ with $g \in C^\infty(\Sigma)$. In particular its Hamiltonian vector field cannot vanish at any point on the critical hypersurface, as it is happening in this example of folded integrable system. Thus, even if the structure $hdh \wedge d\theta$ can be seen as the desingularization of $\frac{1}{h}dh \wedge d\theta$, the dynamics of this folded integrable system cannot be modeled using the b-symplectic structure.

Cotangent lifts for folded symplectic manifolds

In this section we describe the cotangent lift in the set-up of folded symplectic manifolds.

When the group acting on the base is a torus this procedure provides examples of folded integrable systems.

Consider a Lie group G acting on M by an action $\phi : G \times M \longrightarrow M$.

Definition 4.3.8. *The cotangent lift of ϕ is the action on T^*M given by $\hat{\phi}_g := \phi_{g^{-1}}^*$, where $g \in G$.*

The following commuting diagram holds:

$$
\begin{array}{ccc}
T^*M & \xrightarrow{\hat{\phi}_g} & T^*M \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\
M & \xrightarrow{\phi_g} & M
\end{array}
$$

where π is the projection from T^*M to M. The cotangent bundle has the symplectic form $\omega = -d\lambda$ where λ is the Liouville form. This form is defined by the property $\langle \lambda_p, v \rangle = \langle p, (\pi_p)_*(v) \rangle$, where $v \in T(T^*M)$ and $p \in T^*M$. It can be shown easily that the cotangent lift is a Hamiltonian action with momentum map $\mu : T^*M \to \mathfrak{g}^*$ given by

$$\langle \mu(p), X \rangle := \langle \lambda_p, X^\#|_p \rangle = \langle p, X^\#|_{\pi(p)} \rangle.$$

Here $X^{\#}$ denotes the fundamental vector field of X associated to the action. The Liouville form is invariant by the action which implies the invariance of the moment map. In particular, the map is Poisson.

The construction called b-symplectic cotangent lift for b-symplectic manifolds done in [121] can be similarly done in the folded symplectic case which we will do below.

For the standard Liouville form in the folded cotangent bundle, the singularity is in the base space, and we would like to have it on the fiber. A different form, that we call **twisted folded Liouville form** can be defined on $T_V^* S^1$ with coordinates (θ, p):

$$\lambda_{tw} = \frac{p^2}{2} d\theta_1.$$

this way the singularity is in the fiber, and we can apply it to define a folded cotangent lift on the torus. Let $\mathbb{T}^n$ be the manifold and the group acting by translations, and take the coordinates $(\theta_1, ..., \theta_n, a_1, ..., a_n)$ on T^*M. The standard symplectic Liouville form in these coordinates is

$$\lambda = \sum_{i=1}^{n} p_i d\theta_i.$$

The moment map $\mu_{can} : T^*\mathbb{T}^n \to \mathfrak{t}^*$ of the lifted action with respect to the canonical symplectic form is

$$\mu_{can}(\theta, p) = \sum_i p_i d\theta_i$$

where the θ_i are seen as elements of $\mathfrak{t}^*$. In fact, one can identify the moment map as just the projection of $T^*\mathbb{T}^n$ into the second component since $T^*\mathbb{T}^n \cong \mathbb{T}^n \times \mathbb{R}^n$.

This torus action in the cotangent bundle of the torus can be seen as a folded-Hamiltonian action with respect to a folded symplectic form. Similarly to the Liouville one-form we define the following singular form away from the hypersurface $Z = \{p_1 = 0\}$:

$$p_1^2 d\theta_1 + \sum_{i=2}^{n} p_i d\theta_i.$$

The negative differential gives rise to a folded symplectic form called twisted folded symplectic form on $T^*\mathbb{T}^n$:

$$\omega_{tw,f} := p_1 d\theta_1 \wedge dp_1 + \sum_{i=2}^{n} d\theta_i \wedge dp_i.$$

The moment map is then

$$\mu_{tw,f} = (p_1^2, p_2, \ldots, p_n),$$

where we identify $\mathfrak{t}^*$ with $\mathbb{R}^n$ as before.

We call this lift the **folded cotangent lift**. Note that, in analogy to the symplectic case, the components of the moment map define a folded integrable system on $(T^*\mathbb{T}^n, \omega_{tw,f})$.

Remark 4.3.9. As we will see in the next section, the folded cotangent lift does not always serve as the semilocal model for an integrable system in the neighbourhood of a Liouville torus, in contrast to what happens in symplectic and b-symplectic geometry.

4.3.3 Action-angle coordinates and cotangent models

In this section we prove existence of action-angle coordinates for singular symplectic manifolds of order one. One may have the temptation to use the desingularization and the action-angle coordinate theorem proved in [123] to conclude. However, as we saw in previous sections, not every folded integrable system can be seen as a desingularized b-integrable system and thus a complete proof is needed. We end this section by analyzing possible cotangent models of such a normal form, and in particular show that the obtained normal form cannot be interpreted in terms of a canonical cotangent lift model.

Topology of the integrable system

We first show that for a folded integrable system there is a foliation by Liouville tori in the neighborhood of a regular fiber of the integrable system. For this it is important to observe the following:

Arguing as in the proof of Lemma 4.3.5, the kernel of $i^*\omega$ is generated by the joint distribution of the Hamiltonian vector fields of $f_1, ..., f_n$ at every point p of a neighborhood of a regular fiber. Hence, we can assume that $f_1 = t^2/2$ for some semi-local coordinate t defining Z. The foliation given by the Hamiltonian vector fields of F coincides with the foliation described by the level sets of $\bar{F} = (t, f_2, ..., f_n)$ because by definition the Hamiltonian vector field of $t^2/2$ is tangent to the level sets of this $t^2/2$, and hence also to the level sets of t. The same argument in [126] gives a commutative diagram

$$U \xrightarrow{\varphi} \mathbb{T}^n \times B^n$$
$$\bar{F} \searrow \quad \downarrow \pi$$
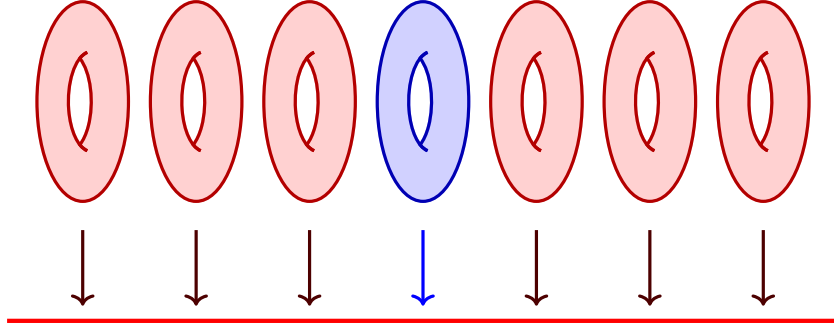$$B^n$$

and proves:

Figure 4.5: Fibration by Liouville tori: In the middle fiber (in blue) of the point $F(p)$, the neighbouring Liouville tori in red.

Proposition 4.3.10. *Let $p \in Z$ be a regular point of a folded-integrable system (M, ω, F). Assume that the integral manifold $\mathcal{F}_p$ is compact. Then there is neighborhood U of $\mathcal{F}_p$ and a diffeomorphism*

$$\varphi : U \cong \mathbb{T}^n \times B^n$$

which takes the foliation $\mathcal{F}$ to the trivial foliation $\{\mathbb{T}^n \times \{b\}\}_{b \in B^n}$.

We now prove a Darboux-Carathéodory theorem for folded symplectic manifolds to (locally) complete a set of folded functions which commute with respect to ω. We do this applying the arguments of the proof of Darboux theorem provided in [8]. The Darboux-Carathéodory theorem will be a key point in the proof of existence of action-angle coordinates.

Theorem 4.3.11 (Folded Darboux-Carathéodory theorem)**.** *Let $p \in Z$ be a point of the folding hypersurface of a folded symplectic manifold (M, ω) and let t be the function defining Z. Consider $f_1, ..., f_n$ to be n folded functions whose Hamiltonian vector fields are smooth, independent at p and commute pairwise with respect to ω. Then in a neighborhood U of p there exists n functions $q_1, ..., q_n$ such that near p the folded symplectic form is written as*

$$\omega = \sum_{i=1}^{n} dq_i \wedge df_i. \tag{4.6}$$

A system of coordinates is given by $q_1, ..., q_n$ and some coordinates $t, y_2, .., y_n$ such that the f_i only depend on the latter.

Remark 4.3.12. Note that the functions $f_1, ..., f_n$ do not define a set of n independent coordinate functions in U and, thus, Equation (4.6) does not correspond to a symplectic form but a folded symplectic form.

Proof. In order to construct this decomposition we first construct the folded symplectic conjugate of the function f_1 following the classical recipe which we can find in [8].

In U the foliation $\mathcal{F}$ induced by the level sets of $(t, f_2, ..., f_n)$ coincides with the one generated by $\mathcal{D} = \langle X_1, ..., X_n \rangle$, where X_i denotes the Hamiltonian vector field of f_i. Take B a submanifold of dimension n containing p and transverse to this foliation. We will now construct a function q_1 such that $X_1 = \frac{\partial}{\partial q_1}$ and $dq_1(X_j) = \delta_{1j}$.

For each point $m \in B$, there is a leaf $L_m \subset U$ of the foliation $\mathcal{F}$ containing p. Each L_m is foliated by $n-1$ dimensional leaves, induced by the foliation integrating $\mathcal{D}' = \langle X_2, ..., X_n \rangle$. Denote by $L'_m \subset L_m$ the $n-1$ dimensional leaf containing M. For some ε, the flow ϕ_1^t of X_1 is defined for a smaller neighborhood U' for $|t| < \varepsilon$ and with its image contained in U. For a point $x \in L_m \cap U'$, there is some $x' \in L'_m \cap U'$ and a t' such that $|t'| < \varepsilon$ and $\phi_1^{t'}(x') = x$. Define the function:

$$q_1 : U' \longrightarrow \mathbb{R}$$
$$x \longmapsto t'(x).$$

We claim that $dq_1(X_j) = \delta_{ij}$. First, observe that the flow ϕ_1^t preserves the foliations induced by D and D' because of the commuting conditions given by the integrable system. This implies that q_1 is constant along such foliation and hence $dq_1(X_j) = 0$ if $j \neq 1$. The definition of t' yields $dq_1(X_1) = 1$.

We can now fix both q_1 and f_1, and apply iteratively this construction for the flow of the Hamiltonian vector field of each of the remaining functions $f_2, ..., f_n$. We get smooth functions $q_2, ..., q_n$ such that $dq_i(X_j) = \delta_{ij}$. At the points in $U \setminus Z$, the functions $f_1, ..., f_n, q_1, ..., q_n$ do form a set of coordinates. This implies that in $U \setminus Z$, where ω is symplectic, we have

$$\omega = \sum_{i=1}^{n} dq_i \wedge df_i.$$

But both ω and the functions q_i, f_i are smooth and defined along U, hence this expression extends to U. Extend the functions q_i to a set of coordinates $(q_1, ..., q_n, y_1, y_2, ..., y_n)$. We can assume that $y_1 = t$ is a defining function of Z, since the q_i are coordinates along the level sets of F: the vector fields $\frac{\partial}{\partial q_i}$ are the Hamiltonian vector fields of the f_i. The fact that $\omega(X_i, X_j) = 0$ implies that $df_i(X_j) = \frac{\partial}{\partial q_j}(f_i) = 0$. This proves that the f_i depend only on $(t, y_2, .., y_n)$. $\square$

In contrast with the Darboux-Carathéodory in symplectic and b-symplectic geometry, one can not obtain a canonical normal form as in Martinet's theorem. This is a consequence of the fact that when you fix several commuting folded functions, various of those functions can have a Hamiltonian vector field with a non-vanishing component in the null line bundle.

Remark 4.3.13. The classical Darboux-Carathéodory theorem considers a set of $k < n$ commuting independent functions $f_1, ..., f_k$. The same proof can be adapted in this situation and the same theorem applies for a set of $k < n$ commuting functions which are independent $f_1, ..., f_k$. We can find then a set of coordinates

$(q_1, ..., q_k, y_1, ..., y_k, x_{k+1}, y_{k+1}, ..., x_n, y_n)$ such that $\omega = \sum_{i=1}^{k} dq_i \wedge df_i + \sum_{i=k+1}^{n} dx_i \wedge dy_i$. This form of Darboux-Carathéodory theorem is convenient for the study of non-commutative integrable systems (see for instance [122]).

Equivariant relative Poincaré's lemma for folded symplectic forms

We start this section with some lemmas which we will need for the proof of the action-angle theorem. They concern Relative Poincaré's lemma for folded symplectic forms and their equivariant versions. Recall from [197, page 25].

Theorem 4.3.14 (Relative Poincaré lemma). *Let $N \subset M$ a closed submanifold and $i : N \hookrightarrow M$ the inclusion map. Let ω a closed k-form on M such that $i^*\omega = 0$. Then there is a $(k-1)$-form α on a neighborhood of N in M such that $\omega = d\alpha$.*

This Relative Poincaré lemma can be used in the particular case in which the form is folded and the submanifold is a Liouville torus.

Proposition 4.3.15. *In a neighborhood $U(L)$ of a Liouville torus the folded symplectic form can be written*

$$\omega = d\alpha.$$

If ω is invariant by a compact group action, α can be assumed to be invariant by the same compact group action.

Proof. Let $i : L \hookrightarrow M$ be the natural inclusion of the Liouville torus on the folded symplectic manifold, since $i^*\omega = 0$ we may apply the following relative Poincaré theorem.

Let us check that the hypotheses of the Relative Poincaré lemma are met. The form is closed and we only need to check $i^*\omega = 0$. Since every Y_i is Hamiltonian with Hamiltonian function σ_i, we obtain that $\iota_{Y_i}\omega = d\sigma_i$. And therefore the tangent space to L is generated by $Y_1, ..., Y_n$. However, we know that $i^*d\sigma_i = 0$, since L is the level set of the integrable system. This implies that $\iota_{Y_i} i^*\omega = 0$ and hence $i^*\omega = 0$.

Now define the averaged $\bar{\alpha}$ as

$$\bar{\alpha} = \int_G \rho_g^* \alpha d\mu,$$

where μ is a Haar measure and ρ_g is the group action. This 1-form is G-invariant, and as ρ preserves ω, we can write,

$$\omega = \int_G \rho_g^* \omega d\mu$$
$$= \int_G d\rho_g^* \alpha d\mu$$

Thus $\omega = d(\int_G \rho_g^*(\alpha) d\mu)$. In particular this proves that the primitive $\bar{\alpha}$ is invariant by the action. i.e, for any Y_i fundamental vector field of the torus action one obtains, $\mathcal{L}_{Y_i} \bar{\alpha} = 0$. Thus finishing the proof of the proposition. $\qquad\square$

Statement and proof of the action-angle coordinate theorem

We proceed now with the statement and the proof of the action-angle theorem.

Theorem 4.3.16. *Let $F = (f_1, ..., f_n)$ be a folded integrable system on (M, ω) and $p \in Z$ a regular point in the folding hypersurface. We assume the integral manifold $\mathcal{F}_p$ containing p is compact. Then there exist an open neighborhood U of the torus $\mathcal{F}_p$ and a diffeomorphism*

$$(\theta_1, ..., \theta_n, t, b_2, ..., b_n) : U \to \mathbb{T}^n \times B^n,$$

where t is a defining function of Z and such that

$$\omega_U = \sum_{i=1}^{n} d\theta_i \wedge dp_i.$$

where the p_i are folded functions which depend only on $(t, b_2, ..., b_n)$ (and so do the f_i).

The S^1-valued functions

$$\theta_1, ..., \theta_n$$

are called angle coordinates and the $\mathbb{R}$-valued folded functions

$$p_1, p_2, ..., p_n$$

are called folded action functions.

Remark 4.3.17. Comparing this theorem with the analog in [123] observe that unlike the b-symplectic case, the expression of ω in a neighborhood of the Liouville torus is not in a folded Darboux-type form.

Besides the lemmas in the former subsection we will need the following technical lemma.

In [126] (see Claim 2 in page 1856) it is shown that given a complete vector field Y of period 1 and a bivector field P such that $\mathcal{L}_Y \mathcal{L}_Y P = 0$ then $\mathcal{L}_Y P = 0$. If instead of a bivector field we take a 2-form, the proof can be easily adapted as follows.

Lemma 4.3.18. *If Y is a complete vector field of period 1 and ω is a 2-form such that $\mathcal{L}_Y \mathcal{L}_Y \omega = 0$ then $\mathcal{L}_Y \omega = 0$.*

Proof. Denote $v = \mathcal{L}_Y \omega$. Denote ϕ_t the flow of Y. For any point p we can write

$$\frac{d}{dt} \left(\phi_t^* \omega_{\phi_{-t}(p)} \right) = (\phi_t)^* (\mathcal{L}_Y \omega_{\phi_{-t}(p)})$$
$$= \phi_t^* v_{\phi_{-t}(p)}$$
$$= v_p$$

In the last equality we used that $\mathcal{L}_Y v = 0$. Integrating we obtain

$$(\phi_t)^* \omega_{\phi_{-t}(p)} = \omega_p + t v_p.$$

At time $t = 1$ the flow is the identity because Y has period 1 and hence $v_p = 0$. $\square$

We now proceed to the action-angle theorem proof.

Proof. The vector fields $X_{f_1}, ..., X_{f_n}$ define a torus action on each Liouville torus $\mathbb{T}^n \times \{b\}_{b \in B^n}$. We would like an action defined in a neighborhood of the type $\mathbb{T}^n \times B^n$. For the first part of the proof we follow the proofs in [126] and [123] and construct a toric action. For this we consider the classical action of the joint-flow (which is an $\mathbb{R}^n$-action) and prove uniformization of periods to induce a $\mathbb{T}^\ltimes$-action.

We denote by φ_i^t the time-t-flow of the Hamiltonian vector fields X_{f_i}. Consider the joint flow of these Hamiltonian vector fields.

$$\varphi : \mathbb{R}^n \times (\mathbb{T}^n \times B^n) \longrightarrow \mathbb{T}^n \times B^n$$
$$\left((t_1, \ldots, t_n), (x, y)\right) \longmapsto \varphi_1^{t_1} \circ \cdots \circ \varphi_n^{t_n}(x, y).$$

The vector fields X_{f_i} are complete and commute with one another so this defines an $\mathbb{R}^n$-action on $\mathbb{T}^n \times B^n$. When restricted to a single orbit $\mathbb{T}^n \times \{b\}$ for some $b \in B^n$, the kernel of this action is a discrete subgroup of $\mathbb{R}^n$, a lattice Λ_b. We call Λ_b the period lattice of the orbit. The rank of Λ_b is n because the orbit is assumed to be compact.

The lattice Λ_b will in general depend on b. The idea of uniformization of periods is to modify the action to get constant isotropy groups such that $\Lambda_b = \mathbb{Z}^n$ for all b. For any $b \in B^{n-1} \times \{0\}$ and any $a_i \in \mathbb{R}$ the vector field $\sum a_i X_{f_i}$ on $\mathbb{T}_n \times \{b\}$ is the Hamiltonian vector field of the function

$$\sum_{i=1}^n a_i f_i.$$

To perform the uniformization we pick smooth functions

$$(\lambda_1, \lambda_2, \ldots, \lambda_n) : B^n \to \mathbb{R}^n$$

such that $(\lambda_1(b), \lambda_2(b), \ldots, \lambda_n(b))$ is a basis for the period lattice Λ_b for all $b \in B^n$. Such functions λ_i exist such that they satisfy this condition (perhaps after shrinking B^n) by the implicit function theorem, using the fact that the Jacobian of the equation $\Phi(\lambda, m) = m$ is regular with respect to the s variables.

We define a uniformized flow using the functions λ_i as

$$\tilde{\Phi} : \mathbb{R}^n \times (\mathbb{T}^n \times B^n) \to \mathbb{T}^n \times B^n$$
$$\left((s_1, \ldots, s_n), (x, b)\right) \mapsto \Phi\left(\sum_{i=1}^n s_i \lambda_i(b), (x, b)\right).$$

The period lattice of this $\mathbb{R}^n$ action is $\mathbb{Z}^n$, and therefore constant hence the initial action clearly descends to the quotient to define a new action of the group $\mathbb{T}^n$.

We want to find now functions $\sigma_1, ..., \sigma_n$ such that their Hamiltonian vector fields are precisely the ones constructed above $Y_i = \sum_{j=1}^{n} \lambda_i^j X_{f_j}$. We compute the Lie derivative of the vector fields Y_i using Cartan's formula:

$$\mathcal{L}_{Y_i}\omega = d\iota_{Y_i}\omega + \iota_{Y_i}d\omega$$
$$= d(-\sum_{j=1}^{n} \lambda_i^j df_j)$$
$$= -\sum_{j=1}^{n} d\lambda_i^j \wedge df_j$$

We deduce that

$$\mathcal{L}_{Y_i}\mathcal{L}_{Y_i}\omega = \mathcal{L}_{Y_i}(-\sum_{j=1}^{n} d\lambda_i^j \wedge df_j).$$

In the last equality we have used the fact that λ_i^j are constant on the level sets of F. Lemma 4.3.18 applied to the vector fields Y_i yields $\mathcal{L}_{Y_i}\omega = 0$ and the folded-symplectic structure is preserved.

The next step is to prove that the collection of 1-forms $\iota_{Y_i}\omega$ are exact in the neighbourhood of a Liouville torus. So the new action is indeed Hamiltonian. We apply proposition 4.3.15 in a neighbourhood of a Liouville torus and the symplectic form ω can be written as $\omega = d\alpha$. Now since $\mathcal{L}_{Y_i}\omega = 0$, consider the toric action generated by the vector fields Y_i. Applying the equivariant version of Proposition 4.3.15 with the group $G = \mathbb{T}^n$ the form ω is G-invariant and we can find a new $\bar{\alpha}$ which is at the same time a primitive for the folded symplectic structure ω and $\mathbb{T}^n$-invariant.

Cartan's formula yields:

$$\iota_{Y_i}\omega = -\iota_{Y_i}d\bar{\alpha}$$
$$= -d\iota_{Y_i}\bar{\alpha}.$$

Thus we deduce that the fundamental vector fields Y_i are indeed Hamiltonian with Hamiltonian folded functions $\iota_{Y_i}\bar{\alpha}$. Denoting by $\sigma_1, ..., \sigma_n$ these Hamiltonian functions, they are now the natural candidates for "action" coordinates. Each of these functions defines a smooth Hamiltonian vector field, so by definition they are all folded functions.

We are under the hypotheses of Theorem 4.3.11 (Darboux-Carathéodory theorem), so we can find a coordinate system

$$(t, y_2, ..., y_n, q_1, ..., q_n)$$

and some folded functions σ_i such that

$$\omega = \sum_{i=1}^{n} d\sigma_i \wedge dq_i,$$

and the σ_i depend only on $(t, ..., y_n)$.   The functions σ_i were defined using an equivariant form α_i which is defined in a neighborhood of the whole regular fiber. Hence the σ_i extend to all $U' = \sigma^{-1}(\sigma(U))$.. For the sake of simplicity we denote these extensions using the same notation. The Hamiltonian vector fields of σ_i have period one, so the functions q_i can be viewed as angle variables θ_i. It remains to check that, in the extended functions, ω can be written in the desired *Darboux-type* form.

Observe that $\omega(\frac{\partial}{\partial\sigma_i}, \frac{\partial}{\partial\theta_i}) = \delta_{ij}$ in U' by the own definition of θ_i. By abuse of notation we denote by X_{θ_i} the vector fields which solve the equations: $\iota_{X_{\theta_i}}\omega = -d\theta_i$. By construction, the equality $\omega(Y_i, Y_j) = 0$ holds in U'. This follows from the fact that ω is symplectic away from Z, and since $[Y_i, Y_j] = 0$ we get that $\omega(Y_i, Y_j) = 0$ in $U' \setminus Z$ and hence the equality extends to all U'.

By the Darboux-Carathéodory coordinates

$$\omega(\frac{\partial}{\partial\sigma_i}, \frac{\partial}{\partial\sigma_j}) = \omega(X_{\theta_i}, X_{\theta_j}) = 0$$

in the neighborhood U of the regular point. Applying the definition of exterior derivative, using that ω is closed and that the vector fields commute we obtain:

$$d\omega(X_{\theta_i}, X_{\theta_j}, X_{\sigma_k}) = X_{\theta_i}(\omega(X_{\theta_j}, X_{\sigma_k})) - X_{\theta_j}(\omega(X_{\theta_i}, X_{\sigma_k}))$$
$$+ X_{\sigma_k}(\omega(X_{\theta_i}, X_{\theta_j}))$$
$$= 0$$

Using that $\omega(X_{\sigma_i}, X_{\theta_j}) = \delta_{ij}$ for all i and j, we obtain

$$X_{\sigma_k}(\omega(X_{\theta_i}, X_{\theta_j})) = 0.$$

In particular, by using the joint flow of the vector fields X_{σ_k} we prove that the relation $\omega(\frac{\partial}{\partial\sigma_i}, \frac{\partial}{\partial\sigma_j}) = \{\theta_i, \theta_j\} = 0$ holds in the whole neighborhood U'. We conclude that ω has the desired form

$$\omega = \sum_{i=1}^{n} d\sigma_i \wedge d\theta_i.$$

We consider the change $p_i := -\sigma_i$ so that we can write ω in the form

$$\omega = \sum_{i=1}^{n} d\theta_i \wedge dp_i.$$

Taking some coordinates $(t, b_2, ..., b_n)$ in B^n, the functions $(t, b_2, ..., b_n, \theta_1, ..., \theta_n)$ form a coordinate system and the f_i only depend on $(t, b_2, .., b_n)$. This concludes the proof. □

Desingularization and equivalence with cotangent models

In [121] several examples of b-integrable systems are provided using the b-cotangent lift. In fact, the construction can be generalized to the context of b^m-symplectic manifolds. From the definition of cotangent lift and the results in [121] which we recalled in subsection 4.1.2 we obtain:

Proposition 4.3.19. *The twisted b-cotangent lift of the action of an abelian group G of rank n on M^{2n} yields a b-Hamiltonian action in T^*M. If the action is free or locally free, the twisted cotangent lift yields a b-integrable system.*

In [95] the desingularization of torus actions was explored in detail. As a consequence of theorem 6.1 in [95] where an equivariant desingularization procedure is established for effective torus actions, we obtain the following *desingularized models.*

Proposition 4.3.20. *The equivariant desingularization of a twisted b-cotangent lift of an action of a torus $\mathbb{T}^n$ on M is a twisted folded cotangent lift model.*

Remark 4.3.21. In [147] explicit desingularization formulae are given for action-angle coordinates of desingularized systems. These are convenient for the refinement of KAM theory for singular symplectic manifolds.

Proof. Denote t the defining function of the critical hypersurface Z. The moment map of the action in the S^1 coordinate is a function of the form $f = c\log(|p|)$ for some constant c, where p denotes the momentum coordinate in T^*S^1. Its Hamiltonian vector field is $X_f = c\frac{\partial}{\partial\theta}$. Take $f' = \frac{cp^2}{2}$ as new momentum map component for the folded symplectic structure in T^*M. $\square$

This construction provides a machinery to produce examples of folded integrable systems via desingularization of b-integrable systems which are given by toric actions. However, we know that not all integrable system on a folded symplectic manifold comes from desingularization. Indeed a folded integrable system may not accept a folded cotangent model as we show in the next section.

About equivalence with cotangent models: a case study

For integrable systems in symplectic and b-symplectic geometry, the action-angle coordinates yields has a Darboux-type expression for the associated geometrical structure in a semi-local neighborhood. This does not always apply in the case of folded integrable systems, since we do not obtain an expression as in Martinet theorem:

$$\omega = tdt \wedge dq + \sum_{i=2}^{n} dx_i \wedge dy_i$$

for some coordinates $(t, q, x_1, y_1, ..., x_n, y_n)$. In general, the previous theorem cannot be simplified to obtain such an expression.

Indeed, assume that we can find some action-angle type coordinates of the form, say, $\omega = t dt \wedge d\theta_1 + \sum_{i=2}^{n} dp_i \wedge d\theta_i$ for some $\mathbb{R}$-valued coordinates p_i and S^1-valued coordinates θ_i. In these coordinates, the null line line bundle $\ker i^* \omega|_Z$ is generated by $\langle \frac{\partial}{\partial \theta_1} \rangle$. This would imply that the null line bundle is a fibration by circles near the regular torus but we know this is not the case in general. We will now show an example of folded integrable system on a folded symplectic manifold whose null bundle has only two closed orbits.

Example 4.3.22. *Take the mapping torus of S^2 by a smooth irrational rotation ϕ, which is a symplectomorphism of S^2 equipped with the symplectic form $dh \wedge d\varphi$ which has only two periodic points. We get $S^2 \times S^1$, and a cosymplectic manifold $(\alpha, \tilde{\omega})$ where the form $\tilde{\omega}$ is obtained by gluing $dh \wedge d\varphi$ with $\phi^*(dh \wedge d\theta)$. It satisfies that $\ker \tilde{\omega}$ is a suspended vector field, which is an irrational rotation on each tori given by $h = c$ where h is the height function on S^2.*

By multiplying by S^1, we get $S^2 \times S^1 \times S^1$, which can be endowed with the folded symplectic form

$$\omega = \sin \theta d\theta \wedge \alpha + \tilde{\omega}.$$

The critical hypersurface is given by two copies of $S^2 \times S^1$, at $\theta = 0, \pi$, where $\ker i^* \omega|_Z = \ker \tilde{\omega}$.

The pair $(f_1, f_2) = (\cos \theta, h)$ defines a folded integrable system in $(S^2 \times S^1 \times S^1, \omega)$. Indeed, we have $df_1 \wedge df_2 = -\sin \theta d\theta \wedge dh$, which is non-vanishing in a dense set of M and Z as a section of $\Lambda^2(T_V^* M)$. The null line bundle of of ω is $\ker \tilde{\omega}|_Z$, which generates a vector field which has only two closed orbits at $h = 1$ and $h = 0$. We deduce that this folded integrable system does not admit a cotangent model. In particular, in the normal form obtained in Theorem 4.3.16, none of the functions p_i is of the form $p_i = t^2$ for some defining function t of Z.

This proves the following proposition.

Proposition 4.3.23. *Folded integrable systems do not admit, in general, cotangent models near a regular point.*

Typically, folded symplectic structures exhibited more flexibility (in the geometrical sense) than b-symplectic structures. This is captured by the fact that they adhere to an existence h-principle as proved Cannas [21] and in particular, all 4-dimensional compact orientable manifolds admit a folded structure. On the other hand, the previous proposition can be seen as a rigidity phenomenon, which arises from considering dynamical aspects rather than geometrical ones. This rigidity arises from the existence of a canonical null foliation on the folding hypersurface. For b-symplectic manifolds, this null foliation is not canonical: it is defined up to Hamiltonian vector fields tangent to the leaves. This explains why from this dynamical point of view, this flexibility allows to obtain canonical normal forms for b-integrable systems.

4.3.4   Constructions of integrable systems

In this section, we study the existence of integrable systems on b-symplectic manifolds and their possible desingularization into folded integrable systems. We construct ad-hoc integrable systems on any 4-dimensional b-symplectic manifold whose critical locus admits a transverse Poisson S^1-action, starting from integrable systems defined on the leaf of a cosymplectic manifolds. In what follows we will always assume that the symplectic foliation on the critical set Z contains a compact leaf, and thus Z is a symplectic mapping torus by [91, Theorem 19]. Furthermore, we will assume that the first singular integral induces an S^1-action in a neighborhood of Z which is transverse to the symplectic foliation on Z. In particular, the monodromy obtained by the first return map of the Hamiltonian vector field of the first integral induces a finite group action on the symplectic leaf of Z. The finite group action detects the points where the initial circle action is not free. This means that topologically the cosymplectic structure on Z is given by the symplectic mapping torus by a periodic symplectomorphism of the leaf.

Structure of a b-integrable system on Z

We start analyzing how a b-integrable system behaves on Z, the critical hypersurface of a b-symplectic manifold (M, ω).

Claim. *Let F stand for a b-integrable system on a b-symplectic manifold (M, ω). Then for a fixed symplectic leaf L of Z there is a dense set of points in L that are regular points of F.*

Proof. Assume that the set of regular points in a fixed leaf L is not dense. Then we can find an open neighbourhood U in L which does not contain any regular point, i.e. $df_1 \wedge \cdots \wedge df_n = 0$ (when seen as a section of $\Lambda^n(^bT^*M)$). However, in order for F to define a b-integrable system, one of the functions has to be a genuine (i.e., non-smooth) b-function in a neighborhood of Z. In other words, $f = c \log|t| + g$ with $c \neq 0$ and g a smooth function. We can assume that $f_1 = f$ is a genuine b-function in a neighborhood U' in Z containing U. Since $c \neq 0$, it defines a Hamiltonian vector field whose flow is transverse to the symplectic leaf L. The function f_1 Poisson commutes with all the other integrals, and so the the flow of f_1 preserves $df_1 \wedge \cdots \wedge df_n$.

Denote by φ_t the flow of X_{f_1}. Then the set $V = \{\varphi_t(U') \mid t \in (0, \varepsilon)\}$ is an open subset of Z where $df_1 \wedge \cdots \wedge df_n = 0$. This is a contradiction with the fact that $F = (f_1, \ldots, f_n)$ defines a b-integrable system. $\square$

Once we take into the account that the first integral $f_1 = c \log|t|$ induces an S^1-action, we can deduce the semi-local structure of the system.

Proposition 4.3.24. *Let (M, ω) be a b-symplectic manifold admitting a b-integrable system such that $f_1 = c \log|t|$ defines an S^1-action in the neighborhood of Z. Then*

$(f_2, ..., f_n)$ *induces an integrable system on each symplectic leaf L on Z which is invariant by the monodromy of the S^1-action.*

Proof. The fact that we may always assume that in a neighborhood of Z the first integral is $f_1 = c \log |t|$ follows from remark 16 in [123](see also Proposition 3.5.3 in [120]), where c is the modular period of that connected component, and $f_2, ..., f_n$ are smooth. Observe that because f_1 is regular everywhere in a neighborhood of Z, the induced S^1-action has no fixed points.

By hypothesis, the Hamiltonian vector field $X_{c \log |t|}$ commutes with the Hamiltonian vector fields $X_{f_2}, ..., X_{f_n}$ which implies that the flow φ_t of the S^1-action preserves each of the functions $f_2, ..., f_n$. The flow also preserves the symplectic foliation in Z. Thus, fixing a symplectic leaf L, the flow φ_t satisfies $\varphi_t(L) \cong L$ and $\varphi_t^*(f_2, ..., f_n) = (f_2, ..., f_n)$. This shows that on each leaf the functions $f_2, ..., f_n$ induce the same integrable system. In particular this integrable system in L is preserved by the first return map of the monodromy in that fixed leaf, implying that the system is invariant by that finite group action. □

Remark 4.3.25. In the jargon of three-dimensional geometry, the connected components of the critical set Z of a 4-dimensional manifold are Seifert manifolds with orientable base and vanishing Euler number. This follows from the fact that Z is a mapping torus and that the first Hamiltonian vector field induces an S^1-action without fixed points.

Construction of b-integrable systems

Taking into the account the last remark, in order to construct b-integrable systems we will assume that Z is the mapping torus of a periodic symplectomorphism of a compact leaf L on Z: that is, there is a choice of modular vector field that defines a transverse b-Hamiltonian S^1-action. This periodic symplectomorphism defines a finite group action on L. This is why in order to construct b-integrable systems on 4 dimensional b-symplectic manifolds, we start by proving that we can always find a non-constant function which is invariant under a symplectic finite group action on a surface.

Claim. *Let $\mathbb{Z}_k$ be a finite group acting of a symplectic surface Σ. Then there exists a non-constant analytic function F invariant by the group action.*

Proof. Take f a generic analytic function in Σ. Consider the averaged function given by the *averaging trick*:

$$F(x) := \sum_{i=1}^{k-1} f(i.x)$$

. By construction this analytic function is invariant by the action of $\mathbb{Z}_k$. Given a point p in Σ, the differential of F vanishes at p if and only if $df_p + df_{2.p} + ... df_{(k-1).p} =$

0. Observe that for a generic f, there exists a point where this condition is not fulfilled. In particular, we deduce that $dF_p \neq 0$ at some point p, and hence F is not a constant function. $\qquad\square$

In the claim above we can replace the analytic condition by a Morse function F. See for instance [195] for a proof of the existence and density of invariant Morse functions by the action of a compact Lie group.

Theorem 4.3.26. *Let (M, ω) be a b-symplectic manifold of dimension 4 with critical set Z which is a mapping torus associated to a periodic symplectomorphism. Then (M, ω) admits a b-integrable system.*

Proof. In this case, a leaf of the critical set is a surface L. Take a neighborhood of Z of the form $U = Z \times (-\varepsilon, \varepsilon)$. Denote by X the Hamiltonian vector field of the function $\log t$ for some defining function of Z. By hypothesis, X defines a Poisson S^1-action in Z transverse to the leaves as studied in [17]. This S^1-action can have some monodromy. Denote by α and β the defining one and two forms of ω at Z. That is, in U we can assume that ω has the form $\omega = \alpha \wedge \frac{dt}{t} + \beta$ with $\alpha \in \Omega^1(Z), \beta \in \Omega^2(Z)$. Recall that both forms are closed and $i_L^* \beta$ is a symplectic form in a leaf L of Z.

The critical set can be described as follows: There is an equivariant cover $L \times S^1 \times (-\varepsilon, \varepsilon)$ of U, and we denote by p the projection to U. This equivariant cover can be equipped with the b-Poisson structure

$$\omega = \pi_{Z_0}^* \tilde{\alpha} \wedge \frac{dt}{t} + \pi_{Z_0}^* \tilde{\beta},$$

where $\tilde{\alpha} = p^* \alpha$ and $\tilde{\beta} = p^* \beta$. Then U is Poisson isomorphic by [17, Corollary 17] to the quotient of the equivariant by the action of a finite group $\mathbb{Z}_k$ in the leaf given by the return time flow of the S^1-action and extended trivially to the neighborhood $L \times S^1 \times (-\varepsilon, \varepsilon)$.

The action of $\mathbb{Z}_k$ acts by symplectomorphisms on L. By Lemma 4.3.4, there is an analytic function F in L which is invariant by the action. In particular, F can be extended to $\tilde{F}$ in all Z by the S^1-action. If π is the projection in $U = Z \times (-\varepsilon, \varepsilon)$ to the first component, then we extend $\tilde{F}$ to U by considering $\pi^* \tilde{F}$ and denote it again $\tilde{F}$.

We construct in the neighborhood U the pair of functions

$$(f_1, f_2) = (\varphi(t) c \log |t|, \varphi(t) \tilde{F})$$

in U. The function $\varphi(t)$ denotes a bump function which is constantly equal to 1 for $t \in (-\delta, \delta)$ and constantly equal to 0 for $|t| > \delta'$, with $\delta < \delta' < \varepsilon$. Observe the functions f_1 and f_2 are linearly independent in $^bT^*M$ in a dense set of $Z \times (-\delta', \delta')$. The Hamiltonian vector field of $\varphi(t) f_1$ generates the transverse S^1 action extended to U, and the Hamiltonian vector field of $\tilde{F}$ is tangent to the symplectic leaves

in each $Z \times \{t_0\}$. Hence $\{f_1, \tilde{F}\} = 0$. Now using the properties of the Poisson bracket we obtain

$$\begin{aligned} \{f_1, \varphi(t)\tilde{F}\} &= -\{\varphi(t)\tilde{F}, f_1\} \\ &= \{\varphi(t), f_1\}\tilde{F} + \{\tilde{F}, f_1\} \\ &= 0 + 0, \end{aligned}$$

where we used that f_1 only depends on the coordinate t. We obtain an integrable system in the neighborhood of the critical locus U. To obtain an integrable system in all M, we do it as in the proof of existence of integrable systems in symplectic manifolds as shown by Brailov (cf. [72]). That is, cover $M \setminus U$ by Darboux balls, each of them equipped with a local integrable system of the form $f_i' = x_i^2 + y_i^2$. By cutting off this system using a function $\varphi(\sum_{i=1}^2 (x_i^2 + y_i^2))$, we can obtain for each Darboux ball a globally defined pair of functions $f_i = \varphi.f_i'$. We can now cover $M \setminus U$ by a finite amount of balls B_i whose intersection is only the union of their boundaries. We choose φ in each ball such that the locally defined integrable systems vanish in all derivatives exactly at these boundaries. The closed set of zero measure where the globally constructed n-tuple of functions are not independent is composed of the boundaries of the balls, and includes $Z \times \{-\varepsilon, \varepsilon\}$. This is illustrated in Figure 4.6, where only some balls are depicted close to the boundary of $Z \times [-\varepsilon, \varepsilon]$. The closed set where the functions vanish are represented by the black-colored boundaries.
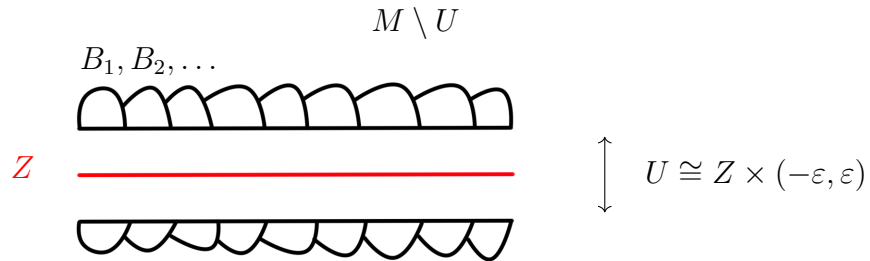

Figure 4.6: Some Darboux balls filling $M \setminus U$.

This allows to glue the system in each ball and with the system we constructed in U, yielding a pair of commuting functions F_1, F_2 such that $dF_1 \wedge dF_2 \neq 0$ in a dense set of M and Z. $\qquad \square$

Remark 4.3.27. The proof generalizes to higher dimensions as long as one can construct an integrable system in the symplectic leaf invariant by the finite group action. This is the content of Claim 4.3.4 for the case of a symplectic surface.

Remark 4.3.28. The original construction of integrable systems in regular symplectic manifolds via the covering of Darboux balls yields an integrable system

without any interesting property. However, the construction in Theorem 4.3.26 gives rise to a lot of examples of b-integrable systems that near the singular set Z can be very rich from a semi-global point of view.

The following theorem is Theorem B in [76]:

Theorem 4.3.29. *Any cosymplectic manifold of dimension 3 is the singular locus of orientable, closed, b-symplectic manifolds.*

In particular, whenever the cosymplectic manifold has periodic monodromy, it can be realized as the critical locus of a b-symplectic manifold with a b-integrable system. Theorem 4.3.29 requires specifically that Z is connected. If we drop that requirement, there is a direct construction (Example 19 in [92]) to realize any cosymplectic manifold as the singular locus of a b-symplectic manifold that we will introduce later.

The proof of Theorem 4.3.26 can be adapted to obtain folded integrable systems in the desingularized folded symplectic manifold resulting from applying Theorem 3.1.17.

Corollary 4.3.30. *Let (M, ω) be a b-symplectic manifold in the hypotheses of Theorem 4.3.26. Then the desingularized folded symplectic manifold (M, ω_ε) admits a folded integrable system.*

Proof. The desingularization given by Theorem 3.1.17 sends ω to ω_ε, which is a folded symplectic structure in M with critical hypersurface Z. The induced structure on Z remains unchanged: it is a cosymplectic manifold with compact leaves whose monodromy is periodic. The S^1-action generated by the modular vector field becomes the null line bundle of ω_ε. Such line bundle is generated in a neighborhood of Z by the Hamiltonian vector field of t^2, where t is defining function of Z. By Claim 4.3.4, there is an analytic function invariant by the first return map X_{t^2}. One can do exactly the same construction as in the proof of Theorem 4.3.26, taking as first function $f_1 = \varphi(t)t^2$ instead of $\varphi(t)\log|t|$ in the neighborhood U of Z. $\square$

4.3.5 About global action-angle coordinates

In this section we extend toric actions on the symplectic leaf on the critical set of a b-symplectic manifold and folded-symplectic manifold to a toric action in the neighbourhood of the critical set Z. Thus obtaining global action-angle coordinates. For certain compact extensions of this neighbourhood we obtain global action-angle coordinates on the compact manifold. In doing so, we explore obstructions for the existence of global-action angle coordinates which lie on the critical set Z.

For a global toric action which we combine with the finite group transversal action given by the cosymplectic structure on Z to produce an example of

integrable system on any b-symplectic/folded symplectic manifold with toric symplectic leaves on the critical set.

By doing so, we explore the limitations that this construction has to admit an extension to a global toric action and thus admit global action-angle coordinates. This limitation lays on the topology of the critical set Z which can be an obstruction for the global existence of action-angle coordinates. In other words, this construction admits global action-angle coordinates if and only if the toric structure of the symplectic leaf of the critical set Z extends to a toric action of the b-symplectic/folded symplectic manifolds. Toric symplectic manifolds are well-understood thanks to [93] and [90].

In this section we will need to following lemma (which is Corollary 16 in [93]):

Lemma 4.3.31. *Let (M^{2n}, Z, ω) be a b-symplectic manifold with a toric action and L a symplectic leaf of Z. Then $Z \cong L \times S^1$.*

Let L be a toric symplectic manifold of dimension $2n - 2$ and $F = (f_2, \ldots, f_n)$ its moment map.

We know from Delzant's theorem [50] that the image of F is a Delzant polytope. From the definition of moment map the components of F Poisson commute and are functionally independent so they form an integrable system on L. Consider now ϕ be a symplectomorphism of L which is equivariant with respect to the toric action and let $Z = L \times [0, 1]/ \sim$ be the cosymplectic manifold associated to it. Extend the integrable system on Z to an integrable system on Z just by observing that by hypothesis the toric action commutes with the symplectomorphism defining the cosymplectic manifold. Observe that the integrable system F on the leaf extends to Z only if Z is a product or F is invariant by the monodromy. Denote by (α, ω) the pair of 1 and 2-forms associated to the cosymplectic structure i.e, ω restricted to the symplectic leaves defines the symplectic structure on Z and α is a closed form defining the codimension one symplectic foliation.

Following the extension theorem (Theorem 50 in [92]) we consider now the open b-symplectic manifold $U = Z \times (-\epsilon, \epsilon)$ with b-symplectic form,

$$\omega = \frac{df}{f} \wedge \pi^*(\alpha) + \pi^*(\omega)$$

where $\pi : U \to Z$ stands for the projection in the first component of U, Z and f is the defining function for the critical set Z.

Consider the map $\hat{F} = (c \log |t|, \pi^*(f_2), \ldots, \pi^*(f_n))$ on with c the modular period of Z where we abuse notation and we write the components on the covering $L \times [0, 1]$ of the mapping torus Z.

In this section we prove,

Theorem 4.3.32. *The mapping $\hat{F} = (c \log |t|, \pi^*(f_2), \ldots, \pi^*(f_n))$ defines a b-integrable system on the open b-symplectic manifold $Z \times (-\epsilon, \epsilon)$ thus extending the integrable system defined by the toric structure of L. The toric structure of L*

extends to a toric structure on the b-symplectic manifold $Z \times (-\epsilon, \epsilon)$ if and only if the cosymplectic structure of Z is trivial, i.e., $Z = L \times [0, 1]$.

Proof. Observe that the functions $f_2, \ldots, f_n$ define an integrable system on the cosymplectic manifold Z as the gluing symplectomorphism that defines the mapping torus commutes with the torus action defined by F. So this torus action descends to the quotient Z and the functions f_i are well-defined on the mapping torus Z. From the definition of b-symplectic form the projection π is a Poisson map and thus $\{\pi^*(f_i), \pi^*(f_j)\} = \{f_i, f_j\} = 0$ for all $i, j \geq 2$. Observe also that functional independence on a dense set W of L, of the functions $f_2, \ldots, f_n$ on L (a factor of U) together with the functional independence of the pure b-function $c \log |t|$ from the functions $\pi^*(f_2), \ldots, \pi^*(f_n)$ implies the functional independence on the dense open set $W \times I$ with the product topology.

Furthermore, the Poisson bracket $\{c \log |t|, \pi^*(f_j)\} = 0$ from the expression of b-symplectic structure. Thus the system $\hat{F}$ defines an integrable system on $Z \times (-\epsilon, \epsilon)$.

To conclude observe that the action-angle coordinates associated to the global toric action on L extends to Z (and thus to a neighborhood $Z \times (-\epsilon, \epsilon)$ if and only if the action extends to a toric action. We now use Lemma 4.3.31 above to conclude that the toric structure extends to Z if and only if the mapping torus is trivial, i.e., $Z = L \times [0, 1]$. This ends the proof of the theorem. □

Observe that given any cosymplectic compact manifold Z, then following the construction from Example 19 in [92], $Z \times S^1$ admits a b-symplectic structure simply by considering the dual b-Poisson structure (where π is the corank regular Poisson structure associated to the cosymplectic structure and X is a Poisson vector field transverse to the symplectic foliation in Z as it was proved in [91]). The function f is a function vanishing linearly. The critical locus of this b-Poisson structure has as many copies of the original Z as zeros of the function f.

$$\Pi = f(\theta)X \wedge \frac{\partial}{\partial \theta} + \pi$$

The theorem above admits its compact version:

Theorem 4.3.33. *The mapping $\hat{F} = (c \log |f(\theta)|, \pi^*(f_2), \ldots, \pi^*(f_n))$ defines a b-integrable system on the b-symplectic manifold $Z \times S^1$ thus extending the integrable system defined by the toric structure of L. The toric structure of L extends to a toric structure on the b-symplectic manifold $Z \times S^1$ if and only if the cosymplectic structure of Z is trivial, i.e., $Z = L \times S^1$.*

As a corollary we can detect situations in which the topological obstruction to global existence of action-angle coordinates lies in the non-triviality of the mapping torus defined by the critical set Z.

Theorem 4.3.34. *Any b-integrable system on b-symplectic manifold extending a toric system on a symplectic leaf of Z does not admit global action-angle coordinates whenever the critical set Z is not a trivial mapping torus $Z = L \times S^1$.*

Below we show an example of a *b*-symplectic manifold M of dimension 6 an admits some *b*-integrable system which is not toric even though the leaves on the critical hypersurface are toric. Observe that the *b*-integrable system cannot define a toric action (and thus admit global action-angle coordinates) because of the topological structure of Z.

Example 4.3.35 (Topological obstructions to semi-local action-angle coordinates). *Consider a product of spheres $S^2 \times S^2$ with coordinates $(h_1, \theta_2, h_2, \theta_2)$ and standard product symplectic form $\omega = dh_1 \wedge d\theta_1 + dh_2 \wedge d\theta_2$. The map*

$$\varphi : S^2 \times S^2 \longrightarrow S^2 \times S^2$$
$$(p, q) \longmapsto (q, p)$$

is a symplectomorphism satisfying that $\varphi^2 = \mathrm{Id}$. The induced map in homology swaps the generators of $H_2(S^2 \times S^2) \cong H_2(S^2) \oplus H_2(S^2)$. This shows that φ is not in the connected component of the identity, as this would imply that induced map in homology would act trivially [102, Theorem 2.10]. Thus, the mapping torus with gluing diffeomorphism φ cannot be a trivial product $S^2 \times S^2 \times S^1$.

The pair of functions $F = (f_1, f_2) = (h_1 + h_2, h_1 h_2)$ are invariant with respect to φ and hence descend to the mapping torus. Furthermore, they define an integrable system (and in fact a toric action) on $S^2 \times S^2$, since they clearly Poisson commute and satisfy that $df_1 \wedge df_2 = (h_2 - h_2)dh_1 \wedge dh_2 \neq 0$ almost everywhere. In particular, by Remark 4.3.27, any b-symplectic manifold with critical set Z admits a b-integrable system. However since the critical hypersurface is not a trivial product, any b-integrable system will not be toric in a neighborhood of Z.

By the discussion before the statement of Theorem 4.3.33, the cosymplectic manifold N can be realized as a connected component of a critical hypersurface of a compact b-symplectic manifold diffeomorphic to $M = N \times S^1$. Thus any b-integrable system in M will not be toric even in a neighborhood of Z.

Observe that with the magic trick of the desingularization we obtain examples of folded-integrable systems without global action-angle coordinates. This is done by applying Theorem 4.3.34 and the behaviour of torus actions under desingularization studied in [95].

Theorem 4.3.36. *Let F be a folded integrable system obtained by desingularization of a b-integrable system, if the critical set Z of the original b-symplectic structure is not a trivial mapping torus, then the folded integrable system F does not admit global action-angle coordinates.*

Let us finish this section with a couple of concluding remarks:

- For symplectic manifolds the obstructions to global action-angle coordinates started with Duistermaat in his seminal paper [52] where Duistermaat related the existence of obstructions to the existence of monodromy which in its turn was naturally associated to the existence of singularities.

  In this section we have concluded that for a singular symplectic manifold there are topological obstructions for existence of global action-angle coordinates that are detectable at first sight: The critical set Z has to be a trivial mapping torus $Z = L \times [0, 1]$ thus the existence of monodromy associated to this mapping torus is also an obstruction.

- Furthermore, the existence of action-angle coordinates yields a free action of a torus in the neighbourhood of a regular torus action thus the existence of isotropy groups for the candidate of torus action defining the system, automatically implies that the locus with non-trivial isotropy groups is singular for the integrable system. The same holds for a sub-circle. In particular:

Corollary 4.3.37. *Let F be a b-integrable system as in Proposition 4.3.24 on a b-symplectic manifold and denote by T the union of the exceptional orbits of the S^1-action defined by $c \log |t|$. Then the system has singularities at the set T.*

Thus not only the topology of the critical set Z yields an obstruction to existence of global action-angle coordinates but it also detects singularities of integrable systems. In particular along the exceptional orbits for the transverse S^1-action given by Proposition 4.3.24. This motivates us to study singularities of integrable systems on singular symplectic manifolds, study which we will pursue in a different work.

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