Exercise 2.3.2

Numbers of the form $t_n = \frac{n(n+1)}{2}$ are called triangular numbers, because marbles arranged in an equilateral triangle, n on a side, will total $\sum_{i=1}^{n} i$ marbles, which we saw in Exercise 2.3.1(a) is t_n marbles. For example, bowling pins are arranged in a triangle 4 on a side and there are $t_4 = \frac{4\times 5}{2} = 10$ pins. Show by induction on n that $\sum_{j=1}^{n} t_j = \frac{n(n+1)(n+2)}{6}$.

Basis case: LHS when n = 1 yields 1.

RHS when n = 1 is 1
LHS = RHS therefore s(n) is true when n is 1.
Inductive step: Assume that n $\geq$ 1 and that s(n) is true as proved above. I must prove that s(n + 1) is true which is:

$$\sum_{j=1}^{n+1} t_j = \frac{(n+1)(n+2)(n+3)}{6} \qquad (2)$$

Replacing the RHS of the above equation with the following:

$$\sum_{j=1}^{n+1} t_j = \left(\sum_{j=1}^{n} t_j\right) + \frac{n(n+1)}{2}$$

which can be rewritten as:

$$\sum_{j=1}^{n+1} t_j = \frac{n(n+1)(n+2)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{2n(n+1)(n+2)+6n(n+1)}{12}$$

$$= \frac{n(n+1)(2(n+2)+6)}{12}$$

$$= \frac{n(n+1)(2n^2+2n+6)}{6}$$

$$= \frac{(n+1)(n^2+2n+6)}{6}$$

The final equation the right is equal to (2) proving that s(n + 1) is also true. Therefore s(n) holds for all n $\geq$ 1