Graph Minors and Tree Decompositions

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Abstract

The field of graph minor theory seeks to understand the structure of graphs that exclude a given minor. A major result within this field is Robertson and Seymour's Graph Minor Theorem that states that every minor-closed class of graphs has a finite list of minimal forbidden minors. In proving this theorem, they developed the tools of path and tree decompositions as well as their corresponding structural parameters: treewidth and pathwidth.

This thesis will survey several important results and properties about path and tree decompositions while also understanding their place within this advanced field of mathematics. It will also investigate the existence of large complete binary tree minors in graphs with bounded treewidth and large pathwidth and state a conjecture for this relationship. It will then move on to discuss tree decompositions indexed by subtrees which were investigated in the hope of resolving the conjecture we had for the large complete binary tree minors. More specifically, it will explore another conjecture of Dvořák which states that for every connected graph with bounded treewidth there exists a tree decomposition indexed by a subtree and with bounded width. We suspect that this conjecture is false and as such we will discuss a family of graphs that we believe do not satisfy the conjecture.

Chapter 1

Introduction

In this thesis, we will only be considering simple graphs; that is finite, undirected graphs with no loops or parallel edges. Let G be a graph. A graph H is a *minor* of G if a graph isomorphic to H can be obtained from G by performing a sequence of the following three operations on G: *vertex deletion*, *edge deletion* and *edge contraction*. For vertex and edge deletion, these operations are self-explanatory. For edge contraction, an edge $uv \in E(G)$ is *contracted* by adding a vertex w to G such that $N(w) = N(u) \cup N(v)$ then deleting u and v from $V(G)$. We denote the graph obtained by contracting an edge by G/uv and the minor relation by $H \leq_M G$. H is called a *proper minor* if at least one of the three operations has been performed. We say that a class of graphs $\mathcal{G}$ is *minor-closed* if for every pair of graphs G and H where $G \in \mathcal{G}$ and $H \leq_M G$ we have $H \in \mathcal{G}$

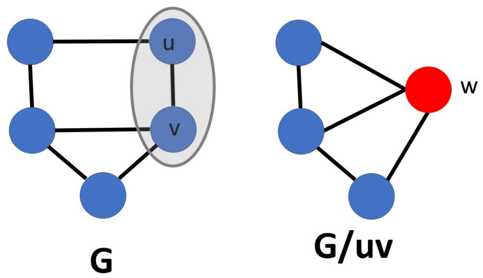


Figure 1.1: Contraction operation.

The field of *Graph Minor Theory* is interested in understanding the structure of graphs that exclude a given minor. In general, it does so by relating the exclusion of a minor to other important structural parameters of graphs such as chromatic number, pathwidth, treewidth, connectivity as well as their topological structure such as planarity and embeddability.

This honours project has two goals. The first is to learn about this advanced field of mathematics. In doing so, we will discuss several important concepts, questions and results that have been obtained within this field. This survey of the field was done in order to give the author a deeper understanding of the key concepts within it. The second goal is to contribute to this field by addressing several open problems within it. Before stating the problems that we address and the results obtained we will first introduce some preliminary concepts.

1.1 Path and Tree Decompositions

Let G be a graph. A *path decomposition* of G is an ordered sequence of sets, $(B_1, \ldots, B_s)$, which satisfies the following three properties:

P1: $B_i \subseteq V(G)$ for all $i \in \{1, \ldots, s\}$;

P2: For all $v \in V(G), \{i : v \in B_i\}$ is a non-empty interval; and

P3: For all $uv \in E(G)$ there exists an $i \in \{1, \ldots, s\}$ such that $u, v \in B_i$.

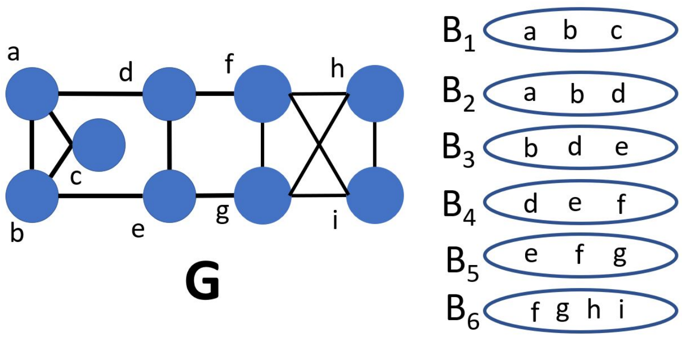


Figure 1.2: Path decomposition of a graph.

The set B_i is called a *bag* and the *width* of a path decomposition is $\max_i |B_i| - 1$. The *pathwidth* of G, $\mathrm{pw}(G)$, is the minimum width of a path decomposition of it. This notion of pathwidth was first introduced by Robertson and Seymour [27].

Similarly, a *tree decomposition* of G is a pair $(T, \mathcal{W})$ where T is a tree and $\mathcal{W} = (W_t : t \in V(T))$ which satisfies the following three properties:

T1: $\bigcup_{t \in V(T)} W_t = V(G)$;

T2: For all $v \in V(G)$, $T_v = T[\{t \in V(T) : v \in W_t\}]$ is a connected subtree of T; and

T3: For all $uv \in E(G)$, there exists $t \in V(T)$ such that $u, v \in W_t$.

The *width* of a tree decomposition is $\max_{t \in V(T)} |W_t| - 1$ and the *treewidth* of G, $\mathrm{tw}(G)$, is the minimum width of a tree decomposition of it. Halin first introduce the concept of tree decompositions in 1976 [14] but it was later rediscovered and substantially developed by Robertson and Seymour in 1984 [28]. These structural parameters, $\mathrm{pw}(G)$ and $\mathrm{tw}(G)$, can be considered to respectively indicate how "path-like" or "tree-like" the graph G is.

1.2 Open Problems Addressed

Both pathwidth and treewidth are central to the two open problems we sought to address. The following theorem is an inspiration to our first problem.

Theorem 1.2.1 (Excluding Forest Minors [4])**.** *For every graph G and forest F, if $\mathrm{pw}(G) \geq |V(F)| - 1$ then $F \leq_M G$.*

We will discuss this theorem further in Section 4.2. A strength of this theorem is its generality in that it requires no structural assumption for G beyond its pathwidth. However this raises the question whether a tighter bound can be found given some structural assumption on G. For our project, we will be considering graphs with bounded treewidth. We will constrain ourselves by only considering the exclusion of the complete binary tree minor with edge-depth h, T_h. This is because their pathwidth increases with h and for every tree T there exists a sufficiently large $h \in \mathbb{N}$ such that $T \leq_M T_h$. As such, there is a universality associated with complete binary trees. Formally, the question we sought to address is the following:

What is the least function f such that every graph with pathwidth at least $f(k, h)$ has either treewidth at least k or contains T_h as a minor?

Currently we have obtained upper bounds on f for several classes of graph. The first is for trees. The second is for connected outer-planar graphs; that is, connected graphs that can be embedded on the plane such that every vertex is on the outer-face. Finally, for weak k-tree which are an edge-maximal class of graphs with treewidth at most k. The following three theorems formally states the results that we have obtained.

Theorem 1.2.2. *For every tree T where $|E(T)| \geq 1$ and $h \in \mathbb{N}$, if $\mathrm{pw}(T) \geq h+1$ then $T_h \leq_M T$.*

Theorem 1.2.3. *For every outer-planar graph G where $|E(G)| \geq 1$ and $h \in \mathbb{N}$, if $\mathrm{pw}(G) \geq 3(h+1)$ then $T_h \leq_M G$.*

Theorem 1.2.4. *For every weak k-tree G where $|E(G)| \geq 1$ and $h \in \mathbb{N}$, if $\mathrm{pw}(G) \geq (k+1)(h+2)-1$ then $T_h \leq_M G$.*

We suspect that Theorem 1.2.4 holds for every graph with treewidth at most k. As such, we have the following conjecture.

Conjecture 1.2.5. *For every $h \in \mathbb{N}$ and graph G where $|E(G)| \geq 2$ and $\mathrm{tw}(G) \leq k$, if $\mathrm{pw}(G) \geq (k+1)(h+2)-1$ then $T_h \leq_M G$.*

In Chapter 6 we will provide a detailed discussion of the results obtained so far on this question. Our main approach to resolve this conjecture was motivated by the following lemma.

Lemma 1.2.6. *Let G be a graph such that $\mathrm{pw}(G) \geq h$. Then for every tree decomposition $(T, \mathcal{W})$ of G with width strictly less than k we have $\mathrm{pw}(T) \geq \lfloor h/k \rfloor$.*

This implies that if G has large pathwidth and there exists a tree decomposition $(T, \mathcal{W})$ of G with small width while also having T be a subtree of G then T will also have large pathwidth. By Theorem 1.2.2, it will then follow that G contains a large complete binary tree minor. This raises the question when does a graph have a tree decomposition with small width indexed by a subtree. This leads to the study of tree decompositions indexed by subtrees. In doing so, we rediscovered the following conjecture by Dvořák.

Conjecture 1.2.7 ([25])**.** *There exists a function f such that every connected graph G has a tree decomposition $(T, \mathcal{W})$ of width at most $f(\mathrm{tw}(G))$ such that T is a subtree of G.*

We have demonstrated that this conjecture holds for trees as well as connected outer-planar graphs. We suspect, however, that this conjecture is false and that we have found a family of graphs that do not satisfy it. In particular, we have the following conjecture.

Conjecture 1.2.8. *For all $k \in \mathbb{N}$ there exists a connected graph G_k with* $\mathrm{tw}(G_k) \leq 2$ *such that every tree decomposition $(T, \mathcal{W})$ of G_k where T is a subtree of G has width at least k.*

In Chapter 7, we will discuss further the family of graphs that we suspect realises Conjecture 1.2.8 as well as the partial results that we have obtained surrounding this topic.

1.3 Structure of the Thesis

This thesis is structured by building up to our original work in order to show its place within the mathematical literature. In Chapter 2, we begin broadly by providing a brief overview to many well-known definitions and theorems in graph theory. This chapter is included for the reader who is unacquainted with the field so that this thesis may be self-contained. As such, we will not prove any of the theorems contained within this chapter. In Chapter 3, we begin to narrow our focus on the sub-field of Graph Minor Theory where we will prove several important theorems related to the structural properties of graphs that exclude particular minors. Many of the proofs incorporated into this chapter are adapted from Norin's lecture notes [24].

In Chapter 4 and 5, we will go into more depth in discussing properties and results about path and tree decompositions. In our final two chapters we discuss the original work that we have done in this field. In Chapter 6, we will go through the partial results obtained in investigating complete binary tree minors in graphs with large pathwidth and small treewidth. In particular, we will prove a bound for trees as well as for weak k-trees.

Finally, in Chapter 7 we will discuss the partial results that we have obtained in studying tree decompositions indexed by subtrees. There we will show that for tree and connected outerplanar graph there exist a tight upper bound for the minimum width of a tree decomposition indexed by a subtree. We will also demonstrate that the existence of a tree decomposition indexed by a minor with width at most k implies the existence of a tree decomposition indexed by a subtree with width at most k. We will conclude the chapter with a discussion of a family of graphs that we suspect realises Conjecture 1.2.8. If this is so, then it will falsify Conjecture 1.2.7.

Chapter 2

Background to Graph Theory

In this chapter we will present several well-known definitions and theorems in Graph Theory that will be drawn upon throughout the thesis. We include this section so that the thesis is self-contained. For a more extensive introduction to this field see Diestel's textbook [12].

2.1 Graphs

A *graph* G consists of a set $V(G)$ whose elements are called *vertices*, and a set $E(G)$ of unordered pairs of vertices called *edges*. The *neighbourhood* of a vertex $v \in V(G)$ is the set $N(v) = \{u : uv \in E(G)\}$ and the *degree* of v, $\deg(v)$, is the size of this set. We denote the minimum and maximum degree of G by $\delta(G)$ and $\Delta(G)$ respectively. We denote the graph obtained by adding an edge uv to the edge set of G by $G \cup \{uv\}$. Similarly, we denote the graph obtained by adding a vertex v to the vertex set of G by $G \cup \{v\}$.

Graphs have a natural visual representation with vertices drawn as dots and edges drawn as curves that connect their respective vertices. We say that graphs G_1 and G_2 are *isomorphic* if there exists a bijection $\phi : V(G_1) \to V(G_2)$ such that $uv \in E(G_1)$ if and only if $\phi(u)\phi(v) \in E(G_2)$. We denote the isomorphism relation by $G_1 \simeq G_2$. For a graph G, we define its *complement*, G^c, by $V(G^c) = V(G)$ and $uv \in E(G^c)$ if and only if $uv \notin E(G)$.

Let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of G. The following notation will be useful for our project. Let $N^-(v)$ and $N^+(v)$ be the

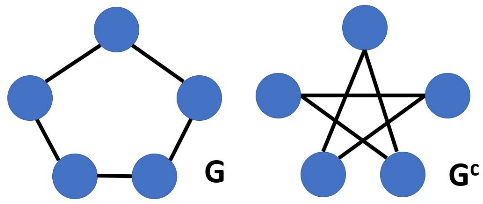


Figure 2.1: Complement of a graph: G^c.

set of neighbours of v in G that are respectively before or after v in the vertex ordering. For the sets that includes v, we will use the notation $N^-[v]$ and $N^+[v]$ respectively.

A graph H is a *subgraph* of G if a graph isomorphic to H may be obtained from G by performing a sequence of the following two operations: *vertex deletion* and *edge deletion*. We denote the subgraph relation by $H \subseteq G$. For vertex deletion, we delete vertex v by removing v from G and every edge incident to it. This operation is denoted by $G-v$. For edge deletion, we remove the edge e from G and we denote this operation by $G-e$. A subgraph H is *proper* if at least one of these operations has been performed on G. For $S \subseteq V(G)$, the subgraph obtained by deleting all the vertices in $V(G) \setminus \{S\}$ is denoted by $G[S]$.

2.2 Important Classes of Graphs

We will now introduce several well-known classes of graphs. The first is the *cycle graph* on $n \geq 3$ vertices, C_n. A characteristic of C_n is that it contains a vertex ordering $v_1, \ldots, v_n$ such that $v_iv_j \in E(C_n)$ if and only if $j = i \pm 1 \mod n$. A graph G is *acyclic* if for every $n \geq 3$, G has no subgraph isomorphic to C_n.

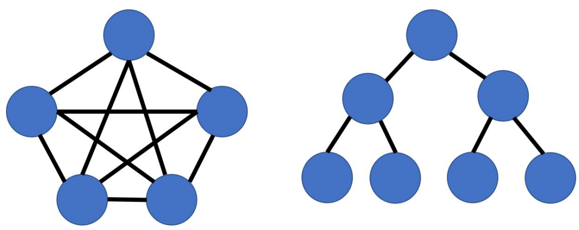


Figure 2.2: A complete graph and a binary tree: K_5 and T_2.

The second class of graphs are the *complete graphs* on n vertices, K_n. This graph is defined with $|V(K_n)| = n$ for some $n \in \mathbb{N}$ and $uv \in E(K_n)$

for every distinct $u, v \in V(K_n)$. We note that every graph G on at most n vertices is a subgraph of K_n.

For a graph G, let $S \subseteq V(G)$. We say that S is a *clique* of G if $G[S] \simeq K_n$ for some $n \in \mathbb{N}$. The *clique number*, $\omega(G)$, is the size of the largest clique of G. Conversely, if $G[S]$ has no edges then S is an *independent set* of G. The independence number, $\alpha(G)$, is the size of the largest independent set of G.

Let G_1 and G_2 be two vertex-disjoint graphs and let $X_1 \subseteq V(G_1)$ and $X_2 \subseteq V(G_2)$ be two cliques with $|X_1| = |X_2| = k$. Let $f : X_1 \to X_2$ be a bijection. A *k-sum* of G_1 and G_2 is obtained from $G_1 \cup G_2$ by identifying x and $f(x)$ for all $x \in X_1$ and possibly deleting some edges with both ends in the clique.

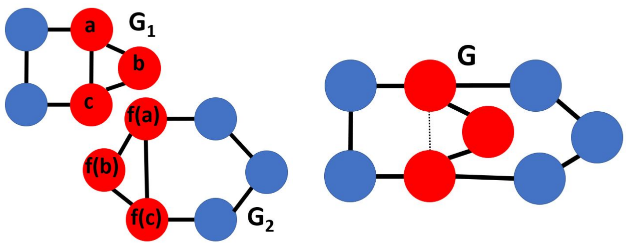


Figure 2.3: G may be obtained by a 3-sum of G_1 and G_2.

Another important class of graphs are bipartite graphs. For a graph G, a *proper bipartition* consists of two vertex sets, (A, B), such that $A \cap B = \emptyset$, $A \cup B = V(G)$, and A and B are independent sets. A graph G is *bipartite* if it contains a proper bipartition. Within the class of bipartite graphs, we have the subclass of the *complete bipartite graphs*, $K_{a,b}$. $K_{a,b}$ is a bipartite graph with $|A| = a$, $|B| = b$ and $uv \in E(K_{a,b})$ if and only if $u \in A$ and $v \in B$.

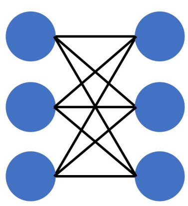


Figure 2.4: The complete bipartite graph: $K_{3,3}$.

The next class of graphs we will consider are forests. A graph G is a *forest* if it is acyclic. If G is also connected, then we say that it is a *tree*. In general, we denote forests and trees by F and T respectively. If

$deg(v) = 1$ for some $v \in V(T)$, then v is a *leaf* of T. If a subgraph of a graph is connected and acyclic then we call it a *subtree*.

A *rooted tree* (T, r) is a tree with a "special" vertex $r \in V(T)$. The *tree order of* (T, r), $\leq_T$, is a relation such that for all $u, v \in V(T)$, we have $u \leq_T v$ if u is on the unique (r, v)-path in T. We say that u is an *ancestor* of v and v is a *descendant* of u. Furthermore, if we also have $uv \in E(T)$, then u is the *parent* of v and v is a *child* of u. The *vertex-depth* of (T, r) is the number of vertices in the longest (r, v)-path in T for some $v \in V(T)$. Similarly, the *edge-depth* of (T, r) is the number of edges in the longest (r, v)-path in T for some $v \in V(T)$. Note that the edge-depth of a rooted tree is one less than the vertex-depth.

There are several types of trees that are noteworthy. The first is the *star graph*, S_k, which is a tree with one internal vertex and k leaves. This graph is isomorphic to $K_{1,k}$. The second tree is a *path*, P_ℓ, which is a tree on ℓ vertices that has at most two vertices of degree 1 with the rest having degree 2 (except when $\ell = 1$ in which case $P_\ell \simeq K_1$).

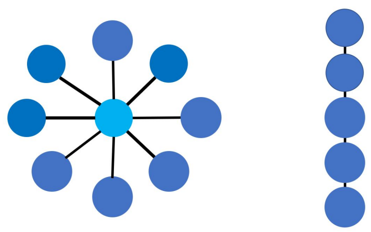


Figure 2.5: A star and path graph: S_8 and P_5.

Another important type of tree that we will consider in this project is the *complete binary tree of edge-depth* h, T_h. This is a rooted tree where each vertex that is not a leaf has exactly two children. See figure 2.2 for T_2, the complete binary tree with edge-depth 2. A similar tree is the *complete ternary tree of edge-depth* h, H_h, which is a rooted tree where each vertex that is not a leaf has exactly three children.

The final class of graphs that we will consider are *planar graphs*. A graph G is *planar* if it can be drawn in the plane without any edges crossing. The *faces* of a drawing of a planar graph are the regions enclosed by edges.

2.3 Vertex Colouring

We will now move on to discuss vertex colouring. A *vertex colouring* of a graph G is defined by a colouring function $\mathrm{col} : V(G) \to C$ where C is a finite set of colours. We say that f is *proper* if $\mathrm{col}(a) \neq \mathrm{col}(b)$ for every $ab \in E(G)$. If there exists a proper colouring of G that uses t colours, then we say that G is *t-colourable*. The minimum $t \in \mathbb{N}$ such that G has a proper colouring function on t colours is called the *chromatic number* of G and is denoted by $\chi(G)$. A simple method to colour a graph G on n vertices is to use *greedy colouring*. The input is an ordering of the vertices $v_1, \ldots, v_n$. For $i \in \{1, \ldots, n\}$, assign the smallest natural number to v_i such that none of its predecessors that are adjacent to it have the same colour. One of the most famous result in Graph Theory involves colouring and is the following theorem.

Theorem 2.3.1 (Four-Colour Theorem [2], [1], [35])**.** *Every planar graph is 4-colourable.*

See [42] for an overview of the history that led to the eventual proof of this theorem.

2.4 Connectivity

A graph G is *connected* if for every $u, v \in V(G)$ there exists a (u, v)-path in G. Furthermore, we say that G is *k-connected* if $|V(G)| \geq k + 1$ and $G \setminus X$ is connected for all $X \subseteq V(G)$ such that $|X| < k$. A *separation* of G is a pair (A, B) such that $A, B \subseteq V(G)$ and $A \cup B = V(G)$, and no edge of G has one end in $A - B$ and the other in $B - A$. The order of a separation is $|A \cap B|$. Similarly for $C, D \subseteq V(G)$, a *C, D-separator* is a set $S \subseteq V(G)$ such that all (C, D)-paths in G uses a vertex in S.

Theorem 2.4.1 (Menger's Theorem)**.** *For every graph G, $k \in \mathbb{N}$, and subsets $Q, R \subseteq V(G)$, either:*

1. *There exists pairwise vertex disjoint paths $P_1, P_2, \ldots, P_k$, each with one end in Q and the other end in R; or*

2. *There exists a separation (A, B) of G of order less than k such that $Q \subseteq A$ and $R \subseteq B$.*

Chapter 3

Excluding Minors

In this chapter we will provide an overview of the structure of graphs that exclude some of the graphs that were introduced in the previous chapter. To commence this chapter, we will present several lemmas that will be used for these proofs.

3.1 Models

A *model of a graph H in a graph G* is a function $\mu : V(H) \to \{\text{subgraphs of } G\}$ such that the following holds:

M1: For each $v \in V(H)$, $\mu(v)$ is connected;

M2: If $uv \in E(H)$ then there exists $u' \in \mu(u)$ and $v' \in \mu(v)$ such that $u'v' \in E(G)$; and

M3: For all distinct $u, v \in V(H)$, $\mu(u)$ and $\mu(v)$ are vertex-disjoint.

We say that μ is a *tree-model of H in G* if $\mu(v)$ is a tree for every $v \in V(H)$. If G contains a model μ' of H then G contains a tree-model μ of H. This can be seen by letting $\mu(v)$ be a subtree of G that spans $\mu'(v)$ for all $v \in V(H)$. Such subtrees exists since $\mu'(v)$ is connected for all $v \in V(H)$. As such, it follows that G has a tree-model of H if and only if G has a model of H. We say that μ *contains* X for some $X \subseteq V(G)$ if for every $x \in X$ there exists $v \in V(H)$ such that $x \in V(\mu(v))$.

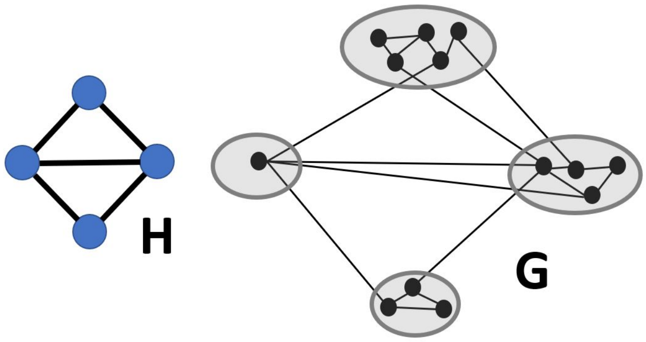


Figure 3.1: Example of a model.

Lemma 3.1.1. *For every connected graph G that contains a model of a graph H, there exists a model of H that contains $V(G)$.*

Proof. For the sake of contradiction, suppose no model of H contains $V(G)$. Let μ be a vertex-maximal model of H in G. Since G is connected, there exists an edge $uv \in E(G)$ such that μ does not contain v but $u \in V(\mu(x))$ for some $x \in V(H)$. By adding uv and v to $\mu(x)$ we maintain μ being a model of H in G. As such, we contradict the maximality of μ. $\square$

Models are an important tool in Graph Minor Theory as they provide a characterisation for graphs that contains another graph as a minor. This is demonstrated by the following lemma.

Lemma 3.1.2. *A graph G contains a model of a graph H if and only if $H \leq_M G$.*

Proof. Suppose there exists a model μ of H in G. For every $v \in V(H)$, we may contract the edges of $\mu(v)$ to identify all the vertices in $V(\mu(v))$ by a single vertex, x_v. By deleting all the other vertices and edges not of the form $x_u x_v$ where $uv \in E(H)$, we obtain a graph isomorphic to H. As such, $H \leq_M G$.

Conversely, suppose that $H \leq_M G$ and $|V(H)| = n_H$, $|V(G)| = n_G$. We claim that there exists a model μ of H in G. We proceed by induction on n_G. For the base case, $n_G = n_H$, H must be obtained by only deleting edges and hence it is a subgraph of G. As such, there exists a bijection $\mu : V(H) \to V(G)$ which defines a model of H in G.

Now suppose that $n_G > n_H$. We note that if a graph K contains H as a model, then any graph G such that $K \subseteq G$ would also contain H as a model. As such, we may assume that H is obtained by performing a sequence of contraction operations only on G by replacing G with a subgraph of itself if necessary.

Let uv be the first edge contracted in the sequence, $G' = G/uv$ and w be the vertex obtained by identifying u and v. Since G' contains H as a minor and $|V(G')| < n_G$, it follows by induction that G' contains a model of H. By Lemma 3.1.1, there exists a model μ' of H in G' which contains $V(G)$. As such, $w \in \mu(x)$ for some $x \in V(H)$. Let μ be obtained from μ' by deleting w and placing u, v and the edge uv in $\mu(x)$ as well as swapping the edges between w and its neighbours in μ with those that correspond to u and v. Then μ is a model of H in G. The result therefore follows by induction. □

We draw upon this lemma regularly to prove other structural properties of graphs that exclude a given minor.

3.2 Subdivisions and Topological Minors

Another concept that is similar to minors is topological minors. We say that J is a *subdivision of a graph* H if a graph isomorphic to J can be obtained from H by replacing some of its edges with internally vertex disjoint paths. If a graph G contains a subdivision of H then H is a *topological minor* of G. Note that a topological minor of G is a minor of G whereas the converse is not always true. For instance, consider the graph in Figure 3.2. By contracting the edge uv we obtain a K_5 minor. However, this graph does not contain a K_5 topological minor (in order to contain a K_5 topological minor, it would require the graph to have at least 5 vertices with degree at least 4, but this graph only contains 4 vertices with degree at least 4).

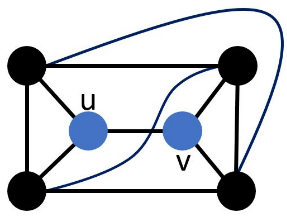


Figure 3.2: G contains a K_5 minor but no K_5 topological minor.

The following lemma provides a case for when the converse does hold.

Lemma 3.2.1. *For all graphs G and H where $\Delta(H) \leq 3$, if $H \leq_M G$ then H is also a topological minor of G.*

Proof. Similar to the previous lemma, we may assume that no proper subgraph of G has H as a minor by replacing G with a subgraph of itself if necessary. From Lemma 3.1.2, G contains a tree-model μ of H. For every $v \in V(H)$, let $T_v = \mu(v)$. If there exists a leaf in T_v that is not adjacent to some T_u for $u, v \in V(H)$, then we can delete that vertex without contradicting μ being a model of H. As such, since no proper subgraph of G has a minor of H, it follows that every leaf of the tree T_v is incident to a vertex in T_u. By a similar reasoning, it follows that there is a unique edge in G between T_u and T_v whenever $uv \in E(H)$. Denote the vertex of T_v incident to this edge by $\ell_{vu} \in V(T_v)$. Since H has maximum degree 3, for every $v \in V(H)$ there exists a vertex $x_v \in V(T_v)$ with $\deg(x_v) = \deg(v)$ and paths that are internally disjoint which connects x_v to ℓ_{vu} for each neighbour u of v. The subgraph induced by the vertices on the (x_v, ℓ_{vu})-paths in G for all $uv \in E(H)$ is a subdivision of H where the vertices $\{x_v : v \in V(H)\}$ correspond to the vertices of H. $\square$

We note that if $\Delta(H) > 3$, then this argument does not hold since $\mu(v)$ may not necessarily contain a vertex x_v that satisfies $\deg(x_v) = \deg(v)$ (see Figure 3.3). We will now move on to consider the structure of graphs

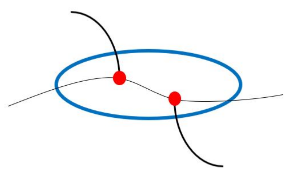


Figure 3.3: Subdivision breaks down for $\Delta(H) > 3$.

that exclude a given minor.

3.3 Excluding Forests

We begin by first considering the structure of graphs that exclude two special types of forests as minors. We first consider excluding paths.

3.3.1 Excluding a Path

Let (T, r) be a rooted tree. The *closure of* (T, r), $\mathrm{clo}(T, r)$, is the graph whose vertex set is $V(T)$ and edge set is $\{uv : u \leq_T v, u \neq v, u, v \in V(T)\}$ (see Figure 3.4 for an example). We note without proof that for a connected graph G, we have $G \subseteq \mathrm{clo}(T_d, r)$ where T_d is a depth-first spanning tree rooted at r. The *tree-depth* of a connected graph, $\mathrm{td}(G)$, is the minimum vertex-depth of a rooted tree (T, r) such that G is a subgraph of $\mathrm{clo}(T, r)$. If G is not connected, then the tree-depth of G is the maximum tree-depth of a component of G. The following theorem shows that if a graph has bounded tree-depth then it does not have a large path as a minor.

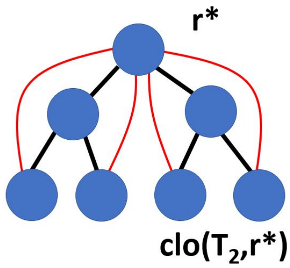


Figure 3.4: Closure of a tree: $\mathrm{clo}(T_2, r^\star)$.

Theorem 3.3.1. *For every graph G with $\mathrm{td}(G) \leq k$, G does not contain any path with 2^k or more vertices as a minor.*

Proof. Without loss of generality, we may assume that G is connected. We proceed by induction on k. For the base case, $k = 1$, $G \simeq K_1$. As such, it does not contain any path with 2 or more vertices.

Now suppose that G has $\mathrm{td}(G) = k > 1$ and let P be the path of maximum length in G. Let (T, r) be a rooted tree of vertex-depth k such that G is a subgraph of $\mathrm{clo}(T, r)$. Let $T_1, T_2, \ldots, T_s$ be the components of $\mathrm{clo}(T, r) - r$ and $P_i = P \cap T_i$ for all $i \in \{1, \ldots, s\}$. Since T_i has depth at most $k - 1$, it follows from induction that $|V(P_i)| < 2^{k-1}$. Furthermore, since P can only go through the root r once it follows that P_i is non-empty for at most two components, say P_1 and P_2. Thus, $|V(P)| \leq |V(P_1)| + |V(P_2)| + 1 < 2 * 2^{k-1} = 2^k$. By the maximality of the choice of P, G therefore does not contain any paths with 2^k or more vertices as a minor. □

Nešetřil and De Mendez have also shown that the converse holds; that is, if the length of the longest path in a graph G is bounded then G has bounded tree-depth [23].

3.3.2 Excluding a Star

Recall that S_k is a tree with one internal vertex and k leaves. We now work towards proving that if a graph is S_k-minor-free then it is a subdivision of a graph with bounded number of vertices.

Lemma 3.3.2. *For every connected graph G and integer $k \geq 3$ the following three statements are equivalent:*

1. $S_k \leq_M G$;
2. *G contains a subtree with at least k leaves; and*
3. *G contains a spanning tree with at least k leaves.*

Proof. Suppose G contains a spanning tree T with j leaves where $j \geq k$. Then T is a subtree of G. By deleting every edge in $E(G) \setminus E(T)$ and repeatedly contracting all the edges in T that are not incident to a leaf we obtain a graph isomorphic to S_j. Hence $S_j \leq_M G$ and thus (3) $\rightarrow$ (2) $\rightarrow$ (1).

It remains to show that (1) $\implies$ (3). Suppose $S_k \leq_M G$. By Lemma 3.1.1 and Lemma 3.1.2, there exists a tree-model μ of S_k in G that contains $V(G)$. Let $t \in V(S_k)$ be such that $\deg(t) = k$. Let X be a set of $u'v' \in E(G)$ such that $|X| = k$ and for every edge $ut \in E(S_k)$ there exists a $u't' \in X$ such that $u' \in V(\mu(u)$ and $t' \in V(\mu(t))$. Let $T = \cup_{v \in V(S_k)} \mu(v) \cup X$. Then T is a spanning tree of G. Furthermore, for every $v \in V(S_k) \setminus \{t\}$, $\mu(v)$ contributes a leaf to T. It therefore follows that T has at least k leaves as required. □

Lemma 3.3.3. *For every connected graph G with n vertices and no vertices of degree 2, there exists a spanning tree T of G with at least $(n + 14)/10$ leaves.*

Proof. Let T be a spanning tree of G with the maximum number of leaves. For the sake of contradiction, suppose T has k leaves with $k < (n+14)/10$. Furthermore, suppose that T has more than $k - 2$ vertices of degree at least 3. Then this will result in T having at least $k + 1$ leaves, a contradiction. As such, T has at most $k - 2$ vertices with degree at most

3. Let T_0 be obtained from T by repeatedly contracting edges that are incident to vertices of degree 2 until T has no vertices of degree 2. Then $|V(T_0)| \leq k+(k-2) = 2k-2$ and $|E(T_0)| = |V(T_0)|-1 \leq 2k-3$. As such, G has at least $n-(2k-2)$ vertices not in T_0. Thus, the average number of vertices that are removed in T for each edge in T_0 is $\frac{n-(2k-2)}{2k-3} > 4$. As such, there exists a path $v_1v_2v_3v_4v_5$ in T which was contracted down to a single vertex-pair in T_0. Now since $\deg(v_3) \geq 3$ in G, there exists a vertex $u \in N(v_3) \setminus \{v_2, v_4\}$. By replacing either v_1v_2 or v_4v_5 with uv_3, we construct a spanning tree of G with at least one more leaf than T. This is because adding the edge uv_3 may result in the removal of a leaf but the deletion of the other edge results in two more leaves being added to T (see Figure 3.5). In doing so, we contradict the maximality of the number of leaves in T. Hence T must have at least $(n+14)/10$ leaves as required.

□

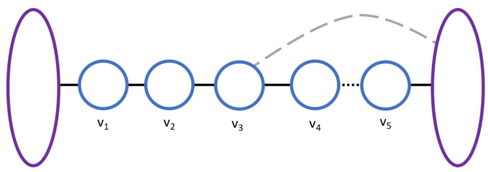


Figure 3.5: Five vertices in a row.

We may now prove that if a graph is S_k-minor-free then it is a subdivision of a graph with bounded number of vertices.

Theorem 3.3.4. *If a connected graph G is S_k-minor-free then G is a subdivision of a graph with strictly less than $10k - 14$ vertices.*

Proof. Fix $k \in \mathbb{N}$ and let G be a connected graph such that $S_k \not\leq_M G$. For the sake of contradiction, suppose G is not a subdivision of any graph with strictly less than $10k-14$ vertices. Let G' be a vertex-minimal graph such that G is a subdivision of it. Then $|V(G')| \geq 10k-14$. Furthermore, by the minimality of G' it follows that it does not contain any vertices of degree 2. By Lemma 3.3.3 there exist a spanning tree T of G' with at least k leaves. Furthermore, by Lemma 3.3.2 we have $S_k \leq_M G'$ and since $G' \leq_M G$ we have $S_k \leq_M G$, a contradiction. □

In Chapter 4, we will present a theorem for a property of graphs that exclude any forest as a minor.

3.4 Excluding K_t Minors

We will now discuss characterisations of graphs that are K_t-minor-free for $t \in \{1, 2, 3, 4, 5\}$. For $t \in \{1, 2\}$, we trivially have the following two theorems.

Theorem 3.4.1. *A graph G is K_1-minor-free if and only if it is the empty graph.*

Theorem 3.4.2. *A graph G is K_2-minor-free if and only if it is a set of isolated vertices.*

We will now move on to the non-trivial cases.

Theorem 3.4.3. *A graph G is K_3-minor-free if and only if G is a forest.*

Proof. If G has a cycle C then by deleting all the vertices in $V(G) - V(C)$ and contracting all the edges except three on C we obtain a K_3 minor. Thus, if G does not contain a K_3 minor it must be acyclic and hence a forest. Conversely, if G has a K_3 minor then by Lemma 3.2.1 it must contain a K_3 subdivision since $\Delta(K_3) = 2$. This subdivision of K_3 defines a cycle in G. As such if G is acyclic it does not contain a K_3 minor. □

Lemma 3.4.4. *Every 3-connected graph contains a K_4 minor.*

Proof. Let G be a 3-connected graph and let $uv \in E(G)$. Since G is 3-connected by Menger's theorem (Theorem 2.4.1) there exists two paths, P and Q, with ends u and v that are internally vertex-disjoint while also not containing the edge uv. Let $p \in V(P) \setminus \{u, v\}$ and $q \in V(Q) \setminus \{u, v\}$. As the graph $G' = G - \{u, v\}$ is connected, G' contains a path S from p to q. By choosing p, q and S such that S is the shortest path between P and Q in G' we may assume that S is internally disjoint from P and Q. Thus, $P \cup Q \cup \{uv\} \cup S$ defines a K_4 subdivision in G where the vertices u, v, p, q correspond to the vertices in K_4 (see Figure 3.6). □

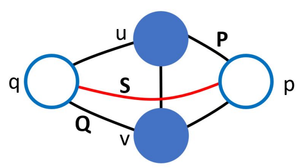


Figure 3.6: K_4 minor in a 3-connected graph.

Recall from Chapter 2 that a k-sum of two vertex-disjoint graphs G_1 and G_2 is obtained from $G_1 \cup G_2$ by identifying the vertices of a clique of size k in both G_1 and G_2 and possibly deleting some edges within that clique. The following two theorems uses k-sums to characterise the structure of graphs that do not contain K_4 and K_5 as minors.

Theorem 3.4.5. *A graph G is K_4-minor-free if and only if G can be obtained by (≤ 2)-sums of graphs on 3 vertices.*

Proof. ($\Longrightarrow$) We proceed by induction on $n = |V(G)|$. Let G be a K_4 minor-free graph on n vertices. For $n < 4$ the claims hold trivially. Suppose that $n \geq 4$. By Lemma 3.4.4, G is not 3-connected. If G is disconnected, then since each component is K_4-minor-free, it follows by induction they can be obtained by (≤ 2)-sums of graphs on at most 3 vertices. As such, since G can be obtained by 0-sums of its component, it follows that G can be obtained by (≤ 2)-sums of graphs on 3 vertices.

If G is connected, then by Menger's theorem (Theorem 2.4.1), there exists an (A, B) separator of G with order at most 2. Let $X = A \cap B$. If $X = \{x\}$ for some $x \in V(G)$ then let $G_1 = G[A]$ and $G_2 = G[B]$. Since both graphs are K_4 minor free it follows by induction that G_1 and G_2 can be obtained by (≤ 2)-sums of graphs on at most 3 vertices. As G can be obtained by a 1-sum of G_1 and G_2 on X, the claim follows.

Otherwise $X = \{x, y\}$ for some distinct $x, y \in V(G)$. Set $G_1 = G[A] \cup \{xy\}$ and $G_2 = G[B] \cup \{xy\}$. By the minimality of the separation, G_1 can be obtained from G by contracting all the edges of the form b_1b_2 where $b_1, b_2 \in B - X$ and hence it is K_4-minor-free. Similarly, G_2 is also K_4 minor-free as it can be obtained from G by contracting all the edges of the form a_1a_2 where $a_1, a_2 \in A - X$. By induction, both G_1 and G_2 can be obtained by (≤ 2)-sums of graphs on 3 vertices. Since G can be obtained from a 2-sum of G_1 and G_2 the claim follows.

($\Longleftarrow$) Suppose that G can be obtained by (≤ 2)-sum of $H_1, H_2, \ldots, H_m$ where $|V(H_j)| \leq 3$ for all $j \in \{1, \ldots, m\}$. Let $G_1 = H_1$ and G_i be obtained by (≤ 2)-sum of G_{i-1} and H_i in such a way that $G_m = G$. Without loss of generality, we may consider only the edge-maximal cases and assume that no edges are deleted in the (≤ 2)-sum operation. We claim that G_i is K_4 minor-free for all $i \in \{1, \ldots, m\}$. We proceed by induction on i. For $i = 1$, since $|V(G_i)| \leq 3$ it trivially does not contain K_4 as a minor.

Now suppose $i > 1$ and for the sake of contradiction suppose that G_i contains a K_4 minor. By Lemma 3.2.1, G_i also contains a K_4 subdivision,

J. Let $Y = \{a, b, c, d\} \subseteq V(J)$ be the set of vertices that have degree 3 in the subdivision. Let $X = \{x, y\}$ be the vertices in which the (≤ 2)-sum of G_{i-1} and H_i was performed on in order to obtain G_i. Let $z \in V(H_i) \setminus \{x, y\}$. Now since $\Delta(H_i) \leq 2$, it follows that $Y \subseteq V(G_{i-1})$. We claim that G_{i-1} must also contain a subdivision of K_4. If $J \subseteq G_{i-1}$ then we are done. Otherwise, J uses the x, z, y path in G. Let J' be obtained from J by replacing the x, z, y path with the x, y path. Then J' is a K_4 subdivision and $J' \subseteq G_{i-1}$. However, by induction G_{i-1} is K_4-minor-free, a contradiction. As such G_i is also K_4-minor-free as required. □

Note that in Chapter 5 we will prove another characterisation of K_4-minor-free graphs which is tied to treewidth. We conclude this section by stating a theorem of Wagner which characterises K_5-minor-free graphs. This theorem uses the Wagner's graph (Figure 3.7).

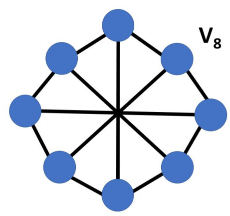


Figure 3.7: Wagner's graph: V_8.

Theorem 3.4.6 ([41])**.** *A graph G is K_5-minor-free if and only if G can be obtained by (≤ 3)-sums of planar graphs and V_8.*

3.5 Planarity and Excluded Minors

We now move on to discuss an important topological characterisation of graphs that excludes particular minors. One of the first major results in Graph Minor Theory is the following.

Theorem 3.5.1 (Wagner's Theorem [41])**.** *A graph is planar if and only if it contains neither K_5 nor $K_{3,3}$ as minors.*

This result is similar to an earlier result by Kuratowski.

Theorem 3.5.2 (Kuratowski's Theorem [19])**.** *A graph is planar if and only if G contain neither K_5 nor $K_{3,3}$ as topological minors.*

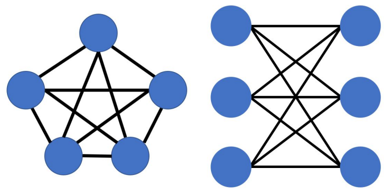


Figure 3.8: Obstacles to planarity: K_5 and $K_{3,3}$ minors.

For this thesis we will not prove these theorems but will simply show that they are equivalent.

Lemma 3.5.3 ([12]). *G has a K_5 or $K_{3,3}$ minor if and only if G has a K_5 or $K_{3,3}$ topological minor.*

Proof. As noted earlier, if H is a topological minor of G then it is also a minor of G. Furthermore, since $\Delta(K_{3,3}) = 3$, if G has a $K_{3,3}$ minor then by Lemma 3.2.1 G contains a $K_{3,3}$ subdivision. Thus, it remains to show that if G has a K_5 minor, then it contains either a K_5 or a $K_{3,3}$ subdivision.

Suppose G has a K_5 minor and let μ be a model of K_5 in G. As we have done previously, we may assume that no proper subgraph of G contains K_5 as a minor by replacing G with a subgraph of itself if necessary. Now for each $a \in V(K_5)$, $\mu(a)$ is a tree which we will denote as T_a. Now for every $a \in V(K_5)$, add the four edges going to the other $T_{a'}$ trees. This gives us a tree with exactly 4 leaves (if it contains other leaves, these could be deleted thus contradicting G being minimal). If each T_a has a vertex of degree 4, then we have a subdivision of K_5. Otherwise, there is some $a \in V(K_5)$ such that T_a has 2 vertices, $\{u, v\}$, where $\deg_{T_a}(u) = \deg_{T_a}(v) = 3$. By Repeatedly contracting all the edges in the other $T_{a'}$ to identify them as a single vertex a' while also contracting all the edges of T_a except for a remaining uv edge, we obtain the graph in Figure 3.9.

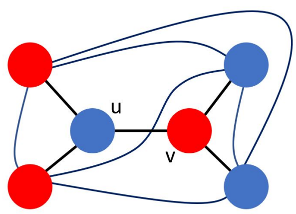


Figure 3.9: Equivalence of Wagner and Kuratowski's theorem.

As this contains a $K_{3,3}$ subgraph it follows that G contains a $K_{3,3}$ minor and hence a $K_{3,3}$ subdivision. $\square$

3.6 Graph Minor Structure Theorem

We now discuss the topological structure of graphs that excludes any given minor. Since every graph on at most n vertices is a minor of K_n, it suffices to consider only the exclusion of the complete graphs as minors. The graph minor structure theorem describes a topological properties of graphs that exclude complete graph minor. Before stating the theorem, we first introduce the relevant background material for it.

A *surface* is a compact connected Hausdorff topological space in which a neighbourhood of every point is homeomorphic to $\mathbb{R}^2$. According to the classification theorem for compact surfaces, we may consider a surface Σ as being constructed from a sphere by adding h handles and c cross-caps where $c \in \{0, 1, 2\}$ [6]. The *Euler genus* of Σ is defined to be $\epsilon(\Sigma) := c + 2h$. If a graph G can be drawn on a surface Σ such that no edges cross, then we say that G can be *embedded* on Σ. The Euler genus of G is the minimum g such that G may be embedded on some surface Σ where $\epsilon(\Sigma) \leq g$.

A *society*, $\Omega = \{v_1, \ldots, v_n\}$ is a cyclic ordering of some set of vertices in G. The pair (G, Ω) is called a *vortex*. A *vortical decomposition* of (G, Ω) is a family of sets $\{B_i : v_i \in V(\Omega)\}$ which satisfy the following:

- V1: For all $i \in \{1, \ldots, n\}$ we have $v_i \in B_i$;
- V2: For all $v \in V(G)$, $\{j : v \in B_j\}$ is a non-empty cyclic interval; and
- V3: For all $vw \in E(G)$, there exists a j such that $v, w \in B_j$.

The *width* of a vortical decomposition is $\max_{i\in\{1,\ldots,n\}} |B_i|$ and the *depth* of a vortex is the minimum width of a vortical decomposition of it. A *multi-vortex* is a tuple $(G, \Omega_1, \ldots, \Omega_p)$ such that (G, Ω_i) is a vortex for every $i \in \{1, \ldots, p\}$ and $V(\Omega_i) \cap V(\Omega_j) = \emptyset$ for every $i \neq j$. The *depth* of a multi-vortex is the maximum width of one of its vortices.

Let $\mathcal{G}_1(g, k, p)$ be the set of graphs such that every graph within the set can be obtained by graphs that can be embedded on a surface of Euler Genus g with at most p vortices added to disjoint faces of the embedding where each vortex has depth at most k. Let $\mathcal{G}_2(g, p, k, a)$ be the set of graphs such that for every $G_2 \in \mathcal{G}_2(g, p, k, a)$ there exists $A \subseteq V(G_2)$ where $|A| \leq a$ such that the connected components of $G - A$ are elements of $\mathcal{G}_1(g, p, k)$. We may now state the graph minor structure theorem.

Theorem 3.6.1 (Graph Minor Structure Theorem [32]). *For every $t \in \mathbb{N}$, there exists $g, p, k, a \in \mathbb{N}$ such that every K_t-minor-free graph G can be obtained from clique-sums of graphs in $\mathcal{G}_2(g, p, k, a)$.*

3.7 Minors and Colouring

For this last section of this chapter, we now discuss the relationship between colouring and minors. One of the most important open problems in Graph Theory may be considered as a generalisation of the four-colour theorem (Theorem 2.3.1). This problem was first proposed in 1943 by Hadwiger and is the following.

Conjecture 3.7.1 (Hadwiger's Conjecture [13]). *If G is a K_{t+1}-minor-free graph, then G is t-colourable.*

Since every graph on n vertices is a subgraph of K_n, we note that this conjecture is equivalent to the following.

Conjecture 3.7.2. *For all graphs G and H where $V(H) = t + 1$, if G is H-minor-free then G is t-colourable.*

When Hadwiger introduced this conjecture, he showed it holds for $t \leq 3$. Wagner has also shown that the four-colour theorem implies it for $t = 4$ [41]. If these conjectures are true, then it will demonstrate a deep interconnection between minors and colouring. Our goal for the rest of this chapter is to show that Hadwiger's conjecture is true for t at most 4 while assuming the four colour theorem (Theorem 2.3.1). Robertson, Seymour and Thomas have also shown that this conjecture holds for $t = 5$ [36] but for any higher t it remains an open question. Kostochka [16] [17] and Thomason [39] [40] have independently prove the best known upper bound for the chromatic number of K_t-minor-free graphs which is $O(t\sqrt{\log(t)})$ For further details of the results obtained surrounding this conjecture see Seymour's survey [37].

3.7.1 Hadwiger's Conjecture for t $\leq$ 4

For $t = 0$, if G is K_1-minor-free, then it is the empty graph and hence 0-colourable. For $t = 1$, if G is K_2-minor-free, then it is a set of isolated vertices and is hence 1-colourable. We now prove the conjecture for $t = 2$.

Theorem 3.7.3. *Every K_3-minor-free graph is 2-colourable.*

Proof. Let G be a K_3-minor-free graph. By Theorem 3.4.3, G is a forest. We claim that every forest F on n vertices is bipartite. We proceed by induction on n. For $n = 1$, the results hold trivially. For $n > 1$, since F is a forest there exists a vertex $v \in V(F)$ with degree at most 1. If v has degree 0, then by induction $F - v$ has a proper bipartition (A', B'). Since v has no neighbours, $A = A' \cup \{v\}$ and $B = B'$ are independent sets and hence (A, B) is a proper bipartition of F as required. Now suppose v has degree 1. Let u be the unique vertex in $N(v)$ and $F' = F - v$. Since F' is a forest it follows by induction that there exists a proper bipartition of its vertices, (A', B'). Without loss of generality, we may assume $u \in A'$. Since $N(v) \cap B = \emptyset$, the bipartition (A, B) where $A = A'$ and $B = B' \cup \{v\}$ is proper. It therefore follows by induction that F is bipartite.

Therefore, G is bipartite with a proper bipartition (A, B) of the vertices. Assign colour 1 to the vertices in A and colour 2 to the vertices in B. Since each set is independent it follows that this is a proper colouring of G and hence G is 2-colourable. □

Theorem 3.7.4. *Every K_4-minor-free graph is 3-colourable.*

Proof. Let G be a K_4-minor-free graph with $n = |V(G)|$. We may assume that G is connected. We proceed by induction on n. For $n \leq 3$, the results hold trivially. For $n \geq 4$, by theorem 3.4.5, G can be obtained by (≤ 2)-sum of connected graphs $H_1, \ldots, H_m$ where $|V(H_i)| \leq 3$ for $i \in \{1, \ldots, m\}$. Let H_m be the last graph added to obtain G, and let G' consists of $G_1, \ldots, G_{m-1}$ such that G is a (≤ 2)-sum of G' and H_m. Since G' is connected and K_4-minor-free, it follows by induction that G' is 3-colourable. Let $\{u, v\}$ be the vertices identified on for the last (≤ 2)-sum. Since $|V(H_m)| \leq 3$, we may assign the colours $\mathrm{col}_{H_m}(v) = \mathrm{col}_{G'}(v)$, $\mathrm{col}_{H_m}(u) = \mathrm{col}_{G'}(u)$, and the other vertex of G_m (if there is one) the remaining colour. Note that $\mathrm{col}_{G'}(v) \neq \mathrm{col}_{G'}(u)$ since $uv \in E(G')$. In doing so, we obtain a 3-colouring of G. □

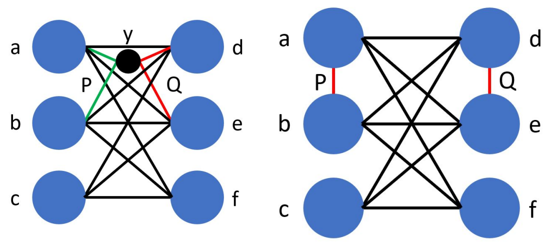
In Chapter 5 we will present two short proofs for these last two theorems using the properties of tree decompositions. We now work towards using the four-colour theorem to prove Hadwiger's conjecture for $t = 4$.

Lemma 3.7.5. *[15] Every non-planar, 4-connected graph contains a K_5 minor.*

Proof. Let G be a non-planar, 4-connected graph. By Theorem 3.5.1 it must contain either K_5 or $K_{3,3}$ as a minor. If it does not contain $K_{3,3}$ as a minor, then the results follow immediately. Thus, we may assume that G has a $K_{3,3}$ minor. Since $\Delta(K_{3,3}) = 3$, by Lemma 3.2.1, $K_{3,3}$ is also a topological minor of G. Let H be a subdivision of $K_{3,3}$ that is a subgraph of G and let $A = \{a, b, c\}$ and $B = \{d, e, f\}$ be the vertices of H of degree 3 that corresponds to the vertices of the proper bipartition of $K_{3,3}$. For each $x \in A$, let H_x denote the component of $H - B$ that contains x. Similarly, for each $x \in B$, let H_x denote the component of $H - A$ that contains x.

Since G is 4-connected, there exists a path P in $G - B$ that joins two vertices in A. Choose P so that $|V(P) \cup V(H)|$ is minimal. Without loss of generality, assume a and b are the endpoints of this path. By minimality, P is a union of a path P_a in H_a, P_b in H_b and another path that joins an end of P_a with an end of P_b that is internally disjoint from H.

Because $|B| = 3 > 2$, there exists $x \in B$ such that $V(P) \cap V(H_x) = \emptyset$. Without loss of generality, assume that $d = x$. Similar to our reasoning above, there exists a path Q in $G - A$ joining d to another vertex in B such that $|V(Q) \cup V(H)|$ is minimal. Without loss of generality, assume e is the second end of Q. By the minimality specification, Q is a union of a path Q_d in H_d, Q_e in H_e and a path joining the end of Q_d with the end of Q_e, which is internally disjoint from H.



(a) Case 1: connected.

(b) Case 2: disconnected.

Figure 3.10: Excluding a K_5 minor for non-planar, 4-connected graphs.

Suppose there exists a $y \in V(P) \cap V(Q)$. By deleting all vertices in $V(G) \setminus \{V(H) \cup V(P) \cup V(Q)\}$ and contracting the edges in H so that only the vertices a, b, c, d, e, f and y remain, we obtain the graph in Figure 3.10a. By contracting the edges af and ce we obtain a K_5 minor in G as required.

Otherwise P and Q are vertex-disjoint. If so, then by deleting all the vertices in $V(G)\setminus\{V(H)\cup V(P)\cup V(Q)\}$ and contracting the edges in H so that only the vertices a, b, c, d, e and f remain, we obtain the graph in Figure 3.10b. By contracting the edge cf we obtain a K_5 minor in G as required. □

Theorem 3.7.6. *Every K_5-minor-free graph is 4-colourable.*

Proof. We proceed by induction on $n = |V(G)|$. For $n \leq 4$, the results hold trivially. Let G be a K_5-minor-free graph and (A, B) be a separation of G with minimum order. If G is 4-connected, then by Lemma 3.7.5 it is planar and thus 4-colourable by the four-colour theorem (Theorem 2.3.1). We may assume then that $|A \cap B| \leq 3$ and let $X = A \cap B$. If $X = \emptyset$ then by induction both $G[A]$ and $G[B]$ are 4-colourable and hence so is G. Otherwise, let G_1 be $G[A] \cup \{z_1\}$ where $N(z_1) = X$. Similarly, let G_2 be $G[B] \cup \{z_2\}$ where $N(z_2) = X$. By the minimality of the separation G_1 and G_2 can be obtained by contracting B and A respectively, thus it follows that both G_1 and G_2 are minors of G.

Suppose that X is an independent set (see Figure 3.11a). Let G'_i be obtained by contracting the edges adjacent to z_i to obtain y_i for $i \in \{1, 2\}$. Since $G'_i \leq_M G$, it is K_5-minor-free and so by induction G'_i is 4-colourable. By permuting the colours of G_1 such that $\mathrm{col}(y_1) = \mathrm{col}(y_2)$ and by assigning this colour to the vertices in X, we obtain a 4-colouring of G.

(a) Case 1: X is an independent set.

(b) Case 2: X is not an independent set.

Figure 3.11: K_5-minor-free graphs are 4-colourable.

Suppose that X is not an independent set (see Figure 3.11b). For $i \in \{1, 2\}$, let $x_i y_i \in E(G)$ for some $x_i, y_i \in X$. Say v is the other vertex in X. Let G'_i be obtained from G_i by contracting $z_i v$ to obtain v_i for $i \in \{1, 2\}$. Then v_i, x_i, y_i is a clique in G'_i. By induction it follows that G'_i is 4-colourable. We may permute the colours in G'_1 such that $\mathrm{col}(x_1) = \mathrm{col}(x_2)$, $\mathrm{col}(y_1) = \mathrm{col}(y_2)$ and $\mathrm{col}(v_1) = \mathrm{col}(v_2)$ which gives a 4-colouring of G. □

Chapter 4

Path Decompositions

Let G be a graph. Recall that a path decomposition of G is an ordered sequence of sets, $(B_1, \dots, B_s)$, which satisfy the following three properties:

P1: $B_i \subseteq V(G)$ for all $i \in \{1, \dots, s\}$;

P2: For all $v \in V(G), \{i : v \in B_i\}$ is a non-empty interval; and

P3: For all $uv \in E(G)$ there exists $i \in \{1, \dots, s\}$ such that $u, v \in B_i$.

In this chapter we will discuss an important theorem for path decompositions. In particular, that only the exclusion of forest minors results in classes of graphs that have bounded pathwidth. We begin by first proving that forests have unbounded pathwidth.

4.1 Forests have Unbounded Pathwidth

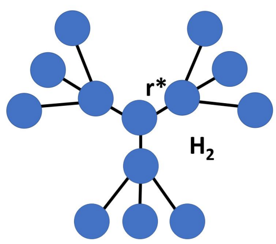


Figure 4.1: Complete ternary tree with edge-depth 2: $(H_2, r^\star)$.

Lemma 4.1.1. *The complete ternary tree of edge-depth d, H_d has* $\mathrm{pw}(H_d) = d$.

Proof. Let $(H_d, r^\star)$ be the complete ternary tree of edge-depth d rooted at $r^\star$. We will first show that $\mathrm{pw}(H_d) \geq d$. We proceed by induction on d. For the base case, $d = 0$, we have $H_0 \simeq K_1$ and as such $\mathrm{pw}(H_0) = 0 = d$.

Now suppose $d > 0$. Let $(H^{(1)}_{d-1}, r_1), (H^{(2)}_{d-1}, r_2), (H^{(3)}_{d-1}, r_2)$ be the three vertex-disjoint connected subgraphs of $H_d - r^\star$ rooted at the neighbours of $r^\star$. Let $B = (B_1, \ldots, B_s)$ be a path decomposition of H_d with minimum width.

We claim that for some $i \in \{1, 2, 3\}$, every bag that contains a vertex of $H^{(i)}_{d-1}$ also contains a vertex in $V(H_d) \setminus V(H^{(i)}_{d-1})$. For each $i \in \{1, 2, 3\}$, let $I_i = \{k : B_k \cap V(H^{(i)}_{d-1}) \neq \emptyset\}$; that is, the set of the indices for the bags of the path decomposition which contains vertices from $H^{(i)}_{d-1}$. As each $H^{(i)}_{d-1}$ is connected, it follows that I_1, I_2 and I_3 are intervals. Let $I = I_1 \cup I_2 \cup I_3, m = \min I$ and $M = \max I$. Without loss of generality, assume that $m = \min(I_2 \cup I_3)$ and $M = \max(I_2 \cup I_3)$. We wish to show that for all $k \in \{m, \ldots, M\}$, B_k contains a vertex of $V(H_d) - V(H^{(1)}_{d-1})$. As $(B_m \cap V(H^{(1)}_{d-1}), .., B_M \cap V(H^{(1)}_{d-1}))$ is a path decomposition of $H^{(1)}_{d-1}$, this will imply that $\mathrm{pw}(H_d) \geq \mathrm{pw}(H^{(1)}_{d-1}) + 1$.

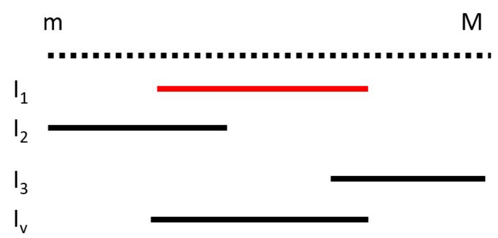


Figure 4.2: Forests have unbounded pathwidth.

Suppose there exists $k \notin I_2 \cup I_3$ for some $k \in \{m, \ldots, M\}$. Then without loss of generality, assume that $I_2 \subseteq [m, k-1]$ and $I_3 \subseteq [k+1, M]$. Let $I_{r^\star} = \{j : r^\star \in B_j\}$. Since $r^\star r_2, r^\star r_3 \in E(H_d)$, it follows that the interval $I_{r^\star}$ overlaps with I_2 and I_3. Therefore $k \in I_{r^\star}$ which proves our claim (see Figure **??**). By induction, we have $\mathrm{pw}(H_d) \geq \mathrm{pw}(H^{(1)}_{d-1}) + 1 \geq (d-1) + 1 = d$.

To show equality, we will now prove that $\mathrm{pw}(H_d) \leq d$. We proceed by induction on d. For $d = 0$, the same argument for the base case above holds. Suppose $d > 0$. Let $H^{(1)}_{d-1}, H^{(2)}_{d-1}, H^{(3)}_{d-1}$ be the three vertex-disjoint connected subgraphs of $H_d - r^\star$. By induction, there exists a path decomposition $B^{(i)}_1, B^{(i)}_2, \dots, B^{(i)}_{q_i}$ of width at most $d-1$ for each $i \in \{1, 2, 3\}$. Then

$$(B^{(1)}_1 \cup \{r\}, \dots, B^{(1)}_{q_1} \cup \{r\}, B^{(2)}_1 \cup \{r\}, \dots, B^{(2)}_{q_2} \cup \{r\}, B^{(3)}_1 \cup \{r\}, \dots, B^{(3)}_{q_3} \cup \{r\})$$

is a path decomposition of H_d. As this has width at most $(d-1)+1 = d$, the result follows by induction. □

4.2 Excluding Forest Minors

In Chapter 1, we introduced the following theorem.

Theorem 4.2.1 (Excluding Forest Minors [4])**.** *For every graph G and forest F, if $\mathrm{pw}(G) \geq |V(F)| - 1$ then $F \leq_M G$.*

This implies that if G does not have F as a minor then $\mathrm{pw}(G) < |V(F)|$. This result is best possible in two senses. First, the value of $|F| - 1$ is sharp because the complete graph K_{n-1} has pathwidth $n-2$ but no F minors on n vertices. Second, since forests are a minor-closed-class of graphs, if H is not a forest then the exclusion of H does not bound the pathwidth of a graph as demonstrated by Lemma 4.1.1.

As mentioned in the introduction, this theorem is an important motivation for our project. We will present a proof of this theorem by Diestel [11]. To do so, we need to first introduce several concepts that will be drawn upon for it.

Terminology Required for the Proof

Let H and G be graphs, $\phi : V(H) \to V(G)$ an injective function. A model μ of H in G is *ϕ-rooted* if $\phi(v) \in V(\mu(v))$ for all $v \in V(H)$. If $X = Im(\phi)$, then we also say that μ is *X-rooted*.

For a set A of vertices in G, let the *boundary* ∂A of A denote the set of vertices in A adjacent to vertices in $V(G) - A$. We say that A has a *H-saturated boundary* if $G[A]$ contains a ∂A-rooted model of H. That is, there exists a model μ of H such that for all $v \in V(H)$, there exists $x \in \partial A \cap \mu(v)$.

The sequence $\mathcal{A} = (A_0, \ldots, A_s)$ is an $\mathcal{A}$-*chain* if $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_s = A$. If $A \subseteq V(G)$, then the *width* of the A-chain is defined as $\max_{i=1}^{s} |(A_i - A_{i-1}) \cup \partial A_{i-1}|$. A is n-*fractured* if there exists an A-chain of width at most n.

A *linkage* $\mathcal{P}$ in a graph G is a collection of pairwise vertex disjoint paths in G. A linkage $\mathcal{P}$ is a (Q, R)-*linkage* for $Q, R \subseteq V(G)$ if every path in $\mathcal{P}$ has one end in Q, the other in R. Q, R are *linked* if $|Q| = |R|$ and there exists a (Q, R)-linkage $\mathcal{P}$ in G with $|\mathcal{P}| = |Q|$.

$\mathcal{A}$-Chains and Path Decompositions

The following lemma shows the connection between $\mathcal{A}$-chains and path decompositions.

Lemma 4.2.2. *For every graph G and integer n, $V(G)$ is n-fractured if and only if* $\mathrm{pw}(G) \leq n - 1$.

Proof. We prove the first direction by construction. Let $B = (B_1, \ldots, B_s)$ be a path decomposition of G with width at most $n - 1$. Define $A_i = B_1 \cup \cdots \cup B_i$ for $i = 0, \ldots, s$. Then $(A_i - A_{i-1}) \cup \partial A_{i-1} \subseteq B_i$. As such, $(A_0, \ldots, A_s)$ is a $V(G)$-chain with width at most n. Thus, $V(G)$ is n-fractured.

For the other direction, if $(A_0, \ldots, A_s)$ is a $V(G)$-chain of width at most n, then

$$(A_1, (A_2 - A_1) \cup \partial A_1, \ldots, (V(G) - A_{s-1}) \cup \partial A_{s-1})$$

is a path decomposition of G with width at most $n - 1$. $\square$

The following lemma will be used to prove the excluding forest minors theorem.

Lemma 4.2.3. *Let $X \subseteq Y \subseteq V(G)$ be such that Y is n-fractured, and ∂X is linked to a subset of ∂Y. Then X is also n-fractured.*

Proof. To show that X is an n-fractured, we construct an X-chain of width at most n. Let $(Y_0, \ldots, Y_s)$ be a Y-chain of width at most n and let $X_i = Y_i \cap X$ for $0 \leq i \leq s$. Then $(X_0, \ldots, X_s)$ is an X-chain. We claim that it has width at most n. Since $X_i - X_{i-1} \subseteq Y_i - Y_{i-1}$, it follows that $|X_i - X_{i-1}| \leq |Y_i - Y_{i-1}|$. Thus to show that X is n-fractured, it suffices to prove that $|\partial X_{i-1}| \leq |\partial Y_{i-1}|$ for $i = 1, .., s$ since Y is n-fractured. Let $\mathcal{P}$ be a $(\partial X, \partial Y)$-linkage which covers ∂X. Let $x \in \partial X_{i-1} - \partial Y_{i-1}$. Then x has neighbour $u \in Y_{i-1} - X_{i-1}$. Since $u \notin X$, it follows that $x \in \partial X$. By assumption, there must exist vertex disjoint paths from $\partial X_{i-1} - \partial Y_{i-1}$ to ∂Y. Since $X_{i-1} \subseteq Y_{i-1}$, it follows that each of these paths contains vertices from $\partial Y_{i-1} - \partial X_{i-1}$. Since the paths are disjoint, this gives an injective function from $\partial X_{i-1} - \partial Y_{i-1}$ to $\partial Y_{i-1} - \partial X_{i-1}$, thus demonstrating that $|\partial X_{i-1}| \leq |\partial Y_{i-1}|$. □

Proof of Theorem 4.2.1. Without loss of generality, we will assume that F is a tree with $|V(F)| = n + 1$ and that G is connected with $\mathrm{pw}(G) \geq n$. Our goal is to show that $F \leq_M G$. Consider a sequence of trees $F_0, F_1, \ldots, F_n, F_{n+1}$ that is constructed with $F_{n+1} = F$ for $i \leq n$ and F_i being obtained from F_{i+1} by deleting a leaf. Let $k \in \{1, \ldots, n\}$ be maximal such that there exists $A \subseteq V(G)$ with the following properties:

A1 A is n-fractured;

A2 A has a F_k-saturated boundary; and

A3 If $A \subset B \subseteq V(G)$, and $|\partial B| \leq k$, then B is not n-fractured.

Note that such a choice is possible since $A = \{v\}$ for some $v \in V(G)$ with $\deg(v) > 0$ satisfies the first two properties for $k = 1$. Furthermore, by Lemma 4.2.2 we have $A \neq V(G)$ since $\mathrm{pw}(G) \geq n$ and thus $V(G)$ does not satisfy A1.

Now by $A2$, $G[A]$ has a ∂A-rooted model of F_k which we denote by μ. Since F is a tree, the unique vertex $v \in V(F_{k+1}) - V(F_k)$ has a unique neighbour $u \in V(F_k)$. Let x be a vertex in $\mu(u) \cap \partial A$. As $x \in \partial A$, it follows that it has a neighbour $y \in V(G) - A$. By setting $\mu(v) = y$, we obtain a model of F_{k+1}. If $k = n$ we are done as it implies $F \leq_M G$.

For the sake of contradiction, assume $k < n$. We aim to contradict the maximality of k by constructing a set that satisfies A1, A2 and A3 for $k + 1$. Let $A' = A \cup \{y\}$. Since A is n-fractured, there exists an A-chain $(A_0, A_1, \ldots, A_s)$ of width at most n . Extending the chain to $(A_0, A_1, \ldots, A_s, A')$, we obtain an A'-chain. This is also n-fractured since $|(A' - A) \cup \partial A| = 1 + k \leq n$.

Choose A'' to be maximal such that $A' \subseteq A''$ and $|\partial A''| \leq k + 1$ and A'' is n-fractured (note that it is valid to ask for a maximal set since A' satisfies those three conditions). Now if $|\partial A''| \leq k$, then this contradicts $A3$, hence $|\partial A''| = k + 1$.

We claim that there exists a $(\partial A', \partial A'')$-linkage. If not then there exists a separation (X, Y) of G of order at most k such that ∂X is linked to a subset of $\partial A''$. Then by Lemma 4.2.3, X is n-fractured and since $|\partial X| \leq k$, this again contradicts $A3$.

Using the paths in the $(\partial A', \partial A'')$-linkage, we may extend the $\partial A'$ rooted model of F_{k+1} in $G[A']$ to be a $\partial A''$-rooted model of F_{k+1} in $G[A'']$. Since A'' satisfies the three properties above with k updated to $k + 1$, this contradicts the maximality of k. $\square$

Chapter 5

Tree Decompositions

Let G be a graph. Recall that a tree decomposition of G is a pair $(T, \mathcal{W})$ where T is a tree and $\mathcal{W} = (W_t : t \in V(T))$ which satisfies the following three properties:

T1: $\bigcup_{t \in V(T)} W_t = V(G)$;

T2: For all $v \in V(G)$, $T_v = T[\{t \in V(T) : v \in W_t\}]$ is a connected subtree of T; and

T3: For all $uv \in E(G)$, there exists a $t \in V(T)$ such that $u, v \in W_t$.

The study of tree decompositions and treewidth have been primarily motivated by their ability to generalise properties of trees to graphs that are "tree-like". In this chapter we will survey various results that involve tree decompositions. We begin by outlining two important areas in which they have been successfully applied. The first being in structural graph theory and the second is in algorithms. We will then move on to prove several properties of tree decompositions follow by an overview of several classes of graphs with bounded treewidth. Following this, we will discuss brambles which are an important tool to determine lower bounds to treewidth. Finally, we conclude this chapter with a discussion of the excluding planar minor theorem.

5.1 Motivation

5.1.1 Graph Minor Theorem

In 1960, Kruskal proved the following theorem for finite trees.

Theorem 5.1.1 (Kruskal [18]). *For every infinite collection of trees $\mathcal{T}$, there exists $T_1, T_2 \in \mathcal{T}$ such that T_1 is a topological minor of T_2.*

This raised the question as to whether a similar result may hold for an infinite collection of finite graphs. Robertson and Seymour set out to generalise this theorem and their project resulted in what is now known as the *graph minor theorem*. This theorem has been described as one of the deepest results in mathematics [12] and its proof spans over 500 pages across 20 papers. The statement of the theorem is as follow.

Theorem 5.1.2 (Graph Minor Theorem [33]). *For every infinite collection of graphs $\mathcal{G}$, there exists distinct $G_1, G_2 \in \mathcal{G}$ such that $G_1 \leq_M G_2$.*

To prove their result, they began by generalising Kruskal's theorem to graphs with bounded treewidth with topological minor replaced by minor. One of their earlier results is the following.

Theorem 5.1.3 ([29]). *For all $k \in \mathbb{N}$, in an infinite collection of graphs $\mathcal{G}$ with treewidth at most k, there exists distinct $G_1, G_2 \in \mathcal{G}$ such that $G_1 \leq_M G_2$.*

This theorem was an important tool that they used to eventually prove the graph minor theorem. As such, their project led to significant theoretical developments in understanding the properties of tree decompositions. As the proof of the graph minor theorem is beyond the scope of this thesis, we will only present a brief sketch of its proof.

Proof of Theorem 5.1.2 (Sketch). For the sake of contradiction, suppose that there exists $G_1, G_2, \ldots,$ such that for all $i < j$ we have $G_i \not\leq_M G_j$. In particular, for all $j \geq 2$ we have $G_1 \not\geq_M G_j$. If G_1 is planar then by Theorem 5.5.1 there exists a $k \in \mathbb{N}$ such that infinitely many of $G_2, G_3, \ldots,$ have treewidth at most k. But by Theorem 5.1.3, we have a contradiction. As such, infinitely many of the graphs have treewidth at least k. Since none of them contains G_1 as a minor, by the graph minor structure theorem (Theorem 3.6.1) it follows that $G_2, G_3, \ldots,$ all have "nice" structure properties. From this "it can be shown" that $G_i \leq_M G_j$ for some $i < j$, a contradiction. □

As noted in the introduction, a family $\mathcal{F}$ of graphs is *minor-closed* if for every graph G and H where $G \in \mathcal{F}$ and $H \leq_M G$ we also have $H \in \mathcal{F}$. A graph H' is a *forbidden minor for* $\mathcal{F}$ if $H' \notin \mathcal{F}$. We say that H' is *minimal* if none of H' proper minors are forbidden minor for $\mathcal{F}$. The following theorem is an important corollary to the graph minor theorem.

Theorem 5.1.4. *Every minor-closed class of graphs has a finite list of minimal forbidden minors.*

Proof. For the sake of contradiction, suppose there exists a minor-closed class $\mathcal{F}$ that has an infinite list, $H'_1, H'_2, \dots$, of distinct minimal forbidden minors. By the graph minor theorem (Theorem 5.1.2), there exists $i < j$ such that $H'_i \leq_M H'_j$. As such, H'_j is not a minimal forbidden minor which is a contradiction. □

This gives a good characterisation of minor-closed graph classes. In particular, one can show that a graph is not an element of a given minor-closed class if it contains a forbidden minor. For forests, the only minimal forbidden minor is K_3 (Theorem 3.4.3). For planar graphs, the minimal forbidden minors are K_5 and $K_{3,3}$ (Theorem 3.5.1).

Robertson and Seymour have also shown that for a fixed graph H and an input graph G there is a polynomial-time algorithm to check whether $H \leq_M G$ [31]. As such, for every minor-closed class there exists a polynomial-time algorithm to check whether an input graph G is an element of that class. One remarkable implication of this theorem is its implication for the family of knotless graphs. A graph is *knotless* if there exists an embedding into the Euclidean space such that no two cycles of the graph are linked. Currently there are no known algorithms that exists to check whether a given graph is knotless. However, as the family of knotless graphs are minor-closed, it follows that there does exists an algorithm and it is a polynomial-time algorithm to check whether a given input graph is knot-less.

We will now move on to proving Kruskal's Theorem (Theorem 5.1.1). To commence, we must first introduce quasi-order relations and some of their properties which will be needed for the proof.

Quasi-Ordering

A quasi-ordering may be considered as a generalisation of the "less than or equal to" relation. A relation $\leq$ is a *quasi-ordering* if it is reflexive (i.e. $x \leq x$) and transitive (i.e. if $x \leq y$ and $y \leq z$, then $x \leq z$). We say that $(X, \leq)$ is a *well-quasi-ordering* for a set X if for all infinite sequences $x_0, x_1, \ldots$ in X there exists some $i < j$ such that $x_i \leq x_j$. In this case, we say that (x_i, x_j) is a *good pair* of this sequence. If an infinite sequence contains a good pair then it is a *good sequence*, otherwise it is a *bad sequence*. If no two elements in the sequence are pairwise comparable by $\leq$, then the bad sequence is an *infinite antichain*.

Lemma 5.1.5. *$(X, \leq)$ is a well-quasi-ordering if and only if X does not contain either an infinite antichain or an infinite strictly decreasing sequence.*

To prove this result, we will use the following theorem which we quote without proof.

Theorem 5.1.6 (Infinite Ramsey [26])**.** *Let $c \in \mathbb{N} \setminus \{0\}$ and G^∞ be the infinite complete graph with vertex set $\mathbb{N}$. Then for every c-colouring of G^∞ there exists an infinite monochromatic subgraph of G^∞.*

Proof of Lemma 5.1.5. Suppose $(X, \leq)$ is a well-quasi-ordering. Then every infinite sequence has a good pair (x_i, x_j) for some $i < j$. As such, X does not contain either an infinite antichain or an infinite strictly decreasing sequence.

Conversely, suppose X does not contain either an infinite antichain or an infinite strictly decreasing sequence. Let $x_0, x_1, \ldots$ be an infinite sequence in X. Let G^∞ be the infinite complete graph on $\mathbb{N} = \{0, 1, \ldots\}$. Colour the edges ij $(i < j)$ with 3 colours: green if $x_i \leq x_j$, red if $x_i > x_j$ and blue if x_i and x_j are incomparable.

By Theorem 5.1.6, G^∞ has an infinite induced subgraph whose edges all have the same colour. Since X has neither an infinite antichain nor an infinite strictly decreasing sequence, this colour must be green. Therefore, there exists an infinite subsequence of $x_0, x_1, \ldots$ such that $x_i \leq x_j$ for all $i < j$ in the subsequence. $\square$

By the proof of this lemma we immediately obtain the following result.

Corollary 5.1.7. *If X is well quasi-ordered, then every infinite sequence in X has an infinite increasing subsequence.*

For a set X, let $X^{(<\omega)}$ be the set of all finite sequences of the elements of X. Given a quasi-ordering $(X, \leq_\alpha)$, we may define a quasi-order on

$(X^{(<\omega)}, \leq_\beta)$ as follow: let $A = (a_1, \ldots, a_k)$ and $B = (b_1, b_2, \ldots, b_l)$ where $a_i, b_j \in X$ for all i, j. We write $A \leq_\beta B$ if there exists an order preserving injection $\phi : [k] \to [l]$ such that $a_i \leq_\alpha b_{\phi(i)}$ for all $i \in [k]$.

Lemma 5.1.8. *If $(X, \leq_\alpha)$ is well-quasi-ordered then $(X^{(<\omega)}, \leq_\beta)$ is also well-quasi-ordered.*

Proof. For the sake of contradiction suppose that there exists a bad sequence $A_0, A_1, \ldots, A_n, \ldots$ in $(X^{(<\omega)}, \leq_\beta)$. Choose such a sequence so that $|A_0|$ is minimal, $|A_1|$ is minimal subject to $|A_0|$ being minimal and so forth.

For every $i \in \mathbb{N}$, let $A_i = (a_i, A'_i)$ where a_i is the first element in the sequence and A'_i is the remaining elements. Since $(a_n)_{n\in\mathbb{N}}$ is a good sequence, by Corollary 5.1.7, there exists an infinite increasing subsequence

$$a_{i_0} \leq_\alpha a_{i_1} \leq_\alpha \cdots \leq_\alpha a_{i_n} \leq_\alpha \ldots$$

of $(a_n)_{n\in\mathbb{N}}$. Now consider the sequence

$$A_0, A_1, \ldots, A_{i_0-1}, A'_{i_0}, \ldots, A'_{i_n}, \ldots$$

By the minimality of $(A_n)_{\in\mathbb{N}}$, the above sequence is good and contains a good pair. As such, we have three possibilities for what the good pair is:

- (A_j, A_k) for some $j < k < i_0$;
- (A_j, A'_{i_k}) in which case we have $A_j \leq_\beta A'_{i_k} \leq_\beta A_{i_k}$; or
- (A'_{i_j}, A'_{i_k}) for some $i_j < i_k$ in which case $A_{i_j} \leq_\beta A_{i_k}$ since $a_{i_j} \leq_\alpha a_{i_k}$.

In each case, we obtain a good pair in $(A_n)_{n\in\mathbb{N}}$ which contradicts our assumption that it is bad. □

Proof of Kruskal's Theorem

Using the language of quasi-ordering we may now express Kruskal's Theorem as follow.

Theorem 5.1.1: *The finite trees are well-quasi-ordered by the topological minor relation.*

We present the proof of this theorem by Nash-Williams [22].

Proof. Let (T, r) and (T', r') be two rooted trees with $\leq_T$ and $\leq_{T'}$ being their corresponding tree orders. We define the quasi-ordering $(T, r) \leq_\gamma (T', r')$ if there exists an isomorphism ϕ from subdivisions of T to subtrees of T' such that for every $s, t \in V(T)$ where $s \leq_T t$, we have $\phi(s) \leq_{T'} \phi(t)$. Clearly then, if $(T, r) \leq_\gamma (T', r')$ then T is a topological minor of T'.

We claim that the finite rooted trees are well-quasi-ordered by $\leq_\gamma$. Suppose not and let

$$(T_0, r_0), \ldots, (T_{n-1}, r_{n-1}), \ldots \tag{1*}$$

be a bad sequence of rooted trees. Let $|V(T_0)|$ be minimal subject to $|V(T_1)|$ being minimal and so forth. For every $i \in \mathbb{N}$, let A_i be a finite sequence of the components of $T_i - r_i$ where each component is rooted at a neighbour of r_i. Let $\mathcal{A}$ denote the union of the finite sequences of the rooted trees in A_i for all $i \in \mathbb{N}$.

We claim that $\leq_\gamma$ is a well-quasi-ordering on $\mathcal{A}$. Suppose not and let

$$(T'_0, r'_0), (T'_1, r'_1), \ldots, (T'_{n-1}, r'_{n-1}), \ldots$$

be the sequence of rooted trees in $\mathcal{A}$. For each i, choose $n(i)$ such that $T'_{i_0} \in A_{n(i)}$. Consider the sequence

$$(T_0, r_0), (T_1, r_1), \ldots, (T_{n(0)-1}, r_{n(0)-1}), (T'_0, r'_0), \ldots, (T'_{n-1}, r'_{n-1})$$

As $T'_0 \in A_{n(0)}$, we have $|V(T'_0)| < |V(T_{n(0)})|$, contradicting the minimality of sequence 1^*.

Since $\mathcal{A}$ is well-quasi-ordered, by lemma 5.1.8 it follows that $\mathcal{A}^{(\omega)}$ is also well-quasi-ordered. We note that $A_i \in \mathcal{A}^{(\omega)}$ for all $i \in \mathbb{N}$. As such, the sequence

$$A_1, A_2 \ldots, A_n, \ldots$$

is a good sequence. Therefore, it contains a good pair (A_i, A_j) for some $i < j$. As such, there exists an injection $\phi : V(A_i) \to V(A_j)$ such that whenever $u \leq_{T_i} v$, we have $\phi(u) \leq_{T_j} \phi(v)$ for all $u, v \in V(T_i) \setminus \{r_i\}$. By extending the domain of ϕ to include $\phi(r_i) = r_j$ it follows that $(T_i, r_i) \leq_\gamma (T_j, r_j)$. This gives us a good pair in 1* contradicting our assumption that it is bad. □

5.1.2 Algorithms

The second important motivation for studying tree decompositions is their algorithmic benefits. Many problems on graphs that are NP-complete may be solved in linear time if the graph has bounded treewidth. The method for solving such problems relies on a principle called *dynamic programming*. In essence, the approach is to first take a tree-decomposition of the graph then run an algorithm which uses this decomposition to solve the problem. One example where this is used is the *Independent Set Problem* (ISP). The problem is as follows: the input is a graph G and integer k and the question is whether G has an independent set of size at least k. This problem is NP-complete, but for graphs with bounded treewidth, we have the following.

Theorem 5.1.9. *For all fixed c and graph G, if $tw(G) \leq c$ then ISP can be solved in time $O(|V(G)|)$.*

Note that the running time of this algorithm to solve the independent set problem on graphs with bounded treewidth is exponential with respect to the treewidth. For an overview of algorithms on graphs with bounded treewidth, see Bodlaender's survey [5]. To convey the intuition for this theorem, we present an algorithm for finding the largest independent set for connected graphs with treewidth 1; that is, trees. The principles behind this algorithm can be applied to graphs with bounded treewidth.

Let (T, r) be a tree rooted at r and for all $v \in V(T)$ let T_v be the subtree of T rooted at v. For every $v \in V(T)$, let I_v be the largest independent set in T_v, $\alpha(T_v) = |I_v|$ and $\beta(T_v)$ be the size of the largest independent set in T_v not including v.

Fix $u \in V(T)$ and let $w_1, \ldots, w_d$ be the children of u in T. We claim that $\alpha(T_u) = \max\{1 + \sum_{i=1}^{d} \beta(T_{w_i}), \sum_{i=1}^{d} \alpha(T_{w_i})\}$ and $\beta(T_u) = \sum_{i=1}^{d} \alpha(T_{w_i})$. Let $(T_{w_1}, w_1), \ldots, (T_{w_d}, w_d)$ be the connected components of $T_u - u$ rooted at the children of u. If $u \in I_u$ then we have $w_i \notin I_u$ for all $i \in \{1, \ldots, d\}$ as otherwise it would contradict I_u being an independent set. As such, we have $I_u \cap V(T_u^{(i)}) = \beta(T_{w_i})$ for all $i \in \{1, \ldots, d\}$ and thus $\alpha(T_u) = 1 + \sum_{i=1}^{d} \beta(T_{w_i})$. Otherwise, if $u \notin I_u$ then $I_u \cap V(T_{w_i}) = \alpha(T_{w_i})$ for all $i \in \{1, \ldots, d\}$ and thus $\alpha(T_u) = \sum_{i=1}^{d} \alpha(T_{w_i})$. Note that this is also equal to $\beta(T_u)$.

This gives us an algorithm to calculate the largest independent set for (T, r) in $O(|V(T)|)$ time. By starting at the leaves of T, we can calculate $\alpha(T_v)$ and $\beta(T_v)$ for each vertex according to the above formula as we work our way towards the root.

5.2 Properties of Tree Decompositions

We will now discuss several properties of tree decompositions. Note that as a path decomposition may be considered to be a type of tree decomposition, the following arguments may be slightly modified to also apply to path decompositions.

5.2.1 Separation

Let G be a graph with a tree decomposition $(T, \mathcal{W})$. For every edge $e = t_1t_2 \in E(T)$, let $W_e = W_{t_1} \cap W_{t_2}$. We call W_e an *adhesion set* of $(T, \mathcal{W})$. For a subtree T' of T, let

$$W_{T'} = \{v \in V(G) : \text{ there exists } t \in V(T') \text{ such that } v \in W_t\}.$$

Lemma 5.2.1. *Let $(T, \mathcal{W})$ be a tree decomposition of a graph G. Let e be an edge of T and let T_1 and T_2 be the two components of $T - e$. Then (W_{T_1}, W_{T_2}) is a separation of G, and $W_{T_1} \cap W_{T_2} = W_e$.*

Proof. For (W_{T_1}, W_{T_2}) to be a separation of G, it must satisfy:

1. $W_{T_1} \cup W_{T_2} = V(G)$; and
2. No edge of G has one end in $W_{T_1} - W_{T_2}$ and the other in $W_{T_2} - W_{T_1}$.

Note that for any $t \in V(T)$ we have $t \in V(T_1) \cup V(T_2)$. Thus, since $\cup_{t \in V(T)} W_t = V(G)$ we have $W_{T_1} \cup W_{T_2} = V(G)$. For the sake of contradiction, suppose that there exists an edge $uv \in E(G)$ where $u \in W_{T_1} - W_{T_2}$ and $v \in W_{T_2} - W_{T_1}$. Then $V(T_v) \subseteq V(T_1)$ and $V(T_u) \subseteq V(T_2)$. However, since $V(T_1) \cap V(T_2) \neq \emptyset$ we have $V(T_u) \cap V(T_v) = \emptyset$. But this contradicts **T3** therefore no such edge can exist.

We are then left to check that $W_{T_1} \cap W_{T_2} = W_e$. By definition, $W_e \subseteq W_{t_1} \cup W_{t_2}$, and as such $W_e \subseteq W_{T_1} \cap W_{T_2}$. Thus, to demonstrate equality we need to show that $W_{T_1} \cap W_{T_2} \subseteq W_e$. Let $v \in W_{T_1} \cap W_{T_2}$ and let $t_i \in V(T_i) \cap V(T_v)$ for $i \in \{1, 2\}$. Let P be the unique (t_1, t_2)-path in T. By **T2**, we have $P \subseteq T_v$. As $e \in E(P)$ it follows that $e \in E(T_v)$ and hence $v \in W_e$ as required. □

Corollary 5.2.2. *If G is k-connected then for every tree decomposition $(T, \mathcal{W})$ of width $\geq k$ such that $W_t \nsubseteq W_{t'}$ for all distinct $t, t' \in V(T)$, we have $|W_e| \geq k$ for all $e \in E(T)$.*

Proof. Let G be a k-connected graph with tree decomposition $(T, \mathcal{W})$ of width $\geq k$. For the sake of contradiction, assume that there exists $e \in E(T)$ such that $|W_e| < k$. Then by Lemma 5.2.1, there exists a separation (A, B) of G such that $|A \cap B| < k$. But by Menger's theorem (Theorem 2.4.1) this contradicts G being k-connected. □

5.2.2 Treewidth Lower Bounds

We will now discuss various properties of graphs that are lower bounds to treewidth. For the first proof, we will use the following notation. Let $\operatorname{argmin}_{x \in S} f(x) := \{x \in S : f(x) \leq f(y) \text{ for all } y \in S\}$; and $\operatorname{argmax}_{x \in S} f(x) := \{x \in S : f(x) \geq f(y) \text{ for all } y \in S\}$.

Cliques

Lemma 5.2.3 (Helly Property)**.** *Let T be a tree. If $T_1, \ldots, T_k$ are subtrees of T and $V(T_i) \cap V(T_j) \neq \emptyset$ for every $i, j \in \{1, \ldots, k\}$ then there exists $t \in V(T)$ such that $t \in V(T_i)$ for every $i \in \{1, \ldots, k\}$.*

Proof. Let T be a tree rooted at some $r \in V(T)$. For each subtree T_i, let $x_i = \operatorname{argmin}_{x \in V(T_i)} \operatorname{dist}(x, r)$. Consider the vertex $x_f = \operatorname{argmax}_{i \in \{1,\ldots,k\}} \operatorname{dist}(x_i, r)$. We claim that $x_f \in V(T_i)$ for all $i \in \{1, \ldots, k\}$. For the sake of contradiction, suppose that there exists a subtree T_j such that $x_f \notin V(T_j)$. By the minimality of the x_i's, there exists a $y \in V(T_j) \cap V(T_f)$ such that $\operatorname{dist}(r, y) > \operatorname{dist}(r, x_f)$. Let P_1 be the unique (r, y)-path in T. Now suppose that $x_f \notin V(P_1)$. Then the graph induced by the union of P_1, the (r, x_f)-path and the (y, x_f)-path contains a cycle, a contradiction. As such, $x_f \in V(P_1)$. Similarly, $x_j \in V(P_1)$. But since $\operatorname{dist}(x_j) \leq \operatorname{dist}(x_f)$, it follows that x_f must lie on the (x_j, y)-path. But since T_j is connected, this path must be contained within T_j, hence contradicting our assumption that $x_f \notin V(T_j)$. As such $x_f \in V(T_i)$ for all $i \in \{1, \ldots, k\}$. □

Corollary 5.2.4. *Let G be a graph and S a clique of G. Then for every tree decomposition $(T, \mathcal{W})$ of G there exists $x \in V(T)$ such that $S \subseteq W_x$.*

Proof. Let (T, W) be a tree decomposition of G. By **T2**, it follows that T_v is a subtree for every $v \in S$. By **T1**, it follows that we have $T_v \cap T_u \neq \emptyset$ for all $v, u \in S$. Therefore, by Lemma 5.2.3 there exists a $x \in V(T)$ such that $x \in T_v$ for all $v \in S$ and hence $S \subseteq W_x$. □

Corollary 5.2.5. *For every graph G, $\omega(G) - 1 \leq \operatorname{tw}(G)$.*

Minors

Lemma 5.2.6. *If H is a minor of G, then* $\mathrm{tw}(H) \leq \mathrm{tw}(G)$.

Proof. Let H and G be graphs such that $H \leq_M G$ and $\mathrm{tw}(G) = k$. Let j be the number of operations in the sequence performed on G to obtain a graph isomorphic to H. We proceed by induction on j. For the base case, $j = 0$, we have $H \simeq G$ and as such we trivially have $\mathrm{tw}(H) = k$.

Suppose that $j > 0$ and let H' be the minor obtained from performing the first $j-1$ operations in the sequence on G. By induction, there exists a tree decomposition $(T', \mathcal{W}')$ of H' with width at most k. We construct a tree decomposition of H as follow: let $T = T'$.

- If H is obtained by contracting the edge $uv \in E(H')$ to obtain a new vertex $w \in V(H) - V(H')$, then for all $t \in V(T)$ let $W_t = (W'_t - \{u, v\}) \cup \{w\}$ if $W_t \cap \{u, v\} \neq \emptyset$ otherwise let $W_t = W'_t$. Since $uv \in E(H')$, by **T2** and **T3** we have $T_w = T'_u \cup T'_v$ being connected;
- If H is obtained by deleting a vertex $v \in V(H')$, then for all $t \in V(T)$ let $W_t = W'_t \setminus \{v\}$; otherwise
- If H is obtained by deleting an edge, then for all $t \in V(T)$ let $W_t = W'_t$.

In each case, $(T, \mathcal{W})$ is a tree decomposition of H with width at most k. By the minimality of the treewidth, we have $\mathrm{tw}(H) \leq \mathrm{tw}(G)$. □

Minimum Degree

Lemma 5.2.7. *For every graph G, $\delta(G) \leq \mathrm{tw}(G)$.*

Proof. Let G be a graph and $(T, \mathcal{W})$ be a tree decomposition of G of minimum width. Let $x \in V(T)$ be a leaf of T. We may assume that $W_x \not\subseteq W_y$ for every $y \in V(T)$ where $y \neq x$. As such, there exists a vertex $v \in W_x$ such that $v \notin W_t$ for every $t \in V(T) \setminus \{x\}$. Since $(T, \mathcal{W})$ is a tree decomposition, it follows that every neighbour of v must also be an element of W_x. Therefore $|W_x| \geq N[v] \geq \delta(G) + 1$ thus $\delta(G) \leq \mathrm{tw}(G)$. □

5.3 Graphs with Bounded Treewidth

We now discuss several classes of graphs with bounded treewidth.

5.3.1 Graphs with $\text{tw}(G) \leq 2$

We begin by first characterising graphs with treewidth at most 1 then graphs with treewidth at most 2.

Theorem 5.3.1. *For every graph G, the following three statements are equivalent:*

1. *$tw(G) \leq 1$;*
2. *G is a forest; and*
3. *G is K_3-minor-free.*

Proof. By Theorem 3.4.3, we have (2) and (3) to be equivalent. For (1) $\implies$ (3), by Corollary 5.2.4 we have $\text{tw}(K_3) \geq 2$. Thus, if G has a K_3 minor we have $\text{tw}(G) \geq 2$ by Lemma 5.2.6 and hence if $\text{tw}(G) \leq 1$ then G has no K_3 minor.

For (2) $\implies$ (1), without loss of generality we may assume that G is connected and is hence a tree. We proceed by induction on $n = V(G)$. For $n = 1$ we have $V(G) = \{x\}$. Let $T = \{t\}$ and $\mathcal{W} = \{W_t\}$ where $W_t = \{x\}$. Then $(T, \mathcal{W})$ is a tree decomposition of G with width at most 1.

Now suppose $n > 1$ and let $uv \in E(G)$ such that $\deg(v) = 1$. Since $G' = G - v$ is also a tree it follows by induction that there exists a tree decomposition $(T', \mathcal{W}')$ of G' with width at most 1. Now for some $t_1 \in V(T')$ we have $u \in W'_{t_1}$. Let $T = T' \cup \{t_2\} \cup \{t_1 t_2\}$ and $\mathcal{W} = \mathcal{W}' \cup \{W_{t_2}\}$ where $W_{t_2} = \{u, v\}$. Then $(T, \mathcal{W})$ is a tree decomposition of G with width at most 1 as required. The result therefore follows by induction.

□

Theorem 5.3.2. *For every graph G, $\text{tw}(G) \leq 2$ if and only if G is K_4-minor-free.*

Proof. Suppose $K_4 \leq_M G$. By Corollary 5.2.5 we have $\mathrm{tw}(K_4) \geq 3$ and thus by Lemma 5.2.6 we have $\mathrm{tw}(G) \geq 3$. Thus, if $\mathrm{tw}(G) \leq 2$ then it is K_4-minor-free.

Suppose $K_4 \not\leq_M G$. By Theorem 3.4.5, G can be obtained (≤ 2)-sum of $H_1, H_2, \ldots, H_m$ where $|V(H_j)| \leq 3$ for all $j \in \{1, \ldots, m\}$. Let $G_1 = H_1$ and G_i be obtained by a (≤ 2)-sum of G_{i-1} and H_i in such a way that $G_m = G$. Furthermore, without loss of generality we may also assume that G_i is connected for all $i \in \{1, \ldots, m\}$. We claim that for all $i \in \{1, \ldots, m\}$ we have $\mathrm{tw}(G_i) \leq 2$.

We proceed by induction on i. For $i = 1$, let $V(T^{(1)}) = \{t\}$ and $\mathcal{W}^{(1)} = \{W_t^{(1)}\}$ where $W_t^{(1)} = V(G_1)$. Since $|V(G_1)| \leq 3$, it follows that $(T^{(1)}, \mathcal{W}^{(1)})$ is a tree decomposition of G_1 with width at most 2. Now suppose $i > 1$. Let $X \subseteq V(G_{i-1})$ be the clique in which the (≤ 2)-sum was performed on to obtain G_i. By induction, there exists a tree decomposition $(T^{(i-1)}, \mathcal{W}^{(i-1)})$ of G_{i-1} with width at most 2. Since X is a clique, by Lemma 5.2.4 there exists a $t \in V(T^{(i-1)})$ such that $X \subseteq W_t^{(i-1)}$. Let $T^{(i)} = T^{(i-1)} \cup \{l\} \cup \{lt\}$ and $\mathcal{W}^{(i)} = \mathcal{W}^{(i-1)} \cup \{W_l^{(i)}\}$ where $W_l^{(i)} = V(H_i)$. Then, $(T^{(i)}, \mathcal{W}^{(i)})$ is a tree decomposition of G_i with width at most 2. It follows by induction that $\mathrm{tw}(G) \leq 2$ as required. $\square$

5.3.2 k-trees

We now describe an edge-maximal class of graphs with treewidth at most k. A vertex v in a graph G is *k-simplical* if $\deg(v) = k$ and $G[N(v)] \simeq K_k$. A graph G is a *k-tree* if either:

- $G = K_{k+1}$ or
- G contains a k-simplical vertex v and $G - v$ is also a k-tree.

Similarly, a graph G is a *weak k-tree* if either

- $G = K_j$ for some $j \in \{1, \ldots, k+1\}$ or
- G contains a $(\leq k)$-simplical vertex v and $G - v$ is also a weak k-tree.

The following characteristic of weak k-trees will be used for our original work.

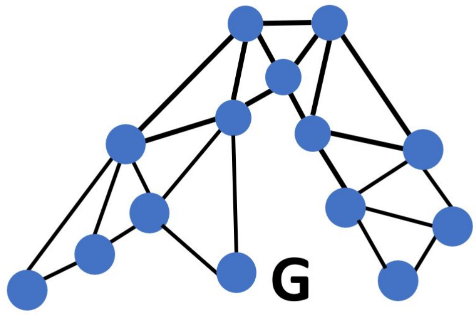


Figure 5.1: Example of a 2-tree.

Lemma 5.3.3. *If G is a weak k-tree, then there exists a vertex ordering $v_1, \ldots, v_n$ such that for all $i \geq 2$ we have $G[N^-(v_i)] \simeq K_j$ for some $j \in \{1, \ldots, k\}$.*

Proof. Let G be a weak k-tree. We proceed by induction on $n = |V(G)|$. For the base case, $n = 2$, any ordering of the vertices will satisfy the condition. Thus, we may assume $n > 2$. Since G is a weak k-tree it contains a $(\leq k)$-simplical vertex v_n such that $G' = G - v_n$ is also a weak k-tree. By induction it follows that there exists a vertex ordering $v_1, \ldots, v_{n-1}$ of G' such that for all $i \in \{1, \ldots, n-1\}$, we have $G[N^-(v_i)] \simeq K_j$ for some $j \in \{1, \ldots, k\}$. Since v_n is $(\leq k)$-simplical, $G[N^-(v_n)] \simeq K_j$ for some $j \in \{1, \ldots, k\}$ and as such the vertex ordering $v_1, \ldots, v_{n-1}, v_n$ satisfies the condition of the lemma. The result therefore follows by induction. □

Lemma 5.3.4. *If G is a weak k-tree, then* $\mathrm{tw}(G) \leq k$.

Proof. Let G be a weak k-tree. We claim that for every weak k-tree there exists a tree decomposition of G with width at most k. We proceed by induction on $n = |V(G)|$. For our base case $n \leq k + 1$, let $V(T) = \{t\}$ and $W_t = V(G)$. As $|V(G)| \leq k + 1$ this gives us a tree decomposition of width at most k.

Now suppose that $n > k + 1$. Since G is a weak k-tree it contains a $(\leq k)$-simplical vertex v such that $G' = G - v$ is also a weak k-tree. By induction there exists a tree decomposition of G', $(T', \mathcal{W}')$, with width at most k. Now since v is $(\leq k)$-simplical, we have $G[N(v)] \simeq K_j$ for some $j \in \{1, \ldots, k\}$. By Corollary 5.2.4, there exists a $x \in V(T)$ such that $N(v) \subseteq W_x$. Let $T = T' \cup \{y\} \cup \{xy\}$ and $\mathcal{W} = \mathcal{W}' \cup \{W_y\}$ where $W_y = N[v]$. Then $(T, \mathcal{W})$ gives a tree decomposition of G. Furthermore, since $|W_y| \leq k+1$ it follows by induction that the width of this tree decomposition is at most k, hence $\text{tw}(G) \leq k$. □

For a connected graph G, let $a(G)$ be the minimum $k \in \mathbb{N}$ such that G is a spanning subgraph of a weak k-tree. The following lemma shows the interconnection between graphs with bounded treewidth and weak k-trees. Together with Lemma 5.3.4, this demonstrates that weak k-trees are a class of edge-maximal graphs with treewidth at most k.

Lemma 5.3.5 ([3]). *For every connected graph G, we have $\text{tw}(G) = a(G)$.*

Proof. Let G be a connected graph with $\text{tw}(G) = k$ for some $k \in \mathbb{N}$. We will first show that $a(G) \leq k$. We proceed by induction on $n = |V(G)|$. Note that for a connected graph to have treewidth k, it must have at least $k+1$ vertices. Thus, our base case is $n = k+1$. Now the only graph with treewidth k and $k+1$ vertices is K_{k+1}. Since K_{k+1} is itself a weak k-tree, it is trivially a spanning subgraph of a weak k-tree. As such, we have $a(G) \leq k$.

Now suppose $n > k+1$. Let $(T, \mathcal{W})$ be a tree decomposition of G with width k where $W_x \nsubseteq W_y$ for all distinct $x, y \in V(T)$. Let W_t be a bag with $|W_t| = k+1$ for some $t \in V(T)$. Since $G \neq K_{k+1}$, it follows that T has at least 2 leaf bags. As such, there exists a leaf bag W_y for some $y \in V(T)$ that contains a vertex v such that $N[v] \in W_y$ but $v \notin W_t$. Let G' be obtained from G by adding edges uw whenever u and w are in the same bag. Since $G' - v$ also has treewidth k, it follows by induction that it is a subgraph of a weak k-tree. Since v is $(\leq k)$-simplical in G', it follows that G' is also a spanning subgraph of a weak k-tree. As such, we have $a(G) \leq a(G') \leq \text{tw}(G)$.

For $\text{tw}(G) \leq a(G)$, let G be a connected graph and let G^* be a weak $k^\star$-tree that realises the minimality of $a(G)$. For the sake of contradiction, suppose $k^\star < \text{tw}(G)$. By Lemma 5.3.4, we have $\text{tw}(G^*) \leq k^\star$. Furthermore, since $G \leq G^*$, by Lemma 5.2.6 we have $\text{tw}(G) \leq \text{tw}(G^*) \leq k^\star < \text{tw}(G)$, a contradiction. Hence $\text{tw}(G) \leq a(G)$ which proves that $\text{tw}(G) = a(G)$.

□

Corollary 5.3.6. *For every graph G, $\chi(G) \leq \text{tw}(G) + 1$.*

Proof. Without loss of generality, we may assume that G is connected with $\text{tw}(G) = k$ for some $k \in \mathbb{N}$. By Lemma 5.3.5, G is a spanning subgraph of some weak k-tree, G^*. As such, by Lemma 5.3.3 there exists a vertex ordering $v_1, \ldots, v_n$ such that $|N^-(v_i)| \leq k$ for all $i \in \{1, \ldots, n\}$. We apply a greedy colouring on this vertex ordering. Since each vertex has at most k neighbours that are already coloured, it follows that greedy will used at most $k+1$ colours and hence $G^\star$ is $(k+1)$-colourable. As such, since $G \subseteq G^*$ we have $\chi(G) \leq k+1$. $\square$

Using this corollary, we will now present another proof of Hadwidger's Conjecture for $t \in \{2, 3\}$.

Corollary 5.3.7. *Every K_3-minor-free graph is 2-colourable.*

Proof. Let G be a K_3-minor-free graph. By Theorem 5.3.1, $\mathrm{tw}(G) \leq 1$. As such, by Corollary 5.3.6 G is 2-colourable. $\square$

Corollary 5.3.8. *Every K_4-minor-free graph is 3-colourable.*

Proof. Let G be a K_4-minor-free graph. By Theorem 5.3.2, $\mathrm{tw}(G) \leq 2$. As such, by Corollary 5.3.6 G is 3-colourable. $\square$

5.4 Brambles and Duality

Let G be a graph. We say that $X, Y \subseteq V(G)$ *touch* if $(X \cup N(X)) \cap Y \neq \emptyset$; that is, if there exists a vertex $u \in X \cap Y$ or an edge xy such that $x \in X$ and $y \in Y$. A *bramble*, $\mathcal{B}$, is a collection of subsets $B \subseteq V(G)$ such that $G[B]$ is a connected subgraph for all $B \in \mathcal{B}$ and any two elements of $\mathcal{B}$ touch. That is $G[B \cup B']$ is connected for all $B, B' \in \mathcal{B}$. A set $S \subseteq V(G)$ is a *cover* of $\mathcal{B}$ if $S \cap B \neq \emptyset$ for all $B \in \mathcal{B}$. The *order* of $\mathcal{B}$ is the minimum size of a cover of $\mathcal{B}$. The *bramble number of a graph* G, $\mathrm{bn}(G)$, is the maximum order of a bramble in G.

Brambles are useful as they allow us to confirm the minimality of a tree decomposition for a graph.

Theorem 5.4.1 (Treewidth Duality Theorem [38])**.** *For every connected graph G,* $\mathrm{bn}(G) = \mathrm{tw}(G) + 1$.

This theorem gives a certificate of optimality as to whether a tree decomposition has minimum width. We present the proof of this theorem by Mazoit [21]. We begin by outlining several important terms for the proof.

Let $k \in \mathbb{N}$. A bag of a tree decomposition is *small* if it has size at most k, otherwise it is *big*. A *partial* $(< k)$*-decomposition* of G is a tree decomposition, $(T, \mathcal{W})$ that has no big internal bags and with at least one small bag (that is the internal bags must be small but the leaf bags may be big). If all the bags are small, then it is a tree decomposition with width less than k. Otherwise, it contains at least one big leaf bag which has a small neighbouring bag. If t is a leaf of T with neighbour u and $|W_t| > k$, then the set $W_t - W_u$ is called a *petal* of T. Note that this set is non-empty since $|W_t| > |W_u|$). A partial $(< k)$-star decomposition induced by $S \subseteq V(G)$ is a tree decomposition $(S_j, \mathcal{W})$ of a connected graph such that the central bag $W_t = S$ and $G[W_u]$ is connected for every $u \in V(T) \setminus t$ where $|S| < k$ and $j \in \mathbb{N}$.

Lemma 5.4.2. *Let X and Y be respectively petals of some partial $(< k)$-decomposition $(T^X, \mathcal{W}^X)$ and $(T^Y, \mathcal{W}^Y)$ of some graph G and let x and y be the leaves of T^X and T^Y such that $X \subseteq W_x^X$, $Y \subseteq W_y^Y$. If X and Y do not touch then there exists a partial $(< k)$-decomposition $(T, \mathcal{W})$ whose petals are subsets of the petals of $(T^X, \mathcal{W}^X)$ and $(T^Y, \mathcal{W}^Y)$ other than X and Y.*

Proof. Since X and Y do not touch, there exists a separation (C, D) such that $X \subseteq C - D$ and $Y \subseteq C - D$. We may choose a separation such that the separator $S = C \cap D$ is minimal. Note that $N(X)$ is a valid separator of X and Y since the sets do not touch. Let u be the unique neighbour of x in T^X. Then $N(X) \subseteq W_u^X$ and since $|W_u^X| \leq k$, it follows from the minimality of S that $|S| \leq N(X) \leq k$.

Claim: *There exists a partial $(< k)$-decomposition of $G[D]$ with S as a leaf and whose petals are subsets of the petals of $(T^X, \mathcal{W}^X)$ other than X.*

Proof of Claim: By the minimality of S and Menger's theorem (Theorem 2.4.1), there exists a family of vertex disjoint paths $\{P_s | s \in S\}$ in $G[C]$ from X to S that only meet D in S. For each $s \in S$, choose a node $t_s \in V(T^X)$ such that $s \in W_{t_s}^X$. For all $t \in V(T^X)$, let $W_t^{X'} = (W_t^X \cap D) \cup \{s \in S : t \in \text{path from } x \text{ to } t_s\}$. Since $S \subset D$, this defines a tree decomposition of $G[D]$ which we will call $(T^X, \mathcal{W}^{X'})$.

Now by construction, only bags that are on the paths between a t_s and x have new vertices added to them. Because T is a tree, if a leaf $y \neq x \in V(T^X)$ is on such a path for a $s \in S$, then it follows that $y = t_s$ and thus $s \in W_y^X$. It therefore follows that the petals of $(T^X, \mathcal{W}^{X'})$ are contained in the petals of $(T^X, \mathcal{W}^X)$ except for x whose bag is replaced by S.

Furthermore, for $t \in T^X$, $W_t^{X'}$ contains $s \in S$ if either W_t^X did or if t is in the path from x to t_s. As such, if $s \notin W_t^X$ then there exists a vertex $v \in P_s \cap W_t^X$ which is not contained in $W_t^{X'}$. Hence, for any internal vertex of the tree t, $|W_t^{X'}| \leq |W_t^X|$ and thus $(T^X, \mathcal{W}^{X'})$ is a partial $(< k)$-decomposition, proving the claim.

Similarly, let $(T^Y, \mathcal{W}^{Y'})$ be constructed in the same way for $G[A]$. By identifying the leaves x and y of T^X and T^Y, we obtain a partial $(< k)$-decomposition $(T, \mathcal{W})$ which satisfies the condition of the lemma. □

Using this lemma, we may now prove the treewidth duality theorem.

Proof of Theorem 5.4.1. Let G be a connected graph. We will first prove that $\text{bn}(G) \leq \text{tw}(G) + 1$ by showing that for any tree decomposition $(T, \mathcal{W})$ and bramble $\mathcal{B}$ of a graph G, there exists a $t \in V(T)$ such that W_t is a cover of $\mathcal{B}$.

Let $e = t_1t_2 \in E(T)$ and $W_e = W_{t_1} \cap W_{t_2}$. If W_e is a cover of $\mathcal{B}$ then the results follow immediately since $W_e \subseteq W_{t_1}$, hence W_{t_1} will also be a cover of $\mathcal{B}$. Thus, we may assume that for any $e \in E(T)$, W_e is not a cover of $\mathcal{B}$. Since W_e is not a cover, there must exists a $B \in \mathcal{B}$ such that $B \cap W_e = \emptyset$. Let T' and T'' be the two connected components of $T - e$. Now since B is connected, it cannot have vertices in both T' and T'' as that would contradict **T3**. Moreover, there cannot be a $B' \in W_{T'} - W_e$ and a $B'' \in W_{T''} - W_e$ as that would contradict $G[B' \cup B'']$ being connected for all $B', B'' \in \mathcal{B}$. Hence there is a unique T^* of $T - e$ such that $B \subseteq W_{T^*} - W_e$ for every B that satisfies $B \cap W_e = \emptyset$.

We orientate e towards T^* and repeat for all $e \in E(T)$. Now for a tree we have $E(T) < V(T)$. As such, there exists a $t \in V(T)$ such that all the edges of T incident to t are orientated towards it. For the sake of contradiction, assume that W_t is not a cover of $\mathcal{B}$. Then there exists a $B \in \mathcal{B}$ such that $B \subseteq W_{T'} - W_t$ for some component T' of $T \setminus t$. Let e be the edge of T joining t to T'. Then by construction, e will be orientated away from t thus contradicting our choice of t.

It therefore follows that W_t is a cover of $\mathcal{B}$. By taking the tree decomposition that realises the minimum treewidth, it follows that the maximum order of a bramble bounded above by $\max_{t \in T} |W_t| = \text{tw}(G) + 1$, thus $\text{bn}(G) \leq \text{tw}(G) + 1$.

We are now left to show that $\text{bn}(G) \geq \text{tw}(G) + 1$. We will argue that if $\text{tw}(G) \geq k$, then G has a bramble of order $> k$ which proves the equality we desire. Since $\mathrm{tw}(G) \geq k$, every tree decomposition $(T, \mathcal{W})$ contains a $t \in T$ such that $|W_t| \geq k+1$. As such, every partial $(< k)$-decomposition must contain a leaf $l \in T$ such that $|W_l| \geq k+1$ and thus contains a petal. Let $\mathcal{B}$ be a set such that:

- $\mathcal{B}$ contains a petal for every partial $(< k)$-decomposition;
- $\mathcal{B}$ is closed under superset; that is, if $C \in \mathcal{B}$ and D is a petal with $C \subseteq D$, then $D \in \mathcal{B}$; and
- Given the above two constraints, $\mathcal{B}$ is the minimal collection of petals that satisfies them.

Furthermore, let $\mathcal{B}^\star$ be the set of connected petals in $\mathcal{B}$. We claim that $\mathcal{B}^\star$ is a bramble of order $\geq k$. By definition, its elements are connected. To see its order is $> k$, let $S \subseteq V(G)$ with $|S| \leq k$. Then $\mathcal{B}^\star$ contains a petal of the star decomposition from S and thus S is not a covering of $\mathcal{B}^\star$. We now prove by contradiction that the elements of $\mathcal{B}$ pairwise touch. Assume that there exists $X, Y \in \mathcal{B}$ that do not touch. Clearly then no subset of X touches a subset of Y. By taking X and Y to be vertex minimal, we have $\mathcal{B} \setminus \{X\}$ and $\mathcal{B} \setminus \{Y\}$ both being closed under supersets while being strict subsets of $\mathcal{B}^\star$. As such, there must exist partial $(< k)$-decompositions, (T_X, l_X) and $(T^Y, \mathcal{W}^Y)$ whose only petals in $\mathcal{B}$ are respectively X and Y.

If this was not the case, then either X or Y may be removed from $\mathcal{B}$ without violating the first two conditions and thus contradicting the minimality of $\mathcal{B}$. By Lemma 5.4.2, we obtain a $(T, \mathcal{W})$ partial $(< k)$-decomposition whose petals are subsets of the petals in $(T^X, \mathcal{W}^X)$ and $(T^Y, \mathcal{W}^Y)$ other than X and Y. Since $\mathcal{B}$ is upwards closed and contains no petals of $(T^X, \mathcal{W}^X)$ and $(T^Y, \mathcal{W}^Y)$ other than X and Y, it does not contain any petals of $(T, \mathcal{W})$. But this contradicts the construction of $\mathcal{B}$. Hence all $X, Y \in \mathcal{B}$ touches and as such all $X, Y \in \mathcal{B}^\star$ also touches and hence we have constructed a bramble of order $> k$. As $\mathrm{bn}(G)$ is the maximum order of a bramble, it follows that $\mathrm{bn}(G) \geq \mathrm{tw}(G) + 1$. □

5.4.1 Grids have Unbounded Treewidth

For $n \in \mathbb{N}$, the $n \times n$ grid, $G_{n \times n}$, is the graph with n^2 vertices $v_{i,j}$ for $(i, j) \in \{1, \dots, n\}^2$ where $v_{i,j}$ and $v_{i',j'}$ are adjacent if and only if $|i - i'| + |j - j'| = 1$. We now use the treewidth duality theorem to show that $G_{n \times n}$ has unbounded treewidth.

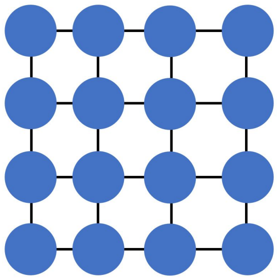


Figure 5.2: The 4×4 grid: $G_{4\times 4}$.

Corollary 5.4.3. *For every* $n \in \mathbb{N} \setminus \{0\}$*, we have* $\mathrm{tw}(G_{n\times n}) \in \{n-1, n\}$.

Proof. We begin by first showing that $\mathrm{tw}(G_{n\times n}) \leq n$. To do so, we will construct a path decomposition of $G_{n\times n}$ with width n. Consider first the bijection $\phi : \{1, \dots, n(n-1)\} \to \{0, \dots, n-2\} \times \{1, \dots, n\}$ where $\phi(i) = (k, j)$ whenever $i = kn + j$. Let $B_i = \{v_{k+1,a} : a \in \{j, \dots, n\}\} \cup \{v_{k+2,b} : b \in \{1, \dots, j\}\}$ for every $i \in \{1, \dots, n(n-1)\}$. We claim that $(B_1, \dots, B_{n(n-1)})$ is a path decomposition of $G_{n\times n}$ with width n.

By construction we have $B_i \subseteq V(G_{n\times n})$ for all $i \in \{1, \dots, n(n-1)\}$ thus **P1** is satisfied. Let $v_{l,m} \in G_{n\times n}$. Then for all $(l, m) \in \{1, \dots, (n)\}^2$ we have

$$\{i : v_{l,m} \in B_i\} = \begin{cases} \{1, \dots, l\}, & \text{if } m = 1; \\ \{\phi^{-1}(l, m-1), \dots, \phi^{-1}(l, m-1) + n\}, & \text{if } m \in \{2, \dots, n-2\}; \\ \{n(n-1) - j, \dots, n(n-1)\} & \text{if } m = n; \end{cases}$$

which is any case is a non-empty interval. As such **P2** is satisfied.

Now $v_{i,j}$ and $v_{i',j'}$ are adjacent if and only if $|i - i'| + |j - j'| = 1$. We therefore have two cases to consider. The first case is if $i' = i$ and $|j - j'| = 1$. Without loss of generality we may assume that $j' = j + 1$. By the construction of the path decomposition, we have $v_{i,j}, v_{i,j+1} \in B_{\phi^{-1}(i-1,1)}$. The second case is if $j' = j$ and $|i - i'| = 1$. Without loss of generality we may assume that $i' = i + 1$. Then we have $v_{i,j}, v_{i+1,j} \in B_{\phi^{-1}(i-1,j)}$. As such, **P3** is satisfied.

Finally, note that for all $i \in \{1, \dots, n(n-1)\}$ we have $|B_i| = (n + 1 - j) + j = n + 1$ and as such the path decomposition has width n. Now since a path decomposition is a type of tree decomposition, it follows that $\mathrm{tw}(G_{n\times n}) \leq \mathrm{pw}(G_{n\times n}) \leq n$.

We will now show $\mathrm{bn}(G_{n\times n}) \geq n$. We construct a bramble $\mathcal{B}$ as follows: let $B_{i,j} = \{v_{i,a} : a \in \{1,\ldots,n\}\} \cup \{v_{b,j} : b \in \{1,\ldots,n\}\}$ for every $i,j \in \{1,\ldots,n\}$. We claim that $\mathcal{B} = \{B_{i,j} : i,j \in \{1,\ldots,n\}\}$ is a bramble of order at least n. Let $i,i',j,j' \in \{1,\ldots,n\}$. Note that $B_{i,j}$ consists of the i^{th} row and j^{th} column of $G_{n\times n}$. As each row and column are connected components that overlap, it follows that $B_{i,j}$ is connected. Furthermore, we also have $v_{i,j'}, v_{i',j} \in B_{i,j} \cap B_{i',j'}$ thus any two elements of $\mathcal{B}$ touches (see Figure 5.3 for an example). As such, $\mathcal{B}$ is indeed a bramble.

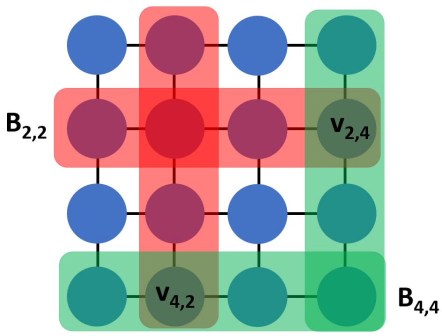


Figure 5.3: Brambles on $G_{4\times 4}$.

Let $S \subseteq V(G_{n\times n})$ be a cover of $\mathcal{B}$ with minimum order. For the sake of contradiction, suppose $|S| < n$. Then there exists a row $i \in \{1,\ldots,n\}$ and a column $j \in \{1,\ldots,n\}$ such that S contains no vertices from them. However, this means that $S \cap B_{i,j} = \emptyset$ which contradicts that S is a cover. Therefore $|S| \geq n$ and as such, we have $\mathrm{bn}(G_{n\times n}) \geq n$. It therefore follows by Theorem 5.4.1 that $\mathrm{tw}(G_{n\times n}) \geq n-1$. □

5.5 Excluding Planar Graphs

In Section 4, we saw that the class of graphs which excludes a forest minor have bounded pathwidth. Furthermore, that only the exclusion of forest minors bounds the pathwidth. The following theorem is analogous for graphs with bounded treewidth.

Theorem 5.5.1 ([30]). *Given any graph H, the graphs that are H-minor-free have bounded treewidth if and only if H is planar.*

By Corollary 5.4.1, the graph $G_{n\times n}$ has unbounded treewidth. Since the class of planar graphs are minor-closed and $G_{n\times n}$ is planar it follows that the exclusion of a nonplanar minor does not bound the treewidth. Furthermore, Robertson, Seymour and Thomas have shown that for every planar graph G where $|V(G)| + 2|E(G)| \leq n$ we have $G \leq_M G_{2n\times 2n}$ [34]. That is, for every planar graph G there exists a sufficiently large $n \in \mathbb{N}$ such that $G \leq_M G_{n\times n}$. The intuition behind this is that we can draw G on a plane with fat vertices and edges then superimposing a sufficiently fine grid on it. As such, Theorem 5.5.1 is implied by the following.

Theorem 5.5.2 (Excluded Grid Minor [30])**.** *For every integer h there exists a function f such that every graph of treewidth at least $f(h)$ contains $G_{h\times h}$ as a minor.*

The first polynomial bound for f was achieved by Chekuri and Chuzhoy [7]. Currently, the best known upper bound is $f(h) = O(h^{20})$ [8] and lower bound is $\Omega(h^2 \log(h))$ [34].

Chapter 6

Complete Binary Tree Minors in Graphs with Bounded Treewidth

In this chapter we will investigate the existence of complete binary tree minors in graphs with large pathwidth and bounded treewidth. We begin by explaining the motivation for the exploration of this topic by discussing a known parallel result for graphs with large treewidth. We then demonstrate the universality of the complete binary tree which is the reason why we are only excluding this particular type of forest. We then discuss a known result for complete binary tree minors in forest followed by a discussion on complete binary tree minors in weak k-trees. We conclude this chapter with a conjecture for the relationship we have for this problem.

6.1 Motivation

One of the main goals of this project is to find the least function f such that every graph with pathwidth at least $f(k, h)$ has treewidth at least k or contains T_h as a minor. This question was inspired by the following theorem which is analogous for graphs with large treewidth where the grid graph plays the role of the complete binary tree.

Theorem 6.1.1 ([10]). *For every $t \in \mathbb{N}$ there exists $c \in \mathbb{N}$ such that if G is a K_t-minor-free graph and $\mathrm{tw}(G) \geq ch$ then $G_{h\times h} \leq_M G$.*

This result gives us a tighter bound to the excluded grid minor theorem (Theorem 5.5.2) for graphs that exclude a K_t minor. This raises the question as to whether a tighter bound may be obtained for the excluded forest minor theorem (Theorem 4.2.1) given some structural assumptions about the graphs. We suspect that bounded treewidth will be the most reasonable assumption to make in producing a meaningful result. The following lemma demonstrates the universality of the complete binary tree minor which is the reason why it is the only forest we are seeking to exclude.

Lemma 6.1.2. *For every tree T there exists $h \in \mathbb{N}$ such that $T \leq_M T_h$.*

This lemma follows as a simple corollary to the following.

Lemma 6.1.3. *For every rooted tree (T, r) there exists $h \in \mathbb{N}$ and an injective function $\phi : V(T) \to V(T_h)$ such that for every $u, v \in V(T)$, $\phi(u) \leq_{T_h} \phi(v)$ if and only if $u \leq_T v$.*

Proof. Let (T, r) be a rooted tree with n vertices. We proceed by induction on n. For $n = 1$, the claim holds trivially with $h = 0$. Suppose $n > 1$. Let $v = \operatorname{argmax}_{v \in V(T)} \operatorname{dist}(v, r)$ and $p_v \in V(T)$ be the parent of v. If v is not a leaf of T, then v has a descendant vertex u such that v is on the unique (u, r)-path in T. But this means that $\operatorname{dist}(u, r) > \operatorname{dist}(v, r)$ which contradicts the maximality of v. As such, v is a leaf and thus $T' = T - v$ is a tree. By induction, there exists a $h \in \mathbb{N}$ and an injective function $\phi' : V(T') \to V(T_h)$ that satisfies the induction hypothesis. We may also assume that $\phi(v')$ is not a leaf of T_h for all $v' \in V(T')$ by considering T_{h+1} if needed.

Let $u_1, \ldots, u_m$ be the descendants of $p_v \in V(T')$. If $p_v u_i \notin E(T')$ for some $i \in \{1, \ldots m\}$, then $\operatorname{dist}(u_i, r) > \operatorname{dist}(v, r)$, a contradiction. As such, $u_1, \ldots, u_m$ are all leaves of T'. If $m = 0$ then there exists a vertex $t \in V(T_h)$ such that $t \geq_{T_h} \phi'(p_v)$. Let $\phi : V(T) \to V(T_h)$ be such that $\phi(u) = \phi'(u)$ for all $u \in V(T')$ and $\phi(v) = t$. By the transitivity of the tree-order it follows that ϕ satisfies the induction hypothesis. Otherwise, let $t_1, t_2 \in \mathcal{V}(T_h)$ be the two children of $\phi(u_1)$ in T_h. Let $\phi(u) = \phi'(u)$ for all $u \in V(T') \setminus \{u_1\}, \phi(u_1) = t_1$ and $\phi(v) = t_2$. By the transitivity of the tree-order it follows that ϕ satisfies the induction hypothesis.

□

6.2 Binary Tree Minors in Forests

We now investigate the existence of complete binary tree minors in graphs with bounded treewidth and large pathwidth. We begin by considering graphs with treewidth at most 1. By Theorem 5.3.1, these graphs are forests. To determine a suitable function f for this class of graphs, it suffices to consider only their connected components; that is, trees. This is because the pathwidth of a disconnected graph is equal to the largest pathwidth of one of its connected components. We will use the following notation: let (T, r) be a tree rooted at r, $f(T)$ be the maximum edge-height of a T_h minor with r in the branch set of the root of T_h and let $\mathrm{pw}(T, r)$ be the minimum width of a path decomposition of T with the additional constraint that r is in either the first or last bag.

Lemma 6.2.1 ([20]). *For every rooted tree (T, r) with $|E(T)| \geq 1$, if $\mathrm{pw}(T, r) \geq h$ then $f(T) \geq h - 1$.*

Proof. Let (T, r) be a rooted tree with $|E(T)| \geq 1$ and $\mathrm{pw}(T, r) \geq h$. We proceed by induction on $n = |V(T)|$. The base case, $n = 2$, holds trivially.

Suppose $n > 2$ and let $d = \deg(r)$. We will first consider the case where $d \geq 2$. Let $(T_1, v_1), (T_2, v_2), \ldots, (T_d, v_d)$ be the connected components of the graph $T - r$ that are rooted at the children of r. Without loss of generality, we may assume that $\mathrm{pw}(T_1, v_1) \geq \mathrm{pw}(T_2, v_2) \geq \cdots \geq \mathrm{pw}(T_d, v_d)$.

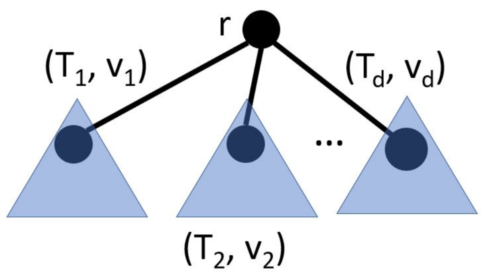


Figure 6.1: Components of (T, r).

If $\mathrm{pw}(T_2, v_2) \geq h-1$, then by induction it follows that both $f(T_1) \geq h-2$ and $f(T_2) \geq h-2$. We can therefore construct a T_{h-1} minor in T with r in the branch set of the root from $\{r, rv_1, rv_2\} \cup T_1 \cup T_2$. If $\mathrm{pw}(T_1, v_1) \geq h$, then by induction we have $f(T_1) \geq h - 1$ and therefore $f(T) \geq h - 1$. Now assume that $\mathrm{pw}(T_1, v_1) \leq h - 1$ and $\mathrm{pw}(T_j, v_j) \leq h - 2$ for all $j \in \{2, \ldots, d\}$. We can construct a path decomposition of (T, r) as done in Figure 6.2.

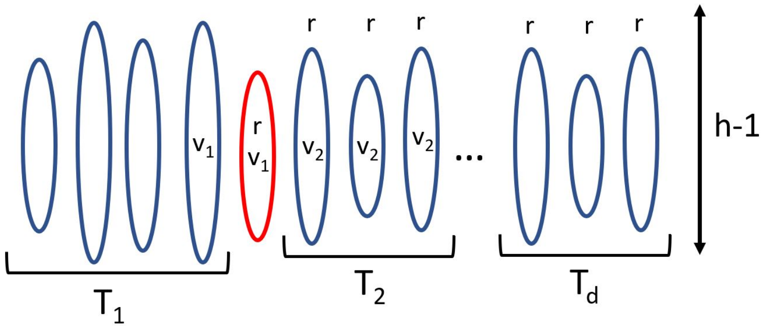


Figure 6.2: Path decomposition of (T, r).

For this decomposition, we have $\max_i |B_i| \le h$ and therefore $\text{pw}(T, r) \le h - 1$, a contradiction.

It remains to consider the case $d = 1$. Let (T_1, v_1) be the tree $T - r$ rooted at the neighbour of r. Take a path decomposition of (T_1, v_1) of minimum width such that v_1 is in either the first or last bag. We may extend this decomposition to that in Figure 6.3 to obtain a path decomposition of (T, r). As such, we have $h \le \text{pw}(T, r) \le \text{pw}(T_1, v_1)$. Since $|V(T_1)| < |V(T)|$ it follows by induction that $f(T_1) \ge h - 1$. By adding r to the branch set of the root bag it follows that $f(T) \ge h - 1$ as required.

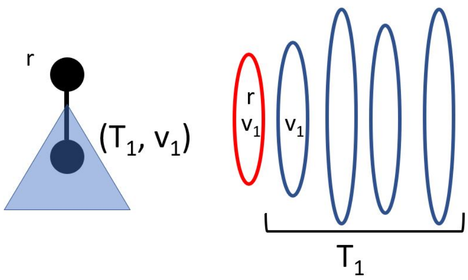


Figure 6.3: Path decomposition of (T, r) where r has one child.

□

We may now prove the first key result.

Corollary 6.2.2. *For every tree T where $|E(T)| \ge 1$ and $h \in \mathbb{N}$, if $\text{pw}(T) \ge h + 1$ then $T_h \le_M T$.*

Proof. Let T be a tree with $|E(T)| \ge 1$ and $\text{pw}(T) \ge h + 1$. Choose $r \in V(T)$ such that $\text{pw}(T, r) \ge h + 1$. By Lemma 6.2.1 it follows that $f(T) \ge h$ and as such $T_h \le_M T$. □

6.3 Binary Tree Minors in Weak k-trees

A goal for this honour's project was to generalise the result of Corollary 6.2.2 to graphs with bounded treewidth and large pathwidth. The main strategy we investigated to do so involved looking for spanning trees that has large pathwidth. Formally, the question we sought to address is the following:

What is the greatest function g such that for every connected graph with pathwidth at least h and treewidth at most k there exist a spanning tree of the graph with pathwidth at least $g(h,k)$?

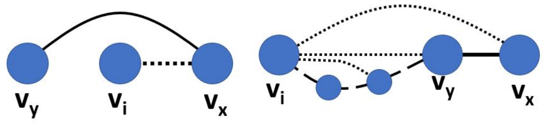
If such a function can be found then together with Corollary 6.2.2 it would give a lower bound to our problem. To investigate this question, we first consider weak k-trees. As we discussed in Section 5.3.2, weak k-trees are an edge-maximal class of connected graphs with treewidth at most k. The following lemma shows that for every weak k-tree, there exist a spanning tree with large pathwidth.

Lemma 6.3.1. *If G is a weak k-tree, then there exists a spanning tree T such that* $\text{pw}(G) \leq (k+1)(\text{pw}(T)+1)-1$.

Proof. Let G be a weak k-tree. By Lemma 5.3.3, there exists a vertex ordering $v_1, v_2, \ldots, v_n$ such that for all $i \geq 2$ we have $G[N^-(v_i)] \simeq K_j$ for some $j \in \{1, \ldots, k\}$. Construct the spanning tree T as follows: let $V(T) = V(G)$ and for all $i \geq 2$, add the edge $v_y v_i \in E(G)$ to T where $y = \max\{j : v_j \in N^-(v_i)\}$. Note that y exists since $|N^-(v_i)| \geq 1$ for all $i \geq 2$. Now suppose that T has a cycle, C. Then the right most vertex would have at least 2 neighbours to its left in T. But this contradicts the construction of T since we only added one edge to the left for each vertex in T. Thus since $|E(T)| = n-1$ and T is acyclic, it follows that T is a spanning tree of G.

Let $S_1, \ldots, S_r$ be a path decomposition of T of minimum width. Let $S'_1, S'_2, .., S'_k$ be obtained by setting $S'_q = S_q \cup \{N^-(v_x) : v_x \in S_q\}$ for all $q \in \{1, \ldots, r\}$. Since $|N^-(v_x)| \leq k$ for $x \in \{1, \ldots, n\}$, it follows that this modification will increase the width of each bag by at most a factor of $(k+1)$.

We claim that $S'_1, S'_2, \ldots, S'_r$ is a path decomposition of G. Since T is a spanning tree, it follows that for all $x \in \{1, \ldots, n\}$, there exists a $d \in \{1, \ldots, r\}$ such that $v_x \in S_d \subseteq S'_d$. Furthermore for all $v_w v_x \in E(G)$ where $w < x$, we have $w \in N^-(v_x)$ and hence $w \in S'_d$. Thus every edge of G is in a bag. It remains to check that $I'_{v_i} = \cup_{v_x \in N^+[v_i]} I_{v_x}$ is an interval. We claim that $T[N^+[v_i]]$ is a connected subtree of T for all $i \in \{1, \dots, n\}$. Suppose not. Let x be minimal such that v_x is disconnected to v_i in $T[N^+[v_i]]$. Now by construction, v_x has an edge to a vertex v_y in T where $y < x$. If $y < i$ then this contradicts the construction of T since the edge $v_x v_i \in E(G)$ should have been chosen (see Figure 6.4a). If $y > i$, then since T is a weak k-tree $v_y \in N^+[v_i]$. But this contradicts v_x being disconnected to v_i since v_y is connected to v_i in $T[N^+[v_i]]$ (see Figure 6.4b). Therefore $N^+[v_i]$ induces a connected subtree of T. Now since the indices of the bags that contains vertices from a connected component of a graph is an interval it follows that I'_{v_i} is an interval for all $i \in \{1, \dots, n\}$. As such, $(S'_1, S'_2, \dots, S'_r)$ is indeed a path decomposition of G.



(a) Case 1: $y < i$. (b) Case 2: $y > i$.

Figure 6.4: Constructing a path decomposition.

□

Corollary 6.3.2. *For every weak k-tree G where $|E(G)| \geq 1$ and $h \in \mathbb{N}$, if* $\mathrm{pw}(G) \geq (k+1)(h+2) - 1$ *then* $T_h \leq_M G$.

Proof. Let G be a weak k-tree with $|E(G)| \geq 2$ and $\mathrm{pw}(G) \geq (k+1)(h+2) - 1$. By Lemma 6.3.1, there exists a spanning tree T such that $\mathrm{pw}(T) \geq h + 1$. By Corollary 6.2.2, we have $T_h \leq_M T$ and hence $T_h \leq_M G$ as required. □

6.4 Open Question

We suspect that the bound achieved for weak k-trees holds for all graphs with treewidth at most k. As such, we have the following conjecture.

Conjecture 6.4.1. *For every integer h and graph G where $|E(G)| \geq 2$ and* $\mathrm{tw}(G) \leq k$, *if* $\mathrm{pw}(G) \geq (k+1)(h+2) - 1$ *then* $T_h \leq_M G$.

If this conjecture holds, then there remains the question as to whether it is best possible. In the next chapter we discuss another conjecture where an answer in the affirmative would give a bound for Conjecture 6.4.1.

Chapter 7

Tree Decompositions Indexed by Subtrees

In this chapter we survey the partial results that we have obtained in investigating tree decompositions indexed by subtrees. We begin by first motivating their study by showing their relationship to complete binary tree minors in graphs with bounded treewidth. We then show that the existence of a tree decomposition with width at most k indexed by a minor implies the existence of a tree decomposition with width at most k indexed by a subtree. This will demonstrate that for our purpose, there is no added value in considering tree decompositions indexed by minors instead of subtrees. We then move on to discuss different weakenings we have establish for conjectures concerning bounds for the minimum width of a tree decomposition indexed by a subtree. Finally, we conclude this chapter with a discussion of a family of graphs that have bounded treewidth, however, we suspect that the minimum width of a tree decomposition indexed by a subtree of the graph is unbounded. If this is so, then it will falsify a conjecture of Dvořák.

7.1 Motivation

We now discuss the connection between tree decompositions indexed by subtrees and complete binary tree minors in graphs with bounded treewidth. Consider the following lemma from Dang and Thomas.

Lemma 7.1.1 ([9]). *Let G be a graph such that $\mathrm{pw}(G) \geq h$. Then for every tree decompositions $(T, \mathcal{W})$ of G with width strictly less than k we have $\mathrm{pw}(T) \geq \lfloor h/k \rfloor$.*

Proof. Let $(T, \mathcal{W})$ be a tree decomposition of G with width strictly less than k. Let $(B'_1, \ldots, B'_m)$ be a path decomposition of T with minimum width. For each $i \in \{1, \ldots, m\}$, let $B_i \subseteq V(G)$ such that $v \in B_i$ if and only if there exists a $t \in B'_i$ such that $v \in W_t$.

We claim that $(B_1, \ldots, B_m)$ is a path decomposition of G. By construction we trivially have **P1** satisfied. Let $v \in V(G)$. Since $(T, \mathcal{W})$ is a tree-decomposition T_v is a non-empty subtree of T. Furthermore, since $(B'_1, B'_2, \ldots, B'_m)$ is a path decomposition of T it follows that $\{i : B'_i \cap V(T_v) \neq \emptyset\}$ is a non-empty interval. By construction of $(B_1, \ldots, B_m)$ it follows that $\{i : v \in B_i\}$ is equal to the previous set and hence is also a non-empty interval. As such **P2** is satisfied. Let $uv \in E(T)$. Then there exists a $t \in V(T)$ such that $u, v \in W_t$. Furthermore, there exists a $j \in \{1, \ldots, m\}$ such that $t \in B'_j$. Thus, we have $u, v \in B_j$ and hence **P3** is satisfied. As such, $(B_1, \ldots, B_m)$ is indeed a path decomposition of G.

Now by construction we have $|B_i| \leq k|B'_i|$ for all $i \in \{1, \ldots, m\}$. Now if for all $i \in \{1, \ldots, m\}$ we have $|B'_i| < h/k$ then $|B_i| < h$. But this contradicts the minimality of the pathwidth. As such there exists a $j \in \{1, \ldots, m\}$ such that $|B_j| \geq h/k$ and hence $\text{pw}(T) \geq \lfloor h/k \rfloor$. □

It follows from this lemma that if a graph G has large pathwidth and there exists a tree decomposition $(T, \mathcal{W})$ of G with small width such that T is a subtree of G then T will also have large pathwidth. The crucial question is when does a graph have a tree decomposition of small width that is indexed by a subtree. This question led to the rediscovery of the following conjecture by Dvořák.

Conjecture 7.1.2 ([25])**.** *There exists a function f such that every connected graph G has a tree decomposition $(T, \mathcal{W})$ of width at most $f(\text{tw}(G))$ for some subtree T.*

Note that the connectivity assumption is essential in order to avoid trivial counterexamples. Now if this conjecture is true then we have a bound for Conjecture 6.4.1 as implied by the following lemma.

Lemma 7.1.3. *Fix $k \in \mathbb{N}$. Suppose there exists a g_k such that every connected graph G with $\text{tw}(G) \leq k$ has a tree decomposition indexed by a subtree T with width strictly less than g_k. Then if $\text{pw}(G) \geq g_k(h+1)$ we have $T_h \leq G$.*

Proof. Let G be a connected graph with $\text{tw}(G) \leq k$ and $\text{pw}(G) \geq g_k(h+1)$. Then by assumption there exists a tree decomposition $(T, \mathcal{W})$ with width strictly less than g_k for some subtree T. As such, by Lemma 7.1.1 we have $\mathrm{pw}(T) \geq \lfloor (g_k(h+1)/g_k) \rfloor \geq h+1$. Hence, by Corollary 6.2.2 we have $T_h \leq_M T$ and since T is a subtree of G we have $T_h \leq_M G$. □

We suspect, however, that Conjecture 7.1.2 is false and that we have found a family of graphs that do not satisfy it. More precisely, we have the following conjecture.

Conjecture 7.1.4. *For every $k \in \mathbb{N}$ there exists a connected graph G_k with $\mathrm{tw}(G_k) \leq 2$ such that every tree decomposition $(T, \mathcal{W})$ of G_k where T is a subtree of G has width at least k.*

7.2 Tree Decompositions Indexed by Minors, Subtrees and Spanning Trees

In the previous section, we argued that if every connected graph has a tree decomposition indexed by a subtree of bounded width then this will give a bound for Conjecture 6.4.1. Now if we consider tree decompositions that are instead indexed by minors, then this will also give a bound for the conjecture. However, the following lemma shows that the existence of a tree decomposition with width at most k indexed by a minor implies the existence of a tree decomposition with width at most k indexed by a subtree. As such, it demonstrates that for our purposes there is no apparent incentive to investigate tree decompositions indexed by minors.

Lemma 7.2.1. *Fix $k \in \mathbb{N}$. For every connected graph G, the following are equivalent:*

1. *There exists a spanning tree $T^{(1)}$ of G and a tree decomposition $(T^{(1)}, \mathcal{W}^{(1)})$ of G with width at most k;*

2. *There exists a subtree $T^{(2)}$ of G and a tree decomposition $(T^{(2)}, \mathcal{W}^{(2)})$ of G with width at most k; and*

3. *There exists a tree $T^{(3)} \leq_M G$ and a tree decomposition $(T^{(3)}, \mathcal{W}^{(3)})$ of G with width at most k.*

Proof. Since a spanning tree of G is also a subtree of G and a subtree of G is also a minor of G we trivially have (1) $\implies$ (2) $\implies$ (3).

It remains to show that (3) $\implies$ (1). Suppose there exists a tree decomposition of G, $(T^{(3)}, \mathcal{W}^{(3)})$, with width at most k such that $T^{(3)} \leq_M G$. By Lemma 3.1.1 and Lemma 3.1.2, there exists a tree-model μ of $T^{(3)}$ in G that contains $V(G)$. Now since μ is a model, for every $t_1t_2 \in E(T^{(3)})$ there exists $x_1 \in \mu(t_1)$ and $x_2 \in \mu(t_2)$ such that $x_1x_2 \in E(G)$. Let X be a set of edges in G that contains an x_1x_2 edge whenever $t_1t_2 \in E(T^{(3)})$ and let $T^{(1)} = \bigcup_{t\in V(T^{(1)})} \mu(t) \cup X$. Then $T^{(1)}$ is a spanning tree of G. We define $\mathcal{W}^{(1)} = \{W_t^{(1)} : t \in V(T^{(1)})\}$ by $W_t^{(1)} = W^{(3)}_{\mu^{-1}(t)}$ for all $t \in V(T^{(1)})$.

We claim that $(T^{(1)}, \mathcal{W}^{(1)})$ is a tree decomposition of G. Now since μ is a model it follows that for every $W_{t_i}^{(3)} \in W^{(3)}$ there exists a $W_{t_j}^{(1)} \in \mathcal{W}^{(1)}$ such that $W_{t_i}^{(3)} = W_{t_j}^{(1)}$. As such, since $(T^{(3)}, \mathcal{W}^{(3)})$ is a tree decomposition of G it follows immediately that $\mathcal{W}^{(1)}$ satisfies **T1** and **T3**. It remains to check whether for all $v \in V(G)$, $T_v^{(1)}$ is an induced subtree of $T^{(1)}$. Note that since $T_v^{(3)}$ is an induced subtree of $T^{(3)}$ and $T_v^{(1)} = T[\{V(\mu(t)) : t \in V(T_v^{(3)})\}]$ it follows that $T_v^{(1)}$ is a subtree of $T^{(1)}$. As such, $(T^{(1)}, \mathcal{W}^{(1)})$ is a tree decomposition of G with width equal to that of $(T^{(3)}, \mathcal{W}^{(3)})$.

□

We now move on to investigate different weakenings of Conjecture 7.1.2.

7.3 Bounds for Graph Classes

The first type of weakening is to consider special classes of connected graphs with bounded treewidth. The first class that we focus on are connected graphs with treewidth at most 1. By Theorem 5.3.1, these graphs are trees. The following result shows that Conjecture 7.1.2 holds for graphs with treewidth at most 1.

Theorem 7.3.1. *For every tree T there exists a tree decomposition of T indexed by itself with width at most 1.*

Proof. We prove the following stronger claim: *for every tree T there exists a tree decomposition $(T, \mathcal{W})$ with width at most 1 such that $t \in W_t$ for all $t \in V(T)$.*

Let T be a tree on n vertices. We proceed by induction on n. For $n = 1$ we have $V(T) = \{t\}$. As such, the tree decomposition $(T, \mathcal{W})$ with $W_t = \{t\}$ satisfies the induction hypothesis.

Now suppose $n > 1$ and let $uv \in E(T)$ where v is a leaf of T. Since $T' = T - v$ is also a tree, it follows by induction that there exists a tree decomposition $(T', \mathcal{W}')$ of width at most 1 such that $t \in W'_t$ for all $t \in V(T')$. In particular, we have $u \in W'_u$. Let $\mathcal{W} = \mathcal{W}' \cup \{W_v\}$ where $W_v = \{u, v\}$. Then $(T, \mathcal{W})$ is a tree decomposition that satisfies the induction hypothesis.

□

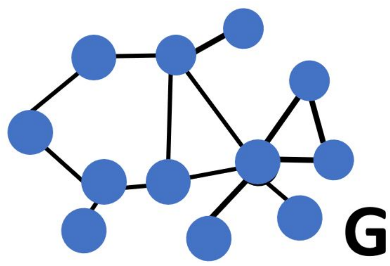


Figure 7.1: Example of a connected outer-planar graph.

The next class of graphs that we now consider are *connected outer-planar graphs*. A graph G is *outer-planar* if there exist an embedding of G on the plane such that every vertex of G lies on the outer-face. We note that the class of outerplanar graphs are minor-closed. We use the following notation to describe the structure of outer-planar graph. We define a *k-pasting* of two graphs, G_1 and G_2, to be a k-sum of G_1 and G_2 where no edges are deleted.

Lemma 7.3.2. *If a graph G is outer-planar then it can be obtained by (≤ 2)-pasting of cycles and paths.*

Proof. Let G be an outer-planar graph. Without loss of generality we may assume that G is connected. We proceed by induction on $n = |V(G)|$. For $n \leq 3$, G is either a path or a cycle so the claims hold.

Suppose $n > 3$. Let (A, B) be a separation of G with the maximum number of edges with ends in the intersection subject to being of minimum order. Let $X = A \cap B$ and for the sake of contradiction suppose $|X| \geq 3$. Then by the minimality of the order and Menger's theorem (Theorem 2.4.1), there exists an $a \in A - B$ and $b \in B - A$ and three internal vertex disjoint paths with ends a and b. By deleting all other vertices not on these paths we obtain a $K_{2,3}$ subdivision. But $K_{2,3}$ is not an outer-planar graph (see Figure 7.2), a contradiction. As such we have $|X| \leq 2$.

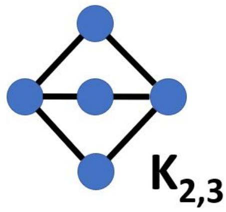


Figure 7.2: $K_{2,3}$ is not an outer-planar graph.

Now if $|X| = 2$ and $x_1, x_2 \in X$ then by the minimality of the separation order G is biconnected. If $x_1x_2 \notin E(G)$ then by the choice of the separation, G is a cycle and as such satisfied our claim.

As such, we may assume that if $|X| = 2$ then $x_1x_2 \in E(G)$. Let $G_1 = G[A]$ and $G_2 = G[A]$. Since $G[X]$ is connected it follows that both G_1 and G_2 are connected subgraphs of G and hence they are both connected outerplanar graphs. By induction, both G_1 and G_2 can be obtained by (≤ 2)-pasting of cycles and paths. As such, since G can be obtained by (≤ 2)-pasting of G_1 and G_2, the result follows by induction. □

Theorem 7.3.3. *For every connected outer-planar graph G, there exists a tree decomposition $(T, \mathcal{W})$ with width at most 2 such that T is a spanning tree of G.*

Proof. Let G be a connected outer-planar graph. By Lemma 7.3.2. G can be obtained by (≤ 2)-pasting of connected graphs $H_1, \dots, H_m$ where either $H_j \simeq C_l$ or $H_j \simeq P_l$ for every $j \in \{1, \dots, m\}$ and some $l \in \mathbb{N}$. Let $G_1 \simeq H_1$ and G_i be obtained as a (≤ 2)-pasting of G_{i-1} and H_i such that $G_m = G$ and G_i is connected for every $i \in \{1, \dots, m\}$.

Claim: *For every $i \in \{1, \dots, m\}$ there exist a rooted spanning tree (T, r) of G_i and a tree decomposition $(T, \mathcal{W})$ of G_i with width at most 2 such that:*

1. *For all $t \in V(G_i)$, we have $t \in W_t$; and*
2. *For all $t \in V(G_i)$, we have $t \notin W_{t'}$ where t' is an ancestor of t in T.*

We proceed by induction on i. For the base case, $i = 1$, G_1 is either a path or a cycle. If $G_1 \simeq P_n$ for some $n \in \mathbb{N}$ with ends $v_1, v_n \in V(G_1)$ then let (P_n, v_1) be the rooted tree to index the tree decomposition. Define the bags as follows: $W_{v_1} = \{v_1\}$ and $W_{v_i} = \{v_{i-1}, v_i\}$ for all $i \in \{2, \dots, n\}$.

By construction, $(T, \mathcal{W})$ where $\mathcal{W} = \{W_t : t \in V(P_n)\}$ satisfies the induction hypothesis.

Otherwise $G_1 \simeq C_n$ for some $n \in \mathbb{N}$. Let $v_1, \ldots, v_n$ the cyclic vertex ordering of G_1. Let (P_n, v_1) be the rooted spanning tree of G_1 to index the tree decomposition. Define the bags as follows: $\mathcal{W}_{v_1} = \{v_1\}$ and $W_{v_i} = \{v_1, v_{i-1}, v_i\}$ for all $i \in \{2, \ldots, n\}$. By construction, $(T, \mathcal{W})$ where $\mathcal{W} = \{W_t : t \in V(P_n)\}$ also satisfies the induction hypothesis.

Now suppose $i > 1$. By construction, G_i is obtained by a (≤ 2)-pasting of G_{i-1} and H_i. By induction, there exists a spanning tree T' of G_{i-1} and a tree decomposition $(T', \mathcal{W}')$ that satisfies the induction hypothesis. There are two cases to consider for H_i. The first is if $H_i \simeq P_n$ for some $n \in \mathbb{N}$. If so, then without loss of generality we may assume that G_i is obtained by a 1-pasting of H_i and G_{i-1}. Let $v^\star \in V(G_{i-1})$ be the vertex to which the 1-pasting was done on. Construct T by a 1-sum of T' and P_n at $v^\star$. Then T is a spanning tree of G_i. Let $\mathcal{W} = \mathcal{W}' \cup \{W_{v_i} : i \in \{2, \ldots, n\}\}$ where $W_{v_i} = \{v_{i-1}, v_i\}$. Then $(T, \mathcal{W})$ is a tree decomposition of G_i that satisfies the induction hypothesis.

Otherwise we have $H_i \simeq C_n$ for some $n \in \mathbb{N}$. Suppose G_i is obtained by a 1-pasting of G_{i-1} and H_i. Similar to our approach before, let $v^\star \in V(G_{i-1})$ be the vertex to which the 1-pasting was performed on and let $v_1, \ldots, v_n$ the cyclic vertex ordering of H_i such that G_i is obtained by identifying v_1 with $v^\star$. Construct T by a 1-sum of T' and P_n at $v^\star$. Then T is a spanning tree of G_i. Let $\mathcal{W} = \mathcal{W}' \cup \{W_{v_i} : i \in \{2, \ldots, n\}\}$ where $W_{v_i} = \{v_{i-1}, v_i, v_1\}$. Then $(T, \mathcal{W})$ is a tree decomposition of G_i that satisfies the induction hypothesis.

The final case to consider is if G_i is obtained by a 2-pasting of G_{i-1} and H_i. Let $u_1, u_2 \in V(G_{i-1})$ be the vertices to which the 2-pasting was performed on. It follows by the induction hypothesis and the fact that $u_1u_2 \in E(G_{i-1})$ that either $u_1 \leq_{T'} u_2$ or $u_2 \leq_{T'} u_1$. Without loss of generality, assume that $u_1 \leq_{T'} u_2$. Then we also have $u_1, u_2 \in W_{u_2}$. Now let $v_1, \ldots, v_n$ be the cyclic vertex ordering of H_i such that G_i is obtained by identifying v_1 with u_1 and v_2 with u_2. Construct T by a 1-sum of T' and P_{n-1} at u_2. In doing so we obtain a spanning tree of G_i. Let $\mathcal{W} = \mathcal{W}' \cup \{W_{v_i} : i \in \{3, \ldots, n\}\}$ where $W_{v_i} = \{v_{i-1}, v_i, v_1\}$. Then $(T, \mathcal{W})$ is a tree decomposition of G_i that satisfies the induction hypothesis. The result therefore follows by induction. □

From this lemma, we obtain an important corollary for the existence of large complete binary tree minors in outer-planar graphs with large pathwidth.

Corollary 7.3.4. *For every outer-planar graph G with $E(G) \geq 1$, if $\mathrm{pw}(G) \geq 3(h+1)$ then $T_h \leq_M G$.*

Proof. Let G be an outer-planar graph with $\mathrm{pw}(G) \geq 3(h+1)$. Without loss of generality we may assume that G is connected. By Lemma 7.3.3, there exists a tree decomposition of G with width at most 2 indexed by a spanning tree T. As such, by Lemma 7.1.1 $\mathrm{pw}(T) \geq \lfloor (3(h+1)/3) \rfloor \geq h+1$. Hence, by Corollary 6.2.2 we have $T_h \leq_M T$ and since T is a spanning tree of G we have $T_h \leq_M G$. □

7.4 Ghost Edges

We now discuss a tool that we suspect may allow us to construct graphs that satisfy Conjecture 7.1.4. Fix $k \in \mathbb{N}$ and let G be a connected graph such that $k \geq \mathrm{tw}(G)$. We say that $uv \in E(G^c)$ is a *k-ghost-edge* of G if for every tree decomposition $(T, \mathcal{W})$ of G with width at most k we have $u, v \in B_t$ for some $t \in V(T)$. If a graph G does not contain any k-ghost-edges, then G is *k-ghost-free*. Let $\mathcal{E}_G^k$ denote the set of ghost edges. The *k-ghosting* of a graph G is $G \cup \mathcal{E}_G^k$.

We call these edges in the complement of G "k-ghost-edges" since they behave like real edges with respect to tree decomposition with width at most k in the sense that the endpoints of a k-ghost-edge always appear in a common bag. The following lemma provides a tool for constructing ghost edges.

Lemma 7.4.1. *Let G be a graph and let $k \in \mathbb{N}$ such that $k \geq tw(G)$. If $uv \in E(G^c)$ and there exists at least $k+1$ internally vertex disjoint paths with ends u and v in G then uv is a k-ghost-edge of G.*

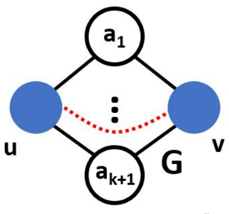


Figure 7.3: Ghost edge: $uv \in E(G^c)$.

Proof. Let $uv \in E(G^c)$ such that there exists at least k internal disjoint paths with the ends u and v. Suppose there exists a tree decomposition

$(T, \mathcal{W})$ of width at most k such that $T_u \cap T_v = \emptyset$. Let $t_1 \in V(T_u)$ and $t_2 \in V(T_v)$ be the vertices that realise the closest distance between T_u and T_v. Let $P^{(1)}, \dots, P^{(k+1)}$ be the internal vertices of the vertex disjoint paths in G between u and v. Let $T_{P^{(i)}} = T[\{t \in V(T) : x \in B_t \cap V(P^{(i)})\}]$ for all $i \in \{1, \dots, k+1\}$. As each path is connected, it follows that $T_P^{(i)}$ is a subtree of T. Now since both u and v are adjacent to a vertex in $P^{(i)}$, it follows that there exists an $x_1^{(i)} \in V(T_{P^{(i)}}) \cap V(T_u)$ and $x_2^{(i)} \in T_{P^{(i)}} \cap V(T_v)$. Since for every $x \in V(T_u)$ and $y \in V(T_v)$, t_1 is in the (x, y)-path in T it follows that $t_1, \in V(T_{P^{(i)}})$ for every $i \in \{1, \dots, k+1\}$. As these paths are vertex internal disjoint, we have $|W_{t_1}| \geq (k+1) + 1 = k+2$, a contradiction. $\square$

We suspect that the converse of this lemma holds as well. That is, if there exists at most k internally vertex disjoints paths with ends u and v in G then uv is not a k-ghost-edge of G. Furthermore, we suspect that Conjecture 7.1.2 holds for all k-ghost-free graphs. In particular, we have the following conjecture.

Conjecture 7.4.2. *Fix $k \in \mathbb{N}$. For every connected k-ghost-free graph G with $\operatorname{tw}(G) \leq k$ there exists a tree decomposition $(T, \mathcal{W})$ with width at most k such that T is a spanning tree of G.*

If this holds then the following statement immediately follows which would be a step towards proving Conjecture 7.1.2.

Corollary 7.4.3. *Fix $k \in \mathbb{N}$. For every connected graph G with $\operatorname{tw}(G) \leq k$ there exists a tree decomposition $(T, \mathcal{W})$ with width at most k such that T is a spanning tree of the k-ghosting of G.*

7.5 Home-Base Assumption

The next form of weakening that we now consider for Conjecture 7.1.2 is with a *home-base assumption*. That is, we are considering tree decompositions that are indexed by a spanning tree such that every vertex is contained within the bag indexed by itself. This is a natural assumption to make to understand the properties of tree decompositions indexed by subtrees. The following result motivated us to suspect that conjecture 7.1.2 is false.

Theorem 7.5.1. *Fix $c \in \mathbb{N}$. There exists a connected graph G_{c+1} such that $\operatorname{tw}(G_{c+1}) \leq 2$ and for every spanning tree T of G_{c+1}, if $(W_x : x \in$*

$V(T))$ is a tree decomposition of G_{c+1} and $x \in W_x$ for all $x \in V(G_{c+1})$, then $(W_x : x \in V(T))$ has width at least c.

Proof. We construct $G_1, \ldots, G_{c+1}$ as follows. Let $G_1 = K_{2,c+1}$ and $X_1 = \{a_1, b_1\}$ be the set of two vertices of G_1 with degree $c+1$. Let $\{u_1, \ldots, u_{c+1}\}$ be the set of vertices of G_1 with degree 2. To construct G_{i+1}, let $G_i^{(1)}, \ldots, G_i^{(c+2)}$ be copies of G_i. G_{i+1} is obtained by the disjoint union of $G_i^{(1)} \cup \cdots \cup G_i^{(c+2)}$ with the added vertices $X_{i+1} = \{a_{i+1}, b_{i+1}\}$ and the edges $\{a_{i+1}a_i^{(j)}, b_{i+1}b_i^{(j)} : j \in \{1, \ldots, c+2\}\}$ (see Figure 7.4). We now show that $\text{tw}(G_i) \leq 2$ for all $i \in \{1, \ldots c+1\}$.

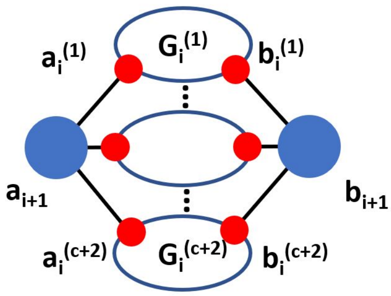


Figure 7.4: Home-base graph.

Claim: *For every $i \in \{1, \ldots, c+1\}$ there exists a tree decomposition $(T^{(i)}, \mathcal{W}^{(i)})$ of G_i with width at most 2 and there exists $t \in V(T^{(i)})$ such that $W_t^{(i)} = \{a_i, b_i\}$.*

We proceed by induction on i. For $i = 1$, Let $T^{(1)} = K_{1,c+1}$ (that is, the star graph with $c+1$ leaves), $v \in V(T^{(1)})$ such that $\deg(v) = c+1$, and $\{t_1, \ldots, t_{c+1}\} = V(T^{(1)}) \setminus \{v\}$. We construct the tree decomposition of G_1 as follows: Let $W_v^{(1)} = \{a_1, b_1\}$ and $W_{t_j}^{(1)} = \{a_1, b_1, u_j\}$ for all $j \in \{1, \ldots c+2\}$. Then $(T^{(1)}, \mathcal{W}^{(1)})$ where $\mathcal{W}^{(1)} = \{W_t : t \in V(T^{(1)})\}$ defines a tree decomposition of width 2 which satisfies the induction hypothesis.

Now suppose $i > 1$. Let $G_{i-1}^{(1)}, \ldots, G_{i-1}^{(c+2)}$ be the connected components of $G_i - X_i$. By induction it follows that for all $j \in \{1, \ldots, c+2\}$ there exists a tree decomposition $(T^{(i-1,j)}, \mathcal{W}^{(i-1,j)})$ of $G_{i-1}^{(j)}$ with width at most 2 and $t_j \in V(T^{(i-1,j)})$ such that $W_{t_j}^{(i-1,j)} = \{a_{i-1}^{(j)}, b_{i-1}^{(j)}\}$. For each $j \in \{1, \ldots, c+2\}$, we modify $T^{(i-1,j)}$ by adding a path of length 4 to $T^{(i-1,j)}$ with the end t_j. Index the following sequence of four bags by the vertices that was added to $T^{(i-1,j)}$: $(\{a_i, b_i\}, \{a_i, a_{i-1}^{(j)}, b_i\}, \{b_i, a_{i-1}^{(j)}, b_{i-1}^{(j)}\}, \{a_{i-1}^{(j)}, b_{i-1}^{(j)}\})$. Let $T^{(i)}$ be obtained from $T^{(i-1,1)} \cup \cdots \cup T^{(i-1,c+1)}$ by identifying all the vertices that indexes the $\{a_i, b_i\}$ bags. Then $(T^{(i)}, \mathcal{W}^{(i)})$ where $\mathcal{W}^{(i)} = (W_t : t \in V(T^{(i)}))$ is a tree decomposition that satisfies the induction hypothesis as required.

We now move on to consider tree decomposition indexes by spanning trees with the home-base assumption.

Claim: *Suppose for every $i \in \{1, \ldots, c+1\}$ there exist a tree decomposition $(T_i, \mathcal{W}_i)$ of G_i of width at most $c+1$ such that T_i is a spanning tree of G_i and $x \in W_x$ for all $x \in V(G)$. Then for every $i \in \{1, \ldots, c+1\}$, there exists a vertex $t^\star$ on the (a_i, b_i)-path in T such that $|W_t| \geq i$.*

We prove the following claim by induction on i. For $i = 1$, the claim holds trivially by the assumption that $x \in W_x$ for every $x \in V(G)$. Now suppose $i > 1$ and let $(T_i, \mathcal{W}_i)$ be a tree decomposition of G_i with width at most $c+1$ such that T_i is a spanning tree of G_i and $x \in W_x$ for all $x \in V(G)$. Now consider the (a_i, b_i)-path in T. As the connected components of $G_i - X_i$ are copies of G_{i-1}, it follows that the internal vertices of this path must be fully contained within $V(G_{i-1}^{(j)})$ for some $j \in \{1, \ldots, c+2\}$. Let $T' = T[V(G_{i-1}^{(j)})]$ and let $W'_t = W_t \cap V(G_{i-1}^{(j)})$ for all $t \in V(T')$.

Claim: *$(T', \mathcal{W}')$ is an induced tree decomposition of $G_{i-1}^{(j)}$ with $x \in W_x$ for all $x \in V(G_{i-1}^{(j)})$.*

Note that by the construction of (T, W), we have $t \in W_t$ for all $t \in V(T)$. As such, we have $t \in W'_t$ for all $t \in V(T')$ and so by the construction of $(T', \mathcal{W}')$, it follows that **T1** holds. Let $uv \in E(G_{i-1}^{(j)})$. Since $(T, \mathcal{W})$ is a tree decomposition we have $V(T_u) \cap V(T_v) \neq \emptyset$. For the sake of contradiction, suppose that $(V(T_u) \cap V(T_v)) \cap V(G_{i-1}^{(j)}) = \emptyset$. Then there exist $y \in V(G_i) \setminus V(G_{i-1}^{(j)})$ such that $u, v \in W_y$. Since T_u is connected, for every vertex t on the (u, y)-path in T we have $u \in W_t$. Similarly, for every t' on the (v, y)-path in T we have $v \in W'_t$. Now as either a_{i-1} or b_{i-1} are on both of those path, it follows that either $u, v \in W_{a_{i-1}}$ or $u, v \in W_{b_{i-1}}$ which contradicts our assumption that $(V(T_u) \cap V(T_v)) \cap V(G_{i-1}^{(j)}) = \emptyset$. As such, **T3** is satisfied.

It remains to check that for every $v \in V(G_{i-1}^{(j)})$, $T'_v = T_v[V(G_{i-1}^{(j)})]$ is a connected subtree of T'. For the sake of contradiction, suppose there exists a $v \in V(G_{i-1}^{(j)})$ such that T'_v is disconnected. Now since T_v is connected, the only way for T'_v to be disconnected is if it contains an (a_{i-1}, b_{i-1})-path that is internally vertex-disjoint from $V(G_{i-1}^{(j)})$. However, by the choice of $j \in \{1, \ldots, c+2\}$, there exist a (a_{i-1}, b_{i-1})-path that is contained within $V(G_{i-1}^{(j)})$. As such, we have two internal disjoint (a_{i-1}, b_{i-1})-paths in T which defines a cycle in T. But this contradicts T being acyclic. It therefore follows that T'_v is connected for all $v \in V(G_{i-1}^{(j)})$ and hence **T2** is satisfied. As such, $(T', \mathcal{W}')$ is an induced tree decomposition of $G_{i-1}^{(j)}$ which satisfies the claim.

Now by induction there exist a vertex $t^\star$ on the (a_{i-1}, b_{i-1})-path in T' such that $|W'_{t^\star}| \geq i-1$. Now since there exists at least $c+1$ internal disjoint paths with ends a_i and b_i in G_i it follows by Lemma 7.4.1 that there exists a $t \in V(T)$ such that $a_i, b_i \in W_t$. As such, since T_{a_i} and T_{b_i} are both connected, it follows that for every t on the (a_i, b_i)-path in T we have $|W_t \cap X_i| \geq 1$. In particular, as $t^\star$ is on that path we have $|W_{t^\star}| \geq i - 1 + 1 \geq i$ as required. □

7.6 Counterexample?

Theorem 7.5.1 motivated us to suspect that Conjecture 7.1.2 is false. We are therefore left with two challenges to meet. The first is finding a suitable family of graphs that do not satisfy the conjecture. This may be done by finding a family of graphs whose tree decompositions of bounded width contains a property similar to the home-base assumption that we had. The second is then proving that it does not satisfy Conjecture 7.1.2. We will conclude this thesis with a discussion of a family of graphs that we suspect realises Conjecture 7.1.4, however, we have not been able to prove it yet.

We construct the family of graphs $G_1, \ldots, G_{k+1}, \ldots$ recursively as follows. For G_1, let $X_1 = \{a_1, b_1, c_1\}$ be a set of vertices. We obtain G_1 by adding $k+1$ vertices $v_1, \ldots, v_{k+1}$ for every distinct vertex pair $x, y \in X$ such that $N(v_j) = \{x, y\}$ for every $j \in \{1, \ldots, k+1\}$. Similarly, to construct G_{i+1}, let $X_{i+1} = \{a_{i+1}, b_{i+1}, c_{i+1}\}$ be a set of vertices. Let $G_i^{(1)}, \ldots, G_i^{(3m)}$ be copies of G_i for some $m > k|V(G_{i-1})|$. We obtain G_{i+1} by adding m vertex disjoint copies of G_i for every distinct vertex pair $x, y \in X$ with the additional edges $\{xa_i^{(j)}, yb_i^{(j)}\}$ for every $j \in \{1, \ldots, m\}$ to obtain G_{i+1} (see Figure 7.5). Note that there is only three distinct vertex pairs in X_i since $\binom{2}{3} = 3$.

We now show that $\mathrm{tw}(G_i) \leq 2$ for all $i \in \mathbb{N} \setminus \{0\}$.

Lemma 7.6.1. *For every $i \in \mathbb{N} \setminus \{0\}$, $\mathrm{tw}(G_i) \leq 2$.*

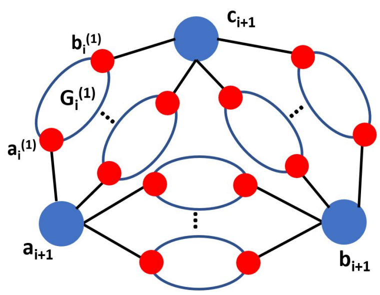


Figure 7.5: Suspected class of graphs that satisfies Conjecture 7.1.4.

Proof. We prove the following claim:

Claim: *For every $i \in \mathbb{N}\setminus\{0\}$ there exists a tree decomposition of G_i, $(T^{(i)}, W^{(i)})$ with width 2 such that there exists $t \in V(T^{(i)})$ such that $a_i, b_i, c_i \in W_t^{(i)}$.*

We proceed by induction on i. For $i = 1$, Let $\{u_1, \dots u_\ell\} = V(G_1) \setminus X_1$, $T^{(1)} = K_{1,\ell}$ and choose $v \in V(T^{(1)})$ such that $\deg(v) = \ell$. Let $\{t_1, \dots t_\ell\} = V(T^{(1)}) \setminus \{v\}$. We construct the tree decomposition of G_1 as follow. Let $W_v^{(1)} = \{a_1, b_1, c_1\}$ and $W_{t_j}^{(1)} = \{x_j, y_j, u_j\}$ for all $j \in \{1, \dots, \ell\}$ where $N(u_j) = \{x_j, y_j\} \subseteq X_1$. Then $(T^{(1)}, \mathcal{W}^{(1)})$ where $\mathcal{W}^{(1)} = \{W_t : t \in V(T^{(1)})\}$ defines a tree decomposition of width 2 which satisfies the induction hypothesis.

Now suppose $i > 1$. Let $G_{i-1}^{(1)}, \dots, G_{i-1}^{(c+1)}$ be the connected components of $G_i - X_i$. By induction it follows that for all $j \in \{1, \dots, c+1\}$ there exists a tree decomposition $(T^{(i-1,j)}, \mathcal{W}^{(i-1,j)})$ of $G_{i-1}^{(j)}$ with width at most 2 such that there exists $t_j \in V(T^{(i-1,j)})$ such that $W_{t_j}^{(i-1,j)} = \{a_{i-1}^{(j)}, b_{i-1}^{(j)}\}$.

For each $j \in \{1, \dots, c+1\}$, we modify $T^{(i-1,j)}$ by adding a path of length 4 to $T^{(i-1,j)}$ with the end t_j. Index the following sequence of four bags by the vertices that was added to $T^{(i-1,j)}$: $(\{a_i, b_i, c_i\}, \{x_i, a_{i-1}^{(j)}, y_i\}, \{y_i, a_{i-1}^{(j)}, b_{i-1}^{(j)}\}, \{a_{i-1}^{(j)}, b_{i-1}^{(j)}\})$. Let $T^{(i)}$ be obtained from $\bigcup_{j \in \{1,\dots,c+1\}} T^{(i-1,j)}$ by identifying all the vertices that indexes the $\{a_i, b_i, c_1\}$ bags. Then $(T^{(i)}, \mathcal{W}^{(i)})$ defines a tree decomposition that satisfies the induction hypothesis as required. □

We explored two approaches to try and prove that $G_1, \dots, G_k, \dots$ satisfies Conjecture 7.1.4. The first approach was to try and answer the following question:

For every $k \in \mathbb{N}$, does there exist a property Y_k such that every tree which indexes a tree decomposition of G_k with width at most k has this property but no subtree of G_k has this property?

If this can be shown then it follows that the set of subtrees of G_k and the set of trees which indexes tree decompositions of G_k with width at most k are mutually exclusive. As such, it will follow that G_k realises Conjecture 7.1.4. This approach, however, did not lead to fruition due to the difficultly of characterising the family of trees that index a tree decomposition of G_k with width at most k. When exploring this question, we found that the trees which indexes a tree decomposition of minimum width may be well characterised. However, for trees which indexes tree decompositions of width much larger than the treewidth of a graph, it does not seem clear what structural properties they will have as there seems to be a lot of fluidity with their structure.

The second approach that we explored parallels the method that we used under the home-base assumption. To apply this method, we assume that a tree decomposition of G_k that is indexed by a subtree with width at most k exist and our goal is to show that there exists an unbounded function h such that the tree decompositions of G_k has a bag of size $h(k)$. To show that such a function exist would require an appropriate induction hypothesis. However, in investigating this we were unable to deduce an appropriate induction hypothesis. We suspect that this approach does have potential in demonstrating that such a function h exists and thus proving that Conjecture 7.1.2 is indeed false.

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